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Discrete Morse Theory

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Contents

Introduction

Topology studies equivalences between topological spaces. In this thesis, equivalence means transforming one space into another by shrinking or expanding, but without cutting, creating holes or filling holes. This notion is often called *continuous deformation*. Morse theory was first developed by Marston Morse in the 1920s for manifolds [13]. The key idea of the theory is to define a smooth function f on a manifold M: the critical points of f relate to the topology of M [12]. Morse theory demonstrated its power when in 1961 Stephen Smale used it to prove the h-Cobordism Theorem [16].

Robin Forman first introduced discrete Morse theory on CW-complexes in 1998 [6]. Over the past two decades, discrete Morse theory has been applied extensively in pure and applied mathematics, including in areas such as combinatorics [2] and computational topology [3]. A relatively recent field of study is the examination of complex systems characterized by "higher-order interactions" [1], including neural systems [15], biomolecular structures [11] and social interactions [9]. Discrete Morse theory can be employed to calculate the topological features of these models.

The objective of this thesis is to present a self-contained introduction to discrete Morse theory and illustrate its application in the analysis of a basic social interaction.

Chapter 1 discusses the objects on which we will develop the theory: simplicial complexes, consisting of points, lines, triangles and their higher-dimensional analogues. Despite Forman originally developed the theory on CW-complexes [6], we formulate the theory for simplicial complexes. The latter are more suited for computations thanks to their combinatorial approach: this makes them a good choice for applications.

Chapter 2 provides an introduction to the algebro-topological tools that are employed for the analysis of the aforementioned structures.

Chapter 3 presents the fundamental concepts of discrete Morse theory and its main theorems, concluding with an illustration of how the theory can markedly reduce the computational burden required for the calculation of topological features.

This thesis is mainly based on the works of Forman [6], Dey and Wang [3], and Scoville [14].

1. Simplicial Complexes

An efficient method for representing a topological space is to break it down into simple pieces. This decomposition is referred to as a *complex* if the intersections of the aforementioned pieces are lower-dimensional pieces of the same kind. In order to align with the computational objectives of this study, we will focus on complexes with many simple components, which are known as *simplicial complexes*. As we will see, the defining characteristics of these complexes are given by the relations between the vertices, and therefore they can be easily represented on a computer.

The present thesis will consider only finite simplicial complexes, as its objective is to develop discrete Morse theory for computational applications. Consequently, the infinite case will not be addressed, as it would introduce unnecessary complexity that would exceed the scope of this study.

There are two ways of talking about a simplicial complex. The first is purely combinatorial (abstract simplicial complex), the second is more geometrical (geometrical simplicial complex). We will show that the two ideas are equivalent, allowing us to discuss the simplicial complex as a unified object.

For this chapter we refer to the work of Dey and Wang [3], and to the work of Edelsbrunner and Harer [5].

1.1 Abstract Simplicial Complex

Definition 1.1.1. An **abstract simplicial complex** is a finite collection of non-empty sets K closed under taking subsets. In other words, given a set S, $K \subseteq \mathcal{P}(S) \setminus \{\varnothing\}$ such that

$$\forall \tau, \forall \sigma, [\varnothing \neq \tau \subseteq \sigma \wedge \sigma \in K] \Rightarrow [\tau \in K].$$

Moreover, $\mathrm{vert}(K) := \bigcup_{\sigma \in K} \sigma$ is the set of **vertices** of K. A subset of $\mathrm{vert}(K)$ of cardinality $n+1$ is called a n-**dimensional simplex** or n-**simplex**. The **dimension** of K, denoted by $\dim(K)$, is the maximum of the dimensions of all its simplices.

Remark 1.1.2. In the literature, one may find a definition of abstract simplicial complex that includes the empty set. The decision to exclude it was made in alignment with the literature on discrete Morse theory, which was considered for this work.

Example 1.1.3. Consider the set $S = \{v_0, v_1, v_2\}$. Examples of 0, 1 and 2-dimensional abstract simplicial complexes are respectively $\{\{v_0\}\}$, $\{\{v_0\}, \{v_1\}, \{v_0, v_1\}\}$, $\{\{v_0\}, \{v_1\}, \{v_2\}, \{v_0, v_1\}, \{v_0, v_2\}, \{v_1, v_2\}, \{v_0, v_1, v_2\}\}$.

Example 1.1.4. An abstract simplicial complex is an appropriate model for social interactions, representing groups and connections within a network. These interactions can occur in real life or on a social media platform. Consider the following scenario, inspired by the work of Kee et al. [10], involving six individuals.

1. **The Flatmates.** Bob (B), Charlie (C), and Dave (D) are flatmates who form a closely connected group. Their strong mutual interaction is represented by the 2-simplex $\{B, C, D\}$. This implies that all pairwise interactions among them also exist: $\{B, C\}$, $\{B, D\}$, and $\{C, D\}$. Furthermore, individual entities $\{B\}$, $\{C\}$, and $\{D\}$ are included. The social structure of this trio forms the abstract simplicial complex:
$$\{\{B\}, \{C\}, \{D\}, \{B, C\}, \{B, D\}, \{C, D\}, \{B, C, D\}\}.$$
2. **Bob and Charlie's Connection with Alice.** Bob and Charlie have a strong group bond with their old friend Alice (A). Their interactions can be represented by the complex:
$$\{\{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}, \{A, B, C\}\}$$
3. **Charlie and Dave's Gaming Circle.** Charlie and Dave often play video games with Frank (F), forming a gaming trio represented by:
$$\{\{C\}, \{D\}, \{F\}, \{C, D\}, \{C, F\}, \{D, F\}, \{C, D, F\}\}.$$
4. **Dave and Eve's Academic Bond.** Dave and Eve (E) study together occasionally, forming a simpler interaction represented by:
$$\{\{D\}, \{E\}, \{D, E\}\}.$$
5. **Eve and Frank's Tennis Games.** Eve and Frank meet weekly to play tennis together. Their connection adds the simplex $\{\{E\}, \{F\}, \{E, F\}\}$.

The social system described above can be represented as the union of the abstract simplicial complexes corresponding to each subgroup, capturing all individual and group interactions:
$$\begin{aligned} K = &\{\{A\}, \{B\}, \{C\}, \{D\}, \{E\}, \{F\}, \\ &\{A, B\}, \{A, C\}, \{B, C\}, \{B, D\}, \\ &\{C, D\}, \{C, F\}, \{D, F\}, \{D, E\}, \{E, F\}, \\ &\{A, B, C\}, \{B, C, D\}, \{C, D, F\}\}. \end{aligned}$$

Definition 1.1.5. Two abstract simplicial complexes K, L are said to be **isomorphic** if there is a bijective function $\varphi : \mathrm{vert}(K) \to \mathrm{vert}(L)$ such that
$$\forall \sigma, [\varphi(\sigma) \in L \iff \sigma \in K].$$
We will denote it by $K \cong L$.

1.2 Geometric Simplicial Complex

Definition 1.2.1. Let $n \in \mathbb{N}$ be a natural number. Let $p_0, \dots, p_n \in \mathbb{R}^d$ be affinely independent points. The **simplex** spanned by $p_0, \dots, p_n$ is the convex hull of $p_0, \dots, p_n$, i.e.

$$\sigma = \operatorname{conv}(\{p_0, \dots, p_n\}) = \left\{ \sum_{i=0}^{n} t_i \; p_i \in \mathbb{R}^d \mid 0 \leq t_i \leq 1 \text{ and } \sum_{i=0}^{n} t_i = 1 \right\} \subseteq \mathbb{R}^d.$$

The **dimension** of σ is n.

Definition 1.2.2. A **geometric simplicial complex** is a set K of a finite number of simplexes such that:

1. $\forall \tau, \forall \sigma, [\varnothing \neq \tau \subseteq \sigma \wedge \sigma \in K] \Rightarrow [\tau \in K]$;
2. $\forall \sigma_1, \sigma_2 \in K, [\sigma_1 \cap \sigma_2 \neq \varnothing \Rightarrow \sigma_1 \cap \sigma_2 \in K]$.

The **dimension** of K is the maximum of the dimensions of all its simplices.

Example 1.2.3. In Figure 1.1 it is given an example of some geometric simplicial complexes.

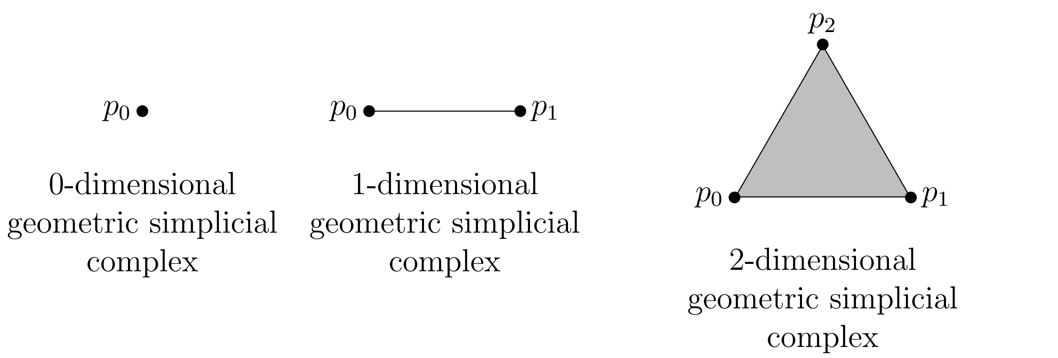

Figure 1.1: 0-dimensional, 1-dimensional and 2-dimensional geometric simplicial complexes.

1.3 Geometric Realization Theorem

This section explains the equivalence between abstract and geometric simplicial complexes.

Remark 1.3.1. Given a geometric simplicial complex K, one can obtain an abstract simplicial complex simply by mapping every geometric simplex spanned by $p_{i_0}, \dots, p_{i_k}$ to the set $\{p_{i_0}, \dots, p_{i_k}\}$. This map is clearly a bijection. The associated abstract simplicial complex is called the **vertex scheme** of K.

Example 1.3.2. The vertex scheme of the 2-dimensional geometric simplicial complex from Example 1.2.3 is

$$\begin{aligned}&\{\{p_0\},\{p_1\},\{p_2\},\\&\{p_0,p_1\},\{p_0,p_2\},\{p_1,p_2\},\\&\{p_0,p_1,p_2\}\}.\end{aligned}$$

Note that the vertex scheme above is isomorphic to the 2-dimensional abstract simplicial complex in Example 1.1.3 accordingly to Definition 1.1.5. This suggests an equivalence between the two structures.

Definition 1.3.3. A **geometric realization** of an abstract simplicial complex K is a geometric simplicial complex $K' \subseteq \mathbb{R}^d$ such that given L the vertex scheme of K', then K is isomorphic to L.

Example 1.3.4. A geometric realization of the abstract simplicial complex of Example 1.1.4 is given in Figure 1.2. To simplify notation, let us denote the vertices as follows: $v_0 := A, v_1 := B, v_2 := C, v_3 := D, v_4 := F, v_5 := E$.

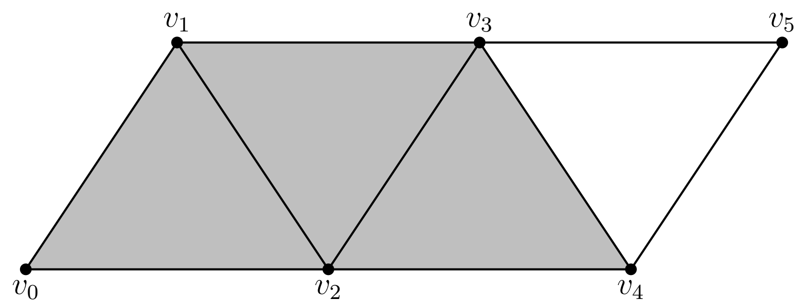

Figure 1.2: A geometric realization of the social system described in Example 1.1.4.

Let K be an abstract simplicial complex with $n := \#\text{vert}(K)$. A geometric realization of K can be easily obtained by identifying the vertices of K in a geometric $(n-1)$-simplicial complex in $\mathbb{R}^n$ and taking the subset $\{\text{conv}(\sigma) \mid \sigma \in K\}$. This is clearly a geometric realization of K, but n dimensions are needed to build it. It is natural to ask wether we can do better and realize K in a smaller dimensional space.

The next theorem states that an abstract d-dimensional simplicial complex has a geometric realization in $\mathbb{R}^{2d+1}$. This can mean a significant reduction in the required dimensions.

Theorem 1.3.5 (Geometric Realization Theorem). Every abstract simplicial complex K of dimension d has a geometric realization in $\mathbb{R}^{2d+1}$.

Proof. Let K be an abstract simplicial complex with $\text{vert}(K) = \{v_0, \dots, v_n\}$ of dimension d. We want to prove that there exists $L \subseteq \mathbb{R}^{2d+1}$ geometric simplicial complex such that its vertex scheme is isomorphic to K.

Consider $p_0, \ldots, p_n \in \mathbb{R}^{2d+1}$ points in general position, meaning that for any subset $\sigma := \{p_{i_0}, \ldots, p_{i_k}\} \subseteq \{p_0, \ldots, p_n\}$ with cardinality $\#\sigma \leq d+2$ we have that $p_{i_0}, \ldots, p_{i_k}$ are affinely independent.

Consider $\varphi : \mathrm{vert}(K) \to \mathbb{R}^{2d+1}$ such that for all $i = 0, \ldots, n$, we have $\varphi(v_i) = p_i$. Notice that φ is bijective by construction.

Let $\sigma = \{v_{i_0}, \ldots, v_{i_k}\} \in K$. Since $\dim(K) = d$, then $\#\sigma \leq d+1$, hence $p_{i_0}, \ldots, p_{i_k}$ are affinely independent, thus

$$\mathrm{conv}(\varphi(\sigma)) = \left\{ \sum_{j=0}^{k} t_j p_{i_j} \mid 0 \leq t_j \leq 1 \text{ and } \sum_{j=0}^{n} t_j = 1 \right\}$$

is a simplex.

