CSC421: Written Homework 5

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Problem 1

VFE/ELBO

(a)

$$\begin{aligned}
F(q) &= E_q[logp(\mathbf{x}|\mathbf{z})] - D_{KL}(q(\mathbf{z})||p(\mathbf{z})) \\
&= E_q[logp(\mathbf{x}|\mathbf{z}) - logq(\mathbf{z}) + logp(\mathbf{z})] \\
&= E_q[log(p(\mathbf{x}|\mathbf{z})p(\mathbf{z})) - logq(\mathbf{z})] \\
&= E_q[log(p(\mathbf{z}|\mathbf{x})p(\mathbf{x})) - logq(\mathbf{z})] \; \#Bayes' \; Rule \\
&= E_q[logp(\mathbf{x}) + logp(\mathbf{z}|\mathbf{x}) - logq(\mathbf{z})] \\
&= logp(\mathbf{x}) - E_q[logq(\mathbf{z}) - logp(\mathbf{z}|\mathbf{x})] \\
&= logp(\mathbf{x}) - D_{KL}(q(\mathbf{z})||p(\mathbf{z}|\mathbf{x}))
\end{aligned} \tag{1}$$

(b)

$$\begin{aligned}
D_{KL}(q(\mathbf{z})||p(\mathbf{z})) &= E_q[logq(\mathbf{z}) - logp(\mathbf{z})] \\
&= E_q[log(\prod_{i=1}^{D} q_i(z_i)) - log(\prod_{i=1}^{D} p_i(z_i))] \\
&= E_q[\sum_{i=1}^{D} log(q_i(z_i)) - \sum_{i=1}^{D} log(p_i(z_i))] \\
&= E_q[\sum_{i=1}^{D}(logq_i(z_i) - logp_i(z_i))] \\
&= \sum_{i=1}^{D} E_q[logq_i(z_i) - logp_i(z_i)] \\
&= \sum_{i=1}^{D} D_{KL}(q_i(z_i)||p_i(z_i))
\end{aligned} \tag{2}$$

(c)

$$
\begin{aligned}
D_{KL}(q(z)||p(z)) &= E_q[logq(z) - logp(z)] \\
&= \int q(z)(logq(z) - logp(z))dz \\
&= \int q(z)log\frac{q(z)}{p(z)}dz \\
&= \int q(z)log\frac{\frac{1}{\sqrt{2\pi}\sigma}exp(-\frac{(z-\mu)^2}{2\sigma^2})}{\frac{1}{\sqrt{2\pi}}exp(-\frac{z^2}{2})}dz \\
&= \int q(z)log\frac{\frac{1}{\sqrt{2\pi}\sigma}}{\frac{1}{\sqrt{2\pi}}}dz + \int q(z)log\frac{exp(-\frac{(z-\mu)^2}{2\sigma^2})}{exp(-\frac{z^2}{2})}dz \\
&= log\frac{1}{\sigma}\int q(z)dz + \int q(z)(-\frac{(z-\mu)^2}{2\sigma^2} + \frac{z^2}{2})dz \\
&= log\frac{1}{\sigma} - \int q(z)\frac{(z-\mu)^2}{2\sigma^2}dz + \int q(z)\frac{z^2}{2}dz \\
&= log\frac{1}{\sigma} - \frac{1}{2\sigma^2}\int q(z)(z-\mu)^2dz + \frac{1}{2}\int q(z)z^2dz \\
&= log\frac{1}{\sigma} - \frac{1}{2\sigma^2}Var(z) + \frac{1}{2}E_q[z^2] \\
&= log\frac{1}{\sigma} - \frac{1}{2} + \frac{1}{2}(Var(z) + E_q[z]^2) \\
&= -\frac{1}{2}log(\sigma^2) - \frac{1}{2} + \frac{1}{2}(\sigma^2 + \mu^2) \\
&= \frac{1}{2}(\sigma^2 + \mu^2 - log(\sigma^2) - 1)
\end{aligned} \tag{3}
$$

