Final Blog Post: Entanglement Distillation

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March 2019

1 Start Here!

Hello there! This report is a summary of what we (Sean, Sahil, and Liang) have learned this quarter in EE 376A. We were studying quantum communication, and more specifically a protocol in quantum communication called *quantum entanglement distillation*. Our overall project goal was to implement a simple simulation of entanglement distillation; you'll find a summary of our progress towards this goal at the end of the report. We've included a **Background** section to bring you up to speed on the basic quantum physics you need to know, and have included a few technical sections describing our chosen distillation protocol in detail. Enjoy!

2 Background: A Crash Course in Quantum Mechanics

Let's start with the basics: what is a qubit? A normal bit can take on two values, 0 or 1; in the qubit case, our state exists as a *superposition* of both bits. We commonly denote a qubit's state with the symbol $|\psi\rangle$, so a superposition over both bits can be written as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

A quick aside: you might wonder why we're writing those funny brackets around the state ψ and the bits 0 and 1. This is called "bra-ket" notation, invented by the physicist Paul Dirac to reduce clutter in complicated quantum mechanics calculations. For our purposes, it will suffice to treat states like $|\psi\rangle$ and $|0\rangle$ as vectors in a complex two dimensional vector space $\mathbb{C}^2$, also called a Hilbert space $\mathcal{H}$. We can write

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

as unit vectors in $\mathbb{C}^2$, so our overall state is given by $|\psi\rangle = (\alpha, \beta)^T$. In this case, the parameters α and β represent the *probability amplitude* that our qubit will be in the state $|0\rangle$ or $|1\rangle$, respectively, upon measurement. To obtain the probability of measuring a given bit, we simply calculate $|\alpha|^2$ or $|\beta|^2$, making sure these parameters are normalized such that $|\alpha|^2 + |\beta|^2 = 1$ (note that α and β are complex numbers, in general).

Of course, we usually want to work with more than one qubit; for the protocol we'll discuss in this report, we'll be considering a canonical example of two shared qubits. We thus need a way to combine two quantum states that each represent a single qubit. It turns out that there are two ways to do this: we can use a *tensor product*, or we can *explicitly construct* the Hilbert space necessary to represent the combined quantum state. Let's first consider the tensor product construction of a two qubit state. We'll let $|\psi\rangle_A$ represent our "A" qubit, and $|\psi\rangle_B$ represent our "B" qubit. We can explicitly write these qubits as

$$|\psi\rangle_A = \alpha|0\rangle + \beta|1\rangle$$
$$|\psi\rangle_B = \gamma|0\rangle + \delta|1\rangle$$

where our states are normalized, as we mentioned above, such that $|\alpha|^2 + |\beta|^2 = 1$ and $|\gamma|^2 + |\delta|^2 = 1$. The tensor product of these two states is denoted by

$$\rho_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B$$

This equation should raise two questions: why do we denote the combined quantum state as ρ_{AB}, and what does this "$\otimes$" operation actually do to our vectors? The notation will become clear once we discuss how the tensor product is implemented, so let's discuss the second question first. In general, the tensor product is defined as an operation to combine vector spaces; for example, the tensor product of $\mathbb{R}^n$ (n-dimensional Euclidean space) and $\mathbb{R}^m$ (m-dimensional Euclidean space) is $\mathbb{R}^{nm}$ ($n \cdot m$-dimensional Euclidean space) [2]. This will be much too abstract for our purposes, as the tensor product of two vectors actually reduces to the *vector outer product*. For two vectors $u, v \in \mathbb{R}^2$, we simply have

$$u \otimes v = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} (v_1 \ v_2) = \begin{pmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \end{pmatrix}$$

