Contents

Pre-requisite :

1. **Differential Equation:**

An equation that represents the relation between the independent variable dependent variable and the derivative of the dependent variable w.r.t the independent variables is called as a *Differential Equation*.

Examples:

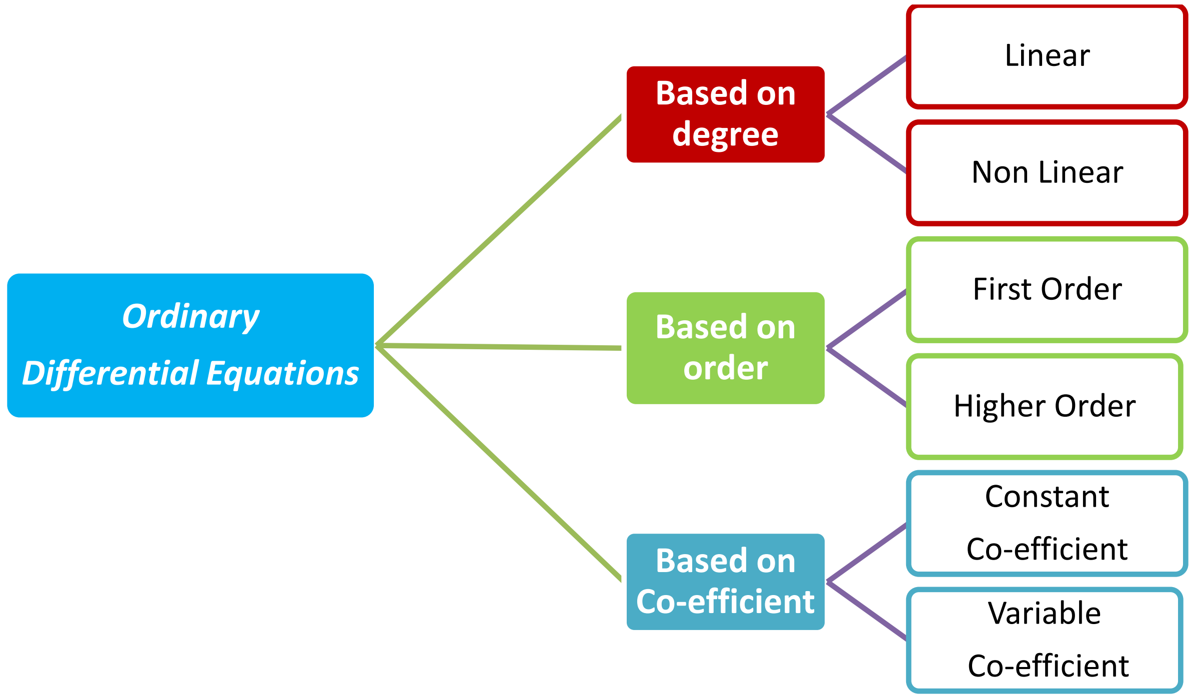
1. $e^{2x}dx + e^{2y}dy=0$
2. $\frac{dy}{dx} = y$
3. $\frac{d^2y}{dx^2} + 4y = 0$

2. Ordinary Differential Equation : A differential equation in which all the differential coefficients or differentials have reference to a *single independent variable* is called an ordinary differential equation.

3. Partial Differential Equation : A differential equation in which there are *two or more independent variables* and the partial differential coefficients are with respect to any one or more of them is called a partial differential equations.

4. Order : The order of a differential equation is the order of the highest derivative in it.

5. Degree : The degree of a differential equation is the exponent of the highest derivative.(*when the derivatives are cleared off radicals and fractions*).



6. **Linear Differential Equation in y :**

A differential equation with dependent variable y is said to be linear in y if

i) y and all its derivatives are of degree one.

ii) No product terms of y and/or any of it derivatives are present

iii) No transcendental functions of y and/or its derivatives occur.

7. **Standard form of a linear differential Equation and its solution :**

Any differential equation of the form $\frac{dy}{dx} + Py = Q$

where ***P and Q are functions of x only*** is called a **linear differential equation in *y***.

