Applied Bayesian Modeling module 3: **Bayesian inference for 1 continuous parameter "Everything's normal"**

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- We draw statistical conclusions about parameters of interest using probability statements.
- General approach for some outcome of interest θ, i.e. regression coefficient, is based on learning via Bayes' rule
  - start off with *prior* probability distribution to quantify information related to θ
  - collect data, and use Bayes' rule to update the prior into the posterior distribution
  - use posterior distribution to draw conclusions

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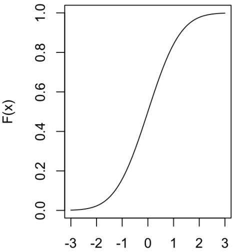

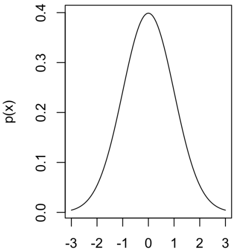

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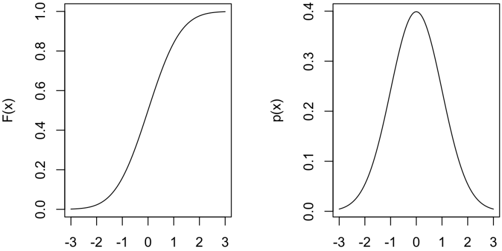

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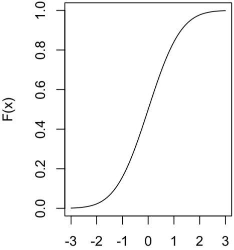

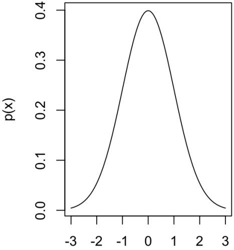

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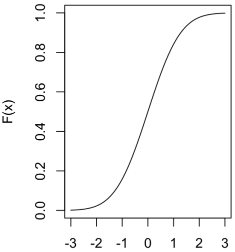

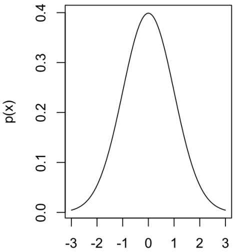

Main differences between discrete and continuous pdfs:

- $p(x)$ is NOT the probability that $X = x$,
- Sums are replaced by integrals

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$$F_{X,Y}(x, y) = Pr(X \leq x \cap Y \leq y) = \int_{x'=-\infty}^{x} \int_{y'=-\infty}^{y} p_{X,Y}(x', y') dx' dy'.$$

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- Conditional pdf is defined as $p(x|y) = p(x,y)/p(y)$
- From the definition of conditional pdf's, we can obtain Bayes' rule:

$$p(x|y) = p(x,y)/p(y) = p(y|x)p(x)/p(y)$$

  (compare to Bayes' rule for events/discrete RV)

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    - i.e., for normal distribution:
    $\mu \sim N(m, s^2)$ is equivalent to $p(\mu) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-1}{2s^2}(\mu - m)^2\right)$

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- From intro:
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  - Radon levels vary greatly across US homes.
  - Data:
    - Radon measurements in over 80K houses throughout the US (we focus on Minnesota)
    - Possible predictors: floor (basement or 1st floor) in the house, soil uranium level at county level.
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- This module: estimate mean radon, assuming that all log-radon measurements are independent draws from a normal distribution.

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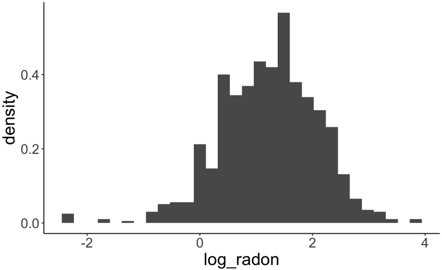

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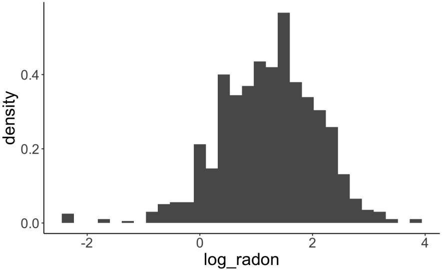

