Notes on Introduction to Quantum Computing

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December 6, 2022

1 Basic Concepts

1.1 Quantum bits (qubits)

Classical bits: $0, 1$
Quantum bit *qubit*: Superposition of 0 and 1:
A quantum state $|\psi\rangle$ is described as

$$|\psi\rangle := \alpha|0\rangle + \beta|1\rangle, \quad \alpha, \beta \in \mathbb{C} \tag{1}$$

where

$$|\alpha|^2 + |\beta|^2 = 1 \quad (normalization). \tag{2}$$

Mathematical description: $|\psi\rangle \in \mathbb{C}^2$ with

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \rightsquigarrow |\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Different from classical bits, cannot (in general) directly observe / measure a qubit (the amplitudes α and β). Instead: "*standard*" measurement will result in

- 0 with probability $|\alpha|^2$

- 1 with probability $|\beta|^2$

The measurement also changes the qubit (*wavefunction collapse*). If measuring 0, the qubit will be $|\psi\rangle = |0\rangle$ directly after the measurement, and likewise if measuring 1, the qubit will be $|\psi\rangle = |1\rangle$.

In practise: Can estimate the probabilities $|\alpha|^2$ and $|\beta|^2$ in experiments by repeating the same experiment many times (i.e via outcome statistics). These repetitions are called *trials* or *shots*.
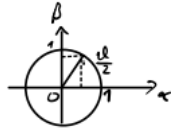

Figure 1: Circuit notation

A useful graphical deputation of a qubit is the Bloch sphere representation: If α and β happen to be real-valued, then can find angle $\vartheta \in \mathbb{R}$ such that

$$\alpha = \cos \frac{\vartheta}{2}, \quad \beta = \sin \frac{\vartheta}{2} \tag{3}$$

$$(\rightsquigarrow |\alpha|^2 + |\beta|^2 = \cos \frac{\vartheta}{2} + \sin \frac{\vartheta}{2}) = 1 \quad \checkmark$$
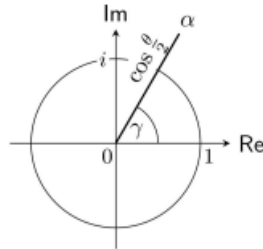
In general: represent

$$\alpha = e^{i\gamma} \cos \frac{\vartheta}{2}$$

$$\beta = e^{i(\varphi+\gamma)} \sin \frac{\vartheta}{2}$$

using so-called phase angles γ for α and $\varphi + \gamma$ for β.

Then:

$$|\psi\rangle = e^{i\psi} \cos \frac{\vartheta}{2} \cdot |0\rangle + \underbrace{e^{i(\varphi+\gamma)}}_{= \, e^{i\varphi} \cdot e^{i\gamma}} \sin \frac{\vartheta}{2} \cdot |1\rangle \tag{4}$$

$$= \underbrace{e^{i\gamma}}_{\text{can be ignored here}} \left(\cos \frac{\vartheta}{2} \cdot |0\rangle + e^{i\varphi} \cdot \sin \frac{\vartheta}{2} \cdot |1\rangle \right) \tag{5}$$

Thus $|\psi\rangle$ is characterized by two angles φ and γ; these specify the point defined as

$$\vec{r} = \begin{pmatrix} \cos \varphi \cdot \sin \vartheta \\ \sin \varphi \cdot \sin \vartheta \\ \cos \vartheta \end{pmatrix}$$
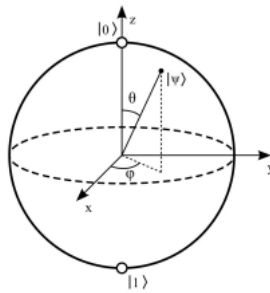
on the surface of a sphere:

3

Figure 2: Bloch Sphere (Felix Bloch)

1.2 Single qubit gates

Principles of <u>time evolution</u>: The quantum state $|\psi\rangle$ at current time point t transitions to a new quantum state $|\psi'\rangle$ at a later time point $t' > t$.
Transition described by a complex unitary matrix U:

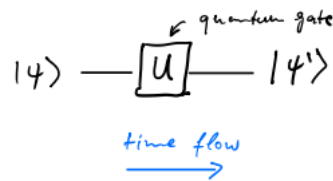$$|\psi'\rangle = U \cdot |\psi\rangle \tag{6}$$

Figure 3: Circuit notation

Notes:

- Circuit is read from left to right, but matrix times vector $(U|\psi\rangle)$ from right to left.

