

# Notes on Introduction to Quantum Computing

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# 1 Basic Concepts

## 1.1 Quantum bits (qubits)

Classical bits: 0, 1

Quantum bit *qubit*: Superposition of 0 and 1:

A quantum state  $|\psi\rangle$  is described as

$$|\psi\rangle := \alpha|0\rangle + \beta|1\rangle, \quad \alpha, \beta \in \mathbb{C} \quad (1)$$

where

$$|\alpha|^2 + |\beta|^2 = 1 \quad (\text{normalization}). \quad (2)$$

Mathematical description:  $|\psi\rangle \in \mathbb{C}^2$  with

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \rightsquigarrow |\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Different from classical bits, cannot (in general) directly observe / measure a qubit (the amplitudes  $\alpha$  and  $\beta$ ). Instead: "*standard*" measurement will result in

- 0 with probability  $|\alpha|^2$
- 1 with probability  $|\beta|^2$

The measurement also changes the qubit (*wavefunction collapse*). If measuring 0, the qubit will be  $|\psi\rangle = |0\rangle$  directly after the measurement, and likewise if measuring 1, the qubit will be  $|\psi\rangle = |1\rangle$ .

In practise: Can estimate the probabilities  $|\alpha|^2$  and  $|\beta|^2$  in experiments by repeating the same experiment many times (i.e via outcome statistics). These repetitions are called *trials* or *shots*.

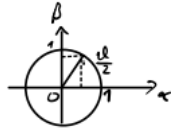


Figure 1: Circuit notation

A useful graphical deputation of a qubit is the Bloch sphere representation: If  $\alpha$  and  $\beta$  happen to be real-valued, then can find angle  $\vartheta \in \mathbb{R}$  such that

$$\alpha = \cos \frac{\vartheta}{2}, \quad \beta = \sin \frac{\vartheta}{2} \quad (3)$$

$$(\leadsto |\alpha|^2 + |\beta|^2 = \cos^2 \frac{\vartheta}{2} + \sin^2 \frac{\vartheta}{2} = 1 \quad \checkmark)$$

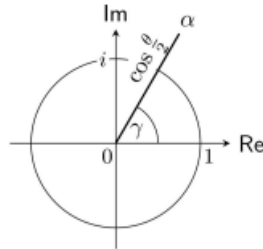


In general: represent

$$\alpha = e^{i\gamma} \cos \frac{\vartheta}{2}$$

$$\beta = e^{i(\varphi+\gamma)} \sin \frac{\vartheta}{2}$$

using so-called phase angles  $\gamma$  for  $\alpha$  and  $\varphi + \gamma$  for  $\beta$ .



Then:

$$|\psi\rangle = e^{i\psi} \cos \frac{\vartheta}{2} \cdot |0\rangle + \underbrace{e^{i(\varphi+\gamma)}}_{= e^{i\varphi} \cdot e^{i\gamma}} \sin \frac{\vartheta}{2} \cdot |1\rangle \quad (4)$$

$$= \underbrace{e^{i\gamma}}_{\text{can be ignored here}} \left( \cos \frac{\vartheta}{2} \cdot |0\rangle + e^{i\varphi} \cdot \sin \frac{\vartheta}{2} \cdot |1\rangle \right) \quad (5)$$

Thus  $|\psi\rangle$  is characterized by two angles  $\varphi$  and  $\gamma$ ; these specify the point defined as

$$\vec{r} = \begin{pmatrix} \cos \varphi \cdot \sin \vartheta \\ \sin \varphi \cdot \sin \vartheta \\ \cos \vartheta \end{pmatrix}$$

on the surface of a sphere:

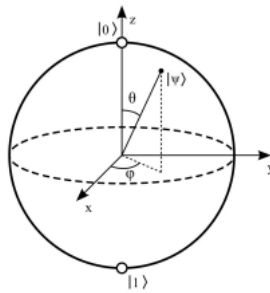


Figure 2: Bloch Sphere (Felix Bloch)

## 1.2 Single qubit gates

Principles of time evolution: The quantum state  $|\psi\rangle$  at current time point  $t$  transitions to a new quantum state  $|\psi'\rangle$  at a later time point  $t' > t$ .

