Notes on Introduction to Quantum Computing

Leon Windheuser

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1 Basic Concepts

1.1 Quantum bits (qubits)

Classical bits: $0, 1$
Quantum bit *qubit*: Superposition of 0 and 1:
A quantum state $|\psi\rangle$ is described as

$$|\psi\rangle := \alpha|0\rangle + \beta|1\rangle, \quad \alpha, \beta \in \mathbb{C} \tag{1}$$

where

$$|\alpha|^2 + |\beta|^2 = 1 \quad (\textit{normalization}). \tag{2}$$

Mathematical description: $|\psi\rangle \in \mathbb{C}^2$ with

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \rightsquigarrow |\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Different from classical bits, cannot (in general) directly observe / measure a qubit (the amplitudes α and β). Instead: "*standard*" measurement will result in

- 0 with probability $|\alpha|^2$
- 1 with probability $|\beta|^2$

The measurement also changes the qubit (*wavefunction collapse*). If measuring 0, the qubit will be $|\psi\rangle = |0\rangle$ directly after the measurement, and likewise if measuring 1, the qubit will be $|\psi\rangle = |1\rangle$.

In practise: Can estimate the probabilities $|\alpha|^2$ and $|\beta|^2$ in experiments by repeating the same experiment many times (i.e via outcome statistics). These repetitions are called *trials* or *shots*.

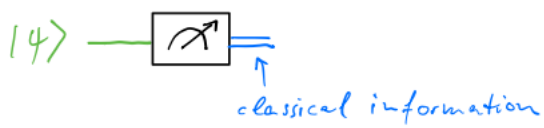
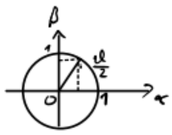


Figure 1: Circuit notation

A useful graphical deputation of a qubit is the <u>Bloch sphere</u> representation: If α and β happen to be real-valued, then can find angle $\vartheta \in \mathbb{R}$ such that

$$\alpha = \cos\frac{\vartheta}{2}, \quad \beta = \sin\frac{\vartheta}{2} \tag{3}$$

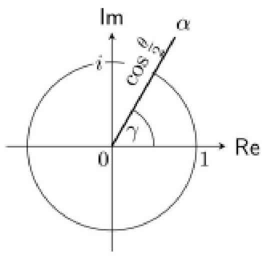
$$(\rightsquigarrow |\alpha|^2 + |\beta|^2 = \cos\frac{\vartheta}{2} + \sin\frac{\vartheta}{2}) = 1 \quad \checkmark$$



In general: represent

$$\alpha = e^{i\gamma}\cos\frac{\vartheta}{2}$$
$$\beta = e^{i(\varphi+\gamma)}\sin\frac{\vartheta}{2}$$

using so-called phase angles γ for α and $\varphi + \gamma$ for β.



Then:

$$|\psi\rangle = e^{i\psi}\cos\frac{\vartheta}{2}\cdot|0\rangle + \underbrace{e^{i(\varphi+\gamma)}}_{=e^{i\varphi}\cdot e^{i\gamma}}\sin\frac{\vartheta}{2}\cdot|1\rangle \tag{4}$$

$$= \underbrace{e^{i\gamma}}_{\text{can be ignored here}}\left(\cos\frac{\vartheta}{2}\cdot|0\rangle + e^{i\varphi}\cdot\sin\frac{\vartheta}{2}\cdot|1\rangle\right) \tag{5}$$

Thus $|\psi\rangle$ is characterized by two angles φ and γ; these specify the point defined as

$$\vec{r} = \begin{pmatrix}\cos\varphi\cdot\sin\vartheta\\ \sin\varphi\cdot\sin\vartheta\\ \cos\vartheta\end{pmatrix}$$

on the surface of a sphere:

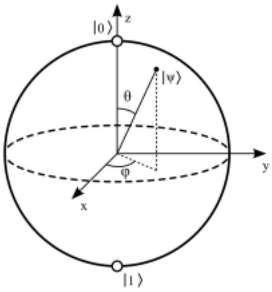


Figure 2: Bloch Sphere (Felix Bloch)