- U preserves normalization

Examples:

- Quantum analogue of the classical NOT gate $(0 \leftrightarrow 1)$ flip $|0\rangle \leftrightarrow |1\rangle$ leads to Pauli-X gate:

$$X \equiv \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7}$$

Check: $X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$ and $X|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$ ✓

4

- Pauli-Y gate:
$$Y \equiv \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{8}$$

- Pauli-Z gate:
$$Z \equiv \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9}$$

Z leaves $|0\rangle$ unchanged, but flips the sign of the coefficient of $|1\rangle$. Recall the Bloch Sphere representation:

$$|\psi\rangle = \cos\frac{\vartheta}{2} \cdot |0\rangle + e^{i\varphi} \sin\frac{\vartheta}{2} \cdot |1\rangle$$

Then

$$Z|\psi\rangle = \cos\frac{\vartheta}{2} \cdot |0\rangle \ - \ e^{i\varphi} \sin\frac{\vartheta}{2} \cdot |1\rangle$$

$$\overset{e^{i\pi}=-1}{=} \cos\frac{\vartheta}{2} \cdot |0\rangle + \underbrace{e^{i\pi} e^{i\varphi}}_{e^{i(\varphi+\pi)}} \sin\frac{\vartheta}{2} \cdot |1\rangle$$

⤳ new Bloch Sphere angles: $\vartheta' = \vartheta, \varphi = \varphi + \pi$ (rotating by $\pi = 180°$ around z-axis)

X, Y, Z gates are called <u>Pauli matrices</u>. The <u>Pauli vector</u> $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) = (X, Y, Z)$ is a vector of 2×2 matrices.

- Hadamard Gate:
$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\alpha \ |0\rangle \ + \ \beta \ |1\rangle \ \text{———}\ \boxed{H}\ \text{———}\ \alpha \ \frac{|0\rangle + |1\rangle}{\sqrt{2}} + \beta \ \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Figure 4: Hadamard Gate

- Phase Gate:
$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

- T Gate:
$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

Note: $T^2 = S$ since $(e^{i\pi/4})^2 = e^{i\pi/2} = i$

Pauli matrices satisfy:

1. $\sigma_j^2 = I$ (identifity) for $j = 1, 2, 3$

2. $\sigma_j \cdot \sigma_k = -\sigma_k \sigma_j$ for all $j \neq k$

3. $[\sigma_j, \sigma_k] := \underbrace{\sigma_j \sigma_k - \sigma_k \sigma_j}_{\text{Commutator}} = 2i\sigma_l$ for (j, k, l) a cyclic permutation of (1,2,3).

General definition of <u>matrix exponential</u>

$$exp(A) \equiv e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k, \quad A \in \mathbb{C}^{n \times n} \tag{10}$$

Special case: $A^2 = I, x \in \mathbb{R}$

$$e^{iAx} = \underbrace{\sum_{k=0}^{\infty} \frac{1}{(2k)!} (ix)^{2k} \underbrace{A^{2k}}_{(A^2)^k = I^k = I}}_{\text{even}} + \underbrace{\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (ix)^{2k+1} \underbrace{A^{2k+1}}_{(A^2)^k \cdot A = I^k \cdot A = A}}_{\text{odd}}$$

$$= \underbrace{\sum_{k=0}^{\infty} \frac{1}{(2k)!} (-1)^k x^{2k} \cdot I}_{=\cos x} + \underbrace{\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (-1)^k x^{2k+1} \cdot A}_{=i \sin x}$$

$$= \cos x \cdot I + i \sin x A$$

(generalizes Euler's formula $e^{ix} = \cos x + i \sin x$)

This can be used to define the following <u>rotation operators</u> via the Pali matrices. Let $\vartheta \in \mathbb{R}$:

$$R_x(\vartheta) := e^{-i\vartheta X/2} = \cos\frac{\vartheta}{2} I - i \sin\frac{\vartheta}{2} X = \begin{pmatrix} \cos\frac{\vartheta}{2} & -i\sin\frac{\vartheta}{2} \\ -i\sin\frac{\vartheta}{2} & \cos\frac{\vartheta}{2} \end{pmatrix} \tag{11}$$

$$R_y(\vartheta) := e^{-i\vartheta Y/2} = \cos\frac{\vartheta}{2} I - i \sin\frac{\vartheta}{2} Y = \begin{pmatrix} \cos\frac{\vartheta}{2} & -\sin\frac{\vartheta}{2} \\ \sin\frac{\vartheta}{2} & \cos\frac{\vartheta}{2} \end{pmatrix} \tag{12}$$

$$R_z(\vartheta) := e^{-i\vartheta Z/2} = \cos\frac{\vartheta}{2} I - i \sin\frac{\vartheta}{2} Z = \begin{pmatrix} e^{-i\vartheta/2} & 0 \\ 0 & e^{i\vartheta/2} \end{pmatrix} \tag{13}$$

General case: Rotation about an axis $\vec{v} \in \mathbb{R}^3$ (normalized such that $\|\vec{v}\|) = \sqrt{v_1^2 + v_2^2 + v_3^2} = 1$):
using the notation:

$$\langle \vec{v}|\vec{\sigma}\rangle = \vec{v} \cdot \vec{\sigma} = v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3 = \begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix} \quad (14)$$

It holds that $(\vec{v} \cdot \vec{\sigma})^2 = I$.
We define the rotation operator around axis $\vec{v}$ as

$$R_v(\vartheta) := e^{-i\vartheta(\vec{v}\cdot\vec{\sigma})/2} = \cos\frac{\vartheta}{2}I - i\sin\frac{\vartheta}{2}(\vec{v} \cdot \vec{\sigma}) \quad (15)$$

Note: R_x, R_y, R_z are special cases corresponding to $\vec{v} = (1,0,0)$, $\vec{v} = (0,1,0)$, and $\vec{v} = (0,0,1)$.

