Notes on Introduction to Quantum Computing

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1 Basic Concepts

1.1 Quantum bits (qubits)

Classical bits: $0, 1$
Quantum bit *qubit*: Superposition of 0 and 1:
A quantum state $|\psi\rangle$ is described as

$$|\psi\rangle := \alpha|0\rangle + \beta|1\rangle, \quad \alpha, \beta \in \mathbb{C} \tag{1}$$

where

$$|\alpha|^2 + |\beta|^2 = 1 \quad (\textit{normalization}). \tag{2}$$

Mathematical description: $|\psi\rangle \in \mathbb{C}^2$ with

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \rightsquigarrow |\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Different from classical bits, cannot (in general) directly observe / measure a qubit (the amplitudes α and β). Instead: "*standard*" measurement will result in

- 0 with probability $|\alpha|^2$
- 1 with probability $|\beta|^2$

The measurement also changes the qubit (*wavefunction collapse*). If measuring 0, the qubit will be $|\psi\rangle = |0\rangle$ directly after the measurement, and likewise if measuring 1, the qubit will be $|\psi\rangle = |1\rangle$.

In practise: Can estimate the probabilities $|\alpha|^2$ and $|\beta|^2$ in experiments by repeating the same experiment many times (i.e via outcome statistics). These repetitions are called *trials* or *shots*.

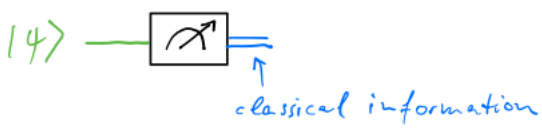
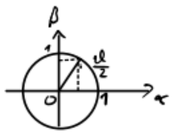


Figure 1: Circuit notation

A useful graphical deputation of a qubit is the Bloch sphere representation: If α and β happen to be real-valued, then can find angle $\vartheta \in \mathbb{R}$ such that

$$\alpha = \cos\frac{\vartheta}{2}, \quad \beta = \sin\frac{\vartheta}{2} \tag{3}$$

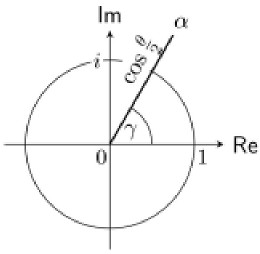
$$(\rightsquigarrow |\alpha|^2 + |\beta|^2 = \cos\frac{\vartheta}{2} + \sin\frac{\vartheta}{2}) = 1 \quad \checkmark$$



In general: represent

$$\alpha = e^{i\gamma}\cos\frac{\vartheta}{2}$$
$$\beta = e^{i(\varphi+\gamma)}\sin\frac{\vartheta}{2}$$

using so-called phase angles γ for α and $\varphi + \gamma$ for β.



Then:

$$|\psi\rangle = e^{i\psi}\cos\frac{\vartheta}{2}\cdot|0\rangle + \underbrace{e^{i(\varphi+\gamma)}}_{=e^{i\varphi}\cdot e^{i\gamma}}\sin\frac{\vartheta}{2}\cdot|1\rangle \tag{4}$$

$$= \underbrace{e^{i\gamma}}_{\text{can be ignored here}}\left(\cos\frac{\vartheta}{2}\cdot|0\rangle + e^{i\varphi}\cdot\sin\frac{\vartheta}{2}\cdot|1\rangle\right) \tag{5}$$

Thus $|\psi\rangle$ is characterized by two angles φ and γ; these specify the point defined as

$$\vec{r} = \begin{pmatrix}\cos\varphi\cdot\sin\vartheta \\ \sin\varphi\cdot\cos\vartheta \\ \cos\vartheta\end{pmatrix}$$

on the surface of a sphere:

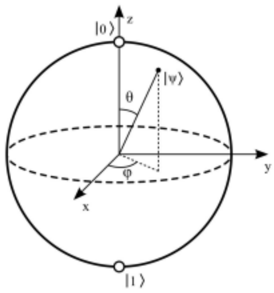


Figure 2: Bloch Sphere (Felix Bloch)

