CSC336: Assignment 2

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Problem 1

prove $||x||_2 \leq ||x||_1$

Choose an arbitrary integer $n \geq 1$ and an arbitrary $x \in R^n$

$$(||x||_2)^2 = \sum_{i=1}^{N} |x_i|^2 \leq \sum_{i=1}^{N} |x_i|^2 + 2 \times \sum_{j=1}^{N} \sum_{k=1}^{j-1} |x_j||x_k| = \left(\sum_{i=1}^{N} |x_i|\right)^2 = (||x||_1)^2$$

Since $(||x||_2)^2 \leq (||x||_1)^2$, $||x||_2 \leq ||x||_1$

Problem 2

Norm question

It is possible for some $x, y \in R^2$ such that $||x||_1 > ||y||_1$, $||x||_2 < ||y||_2$.

For example:

$$x = \begin{bmatrix} 0.5 & 0.6 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$||x||_1 = |0.5| + |0.6| = 1.1, \quad ||y||_1 = |0| + |1| = 1, \quad ||x||_1 > ||y||_1$$

$$||x||_2 = \sqrt{|0.5|^2 + |0.6|^2} = \sqrt{0.61}, \quad ||y||_2 = \sqrt{|0|^2 + |1|^2} = 1, \quad ||x||_2 < ||y||_2$$

Problem 3

Prove $||Ax|| \leq ||A|| \cdot ||x||$

To prove that $||Ax||_v \leq ||A||_m \cdot ||x||_v$ for all $x \in R^n$ and all real $n \times n$ matrix, if $||A||_m$ is subordinate to a vector norm, we do it in cases:
Case 1:
x is the zero vector. Ax would also be a zero vector. $||Ax||_v = 0 \leq 0 = ||A||_m \cdot ||x||_v$
Case 2:
x is not a zero vector. Proving $||Ax||_v \leq ||A||_m \cdot ||x||_v$ is the same as proving $\frac{||Ax||_v}{||x||_v} \leq ||A||_m$

$$\frac{||Ax||_v}{||x||_v} \leq \max_{x \neq 0} \frac{||Ax||_v}{||x||_v} = ||A||_m$$

So $||Ax||_v \leq ||A||_m \cdot ||x||_v$

Problem 4

Solve system of equations

Code:

```
A = zeros(13);
b = zeros(13,1);
alpha = sqrt(2)/2;

A(1,2) = 1; A(1,6) = -1;
A(2,3) = 1; b(2) = 10;
A(3,1) = alpha;A(3,4) = -1;A(3,5) = -alpha;
A(4,1) = alpha;A(4,3) = 1;A(4,5) = alpha;
A(5,4) = 1;A(5,8) = -1;
A(6,7) = 1;
A(7,5) = alpha;A(7,6) = 1;A(7,9) = -alpha;A(7,10) = -1;
A(8,5) = alpha;A(8,7) = 1;A(8,9) = alpha;b(8) = 15;
A(9,10) = 1;A(9,13) = -1;
A(10,11) = 1;b(10) = 20;
A(11,8) = 1;A(11,9) = alpha;A(11,12) = -alpha;
A(12,9) = alpha;A(12,11) = 1;A(12,12) = alpha;
A(13,13) = 1;A(13,12) = alpha;

f = A\b;

for i = 1:13
    fprintf('f%d = %f\n', i, f(i))
end

condA = cond(A);
r = b - A * f;
relative_residual = norm(r)/norm(b);
bound = condA * relative_residual;

fprintf('The condition number is %e\n', condA);
fprintf('The relative residual is %e\n', relative_residual);
fprintf('The bound is %e\n', bound);
```

Output:

```
f1 = -28.284271
f2 = 20.000000
f3 = 10.000000
```

```
f4 = -30.000000
f5 = 14.142136
f6 = 20.000000
f7 = 0.000000
f8 = -30.000000
f9 = 7.071068
f10 = 25.000000
f11 = 20.000000
f12 = -35.355339
f13 = 25.000000
The condition number is 1.041221e+01
The relative residual is 2.012275e-16
The bound is 2.095224e-15
```

a)
The system is set up by first transfroming the functions to the form with all fi in left side and constant in the right side.
The solved value of f1 to f13 are displayed in the output.

b)
As we have shown in the lecture, given Af = b, compute $\hat{f}$, $r = b - A\hat{f}$.
$A\hat{f} = b - r = \hat{b}$
$\Delta b = r = b - \hat{b}$
$\Delta f = f - \hat{f}$
In previous lecture note, we have proved that: $\frac{||\Delta f||}{||f||} \leq cond(A) \cdot \frac{||\Delta b||}{||b||}$
So $\frac{||\Delta f||}{||f||} \leq cond(A) \cdot \frac{||r||}{||b||}$
We can bound the relative error $\frac{||f-\hat{f}||}{||f||}$ by $cond(A) \cdot \frac{||r||}{||b||}$
The code for calcuting the bound is in above code section. The value of the bound on $\frac{||f-\hat{f}||}{||f||}$ is 2.095224e-15