Note that we construct a 2×2 matrix as a result of the outer product operation. This is why we denoted the outer product of our quantum states with ρ_{AB}: we don't end up with a state vector, but a state matrix. This matrix is called the *density matrix*, and is extremely useful for describing quantum states that contain multiple qubits. Our below discussion will rely heavily on the density matrix formalism; although we don't have the space to discuss this formalism in detail, excellent information can be found in [3] for interested readers. We can explicitly construct the density matrix for our system consisting of the "A" and "B" qubits above:

$$\begin{aligned} \rho_{AB} &= (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle) \\ &= \alpha\gamma(|0\rangle \otimes |0\rangle) + \alpha\delta(|0\rangle \otimes |1\rangle) + \beta\gamma(|1\rangle \otimes 0\rangle) + \beta\delta(|1\rangle \otimes |1\rangle) \end{aligned}$$

Since we know how to write the $|0\rangle$ and $|1\rangle$ states as vectors, we can perform the outer product operation to obtain an explicit form for the density matrix. Try it!

A brief aside: we can also apply the tensor product to matrices, where it is referred to as the *matrix Kronecker product*. We can thus define a hierarchy of tensor products, depending on the objects we're working with:

$$\underbrace{\text{Tensor Product}}_{\text{Tensors}} \rightarrow \underbrace{\text{Kronecker Product}}_{\text{Matrices}} \rightarrow \underbrace{\text{Outer Product}}_{\text{Vectors}}$$

In summary, then, we can use the tensor product to combine two qubits into a density matrix, which gives information about the combined system. Although the density matrix formalism is generally more useful for quantum computation, we can also explicitly construct the complex vector space (Hilbert space) of the two-qubit systm, which allows us to work with state vectors rather than density matrices (while conceptually simpler, this approach is much more tedious, as we'll soon see).

How can we understand the combined qubit system in the state vector formalism? When we measure one qubit, we can obtain either a $|0\rangle$ state (0 bit) with some probability, or a $|1\rangle$ state (1 bit) with some probability. What happens when we have two qubits? If they are independent (as we're considering here), we simply measure both qubits and obtain either a $|0\rangle$ or $|1\rangle$ state for each. The set of possible measurement outcomes is then given by

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$$

The notation $|ij\rangle$, for $i, j \in \{0, 1\}$, means the first qubit is in state $|i\rangle$, and the second qubit is in state $|j\rangle$. A single qubit lives in a two-dimensional complex vector space, as there were two possible measurement outcomes; by analogy, two qubits should live in a four-dimensional complex vector space. That is, we *define* the above states as corresponding to basis vectors in $\mathbb{C}^4$:

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Note that these form an orthonormal basis for $\mathbb{C}^4$, as needed. So we can write our two-qubit state vector $|\psi\rangle_{AB}$ as:

$$|\psi\rangle_{AB} = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

Note that the probability amplitudes here are distinct from those used above; in particular, they are normalized such that $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$.

Now that we understand how to use the density matrix and state vector formalisms, we have one last piece to understand: entanglement. Consider the state $|\psi\rangle_{AB}$ above; we can factor this state to obtain

$$|\psi\rangle_{AB} = (a_1|0\rangle + a_2|1\rangle) \otimes (b_1|0\rangle + b_2|1\rangle)$$

where $a_1 \cdot b_1 = \alpha$, and so on. We call this a *product state*, because the state can be decomposed into a (tensor) product of two "smaller" systems (qubits). It's natural to ask whether *any* state of n qubits can be decomposed into a product state, and the answer is an emphatic **no**. Consider the following state, which we call (for reasons we won't discuss here) the *singlet Bell state*:

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Try as you might, you will never find a product decomposition of this state. This is an *entangled state*; formally, a state is entangled if it cannot be written as a product state. What makes these states so weird? Suppose we prepare the entangled state above, and keep one qubit on Earth while (very carefully) transporting the other qubit far from our galaxy. If we measure the Earth qubit, it will return either $|0\rangle$ or $|1\rangle$, but we know the other qubit *must* be in the opposite state based on the form of $|\Psi^-\rangle$. That is, if you measure the Earth qubit and obtain the $|0\rangle$ state, the other qubit must be in the $|1\rangle$ state *immediately* upon measurement of the Earth qubit. But the non-Earth qubit is outside the galaxy, where it would take a light signal hundreds of thousands of years to reach! This "spooky action at a distance" has perplexed many generations of physicists, and even now we don't have a proper answer for why this is possible. But it certainly exists, and entangled states are incredibly important for quantum communication, as we'll see in the sections below.

