Introduction to higher order differential equations, complementary function and particular integral

Linear Differential Equations are those in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together.

The general linear differential equation of the nth order is of the form

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + P_n y = X \qquad \text{---------(I)}$$

where P_1 , P_2, ... , P_n and X are functions only of x. If P_1, P_2 , ..., P_n are all constants then it is called a linear differential equation with constant coefficients.

If X = 0, the above equation is said to be homogeneous. Otherwise, it is called non homogeneous.

Such equations are most important in the study of electro mechanical vibrations and other engineering problems.

Examples :

1. $\frac{d^3 y}{dx^3} + 2\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} = e^x + \sin 2x + x^3$

2. $x^2 \frac{d^2 y}{dx^2} + 3x\frac{dy}{dx} + 2y = \log x$

Solution of homogeneous Linear differential equations with constant coefficients:

Theorem : If y_1 and y_2 are two solutions of the equation

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + P_n y = 0 \qquad \text{..........................(1)}$$

then u = $c_1y_1 + c_2y_2$ is also its solution.

Since the general solution of a differential equation of nth order contains n arbitrary constants it follows from the above theorem that, if y_1 , y_2 , ... , y_n are n independent solutions of equation (1) then $u = c_1y_1 + c_2y_2 + ... + c_ny_n$ is its complete solution.

Hence $$\frac{d^n u}{dx^n} + P_1 \frac{d^{n-1}u}{dx^{n-1}} + P_2 \frac{d^{n-2}u}{dx^{n-2}} + ... + P_n u = 0 \qquad ....(2)$$

If y = v is any particular solution of

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + ... + P_n y = X \qquad ....(3)$$

then

$$\frac{d^n v}{dx^n} + P_1 \frac{d^{n-1}v}{dx^{n-1}} + P_2 \frac{d^{n-2}v}{dx^{n-2}} + ... + P_n v = X \qquad ....(4)$$

Adding (2) and (4) we get

$$\frac{d^n (u+v)}{dx^n} + P_1 \frac{d^{n-1}(u+v)}{dx^{n-1}} + P_2 \frac{d^{n-2}(u+v)}{dx^{n-2}} + ... + P_n (u+v) = X$$

This shows that y = u + v is the complete solution of (3).

Therefore the general solution of a differential equation of nth order with constant coefficients is of the form y = u + v where u is called the **Complementary Function** (C.F) and v is called the **Particular Integral** (P. I) .

The Operator D

For convenience, we shall denote $\frac{dy}{dx}, \frac{d^2y}{dx^2}, ...$ by D , D^2 , ... so that the general differential equation can be written as

f (D) y = X where

$$f(D) = D^n + P_1 D^{n-1} + ...+ P_{n-1}D + P_n$$

is a polynomial in D.

Thus, the symbol D stands for the operation of differentiation and can be treated much the same as an algebraic quantity. Thus, f (D) can be factorized by ordinary rules of algebra and the factors may be taken in any order.

For example,

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = (D^2 + 5D + 6)y = (D+2)(D+3)y = (D+3)(D+2)y$$

Rules for Finding the Complementary Function

The equation

$$\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots P_{n-1}\frac{dy}{dx} + P_n y = 0$$

in symbolic form is

$$(D^n + P_1 D^{n-1} + P_2 D^{n-2} \ldots\ldots + P_{n-1}D + P_n)y = 0$$

Its symbolic coefficient equated to zero, that is

$$D^n + P_1 D^{n-1} + P_2 D^{n-2} \ldots\ldots + P_{n-1}D + P_n = 0$$

is called the **Auxiliary Equation (AE)**. Let the roots of this equation be m_1 , m_2 , ... m_n.

Case 1: If all the roots are real and different then

$$CF = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots\ldots + c_n e^{m_n x}$$

where c_1 , c_2, ... , c_n are arbitrary constants.