Can derive that the Bloch Sphere representaiton of $R_{\vec{v}}(\vartheta)$ is a "conventional" rotation (in three dimensions) by angle ϑ about axis $\vec{v}$.
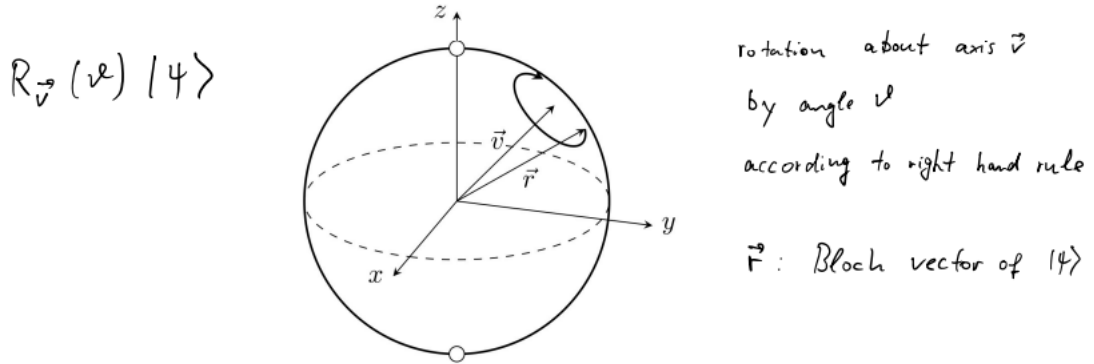

Figure 5: Circuit notation

Z-Y decomposition of an arbitrary 2×2 unitary matrix:
For any unitary matrix $U \in \mathbb{C}^{n\times n}$ there exist real numbers $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$U = e^{i\alpha} \underbrace{\begin{pmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{pmatrix}}_{R_z(\beta)} \cdot \underbrace{\begin{pmatrix} \cos\frac{\gamma}{2} & -\sin\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{pmatrix}}_{R_y(\gamma)} \cdot \underbrace{\begin{pmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{pmatrix}}_{R_z(\delta)} \quad (16)$$

1.3 Multiple qubits

So far: Single qubits, superposition of basis states $|0\rangle$ and $|1\rangle$. For two qubits, this generalizes to $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

General two-qubit state:

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle \tag{17}$$

with amplitudes $\alpha_{ij} \in \mathbb{C}$ such that

$$|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1 \quad \text{(normalization)}. \tag{18}$$

Can identify the basis states with unit vectors:

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{19}$$

Thus:

$$|\psi\rangle = \begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix} \in \mathbb{C}^4 \tag{20}$$

What happens if we measure only one qubit of a two-qubit state? Say we measure the first qubit: Obtain result

$$
\begin{array}{lll}
0 & \text{with probability} & |\alpha_{00}|^2 + |\alpha_{01}|^2 \\
1 & \text{with probability} & |\alpha_{10}|^2 + |\alpha_{11}|^2
\end{array}
$$

Wavefunction directly after measurement:

$$\text{if measured 0:} \quad |\psi'\rangle = \frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}}$$

$$\text{if measured 1:} \quad |\psi'\rangle = \frac{\alpha_{10}|10\rangle + \alpha_{11}|11\rangle}{\sqrt{|\alpha_{10}|^2 + |\alpha_{11}|^2}}$$

Mathematical formalism for constructing two qubit states: Tensor product of vector space.
Can combine two (arbitrary) vector spaces V and W to form the <u>tensor product</u> $V \otimes W$.

The elements of $V \otimes W$ are linear combinations of "tensor products" $|v\rangle \otimes |w\rangle$ consisting of elements $|v\rangle \in V$ and $|w\rangle \in W$.

