Notes on Introduction to Quantum Computing

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1 Basic Concepts

1.1 Quantum bits (qubits)

Classical bits: $0, 1$
Quantum bit *qubit*: Superposition of 0 and 1:
A quantum state $|\psi\rangle$ is described as

$$|\psi\rangle := \alpha|0\rangle + \beta|1\rangle, \quad \alpha, \beta \in \mathbb{C} \tag{1}$$

where

$$|\alpha|^2 + |\beta|^2 = 1 \quad (\textit{normalization}). \tag{2}$$

Mathematical description: $|\psi\rangle \in \mathbb{C}^2$ with

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \rightsquigarrow |\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Different from classical bits, cannot (in general) directly observe / measure a qubit (the amplitudes α and β). Instead: "*standard*" measurement will result in

- 0 with probability $|\alpha|^2$
- 1 with probability $|\beta|^2$

The measurement also changes the qubit (*wavefunction collapse*). If measuring 0, the qubit will be $|\psi\rangle = |0\rangle$ directly after the measurement, and likewise if measuring 1, the qubit will be $|\psi\rangle = |1\rangle$.

In practise: Can estimate the probabilities $|\alpha|^2$ and $|\beta|^2$ in experiments by repeating the same experiment many times (i.e via outcome statistics). These repetitions are called *trials* or *shots*.

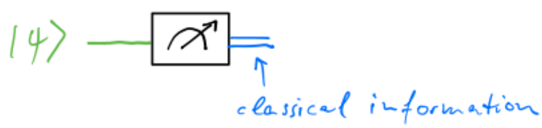
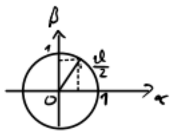


Figure 1: Circuit notation

A useful graphical deputation of a qubit is the Bloch sphere representation: If α and β happen to be real-valued, then can find angle $\vartheta \in \mathbb{R}$ such that

$$\alpha = \cos\frac{\vartheta}{2}, \quad \beta = \sin\frac{\vartheta}{2} \tag{3}$$

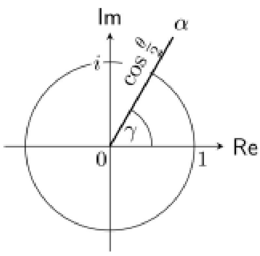
$$(\rightsquigarrow |\alpha|^2 + |\beta|^2 = \cos\frac{\vartheta}{2} + \sin\frac{\vartheta}{2}) = 1 \quad \checkmark$$



In general: represent

$$\alpha = e^{i\gamma}\cos\frac{\vartheta}{2}$$
$$\beta = e^{i(\varphi+\gamma)}\sin\frac{\vartheta}{2}$$

using so-called phase angles γ for α and $\varphi + \gamma$ for β.



Then:

$$|\psi\rangle = e^{i\psi}\cos\frac{\vartheta}{2}\cdot|0\rangle + \underbrace{e^{i(\varphi+\gamma)}}_{=e^{i\varphi}\cdot e^{i\gamma}}\sin\frac{\vartheta}{2}\cdot|1\rangle \tag{4}$$

$$= \underbrace{e^{i\gamma}}_{\text{can be ignored here}}\left(\cos\frac{\vartheta}{2}\cdot|0\rangle + e^{i\varphi}\cdot\sin\frac{\vartheta}{2}\cdot|1\rangle\right) \tag{5}$$

Thus $|\psi\rangle$ is characterized by two angles φ and γ; these specify the point defined as

$$\vec{r} = \begin{pmatrix}\cos\varphi\cdot\sin\vartheta \\ \sin\varphi\cdot\cos\vartheta \\ \cos\vartheta\end{pmatrix}$$

on the surface of a sphere:

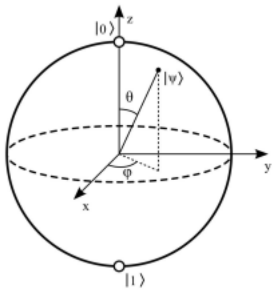


Figure 2: Bloch Sphere (Felix Bloch)

1.2 Single qubit gates