Define

$$L := \bigcup_{\sigma \in K} \mathrm{conv}(\varphi(\sigma)).$$

We prove that L is a geometric simplicial complex.

1. Let σ, τ such that $\varnothing \neq \tau \subseteq \sigma \wedge \sigma \in L$. Consider $p_{i_0}, \ldots, p_{i_k}$ such that $\sigma = \mathrm{conv}(\{p_{i_0}, \ldots, p_{i_k}\})$. Up to reordering, suppose $\tau = \mathrm{conv}(\{p_{i_0}, \ldots, p_{i_t}\})$ with $t \leq k$. Hence $\tau = \mathrm{conv}(\{\varphi(v_{i_0}), \ldots, \varphi(v_{i_t})\}) = \mathrm{conv}(\varphi(\{v_{i_0}, \ldots, v_{i_t}\}))$. Since $\{v_{i_0}, \ldots, v_{i_t}\}$ is a simplex in K, it follows that $\tau \in L$.

2. Let $\sigma_1, \sigma_2 \in L$ be simplexes with $\sigma_1 \cap \sigma_2 \neq \varnothing$. We will prove that $\sigma_1 \cap \sigma_2 \in L$ by proving that there exists $\sigma \in K$ such that $\mathrm{conv}(\varphi(\sigma)) = \sigma_1 \cap \sigma_2$. Consider $\alpha, \beta \in K$ such that $\sigma_1 = \mathrm{conv}(\varphi(\alpha)), \sigma_2 = \mathrm{conv}(\varphi(\beta))$. If we prove that $\sigma_1 \cap \sigma_2 = \mathrm{conv}(\varphi(\alpha \cap \beta))$, since $\alpha \cap \beta \in K$, it follows $\sigma_1 \cap \sigma_2 \in L$.

   We have $\sigma_1 \cap \sigma_2 = \mathrm{conv}(\varphi(\alpha)) \cap \mathrm{conv}(\varphi(\beta))$. Since $\#(\alpha \cup \beta) \leq \#\alpha + \#\beta \leq 2d+2$, then $\varphi(\alpha \cup \beta)$ is a set of affinely independent points, thus the function

   $$\begin{aligned} \mathrm{conv} : \mathcal{P}(\varphi(\alpha \cup \beta)) &\longrightarrow \{\mathrm{conv}(\varphi(\tau)) \mid \tau \in \alpha \cup \beta\} \\ \{p_{k_0}, \ldots, p_{k_s}\} &\longmapsto \mathrm{conv}(\{p_{k_0}\}, \ldots, p_{k_s}\}) \end{aligned}$$

   is a bijection, hence

   $$\mathrm{conv}(\varphi(\alpha)) \cap \mathrm{conv}(\varphi(\beta)) = \mathrm{conv}(\varphi(\alpha) \cap \varphi(\beta)) = \mathrm{conv}(\varphi(\alpha \cap \beta)).$$

   So we have $\sigma_1 \cap \sigma_2 = \mathrm{conv}(\varphi(\alpha \cap \beta))$ with $\alpha \cap \beta \in K$ i.e. $\sigma_1 \cap \sigma_2 \in L$.

Finally, we prove that $K \cong L'$ where L' is the vertex scheme of L.

Let $\psi' : L \to L'$ be the map described in Remark 1.3.1. Recall that ψ' is a bijection. Consider $\psi := \psi'|_{\{\{p_0,\ldots,p_n\}\}} : \{\{p_0, \ldots, p_n\}\} \to \mathrm{vert}(L')$ which is as well a bijection. Define $\Phi := \psi \circ \varphi : \mathrm{vert}(K) \to \mathrm{vert}(L')$ a composition of bijections.

Let $\sigma = \{v_{i_0}, \ldots, v_{i_k}\} \in K$. Then $\Phi(\sigma) = \psi(\{p_{i_0}, \ldots, p_{i_k}\}) \in L$ by construction of ψ.

Let $\Phi(\sigma) \in L'$. Then, by construction of Φ, there exists $\{p_{i_0}, \dots, p_{i_k}\} \subseteq \mathbb{R}^{2d+1}$ such that $\Phi(\sigma) = \psi(\{p_{i_0}, \dots, p_{i_k}\})$, hence

$$\sigma = \varphi^{-1}(\psi^{-1}(\psi(\{p_{i_0}, \dots, p_{i_k}\}))) = \{v_{i_0}, \dots, v_{i_k}\} \in K.$$

$\square$

Example 1.3.6. At this point, one might wonder whether every abstract simplicial complex of dimension d can be geometrically realized in $\mathbb{R}^{2d}$. In general, this is not true. For instance, consider the following 1-dimensional simplicial complex:

$$\begin{aligned} K_5 = \{&\{v_0, v_1\}, \{v_0, v_2\}, \{v_0, v_3\}, \{v_0, v_4\}, \{v_1, v_2\}, \\ &\{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\}. \end{aligned}$$

Notice that K_5 is homeomorphic to a graph. According to Kuratowski's Theorem (see [17]), K_5 does not admit a geometric realization in $\mathbb{R}^2$ — or, in the language of graph theory, K_5 is non-planar.

1.4 Other Basic Definitions

We have shown that there is a correspondence between abstract and geometric simplicial complexes. Therefore, when it is not necessary to distinguish between the two notions, we refer to simplicial complex to talk about both an abstract and the vertex set of a geometrical simplicial complex. This section introduces additional concepts related to simplicial complexes.

Definition 1.4.1. Let K be a simplicial complex. If $\sigma, \tau \in K$ with $\tau \subseteq \sigma$, then τ is a **face** of σ and σ is a **coface** of τ. We denote it with $\tau \leq \sigma$, or $\tau < \sigma$ when we know that $\tau \neq \sigma$. The number $\dim(\sigma) - \dim(\tau)$ is the **codimension of τ with respect to σ**. A **facet** of K is a simplex that is not properly contained in any other simplex of K.

Example 1.4.2. Consider the simplicial complex K from Example 1.1.4. Then $\{A\}$ is a face of $\{A, B, C\}$ of codimension 2 and $\{A, B, C\}$ is a facet of K.

Remark 1.4.3. A simplicial complex can be easily stored on a computer just by storing list of faces through the corresponding list of vertices, namely the abstract simplicial complex. Since a simplicial complex is closed under taking subsets, it is enough to store its facets. The simplicial complex from Example 1.1.4 can simply be stored as $\{\{A, B, C\}, \{B, C, D\}, \{C, D, F\}, \{D, E\}, \{E, F\}\}$. Compared to the description given in the mentioned example, this representation requires much less space.

Definition 1.4.4. The **underlying space** of a geometric simplicial complex is the topological space given by the union of its faces with the topology induced by the Euclidean topology of $\mathbb{R}^{2d+1}$. The **triangulation** of a topological space X is a simplicial complex K such that its underlying space is homeomorphic to X. A topological space is said to be **triangulable** if it admits a triangulation.

Example 1.4.5. Consider the Möbius strip M, the topological surface that is obtained by identifying opposite sides of a strip with a half-twist, as depicted in Figure 1.3.

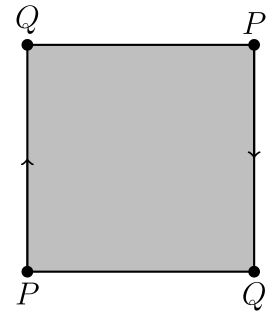

Figure 1.3: The Möbius strip.

A possible triangulation of the Möbius strip is depicted in Figure 1.4.

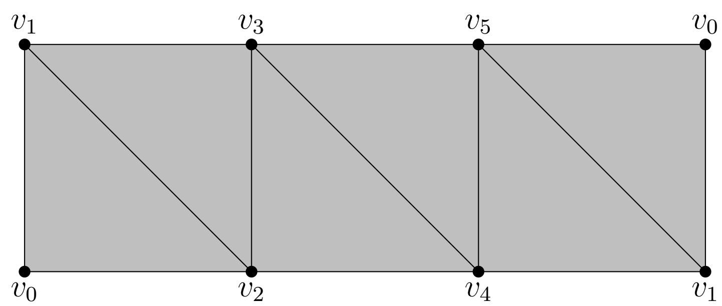

Figure 1.4: A triangulation of the Möbius strip.

As we have shown, the theory that has been developed thus far permits us to describe a model of a smooth (continuous) manifold on a computer by storing the facets of its triangulation. For example, the Möbius strip can be stored as $\{\{v_0, v_1, v_2\}, \{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}, \{v_3, v_4, v_5\}, \{v_4, v_5, v_1\}, \{v_5, v_1, v_0\}\}$.

Definition 1.4.6. Let K be an n-dimensional simplicial complex. We denote an i-simplex as $\sigma^{(i)}$ and $F_i(K)$ as the set of i-dimensional faces of K for $i = 0, \ldots, n$. The f-**vector** of K is the vector $(f_0, \ldots, f_n)$ where $f_i = \#F_i$. A **subcomplex** of K is a simplicial complex $L \subseteq K$. The i-**skeleton** of K is the subcomplex $K^i := \{\sigma \in K \mid \dim(\sigma) \leq i\}$.

Example 1.4.7. The f-vector of the simplicial complex from Example 1.3.4 is $(6, 9, 3)$. In Figure 1.5 it is shown its 1-skeleton.

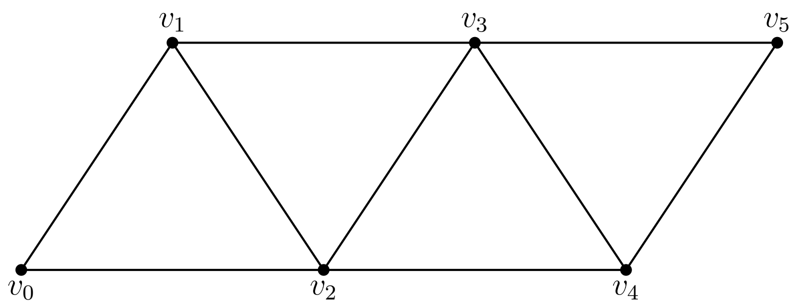

Figure 1.5: 1-skeleton of the social system from Example 1.1.4.

Definition 1.4.8. Let K be an n-dimensional simplicial complex. Let f_i denote the number of i-simplexes of K, for $i = 0, \ldots, n$. The **Euler characteristic** of K is defined by $\chi(K) := \sum_{i=0}^{n}(-1)^i f_i$.

Example 1.4.9. The Euler characteristic of the simplicial complex K from Example 1.3.4 is $\chi(K) = 6 - 9 + 3 = 0$.

2. Algebraic Topology

In topology, a central goal is to identify properties that remain invariant under homeomorphism or weaker equivalences. Two of these features are homotopy and homology. We will study them in this chapter. On one side simplicial complexes can be viewed as topological spaces, allowing the direct application of general homotopy and homology concepts. On the other side, their combinatorial structure offers significant computational advantages. This combinatorial nature enables direct manipulation of simplicial complexes making the evaluation of connectivity, the detection of cycles, and the identification of higher-dimensional voids more accessible, transforming complex problems into manageable algebraic or combinatorial ones.

In this chapter, we will explore specialized adaptations of these concepts tailored to simplicial complexes: *simple homotopy* and *simplicial homology*.

For this chapter, we draw primarily from the works of Dey and Wang [3], Scoville [14] and Hatcher [8].

2.1 Simple Homotopy

This section introduces the concept of simple homotopy, a notion of equivalence between simplicial complexes first proposed by J. H. C. Whitehead [18]. Simple homotopy captures an idea of equivalence analogous to that of homotopy for manifolds in the smooth setting.

Definition 2.1.1. Let K be a simplicial complex and $\{\sigma^{(p-1)}, \tau^{(p)}\} \subseteq K$ such that $\sigma < \tau$ and σ has no other cofaces. Then σ is called a **free face**. An **elementary collapse** of K is a simplicial complex $K' := K \setminus \{\sigma, \tau\}$. The set $\{\sigma, \tau\}$ is said to be a **free pair**. If there is a sequence of elementary collapses from K to L, we say that K collapses to L and we write $K \searrow L$.

Example 2.1.2. Consider the simplicial complex K from Example 1.3.4. Figure 2.1 illustrates an example of elementary collapse of K. In this example, the free pair is $\{\{v_0, v_2\}, \{v_0, v_1, v_2\}\}$.

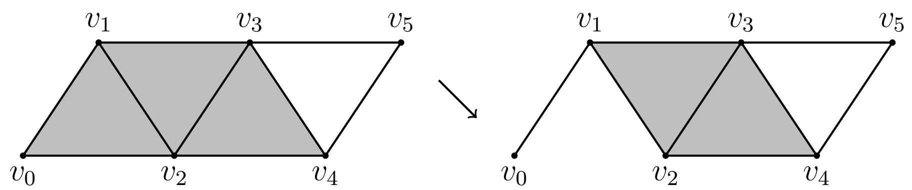

Figure 2.1: An elementary collapse.

Definition 2.1.3. Let K be a simplicial complex and $\{\sigma^{(p-1)}, \tau^{(p)}\} \cap K = \varnothing$ such that $\sigma < \tau$ and all other faces of τ are in K (and, consequently, all faces of σ are also in K). An **elementary expansion** of K is a simplicial complex $K' := K \cup \{\sigma^{(p-1)}, \tau^{(p)}\}$, denoted by $K \nearrow K'$. As above, $\{\sigma^{(p-1)}, \tau^{(p)}\}$ is said to be a **free pair**.

Example 2.1.4. Figure 2.2 shows an elementary expansion of K from Example 1.3.4.