Example: Let $V = \mathbb{C}^2$ and $W = \mathbb{C}^2$ be the single qubit spaces with basis $\{|0\rangle, |1\rangle\}$, then

$$\underbrace{\frac{1}{2}|0\rangle \otimes |0\rangle}_{=|00\rangle} + \underbrace{\frac{5i}{7}|1\rangle \otimes |0\rangle}_{=|10\rangle} \in V \otimes W$$

Let $\{|i\rangle_v : i = 1, ..., m\}$ be a basis of of V, and let $\{|j\rangle_w : j = 1, ..., n\}$ be a basis of of W, then

$$\{|i\rangle_v \otimes |j\rangle_w : i = 1, ..., m, j = 1, ..., n\}$$

is a basis of $V \otimes W$. In particular, $dim(V \otimes W) = dim(V) \cdot dim(W)$.
Note: $|i\rangle_v \otimes |j\rangle_w$ is also written as $|ij\rangle$.

Basic properties of tensor product:

- $\forall |v\rangle \in V, |w\rangle \in W \wedge \alpha \in \mathbb{C}:$

$$\alpha(|v\rangle \otimes |w\rangle) = (\alpha|0\rangle) \otimes |w\rangle = |v\rangle \otimes (\alpha|w\rangle) \quad (21)$$

- $\forall |v_1\rangle, |v_2\rangle \in V \wedge |w\rangle \in W:$

$$(|v_1\rangle + |v_2\rangle) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle \quad (22)$$

- $\forall |v\rangle \in V \wedge |w_1\rangle, |w_2\rangle \in W:$

$$|v\rangle \otimes (|w_1\rangle + |w_2\rangle) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle \quad (23)$$

Vector notation using standard basis, e.g.

$$|v\rangle = v_1|0\rangle + v_2|1\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$|w\rangle = w_1|0\rangle + w_2|1\rangle = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$|v\rangle \otimes |w\rangle = (v_1|0\rangle + v_2|1\rangle) \otimes (w_1|0\rangle + w_2|1\rangle)$$
$$= v_1 w_1|00\rangle + v_1 w_2|01\rangle + v_2 w_1|10\rangle + v_2 w_2|11\rangle$$

Thus:

$$
\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \otimes \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 w_1 \\ v_1 w_2 \\ v_2 w_1 \\ v_2 w_2 \end{pmatrix}
$$

Note: Not every element of $V \otimes W$ can be written in the form $|v\rangle \otimes |w\rangle$, for example the Bell state

$$
|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).
$$

Assuming that V and W have an inner product $\langle \cdot | \cdot \rangle$, define inner product on $V \otimes W$ by

$$
\langle \sum_j \alpha_j |v_j\rangle \otimes |w_j\rangle | \sum_k \beta_k |v_k\rangle \otimes |w_k\rangle \rangle := \sum_j \sum_k \alpha_j^* \beta_k \langle v_j | v_k \rangle \cdot \langle w_j | w_k \rangle \quad (24)
$$

Generalization to n qubits: 2^n computational basis states

$$
\{\underbrace{|0, ..., 0\rangle}_{\text{length } n}, |0, ..., 0, 1\rangle, ... |1, ..., 1\rangle\}
$$

Thus: General n-qubit quantum state, also denoted as "quantum register", given by:

$$
|\psi\rangle = \sum_{x_0=0}^{1} \sum_{x_1=0}^{2} \cdots \sum_{x_{n-1}}^{1} \alpha_{x_{n-1}, ..., x_1, x_0} \cdot |x_{n-1}, ..., x_1 x_0\rangle \quad (25)
$$

with $\alpha_x \in \mathbb{C}$ for all $x \in \{0, ..., 2^n - 1\}$, such that $\|\psi\|^2 = \sum_{x=0}^{2^n-1} |\alpha_x|^2 = 1$ (normalization).

$\rightsquigarrow$ In general *"hard"* to simulate on classical computer (for large n) due to "curse of dimensionality".

Vector space as tensor products: $\underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}} = (\mathbb{C}^2)^{\otimes n} = \mathbb{C}^{(2^n)}$

1.4   Multiple qubit gates

As for single qubits, an operation on multiple qubits is described by an unitary matrix U. For n qubits: $U \in \mathbb{C}^{2^n \times 2^n}$

Example: <u>controlled-NOT</u> gate (also CNOT):
two qubits: control and target, target qubit gets flipped if control is 1:

$$|00\rangle \mapsto |00\rangle, \quad |01\rangle \mapsto |01\rangle, \quad |10\rangle \mapsto |11\rangle, \quad |11\rangle \mapsto |11\rangle$$

Can be expressed as

$$|a, b\rangle \mapsto |a, a \oplus b\rangle \quad \forall a, b \in \{0, 1\} \tag{26}$$
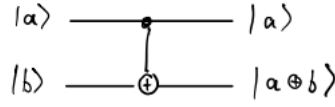
, where $\oplus$ is the addition modulo 2.