1.2 Single qubit gates

Principles of time evolution: The quantum state $|\psi\rangle$ at current time point t transitions to a new quantum state $|\psi'\rangle$ at a later time point $t' > t$.
Transition described by a complex unitary matrix U:

$$|\psi'\rangle = U \cdot |\psi\rangle \tag{6}$$

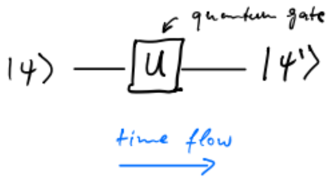


Figure 3: Circuit notation

Notes:

- Circuit is read from left to right, but matrix times vector ($U|\psi\rangle$) from right to left.
- U preserves normalization

Examples:

- Quantum analogue of the classical NOT gate ($0 \leftrightarrow 1$) flip $|0\rangle \leftrightarrow |1\rangle$ leads to Pauli-X gate:

$$X \equiv \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7}$$

Check: $X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$ and $X|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$ ✓

- Pauli-Y gate:
$$Y \equiv \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{8}$$
- Pauli-Z gate:
$$Z \equiv \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9}$$
  Z leaves $|0\rangle$ unchanged, but flips the sign of the coefficient of $|1\rangle$. Recall the Bloch Sphere representation:
$$|\psi\rangle = \cos\frac{\vartheta}{2} \cdot |0\rangle + e^{i\varphi} \sin\frac{\vartheta}{2} \cdot |1\rangle$$
  Then
$$\begin{aligned} Z|\psi\rangle &= \cos\frac{\vartheta}{2} \cdot |0\rangle - e^{i\varphi} \sin\frac{\vartheta}{2} \cdot |1\rangle \\ &\overset{e^{i\pi}=-1}{\equiv} \cos\frac{\vartheta}{2} \cdot |0\rangle + \underbrace{e^{i\pi} e^{i\varphi}}_{e^{i(\varphi+\pi)}} \sin\frac{\vartheta}{2} \cdot |1\rangle \end{aligned}$$
  $\rightsquigarrow$ new Bloch Sphere angles: $\vartheta' = \vartheta, \varphi = \varphi + \pi$ (rotating by $\pi = 180°$ around z-axis)
  X, Y, Z gates are called <u>Pauli matrices</u>. The <u>Pauli vector</u> $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) = (X, Y, Z)$ is a vector of 2×2 matrices.
- Hadamard Gate:
$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

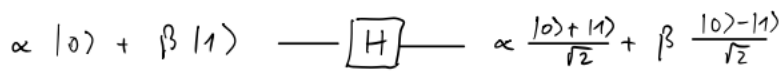


Figure 4: Hadamard Gate

- Phase Gate:
$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$
- T Gate:
$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$
  Note: $T^2 = S$ since $(e^{i\pi/4})^2 = e^{i\pi/2} = i$

Pauli matrices satisfy:

1. $\sigma_j^2 = I$ (identifity) for $j = 1, 2, 3$
2. $\sigma_j \cdot \sigma_k = -\sigma_k \sigma_j$ for all $j \neq k$
3. $[\sigma_j, \sigma_k] := \underbrace{\sigma_j \sigma_k - \sigma_k \sigma_j}_{\text{Commutator}} = 2i\sigma_l$ for (j, k, l) a cyclic permutation of (1,2,3).

General definition of matrix exponential

$$exp(A) \equiv e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k, \quad A \in \mathbb{C}^{n \times n} \tag{10}$$

Special case: $A^2 = I, x \in \mathbb{R}$

$$\begin{aligned} e^{iAx} &= \underbrace{\sum_{k=0}^{\infty} \frac{1}{(2k)!}(ix)^{2k} \underbrace{A^{2k}}_{(A^2)^k = I^k = I}}_{\text{even}} + \underbrace{\sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(ix)^{2k+1} \underbrace{A^{2k+1}}_{(A^2)^k \cdot A = I^k \cdot A = A}}_{\text{odd}} \\ &= \underbrace{\sum_{k=0}^{\infty} \frac{1}{(2k)!}(-1)^k x^{2k}}_{= \cos x} \cdot I + \underbrace{\sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(-1)^k x^{2k+1}}_{= i \sin x} \cdot A \\ &= \cos x \cdot I + i \sin x A \end{aligned}$$