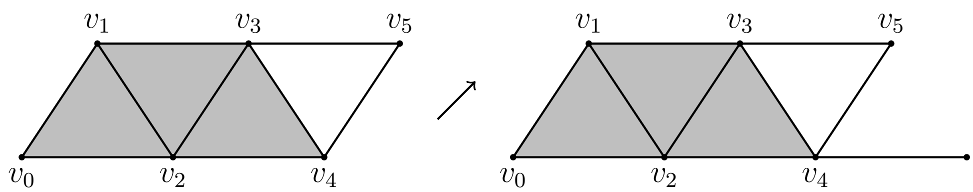

Figure 2.2: An elementary expansion.

Definition 2.1.5. Two simplicial complexes K, L are said to be of the same **simple homotopy type** if there is a sequence of elementary collapses and expansions from K to L. We denote it by $K \sim L$.

Proposition 2.1.6. Simple homotopy is an equivalence relation on the set of all simplicial complexes.

Proof. Let K, L, M be simplicial complexes. Since we do not need any expansions nor collapses to get K from K, then $K \sim K$.
Suppose $K \sim L$. Then we have a sequence of elementary collapses and expansions, say $(e_1, \ldots, e_n)$, from K to L. Consider $(d_n, \ldots, d_1)$ where d_i is the inverse of e_i, in the sense that if the latter is an elementary expansion, then the former is an elementary collapse, or vice versa. Then this is a sequence from L to K, i.e. $L \sim K$.
Suppose $K \sim L$ and $L \sim M$. Then we just need to concatenate the sequence from K to L and the sequence from L to M to get $K \sim M$. $\square$

An interesting question is what invariant of a simplicial complex are preserved under simple homotopy equivalence.

Example 2.1.7. Let $K' := K \backslash \{\{v_0, v_1\}, \{v_0, v_1, v_2\}\}$ be the simplicial complex shown in Figure 2.1, obtained via an elementary collapse of K. Next, perform two additional elementary collapses as follows:

$$K' \searrow K'' := K \setminus \{\{v_1, v_2\}, \{v_1, v_2, v_3\}\},$$
$$K'' \searrow K''' := K'' \setminus \{\{v_2, v_3\}, \{v_2, v_3, v_4\}\}.$$

The resulting complex K''' is of dimension 1, yet it shares the same simple homotopy type as K', which has dimension 2. This demonstrates that dimension is not an invariant under simple homotopy equivalence.

Proposition 2.1.8. The Euler characteristic is a simple homotopy invariant. In other words, if K, L are simplicial complex such that $K \sim L$, then $\chi(K) = \chi(L)$.

Proof. Since $K \sim L$, there exists a sequence of elementary collapses and expansions transforming K into L. We will show that neither a collapse nor an expansion affects the Euler characteristic, implying $\chi(K) = \chi(L)$.
An elementary collapse involves removing a free pair, which consists of a p-simplex and its free face, a $(p-1)$-simplex. Similarly, an expansion adds such a free pair. In both cases, the contribution to the Euler characteristic is $(+1) + (-1) = 0$. Thus, the Euler characteristic remains unchanged after each collapse or expansion. By induction on the sequence of operations, $\chi(K) = \chi(L)$. □

Example 2.1.9. This property allows us to distinguish between simplicial complexes that are not of the same simple homotopy type. If two simplicial complexes have different Euler characteristics, they cannot be simple homotopy equivalent. This is an important tool because, a priori, it might be difficult to determine whether two simplicial complexes are equivalent: there could exist a sequence of elementary collapses and expansions transforming one into the other.

2.2 Simplicial Homology

A key feature of topological spaces is the number of n-dimensional holes they contain. One of the principal methodologies for analyzing holes is through the lens of homotopy theory. In this context, the so-called homotopy groups are defined as higher-dimensional analogues of the fundamental group. However, the higher-dimensional homotopy groups are extremely difficult to compute. An alternative that is much easier to compute goes by the homology groups.

In this section we present simplicial homology, a homology theory specifically designed for simplicial complexes. While it is tailored for these combinatorial structures, it remains equivalent to singular homology when applied to simplicial complexes (see Hatcher [8] for details). Later, in Section 3.4, we will show the power of discrete Morse theory as a method for efficiently computing the homology groups of simplicial complexes.

Definition 2.2.1. Let K be a simplicial complex with $F_p(K) = \{\sigma_1, \ldots, \sigma_{n_p}\}$, for $0 \leq p \leq \dim(K)$. A p**-chain** in K is a formal sum c of p-simplices with coefficients in a ring R:

$$c = \sum_{i=1}^{n_p} t_i \sigma_i^{(p)}$$

where $\sigma_i^{(p)} \in K$ and $t_i \in R$.

Definition 2.2.2. Let K be a simplicial complex with $F_p(K) = \{\sigma_1, \ldots, \sigma_{n_p}\}$, for $0 \leq p \leq \dim(K)$. Let R be a ring. Let $c = \sum_{i=1}^{n_p} t_i\sigma_i, d = \sum_{i=1}^{n_p} s_i\sigma_i$ be two p-chains in K, where $t_i, s_i \in R$. Then we define the **addition** between them as

$$c + d = \sum_{i=0}^{n_p} (t_i + s_i)\sigma_i.$$

Definition 2.2.3. Let K be a simplicial complex with $F_p(K) = \{\sigma_1, \ldots, \sigma_{n_p}\}$, for $p \geq 0$. Let R be a ring. The p-chains with the addition inherit a structure of group from the ring, where the identity is the chain $0 = \sum_{i=1}^{n_p} 0\sigma_i$ and the inverse of a chain $c = \sum_{i=1}^{n_p} t_i\sigma_i^{(p)}$ is

$$-c := \sum_{i=1}^{n_p} -t_i\sigma_i^{(p)},$$

where $-t_i$ is the inverse of t_i in R. This group is called the p**-th chain group** and is denoted as $C_p(K)$ or simply C_p. We define $C_{-1} = \{0\}$ and for any $p > \dim(K)$, $C_p = \{0\}$.

In this thesis we will use coefficients from the field $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. Recall that $\mathbb{Z}_2 = \{0, 1\}$, with the property that $1 + 1 = 0$. From a computational perspective, $\mathbb{Z}_2$ is an optimal choice, as it simplifies calculations and it is commonly used in practical applications. Moreover, employing other fields introduces further complexity which goes beyond the scope of this work.

Remark 2.2.4. Since we are working over $\mathbb{Z}_2$, a p-chain can be thought of as a finite collection of p-simplicies in K with multiplicities. Alternatively, this can be viewed as constructing a vector space over the field $\mathbb{Z}_2$. From this perspective, every p-chain that has a unique non-zero coefficient corresponding to a single p-simplex consists of a basis of the vector space. In this sense, with a slight abuse of notation, we can say that a basis of the vector space is formed by the p-simplices of the simplicial complex, i.e. $F_p(K)$.

Notation 2.2.5. Consider a simplicial complex K with $\text{vert}(K) = \{v_0, \ldots, v_n\}$. From now on, we will denote a k-simplex $\{v_0, v_1, \ldots, v_k\}$ as $v_0v_1 \ldots v_k$ for brevity.

Example 2.2.6. Consider the simplicial complex from Example 1.3.4. When dealing with chains, terms with a coefficient of 0 will be omitted, and we will write v_0v_1 instead of $1 \cdot v_0v_1$. As an example, consider the addition of two chains:

$$(v_0v_1 + v_1v_3 + v_2v_3) + (v_2v_3 + v_3v_5 + v_4v_5) = (v_0v_1 + v_1v_3 + v_3v_5 + v_4v_5).$$

This illustrates how chains are combined by summing their simplices, eliminating terms with coefficients that cancel out.

Example 2.2.7. Consider the simplicial complex K from Example 1.3.4. A basis of the second chain group of K is the following set of 2-chains:

$$\{0, v_0v_1v_2, v_1v_2v_3, v_2v_3v_4\}.$$

Definition 2.2.8. Let K be a simplicial complex with $F_p(K) = \{\sigma_1, \dots, \sigma_{n_p}\}$, for $p \geq 0$. For any p-simplex $\sigma = \{v_0, \dots, v_p\}$, the **boundary** of σ is the $(p-1)$-chain defined by:

$$\partial_p \sigma = \sum_{i=0}^{p} \{v_0, \dots, \hat{v}_i, \dots, v_p\}$$

where $\hat{v}_i$ indicates that the vertex v_i is omitted. In other words, ∂_p is the $(p-1)$-chain that has non-zero coefficients only on σ's $(p-1)$-faces. The **boundary operator** $\partial_p : C_p \to C_{p-1}$ is then defined on a p-chain $c = \sum_{i=1}^{n_p} t_i\sigma_i \in C_p$ by linearity, namely:

$$\partial_p(c) = \sum_{i=1}^{n_p} t_i(\partial_p \sigma_i).$$

Example 2.2.9. Consider the simplicial complex K from Example 1.3.4. We have

$$\partial_2(v_1v_2v_3) = v_1v_2 + v_1v_3 + v_2v_3$$

and for $c = v_1v_2 + v_2v_3 + v_3v_4$:

$$\partial_1(c) = v_1 + v_2 + v_2 + v_3 + v_3 + v_4 = v_1 + v_4.$$

Remark 2.2.10. Let $\sigma = \{v_0\}$ be a 0-simplex. Then its boundary is zero: $\partial_0(\sigma) = 0$. Furthermore, for any $c \in C_0$ we have that $\partial_0(c) = 0$.

Proposition 2.2.11. Let K be a simplicial complex with $F_p(K) = \{\sigma_1, \dots, \sigma_{n_p}\}$, for $p \geq 0$. Then the boundary operator $\partial_p : C_p \to C_{p-1}$ is a group homomorphism.

Proof. Let $c = \sum_{i=1}^{n_p} t_i\sigma_i, d = \sum_{i=1}^{n_p} s_i\sigma_i$ be two p-chains in K. We have

$$\begin{aligned}
\partial_p(c + d) &= \partial_p\left(\sum_{i=0}^{n_p}(t_i + s_i)\sigma_i\right) \\
&= \sum_{i=0}^{n_p}(t_i + s_i)(\partial_p\sigma_i) \\
&= \sum_{i=0}^{n_p} t_i(\partial_p\sigma_i) + \sum_{i=0}^{n_p} s_i(\partial_p\sigma_i) \\
&= \partial_p(c) + \partial_p(d).
\end{aligned}$$

□

Proposition 2.2.12. Let K be a simplicial complex. Let $p \geq 1$ be an integer. Then $\partial_{p-1} \circ \partial_p = 0$.

Proof. In Remark 2.2.10 we saw that ∂_0 is a zero map. For $p > \dim(K)$, we have that $C_p = \{0\}$, hence ∂_p is also a zero map. It is therefore sufficient to show that for $1 \leq p \leq \dim(K), \partial_{p-1} \circ \partial_p(\sigma) = 0$ for a p-simplex $\sigma \in K$. Observe that $\partial_p(\sigma)$ has non-zero coefficients only on σ's $(p-1)$-faces and every $(p-2)$-face of σ is contained in exactly two $(p-1)$-faces of σ. Since we are working with coefficients in $\mathbb{Z}_2$, then 2 is equivalent to 0, hence $\partial_{p-1} \circ \partial_p(\sigma) = 0$. □

Example 2.2.13. Let K be the simplicial complex from Example 1.3.4. Consider the computation of $\partial_1 \circ \partial_2(v_0v_1v_2)$:

$$\begin{aligned} \partial_1 \circ \partial_2(v_0v_1v_2) =& \partial_1(v_0v_1 + v_1v_2 + v_0v_2) \\ =& v_0 + v_1 + v_1 + v_2 + v_0 + v_2 = 0. \end{aligned}$$

This computation verifies the property for this specific 2-chain.

Definition 2.2.14. Let K be a simplicial complex. Let $p \geq 0$ be an integer. Let c be a p-chain. If $\partial(c) = 0$, then c is called a p-**cycle**. The set of all p-cycles in K form a subgroup of C_p that is called the p-**th cycle group** $Z_p(K)$ or Z_p when K is clear from the context. In terms of the boundary operator ∂_p, notice that $Z_p(K) = \ker \partial_p$.

Example 2.2.15. Let K be the simplicial complex from Example 1.3.4. Consider the 1-chains $c_1 = v_0v_1 + v_1v_2 + v_0v_2$ and $c_2 = v_3v_4 + v_3v_5 + v_4v_5$. These chains are cycles because:

$$\begin{aligned} \partial_p(v_0v_1 + v_1v_2 + v_0v_2) = 0, \\ \partial_p(v_3v_4 + v_3v_5 + v_4v_5) = 0. \end{aligned}$$

Geometrically, c_1 is the boundary of a higher-dimensional face, whereas c_2 encircles a "hole". This illustrates the concept of a cycle in homology, where a p-chain captures the boundary of a higher-dimensional feature or an enclosed void.

The first cycle group $Z_1(K)$ is generated by the following set of 1-cycles:

$$\begin{aligned} &\{v_0v_1 + v_1v_2 + v_2v_0, v_1v_2 + v_2v_3 + v_3v_1, \\ &v_2v_3 + v_3v_4 + v_2v_4, v_3v_4 + v_4v_5 + v_3v_5\} \end{aligned}$$

Remark 2.2.16. Notice that thanks to Proposition 2.2.12, the boundary of a p-chain is a $(p-1)$-cycle.

Definition 2.2.17. Let K be a simplicial complex. Let $p \geq 0$ be an integer. The p-**th boundary group** is the subgroup $B_p(K) = \operatorname{Im} \partial_{p+1} = \partial_{p+1}(C_{p+1})$. For simplicity, when the context makes K clear, we denote it as B_p.