Case 2: If two of the roots are equal say $m_1 = m_2 = m$ then

$$CF = (c_1 + c_2 x)e^{mx} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

If, however, the AE has three equal roots $m_1 = m_2 = m_3 = m$ then

$$CF = (c_1 + c_2 x + c_3 x^2)e^{mx} + c_4 e^{m_4 x} + \ldots\ldots + c_n e^{m_n x}$$

Case 3: If two of the roots are complex say $\alpha \pm i\beta$ and the remaining roots are all real and different then

$$CF = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

Case 4: If two pairs of complex roots are equal say $m_1 = m_2 = \alpha + i\beta$ and $m_3 = m_4 = \alpha - i\beta$ and the remaining roots are all real and different then

$$CF = [e^{\alpha x}(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x] + c_5 e^{m_5 x} + \ldots\ldots + c_n e^{m_n x}$$

Examples:

1. Solve the differential equation $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$

Solution : The given equation can be written as

$$(D^3 + 6D^2 + 11D + 6)\, y = 0$$

The auxiliary equation is $D^3 + 6D^2 + 11D + 6 = 0$

The roots are $m_1 = -1$, $m_2 = -2$ and $m_3 = -3$ which are real and different.

Hence CF $= c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$

The Complete Solution is y $= c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$ where c_1, c_2 and c_3 are arbitrary constants.

2. Solve the differential equation $\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} - 11\frac{dy}{dx} - 4y = 0$

Solution : The given equation can be written as

$$(D^4 - D^3 - 9D^2 - 11D - 4)y = 0$$

The auxiliary equation is $D^4 - D^3 - 9D^2 - 11D - 4 = 0$ or $(D+1)^3(D-4) = 0$

The roots are $m_1 = m_2 = m_3 = -1$ and $m_4 = 4$ which are real but repeated.

Hence CF = $(c_1 + c_2\,x + c_3\,x^2)e^{-x} + c_4\,e^{4x}$

The Complete Solution is y = $(c_1 + c_2\,x + c_3\,x^2)e^{-x} + c_4\,e^{4x}$ where c_1 , c_2 , c_3 and c_4 are arbitrary constants.

3. Solve the differential equation $\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 8\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = 0$

Solution : The given equation can be written as

$$(D^4 - 4D^3 + 8D^2 - 8D + 4)y = 0$$

The auxiliary equation is $D^4 - 4D^3 + 8D^2 - 8D + 4 = 0$ or $(D^2 - 2D + 2)^2 = 0$

The roots are $m_1 = m_2 =$ $1 \pm i$ which are complex repeated roots.

Hence CF = $e^x\,\{\,(c_1 + c_2 x)\cos x + (c_3 + c_4\,x)\sin x\,\}$

The Complete Solution is y = $e^x\,\{\,(c_1 + c_2 x)\cos x + (c_3 + c_4\,x)\sin x\,\}$ where c_1 , c_2 , c_3 and c_4 are arbitrary constants.

4. Solve $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$ given that x (0) = 0, and $\frac{dx}{dt}(0) = 15$

Solution : The auxiliary equation is $D^2 + 5D + 6 = 0$ whose roots are -2 and -3.

Therefore, CF = x = $c_1\,e^{-2t} + c_2\,e^{-3t}$ ....(1)

Differentiating with respect to t we get

$$\frac{dx}{dt} = -2c_1\,e^{-2t} - 3c_2\,e^{-3t} \qquad \text{....................................(2)}$$

Given, x = 0 when t = 0. Substituting in equation (1) we get

$$c_1 + c_2 = 0 \qquad \text{....................................(3)}$$

Also, $\frac{dx}{dt} = 15$ when t = 0. Substituting in equation (2) we get

$$-2c_1 - 3c_2 = 15 \qquad \text{..................................(4)}$$

Solving (3) and (4) simultaneously we get $c_1 = 15$, $c_2 = -15$

Hence, the complete solution is $x = 15\,(\,e^{-2t} - c_2\,e^{-3t}\,)$.

Solution of non homogeneous Linear differential equations with constant coefficients:

The Inverse Operator

The function $\frac{X}{f(D)}$ is a function of x which when operated upon f (D) gives X. That is ,

$$f(D)\left(\frac{X}{f(D)}\right) = X$$

Thus, the function $\frac{X}{f(D)}$ satisfies the equation f (D) y = X and is therefore its Particular Integral.

Obviously, f (D) and 1 / f (D) are Inverse Operators.