(generalizes Euler's formula $e^{ix} = \cos x + i \sin x$)

This can be used to define the following rotation operators via the Pali matrices. Let $\vartheta \in \mathbb{R}$:

$$R_x(\vartheta) := e^{-i\vartheta X/2} = \cos\frac{\vartheta}{2} I - i \sin\frac{\vartheta}{2} X = \begin{pmatrix} \cos\frac{\vartheta}{2} & -i\sin\frac{\vartheta}{2} \\ -i\sin\frac{\vartheta}{2} & \cos\frac{\vartheta}{2} \end{pmatrix} \tag{11}$$

$$R_y(\vartheta) := e^{-i\vartheta Y/2} = \cos\frac{\vartheta}{2} I - i \sin\frac{\vartheta}{2} Y = \begin{pmatrix} \cos\frac{\vartheta}{2} & -\sin\frac{\vartheta}{2} \\ \sin\frac{\vartheta}{2} & \cos\frac{\vartheta}{2} \end{pmatrix} \tag{12}$$

$$R_z(\vartheta) := e^{-i\vartheta Z/2} = \cos\frac{\vartheta}{2} I - i \sin\frac{\vartheta}{2} Z = \begin{pmatrix} e^{-i\vartheta/2} & 0 \\ 0 & e^{i\vartheta/2} \end{pmatrix} \tag{13}$$

General case: Rotation about an axis $\vec{v} \in \mathbb{R}^3$ (normalized such that $\|\vec{v}\|) = \sqrt{v_1^2 + v_2^2 + v_3^3} = 1$):
using the notation:

$$\langle \vec{v} | \vec{\sigma} \rangle = \vec{v} \cdot \vec{\sigma} = v_1 \sigma_1 + v_2 \sigma_2 + v_3 \sigma_3 = \begin{pmatrix} v_3 & v_1 - i v_2 \\ v_1 + i v_2 & -v_3 \end{pmatrix} \tag{14}$$

It holds that $(\vec{v} \cdot \vec{\sigma})^2 = I$.

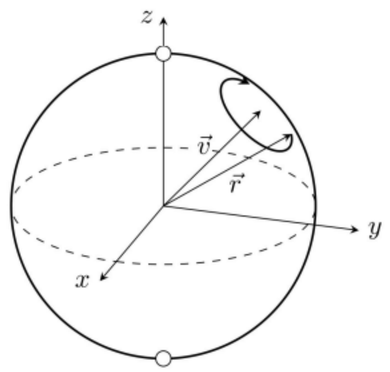
We define the rotation operator around axis $\vec{v}$ as

$$R_v(\vartheta) := e^{-i\vartheta(\vec{v}\cdot\vec{\sigma})/2} = \cos\frac{\vartheta}{2} I - i \sin\frac{\vartheta}{2} (\vec{v} \cdot \vec{\sigma}) \tag{15}$$

Note: R_x, R_y, R_z are special cases corresponding to $\vec{v} = (1,0,0)$, $\vec{v} = (0,1,0)$, and $\vec{v} = (0,0,1)$.

Can derive that the Bloch Sphere representaiton of $R_{\vec{v}}(\vartheta)$ is a "conventional" rotation (in three dimensions) by angle ϑ about axis $\vec{v}$.

$R_{\vec{v}}(\vartheta) |\psi\rangle$



rotation about axis $\vec{v}$
by angle ϑ
according to right hand rule

$\vec{r}$: Bloch vector of $|\psi\rangle$

Figure 5: Circuit notation

Z-Y decomposition of an arbitrary 2×2 unitary matrix:
For any unitary matrix $U \in \mathbb{C}^{n \times n}$ there exist real numbers $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$U = e^{i\alpha} \underbrace{\begin{pmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{pmatrix}}_{R_z(\beta)} \cdot \underbrace{\begin{pmatrix} \cos\frac{\gamma}{2} & -\sin\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{pmatrix}}_{R_y(\gamma)} \cdot \underbrace{\begin{pmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{pmatrix}}_{R_z(\delta)} \tag{16}$$

1.3 Multiple qubits

So far: Single qubits, superposition of basis states $|0\rangle$ and $|1\rangle$. For two qubits, this generalizes to $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

