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A Time-Optimal Control Strategy for Pursuit-Evasion Games Problems

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***Abstract*—This paper presents a control strategy for the pursuer in the pursuit-evasion game problem when the evader behaves intelligently. The pursuer in the proposed technique does not try to react to the evader's behavior instantaneously. The proposed technique therefore does not yield instantaneous optimality but capture the evader in a time-efficient and robust fashion even when the evader is intelligent. The proposed technique was applied to two numerical examples and the results were compared to those by the conventional motion tracking algorithms. The results and comparison show that the proposed technique could capture the evader faster than the conventional motion tracking algorithms in both the examples.**

I. Introduction

The pursuit-evasion game problem is concerned with finding the control actions of two different characters, the pursuer and the evader. While the pursuer attempts to capture the evader as quickly as possible, the evader tries to escape from the pursuer as long as possible. If the capture takes place, this fundamental control rule of these characters brings an interesting dilemma, i.e., the pursuer tries to minimize the capture time while the evader tries to maximize the capture time.

The study of pursuit-evasion problems has a long history, and there are many variations in pursuit-evasion problems [1]-[9]. They can be first classified in terms of whether the control actions to be found are continuous ones until the capture takes place or instantaneous ones which are determined based on the sensor readings. The former corresponds to the optimal feedforward control or optimal trajectory planning, whilst the latter corresponds to the optimal feedback control or the optimal sensor-based control.

The former can be further divided into three types depending on the type of evader' action, i.e., whether the evader (1) maximizes the capture time, (2) observes the pursuer's action or (3) follows fixed rules. The latter can also be divided into three types, depending on whether the evader's control action is (4) predictable and deterministic, (5) predictable and stochastic or (6) unpredictable as it is based on some intelligence [10-14].

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This paper presents a control strategy, namely evader's predictor, to find the control action of the pursuer when the evader is of type (6), or intelligent. As the evader's action is unpredictable, the technique proposed here does not attempt to optimally react to every sensor reading. The pursuer rather reacts by taking the worst scenario into account. The instantaneous control action is not thus optimal, but the advantage of the proposed technique is that the pursuit can capture the evader in a time-efficient and robust manner even when he behaves intelligently.

The paper is organized as follows. The next section describes the pursuer and evader models used in this paper, whereas the canonical time-optimal control formulation is presented in the third section. The fourth section presents the proposed control strategy, and the numerical examples are referred to in the fifth section. Conclusions are summarized in the final section.

II. Pursuer and Evader Models

A. Pursuer Model

For simplicity, the pursuer and evader $i \in \{p,e\}$ are each given by the tricycle model in two-dimensional space described as

$$\dot{x}_i(t) = v_i(t)\cos\theta_i(t), \tag{1a}$$

$$\dot{y}_i(t) = v_i(t)\sin\theta_i(t), \tag{1b}$$

$$\dot{\theta}_i(t) = \frac{v_i(t)}{l_i}\tan\gamma_i(t), \tag{1c}$$

where $[x_i, y_i]$ and θ_i are the position and orientation respectively, representing the state of the tricycle, v_i and γ_i are the velocity and steering angle respectively, representing the control inputs to the tricycle, and l_i is the tricycle wheel base.

The pursuer and evader's actions are at least constrained by the control bounds:

$$(v_i)_{\min} \le v_i(t) \le (v_i)_{\max}, \tag{2a}$$

$$(\gamma_i)_{\max} \le \gamma_i(t) \le (\gamma_i)_{\max}. \tag{2a}$$

As an additional control constraint, the actual maximum velocity of each character is governed by the current steering angle:

$$v_i(t) \le (v_i)_{\max} \cos\gamma_i(t). \tag{3}$$

This constraint, in the case of a ground car for instance, is effectively used to avoid slips when the car is turning sharply.