Example 2.2.18. Consider the simplicial complex K from Example 1.3.4. The second chain group $C_2(K)$ is given by the following set of 2-chains:

$$\begin{aligned}C_2(K) = &\{0, v_0v_1v_2, v_1v_2v_3, v_2v_3v_4, v_0v_1v_2 + v_1v_2v_3, v_0v_1v_2 + v_2v_3v_4,\\ &v_1v_2v_3 + v_2v_3v_4, v_0v_1v_2 + v_1v_2v_3 + v_2v_3v_4\}.\end{aligned}$$

Applying the boundary operator ∂_2 to C_2, we obtain the first boundary group $B_1 = \partial_2(C_2)$, which consists of the following 1-chains:

$$\begin{aligned}&\{0, v_0v_1 + v_1v_2 + v_0v_2, v_1v_2 + v_2v_3 + v_1v_3, v_2v_3 + v_3v_4 + v_2v_4\\ &v_0v_1 + v_1v_3 + v_2v_3 + v_0v_2, v_1v_2 + v_1v_3 + v_3v_4 + v_2v_4,\\ &v_0v_1 + v_1v_2 + v_0v_2 + v_2v_3 + v_3v_4 + v_2v_4, v_0v_1 + v_1v_3 + v_3v_4 + v_2v_4 + v_0v_2\}.\end{aligned}$$

The elements of B_1 represent all 1-cycles that are the boundaries of 2-chains in C_2. Geometrically, these are the 1-cycles that are "filled in" by 2-simplices, meaning they do not correspond to holes in the simplicial complex.

Remark 2.2.19. Notice that due to Proposition 2.2.12, for $p \geq 1$ we have $\partial_{p-1}(B_{p-1}) = \{0\}$, hence $B_{p-1} \subseteq Z_{p-1}$. Therefore, for $p \geq 0$ we have $B_p \subseteq Z_p \subseteq C_p$. Since C_p is a vector space, then B_p and Z_p are also vector spaces.

Definition 2.2.20. Let K be a simplicial complex and let $p \geq 0$ be an integer. The **p-th homology group** is defined as the quotient group

$$H_p(K) = Z_p(K)/B_p(K) = \ker \partial_p / \mathrm{Im}\ \partial_{p+1}.$$

Since we are working over a field, H_p inherits the structure of vector space. Its dimension is called the **p-th Betti number**, denoted by $\beta_p := \dim H_p$. By the rank-nullity theorem it follows that

$$\beta_p = \dim Z_p - \dim B_p.$$

Each element of H_p is a coset of B_p in Z_p and can be written as $c + B_p$, where $c \in Z_p$ is a p-cycle. The set of all cycles obtained by adding an element of B_p to c forms the coset $[c]$, called the **homology class of** c. Two cycles that belong to the same homology class are said to be **homologous**.

Remark 2.2.21. Although we work over Z_2, the formula for computing Betti numbers remains valid in the general construction over rings.

Example 2.2.22. When performing the quotient $H_p = Z_p/B_p$, we identify every chain in $B_p = B_p \cap Z_p$ with the zero chain. Geometrically, this process removes all p-cycles that are "filled in" by $(p+1)$-chains, as described in Example 2.2.18. The resulting homology group H_p comprises p-cycles that enclose voids and are not boundaries of any higher-dimensional chains. In other words, the p-th homology group H_p encodes the p-dimensional "holes" of the simplicial complex K. Additionally, the 0-th homology group H_0 can be interpreted as the number of connected components of K, offering a fundamental measure of its overall structure.

Example 2.2.23. Consider the simplicial complex K from Example 1.3.4. In Example 2.2.15, we observed that a basis for $Z_1(K)$ is:

$$\{v_0v_1 + v_1v_2 + v_2v_0, v_1v_2 + v_2v_3 + v_3v_1,$$
$$v_2v_3 + v_3v_4 + v_2v_4, v_3v_4 + v_4v_5 + v_3v_5\}.$$

In Example 2.2.18, we computed $B_1(K)$, which has a basis given by:

$$\{v_0v_1 + v_1v_2 + v_0v_2, v_1v_2 + v_2v_3 + v_1v_3, v_2v_3 + v_3v_4 + v_2v_4\}.$$

Notice that the basis of $Z_1(K)$ and $B_1(K)$ differ by a single 1-chain, namely $c := v_3v_4 + v_4v_5 + v_3v_5$. This implies that $H_p(K) = Z_1(K)/B_1(K)$ has a basis consisting of the coset of c, representing the single 1-dimensional "hole" in K. Therefore, the first Betti number of K is $\beta_1 = 1$.

This computation can be greatly simplified using discrete Morse theory, which allows us to reduce the simplicial complex while preserving its homology groups. By applying this method, the same homological information can be obtained more efficiently.

3. Discrete Morse Theory

This chapter introduces discrete Morse theory, covering fundamental concepts and presenting the key theorems. The power of these results is shown through a final computation.

This chapter is primarily based on the works of Forman [6], Dey and Wang [3], Scoville [14] and Gallais [7].

3.1 Discrete Morse Function

Definition 3.1.1. Let K be a simplicial complex. A **discrete Morse function** is a map $f : K \to \mathbb{R}$ such that for all $\sigma^{(p)} \in K$ we have:

1. $\#\{\tau^{(p-1)} < \sigma \mid f(\tau) \geq f(\sigma)\} \leq 1$, and

2. $\#\{\tau^{(p+1)} > \sigma \mid f(\tau) \leq f(\sigma)\} \leq 1$.

Example 3.1.2. We represent a function f on a simplicial complex by annoting each face σ with its corresponding value $f(\sigma)$. Figure 3.1 illustrates a discrete Morse function for the simplicial complex described in Example 1.1.4.

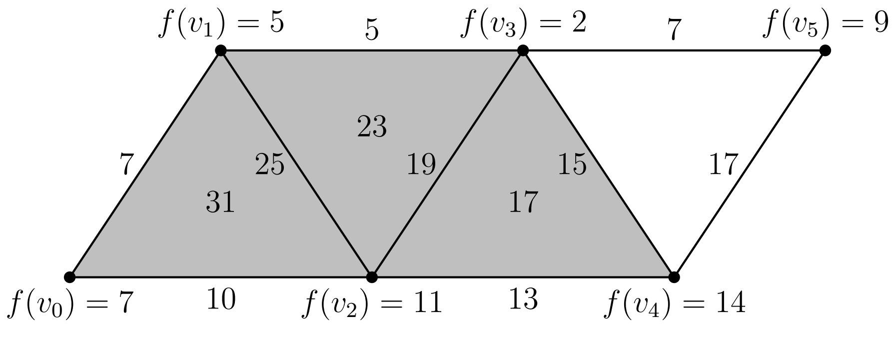

Figure 3.1: A discrete Morse function.

Example 3.1.3. In Figure 3.2 we show a function that is not a discrete Morse function. Indeed, it does not satisfy Condition 2 of Definition 3.1.1 because the 0-simplex v_5 violates the required inequality.

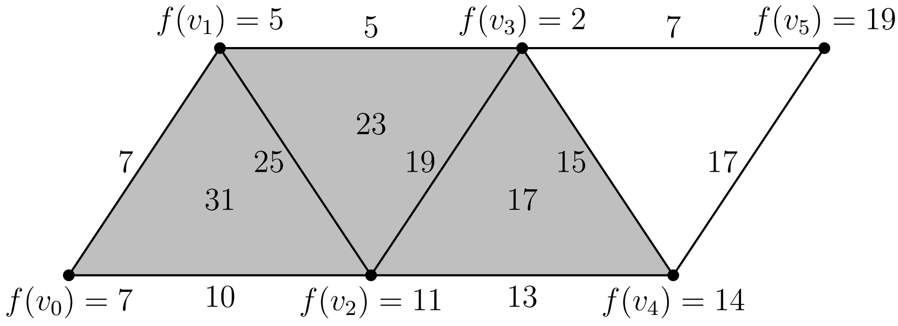

Figure 3.2: A function that is not a discrete Morse function.

Definition 3.1.4. Let K be a simplicial complex. Let $f : K \to \mathbb{R}$ be a discrete Morse function. A simplex $\sigma \in K$ is a **critical simplex** if both sets in Definition 3.1.1 are empty. In other words, $\sigma^{(p)} \in K$ is critical if both of the following conditions hold:

1. $\forall \tau^{(p-1)} < \sigma,\ f(\tau) < f(\sigma)$;

2. $\forall \tau^{(p+1)} > \sigma,\ f(\tau) > f(\sigma)$.

In this case, $f(\sigma)$ is a **critical value**. A simplex that is not critical is a **regular simplex**, and its value through f is a **regular value**.

Example 3.1.5. Consider the discrete Morse function in Figure 3.1. In this setting, v_3 is a critical simplex and $2 = f(v_3)$ is a critical value (recall that $\varnothing \notin K$, thus condition 1 of Definition 3.1.4 if always true). Instead, v_0v_2 is a regular simplex and $10 = f(v_0v_2)$ is a regular value.

Lemma 3.1.6 (**Exclusion Lemma**)**.** Let K be a simplicial complex. Let $f : K \to \mathbb{R}$ be a discrete Morse function and $\sigma \in K$ a regular simplex. Then the sets in Definition 3.1.1 cannot both have cardinality 1.

Proof. We proceed by contradiction. Let us assume that in K there are $\nu^{(p-1)} < \sigma^{(p)} < \tau^{(p+1)}$ such that $f(\nu) \geq f(\sigma) \geq f(\tau)$. Without loss of generality, assume up to reordering $\nu = \{a_0, \ldots, a_{p-1}\}, \sigma = \{a_0, \ldots, a_{p-1}, a_p\}, \tau = \{a_0, \ldots, a_{p-1}, a_p, a_{p+1}\}$. Define $\sigma' := \{a_0, \ldots, a_{p-1}, a_{p+1}\}$. Then $\nu < \sigma' < \tau$. Since $\nu < \sigma'$ and $\nu < \sigma$ with $f(\nu) \geq f(\sigma)$, then $f(\nu) < f(\sigma')$. Since $\sigma' < \tau$ and $\sigma < \tau$ with $f(\sigma) \geq f(\tau)$ then $f(\sigma') < f(\tau)$. It follows that $f(\nu) < f(\sigma') < f(\tau) \leq f(\nu)$, which is a contradiction. $\square$

Example 3.1.7. It is possible to have a pair of simplices $\tau^{(i)} < \omega^{(p)}$ in K with $i < p - 1$ such that $f(\tau) > f(\omega)$. Consider as an example the discrete Morse function in Figure 3.3.

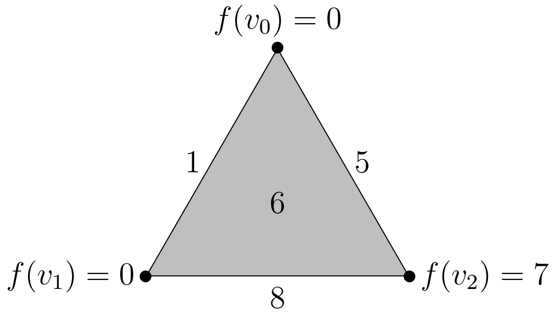

Figure 3.3: A discrete Morse function where $v_2^{(i)} < v_0v_1v_2^{(p)}$ with $i < p-1$ such that $f(v_2) > f(v_0v_1v_2)$.

3.2 Gradient Vector Field

In this section, we introduce a feature of a discrete Morse function which encapsulates the behavior of the function: the gradient vector field. Working with the gradient vector field often simplifies analysis and computation compared to working directly with the function. We conclude the section by presenting a fundamental result that facilitates the construction of a discrete Morse function with the minimum possible number of critical simplexes.

Definition 3.2.1. Let K be a simplicial complex and let $f : K \to \mathbb{R}$ be a discrete Morse function. The **induced gradient vector field of** f, or just **gradient vector field**, is the set

$$V_f := \{(\sigma^{(p)}, \tau^{(p+1)}) \mid \sigma < \tau, f(\sigma) \geq f(\tau)\}.$$

A pair $(\sigma, \tau) \in V_f$ is called **vector** or **arrow**. In this case σ is called a **tail** and τ a **head**. We will sometimes denote this by $\sigma \to \tau$.

Remark 3.2.2. Notice that a simplex is critical if and only if it is neither the tail nor the head of an arrow.

Example 3.2.3. Figure 3.4 shows the gradient vector field on the discrete Morse function from Example 3.1.2. Notice, for example, that the critical simplex v_3 is neither the tail nor the head of any arrow in the figure.

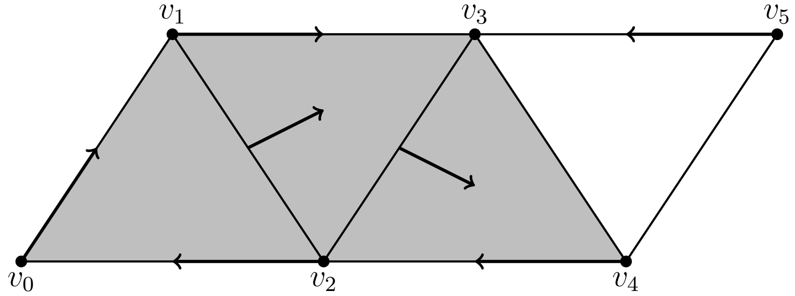

Figure 3.4: A gradient vector field.

Notice that Exclusion Lemma (Lemma 3.1.6) implies that for any simplex σ, exactly one of the following conditions hold:

- σ is the head of exactly one arrow;
- σ is the tail of exactly one arrow;
- σ is a critical simplex.

This conditions give us a partition of the set of vertices in three sets. We can define a general notion of *discrete vector field* as a such partition, that is not necessarily associated with any discrete Morse function. While every gradient vector field is a discrete vector field, the converse does not hold in general.