Rules For Finding The Particular Integral

Consider the equation

$$\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + P_{n-1}\frac{dy}{dx} + P_n y = X$$

which in symbolic form is $(D^n + P_1 D^{n-1} + P_2 D^{n-2} \ldots\ldots + P_{n-1}D + P_n)\,y = X$

or , f (D) y = X

Case 1 : When $X = e^{ax+b}$ where a and b are constants then

PI = $e^{ax+b} / f(a)$ provided $f(a) \neq 0$. If $f(a) = 0$ then

PI = $x\,e^{ax+b} / f'(a)$ provided $f'(a) \neq 0$. If $f'(a) = 0$ then

PI = $x^2 e^{ax+b} / f''(a)$ provided $f''(a) \neq 0$ and so on.

Case 2 : When X = sin (ax + b) or cos (ax + b) where a and b are constants then

PI = X / f (D^2) = X / f ($-a^2$) provided f ($-a^2$) ≠ 0.

If f ($-a^2$) = 0 then PI = $x \frac{1}{f'(-a^2)} X$ provided f ' ($-a^2$) ≠ 0 and so on.

Case 3 : When X = x^m , a polynomial in x of degree m then

PI = x^m / f (D) = [f (D) $]^{-1}$ x^m.

We now expand [f (D) $]^{-1}$ in ascending powers of D as far as D^m and then operate on x^m term by term. Since the (m+1)th and higher derivatives of x^m are all zero we need not consider beyond D^m.

Case 4 : When X = e^{ax+b} V where V is a function of x. Then

$$PI = e^{ax+b} \frac{V}{f(D+a)}$$

The function V could be sin (ax +b) or cos (ax+b) in which case we use Rule 2 or it could be a polynomial in x in which case we use Rule 3.

Working procedure to solve the equation:

Given a differential equation of the form f (D) y = X ,

Step 1 : Write the Auxiliary Equation f (D) = 0 and solve it for D.

Step 2 : Based on the nature of the roots obtained in step 1, write the Complementary Function (CF)

Step 3 : Depending on the function X, find the Particular Integral (PI).

Step 4 : The Complete Solution (CS) of the given differential equation is given by

CS = CF + PI

Note : When X = xV where V is any function of x then we can find the PI using the formula

$$\text{PI} = x \frac{V}{f(D)} - \frac{f'(D)}{[f(D)]^2} V$$

Example: The Particular integral of the differential equation $(D^2-1)y = xsinx$ can be found using above formula

We know that C.F $= c_1e^t + c_2e^{-t}$

And $P.I = \frac{xsinx}{D^2-1} = x\frac{sinx}{D^2-1} - \frac{2D}{(D^2-1)^2}sinx = \frac{xsinx}{2} - \frac{cosx}{2}$ using above mentioned formula.

Therefore, solution y = C.F + P.I

Problems:

5. Solve : $(D^2-5D+6)\,y = e^{4x} + \sin x + x^2$

Solution : The auxiliary equation is $D^2-5D+6 = 0$

The roots are $m_1 = 2$, $m_2 = 3$ which are real and different.

Hence CF = $c_1e^{2x} + c_2e^{3x}$

Since there are three functions on the right side of the given differential equation we break the particular integral into three parts and evaluate them separately.

PI_1 = $\frac{e^{4x}}{D^2-5D+6} = \frac{e^{4x}}{4^2-(5x4)+6} = \frac{e^{4x}}{2}$ (we have substituted D = 4 in the denominator)

PI_2 = $\frac{\sin x}{D^2-5D+6} = \frac{\sin x}{(-1)-5D+6} = \frac{\sin x}{5-5D} = \frac{-\sin x}{5(D-1)}$ (we have replaced D^2 by -1)

In order to get D^2 function in the denominator we now multiply and divide by D+1. This gives

PI_2 = $\frac{-(D+1)\sin x}{5(D^2-1)} = \frac{-D(\sin x)-\sin x}{5(-1-1)} = \frac{\cos x+\sin x}{10}$

PI_3= $\frac{x^2}{D^2-5D+6} = \frac{x^2}{6\left[1+\frac{D^2-5D}{6}\right]} = \frac{1}{6}\left[1+\frac{D^2-5D}{6}\right]^{-1}x^2$

We now expand this function of D in powers of D upto D^2 since the higher order derivatives of x^2 are all zero.

$$PI_3 = \frac{1}{6}\left[1-\frac{D^2-5D}{6}+\left(\frac{D^2-5D}{6}\right)^2\right]x^2 = \frac{1}{6}\left[1-\frac{D^2}{6}+\frac{5D}{6}+\frac{25D^2}{36}\right]x^2$$

$$= \frac{1}{6}\left[x^2+\frac{5x}{3}+\frac{4}{3}\right]$$

Hence, PI = $PI_1 + PI_2 + PI_3$. The Complete Solution is y = CF + PI

6. Solve $\left(D^4+2D^3-3D^2\right)y = 3e^x+4\sin x+x$

Solution : The auxiliary equation is $\left(D^4+2D^3-3D^2\right) = 0$ or $D^2(D^2+2D-3) = 0.$

The roots are m = 0, 0, 1 and -3.