Integrating Factor = $e^{\int P dx}$

Solution : $y(IF) = \int Q.(IF)dx + c$

8. Standard form of a linear differential Equation and its solution :

Any differential equation of the form $\frac{dx}{dy} + Px = Q$

where ***P and Q are functions of y only*** is called a **linear differential equation in *x***.

Integrating Factor = $e^{\int Pdy}$

Solution : $x(IF) = \int Q.(IF)dy + c$

Problem:

1. **Solve** $\frac{dy}{dx} + y\cot x = \cos x.$

Answer:

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Where P = cot x and Q = cos x

$$\mathrm{IF} = e^{\int P dx} = e^{\int \cot x\, dx} = \sin x.$$

Hence, the general solution is $y e^{\int P dx} = \int Q.e^{\int P dx} dx + c$

$y \sin x = \int \cos x \sin x \, dx + c$

$y = \frac{1}{2}\sin x + c \operatorname{cosec} x$ is the required solution

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1. Bernoulli's Differential Equation:

Any differential equation of the form $\frac{dy}{dx} + Py = Qy^n$
where P and Q are functions of x only is called as Bernoulli's Differential equation in y.

To reduce this to the linear equation in standard form,

Step 1 : Divide throughout by y^n to obtain $\frac{1}{y^n}.\frac{dy}{dx} + Py^{1-n} = Q$

Step 2 : Take the substitution $y^{1-n} = z$, then $(1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$

and $\frac{dz}{dx} + P't = Q'$ which is linear in z

Integrating Factor = $e^{\int P'dx}$

Solution : $z(IF) = \int Q'.(IF)dx + c$

Any differential equation of the form $\frac{dx}{dy} + Px = Qx^n$
where P and Q are functions of y only is called as Bernoulli's Differential equation in x.

To reduce this to the linear equation in standard form,

Step 1 : Divide throughout by x^n to obtain $\frac{1}{x^n} . \frac{dx}{dy} + Px^{1-n} = Q$

Step 2 : Take the substitution $x^{1-n} = z$, then $(1-n)x^{-n}\frac{dx}{dy} = \frac{dz}{dy}$

and $\frac{dz}{dy} + P'z = Q'$ which is linear in z

Integrating Factor = $e^{\int P'dy}$

Solution : $z(IF) = \int Q'.(IF)dy + c$

………

Problems:

1. **Solve** $\frac{dy}{dx} + \frac{y}{x} = y^2x$

Answer : The given equation is of the form,

$$\frac{dy}{dx} + Py = Qy^n$$

where $P = \frac{1}{x}$, $Q = x$ and $n = 2$.

Dividing throughout by y^2,

$$\frac{1}{y^2}\frac{dy}{dx} + \left(\frac{1}{xy}\right) = x$$

Taking the substitution, $y^{-1} = z$, we obtain $\frac{dz}{dx} = \frac{-1}{y^2} . \frac{dy}{dx}$

Therefore - $\frac{dz}{dx} + \frac{1}{x}.z = x$ or $\frac{dz}{dx} - \frac{z}{x} = -x$

This is a linear differential equation of the form $\frac{dz}{dx} + Pz = Q$

Where P = $-\frac{1}{x}$ and Q = -x

$$IF = e^{-\int Pdx} = e^{-\int\frac{1}{x}dx} = \frac{1}{x}$$

$te^{\int Pdx} = \int Q.e^{\int Pdx}dx + c$

$z\frac{1}{x} = \int -x\,\frac{1}{x}dx + c$

$z\frac{1}{x} = -x + c$

$$\frac{1}{xy} = -x + c \text{ is the required solution.}$$

....

2. **Solve :** $\frac{\mathbf{dy}}{\mathbf{dx}} + \frac{\mathbf{2y}}{\mathbf{x}} = \mathbf{y^2x^2}$

Answer : The given equation is of the form,

$$\frac{dy}{dx} + Py = Qy^n$$

where $P = \frac{2}{x}$, $Q = x^2$ and $n = 2$.