Okay, that's the bare minimum you need to start reading what we've learned this quarter! Have fun!

3 Entanglement Distillation: Introduction & Setup (Sean)

3.1 Teleportation → Distillation

How can we accomplish reliable quantum communication? This is one of the central questions of quantum information theory, and turns out to be particularly difficult to answer. One fundamental principle in QIT is that entanglement is essential for any type of reliable quantum communication, to the point that entanglement can be considered a *physical resource* as much as classical bits. In fact, of the two general methods used to accomplish reliable communication, the method of teleporation is based solely on the entanglement of a pair of spins (the other method, quantum error correction, will not be considered in our project). For teleportation to succeed, two spins of maximum entanglement are required. While we're able to locally construct maximally entangled pairs like the Bell singlet state

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle),$$

distributing one spin of this entangled system from one party (Alice) to another (Bob) ultimately results in decoherence and thus a reduction in the entanglement. That is, if we let ρ denote the density matrix of the state originally in a maximally entangled singlet, the *fidelity* of the state

$$F \equiv \langle\Psi^-|\rho|\Psi^-\rangle$$

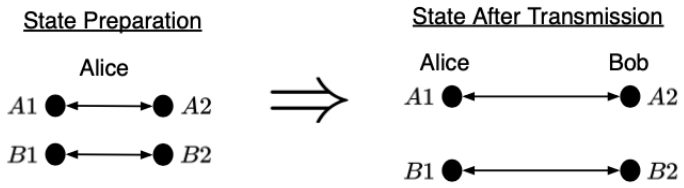


Figure 1: Alice prepares two pure singlets, consisting of qubits (A1, A2) and (B1, B2). Note that the singlets are independent. After transmitting one qubit in each singlet to Bob over a noisy channel, Alice has qubits A1 and B1, while Bob has qubits A2 and B2. The density matrix ρ_A (ρ_B) refers to the entangled spins A1 and A2 (B1 and B2).

decreases when one spin is transported to Bob. Note that $F = 1$ corresponds to a pure singlet state, $\rho = |\Psi^-\rangle\langle\Psi^-|$. We can interpret the fidelity as a measure of "overlap" between our reference state (in this case, the maximally entangled singlet) and current state ρ. To accomplish quantum teleportation, we need to overcome this drop in fidelity to maintain highly entangled states that are shared between Alice and Bob. This is the task of *entanglement distillation*: given N mixed entangled states, generate (or "distill") M entangled states of arbitrarily high fidelity, so $F \to 1$. The BBPSSW distillation protocol seeks to achieve entanglement distillation through a recurrent algorithm that discards some entangled states to increase the fidelity of others. In the next subsection, we'll discuss how the protocol is set up, and then discuss its execution in the next two sections.

3.2 BBPSSW Setup

To explain the BBPSSW protocol proposed in [1], we'll focus on only two entangled pairs shared between Alice and Bob, although a general application of the algorithm requires many more states to achieve a good fidelity. Alice generates two singlet states in her laboratory, so the state of the Alice-Bob system is

$$\rho_{AB} = (|\Psi_A^-\rangle \otimes |\Psi_B^-\rangle)(\langle\Psi_A^-| \otimes \langle\Psi_B^-|)$$