General two-qubit state:

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle \tag{17}$$

with amplitudes $\alpha_{ij} \in \mathbb{C}$ such that

$$|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1 \quad \text{(normalization).} \tag{18}$$

Can identify the basis states with unit vectors:

$$|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad |10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad |11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \tag{19}$$

Thus:

$$|\psi\rangle = \begin{pmatrix}\alpha_{00}\\\alpha_{01}\\\alpha_{10}\\\alpha_{11}\end{pmatrix} \in \mathbb{C}^4 \tag{20}$$

What happens if we measure only one qubit of a two-qubit state? Say we measure the first qubit: Obtain result

$$\begin{array}{lll} 0 & \text{with probability} & |\alpha_{00}|^2 + |\alpha_{01}|^2 \\ 1 & \text{with probability} & |\alpha_{10}|^2 + |\alpha_{11}|^2 \end{array}$$

Wavefunction directly after measurement:

$$\begin{aligned} \text{if measured 0:} \quad |\psi'\rangle &= \frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}} \\ \text{if measured 1:} \quad |\psi'\rangle &= \frac{\alpha_{10}|10\rangle + \alpha_{11}|11\rangle}{\sqrt{|\alpha_{10}|^2 + |\alpha_{11}|^2}} \end{aligned}$$

Mathematical formalism for constructing two qubit states: Tensor product of vector space.
Can combine two (arbitrary) vector spaces V and W to form the <u>tensor product</u> $V \otimes W$.

The elements of $V \otimes W$ are linear combinations of "tensor products" $|v\rangle \otimes |w\rangle$ consisting of elements $|v\rangle \in V$ and $|w\rangle \in W$.

Example: Let $V = \mathbb{C}^2$ and $W = \mathbb{C}^2$ be the single qubit spaces with basis $\{|0\rangle, |1\rangle\}$, then

$$\underbrace{\frac{1}{2}|0\rangle \otimes |0\rangle}_{=|00\rangle} + \underbrace{\frac{5i}{7}|1\rangle \otimes |0\rangle}_{=|10\rangle} \in V \otimes W$$

Let $\{|i\rangle_v : i = 1, ..., m\}$ be a basis of of V, and let $\{|j\rangle_w : j = 1, ..., n\}$ be a basis of of W, then

$$\{|i\rangle_v \otimes |j\rangle_w : i = 1, ..., m, j = 1, ..., n\}$$

is a basis of $V \otimes W$. In particular, $dim(V \otimes W) = dim(V) \cdot dim(W)$.
Note: $|i\rangle_v \otimes |j\rangle_w$ is also written as $|ij\rangle$.

Basic properties of tensor product:

- $\forall |v\rangle \in V, |w\rangle \in W \wedge \alpha \in \mathbb{C}$:
$$\alpha(|v\rangle \otimes |w\rangle) = (\alpha|0\rangle) \otimes |w\rangle = |v\rangle \otimes (\alpha|w\rangle) \tag{21}$$
- $\forall |v_1\rangle, |v_2\rangle \in V \wedge |w\rangle \in W$:
$$(|v_1\rangle + |v_2\rangle) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle \tag{22}$$
- $\forall |v\rangle \in V \wedge |w_1\rangle, |w_2\rangle \in W$:
$$|v\rangle \otimes (|w_1\rangle + |w_2\rangle) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle \tag{23}$$

Vector notation using standard basis, e.g.

$$\begin{aligned}
|v\rangle &= v_1|0\rangle + v_2|1\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
|w\rangle &= w_1|0\rangle + w_2|1\rangle = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\
|v\rangle \otimes |w\rangle &= (v_1|0\rangle + v_2|1\rangle) \otimes (w_1|0\rangle + w_2|1\rangle) \\
&= v_1w_1|00\rangle + v_1w_2|01\rangle + v_2w_1|10\rangle + v_2w_2|11\rangle
\end{aligned}$$

Thus:

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \otimes \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 w_1 \\ v_1 w_2 \\ v_2 w_1 \\ v_2 w_2 \end{pmatrix}$$