Therefore CF = $(c_1+c_2x)+c_3e^x+c_4e^{-3x}$

$$PI_1 = \frac{3e^x}{D^4+2D^3-3D^2} = \frac{3e^x}{1+2-3} = \frac{3xe^x}{4D^3+6D^2-6D} = \frac{3xe^x}{4}$$

$$PI_2 = \frac{4\sin x}{D^4+2D^3-3D^2} = \frac{4\sin x}{(-1)^2+2D(-1)-3(-1)} = \frac{4\sin x}{4-2D} = \frac{2(2+D)\sin x}{4-D^2}$$

$$= \frac{4\sin x+2\cos x}{5}$$

$$PI_3 = \frac{x}{D^4+2D^3-3D^2} = \frac{1}{-3D^2}\frac{x}{\left[1-\frac{D^2+2D}{3}\right]} = \frac{-1}{3D^2}\left[1-\frac{D^2+2D}{3}\right]^{-1}x$$

$$= \frac{-1}{3D^2}\left[1+\left(\frac{D^2+2D}{3}\right)+\left(\frac{D^2+2D}{3}\right)^2+\left(\frac{D^2+2D}{3}\right)^3\right]x$$

We now consider terms up to D^3 since there is a D^2 in the denominator.

$PI_3 = \frac{-1}{3D^2}\left[1+\frac{D^2}{3}+\frac{2D}{3}+\frac{4D^3}{9}+\frac{4D^2}{9}+\frac{8D^3}{27}\right]x$

$= \frac{-1}{3}\left[\frac{1}{D^2}+\frac{1}{3}+\frac{2}{3D}+\frac{4D}{9}+\frac{4}{9}+\frac{8D}{27}\right]x = \frac{-1}{3}\left[\frac{x^3}{6}+\frac{x^2}{3}+\frac{7x}{9}+\frac{20}{27}\right]$

Therefore PI = PI_1 + PI_2 + PI_3 . The complete solution is y = CF + PI.

7. Solve : $(D^2+2)\, y = x^2 e^{3x}$

Solution : The auxiliary equation is $D^2 + 2 = 0$. The roots are D = $\pm i\sqrt{2}$

Therefore CF = $c_1 \cos\sqrt{2}\,x + c_2 \sin\sqrt{2}\,x$

Now, PI = $\frac{x^2e^{3x}}{D^2+2} = e^{3x}\frac{x^2}{(D+3)^2+2} = e^{3x}\frac{x^2}{D^2+6D+11}$

Proceeding as we did in the previous example we get

PI = $\frac{e^{3x}}{11}\left[x^2-\frac{12}{11}x+\frac{50}{121}\right]$

The Complete Solution is y = CF + PI.

Equations Reducible to Linear Equations with Constant Coefficients

1. Cauchy's Homogeneous Linear Equation: An equation of the form

$$x^n\frac{d^ny}{dx^n}+P_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+P_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}}+\dots+P_{n-1}x\frac{dy}{dx}+P_ny = X$$

where X is a function of x is called **Cauchy's Homogeneous DE**.

Such equations can be reduced to Linear DE with constant coefficients by substituting $x = e^t$ or $t = \log x$.

Then, if $D=\frac{d}{dt}$ then $x\frac{dy}{dx}=\frac{dy}{dt}=Dy$ and $x^2\frac{d^2y}{dx^2}=D(D-1)y$

Similarly , $x^3\frac{d^3y}{dx^3}=D(D-1)(D-2)y$ and so on.

2. Legendre's Linear Equation : An equation of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + P_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + P_2(ax+b)^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + P_{n-1}(ax+b)\frac{dy}{dx} + P_n y = X$$

is called **Legendre's DE**. Such equations can be reduced to linear DE with constant coefficients by the substitution $ax + b = e^t$ or $t = \log(ax+b)$.