Problem 5

Hilbert Matrix

Part 1)
Code:

```
fprintf('%4s   %12s\n', 'n', 'Error');
max = 15;
x = ones(max, 1);
for n = 1:max
    cur_x = x(1:n);
    H = hilb(n);
    b = H * cur_x;
    x_hat = H\b;
    error = norm(cur_x - x_hat, Inf)/norm(cur_x, Inf);
    fprintf("%4d    %12e\n", n, error);
end
```

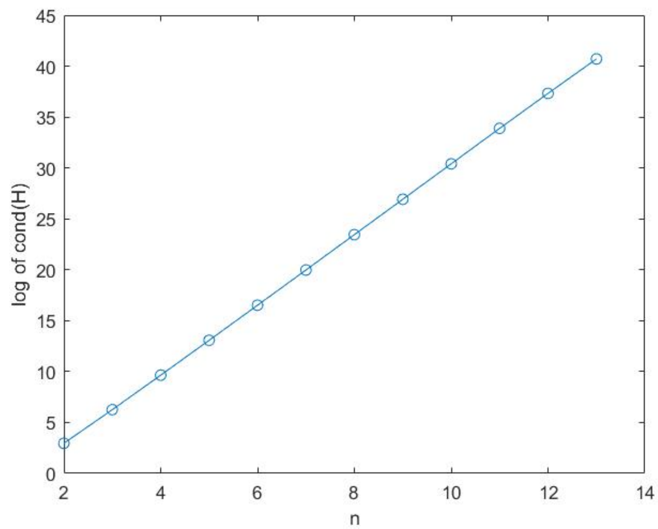
Output:

```
   n         Error
   1  0.000000e+00
   2  7.771561e-16
   3  7.438494e-15
   4  4.518608e-13
   5  4.000800e-12
   6  5.724005e-10
   7  2.018865e-08
   8  2.046369e-07
   9  1.208404e-05
  10  3.592619e-04
  11  5.105110e-04
  12  4.996064e-01
  13  2.979912e+00
  14  5.019723e+00
  15  1.507535e+01
```

From the output, we can see that when $n \leq 12$, the relative error in the solution is smaller than 1. When $n > 12$, the relative error in the solution is greater than 1.

Part 2)

```
ns = 2:13;
cond_logs = zeros(1,12);
for i=1:12
    n = i + 1;
    cond_logs(i) = log(cond(hilb(n)));
end
plot(ns,conds,"-o");
```



From the graph, we can see that the log of the condition number is in a roughly linear relationship with n. So condition number and n have a exponential relationship. If n is doubled, cond(H) is squared, etc.

Take two points from graph, say (2,2.9591) and (13, 40.7097), we can estimate the line by $log(cond(hilb(n))) \approx -3.9047 + 3.4319 \times n$

$cond(hilb(n)) \approx f(n) = e^{-3.9047+3.4319\times n}$

Part 3)

```
fprintf('%4s   %12s   %12s\n', 'n', 'Cond(H)', 'Max Error');
max_n = 15;
x = ones(max_n, 1);
for n = 1:max_n
    cur_x = x(1:n);
    H = hilb(n);
```

```
    b = H * cur_x;
    x_hat = H\b;
    fprintf("%4d    %12e    %12e\n", n, cond(H), max(abs(cur_x-x_hat)));
end
```

```
   n          Cond(H)         Max Error
   1     1.000000e+00    0.000000e+00
   2     1.928147e+01    7.771561e-16
   3     5.240568e+02    7.438494e-15
   4     1.551374e+04    4.518608e-13
   5     4.766073e+05    4.000800e-12
   6     1.495106e+07    5.724005e-10
   7     4.753674e+08    2.018865e-08
   8     1.525758e+10    2.046369e-07
   9     4.931534e+11    1.208404e-05
  10     1.602503e+13    3.592619e-04
  11     5.220207e+14    5.105110e-04
  12     1.621164e+16    4.996064e-01
  13     4.786392e+17    2.979912e+00
  14     2.551499e+17    5.019723e+00
  15     2.495952e+17    1.507535e+01
```

The 'max error' row is the maximum absolute value of $x - \hat{x}$ at each value of n.
Since the components of x are all 1, we can easily observe the number of correct digits through this table. For example, if max error is 7.771561e-16, then we can know that at least the first 16 digits of any component in $\hat{x}$ is the same as the correspond component in x because the 17th digit is the first different digit.
With this logic, we can then see from the table that when cond(hilb(n)) is multiplied by about 10 times, the number of correct digits decrease by 1. And, suppose the condition number is $num \cdot 10^i$, the number of correct digits is roughly 17 - i.