Alice then needs to send one spin from each singlet, labeled spin A2 and spin B2, to Bob (see Figure 1). Unfortunately, she must do so through a noisy quantum channel $\mathcal{E}$, which introduces some noise and reduces the entanglement; we can model the action of the channel as

$$\rho'_{AB} \equiv \mathcal{E}\rho_{AB} = \sum_{k=0}^{3} p_k(\sigma_k \otimes \sigma_k)\rho_{AB}(\sigma_k \otimes \sigma_k)$$

where σ_k are the spin matrices for a spin-1/2 system and the p_k are normalized such that $\sum_k p_k = 1$. Note that $\rho'_{AB} = \rho'_A \otimes \rho'_B$ represents some mixed state of *independent* qubit pairs, where each qubit pair has some fidelity $F < 1$ (this fidelity is not necessarily the same for both pairs). We can, of course, expand this mixed state in the Bell basis; however, we'd like for this mixed state to be *diagonal* in the Bell basis, to allow for easier computations. Fortunately, we can reduce (or *depolarize*) any mixed state ρ' to a state diagonal in the Bell basis, called a *Werner state*, by the random application of bilateral $\pi/2$ rotations about the x, y, or z axes. A bilateral rotation $\{B_x, B_y, B_z\}$ performs a rotation on both spins in an entangled pair, and maps the Bell states onto one another as follows:

$$B_x : |\Phi^+\rangle \leftrightarrow |\Psi^+\rangle$$
$$B_y : |\Phi^-\rangle \leftrightarrow |\Psi^+\rangle$$
$$B_z : |\Phi^+\rangle \leftrightarrow |\Phi^-\rangle$$

Note in particular that the singlet state $|\Psi^-\rangle$ is left invariant by all bilateral rotations, while one of the triplet states is also left invariant for each rotation (for example, the $|\Phi^-\rangle$ triplet is left invariant under a

B_x rotation). While we don't have the space here to complete the derivation, the result of applying these bilateral rotations to a pair of qubits with density matrix ρ' is the Werner state

$$\rho_W = F|\Psi^-\rangle\langle\Psi^-| + \frac{1-F}{3}\left(|\Psi^+\rangle\langle\Psi^+| + |\Phi^-\rangle\langle\Phi^-| + |\Phi^+\rangle\langle\Phi^+|\right),$$

where $F \equiv \langle\Psi^-|\rho'|\Psi^-\rangle$ is the fidelity of the original mixed state. We first note that this state is diagonal in the Bell basis, as we wanted. It also has the same fidelity as the mixed state ρ':

$$\begin{aligned} F_W &= \langle\Psi^-|\rho_W|\Psi^-\rangle \\ &= F \cdot \underbrace{\langle\Psi^-|\Psi^-\rangle\langle\Psi^-|\Psi^-\rangle}_{=1} + \frac{1-F}{3} \cdot \underbrace{\langle\Psi^-|\left(|\Psi^+\rangle\langle\Psi^+| + |\Phi^-\rangle\langle\Phi^-| + |\Phi^+\rangle\langle\Phi^+|\right)|\Psi^-\rangle}_{=0} \\ &= F \Rightarrow F_W = F \end{aligned}$$

This follows because the Bell states form an orthonormal basis for the Hilbert space of two qubits $\mathcal{H}^{\otimes 2}$.

Why is this Werner state useful? We've already noted that is has a simple expansion in the Bell basis, and that *any* mixed state can be brought to Werner form by suitable application of the bilateral rotation operators. Finally, we see that this Werner state has the same fidelity as our original mixed state, which is the only parameter that's important for our purposes. We don't care that the Werner state is a fundamentally different state than the mixed state ρ' (in other words, we don't care that their density matrices are different); as long as the fidelity remains the same, either state is useful for the protocol. Because of this, it suffices to develop a distillation protocol only for Werner states. In the following sections, we'll see how the BBPSSW protocol can operate on these Werner states to improve the fidelity of one qubit in the Alice-Bob system.