B. *Canonical Model Description*

Describe the tricycle model in a canonical form with the state $\mathbf{x}_i(t) \in R^{n_i}$ subject to control inputs $\mathbf{u}_i(t) \in R^{r_i}$:

$$\dot{\mathbf{x}}_i(t) = \mathbf{f}_i(t, \mathbf{x}_i(t), \mathbf{u}_i(t)) \tag{4}$$

and define the *continuous control action* as the complete control function from a time $t = 0$ to a terminal time T as

$$U_i = \{\mathbf{u}_i(t) \mid \forall t \in [0, T)\}. \tag{5}$$

Given a continuous control action U_i and the initial state $\mathbf{x}_i(0) = (\mathbf{x}_i)_0$, the state of the system at any time $t \in (0, T]$ can be determined uniquely in the form

$$\mathbf{x}_i(t) = (\mathbf{x}_i)_0 + \int_0^t \mathbf{f}_i(\tau, \mathbf{x}_i(\tau), \mathbf{u}_i(\tau))\, d\tau. \tag{6}$$

This means that, given the initial state $(\mathbf{x}_i)_0$, the state $\mathbf{x}_i(t)$ at any time t can be specified only by the control function U_i. When this needs to emphasized, the state $\mathbf{x}_i(t)$ given U_i will be written in the form $\mathbf{x}_i(t \mid U_i)$.

C. *Evader's Control Action*

The primary assumption to be made such that the existence of a solution is guaranteed for this class of problem is that the maximum velocity of the evader is less than the maximum velocity of the pursuer:

$$(v_e)_{\max} < (v_p)_{\max}. \tag{7}$$

Often, the structure of the system with a smaller maximum velocity advantageously results in producing a smaller non-holonomic constraint, i.e., a larger steering range:

$$(\gamma_e)_{\max} > (\gamma_p)_{\max}. \tag{8}$$

The evader must prolong the capture time within the velocity constraint by steering intelligently.

Steering of the evader is a trivial action for the pursuer when the distance between the evader and the pursuer is large. It simply worsens the situation for the evader by reducing his speed via the constraint (3). The basic control strategy of the evader therefore consists of two control rules. At every sensor reading, the evader moves:

(1) Such that the distance from the present position of the pursuer is maximized, when the pursuer is far enough from the evader,

(2) To enter into the pursuer's non-holonomic constraint region to force the pursuer to reroute his path to continue pursuing the evader, when the pursuer is close to the evader. In other words, the evader steers such that the orientation of the evader ultimately (after minimum Δt) becomes opposite to the present orientation of the pursuer:

$$\theta_e(t + \Delta t) = \theta_p(t) + \pi. \tag{9}$$

The rule (2) is introduced, to some degree, as the intelligence of the evader. If the pursuer simply approaches by tracking the path of the evader, the pursuer would not only able to capture the evader but also end up in a configuration that requires the maximum time to approach to the evader again.

III. Time-Optimal Control

A. *Canonical Optimal Control Problem Formulation*

For the pursuer, the optimal control problem can be formulated in canonical form as finding the continuous control actions U that optimize a payoff function $g_0(U_p)$,

$$g_0(U_p) \to \min_{U_p, T}. \tag{10}$$

It is well known that the canonical form of the payoff function can be expressed as

$$\begin{aligned} g_0(U_p) &\equiv \Phi_0(\mathbf{x}_p(T \mid U_p)) \\ &+ \int_0^T \Im_0(t, \mathbf{x}_p(t \mid U_p), \mathbf{u}_p(t))\, dt \end{aligned} \tag{11}$$

where the first and second terms of the right-hand side are the terminal payoff and integral payoff respectively [2].

The control actions U_p and time T are normally constrained at least so that

$$(U_p)_{\min} \le (U_p)(T) \le (U_p)_{\min}, \forall t \in [0, T], 0 < T. \tag{12}$$

The problem may also be subject to a variety of other constraints, generally in the form

$$g_i(U_p) \equiv \Phi_i(\mathbf{x}_p(\tau_i \mid U_p)) + \int_0^{\tau_i} \Im_i(t, \mathbf{x}_p(t \mid U_p), \mathbf{u}_p(t))\, dt = 0,$$

$$\forall i \in \{1, \ldots, M_e\}, \tag{13}$$

for equality constraints and

$$g_i(U_p) \equiv \Phi_i(\mathbf{x}_p(\tau_i \mid U_p)) + \int_0^{\tau_i} \Im_i(t, \mathbf{x}_p(t \mid U_p), \mathbf{u}_p(t))\, dt \le 0,$$

$$\forall i \in \{M_e+1,...,M\}, \tag{14}$$

for inequality constraints [3]. Time τ_i ($0 < \tau_i \neq T$) is referred to as the characteristic time for the i th constraint with $\tau_0 = T$ by convention.