Transition described by a complex unitary matrix  $U$ :

$$|\psi'\rangle = U \cdot |\psi\rangle \quad (6)$$

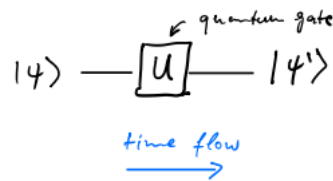


Figure 3: Circuit notation

Notes:

- Circuit is read from left to right, but matrix times vector ( $U|\psi\rangle$ ) from right to left.
- $U$  preserves normalization

Examples:

- Quantum analogue of the classical NOT gate ( $0 \leftrightarrow 1$ ) flip  $|0\rangle \leftrightarrow |1\rangle$  leads to Pauli-X gate:

$$X \equiv \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (7)$$

$$\text{Check: } X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle \text{ and } X|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle \quad \checkmark$$

- Pauli-Y gate:

$$Y \equiv \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (8)$$

- Pauli-Z gate:

$$Z \equiv \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (9)$$

Z leaves  $|0\rangle$  unchanged, but flips the sign of the coefficient of  $|1\rangle$ . Recall the Bloch Sphere representation:

$$|\psi\rangle = \cos \frac{\vartheta}{2} \cdot |0\rangle + e^{i\varphi} \sin \frac{\vartheta}{2} \cdot |1\rangle$$

Then

$$\begin{aligned} Z|\psi\rangle &= \cos \frac{\vartheta}{2} \cdot |0\rangle - e^{i\varphi} \sin \frac{\vartheta}{2} \cdot |1\rangle \\ &\stackrel{e^{i\pi} \equiv -1}{=} \cos \frac{\vartheta}{2} \cdot |0\rangle + \underbrace{e^{i\pi} e^{i\varphi}}_{e^{i(\varphi+\pi)}} \sin \frac{\vartheta}{2} \cdot |1\rangle \end{aligned}$$

$\rightsquigarrow$  new Bloch Sphere angles:  $\vartheta' = \vartheta, \varphi = \varphi + \pi$  (rotating by  $\pi = 180^\circ$  around z-axis)

X, Y, Z gates are called Pauli matrices. The Pauli vector  $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) = (X, Y, Z)$  is a vector of  $2 \times 2$  matrices.

- Hadamard Gate:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\propto |0\rangle + \beta |1\rangle \quad \longrightarrow \quad \boxed{H} \quad \longrightarrow \quad \propto \frac{|0\rangle + |1\rangle}{\sqrt{2}} + \beta \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Figure 4: Hadamard Gate

- Phase Gate:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

- T Gate:

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

Note:  $T^2 = S$  since  $(e^{i\pi/4})^2 = e^{i\pi/2} = i$

Pauli matrices satisfy:

1.  $\sigma_j^2 = I$  (identity) for  $j = 1, 2, 3$
2.  $\sigma_j \cdot \sigma_k = -\sigma_k \sigma_j$  for all  $j \neq k$
3.  $[\sigma_j, \sigma_k] := \underbrace{\sigma_j \sigma_k - \sigma_k \sigma_j}_{\text{Commutator}} = 2i\sigma_l$  for  $(j, k, l)$  a cyclic permutation of  $(1, 2, 3)$ .