Definition 3.2.4. Let K be a simplicial complex. A **discrete vector field** is a set

$$V := \{(\sigma^{(p)}, \tau^{(p+1)}) \mid \sigma < \tau \wedge \nexists \nu \in K \text{ s.t. } [(\nu, \tau) \in V \vee (\sigma, \nu) \in V]\}.$$

Notice that the set of critical simplices of K in the partition discussed above is the set of the simplices of K not being in V.

Proposition 3.2.5. Every gradient vector field is also a discrete vector field.

Proof. Let $f : K \to \mathbb{R}$ be a discrete Morse function and consider $(\sigma^{(p)}, \tau^{(p+1)}) \in V_f$, thus $\sigma < \tau \wedge f(\sigma) \geq f(\tau)$. If there is a $\nu^{(p+1)} \in K$ such that $(\sigma, \nu) \in V_f$, then we have a contradiction with condition 2 of Definition 3.1.1. If there is a $\nu^{(p-1)} \in K$ such that $(\nu, \sigma) \in V_f$, then we have a contradiction with Lemma 3.1.6 (Exclusion Lemma). If there is a $\nu^{(p)} \in K$ such that $(\nu, \tau) \in V_f$, then we have a contradiction with condition 2 of Definition 3.1.1. If there is a $\nu^{(p+2)} \in K$ such that $(\tau, \nu) \in V_f$, then we have a contradiction with Lemma 3.1.6 again. It follows that every simplex in K can be in at most one pair. □

Example 3.2.6. There are discrete vector fields that are not induced by any discrete Morse function. Consider for example the discrete vector field V in Figure 3.5. For it to be a gradient vector field, we would need a discrete Morse function f such that $f(v_0) \geq f(v_0v_1) > f(v_1) \geq f(v_1v_2) > f(v_2) \geq f(v_2v_0) > f(v_0)$ which is a contradiction. Hence, there is no discrete Morse function f such that $V = V_f$.

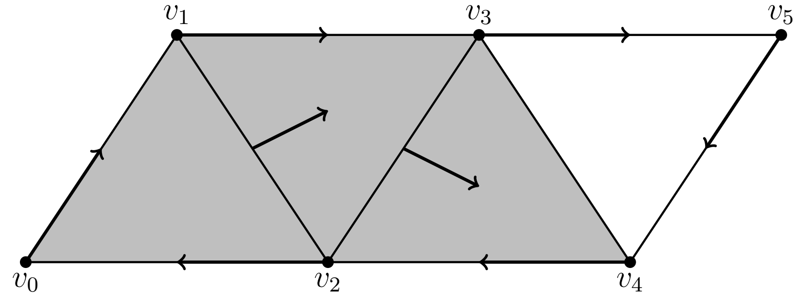

Figure 3.5: A discrete vector field that is not induced by any discrete Morse function.

We may naturally ask under what conditions a discrete vector field qualifies as a gradient vector field. To establish a sufficient condition for this, we must first introduce an additional concept that will provide the necessary foundation.

Definition 3.2.7. Let K be a simplicial complex and V a gradient vector field on K. A V**-path** is a sequence

$$(\sigma_0^{(p)}, \tau_0^{(p+1)}, \ldots, \sigma_{n-1}^{(p)}, \tau_{n-1}^{(p+1)}, \sigma_n^{(p)})$$

of simplices in K such that:

1. $(\sigma_i^{(p)}, \tau_i^{(p+1)}) \in V$ for $i = 0, \ldots, n-1$
2. $\tau_{i-1}^{(p+1)} > \sigma_i^{(p)} \neq \sigma_{i-1}^{(p)}$ for $i = 1, \ldots, n$.

If $n \geq 1$, then the V-path is said to be **non-trivial**. A V-path beginning at $\sigma_0^{(p)}$ and ending at $\sigma_n^{(p)} = \sigma_0^{(p)}$ is said to be **closed**. A V-path can also start with $\tau_{-1}^{(p+1)} \in K$ such that $(\sigma_0^{(p)}, \tau_{-1}^{(p+1)}) \notin V$, thus being

$$(\tau_{-1}^{(p+1)}, \sigma_0^{(p)}, \tau_0^{(p+1)}, \ldots, \sigma_{n-1}^{(p)}, \tau_{n-1}^{(p+1)}, \sigma_n^{(p)}).$$

We may denote a V-path in the following way:

$$\tau_{-1}^{(p+1)} > \sigma_0^{(p)} \to \tau_0^{(p+1)} > \cdots > \sigma_{n-1}^{(p)} \to \tau_{n-1}^{(p+1)} > \sigma_n^{(p)}.$$

Example 3.2.8. Consider the gradient vector field shown in Figure 3.4. A V-path from $v_0v_1v_2$ to v_3v_4 is given by:

$$v_0v_1v_2 > v_1v_2 \to v_1v_2v_3 > v_2v_3 \to v_2v_3v_4 > v_3v_4.$$

Here, the symbol $\to$ represents an arrow in the gradient vector field, while $>$ indicates an inclusion between simplices.

The following result provides a characterization of a discrete vector field that can serve as the gradient vector field of an associated discrete Morse function. We omit the proof here because it lies beyond the scope of this work.

Proposition 3.2.9. Let V be a discrete vector field. Then there is a discrete Morse function f such that $V = V_f$ if and only if every closed V-path in V is trivial.

Proof. See [14, Theorem 2.51]. $\square$

We now present a fundamental result that provides a method for minimizing the number of critical simplices in a discrete Morse function.

Proposition 3.2.10 (Cancellation of critical simplices). Let K be a simplicial complex, V a gradient vector field on K and $\sigma^{(p)}, \tau^{(p+1)} \in K$ critical simplices such that exists a unique V-path

$$\gamma := (\tau^{(p+1)}, \sigma_0^{(p)}, \tau_0^{(p+1)}, \sigma_1^{(p)}, \ldots, \sigma_{n-1}^{(p)}, \tau_{n-1}^{(p+1)}, \sigma_n^{(p)} = \sigma).$$

Define W such that:

1. $W \setminus \gamma = V \setminus \gamma$
2. $(\sigma_0, \tau) \in W$
3. $(\sigma_{i+1}, \tau_i) \in W$ for $i = 0, \dots, n-1$ (in particular $(\sigma_n, \tau_{n-1}) \in W$).

Then W is a gradient vector field, and exists a unique W-path from σ to σ_0.

Example 3.2.11. We illustrate the procedure described in Proposition 3.2.10 with an example. Consider the gradient vector field V shown in Figure 3.4 and the V-path γ described in Example 3.2.8. Let W be the gradient vector field depicted in Figure 3.6. Observe that W differs from V only along γ. Specifically, all arrows in γ have been "reversed", and the initial and terminal simplices of the path are no longer critical. In this way, the pair of critical simplices $(v_0v_1v_2, v_3v_4) = (\tau_{-1}^{(p+1)}, \sigma_n^{(p)})$ has been canceled.

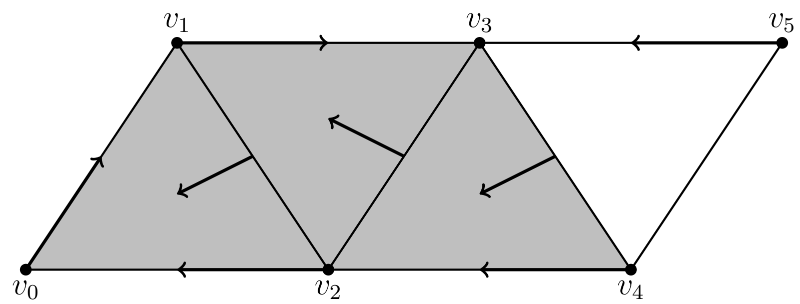

Figure 3.6: An example of Cancellation of critical simplices.

Proof. Notice that the unique V-path γ is

$$\tau > \sigma_0 \rightarrow \tau_0 > \sigma_1 \rightarrow \tau_1 > \cdots > \sigma_{n-1} \rightarrow \tau_{n-1} > \sigma_n = \sigma$$

whereas the corresponding W-path γ' will be

$$\tau \leftarrow \sigma_0 < \tau_0 \leftarrow \sigma_1 < \tau_1 \leftarrow \cdots \leftarrow \sigma_{n-1} < \tau_{n-1} \leftarrow \sigma_n = \sigma.$$

Since $W \setminus \gamma = V \setminus \gamma$, then W is a discrete vector field. We prove that W has no non-trivial closed W-paths, and from Proposition 3.2.9 it follows that W is a gradient vector field.
Suppose W contains a closed path μ. Since V is a gradient vector field, μ cannot be contained only in $W \setminus \gamma = V \setminus \gamma$. Furthermore, we shall clearly see that γ' is not a closed W-path. Hence, μ must be contained in both $W \setminus \gamma$ and γ'. Therefore, μ must contain a sequence of elements in $W \setminus \gamma$ and a sequence of elements in γ'. We have four possible cases:

$$\mu = (\sigma_j, \dots, \sigma_i, \delta_0, \dots, \delta_r, \sigma_j)$$
$$\mu = (\sigma_j, \dots, \tau_i, \delta_0, \dots, \delta_r, \sigma_j)$$
$$\mu = (\tau_j, \dots, \sigma_i, \delta_0, \dots, \delta_r, \tau_j)$$
$$\mu = (\tau_j, \dots, \tau_i, \delta_0, \dots, \delta_r, \tau_j)$$

where $\delta_0, \dots, \delta_r \in W \setminus \gamma$, $r \geq 1$ and $n \geq j > i \geq 0$. Bearing in mind Definition 3.2.7, notice that only the second one is a W-path.

$$\begin{array}{c}
\tau \leftarrow \sigma_0 < \tau_0 \leftarrow \cdots \leftarrow \sigma_i < \tau_i \leftarrow \cdots \leftarrow \sigma_j < \tau_j \leftarrow \cdots \leftarrow \sigma_n = \sigma \\
\vee \qquad\qquad \wedge \\
\delta_0 \to \cdots \to \delta_r
\end{array}$$

Then we have that $(\tau, \sigma_0, \dots, \sigma_i, \tau_i, \delta_0, \dots, \delta_r, \sigma_j, \tau_j, \dots, \sigma_n)$ is a V-path from τ to σ that is different from γ, contradicting the uniqueness of γ.

$$\begin{array}{c}
\tau > \sigma_0 \to \tau_0 > \cdots > \sigma_i \to \tau_i > \cdots > \sigma_j \to \tau_j > \cdots > \sigma_n = \sigma \\
\vee \qquad\qquad \wedge \\
\delta_0 \to \cdots \to \delta_r
\end{array}$$

Now we prove that γ' is the only W-path from σ to σ_0. Suppose there is another such W-path μ. Since V has no other paths from σ_0 to σ, then μ cannot be contained only in $W \setminus \gamma = V \setminus \gamma$. Since $\mu \neq \gamma'$, then μ must contain a sequence of elements in $W \setminus \gamma$ and a sequence of elements in γ'. Analogously as above, the only possible case is the following.

$$\begin{array}{c}
\tau \leftarrow \sigma_0 < \tau_0 \leftarrow \cdots \leftarrow \sigma_i < \tau_i \leftarrow \cdots \leftarrow \sigma_j < \tau_j \leftarrow \cdots \leftarrow \sigma_n = \sigma \\
\wedge \qquad\qquad\qquad\qquad \vee \\
\pi_s \leftarrow \qquad \dots \qquad \leftarrow \pi_0
\end{array}$$

where $\pi_0, \dots, \pi_s \in W \setminus \gamma$. Then we have that $(\sigma_i, \tau_i, \dots, \sigma_j, \tau_j, \pi_0, \dots, \pi_s, \sigma_i)$ is a closed V-path, contradicting the fact that V is a gradient vector field.

$$\begin{array}{c}
\tau > \sigma_0 \to \tau_0 > \cdots > \sigma_i \to \tau_i > \cdots > \sigma_j \to \tau_j > \cdots > \sigma_n = \sigma \\
\wedge \qquad\qquad\qquad\qquad \vee \\
\pi_s \leftarrow \qquad \dots \qquad \leftarrow \pi_0
\end{array}$$

□

3.3 Main Theorems

In this section, we explore the most significant results of discrete Morse theory, highlighting both its theoretical significance and its practical applications. These results reveal the remarkable power and flexibility of discrete Morse theory in simplifying and analyzing the structure of simplicial complexes. Notably, these findings are closely analogous to the fundamental results of classical Morse theory for smooth manifolds, as Forman discusses in his work [6].

We start with the exposition of *Collapse Theorem* (Theorem 3.3.6), which provides a way for simplifing a simplicial complex without changing its topology, via elementary collapses. We first need to introduce the concept of *level subcomplex* and two technical lemmas required to prove the theorem.

Definition 3.3.1. Let K be a simplicial complex $f : K \to \mathbb{R}$ be a discrete Morse function and $c \in \mathbb{R}$. The **level subcomplex** $K_f(c)$ is the subcomplex of K consisting of all simplexes $\sigma \in K$ with $f(\sigma) \leq c$ and all their faces. In other words,

$$K_f(c) := \bigcup_{\substack{\sigma \in K \\ f(\sigma) \leq c}} \bigcup_{\tau \leq \sigma} \{\tau\}.$$

Example 3.3.2. Figure 3.7 shows a discrete Morse function associated with the gradient vector field illustrated in Figure 3.6.