B. Time-Optimal Control Problem Formulation

The payoff function of a time-optimal control problem is the capture time T:

$$g_0(U_p) = \int_0^T dt = T, \tag{15}$$

which corresponds to the canonical from of Equation (11) with $\Phi_0(\mathbf{x}_p(T|U_p)) = 0$ and $\Im_0(t, \mathbf{x}_p(t|U_p), \mathbf{u}_p(t)) = 1$. Thus, the time-optimal control problem is defined as

$$T \to \min_{U_p, T}, \tag{16}$$

subject to the primary constraints

$$(U_p)_{\min} \leq (U_p)(T) \leq (U_p)_{\min}, \forall t \in [0,T], 0 < T. \tag{17}$$

Without further constraints, the time-optimal control problem becomes uninteresting as T simply goes to zero. The most common type of constraint is on the value of the terminal state:

$$g_1(U_p) = \Phi_1(\mathbf{x}_p(T|U_p)) = 0. \tag{18}$$

This problem is concerned with the pursuer reaching the evader. Therefore, the constraint is given by

$$g_1(U_p) = [x_p(T|U_p) - x_e(T)]^2 + [y_p(T|U_p) - y_e(T)]^2 = 0. \tag{19}$$

The continuous inequality constraint (3) is normally handled by converting it to an integrated equality constraint:

$$g_2(U_p) = \int_0^T \max\left\{\left[v_p(t) - (v_p)_{\max} \cos\gamma_p(t)\right], 0\right\}^2 dt = 0, \tag{20}$$

where $\max\{.,.\}$ takes the larger value of its elements.

The problem formulations described above are for finding continuous control actions, which is not the strategy proposed in this paper. However, these brings the following conclusions in relation to the technique proposed in this paper:

- The solution of the problem depends on the instantaneous control action of the evader,
- The optimal continuous control action derived by solving the problem is not instantaneously optimal.

IV. Evader's Predictor

The predictor basically takes the two fundamental processes:

(1) The pursuer predicts the future location of the evader based on the worst-case scenario, i.e., the evader attempts to intrude into the non-holonomic constraint region of the pursuer,

(2) The pursuer takes the best instantaneous control action in sanction to the worst-case scenario.

The possible increase of capture time by the proposed technique contradicts to the instantaneous optimality but contributes to the terminal optimality in a sense that the pursuer moves to a position robust to the unpredictable evader's behavior. In other words, this control strategy may be illustrated as the "careful approaching technique" to overcome the intelligence of the evader and capture him in minimum time.

The pursuer therefore predicts the future location of the evader, $\mathbf{x}_e(t+\Delta t_p)$, from Equation (9). Note that Δt_p should be less than Δt in Equation (9) but can be much larger than Δt_e, which is the sampling time interval for the evader. The steering angle of the pursuer is then determined from

$$\gamma_p(t) = \begin{cases} (\gamma_p)_{\max} & \text{if} \quad \delta(t) \geq (\gamma_p)_{\max} \\ \delta(t) & \text{if} \quad -(\gamma_p)_{\max} \leq \delta(t) < (\gamma_p)_{\max} \\ -(\gamma_p)_{\max} & \text{if} \quad \delta(t) < -(\gamma_p)_{\max} \end{cases}, \tag{21}$$

where

$$\delta(t) = \tan^{-1}\frac{y_e(t+\Delta t_p) - y_p(t)}{x_e(t+\Delta t_p) - x_p(t)}, \tag{22}$$

followed by the computation of the maximum possible velocity with this steering increase from Equation (3).

V. Numerical Examples

A. Control Action of the Evader

Table I lists the maximum velocities and steering angles of the evader and the pursuer, which were used throughout the paper. As listed, the evader has a smaller maximum velocity, the non-holonomic constraint is less.