If $D = \frac{d}{dt}$ then $(ax+b)\frac{dy}{dx} = a\frac{dy}{dt} = aDy$ and $(ax+b)^2\frac{d^2y}{dx^2} = a^2 D(D-1)y$

Similarly $(ax+b)^3\frac{d^3y}{dx^3} = a^3 D(D-1)(D-2)y$ and so on.

After making these substitutions in the given equation there results a linear equation with constant coefficients which can be solved as before.

Examples :

8. Solve the following differential equations

a) $(x^2D^2 - xD + 4)y = \cos(\log x) + x\sin(\log x)$

Solution : This is Cauchy's linear Differential Equation. Substituting $x = e^t$ or $t = \log x$

we get $(D'(D'-1) - D' + 4)\, y = \cos t + e^t \sin t$ where $D' = d/dt$.

The auxiliary equation is $D'^2 - 2D' + 4 = 0$. The roots are $1 \pm i\sqrt{3}$

Therefore CF $= e^t[c_1 \cos\sqrt{3}t + c_2 \sin\sqrt{3}t]$ where $t = \log x$

$PI_1 = \frac{\cos t}{D'^2 - 2D' + 4} = \frac{3\cos t - 2\sin t}{13}$ and $PI_2 = \frac{e^t \sin t}{2}$ where $t = \log x$

The Complete Solution is $y = CF + PI$.

b) $x^2y'' + xy' + y = \sin(log(x^2))$

Solution: This is Cauchy's linear Differential Equation. Substituting x = e^t or t = log x

we get, (D (D - 1) + D + 1) y = sin2 t where D = d / dt.

The auxiliary equation is $D^2 + 1 = 0$. The roots are ± 1

Therefore CF = $c_1e^t + c_2e^t$ where t = log x

PI = $\frac{1}{D^2+1}sin2t = -\frac{1}{3}sin2t$ where t = log x

The Complete Solution is y = CF + PI.

9. Solve the following differential equations

a) $(3x+2)^2\, y'' + 3(3x+2\,)\,y' \; - 36\; y \;\; = 3x^2 + 4x + 1$

Solution : This is Legendre's linear differential equation. Substituting (3x + 2) = e^t or t = log(3x + 2)

we get $[\ 9D(D-1)\ + 9D - 36\ ]\ y\ = \frac{1}{3}[t^2 - 1]$

where D = d/dt

Or $[D^2 - 4]\,y\ = \frac{1}{27}[t^2 - 1]$

On solving auxiliary equation we get CF = $c_1\,e^{2t}\ + c_2\,e^{-2t}$

PI $= \frac{1}{27}\frac{1}{D^2-4}(t^2 + 1) = \ \frac{1}{108}\left[t^2 \log t\ + 1\right]$

The Complete Solution is y = CF + PI. Where t = log(3x + 2)

b) $(2x + 3)^2y'' + 2(2x + 3)y' + 4y = \sin\,(\log(2x + 3))$

Solution: This is Legendre's linear differential equation. Substituting (2x + 3) = e^t or t = log(3x + 2)

we get $(4D(D-1) + 2.2D + 4) = sint$

where D = d/dt

Or $[D^2+1]\,y = \sin t$

On solving auxiliary equation we get CF = $c_1 \cos t + c_2 \sin t$

PI $= \frac{1}{D^2+1}(sint) = -\frac{t}{2}sint$

The Complete Solution is y = CF + PI. Where t = log(2x + 3)

Method of variation of Parameters

This method is quite general and applies to equations of the form

y ʺ + p y ʹ + q y = X where p , q and X are functions of x.

Let the CF of the above equation be CF = $c_1 y_1 + c_2 y_2$

We replace c_1 and c_2 (regarded as parameters) by unknown functions u (x) and v (x) and write the PI as PI = $u y_1 + v y_2$

It gives PI = $-y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is called the Wronskian of y_1 and y_2.

Example :

10. Solve by the method of variation of parameters the differential equation

a) $(D^2 + 4)y = \tan 2x$.

Solution : The auxiliary equation is $D^2 + 4 = 0$. Its roots are $\pm 2i$

Therefore CF = $c_1 \cos 2x + c_2 \sin 2x$.