4 First Protocol (Sahil)

4.1 Protocol

Consider an arbitrary state M that consists of an ensemble of spin-$\frac{1}{2}$ particle pairs, and suppose that two separated observers, Alice and Bob, are in possession of one particle from each of these pairs (see Figure 1). We can expand M into a linear combination of orthogonal Bell states:

$$M = p_{\Psi^-}|\Psi^-\rangle\langle\Psi^-| + p_{\Psi^+}|\Psi^+\rangle\langle\Psi^+| + p_{\Phi^-}|\Phi^-\rangle\langle\Phi^-| + p_{\Phi^+}|\Phi^+\rangle\langle\Phi^+|,$$

where p_x denotes the probability of being in the state x, normalized such that $\sum_x p_x = 1$. Among the ensemble of particles shared by Alice and Bob, we would like to increase the probability of being in the singlet state $|\Psi^-\rangle$ relative to the other three Bell states. The first step in this process is to *depolarize* the state M; as noted above, by applying random bilateral rotations, we can reduce M to the Werner state. This state is the basis for the next steps of the protocol.

4.1.1 Unilateral y-rotation

Next, we perform a unilateral π rotation in the y direction on each of the pairs. This amounts to operating on either the first or second particle (but not both) by the Pauli operator σ_y, which is just a permutation on the set of Bell states:

$$M'' = F|\Phi^+\rangle\langle\Phi^+| + \frac{1-F}{3}\left(|\Phi^-\rangle\langle\Phi^-| + |\Psi^+\rangle\langle\Psi^+| + |\Psi^-\rangle\langle\Psi^-|\right).$$

Note that the fidelity of this state is still F.

4.1.2 BXOR

The bilateral XOR, or BXOR, operator takes two spin pairs $|a_1a_2\rangle, |b_1b_2\rangle$ as input and returns $|a_1a_2\rangle, |b_1 \oplus a_1, b_2 \oplus a_2\rangle$ as output. We'll partition our particles into two halves: the sources, and the targets. After pairing each source with each target, we then perform the BXOR operation onto each source-target pair, which in the space of Bell states may or may not change both of them, as opposed to just the target. We then measure the target spins' projections on the z axis, and keep the corresponding source only if both of the target particles' spins are parallel. This is only possible if the target is now in either a $|\Phi^+\rangle$ or $|\Phi^-\rangle$ state, and the probability of this happening is given by

$$F^2 + \frac{2}{3}F(1-F) + \frac{5}{9}(1-F)^2$$

Moreover, the probability that the target is in either of these two states *while* the corresponding source is $|\Phi^+\rangle$ is

$$F^2 + \frac{1}{9}(1-F)^2$$

Therefore, the conditional probability that the target is $|\Phi^+\rangle$ given that the source is $|\Phi^-\rangle$ or $|\Phi^+\rangle$, or the ensemble's new fidelity after dropping the "failed" source-target pairs, and reversing the σ_y rotation, is:

$$F'(F) = \frac{F^2 + \frac{1}{9}(1-F)^2}{F^2 + \frac{2}{3}F(1-F) + \frac{5}{9}(1-F)^2}.$$

At this point, we reverse the $|\sigma_y\rangle$ rotation to convert the $|\Phi^+\rangle$ states back to the $|\Psi^-\rangle$ states. Afterwards, we can either exit the protocol, or convert the result into a Werner state and perform another iteration.

4.2 Analysis

Some notes about the above procedure:

- It will only increase the fidelity if the original fidelity is greater than 0.5.
- The drop rate following the BXOR operation is fairly high, starting at about 75% when the original fidelity is 0.5.

The latter of these two reasons makes this protocol ineffective in practice, but we will discuss an improvement in the following section.