1.2 Single qubit gates

Principles of time evolution: The quantum state $|\psi\rangle$ at current time point t transitions to a new quantum state $|\psi'\rangle$ at a later time point $t' > t$.
Transition described by a complex unitary matrix U:

$$|\psi'\rangle = U \cdot |\psi\rangle \tag{6}$$

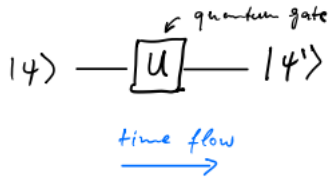


Figure 3: Circuit notation

Notes:

- Circuit is read from left to right, but matrix times vector ($U|\psi\rangle$) from right to left.
- U preserves normalization

Examples:

- Quantum analogue of the classical NOT gate ($0 \leftrightarrow 1$) flip $|0\rangle \leftrightarrow |1\rangle$ leads to Pauli-X gate:

$$X \equiv \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7}$$

Check: $X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$ and $X|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$ ✓

- Pauli-Y gate:
$$Y \equiv \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{8}$$
- Pauli-Z gate:
$$Z \equiv \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9}$$
Z leaves $|0\rangle$ unchanged, but flips the sign of the coefficient of $|1\rangle$. Recall the Bloch Sphere representation:
$$|\psi\rangle = \cos\frac{\vartheta}{2} \cdot |0\rangle + e^{i\varphi} \sin\frac{\vartheta}{2} \cdot |1\rangle$$
Then
$$\begin{aligned} Z|\psi\rangle &= \cos\frac{\vartheta}{2} \cdot |0\rangle \; - \; e^{i\varphi} \sin\frac{\vartheta}{2} \cdot |1\rangle \\ &\overset{e^{i\pi}=-1}{\equiv} \cos\frac{\vartheta}{2} \cdot |0\rangle + \underbrace{e^{i\pi} e^{i\varphi}}_{e^{i(\varphi+\pi)}} \sin\frac{\vartheta}{2} \cdot |1\rangle \end{aligned}$$
$\rightsquigarrow$ new Bloch Sphere angles: $\vartheta' = \vartheta, \varphi = \varphi + \pi$ (rotating by $\pi = 180°$ around z-axis)
X, Y, Z gates are called <u>Pauli matrices</u>. The <u>Pauli vector</u> $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) = (X, Y, Z)$ is a vector of 2×2 matrices.
- Hadamard Gate:
$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\alpha\,|0\rangle + \beta\,|1\rangle \quad -\boxed{H}- \quad \alpha\,\frac{|0\rangle + |1\rangle}{\sqrt{2}} + \beta\,\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Figure 4: Hadamard Gate

- Phase Gate:
$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$
- T Gate:
$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$
Note: $T^2 = S$ since $(e^{i\pi/4})^2 = e^{i\pi/2} = i$

Pauli matrices satisfy:

1. $\sigma_j^2 = I$ (identifity) for $j = 1, 2, 3$
2. $\sigma_j \cdot \sigma_k = -\sigma_k \sigma_j$ for all $j \neq k$
3. $[\sigma_j, \sigma_k] := \underbrace{\sigma_j \sigma_k - \sigma_k \sigma_j}_{\text{Commutator}} = 2i\sigma_l$ for (j, k, l) a cyclic permutation of (1,2,3).

General definition of matrix exponential

$$exp(A) \equiv e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k, \quad A \in \mathbb{C}^{n \times n} \tag{10}$$

Special case: $A^2 = I, x \in \mathbb{R}$

$$\begin{aligned} e^{iAx} &= \underbrace{\sum_{k=0}^{\infty} \frac{1}{(2k)!} (ix)^{2k} \underbrace{A^{2k}}_{(A^2)^k = I^k = I}}_{\text{even}} + \underbrace{\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (ix)^{2k+1} \underbrace{A^{2k+1}}_{(A^2)^k \cdot A = I^k \cdot A = A}}_{\text{odd}} \\ &= \underbrace{\sum_{k=0}^{\infty} \frac{1}{(2k)!} (-1)^k x^{2k}}_{=\cos x} \cdot I + \underbrace{\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (-1)^k x^{2k+1}}_{=i \sin x} \cdot A \\ &= \cos x \cdot I + i \sin x A \end{aligned}$$