General definition of matrix exponential

$$\exp(A) \equiv e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k, \quad A \in \mathbb{C}^{n \times n} \quad (10)$$

Special case:  $A^2 = I, x \in \mathbb{R}$

$$\begin{aligned} e^{iAx} &= \underbrace{\sum_{k=0}^{\infty} \frac{1}{(2k)!} (ix)^{2k} \underbrace{A^{2k}}_{(A^2)^k = I^k = I}}_{\text{even}} + \underbrace{\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (ix)^{2k+1} \underbrace{A^{2k+1}}_{(A^2)^k \cdot A = I^k \cdot A = A}}_{\text{odd}} \\ &= \underbrace{\sum_{k=0}^{\infty} \frac{1}{(2k)!} (-1)^k x^{2k} \cdot I}_{=\cos x} + \underbrace{\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (-1)^k x^{2k+1} \cdot A}_{=i \sin x} \\ &= \cos x \cdot I + i \sin x A \end{aligned}$$

(generalizes Euler's formula  $e^{ix} = \cos x + i \sin x$ )

This can be used to define the following rotation operators via the Pauli matrices. Let  $\vartheta \in \mathbb{R}$ :

$$R_x(\vartheta) := e^{-i\vartheta X/2} = \cos \frac{\vartheta}{2} I - i \sin \frac{\vartheta}{2} X = \begin{pmatrix} \cos \frac{\vartheta}{2} & -i \sin \frac{\vartheta}{2} \\ -i \sin \frac{\vartheta}{2} & \cos \frac{\vartheta}{2} \end{pmatrix} \quad (11)$$

$$R_y(\vartheta) := e^{-i\vartheta Y/2} = \cos \frac{\vartheta}{2} I - i \sin \frac{\vartheta}{2} Y = \begin{pmatrix} \cos \frac{\vartheta}{2} & -\sin \frac{\vartheta}{2} \\ \sin \frac{\vartheta}{2} & \cos \frac{\vartheta}{2} \end{pmatrix} \quad (12)$$

$$R_z(\vartheta) := e^{-i\vartheta Z/2} = \cos \frac{\vartheta}{2} I - i \sin \frac{\vartheta}{2} Z = \begin{pmatrix} e^{-i\vartheta/2} & 0 \\ 0 & e^{i\vartheta/2} \end{pmatrix} \quad (13)$$

General case: Rotation about an axis  $\vec{v} \in \mathbb{R}^3$  (normalized such that  $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = 1$ ):  
using the notation:

$$\langle \vec{v} | \vec{\sigma} \rangle = \vec{v} \cdot \vec{\sigma} = v_1 \sigma_1 + v_2 \sigma_2 + v_3 \sigma_3 = \begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix} \quad (14)$$

It holds that  $(\vec{v} \cdot \vec{\sigma})^2 = I$ .

We define the rotation operator around axis  $\vec{v}$  as

$$R_v(\vartheta) := e^{-i\vartheta(\vec{v} \cdot \vec{\sigma})/2} = \cos \frac{\vartheta}{2} I - i \sin \frac{\vartheta}{2} (\vec{v} \cdot \vec{\sigma}) \quad (15)$$

Note:  $R_x, R_y, R_z$  are special cases corresponding to  $\vec{v} = (1, 0, 0)$ ,  $\vec{v} = (0, 1, 0)$ , and  $\vec{v} = (0, 0, 1)$ .

Can derive that the Bloch Sphere representation of  $R_{\vec{v}}(\vartheta)$  is a "conventional" rotation (in three dimensions) by angle  $\vartheta$  about axis  $\vec{v}$ .