Figures 1 and 2, respectively, present the flexibility of the evader and how the evader enters the non-holonomic constraint region of the pursuer successfully. The pursuer is illustrated by a blue missile while the evader is marked by a red airplane. Figure 2 further indicates that the capture time is lengthened if the evader successfully enters the non-holonomic constraint region.

TABLE I. MAXIMUM VELOCITY AND STEERING ANGLES OF PURSUER AND EVADER

Parameters	Pursuer	Evader
$\mathbf{u}_{max}$ [m/s, °]	[2, 30]	[1, 50]

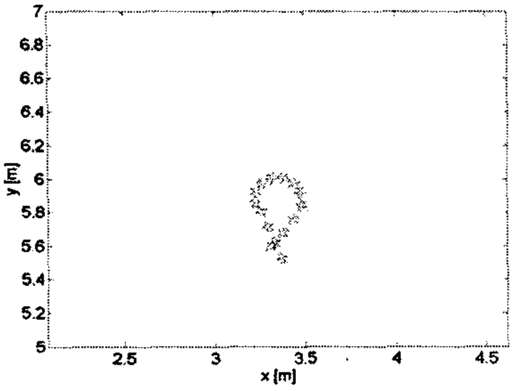


Figure 1. Control action of the evader

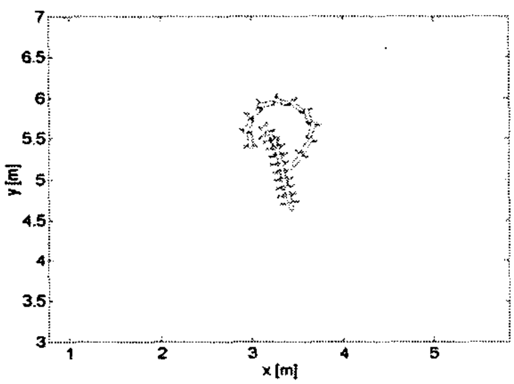


Figure 2. Evader in non-holonomic constraint region of the pursuer

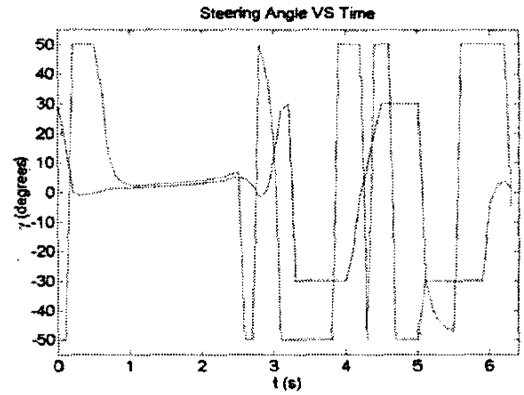
The subsequent two subsections present results of pursuit-evasion game of two cases where the evader starts from a different initial location.

B. Case 1

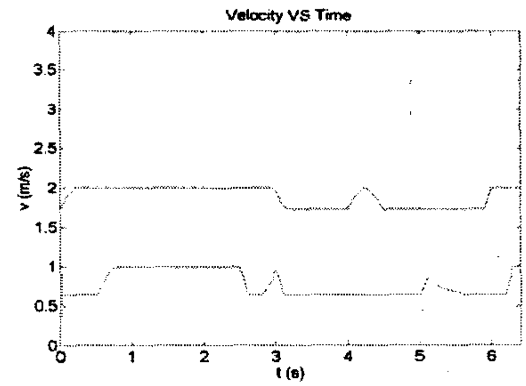
The first item of Table II shows the initial location of the pursuer and evader. Figure 3 shows the transitions of the velocity and steering angle with respect to time when the pursuer used the proposed evader's predictor to capture the evader.