Note: Not every element of $V \otimes W$ can be written in the form $|v\rangle \otimes |w\rangle$, for example the Bell state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Assuming that V and W have an inner product $\langle \cdot | \cdot \rangle$, define inner product on $V \otimes W$ by

$$\langle \sum_j \alpha_j |v_j\rangle \otimes |w_j\rangle | \sum_k \beta_k |v_k\rangle \otimes |w_k\rangle\rangle := \sum_j \sum_k \alpha_j^* \beta_k \langle v_j | v_k \rangle \cdot \langle w_j | w_k \rangle \quad (24)$$

Generalization to n qubits: 2^n computational basis states

$$\{\underbrace{|0, ..., 0\rangle}_{\text{length } n}, |0, ..., 0, 1\rangle, ...|1, ..., 1\rangle\}$$

Thus: General n-qubit quantum state, also denoted as "quantum register", given by:

$$|\psi\rangle = \sum_{x_0=0}^{1} \sum_{x_1=0}^{2} \cdots \sum_{x_{n-1}}^{1} \alpha_{x_{n-1}, ..., x_1, x_0} \cdot |x_{n-1}, ..., x_1 x_0\rangle \quad (25)$$

with $\alpha_x \in \mathbb{C}$ for all $x \in \{0, ..., 2^n - 1\}$, such that $\|\psi\|^2 = \sum_{x=0}^{2^n-1} |\alpha_x|^2 = 1$ (normalization).

$\rightsquigarrow$ In general *"hard"* to simulate on classical computer (for large n) due to "curse of dimensionality".

Vector space as tensor products: $\underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}} = (\mathbb{C}^2)^{\otimes n} = \mathbb{C}^{(2^n)}$

1.4 Multiple qubit gates

As for single qubits, an operation on multiple qubits is described by an unitary matrix U. For n qubits: $U \in \mathbb{C}^{2^n \times 2^n}$

Example: controlled-NOT gate (also CNOT):
two qubits: control and target, target qubit gets flipped if control is 1:

$$|00\rangle \mapsto |00\rangle, \quad |01\rangle \mapsto |01\rangle, \quad |10\rangle \mapsto |11\rangle, \quad |11\rangle \mapsto |11\rangle$$

Can be expressed as

$$|a, b\rangle \mapsto |a, a \oplus b\rangle \quad \forall a, b \in \{0, 1\} \tag{26}$$

, where $\oplus$ is the addition modulo 2.

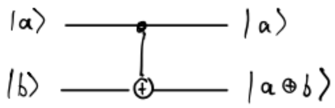


Figure 6: CNOT circuit notation

Matrix representation:

$$U_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{27}$$

, with the Pauli-X matrix $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Figure 7: Alternative CNOT circuit notation

Can generalize Pauli-X to any unitary operator U acting on target qubit
$\rightsquigarrow$ **controlled-U gate**:

$$|00\rangle \mapsto |00\rangle, \quad |01\rangle \mapsto |01\rangle, \quad |10\rangle \mapsto |1\rangle \otimes (U|0\rangle), \quad |11\rangle \mapsto |1\rangle \otimes (U|1\rangle)$$

Figure 8: Controlled-U gate

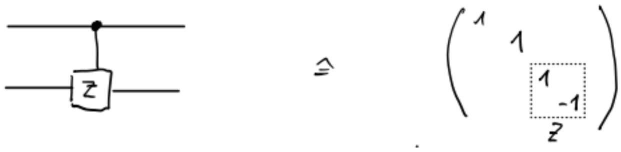


Figure 9: Example: Controlled-Z gate

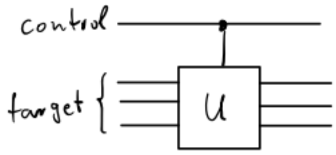


Figure 10: Controlled-U for multiple target qubits

Note: Single qubit and CNOT gates are universal: They can be used to implement an arbitraty unitray operation on n qubits (Quantum analogue of univerisity of classical NAND gate). Proof in Nielsen and Chuang section 4.5.