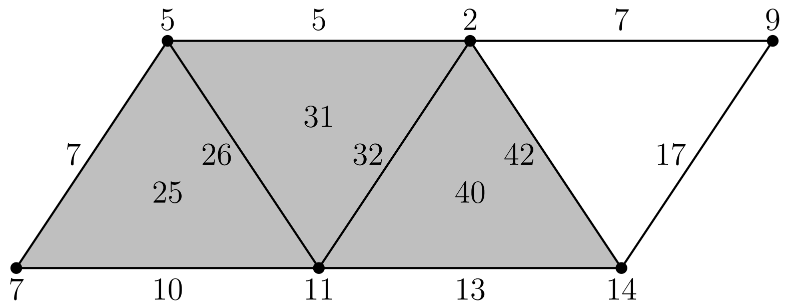

Figure 3.7: A discrete Morse function associated to the gradient vector field from Figure 3.6.

In Figure 3.8 it is depicted the level subcomplex $K(10)$. Notice that there is a vertex with value 11 because it is contained in the edge v_0v_2 with value 10.

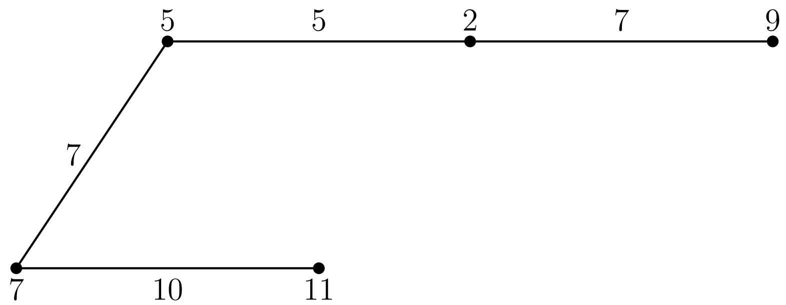

Figure 3.8: A level subcomplex $K(10)$.

The following lemma states that a discrete Morse function with no critical values in a closed interval $[a, b] \subseteq \mathbb{R}$ can be assumed to be injective.

Lemma 3.3.3. Let K be a simplicial complex and $f : K \to \mathbb{R}$ be a discrete Morse function with no critical values in $[a, b] \subseteq \mathbb{R}$. Then there exists a discrete Morse function $g : K \to \mathbb{R}$ such that:

1. g is injective on $[a, b]$;
2. g has no critical value in $[a, b]$;
3. $K_f(b) = K_g(b)$ and $K_f(a) = K_g(a)$;
4. $f(x) = g(x),\ \forall x \in \mathbb{R} \setminus [a, b]$.

Proof. Assume there are two simplices $\sigma, \tau \in K$ with $\sigma \neq \tau$ and $f(\sigma) = f(\tau)$.

(i) Suppose $\tau^{(p+1)} > \sigma^{(p)}$ satisfies $f(\tau) = f(\sigma)$. According to Exclusion Lemma (Lemma 3.1.6), every other face of σ has value strictly greater that $f(\sigma)$ and every other facet of τ has value strictly lesser than $f(\tau)$. Therefore there exists a $\varepsilon > 0$ small enough such that defining $g(\tau) = f(\tau) - \varepsilon$, or $g(\sigma) = f(\sigma) + \varepsilon$, doesn't change what cells are critical and keeps $g(\sigma)$ and $g(\tau)$ in $[a, b]$, thus respecting conditions 3. and 4.

(ii) On the other end, suppose σ is not a face of τ and $f(\sigma) = f(\tau)$. Notice that for every $\pi^{(p+1)} < \nu^{(p)} < \zeta^{(p-1)}$ with $f(\pi) \neq f(\nu) \neq f(\zeta)$ we can change the value of ν for a $\varepsilon > 0$ small enough without changing what cells are critical and respecting conditions 3. and 4. Therefore, we can perform this changes on σ or τ.

(iii) Finally, suppose we have $\tau^{(p)} > \sigma^{(q)}$ with $f(\tau) = f(\sigma)$ and $p > i + 1$. Then, for the same argument used in (ii), we can change the value of σ or τ in order to obtain an injective function as requested.

Since f has a finite image, we can repeat the operations described above until the function is injective on $[a, b]$ and still satisfying conditions 2., 3. and 4. □

Example 3.3.4. Consider the discrete Morse function f described in Figure 3.7. The critical values of f are 2 and 17. Let $[a, b] = [4, 15]$, that has no critical values. Observe that f is not injective on $f(v_1) = f(v_1v_3) = 5$ and on $f(v_0) = f(v_0v_1) = f(v_3v_5) = 7$. We apply operation (a) from the proof of Lemma 3.3.3 on the pair $\{v_0, v_0v_1\}$, defining $g(v_0v_1) = 7-\varepsilon_1$, where $\varepsilon_1 > 0$. We apply operation (b) on pairs $\{v_1, v_1v_3\}$ and $\{v_0, v_3v_5\}$ defining $g(v_1v_3) = 5 - \epsilon_2$ and $g(v_3v_5) = 7 + \varepsilon_3$, where $\varepsilon_1, \varepsilon_2 > 0$. The injective discrete Morse function g that results from these operations is illustrated in Figure 3.9.

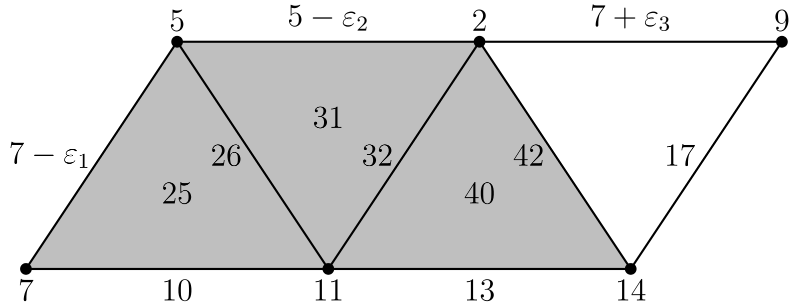

Figure 3.9: An injective discrete Morse function.

Lemma 3.3.5. Let K be a simplicial complex and let σ be a p-simplex of K. Then, for any $\omega > \sigma$, there exists a $(p+1)$-simplex τ such that $\sigma < \tau \leq \omega$ and $f(\tau) \leq f(\omega)$.

Proof. Since $\omega > \sigma$, it follows that $\dim(\omega) > \dim(\sigma)$. We proceed by induction on $r = \dim(\omega) - \dim(\sigma)$.
Base Case ($r = 1$). In this case, let $\tau := \omega$. Clearly, $\sigma < \tau \leq \omega$, and $f(\tau) = f(\omega)$ satisfies the desired condition.
Inductive Step ($r > 1$). By the definition of a simplex, ω has at least two $(p+r-1)$-faces, say ν_1 and ν_2, such that $\sigma < \nu_1 < \omega$ and $\sigma < \nu_2 < \omega$. From condition 2. of Definition 3.1.1, it follows that

$$f(\nu_1) < f(\omega) \text{ or } f(\nu_2) < f(\omega).$$

Assume without loss of generality that $f(\nu_1) < f(\omega)$. By the inductive hypothesis, there exists a $(p+1)$-simplex τ such that $\sigma < \tau \leq \nu_1$ and $f(\tau) \leq f(\nu_1)$. Therefore, $\sigma < \tau \leq \omega$ and $f(\tau) \leq f(\omega)$, completing the proof. □

Theorem 3.3.6 (**Collapse Theorem**)**.** Let K be a simplicial complex and let $f : K \to \mathbb{R}$ be a discrete Morse function with no critical values in $[a, b] \subseteq \mathbb{R}$. Then $K(b)$ collapses to $K(a)$.

Proof. By Lemma 3.3.3, we may assume f to be injective on $[a, b]$. If $f^{-1}([a, b]) = \varnothing$, then $K(b) = K(a)$. Otherwise, since f is injective and has a finite image, we can partition $[a, b]$ into subintervals, each containing exactly one regular value. Thus, up to repeating the following for every simplex in $[a, b]$, we may assume without loss of generality $f^{-1}([a, b]) = \{\sigma\}$, where σ is a regular p-simplex. By the Exclusion Lemma (Lemma 3.1.6), exactly one of the following conditions holds.
Case 1. There is $\tau^{(p+1)} > \sigma^{(p)}$ such that $f(\tau) \leq f(\sigma)$. Since $f(\tau) < a$, it follows that $\tau \in K(a)$. As $\sigma < \tau$, then $\sigma \in K(a)$. Therefore, $K(b) = K(a)$.
Case 2. There is $\nu^{(p-1)} < \sigma^{(p)}$ such that $f(\nu) \geq f(\sigma)$. We show that $\{\nu, \sigma\}$ forms a free pair, implying

$$K(b) \searrow K(b) \setminus \{\nu, \sigma\} = K(a).$$

From Lemma 3.1.6, for every $\tau^{(p+1)} > \sigma$, we have $f(\tau) > f(\sigma)$, hence $f(\tau) > b$. By Lemma 3.3.5, for any $\omega > \sigma$ of any dimension, there exists $\tau^{(p+1)} > \sigma$ such that $\sigma < \tau \leq \omega$ and $f(\tau) \leq f(\omega)$. Since $f(\tau) > b$, it follows that $f(\omega) > b$. Due to Definition 3.3.1 we conclude that $\sigma \notin K(a)$.
By Definition 3.1.1, for any $\pi^{(p-1)} < \sigma$ with $\pi \neq \nu$, we have $f(\pi) < f(\sigma)$, hence $f(\pi) < a$, so $\pi \in K(a)$. Since $f(\nu) \geq f(\sigma)$ and $f^{-1}([a, b]) = \{\sigma\}$, it follows that $f(\nu) > b$. Similarly, for any $\eta^{(p)} > \nu$ with $\eta \neq \sigma$, we have $f(\eta) > f(\nu)$, hence $f(\eta) > b$. Analogously as above, by Lemma 3.3.5 it follows that $\forall \eta > \nu$ of any dimension with $\eta \neq \sigma$, $f(\nu) > b$, hence $\eta \notin K(b)$. Since $\sigma \notin K(a)$, then $\nu \notin K(a)$. Therefore we can write $K(b)$ as a disjoint union $K(b) = K(a) \cup \{\nu, \sigma\}$ where ν has no other cofaces in $K(b)$. We conclude that $\{\nu, \sigma\}$ is a free pair and $K(b) \searrow K(a)$. □

Example 3.3.7. Consider the injective discrete Morse function f from Figure 3.9 and the interval $[18, 40]$ where f has no critical values. The Collapse Theorem implies that $K = K(40) \searrow K(18)$. The following is a sequence of elementary collapses from $K(40)$ to $K(18)$:

$$\begin{aligned} K(40) \searrow K' &:= K(40) \setminus \{v_3v_4, v_2v_3v_4\}, \\ K' \searrow K'' &:= K' \setminus \{v_2v_3, v_1v_2v_3\}, \\ K'' \searrow K'' &\setminus \{v_1v_2, v_0v_1v_2\} = K(18). \end{aligned}$$

Figure 3.10 shows the resulting level subcomplex. We conclude that K and $K(18)$ are of the same simple homotopy type. Indeed, observe that $K(18)$ has a single 1-dimensional "hole", like K, as we computed in Example 2.2.23.

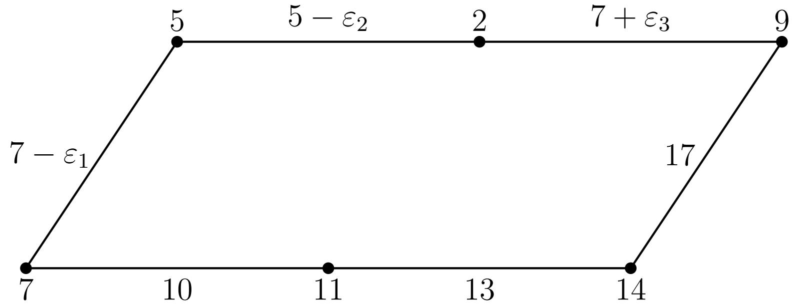

Figure 3.10: $K(18)$.

Now we present the discrete analogue to a classic result in Morse theory according to which the only "significant" simplices are the critical ones. This result requires us to introduce some new terminology. The theorem holds for a general smooth manifold: it is not just a special case for simplicial complexes.

Definition 3.3.8. Let K be a simplicial complex and $f : K \to \mathbb{R}$ be a discrete Morse function. The **Morse numbers of** f are, for every $0 \leq p \leq \dim(K)$, the number of critical p-simplices $m_p(f)$. We denote it m_p when f is clear from the context.

To state the next theorem, we need to introduce the concept of **CW complex**. Here, we provide a partial description, because a comprehensive discussion exceeds the scope of this work. For a detailed treatment we refer to Hatcher [8].

Notation 3.3.9. Here $\mathbb{S}^n$ represents the n-dimensional sphere and $\mathbb{D}^n$ represents the n-dimensional disk or ball.

Definition 3.3.10. A finite CW complex X of dimension N is a topological space constructed in the following way:

- Start with a finite set of points, denoted X^0.
- For each n (where $0 < n \leq N$), construct X^n by attaching a finite number of n-dimensional disks $\mathbb{D}_i^n$ to X^{n-1}. Each disk is attached along its boundary $\mathbb{S}_i^{n-1}$ through a continuous map $\varphi_i^n : \mathbb{S}_i^{n-1} \to X^{n-1}$, called an *attaching maps*.

- The final space, X, is defined as $X = X^N$.

Intuitively, a CW complex can be tought of as a generalization of a simplicial complex, where the role of simplices (such as triangles and tetrahedra) is replaced by geometric objects like disks. For example, a 2-simplex (triangle) in a simplicial complex corresponds to a 2-disk ($\mathbb{D}^2$) in a CW complex, with its boundary (a circle, $\mathbb{S}^1$) attached to the complex. As triangles in a simplicial complex are "glued" along their edges, disks in a CW complex are attached to their corresponding boundaries, providing a flexible framework for constructing (smooth) topological spaces.