5 Second Protocol (Liang)

In [1], Bennett *et al.* present two protocols of recurrent entanglement distillation process. Protocol A chooses the source pair and the target pair arbitrarily, so the limit of purified output pairs by impure input pairs tends to zero when the output fidelity approaches to one (this protocol was discussed in Sahil's section above). A higher yield can be obtained by using previously purified bell state pairs as target pairs. The protocol works as follows:

1. Alice and Bob perform the recurrent method described by protocol A on n impure pairs, resulting in $n \cdot S(W)$ prepurified bell states in the limit of large n. Here, $S(W)$ is the von Neumann entropy on any Bell-diagonal density matrix.
2. Alice and Bob then perform the second step of protocol A (BXOR) on impure pairs to extract all $|\Psi\rangle$ states with target pairs as the prepurified bell states above. From [1], we know that the source pair is $|\Psi\rangle$ (both + and -) with the bell state as the target if the measurement of the target pair after

performing BXOR corresponds to the state of $|\Psi^+\rangle$. Then the impure states which consist only of $|\Phi^\mp\rangle$ can be obtained by applying a unilateral σ_y rotation on $|\Psi^\pm\rangle$. We can write a rotation of angle θ about the y-axis as

$$R_y(\theta) = e^{-i\frac{\theta}{2}\sigma_y} = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

For $\theta = \pi$, we have:

$$R_y(\pi) \otimes \mathbb{I} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

So the density matrix of the $|\Phi^\pm\rangle\langle\Phi^\mp|$ state, which we'll denote by $\rho_\pm$, after rotation is given by

$$\rho_\pm = (R_y(\pi) \otimes \mathbb{I})\rho_\pm(R_y(\pi) \otimes \mathbb{I})^\dagger$$

3. Alice and bob convert $|\Phi^-\rangle$ to $|\Psi^+\rangle$ by using a bilateral B_y rotation (invariant on $|\Phi^+\rangle$ and $|\Psi^-\rangle$). Then, they extract all the $|\Phi^+\rangle$ states by performing BXOR tests. After, they convert $|\Phi^+\rangle$ to the desired $|\Phi^+\rangle$ states by applying unilateral σ_x rotations. We can write a bilateral $\pi/2$ rotation B_y as:

$$B_y = R_y\left(\frac{\pi}{2}\right) \otimes R_y\left(\frac{\pi}{2}\right) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

In this protocol, two rounds of BXOR tests are performed. In the first round, if r_1 tests are performed, the expected number of strings in the typical set but not the correct x is $\leq N_1 2^{-r_1}$. In the second round, if r_2 tests are performed, it's $\leq N_2 2^{-r_2}$. As a result, we have

$$\log N_1 N_2 = nS(W) + O(\sqrt{n}),$$

and it requires $S(W)$ BXOR tests per impure pair to find all the errors. The yield of for Werner states

$$1 - S(W_F) = 1 + F\log_2 F + (1-F)\log_2\frac{1-F}{3}$$

is positive if $F > 0.8107$. The quality of entanglement can be measured by distillable entanglement $D(M)$ and entanglement of formation $E(M)$. For pure state, $D(M)$ and $E(M)$ are the same. For mixed state, the upper bound of the entanglement of formation for Werner states is given by