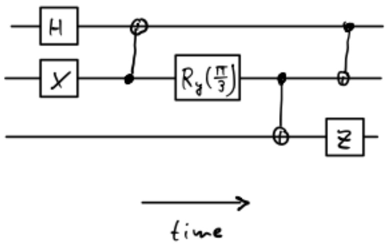
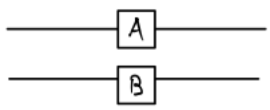


Figure 11: Example of a circuit lousisting only of single qubit gates and CNOTs

1.4.1 Matrix Kronecker Products

Matrix representation of single qubit gates acting in parallel:



Operation on basis states: $a, b \in \{0, 1\}$:

$$|a, b\rangle \mapsto (A|a\rangle) \otimes (B|b\rangle) \tag{28}$$

Example: $A = I$ (identity), $B = Y$

$$\begin{aligned}
|00\rangle &\mapsto |0\rangle \otimes (Y|0\rangle) = i|01\rangle \\
|01\rangle &\mapsto |0\rangle \otimes (Y|1\rangle) = -i|00\rangle \\
|10\rangle &\mapsto |1\rangle \otimes (Y|0\rangle) = i|11\rangle \\
|11\rangle &\mapsto |1\rangle \otimes (Y|1\rangle) = -i|10\rangle
\end{aligned}$$

Matrix representation:

$$\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} = \begin{pmatrix} Y & 0 \\ 0 & Y \end{pmatrix} = I \otimes Y$$

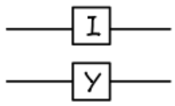


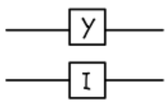
Figure 12: Circuit notation

General formula: Kronecker product (matrix representation of tensor products of operators)

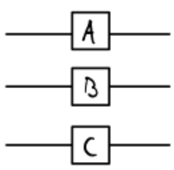
$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix} \in \mathbb{C}^{mp \times nq} \tag{29}$$

for all $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{p \times q}$.

Another example:



$$\widehat{=}\ Y \otimes I = \begin{pmatrix} 0 \cdot I & -i \cdot I \\ i\, I & 0 \cdot I \end{pmatrix} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$



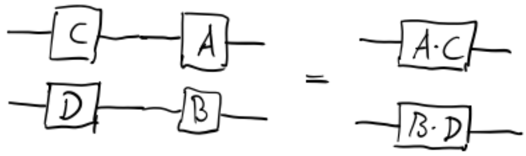
$$\widehat{=}\ A \otimes B \otimes C$$

$$= (A \otimes B) \otimes C = A \otimes (B \otimes C)$$

Figure 13: Generalization to arbitrary number of tensor factors possible

Basic properties:

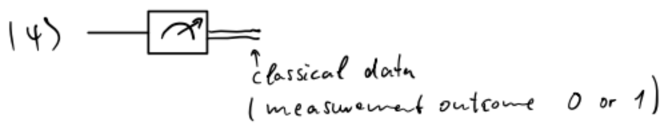
1. $(A \otimes B)^* = A^* \otimes B^*$ (elementwise complex conjugation)
2. $(A \otimes B)^T = A^T \otimes B^T$ (transposition)
3. $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$
4. $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ (associative property)
5. $(A \otimes B) \cdot (C \otimes D) = (A \cdot C) \otimes (B \cdot D)$ (for matrix of compatible dimensions)



6. Kronecker product of Hermitian matrices is Hermitian.
7. Kronecker product of unitary matrices is unitary (follows from 3. and 5.)

1.5 Quantum measurement

Review: measurement of a single qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with resepct to the computational basis $\{|0\rangle, |1\rangle\}$.



Linear algebra: Can switch to a different (orthonormal) basis to represent a qubit, e.g.

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

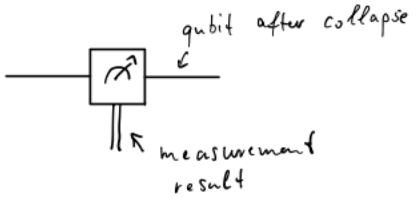
Representation of $|\psi\rangle$ w.r.t $\{|+\rangle, |-\rangle\}$ basis:

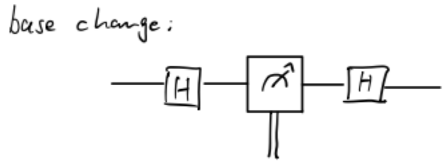
$$\alpha|0\rangle + \beta|1\rangle = \alpha \frac{|+\rangle + |-\rangle}{\sqrt{2}} + \beta \frac{|+\rangle - |-\rangle}{\sqrt{2}} = \frac{\alpha + \beta}{\sqrt{2}} |+\rangle + \frac{\alpha - \beta}{\sqrt{2}} |-\rangle$$