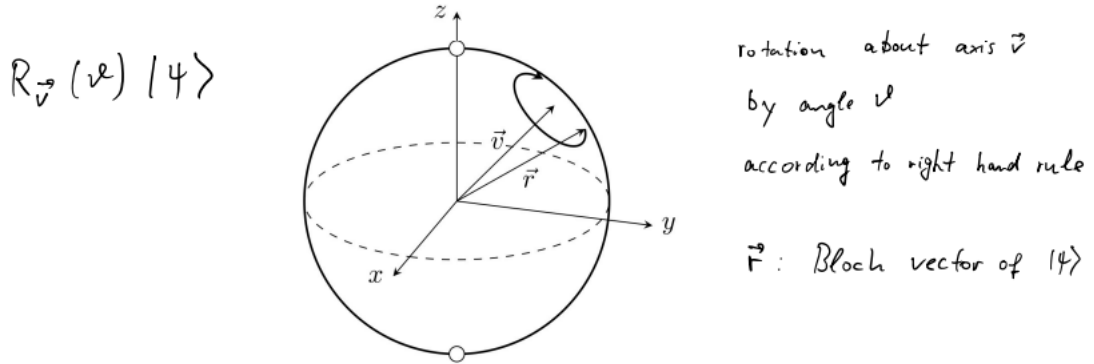


Figure 5: Circuit notation

Z-Y decomposition of an arbitrary  $2 \times 2$  unitary matrix:  
For any unitary matrix  $U \in \mathbb{C}^{n \times n}$  there exist real numbers  $\alpha, \beta, \gamma, \delta \in \mathbb{R}$  such that

$$U = e^{i\alpha} \underbrace{\begin{pmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{pmatrix}}_{R_z(\beta)} \cdot \underbrace{\begin{pmatrix} \cos \frac{\gamma}{2} & -\sin \frac{\gamma}{2} \\ \sin \frac{\gamma}{2} & \cos \frac{\gamma}{2} \end{pmatrix}}_{R_y(\gamma)} \cdot \underbrace{\begin{pmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{pmatrix}}_{R_z(\delta)} \quad (16)$$

### 1.3 Multiple qubits

So far: Single qubits, superposition of basis states  $|0\rangle$  and  $|1\rangle$ . For two qubits, this generalizes to  $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ .

General two-qubit state:

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle \quad (17)$$

with amplitudes  $\alpha_{ij} \in \mathbb{C}$  such that

$$|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1 \quad (\text{normalization}). \quad (18)$$

Can identify the basis states with unit vectors:

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (19)$$

Thus:

$$|\psi\rangle = \begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix} \in \mathbb{C}^4 \quad (20)$$

What happens if we measure only one qubit of a two-qubit state? Say we measure the first qubit: Obtain result

$$\begin{array}{ll} 0 & \text{with probability } |\alpha_{00}|^2 + |\alpha_{01}|^2 \\ 1 & \text{with probability } |\alpha_{10}|^2 + |\alpha_{11}|^2 \end{array}$$

Wavefunction directly after measurement:

$$\begin{array}{ll} \text{if measured 0: } |\psi'\rangle &= \frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}} \\ \text{if measured 1: } |\psi'\rangle &= \frac{\alpha_{10}|10\rangle + \alpha_{11}|11\rangle}{\sqrt{|\alpha_{10}|^2 + |\alpha_{11}|^2}} \end{array}$$

Mathematical formalism for constructing two qubit states: Tensor product of vector space.

Can combine two (arbitrary) vector spaces  $V$  and  $W$  to form the tensor product  $V \otimes W$ .



The elements of  $V \otimes W$  are linear combinations of "tensor products"  $|v\rangle \otimes |w\rangle$  consisting of elements  $|v\rangle \in V$  and  $|w\rangle \in W$ .

Example: Let  $V = \mathbb{C}^2$  and  $W = \mathbb{C}^2$  be the single qubit spaces with basis  $\{|0\rangle, |1\rangle\}$ , then

$$\underbrace{\frac{1}{2}|0\rangle \otimes |0\rangle}_{=|00\rangle} + \underbrace{\frac{5i}{7}|1\rangle \otimes |0\rangle}_{=|10\rangle} \in V \otimes W$$

Let  $\{|i\rangle_v : i = 1, \dots, m\}$  be a basis of  $V$ , and let  $\{|j\rangle_w : j = 1, \dots, n\}$  be a basis of  $W$ , then

$$\{|i\rangle_v \otimes |j\rangle_w : i = 1, \dots, m, j = 1, \dots, n\}$$

is a basis of  $V \otimes W$ . In particular,  $\dim(V \otimes W) = \dim(V) \cdot \dim(W)$ .  
Note:  $|i\rangle_v \otimes |j\rangle_w$  is also written as  $|ij\rangle$ .