Writing CF = $c_1 y_1 + c_2 y_2$ we have $y_1 = \cos 2x$ and $y_2 = \sin 2x$.

The wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 2$

Hence, $PI = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$

$$= -\cos 2x \int \frac{\sin 2x \tan 2x}{2} dx + \sin 2x \int \frac{\cos 2x \tan 2x}{2} dx$$

$$= \frac{-1}{4} \cos 2x \log(\sec 2x + \tan 2x)$$

The Complete Solution is y = CF + PI.

b) $(D^2 + 4)y = secx$

Solution: The auxiliary equation is $D^2 + 4 = 0$. Its roots are $\pm 2i$

Therefore CF = $c_1 \cos 2x + c_2 \sin 2x$.

Writing CF = $c_1 y_1 + c_2 y_2$ we have $y_1 = \cos 2x$ and $y_2 = \sin 2x$.

The wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 2$

Hence, $PI = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$

$$= \quad -\cos 2x \int \frac{\sin 2x \sec x}{2}\, dx \; + \; \sin 2x \int \frac{\cos 2x \sec x}{2}\, dx$$

$$= \quad cosxcos2x + (sinx - \tfrac{1}{2}\ln|secx + tanx|)\ \ sin2x$$

The Complete Solution is y = CF + PI.

Applications of Linear Differential Equations

The linear differential equations with constant coefficients find their most important applications in the study of electrical, mechanical and other linear systems. Such equations play a dominant role in unifying the theory of electrical and mechanical oscillatory systems. We take up two such applications of linear differential equations in this section.

1. Simple Harmonic Motion

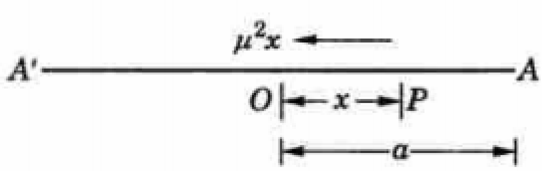
When the acceleration of a particle is proportional to its displacement from a fixed point and is always directed towards it then the motion is said to be simple harmonic.

If the displacement of the particle at any time t from the fixed point O is x then

$$\frac{d^2x}{dt^2} = -\mu^2 x$$

(The negative sign indicates that the force is acting on P towards O, that is in the direction of x decreasing)

Or $(D^2 + \mu^2)\, x = 0$ whose solution is $x = c_1 \cos \mu t + c_2 \sin \mu t$



Therefore, its velocity at P = $\frac{dx}{dt} = \mu\,(-c_1 \sin \mu t + c_2 \cos \mu t\,)$

If the particle starts from rest at A where OA = a then c_1 = a and c_2 = 0.

Hence x = $a \cos \mu t$ and $\frac{dx}{dt} = -a\mu \sin \mu t = -\mu\sqrt{a^2 - x^2}$ which give the displacement and the velocity of the particle at any time t.

Nature of Motion: The particle starts from A towards O under acceleration directed towards O which gradually decreases until it vanishes at O , when the particle has gained the maximum velocity. On passing through O, retardation begins and the particle comes to an instantaneous rest at A' where OA = OA'. It then retraces its path and goes on oscillating between the two points A and A'.

Note :

1. The **amplitude** or the maximum displacement from the centre is a.

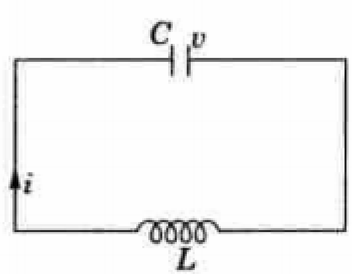
2. The **period time** , that is, the time of complete oscillation is $\frac{2\pi}{\mu}$.

3. The **frequency** or the number of oscillations per second is $\frac{\mu}{2\pi}$.

2. **Oscillatory Electrical Circuit**

(i) L-C Circuit

Consider an electrical circuit containing an inductance L and capacitance C.



Let i be the current and q the charge in the condenser plate at any time t so that the voltage drop across L is $L\frac{di}{dt} = L\frac{d^2q}{dt^2}$ and the voltage drop across C is q / C.