Can perform measurement with resepct to orthonormal basis $\{|+\rangle, |-\rangle\}$, will obtain result

$$\begin{aligned} + \quad & \text{with probability} \quad \frac{|\alpha+\beta|^2}{2} \\ - \quad & \text{with probability} \quad \frac{|\alpha-\beta|^2}{2} \end{aligned}$$

Wavefunction collapse: immediately after the measurement, qubit will be in the state $|+\rangle$ if measured "+", likewise in the state $|-\rangle$ if measured "-".





In general given an orthonormal basis $\{|u_1\rangle, |u_2\rangle\}$, one can represent a qubit as $|\psi\rangle = \alpha_1|u_1\rangle + \alpha_2|u_1\rangle$ and measure with respect to this orthonormal basis; will obtain measurement result u_1 or u_2 with respective probabilities $|\alpha_1|^2$ and $|\alpha_2|^2$.

1.5.1 Abstract general definition of quantum measurements

Quantum measurements are described by a collection $\{M_m\}$ of measurement operators acting on the quantum system, with the index m labelling possible measurement outcomes.
Denoting the quantum state before measurement $|\psi\rangle$, result m occurs with probability

$$p(m) = \left\langle\psi\middle|M_m^\dagger M_m\middle|\psi\right\rangle = \||M_m|\psi\rangle\|^2 \tag{30}$$

, state after measurement is:

$$\frac{M_m|\psi\rangle}{\|M_m|\psi\rangle\|} \tag{31}$$

The measurement operators satisfy the completeness relation

$$\sum_m M_m^\dagger M_m = I \tag{32}$$

such that probabilities sum to 1:

$$\sum_m p(m) = \sum_m \left\langle\psi\middle|M_m^\dagger M_m\middle|\psi\right\rangle = \left\langle\psi\middle|\sum_m M_m^\dagger M_m\middle|\psi\right\rangle \langle\psi|\psi\rangle = 1 \tag{33}$$

since $\sum_m M_m^\dagger M_m = I$.

Example: measurement of a qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with respect to computational basis $\{|0\rangle, |\psi\rangle\}$.

$$
\begin{aligned}
M_0 := |0\rangle\langle 0| &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1,0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
M_1 := |1\rangle\langle 1| &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0,1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
\rightsquigarrow p(0) &= \left\langle \psi \middle| M_0^\dagger M_0 \middle| \psi \right\rangle = \langle \psi | M_0 | \psi \rangle = |\alpha|^2 \\
p(1) &= \left\langle \psi \middle| M_1^\dagger M_1 \middle| \psi \right\rangle = \langle \psi | M_1 | \psi \rangle = |\beta|^2
\end{aligned}
$$

1.5.2 Projective Measurements

Projector onto subspace V with orthonormal basis $\{|u_1\rangle, ..., |u_m\rangle\}$:

$$P = \sum_{j=1}^{m} |u_j\rangle\langle u_j| \tag{34}$$

$$P|w\rangle = \sum_{j=1}^{m} \underbrace{\langle u_j|w\rangle}_{\text{inner product}} \tag{35}$$

Relation to spectral decomposition of a normal matrix $A \in \mathbb{C}^{n\times n}$:

$$A = U \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U^\dagger = \sum_{j=1}^{n} \lambda_j |u_j\rangle\langle u_j| \tag{36}$$

$$= \sum_{k=1}^{m} \widetilde{\lambda}_k P_k \quad \text{with } \{\widetilde{\lambda_1}...\widetilde{\lambda_m}\} \text{ the distinct eigenvalues} \tag{37}$$

Definition: A <u>projective measurement</u> is described by an <u>observable</u> M, a Hermitian operator acting on the quantum system. Spectal decomposition:

$$M = \sum_m \lambda_m P_m \tag{38}$$

with P_m: projection onto eigenspace with eigenvalue λ_m. The possible outcomes of the measurement correspond to the eigenvalues λ_m.
Probability of getting result λ_m when measuring a quantum state $|\psi\rangle$:

$$p(\lambda_m) = \langle\psi|P_m|\psi\rangle \tag{39}$$

State of the quantum system directly after the measurement:

$$\frac{P_m|\psi\rangle}{\|P_m|\psi\rangle\|} = \frac{P_m|\psi\rangle}{\sqrt{p(\lambda_m)}} \tag{40}$$

Remarks:

- Projective measurements are special cases of general measurement framework
- Projective measurements combined with unitary transformations are equivalent to general measurement framework, see pages 94, 95 in Nielsen and Chuang.