Basic properties of tensor product:

- $\forall |v\rangle \in V, |w\rangle \in W \wedge \alpha \in \mathbb{C} :$

$$\alpha(|v\rangle \otimes |w\rangle) = (\alpha|v\rangle) \otimes |w\rangle = |v\rangle \otimes (\alpha|w\rangle) \quad (21)$$

- $\forall |v_1\rangle, |v_2\rangle \in V \wedge |w\rangle \in W :$

$$(|v_1\rangle + |v_2\rangle) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle \quad (22)$$

- $\forall |v\rangle \in V \wedge |w_1\rangle, |w_2\rangle \in W :$

$$|v\rangle \otimes (|w_1\rangle + |w_2\rangle) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle \quad (23)$$

Vector notation using standard basis, e.g.

$$\begin{aligned} |v\rangle &= v_1|0\rangle + v_2|1\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ |w\rangle &= w_1|0\rangle + w_2|1\rangle = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ |v\rangle \otimes |w\rangle &= (v_1|0\rangle + v_2|1\rangle) \otimes (w_1|0\rangle + w_2|1\rangle) \\ &= v_1w_1|00\rangle + v_1w_2|01\rangle + v_2w_1|10\rangle + v_2w_2|11\rangle \end{aligned}$$

Thus:

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \otimes \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 w_1 \\ v_1 w_2 \\ v_2 w_1 \\ v_2 w_2 \end{pmatrix}$$

Note: Not every element of  $V \otimes W$  can be written in the form  $|v\rangle \otimes |w\rangle$ , for example the Bell state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Assuming that  $V$  and  $W$  have an inner product  $\langle \cdot | \cdot \rangle$ , define inner product on  $V \otimes W$  by

$$\langle \sum_j \alpha_j |v_j\rangle \otimes |w_j\rangle | \sum_k \beta_k |v_k\rangle \otimes |w_k\rangle \rangle := \sum_j \sum_k \alpha_j^* \beta_k \langle v_j | v_k \rangle \cdot \langle w_j | w_k \rangle \quad (24)$$

Generalization to  $n$  qubits:  $2^n$  computational basis states

$$\{\underbrace{|0, \dots, 0\rangle}_{\text{length } n}, |0, \dots, 0, 1\rangle, \dots, |1, \dots, 1\rangle\}$$

Thus: General  $n$ -qubit quantum state, also denoted as "quantum register", given by:

$$|\psi\rangle = \sum_{x_0=0}^1 \sum_{x_1=0}^1 \cdots \sum_{x_{n-1}=0}^1 \alpha_{x_{n-1}, \dots, x_1, x_0} \cdot |x_{n-1}, \dots, x_1, x_0\rangle \quad (25)$$

with  $\alpha_x \in \mathbb{C}$  for all  $x \in \{0, \dots, 2^n - 1\}$ , such that  $\|\psi\|^2 = \sum_{x=0}^{2^n-1} |\alpha_x|^2 = 1$  (normalization).

$\leadsto$  In general "*hard*" to simulate on classical computer (for large  $n$ ) due to "curse of dimensionality".

Vector space as tensor products:  $\underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}} = (\mathbb{C}^2)^{\otimes n} = \mathbb{C}^{(2^n)}$

## 1.4 Multiple qubit gates

As for single qubits, an operation on multiple qubits is described by an unitary matrix  $U$ . For  $n$  qubits:  $U \in \mathbb{C}^{2^n \times 2^n}$

Example: controlled-NOT gate (also CNOT):  
two qubits: **control** and target, target qubit gets flipped if **control** is 1:

$$|00\rangle \mapsto |00\rangle, \quad |01\rangle \mapsto |01\rangle, \quad |10\rangle \mapsto |11\rangle, \quad |11\rangle \mapsto |11\rangle$$

Can be expressed as

$$|a, b\rangle \mapsto |a, a \oplus b\rangle \quad \forall a, b \in \{0, 1\} \quad (26)$$

, where  $\oplus$  is the addition modulo 2.