(generalizes Euler's formula $e^{ix} = \cos x + i \sin x$)

This can be used to define the following rotation operators via the Pali matrices. Let $\vartheta \in \mathbb{R}$:

$$R_x(\vartheta) := e^{-i\vartheta X/2} = \cos \frac{\vartheta}{2} I - i \sin \frac{\vartheta}{2} X = \begin{pmatrix} \cos \frac{\vartheta}{2} & -i \sin \frac{\vartheta}{2} \\ -i \sin \frac{\vartheta}{2} & \cos \frac{\vartheta}{2} \end{pmatrix} \tag{11}$$

$$R_y(\vartheta) := e^{-i\vartheta Y/2} = \cos \frac{\vartheta}{2} I - i \sin \frac{\vartheta}{2} Y = \begin{pmatrix} \cos \frac{\vartheta}{2} & -\sin \frac{\vartheta}{2} \\ \sin \frac{\vartheta}{2} & \cos \frac{\vartheta}{2} \end{pmatrix} \tag{12}$$

$$R_z(\vartheta) := e^{-i\vartheta Z/2} = \cos \frac{\vartheta}{2} I - i \sin \frac{\vartheta}{2} Z = \begin{pmatrix} e^{-i\vartheta/2} & 0 \\ 0 & e^{i\vartheta/2} \end{pmatrix} \tag{13}$$

General case: Rotation about an axis $\vec{v} \in \mathbb{R}^3$ (normalized such that $\|\vec{v}\|) = \sqrt{v_1^2 + v_2^2 + v_3^3} = 1)$:
using the notation:

$$\langle \vec{v} | \vec{\sigma} \rangle = \vec{v} \cdot \vec{\sigma} = v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3 = \begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix} \tag{14}$$

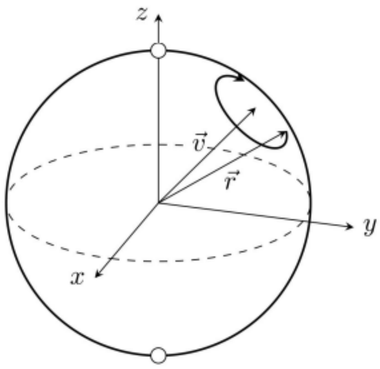
It holds that $(\vec{v} \cdot \vec{\sigma})^2 = I$.
We define the rotation operator around axis $\vec{v}$ as

$$R_v(\vartheta) := e^{-i\vartheta(\vec{v}\cdot\vec{\sigma})/2} = \cos\frac{\vartheta}{2} I - i \sin\frac{\vartheta}{2}(\vec{v} \cdot \vec{\sigma}) \tag{15}$$

Note: R_x, R_y, R_z are special cases corresponding to $\vec{v} = (1,0,0)$, $\vec{v} = (0,1,0)$, and $\vec{v} = (0,0,1)$.

Can derive that the Bloch Sphere representaiton of $R_{\vec{v}}(\vartheta)$ is a "conventional" rotation (in three dimensions) by angle ϑ about axis $\vec{v}$.

$R_{\vec{v}}(\vartheta)\,|\psi\rangle$



rotation about axis $\vec{v}$
by angle ϑ
according to right hand rule

$\vec{r}$: Bloch vector of $|\psi\rangle$

Figure 5: Circuit notation

<u>Z-Y decomposition</u> of an arbitrary 2×2 unitary matrix:
For any unitary matrix $U \in \mathbb{C}^{n \times n}$ there exist real numbers $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$U = e^{i\alpha} \underbrace{\begin{pmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{pmatrix}}_{R_z(\beta)} \cdot \underbrace{\begin{pmatrix} \cos\frac{\gamma}{2} & -\sin\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{pmatrix}}_{R_y(\gamma)} \cdot \underbrace{\begin{pmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{pmatrix}}_{R_z(\delta)} \tag{16}$$

1.3 Multiple qubits