TABLE II. CASE 1

Parameters	Pursuer	Evader
$\mathbf{x}_0^T$ [m, m, °]	[0, 0, 0]	[3, 3, 0]
T [sec]	6.4	18.3



(a) Velocity



(b) Steering angle

Figure 3. Results using evader's predictor for Case 1

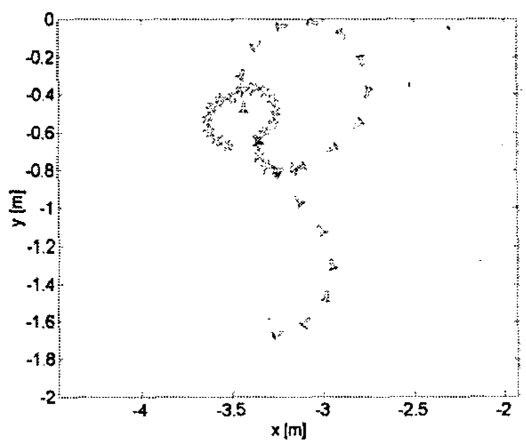
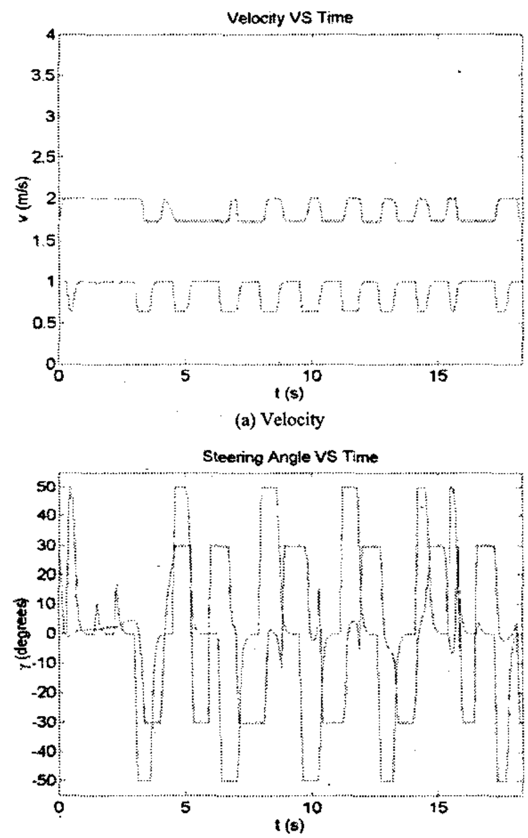


Figure 4. Resultant motions using the evader's predictor for Case 1

The resultant motions of the pursuer and evader are shown in Figure 4. The figure shows that the pursuer successfully captures the evader. The motions may not be seen as optimal at the first glance. However, it is seen that the evader drives at a maximum steering angle a number of times, which is 20 degrees more than the maximum steering angle of the pursuer. The effectiveness of the proposed technique is that pursuer can still capture the evader efficiently even when the evader attempted to enter the non-holonomic constraint region of the pursuer intelligently.

For comparison, the conventional motion tracking algorithms were also implemented and tested. In the conventional motion tracking, the pursuer simply tries to follow the path of the evader. Figure 5 shows the resultant velocity and steering angle variations by the motion tracking algorithms. In comparison to the evader's predictor, it is easily seen that the steering angle change of the pursuer in the motion tracking algorithms simply follows that of the evader. On the other hand, the pursuer with the evader's predictor steers irrelevantly to the evader's steering action.