Figure 6: CNOT circuit notation

Matrix representation:

$$U_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{27}$$

, with the Pauli-X matrix $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Figure 7: Alternative CNOT circuit notation

Can generalize Pauli-X to any unitary operator U acting on target qubit
⤳ **controlled-U gate**:

$$|00\rangle \mapsto |00\rangle, \quad |01\rangle \mapsto |01\rangle, \quad |10\rangle \mapsto |1\rangle \otimes (U|0\rangle), \quad |11\rangle \mapsto |1\rangle \otimes (U|1\rangle)$$

Figure 8: Controlled-U gate
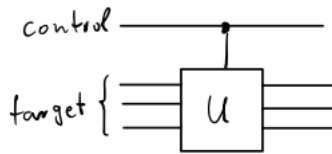
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Figure 9: Example: Controlled-Z gate

Figure 10: Controlled-U for multiple target qubits

Note: Single qubit and CNOT gates are <u>universal</u>: They can be used to implement an arbitraty unitray operation on n qubits (Quantum analogue of univerisity of classical NAND gate). Proof in Nielsen and Chuang section 4.5.
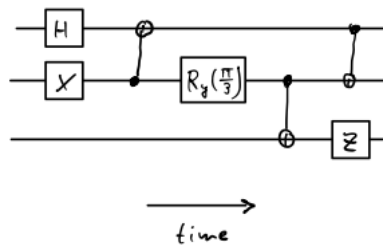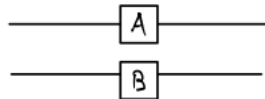

Figure 11: Example of a circuit lousisting only of single qubit gates and CNOTs

1.4.1 Matrix Kronecker Products

Matrix representation of single qubit gates acting in parallel:

Operation on basis states: $a, b \in \{0, 1\}$:

$$|a, b\rangle \mapsto (A|a\rangle) \otimes (B|b\rangle) \tag{28}$$

12

Example: $A = I$ (identity), $B = Y$

$$|00\rangle \mapsto |0\rangle \otimes (Y|0\rangle) = i|01\rangle$$
$$|01\rangle \mapsto |0\rangle \otimes (Y|1\rangle) = -i|00\rangle$$
$$|10\rangle \mapsto |1\rangle \otimes (Y|0\rangle) = i|11\rangle$$
$$|11\rangle \mapsto |1\rangle \otimes (Y|1\rangle) = -i|10\rangle$$

Matrix representation:

$$\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} = \begin{pmatrix} Y & 0 \\ 0 & Y \end{pmatrix} = I \otimes Y$$
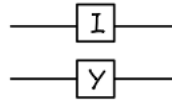

Figure 12: Circuit notation

General formula: <u>Kronecker product</u> (matrix representation of tensor products of operators)

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix} \in \mathbb{C}^{mp \times nq} \qquad (29)$$

for all $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{p \times q}$.
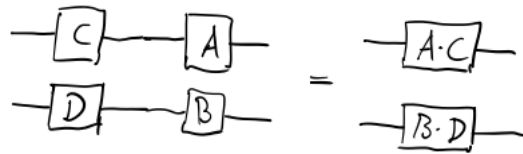
Another example:

$$\hat{=} \quad Y \otimes I = \begin{pmatrix} 0 \cdot I & -i \cdot I \\ i I & 0 \cdot I \end{pmatrix} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\hat{=} \quad A \otimes B \otimes C$$
$$= (A \otimes B) \otimes C = A \otimes (B \otimes C)$$

Figure 13: Generalization to arbitrary number of tensor factors possible

Basic properties:

1. $(A \otimes B)^* = A^* \otimes B^*$ (elementwise complex conjugation)

2. $(A \otimes B)^T = A^T \otimes B^T$ (transposition)

3. $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$

4. $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ (associative property)

5. $(A \otimes B) \cdot (C \otimes D) = (A \cdot C) \otimes (B \cdot D)$ (for matrix of compatible dimensions)

6. Kronecker product of Hermitian matrices is Hermitian.

7. Kronecker product of unitary matrices is unitary (follows from 3. and 5.)

1.5 Quantum measurement

Review: measurement of a single qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with resepct to the computational basis $\{|0\rangle, |1\rangle\}$.
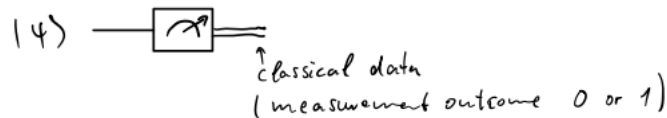
Linear algebra: Can switch to a different (orthonormal) basis to represent a qubit, e.g.