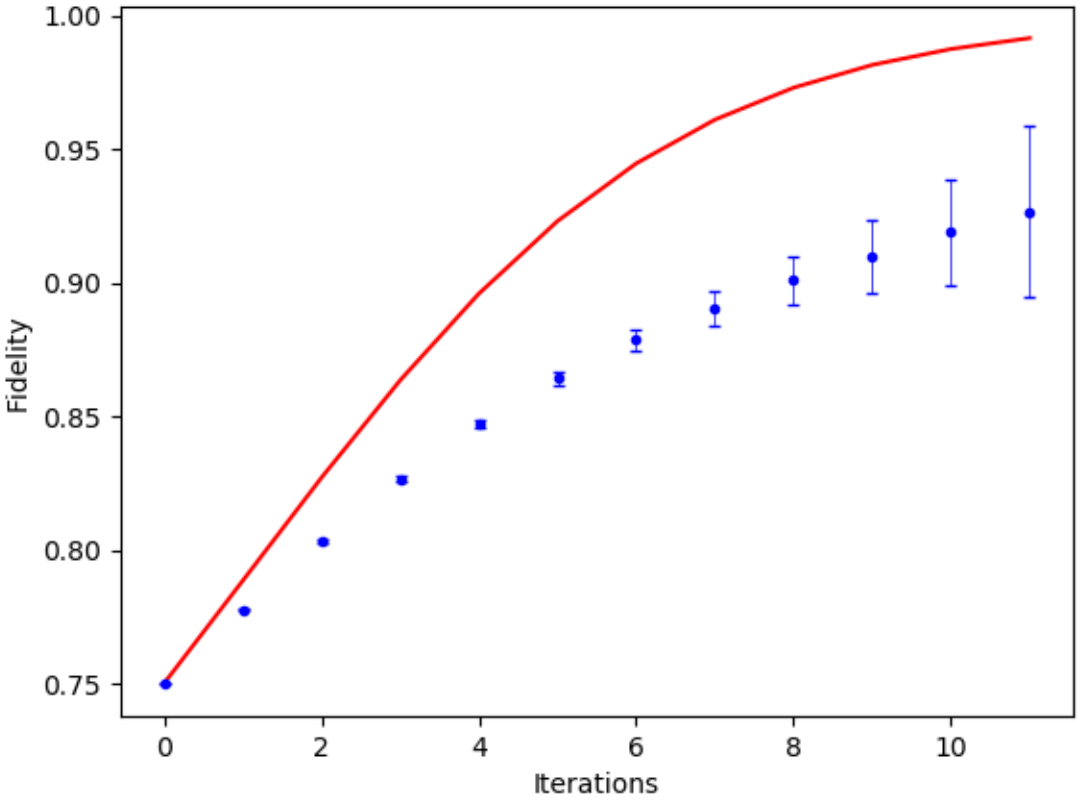
$$E(W_F) = H_2\left(\frac{1}{2} + \sqrt{F(1-F)}\right),$$

and the distillable entanglement is limited by LOCC, thus cannot exceed its entanglement of formation when $F > 1/2$. In this case, the mixted state can be written as the Werner state. For $F \leq 1/2$, the Werner state lose its entanglement and can be expressed by a mixture of product states $|10\rangle, |01\rangle, |00\rangle, |11\rangle$; thus, the entanglement of formation in this case is zero, which means it loses all the entanglement.

6 Simulation & Code

See our Github: `https://github.com/seandiscovery/entanglement-distillation-sim`

7 Plots



The fidelity plot with respect to iterations is shown in the above figure. The red line represents the recurrent fidelity described in the section 4.1.2. The blue dots represent the fidelities of simulated result. The simulation is performed according to the protocol A mentioned in the paper with 500k randomly generated initial mixed states and 100 trials. The error bars represent the standard deviations of average fidelities over 100 trials. The simulated fidelities is calculated by averaging over each iteration and over all trials. Note that the standard deviation increases significantly with higher iterations. It is because for each iteration, more than half of the samples are discarded either due to source/target states not in Φ^+ state or both. The average fidelity of higher iterations fluctuates due to lack of remaining sample space. Compared to the theoretical interpretation (red line), the simulated fidelity is lower than the recurrent fidelity since the latter targets to a single example where the former is averaging over all samples. Overall, we can see that the simulated result lines up with the theoretical interpretation in terms of growth rate. In other word, they both approach to a finite limit, one in this case.

References

[1] Charles H. Bennett, Gilles Brassard, Sandu Popescu, Benjamin Schumacher, John A. Smolin, and William K. Wootters. Purification of noisy entanglement and faithful teleportation via noisy channels. *Physical Review Letters*, 76(5), 1996.

[2] Todd Rowland. Vector space tensor product. Available at `http://mathworld.wolfram.com/VectorSpaceTensorProduct.html` (2019/03/22).

[3] Leonard Susskind and Art Friedman. *Quantum Mechanics*. Basic Books, New York, 2014.