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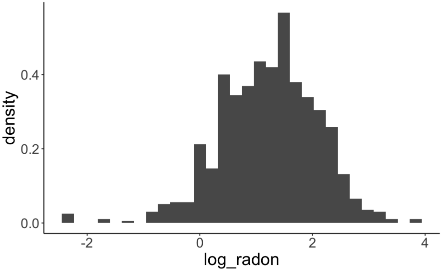
$$p(y_i|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-1}{2\sigma^2}(\mu - y_i)^2\right). \quad (2)$$

- Goal: estimate mean log-radon level μ, assume that σ^2 is known.



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- Details in next module, current focus is on seeing how posterior depends on prior and data, and what to do with it.

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$y_i|\mu, \sigma^2 \sim N(\mu, \sigma^2)$; $\mu \sim N(m_0, s_{\mu 0}^2)$; $\mu|\boldsymbol{y}, \sigma^2 \sim N\left(\frac{m_0/s_{\mu 0}^2 + n \cdot \bar{y}/\sigma^2}{1/s_{\mu 0}^2 + n/\sigma^2}, \frac{1}{1/s_{\mu 0}^2 + n/\sigma^2}\right)$

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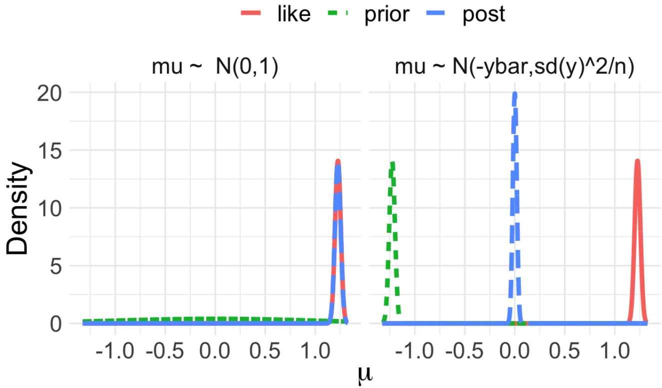
- Use radon data and set $\sigma = s\{y\} = 0.86$, the st.dev. of the y_i's
- Results based on 2 different priors:
  $\mu \sim N(0,1)$ [LEFT] and $\mu \sim N(-\bar{y}, s\{y\}^2/n)$ [RIGHT]



Bayesian inference for μ: Examples with different priors

$y_i|\mu,\sigma^2 \sim N(\mu,\sigma^2)$; $\mu \sim N(m_0, s^2_{\mu 0})$; $\mu|\boldsymbol{y},\sigma^2 \sim N\left(\frac{m_0/s^2_{\mu 0}+n\cdot\bar{y}/\sigma^2}{1/s^2_{\mu 0}+n/\sigma^2}, \frac{1}{1/s^2_{\mu 0}+n/\sigma^2}\right)$

- Use radon data and set $\sigma = s\{y\} = 0.86$, the st.dev. of the y_i's
- Results based on 2 different priors:
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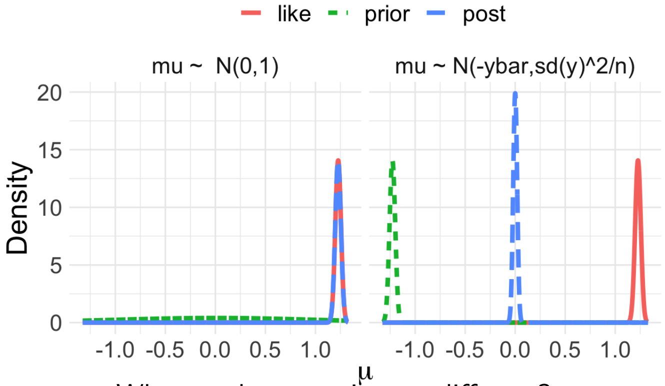

Why are the posteriors so different?