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

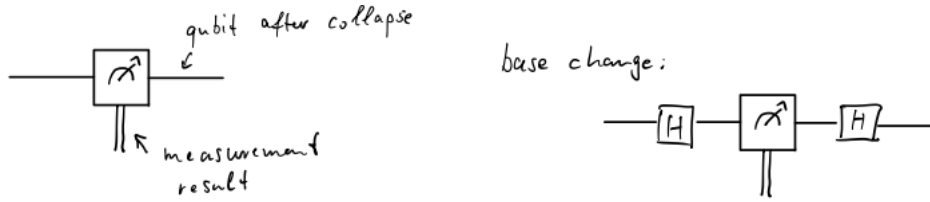$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Representation of $|\psi\rangle$ w.r.t $\{|+\rangle, |-\rangle\}$ basis:

$$\alpha|0\rangle + \beta|1\rangle = \alpha\frac{|+\rangle + |-\rangle}{\sqrt{2}} + \beta\frac{|+\rangle - |-\rangle}{\sqrt{2}} = \frac{\alpha + \beta}{\sqrt{2}}|+\rangle + \frac{\alpha - \beta}{\sqrt{2}}|-\rangle$$

Can perform measurement with resepct to orthonormal basis $\{|+\rangle, |-\rangle\}$, will obtain result

$$+ \quad \text{with probability} \quad \frac{|\alpha + \beta|^2}{2}$$

$$- \quad \text{with probability} \quad \frac{|\alpha - \beta|^2}{2}$$

Wavefunction collapse: immediately after the measurement, qubit will be in the state $|+\rangle$ if measured "+", likewise in the state $|-\rangle$ if measured "-".

In general given an orthonormal basis $\{|u_1\rangle, |u_2\rangle\}$, one can represent a qubit as $|\psi\rangle = \alpha_1|u_1\rangle + \alpha_2|u_1\rangle$ and measure with respect to this orthonormal basis; will obtain measurement result u_1 or u_2 with respective probabilities $|\alpha_1|^2$ and $|\alpha_2|^2$.

1.5.1 Abstract general definition of quantum measurements

Quantum measurements are described by a collection $\{M_m\}$ of <u>measurement operators</u> acting on the quantum system, with the index m labelling possible measurement outcomes.
Denoting the quantum state before measurement $|\psi\rangle$, result m occurs with probability

$$p(m) = \langle \psi | M_m^\dagger M_m | \psi \rangle = \|M_m|\psi\rangle\|^2 \tag{30}$$

, state after measurement is:

$$\frac{M_m|\psi\rangle}{\|M_m|\psi\rangle\|} \tag{31}$$

The measurement operators satisfy the completeness relation

$$\sum_m M_m^\dagger M_m = I \tag{32}$$

such that probabilities sum to 1:

$$\sum_m p(m) = \sum_m \langle \psi | M_m^\dagger M_m | \psi \rangle = \left\langle \psi \left| \sum_m M_m^\dagger M_m \right| \psi \right\rangle \langle \psi | \psi \rangle = 1 \tag{33}$$

since $\sum_m M_m^\dagger M_m = I$.

Example: measurement of a qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with respect to computational basis $\{|0\rangle, |\psi\rangle\}$.

$$M_0 := |0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$M_1 := |1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0, 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rightsquigarrow p(0) = \left\langle \psi \middle| M_0^\dagger M_0 \middle| \psi \right\rangle = \langle \psi | M_0 | \psi \rangle = |\alpha|^2$$

$$p(1) = \left\langle \psi \middle| M_1^\dagger M_1 \middle| \psi \right\rangle = \langle \psi | M_1 | \psi \rangle = |\beta|^2$$

1.5.2   Projective Measurements

Projector onto subspace V with orthonormal basis $\{|u_1\rangle, ..., |u_m\rangle\}$:

$$P = \sum_{j=1}^m |u_j\rangle\langle u_j| \tag{34}$$

$$P|w\rangle = \sum_{j=1}^m \underbrace{\langle u_j | w \rangle}_{\text{inner product}} \tag{35}$$

Relation to spectral decomposition of a normal matrix $A \in \mathbb{C}^{n\times n}$:

$$A = U \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U^\dagger = \sum_{j=1}^n \lambda_j |u_j\rangle\langle u_j| \tag{36}$$

$$= \sum_{k=1}^m \widetilde{\lambda_k} P_k \quad \text{with } \{\widetilde{\lambda_1}...\widetilde{\lambda_m}\} \text{ the distinct eigenvalues} \tag{37}$$

Definition: A <u>projective measurement</u> is described by an <u>observable</u> M, a Hermitian operator acting on the quantum system. Spectal decomposition:

$$M = \sum_m \lambda_m P_m \tag{38}$$

with P_m: projection onto eigenspace with eigenvalue λ_m. The possible outcomes of the measurement correspond to the eigenvalues λ_m.

Probability of getting result λ_m when measuring a quantum state $|\psi\rangle$:

$$p(\lambda_m) = \langle\psi|P_m|\psi\rangle \tag{39}$$

State of the quantum system directly after the measurement:

$$\frac{P_m|\psi\rangle}{\||P_m|\psi\rangle\|} = \frac{P_m|\psi\rangle}{\sqrt{p(\lambda_m)}} \tag{40}$$

Remarks:

- Projective measurements are special cases of general measurement framework

- Projective measurements combined with unitary transformations are equivalent to general measurement framework, see pages 94, 95 in Nielsen and Chuang.