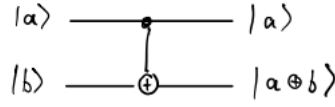


Figure 6: CNOT circuit notation

Matrix representation:

$$U_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (27)$$

, with the Pauli-X matrix  $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .



Figure 7: Alternative CNOT circuit notation

Can generalize Pauli-X to any unitary operator  $U$  acting on target qubit  
 $\rightsquigarrow$  **controlled-U gate**:

$$|00\rangle \mapsto |00\rangle, \quad |01\rangle \mapsto |01\rangle, \quad |10\rangle \mapsto |1\rangle \otimes (U|0\rangle), \quad |11\rangle \mapsto |1\rangle \otimes (U|1\rangle)$$



Figure 8: Controlled-U gate



Figure 9: Example: Controlled-Z gate

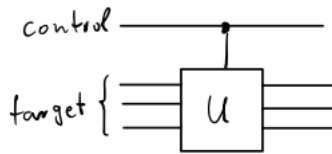


Figure 10: Controlled-U for multiple target qubits

Note: Single qubit and CNOT gates are universal: They can be used to implement an arbitrary unitary operation on  $n$  qubits (Quantum analogue of universality of classical NAND gate). Proof in Nielsen and Chuang section 4.5.

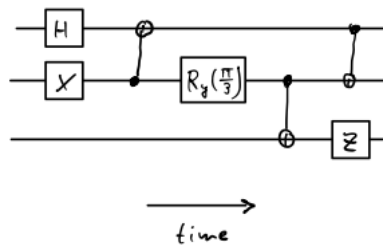
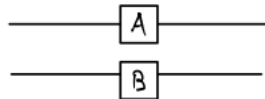


Figure 11: Example of a circuit consisting only of single qubit gates and CNOTs

#### 1.4.1 Matrix Kronecker Products

Matrix representation of single qubit gates acting in parallel:



Operation on basis states:  $a, b \in \{0, 1\}$ :

$$|a, b\rangle \mapsto (A|a\rangle) \otimes (B|b\rangle) \quad (28)$$

Example:  $A = I$  (identity),  $B = Y$

$$\begin{aligned} |00\rangle &\mapsto |0\rangle \otimes (Y|0\rangle) = i|01\rangle \\ |01\rangle &\mapsto |0\rangle \otimes (Y|1\rangle) = -i|00\rangle \\ |10\rangle &\mapsto |1\rangle \otimes (Y|0\rangle) = i|11\rangle \\ |11\rangle &\mapsto |1\rangle \otimes (Y|1\rangle) = -i|10\rangle \end{aligned}$$

Matrix representation:

$$\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} = \begin{pmatrix} Y & 0 \\ 0 & I \end{pmatrix} = I \otimes Y$$

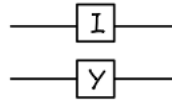


Figure 12: Circuit notation

General formula: Kronecker product (matrix representation of tensor products of operators)

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix} \in \mathbb{C}^{mp \times nq} \quad (29)$$

for all  $A \in \mathbb{C}^{m \times n}$  and  $B \in \mathbb{C}^{p \times q}$ .

Another example:

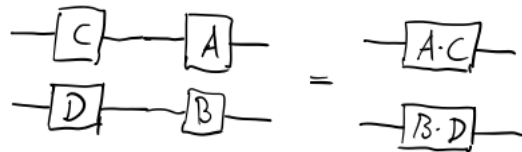
$$\begin{array}{c} \text{---} \boxed{Y} \text{---} \\ \text{---} \boxed{I} \text{---} \end{array} \cong Y \otimes I = \begin{pmatrix} 0 \cdot I & -i \cdot I \\ i \cdot I & 0 \cdot I \end{pmatrix} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\begin{array}{c} \text{---} \boxed{A} \text{---} \\ \text{---} \boxed{B} \text{---} \\ \text{---} \boxed{C} \text{---} \end{array} \cong A \otimes B \otimes C = (A \otimes B) \otimes C = A \otimes (B \otimes C)$$