Theorem 3.3.11. Let K be a simplicial complex and $f : K \to \mathbb{R}$ be a discrete Morse function. Then K is homotopy equivalent to a CW complex with exactly $m_p(f)$ cells of dimension p.

Proof. See [6, Corollary 3.5]. □

The next two theorems establish a connection between the Betti numbers of a simplicial complex and the discrete Morse functions defined on it. These theorems serve as powerful tools for computing the homology of a simplicial complex within the framework of discrete Morse theory. While we will not provide a proof of the first theorem, as it lies beyond the scope of this work, we will demonstrate how the second theorem can be derived from the first.

Theorem 3.3.12 (Strong Discrete Morse Inequalities). Let K be a simplicial complex of dimension n. Let $f : K \to \mathbb{R}$ be a discrete Morse function. Then for any $p = 0, 1, \dots, n$:

$$\beta_p - \beta_{p-1} + \cdots \pm \beta_0 \leq m_p - m_{p-1} + \cdots \pm m_0.$$

Proof. See [14, Section 4.1]. □

Theorem 3.3.13 (Weak Discrete Morse Inequalities). Let K be a simplicial complex of dimension n. Let $f : K \to \mathbb{R}$ be a discrete Morse function. Then:

1. $$\beta_p \leq m_p$$
   for every $p = 0, 1, \dots, n$;
2. $$\sum_{p=0}^{n} (-1)^p m_p = \chi(K),$$
   where $\chi(K)$ is the Euler characteristic of K.

Proof. Let $0 \leq p \leq n$ be an integer. By Theorem 3.3.12, the following two inequalities hold:

$$\beta_p - \beta_{p-1} + \cdots \pm \beta_0 \leq m_p - m_{p-1} + \cdots \pm m_0;$$
$$\beta_{p-1} - \cdots \mp \beta_0 \leq m_{p-1} - \cdots \mp m_0.$$

Adding these inequalities term by term eliminates all alternating terms on both sides, yielding $\beta_p \leq m_p$. This proves the first inequality.

For the second part, recall that the Euler Characteristic $\chi(K)$ is defined as

$$\chi(K) = \sum_{p=0}^{n} (-1)^p n_p$$

where n_p is the number of p-simplices in K. Now consider the gradient vector field V associated with f. The field V is defined as a collection of pairs $(\sigma^{(k)}, \tau^{(k+1)})$ of simplices where $\sigma^{(k)}$ is a codimension-1 face of $\tau^{(k+1)}$. This means that each pair in V does not contribute to the Euler characteristic. As noted in Remark 3.2.2, all and only the simplices that are critical, i.e., those not part of the gradient vector field, contribute to the computation of $\chi(K)$. Therefore, the number of critical simplices determines the Euler characteristic, which is expressed as

$$\sum_{p=0}^{n} (-1)^p m_p = \chi(K).$$

□

Remark 3.3.14. Recall that homology is a topological invariant. Euler characteristic can be equivalently defined as the alternating sum of the Betti numbers, namely

$$\chi(K) = \sum_{p=0}^{n} (-1)^p \beta_p$$

for a n-dimensional simplicial complex (see Hatcher [8] for a complete discussion). Therefore, Euler characteristic is a topological invariant. From this point of view, the Weak Discrete Morse Inequalities give us an important insight on the discrete Morse functions that can be defined on a simplicial complex: they depend on the topological structure of K.

Example 3.3.15. Consider the function f defined in Example 3.2. Its Morse numbers are $m_0 = 1, m_1 = 0, m_2 = 1$. Recall that from Example 1.4.9 that $\chi(K) = 0$. Then we have that $\chi(K) \neq -2 = \sum_{p=0}^{2} (-1)^p m_p$. This tells us from another perspective that f is not a discrete Morse function.

3.4 Critical Complex

We introduce the concept of a critical complex, which offers a method for computing the homology of a simplicial complex by focusing solely on the critical simplices associated with a discrete Morse function.

Definition 3.4.1. Let K be a simplicial complex and let $C_p(K)$ be its p-th chain group for $0 \leq p \leq n = \dim(K)$. Let $f : K \to \mathbb{R}$ be a discrete Morse function with associated gradient vector field V. The **p-th critical group of**

K **with respect to** V is the subgroup $C_p^V(K)$ of $C_p(K)$ generated by the p-chains consisting of the critical p-simplices of f. In other words:

$$C_p^V(K) = \left\{ \sum_{\sigma \in F_p} t_\sigma \sigma \;\middle|\; t_\sigma \in \mathbb{Z}_2 \text{ and } \Big[\; t_\sigma = 1 \Longrightarrow \sigma \text{ is critical} \;\Big] \right\}.$$

We define $C_{-1}^V(K) = \{0\}$ and for any $p > \dim(K), C_p^V(K) = \{0\}$. Furthermore, we write C_p^V when K is clear from the context.

Notation 3.4.2. Let $\tau^{(p)} \in K$ be a p-simplex. Consider the p-chain $c_\tau \in C_p^V$ with all zero coefficients except for $t_\tau = 1$. We will refer to c_τ simply as τ even though c_τ is a p-chain and τ is a p-simplex.

Remark 3.4.3. Notice that C_p^V has a structure of vector space over the field $\mathbb{Z}_2$. Furthermore, notice that

$$\dim(C_p^V) = m_p(f).$$

Definition 3.4.4. Let K be a simplicial complex, f be a discrete Morse function on K with induced gradient vector field V. Let $C_p^V(K)$ be the p-th critical group of K with respect to V, for each $0 \leq p \leq n = \dim(K)$. Define the **boundary operator of** $C_p^V(K)$ as the function $\partial_p^V : C_p^V \to C_{p-1}^V$ with

$$\partial_p^V(\tau) = \sum_{\substack{\sigma \text{ critical} \\ (p-1)-\text{simplex}}} \delta_{\tau,\sigma}\, \sigma$$

where

$$\delta_{\tau,\sigma} = \begin{cases} 0 & \text{if } \gamma(\tau,\sigma) \text{ is even} \\ 1 & \text{if } \gamma(\tau,\sigma) \text{ is odd} \end{cases}$$

and $\gamma(\tau, \sigma)$ is the number of V-paths from τ to σ.

Before we define the homology groups defined on the critical groups, we need to show the following result on the composition of boundary operators.

Proposition 3.4.5. Let K be a simplicial complex and f be a discrete Morse function on K with induced gradient vector field V. Let q be an arbitrary integer such that $0 \leq q \leq n = \dim(K)$. Then $\partial_q^V \circ \partial_{q+1}^V = 0$.

Proof. The proof is by induction on the number of pairs belonging to the gradient vector field V.
Base case (V has no pairs). In this case all simplices are critical, so the number of V-paths from a q-simplex $\sigma^{(q)}$ to a $(q-1)$-simplex $\tau^{(q-1)}$ is 1 if and only if $\tau < \sigma$; otherwise the are no V-paths from σ to τ. Therefore the operator ∂_q^V is equal to the corresponding operator from simplicial homology ∂_q for every $0 \leq q \leq n+1$. Then the postulate follows from Proposition 2.2.12.
Inductive step (V has $k \geq 1$ pairs, the postulate holds for $0 \leq l < k$). Let p be an integer such that $0 \leq p \leq n$ and $(\sigma^{(p)}, \tau^{(p+1)}) \in V$ and $\overline{V} := V \setminus \{\sigma, \tau\}$. By inductive hypothesis, for all $0 \leq q \leq n$ we have $\partial_q^{\overline{V}} \circ \partial_{q+1}^{\overline{V}} = 0$. Notice that

$C_i^V = C_i^{\overline{V}}$ for $i \notin \{p, p+1\}$ and $\partial_i^V = \partial_i^{\overline{V}}$ for $i \notin \{p, p+1, p+2\}$. Then we only need to show that $\partial_{i-1}^V \circ \partial_i^V = 0$ for $i \in \{p, p+1, p+2\}$. To prove this, we fix an arbitrary $\alpha \in C_i^V$ and we prove that the coefficient of each $(i-2)$-simplex $\nu^{(i-2)}$ in the expression $\partial_{i-1}^V \circ \partial_i^V(\alpha)$ is zero. In order to prove the latter, we are going to use the following equality:

$$\begin{aligned}
\partial_{i-1}^V \circ \partial_i^V &= \sum_{\substack{\beta \text{ critical} \\ (i-1)\text{-simplex}}} \Big(\delta_{\alpha,\beta}\ \partial_{i-1}^V(\beta)\Big) = \\
&= \sum_{\substack{\beta \text{ critical} \\ (i-1)\text{-simplex}}} \delta_{\alpha,\beta} \left(\sum_{\substack{\nu \text{ critical} \\ (i-2)\text{-simplex}}} \delta_{\beta,\nu}\ \nu \right) = \\
&= \sum_{\substack{\nu \text{ critical} \\ (i-2)\text{-simplex}}} \left(\sum_{\substack{\beta \text{ critical} \\ (i-1)\text{-simplex}}} \delta_{\alpha,\beta}\ \delta_{\beta,\nu} \right) \nu = \\
&= \sum_{\substack{\nu \text{ critical} \\ (i-2)\text{-simplex}}} \delta_{\alpha,\nu}\ \nu
\end{aligned}$$

where

$$\delta_{\alpha,\nu} = \begin{cases} 0 & \text{if } \lambda(\alpha,\nu) \text{ is even} \\ 1 & \text{if } \lambda(\alpha,\nu) \text{ is odd} \end{cases}$$

and $\lambda(\alpha,\nu)$ is the number of juxtaposition of V-paths from α to a critical $(i-1)$-simplex β and V-paths from β to ν. We denote these juxtapositions as $\alpha \rightharpoondown \beta \rightharpoondown \nu$. Notice that the latter is valid for any gradient vector field.

- **Case 1** ($i = p$). Let $\alpha^{(p)} \in C_p^V$. Fix an arbitrary $\nu^{(p-1)} \in C_{p-1}^V$. By inductive hypothesis, we have $\partial_{p-1}^{\overline{V}} \circ \partial_p^{\overline{V}}(\alpha) = 0$. We only need to show that $\partial_p^V = \partial_p^{\overline{V}}$, since we already have $\partial_{p-1}^V = \partial_{p-1}^{\overline{V}}$. If $\alpha = \sigma$, then the V-paths from $\sigma^{(p)}$ to $\nu^{(p-1)}$ are the same as the $\overline{V}$-paths from $\sigma^{(p)}$ to $\nu^{(p-1)}$ because $\overline{V} = V \setminus \{(\sigma^{(p)}, \tau^{(p+1)})\}$. Instead, if $\alpha \neq \sigma$, then no $\overline{V}$-path from $\alpha^{(p)}$ to $\nu^{(p-1)}$ goes through $\sigma^{(p)}$ because $\sigma^{(p)}$ is critical for $\overline{V}$. Moreover, no V-path from $\alpha^{(p)}$ to $\nu^{(p-1)}$ passes through $\sigma^{(p)}$ since $(\sigma^{(p)}, \tau^{(p+1)}) \in V$. Therefore, in both cases we have $\partial_{p-1}^V \circ \partial_p^V(\alpha) = \partial_{p-1}^{\overline{V}} \circ \partial_p^{\overline{V}}(\alpha) = 0$.

- **Case 2** ($i = p+1$). Let $\alpha \in C_{p+1}^V$. Fix an arbitrary $\nu^{(p-1)} \in C_{p-1}^V$. By inductive hypothesis, we have $\partial_p^{\overline{V}} \circ \partial_{p+1}^{\overline{V}}(\alpha) = 0$. From the equalities above, it follows that for every juxtaposition of $\overline{V}$-paths $\alpha^{(p+1)} \rightharpoondown \beta_1^{(p)} \rightharpoondown \nu^{(p-1)}$ with β_1 an arbitrary critical simplex, there is another juxtaposition $\alpha^{(p+1)} \rightharpoondown \beta_2^{(p)} \rightharpoondown \nu^{(p-1)}$ (with β_2 critical complex that is possibly equal to β_1) that makes the total number of juxtapositions even: in other words, $\alpha^{(p+1)} \rightharpoondown \beta_1^{(p)} \rightharpoondown \nu^{(p-1)}$ cancels with $\alpha^{(p+1)} \rightharpoondown \beta_2^{(p)} \rightharpoondown \nu^{(p-1)}$. If both $\beta_1 \neq \sigma$ and $\beta_2 \neq \sigma$, then nothing changes when considering the juxtapositions of V-paths. Instead, consider the case when at least one of β_1, β_2 is equal to σ: we show the case $\beta_1 = \sigma \neq \beta_2$, but the same procedure holds for $\beta_1 = \sigma = \beta_2$. Now, we have that the juxtaposition of $\alpha^{(p+1)} \rightharpoondown \sigma^{(p)} \rightharpoondown \nu^{(p-1)}$ cancels with $\alpha^{(p+1)} \rightharpoondown \beta_2^{(p)} \rightharpoondown \nu^{(p-1)}$. Since $\partial_p^{\overline{V}} \circ \partial_{p+1}^{\overline{V}}(\tau) = 0$, then there must be a critical simplex $\beta_3^{(p)}$ such that $\tau^{(p+1)} \rightharpoondown \sigma^{(p)} \rightharpoondown \nu^{(p-1)}$ cancels with $\tau^{(p+1)} \rightharpoondown \beta_3^{(p)} \rightharpoondown \nu^{(p-1)}$. When considering $V = \overline{V} \cup \{(\sigma, \tau)\}$, since σ and τ are not critical anymore, then the $\overline{V}$-paths starting or ending with σ or τ are not V-paths. Then the only juxtaposition of $\overline{V}$-paths that remains also for V is $\alpha^{(p+1)} \rightharpoondown \beta_2^{(p)} \rightharpoondown \nu^{(p-1)}$. This cancels with a new juxtaposition of V-paths, namely $\alpha^{(p+1)} \rightharpoondown (\sigma^{(p)} \rightarrow \tau^{(p+1)}) \rightharpoondown \beta_3^{(p)} \rightharpoondown \nu^{(p-1)}$ that is created with the only $\overline{V}$-paths that "survive" also in V. Of course, if $\alpha = \tau$, the problem doesn't even arise: All juxtapositions starting from τ don't "survive" in V. We conclude that $\partial_p^{\overline{V}} \circ \partial_{p+1}^{\overline{V}}(\alpha) = 0$ implies $\partial_p^V \circ \partial_{p+1}^V(\alpha) = 0$.