Average value of a projective measurement:

$$\mathbb{E}[M] = \sum_m \lambda_m p(\lambda_m) = \sum_m \lambda_m \langle\psi|P_m|\psi\rangle \tag{41}$$

$$= \left\langle\psi\left|\sum_m \lambda_m P_m\right|\psi\right\rangle = \langle\psi|M|\psi\rangle = \langle M\rangle \tag{42}$$

Corresponding <u>standard deviation</u>:

$$\Delta(M) := \sqrt{\langle M^2\rangle - \langle M\rangle^2} = \sqrt{\langle(M - \langle M\rangle)^2\rangle} \tag{43}$$

Examples:

- Measuring a qubit w.r.t computational basis $\{|0\rangle, |1\rangle\}$ is actually a projective measurement.

- In general: Measurement w.r.t orthonormal basis $\{|u_1\rangle, |u_2\rangle\}$ is a projective measurement: Set

$$P_m = |u_m\rangle\langle u_m| \quad \text{for } m = 1, 2$$

Define observable M by

$$M := \sum_{m=1}^{2} \lambda_m P_m \quad \text{with arbitrary } \lambda_1, \lambda_2 \in \mathbb{R}; \lambda_1 \neq \lambda_2$$

- Measuring Pauli-Z

$$Z = 1 \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{P_1} + (-1) \cdot \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{P_2}$$

agrees with standard measurement w.r.t computational basis $\{|0\rangle, |1\rangle\}$

1.6 The Heisenberg uncertainty principle

Suppose A and B are Hermitian operators, and $|\psi\rangle$ a quantum state. Write

$$\langle\psi|AB|\psi\rangle = x + iy, \quad x, y \in \mathbb{R} \tag{44}$$

$$\langle\psi|AB|\psi\rangle^* = \langle\psi|(AB)^\dagger|\psi\rangle = \langle\psi|B^\dagger A^\dagger|\psi\rangle = \langle\psi|BA|\psi\rangle \tag{45}$$

Thus

$$\langle\psi|[A,B]|\psi\rangle = 2iy \quad \text{and} \quad \langle\psi|\{A,B\}|\psi\rangle = 2x \tag{46}$$

where $\{A, B\} := AB + BA$ is the anti-Commutator.

$$|\langle\psi|[A,B]|\psi\rangle|^2 + |\langle\psi|\{A,B\}|\psi\rangle|^2 = 4 \cdot \underbrace{|\langle\psi|AB|\psi\rangle|^2}_{x^2 + y^2} \tag{47}$$

Cauchy-Schwarz inequality applied to $|v\rangle = A|\psi\rangle, |w\rangle = B|\psi\rangle$:

$$|\langle\psi|[A,B]|\psi\rangle|^2 \overset{(47)}{\leq} 4 \cdot |\langle\psi|AB|\psi\rangle|^2 \leq 4 \cdot \langle\psi|A^2|\psi\rangle \cdot \langle\psi|B^2|\psi\rangle \tag{48}$$

Suppose C and D are two observables: substitute $A = C - \langle C \rangle$ and $B = D - \langle D \rangle$ leads to Heisenberg uncertainty principle:

$$\Delta(C) \cdot \Delta(D) \geq \frac{|\langle\psi|[C,D]|\psi\rangle|}{2} \tag{49}$$

Interpretation for experiments: Repeated preparation of $|\psi\rangle$, measure C in some cases, D in other cases to obtain standard deviations ΔC and $\Delta(D)$.

2 Entanglement and its applications

A n-qubit state $|\psi\rangle$ ($n \geq 2$) is called entangled if it cannnot be written as a tensor product of single-qubit states; i.e

$$|\psi\rangle \neq |\psi_{n-1}\rangle \otimes \cdots \otimes |\psi_0\rangle \tag{50}$$

Example: Bell states, also denoted EPR states (Einstein, Podolsky, Rosen):

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

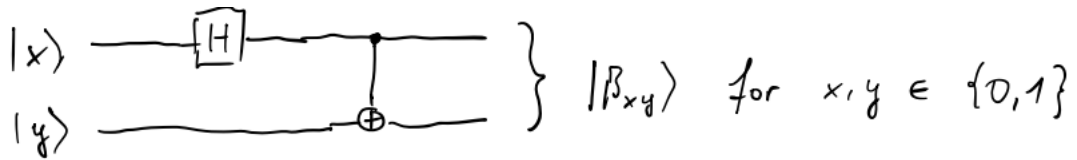$$|\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Figure 14: Quantum circuit to create Bell states

2.1 Quantum teleportation

Scenario: two (experimental physicists) Alice and Bob, are far away from each other