Figure 13: Generalization to arbitrary number of tensor factors possible

Basic properties:

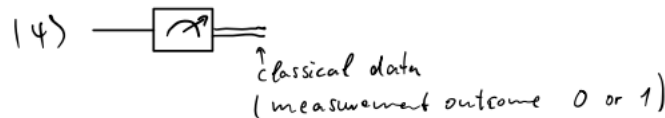
1.  $(A \otimes B)^* = A^* \otimes B^*$  (elementwise complex conjugation)
2.  $(A \otimes B)^T = A^T \otimes B^T$  (transposition)
3.  $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$
4.  $(A \otimes B) \otimes C = A \otimes (B \otimes C)$  (associative property)
5.  $(A \otimes B) \cdot (C \otimes D) = (A \cdot C) \otimes (B \cdot D)$  (for matrix of compatible dimensions)



6. Kronecker product of Hermitian matrices is Hermitian.
7. Kronecker product of unitary matrices is unitary (follows from 3. and 5.)

## 1.5 Quantum measurement

Review: measurement of a single qubit  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$  with respect to the computational basis  $\{|0\rangle, |1\rangle\}$ .



Linear algebra: Can switch to a different (orthonormal) basis to represent a qubit, e.g.

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

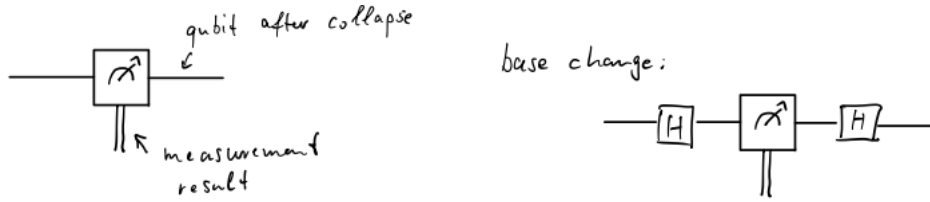
Representation of  $|\psi\rangle$  w.r.t  $\{|+\rangle, |-\rangle\}$  basis:

$$\alpha|0\rangle + \beta|1\rangle = \alpha \frac{|+\rangle + |-\rangle}{\sqrt{2}} + \beta \frac{|+\rangle - |-\rangle}{\sqrt{2}} = \frac{\alpha + \beta}{\sqrt{2}} |+\rangle + \frac{\alpha - \beta}{\sqrt{2}} |-\rangle$$

Can perform measurement with respect to orthonormal basis  $\{|+\rangle, |-\rangle\}$ , will obtain result

$$\begin{aligned} + & \text{ with probability } \frac{|\alpha + \beta|^2}{2} \\ - & \text{ with probability } \frac{|\alpha - \beta|^2}{2} \end{aligned}$$

Wavefunction collapse: immediately after the measurement, qubit will be in the state  $|+\rangle$  if measured "+", likewise in the state  $|-\rangle$  if measured "-".



In general given an orthonormal basis  $\{|u_1\rangle, |u_2\rangle\}$ , one can represent a qubit as  $|\psi\rangle = \alpha_1|u_1\rangle + \alpha_2|u_2\rangle$  and measure with respect to this orthonormal basis; will obtain measurement result  $u_1$  or  $u_2$  with respective probabilities  $|\alpha_1|^2$  and  $|\alpha_2|^2$ .

### 1.5.1 Abstract general definition of quantum measurements

Quantum measurements are described by a collection  $\{M_m\}$  of measurement operators acting on the quantum system, with the index  $m$  labelling possible measurement outcomes.