Average value of a projective measurement:

$$\mathbb{E}[M] = \sum_m \lambda_m p(\lambda_m) = \sum_m \lambda_m \langle\psi|P_m|\psi\rangle \tag{41}$$

$$= \left\langle \psi \middle| \sum_m \lambda_m P_m \middle| \psi \right\rangle = \langle\psi|M|\psi\rangle = \langle M\rangle \tag{42}$$

Corresponding standard deviation:

$$\Delta(M) := \sqrt{\langle M^2\rangle - \langle M\rangle^2} = \sqrt{\langle (M - \langle M\rangle)^2\rangle} \tag{43}$$

Examples:

- Measuring a qubit w.r.t computational basis $\{|0\rangle, |1\rangle\}$ is actually a projective measurement.
- In general: Measurement w.r.t orthonormal basis $\{|u_1\rangle, |u_2\rangle\}$ is a projective measurement: Set

  $$P_m = |u_m\rangle\langle u_m| \quad \text{for } m = 1, 2$$

  Define observable M by

  $$M := \sum_{m=1}^{2} \lambda_m P_m \quad \text{with arbitrary } \lambda_1, \lambda_2 \in \mathbb{R}; \lambda_1 \neq \lambda_2$$

- Measuring Pauli-Z

$$Z = 1 \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{P_1} + (-1) \cdot \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{P_2}$$

agrees with standard measurement w.r.t computational basis $\{|0\rangle, |1\rangle\}$

1.6 The Heisenberg uncertainty principle

Suppose A and B are Hermitian operators, and $|\psi\rangle$ a quantum state. Write

$$\langle\psi|AB|\psi\rangle = x + iy, \quad x, y \in \mathbb{R} \tag{44}$$

$$\langle\psi|AB|\psi\rangle^* = \left\langle\psi\middle|(AB)^\dagger\middle|\psi\right\rangle = \left\langle\psi\middle|B^\dagger A^\dagger\middle|\psi\right\rangle = \langle\psi|BA|\psi\rangle \tag{45}$$

Thus

$$\langle\psi|[A, B]|\psi\rangle = 2iy \quad \text{and} \quad \langle\psi|\{A, B\}|\psi\rangle = 2x \tag{46}$$

where $\{A, B\} := AB + BA$ is the anti-Commutator.

$$|\langle\psi|[A, B]|\psi\rangle|^2 + |\langle\psi|\{A, B\}|\psi\rangle|^2 = 4 \cdot \underbrace{|\langle\psi|AB|\psi\rangle|^2}_{x^2+y^2} \tag{47}$$

Cauchy-Schwarz inequality applied to $|v\rangle = A|\psi\rangle, |w\rangle = B|\psi\rangle$:

$$|\langle\psi|[A, B]|\psi\rangle|^2 \overset{(47)}{\leq} 4 \cdot |\langle\psi|AB|\psi\rangle|^2 \leq 4 \cdot \left\langle\psi\middle|A^2\middle|\psi\right\rangle \cdot \left\langle\psi\middle|B^2\middle|\psi\right\rangle \tag{48}$$

Suppose C and D are two observables: substitute $A = C - \langle C\rangle$ and $B = D - \langle D\rangle$ leads to Heisenberg uncertainty principle:

$$\Delta(C) \cdot \Delta(D) \geq \frac{|\langle\psi|[C, D]|\psi\rangle|}{2} \tag{49}$$

Interpretation for experiments: Repeated preparation of $|\psi\rangle$, measure C in some cases, D in other cases to obtain standard deviations ΔC and $\Delta(D)$.

2 Entanglement and its applications