When visiting each other a long time ago, they generated the EPR pair $|\beta_{00}\rangle$ each keeping on qubit of the pair. Alice's task is to send another (unkown) qubit $|\psi\rangle$ to Bob. Note: Measurement is not an option.
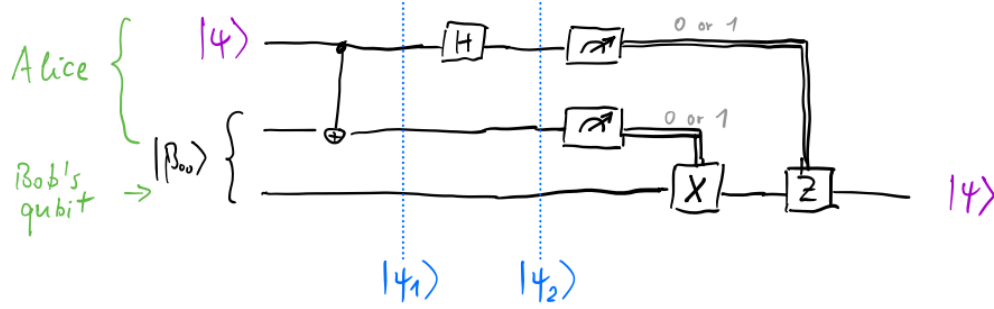
Figure 15: Quantum circuit for teleporting $|\psi\rangle$

Input:

$$|\psi\rangle|\beta_{00}\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes |\beta_{00}\rangle = \frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta\underbrace{|1\rangle(|00\rangle + |11\rangle)}_{\text{CNOT}}\right]$$

after CNOT:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|10\rangle + |01\rangle)\right]$$

after Hadamard:

$$|\psi_2\rangle = \frac{1}{2}[\alpha(|0\rangle + |1\rangle)\cdot(|00\rangle + |11\rangle)] + \beta(|0\rangle - |1\rangle)\cdot((|10\rangle + |01\rangle))$$

$$= \frac{1}{2}(\alpha|000\rangle + \alpha|011\rangle\alpha|100\rangle + \alpha|111\rangle) + \beta|010\rangle + \beta|001\rangle - \beta|110\rangle - \beta|101\rangle)$$

$$= \frac{1}{2}(\alpha|000\rangle + \alpha|011\rangle + \alpha|100\rangle + \alpha|111\rangle) + \beta|010\rangle + \beta|001\rangle - \beta|110\rangle - \beta|101\rangle$$

$$= \frac{1}{2}(|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle)) + |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle))$$

Now Alice measures her qubits w.r.t computational basis, e.g. projective measurement with

$$P_1 = |00\rangle\langle00| \otimes I, \quad P_2 = |01\rangle\langle10| \otimes I$$
$$P_3 = |10\rangle\langle10| \otimes I, \quad P_2 = |11\rangle\langle11| \otimes I$$

If Alice measures $|00\rangle$, then $|\psi_2\rangle$ will collapse to

$$|00\rangle(\alpha|0\rangle + \beta|1\rangle) = |00\rangle \underbrace{|\psi\rangle}_{\text{Qubit at Bob's place}}$$

20

similarly:

$$00 \mapsto \alpha|0\rangle + \beta|1\rangle$$
$$01 \mapsto \alpha|1\rangle + \beta|0\rangle$$
$$10 \mapsto \alpha|0\rangle - \beta|1\rangle$$
$$11 \mapsto \alpha|1\rangle - \beta|0\rangle$$

Alice transmits her measurement result to Bob (classical information), Bob then applies Pauli-X and / or Pauli-Z to recover $|\psi\rangle$. Even though wavefunction collapse is instantaneous, no faster-than-light transfer possible due to required classical communication.

2.2   EPR and the Bell inequality

EPR: Einstein, Podolsky, Rosen EPR paper: *"Can quantum mechanical description of physical relatiy be considered complete?"* (1935)
The another argue that quantum mechanics is incomplete since it lacks certain "elements of reality" (property can be predicted with certainity).

Scenario: Alice and Bob are far from each other, but share the entangled two-qubit "spin-singlet" state

$$|\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Alice and Bob measure the observable $\vec{v} \circ \vec{\sigma} = v_1 X + v_2 Y + v_3 Z$ (with $v \in \mathbb{R}^3, \|\vec{v}\| = 1$) on their respective qubit. (Recall $\vec{v} \circ \vec{\sigma}$ is Hermitian and unitary, and has eigenvalues ± 1)

Alice performs her measurement immediately before Bob. Example:

- $\vec{v} = (0, 0, 1)^T$, observable $Z = 1 \cdot |0\rangle\langle 0| + (-1) \cdot |1\rangle\langle 1|$ (standard measurement)
  if alice measures eigenvalue

$$1 : \quad \text{wavefunction collapses to } |01\rangle$$
$$0 : \quad \text{wavefunction collapses to } |10\rangle$$

$\rightsquigarrow$ Bob will always obtain the opposite measurement result.