Dividing throughout by y^2,

$$\frac{1}{y^2}\frac{dy}{dx} + \left(\frac{1}{xy}\right) = x^2$$

Taking the substitution, $y^{-1} = z$, we obtain $\frac{dz}{dx} = \frac{-1}{y^2}.\frac{dy}{dx}$

Therefore - $\frac{dz}{dx} + \frac{2}{x}.z = x^2$ or $\frac{dt}{dx} - \frac{2z}{x} = -x^2$

This is a linear differential equation of the form $\frac{dz}{dx} + Pz = Q$

Where P = $-\frac{2}{x}$ and Q = $-x^2$

$$IF = e^{-\int Pdx} = e^{-\int \frac{2}{x}dx} = \frac{1}{x^2}$$

$$ze^{\int Pdx} = \int Q.e^{\int Pdx}dx + c$$

$$z\frac{1}{x^2} = \int -x^2\frac{1}{x^2}dx + c$$

$$z\frac{1}{x^2} = -x + c$$

$\frac{1}{y} = -x^3 + cx^2$ is the required solution.

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3. Solve : $\boldsymbol{\frac{dr}{d\theta} = r\tan\theta - \frac{r^2}{\cos\theta}}$

Solution : The given equation can be written as

$$\frac{dr}{d\theta} - r\tan\theta = -\frac{r^2}{\cos\theta}$$

This is the Bernoulli's equation linear in r

where $P = -\tan\theta$, $Q = -1/\cos\theta$ and $n = 2$.

Dividing throughout by r^2,

$$\frac{1}{r^2}\frac{dr}{d\theta} - r^{-1}\tan\theta = -\frac{1}{\cos\theta} = -\sec\theta$$

Taking the substitution, $r^{-1} = t$, we obtain $\frac{dt}{d\theta} = \frac{-1}{r^2}.\frac{dr}{d\theta}$

Therefore $-\frac{dt}{d\theta} - t.\tan\theta = -\sec\theta$ or $\frac{dt}{d\theta} + t.\tan\theta = \sec\theta$

$$IF = e^{\int Pd\theta} = e^{\int \tan\theta d\theta} = e^{\log(\sec\theta)} = \sec\theta$$

Solution :

$$te^{\int Pd\theta} = \int Q.e^{\int Pd\theta}d\theta + c$$

$$t\sec\theta = \int \sec^2\theta d\theta + c$$

$$t\sec\theta = \tan\theta + c$$

$$\frac{\sec\theta}{r} = \tan\theta + c$$

....

4. Solve : $\boldsymbol{y^4dx = \left(x^{-\frac{3}{4}} - y^3x\right)dy}$

Answer : The given equation can be written as

$$\frac{dx}{dy} + \frac{x}{y} = \frac{x^{-3/4}}{y^4}\text{............(1)}$$

This equation is of the form, $\frac{dx}{dy} + Px = Qx^n$

Dividing throughout by $x^{-3/4}$,

$$x^{3/4}\frac{dx}{dy} + x^{7/4}\left(\frac{1}{y}\right) = \frac{1}{y^4}$$

Taking the substitution, $x^{7/4} = v$,

Then $\frac{7}{4}x^{3/4}\frac{dx}{dy} = \frac{dv}{dy}$

Substituting in Eq(1),

$$\frac{4}{7}\frac{dv}{dx} + \left(\frac{1}{y}\right)v = \frac{1}{y^4}$$

$$\frac{dv}{dy} + \left(\frac{7}{4y}\right)v = \frac{7}{4y^4}\text{.........................(2)}$$

$$\textit{This equation is linear in } v.$$

$$P = \frac{7}{4y}, \qquad Q = \frac{7}{4y^4}$$

$$Integrating\ Factor = e^{\int\frac{7}{4y}dy} = e^{\frac{7}{4}\log y} = y^{\frac{7}{4}}$$

Solution of Eq(2) is

$$y^{\frac{7}{4}}v = \int y^{\frac{7}{4}}.\frac{7}{4y^4}dy + c$$

$$y^{\frac{7}{4}}v = -\frac{7}{5}y^{\frac{-5}{4}} + c$$

Hence

$$x^{\frac{7}{4}}y^{3} = -\frac{7}{5} + cy^{\frac{5}{4}}$$

… …

Exact Differential Equation

Definition : A differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0$$

is said to be exact if its left hand member is the exact differential of some function $u(x,y)$.