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- More details on this in the next module

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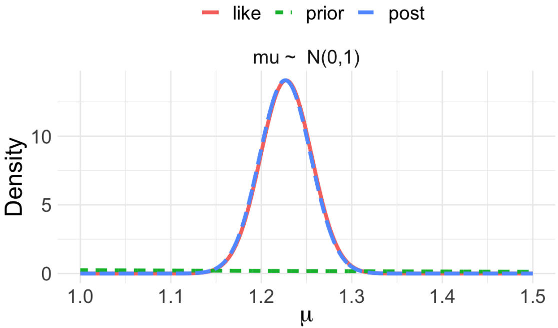

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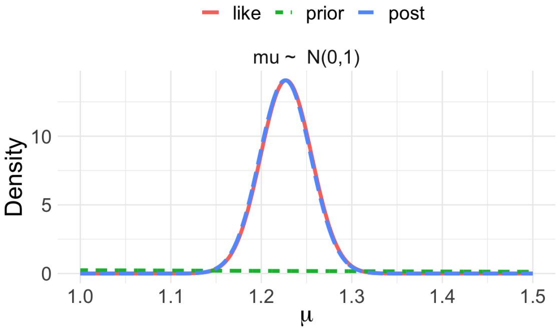

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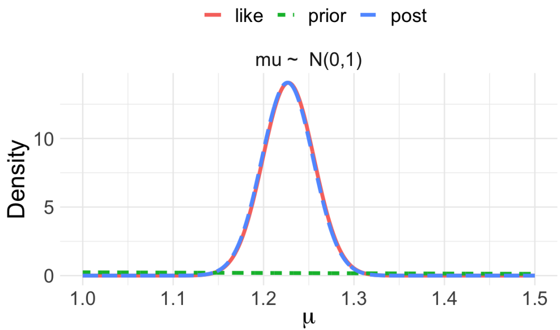

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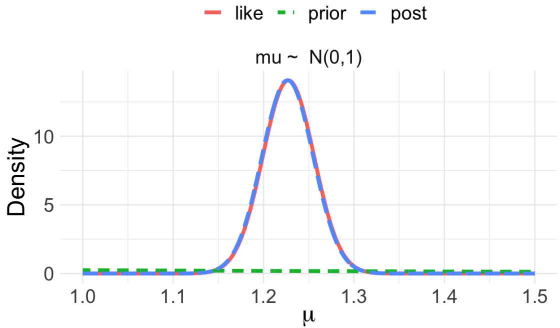

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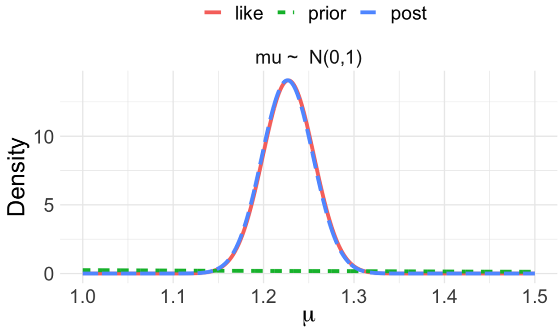

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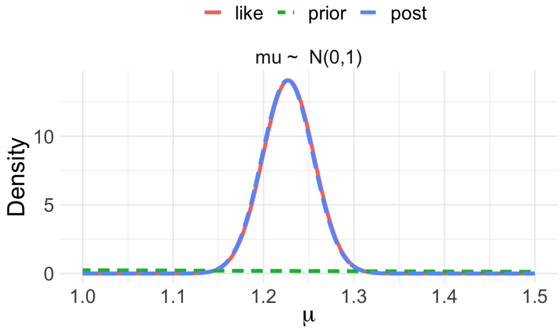

- Here $E(\mu|\boldsymbol{y}) =$ median $= 1.23$.

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- For the radon example with $\mu \sim N(0,1)$, the quantile-based 95% CI is (1.17, 1.28).

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  (1) relate data $\boldsymbol{y}$ to μ through a likelihood function $p(\boldsymbol{y}|\mu)$
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- Model set-up in this module: everything's normal;
  When data and prior are normal, the posterior is normal too.
- Next module: derive the posterior using Bayes' rule, discuss the role of prior information, bias-variance trade-off.