(a) Velocity

(b) Steering angle

Figure 5. Results using motion tracking for Case 1

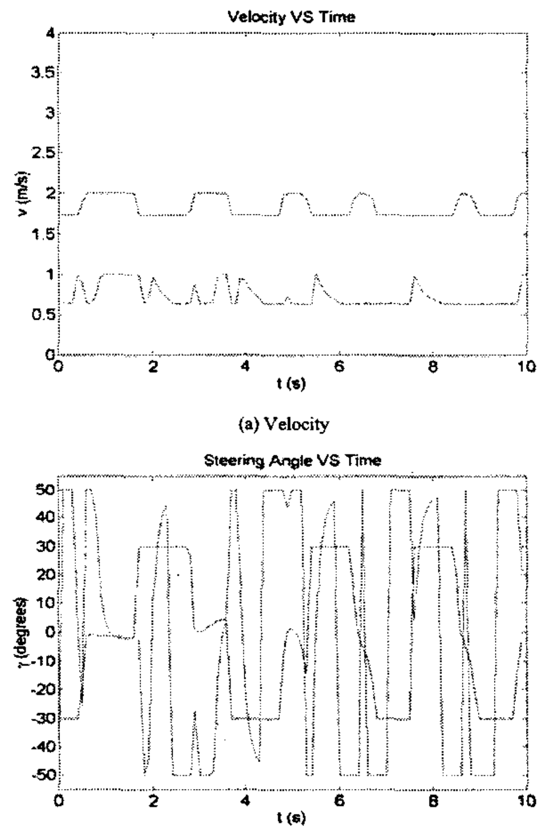
As listed in the second item of Table II, the terminal time for the evader's predictor is 6.4 seconds while the terminal time for the motion tracking is 18.3 seconds. The reason for the delay of capture time in the motion tracking algorithms is that the evader successfully entered the non-holonomic constraint region of the evader a number of times.

C. Case 2

The initial locations of the pursuer and evader are listed in Table III. Similarly, Figures 6 and 7 show the transitions of the velocity and steering angle with respect to time when the pursuer used the proposed evader's predictor and the conventional motion tracking algorithms, respectively.

TABLE III. CASE 2

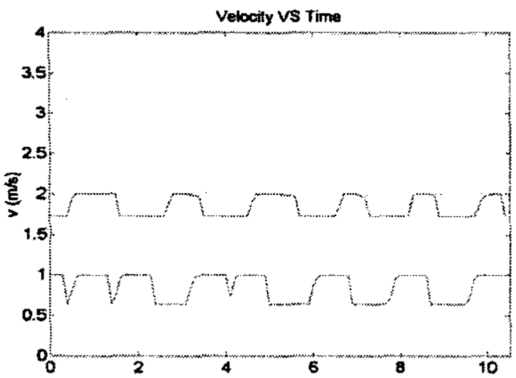
Parameters	Pursuer	Evader
$\mathbf{x}_0^T$ [m, m, °]	[0, 0, 0]	[-4, -2, 0]
T [sec]	9.9	10.5



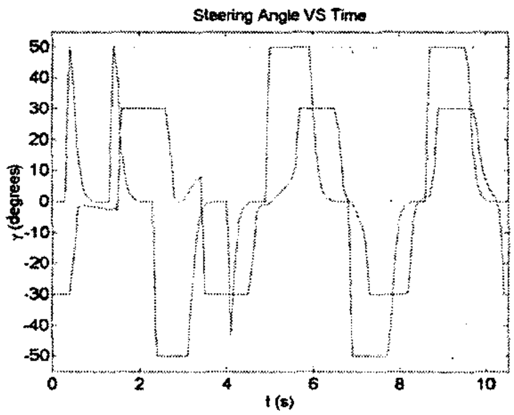
(a) Velocity

(b) Steering angle

Figure 6. Results using evader's predictor for Case 2



(a) Velocity



(b) Steering angle

Figure 7. Results using motion tracking for Case 2

Table III also shows that the capture time using the evader's predictor and the motion tracking algorithms are 9.9 seconds and 10.5 seconds respectively. The proposed technique was again superior in capture time, but the result is quite similar in this case. The capture time is, indeed, heavily dependent on the evader's control strategy, initial location, parameters and many others. However, the success of capture by the proposed technique in this second case has demonstrated its applicability in various situations, in addition to its strong feature against intelligence of the evader.

VI. Conclusion And Future Work

A control strategy to capture an intelligent evader in a time-efficient and robust manner has been proposed. The proposed technique finds instantaneous control actions of the pursuer that may delay the capture time but is formulated such that the intelligent evader can be captured time-efficiently in the long run. The technique was applied to two cases, and the proposed technique could capture the evader faster than the conventional motion tracking algorithms in both the cases.

This research is just the first step toward the time-optimal control of pursuit-evasion game problems when the evader is intelligent, and many research topics are still open. The authors are continuously working on the further theoretical and numerical developments and implementation into the indoor robots.

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