As there is no applied emf in the circuit, by Kirchhoff's first law we have

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0$$

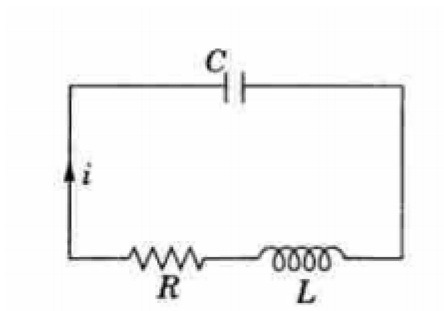
On dividing by L and writing $1 / LC = \mu^2$ we get

$$\frac{d^2q}{dt^2} + \mu^2 q = 0$$

which is the same as the one that we had in the previous application and hence it represents free electrical oscillations of the current having period $\frac{2\pi}{\mu} = 2\pi\sqrt{LC}$.

(ii) L-C-R Circuit

Now consider the discharge of a condenser C through an inductance L and the resistance R.



The voltage drop across L, C and R are respectively

$$L\,\frac{d^2q}{dt^2},\ q/C \text{ and } R\,\frac{dq}{dt}$$

Therefore by Kirchhoff's law we have

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

On writing $R/L = 2\lambda$ and $1 / LC = \mu^2$ we get

$$\frac{d^2q}{dt^2} + 2\lambda\frac{dq}{dt} + \mu^2 q = 0$$

The auxiliary equation $D^2 + 2\lambda D + \mu^2 = 0$ or $D = -\lambda \pm \sqrt{\lambda^2 - \mu^2}$

Case (i) When $\lambda > \mu$ the roots of the auxiliary equation are real and distinct say u and v. The solution in this case is

$$q = c_1\, e^{ut} + c_2 e^{vt}$$

Case (ii) When λ = μ the roots are real and equal each being equal to – λ. The general solution in this case is

$$q = (\, c_1 + c_2\, t\,)\, e^{-\lambda t}$$

Case (iii) When λ < μ the roots are imaginary. That is D = - λ ± iα where $\alpha^2 = \mu^2 - \lambda^2$

The general solution in this case is

$$q = (\, c_1 \cos \alpha t + c_2 \sin \alpha t\,)\, e^{-\lambda t}$$

Examples:

11. Show that if the displacement of a particle in a straight line is expressed by the equation x = a cos nt + b sin nt, it describes a simple harmonic equation with amplitude $\sqrt{a^2+b^2}$ and period 2π / n.

Solution: Since x (t) = a cos nt + b sin nt we have

x'(t) = - an sin nt + bn cos nt and

x '' (t) = - n^2x

Comparing this equation with $\frac{d^2x}{dt^2} = -\mu^2 x$, we see that the motion is simple harmonic with period 2π / n.

Further, amplitude is that value of x where x'(t) = 0.

This gives tan nt = b/a.

Hence , sin nt = $\frac{b}{\sqrt{a^2+b^2}}$ and cos nt = $\frac{a}{\sqrt{a^2+b^2}}$

Thus, the amplitude x (t) = $\sqrt{a^2+b^2}$

12. A particle is executing simple harmonic motion with amplitude 20 cm and time 4 seconds. Find the time required by the particle in passing between points which are at distances 15 cm and 5 cm from the centre of force and are on the same side of it.

Solution: Given a =20, T = 4 seconds.

As T = $\frac{2\pi}{\mu}$, we get $\mu = \pi/2$.

Let t_1 and t_2 be the time in seconds when the particle is at distance 15 cm and 5 cm respectively from the centre of force.

From x = $a \cos \mu t$, we can write $t = \frac{1}{\mu} \cos^{-1}\frac{x}{a}$

Therefore the required time is $t_2 - t_1$ = $\frac{2}{\pi}\left[\cos^{-1}\frac{1}{4} - \cos^{-1}\frac{3}{4}\right]$ = 0. 38 seconds.

13. Show that the frequency of forced vibrations in a closed electrical circuit with inductance L and capacitance C in series is $\frac{30}{\pi\sqrt{LC}}$ per minute.

Solution: This case is comparable to the equation $\frac{d^2q}{dt^2} + \mu^2 q = 0$ and hence it represents free electrical oscillations of current leaving period

$$T = \frac{2\pi}{\mu} = 2\pi\sqrt{LC}$$

and frequency = 1/ T per second

= 60 /T per minute = $\frac{30}{\pi\sqrt{LC}}$ per minute.

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