Denoting the quantum state before measurement  $|\psi\rangle$ , result  $m$  occurs with probability

$$p(m) = \langle \psi | M_m^\dagger M_m | \psi \rangle = \|M_m |\psi\rangle\|^2 \quad (30)$$

, state after measurement is:

$$\frac{M_m |\psi\rangle}{\|M_m |\psi\rangle\|} \quad (31)$$

The measurement operators satisfy the completeness relation

$$\sum_m M_m^\dagger M_m = I \quad (32)$$

such that probabilities sum to 1:

$$\sum_m p(m) = \sum_m \langle \psi | M_m^\dagger M_m | \psi \rangle = \left\langle \psi \left| \sum_m M_m^\dagger M_m \right| \psi \right\rangle \langle \psi | \psi \rangle = 1 \quad (33)$$

since  $\sum_m M_m^\dagger M_m = I$ .

Example: measurement of a qubit  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$  with respect to computational basis  $\{|0\rangle, |1\rangle\}$ .

$$\begin{aligned} M_0 &:= |0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ M_1 &:= |1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0, 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \rightsquigarrow p(0) &= \langle \psi | M_0^\dagger M_0 | \psi \rangle = \langle \psi | M_0 | \psi \rangle = |\alpha|^2 \\ p(1) &= \langle \psi | M_1^\dagger M_1 | \psi \rangle = \langle \psi | M_1 | \psi \rangle = |\beta|^2 \end{aligned}$$

### 1.5.2 Projective Measurements

Projector onto subspace  $V$  with orthonormal basis  $\{|u_1\rangle, \dots, |u_m\rangle\}$ :

$$P = \sum_{j=1}^m |u_j\rangle\langle u_j| \quad (34)$$

$$P|w\rangle = \sum_{j=1}^m \underbrace{\langle u_j | w \rangle}_{\text{inner product}} |u_j\rangle \quad (35)$$

Relation to spectral decomposition of a normal matrix  $A \in \mathbb{C}^{n \times n}$ :

$$A = U \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U^\dagger = \sum_{j=1}^n \lambda_j |u_j\rangle\langle u_j| \quad (36)$$

$$= \sum_{k=1}^m \tilde{\lambda}_k P_k \quad \text{with } \{\tilde{\lambda}_1, \dots, \tilde{\lambda}_m\} \text{ the distinct eigenvalues} \quad (37)$$

Definition: A projective measurement is described by an observable  $M$ , a Hermitian operator acting on the quantum system. Spectral decomposition:

$$M = \sum_m \lambda_m P_m \quad (38)$$



with  $P_m$ : projection onto eigenspace with eigenvalue  $\lambda_m$ . The possible outcomes of the measurement correspond to the eigenvalues  $\lambda_m$ . Probability of getting result  $\lambda_m$  when measuring a quantum state  $|\psi\rangle$ :

$$p(\lambda_m) = \langle\psi|P_m|\psi\rangle \quad (39)$$

State of the quantum system directly after the measurement:

$$\frac{P_m|\psi\rangle}{\|P_m|\psi\rangle\|} = \frac{P_m|\psi\rangle}{\sqrt{p(\lambda_m)}} \quad (40)$$

Remarks:

- Projective measurements are special cases of general measurement framework
- Projective measurements combined with unitary transformations are equivalent to general measurement framework, see pages 94, 95 in Nielsen and Chuang.

Average value of a projective measurement:

$$\mathbb{E}[M] = \sum_m \lambda_m p(\lambda_m) = \sum_m \lambda_m \langle\psi|P_m|\psi\rangle \quad (41)$$

$$= \left\langle \psi \left| \sum_m \lambda_m P_m \right| \psi \right\rangle = \langle\psi|M|\psi\rangle = \langle M \rangle \quad (42)$$

Corresponding standard deviation:

$$\Delta(M) := \sqrt{\langle M^2 \rangle - \langle M \rangle^2} = \sqrt{\langle (M - \langle M \rangle)^2 \rangle} \quad (43)$$