(d)
As defined in the question, to find D_{KL}, first, we want to find $\partial t/\partial\theta = \bar{\theta} = [\bar{\mu}, \bar{\sigma}]$

$$\begin{aligned}
\bar{t} &= 1 \\
\bar{r} &= 1 \\
\bar{s} &= -1 \\
\bar{z} &= \bar{r}\frac{\partial r}{\partial z} + \bar{s}\frac{\partial s}{\partial z} \\
&= 1\frac{\partial}{\partial z}logq(z) + (-1)\frac{\partial}{\partial z}logp(z) \\
&= \frac{\partial}{\partial z}logq(z) - \frac{\partial}{\partial z}logp(z) \\
\frac{\partial}{\partial z}logq(z) &= \frac{\partial}{\partial z}log(\frac{1}{\sqrt{2\pi}\sigma}exp(-\frac{(z-\mu)^2}{2\sigma^2})) \\
&= \frac{\partial}{\partial z}(log(\frac{1}{\sqrt{2\pi}\sigma}) - \frac{(z-\mu)^2}{2\sigma^2}) \\
&= \frac{\partial}{\partial z}(-\frac{(z-\mu)^2}{2\sigma^2}) \\
&= -\frac{(z-\mu)}{\sigma^2} \\
\frac{\partial}{\partial z}logp(z) &= \frac{\partial}{\partial z}log(\frac{1}{\sqrt{2\pi}}exp(-\frac{z^2}{2})) \\
&= \frac{\partial}{\partial z}(-\frac{z^2}{2}) \\
&= -z \\
\bar{z} &= -\frac{(z-\mu)}{\sigma^2} + z \\
\bar{\mu} &= \bar{z} = -\frac{(z-\mu)}{\sigma^2} + z \\
\bar{\sigma} &= \bar{z}\epsilon = \epsilon(-\frac{(z-\mu)}{\sigma^2} + z)
\end{aligned} \tag{4}$$

Since $\nabla_\theta D_{KL}(q(z)||p(z)) = E_\epsilon[\nabla_\theta t]$ where $\epsilon \sim N(0,1)$:

$$\begin{aligned}
\frac{\partial}{\partial\mu}D_{KL}(q(z)||p(z)) &= E_\epsilon[\bar{\mu}] \\
&= E_\epsilon[-\frac{(z-\mu)}{\sigma^2} + z] \\
&= E_\epsilon[-\frac{(\mu+\sigma\epsilon-\mu)}{\sigma^2} + \mu + \sigma\epsilon] \\
&= E_\epsilon[-\frac{(\sigma\epsilon)}{\sigma^2}] + E_\epsilon[\mu] + E_\epsilon[\sigma\epsilon] \\
&= \mu \quad \#since\ \mu_\epsilon = 0
\end{aligned} \tag{5}$$

$$\begin{aligned}
\frac{\partial}{\partial \sigma} D_{KL}(q(z)||p(z)) &= E_\epsilon[\bar{\sigma}] \\
&= E_\epsilon[\epsilon(-\frac{(z-\mu)}{\sigma^2} + z)] \\
&= E_\epsilon[\epsilon(-\frac{(\mu + \sigma\epsilon - \mu)}{\sigma^2} + \mu + \sigma\epsilon)] \\
&= E_\epsilon[\epsilon(-\frac{\epsilon}{\sigma} + \mu + \sigma\epsilon)] \\
&= E_\epsilon[-\frac{\epsilon^2}{\sigma}] + E_\epsilon[\epsilon\mu] + E_\epsilon[\sigma\epsilon^2] \\
&= -\frac{1}{\sigma} E_\epsilon[\epsilon^2] + \mu E_\epsilon[\epsilon] + \sigma E_\epsilon[\epsilon^2] \\
&= -\frac{1}{\sigma}(Var[\epsilon] + E_\epsilon[\epsilon]^2) + \mu * 0 + \sigma(Var[\epsilon] + E_\epsilon[\epsilon]^2) \\
&= -\frac{1}{\sigma}(1 + 0^2) + \mu * 0 + \sigma(1 + 0^2) \\
&= \sigma - \frac{1}{\sigma}
\end{aligned} \tag{6}$$