- **Case 3** ($i = p+2$). This case is similar to the previous one. Let $\alpha \in C_{p+2}^V$. There are two cases to consider. The first is when the two juxtapositions of $\overline{V}$-paths that cancel each other don't go through τ. Then, like above, nothing changes when considering the juxtapositions of V-paths. The second case is when the juxtaposition of $\overline{V}$-paths that doesn't survive in V is replaced by a new one which goes through the pair $(\sigma \rightarrow \tau)$. Both cases are analogous to the above procedure, implying $\partial_{p+1}^V \circ \partial_{p+2}^V(\alpha) = 0$.

□

Thanks to Proposition 3.4.5 we have $\mathrm{Im}(\partial_{p+1}^V) \subseteq \ker(\partial_p^V) \subseteq C_P^V$. Therefore, the following are well defined.

Definition 3.4.6. The chain complex

$$C_n^V \xrightarrow{\partial_n^V} C_{n-1}^V \xrightarrow{\partial_{n-1}^V} \dots \xrightarrow{\partial_1^V} C_0^V \xrightarrow{\partial_0^V} \{0\}$$

is called the **critical complex of K with respect to V** and it is denoted (C_*^V, ∂^V). The p**-th critical homology group** is the quotient group

$$H_p^V(K) = \ker(\partial_p^V)/\mathrm{Im}(\partial_{p+1}^V).$$

It follows the final and most important result of the section, which tells us that the homology groups and the critical homology groups are isomorphic.

Theorem 3.4.7. Let K be a simplicial complex, f be a discrete Morse function on K with induced gradient vector field V and $C_p^V(K)$ be the critical complex of K with respect to V, for each $0 \leq p \leq n = \dim(K)$. Then $H_p^V(K)$ is isomorphic to $H_p(K)$.

Proof. The proof of this theorem goes beyond the scope of this thesis. For a complete discussion of the result we refer to [14, Chapter 8]. □

The previous theorem informs us that the homology group of a simplicial complex can also be computed by determining the critical homology group. We now provide two examples of computations carried out in this way: the first on the *torus*, a classic example in topology, and the second on the simplicial complex from Example 1.1.4.

Example 3.4.8. Consider the gradient vector field V shown in Figure 3.11, defined on a triangulation of the torus. Then the critical simplices are as follows: $v_0, v_3v_4, v_2v_7, v_6v_8, v_3v_5v_7$ and $v_1v_2v_6$. From this, we obtain the critical groups:

$$\begin{aligned}
C_0^V =& \{0, v_0\}\\
C_1^V =& \{0, v_3v_4, v_2v_7, v_6v_8, v_3v_4 + v_2v_7, v_3v_4 + v_6v_8, v_2v_7 + v_6v_8, v_2v_7 + v_6v_8 + v_3v_4\}\\
C_2^V =& \{0, v_3v_5v_7, v_1v_2v_6, v_3v_5v_7 + v_1v_2v_6\}
\end{aligned}$$

We now compute the boundary operators. Clearly, $\ker(\partial_0^V) = \{v_0\}$. We only need to evaluate ∂_1^V on v_3v_4, v_2v_7 and v_6v_8 as the others can be derived from these. To count the number of V-paths starting from v_3v_4 and ending at a critical 0-simplex, we simply follow the arrows in Figure 3.11 beginning at v_3v_4. If this leads to a critical 0-simplex, we count it as 1; if it leads to a 1-simplex with no further continuation, we disregard the path. By following this procedure, we obtain the following V-paths, where $\Gamma(\tau, \sigma)$ denotes the set of V-paths from τ to σ:

$$\begin{aligned}
\Gamma(v_3v_4, v_0) &= \{v_3v_4 > v_3 \to v_0v_3 > v_0,\\
&\qquad v_3v_4 > v_4 \to v_0v_4 > v_0\}\\
\Gamma(v_2v_7, v_0) &= \{v_2v_7 > v_2 \to v_2v_8 > v_8 \to v_5v_8 > v_5 \to v_0v_5 > v_0,\\
&\qquad v_2v_7 > v_7 \to v_4v_7 > v_4 \to v_0v_4 > v_0\}\\
\Gamma(v_6v_8, v_0) &= \{v_6v_8 > v_6 \to v_4v_6 > v_4 \to v_0v_4 > v_0,\\
&\qquad v_6v_8 > v_8 \to v_5v_8 > v_5 \to v_0v_5 > v_0\}
\end{aligned}$$

Using the definition of boundary operator, we compute $\partial_1^V(v_3v_4) = \partial_1^V(v_2v_7) = \partial_1^V(v_6v_8) = 0$, which implies that $\ker(\partial_1^V) = C_1^V$ and $\mathrm{Im}(\partial_1^V) = \{0\}$. We apply the same procedure for ∂_2^V. Notice that following the path $v_3v_5v_7 > v_3v_5 \to v_0v_3v_5$ leads us to a 2-simplex with no further continuation, so we disregard it. Below are the sets of V-paths that terminate at a generic critical edge $\sigma^{(1)}$:

$$\begin{aligned}
\Gamma(v_3v_5v_7, \sigma) &= \{v_3v_5v_7 > v_3v_7 \to v_3v_4v_7 > v_3v_4,\\
&\qquad v_3v_5v_7 > v_5v_7 \to v_5v_7v_8 > v_7v_8 \to v_2v_7v_8 \to v_2v_7\}\\
\Gamma(v_1v_2v_6, \sigma) &= \{v_1v_2v_6 > v_1v_2 \to v_1v_2v_7 > v_2v_7,\\
&\qquad v_1v_2v_6 > v_2v_6 \to v_2v_3v_6 > v_3v_6 \to v_3v_4v_6 > v_3v_4,\\
&\qquad v_1v_2v_6 > v_2v_6 \to v_2v_3v_6 > v_2v_3 \to v_0v_2v_3 > v_0v_2 \to\\
&\qquad \to v_0v_2v_8 > v_0v_8 \to v_0v_4v_8 > v_4v_8 \to v_4v_6v_8 > v_6v_8,\\
&\qquad v_1v_2v_6 > v_1v_6 \to v_1v_5v_6 > v_5v_6 \to v_5v_6v_8 > v_6v_8\}
\end{aligned}$$

It follows that $\partial_2^V(v_3v_5v_7) = \partial_2^V(v_1v_2v_6) = v_3v_4 + v_2v_7$, hence $\ker(\partial_2^V) = \{0, v_3v_5v_7 + v_1v_2v_6\}$ and $\mathrm{Im}(\partial_2^V) = \{0, v_3v_4 + v_2v_7\}$. Finally, we have $\mathrm{Im}(\partial_3^V) =$

$\{0\}$. We can obtain the critical homology groups:

$$\begin{aligned}
H_0^V &= \ker(\partial_0^V)/\mathrm{Im}(\partial_1^V) = \{0, v_0\}/\{0\} = \{[0], [v_0]\} \cong \mathbb{Z}_2 \\
H_1^V &= \ker(\partial_1^V)/\mathrm{Im}(\partial_2^V) = C_1^V/\{0, v_3v_4 + v_2v_7\} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\
H_2^V &= \ker(\partial_2^V)/\mathrm{Im}(\partial_3^V) = \{0, v_3v_5v_7 + v_1v_2v_6\}/\{0\} \cong \mathbb{Z}_2
\end{aligned}$$

where we denoted with $[\sigma]$ the equivalence class containing σ with respect to the quotient equivalence relation. From Theorem 3.4.7 it follows that the Betti numbers of the torus are $\beta_0 = 1, \beta_1 = 2$ and $\beta_2 = 1$.

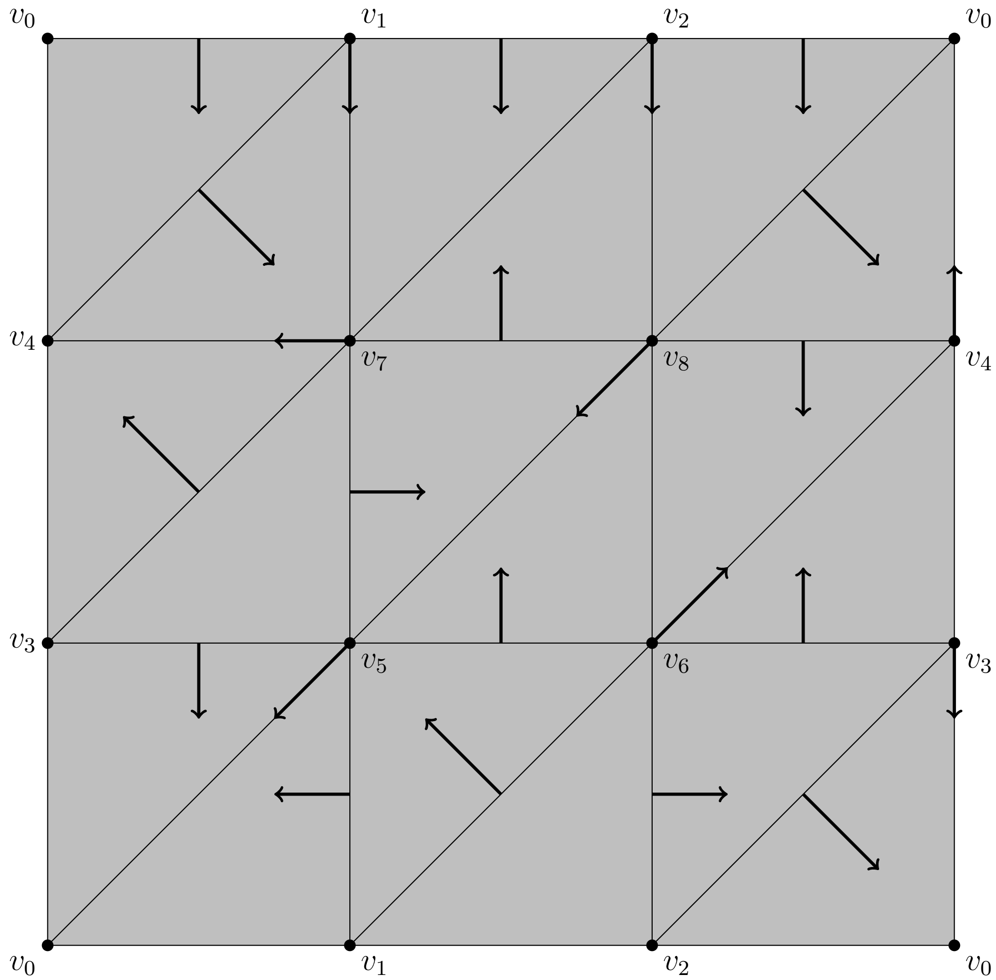

Figure 3.11: A gradient vector field on the triangulation of the torus.

Example 3.4.9. Consider the gradient vector field V from Example 3.2.11. The critical simplices of V are v_3 and v_4v_5. Then we obtain the critical groups $C_0^V = \{0, v_3\}$ and $C_1^V = \{0, v_4v_5\}$. Now we have that $\partial_0^V(v_3) = 0$ and since there are two V-paths from v_4v_5 to v_3, we also have $\partial_1^V(v_4v_5) = 0$. From this, we can compute the homology groups:

$$\begin{aligned}
&\ker(\partial_0^V) = M_0,\ \mathrm{Im}(\partial_1^V) = \{0\} && \Rightarrow H_0^V = C_0^V, \\
&\ker(\partial_1^M) = M_1,\ \mathrm{Im}(\partial_2^M) = \partial_2^M(\{0\}) = \{0\} && \Rightarrow H_1^M = M_1.
\end{aligned}$$

By Theorem 3.4.7, we conclude that the Betti numbers of K are $\beta_0 = 1, \beta_1 = 1$.

This example highlights the power of discrete Morse theory: it allows us to reduce a complicated simplicial complex into a much smaller "critical complex" while preserving its homological properties. Using Theorem 3.4.7, we can compute the homology of the critical complex and directly infer the homology of the original simplicial complex.

Example 3.4.10. We computed the homological properties of the social system from Example 1.1.4, but what do these results actually mean to us? The 0-th Betti number, $\beta_0 = 1$, indicates that the system is a single connected component. In other words, everyone is either directly or indirectly connected through mutual friends. This is useful information, for example, because it suggests that a rumor (or an opinion) can spread from any individual to anyone else in the group. Meanwhile, the 1-st Betti number being 1 indicates the presence of a 1-dimensional "hole" between Dave, Eve and Frank. This suggests a lack of a higher-order interaction between individuals. Specifically, the hole reflects that while Dave, Eve, and Frank interact in pairs, they do not engage with each other as a whole group.

Conclusions. Although the previous example is simplistic, analyzing a larger social system can provide valuable insights into the social dynamics of an event [19], the spread of opinions [4], and the adoption of innovations [9]. Discrete Morse theory provides an alternative method for calculating the homology of a simplicial complex, as we showed with Theorem 3.4.7. Directly computing homology can prove to be quite complex when dealing with big data clouds, as it involves matrices whose dimensions are the number of simplices. This is where the theory we exposed comes in handy: it allows us to compute homology by considering only critical simplices, which can be significantly less numerous, resulting in much lighter matrices, hence simpler computations.

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