That is, $\mathrm{du} = M(x,y)dx + N(x,y)dy = 0$

Therefore, its solution is $u(x,y) = c$

Example : Consider, the differential equation,

$ydx + xdy = 0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .(1)$

Note that

$$\mathrm{d(xy)} = ydx + xdy = 0$$

Therefore, the solution of eq. 1 is $xy = c$

Theorem : The necessary and sufficient condition for the differential equation $M(x,y)dx + N(x,y)dy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Note : The solution of an exact differential equation is given by,

$$\int_{y\ constant} Mdx + \int N(y)dy = c$$

where $N(y)$ = *terms of N which contain y alone*.

………

Problems:

1. Test the differential equation for exactness & solve $(x^2-4xy-2y^2)dx+(y^2-4xy-2x^2)dy=0$

Answer : The given equation is of the form $Mdx+Ndy=0$, where

$M=x^2-4xy-2y^2$ and $N=y^2-4xy-2x^2$.

Then, $\frac{\partial M}{\partial y}=-4x-4y$; $\frac{\partial N}{\partial x}=-4x-4y$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ the given equation is exact.

Solution : $\int Mdx+\int N(y)dy=C$

$$\int x^2-4xy-2y^2dx+\int y^2dy=C$$

$$\frac{x^3}{3}-\frac{4x^2y}{2}-2xy^2+\frac{y^3}{3}=\mathrm{c}$$

……

PROBLEMS ON EQUATIONS REDUCIBLE TO EXACT:

Sometimes a differential equation which is not exact may become so, on multiplication by a suitable function known as the integrating factor (IF).

The integrating factor can be obtained as follows :

Case 1. *If* $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=g(x)$ *(function of x alone) then, IF =* $e^{\int g(x)dx}$.

Case 2. *If* $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=h(y)$ *(function of y alone) then, IF =* $e^{\int h(y)dy}$.

Case 3. *In the given differential equation* $Mdx+Ndy=0$ *, if* $M(x,y)$ *and* $N(x,y)$ *is homogeneous of the same degree then,*

IF = $\frac{1}{Mx+Ny}$, *provided that* $Mx+Ny\neq 0$.

Note : If $Mx+Ny=0$ *then IF =* $\frac{1}{x^2}$ *or* $\frac{1}{y^2}$ *or* $\frac{1}{xy}$.

Case 4. *If the differential equation is of the form*

$f_1(xy)ydx+f_2(xy)xdy=0$, *then IF =* $\frac{1}{Mx-Ny}$,

where $M=f_1(xy)y$ *&* $N=f_2(xy)x$, *provided that* $Mx-Ny\neq 0$.

Note : If $Mx - Ny = 0$ then $\frac{M}{N} = \frac{y}{x}$ and the given differential equation reduces to $xdy + ydx = 0$ and its solution is $xy = c$.

....

Problems:

1. $[x^2y - 2xy^2]dx - [x^3 - 3x^2y]dy = 0$

Solution : The given equation is of the form,

$$Mdx + Ndy = 0$$

$M = x^2y - 2xy^2;\ N = -x^3 + 3x^2y$

$$\frac{\partial M}{\partial y} = x^2 - 4xy; \frac{\partial N}{\partial x} = -3x^2 + 6xy$$

The given equation is not exact.

Consider, $IF = \frac{1}{Mx+Ny} = \frac{1}{x^3y-2x^2y^2-x^3y+3x^2y^2} = \frac{1}{x^2y^2}$

Multiplying the given equation by the IF we have,

$M = \frac{1}{y} - \frac{2}{x}\ , N = -\frac{x}{y^2} + \frac{3}{y}$

$$\int Mdx + \int N(y)dy = C$$

$$\int \frac{1}{y} - \frac{2}{x}dx + \int \frac{3}{y}dy = C$$

$$\frac{x}{y} - 2logx + 3logy = c$$

$$\frac{x}{y} + log\left(\frac{y^3}{x^2}\right) = c$$

....

2. $\left[\boldsymbol{xysinxy + cosxy}\right]\boldsymbol{ydx} + \left[\boldsymbol{xysinxy - cosxy}\right]\mathbf{dy} = \mathbf{0}$

Solution : The given equation is of the form,

$$yf(xy)dx + xg(xy)dy = 0$$

$M = xy^2sinxy + ycosxy, N = x^2ysinxy - xcosxy$

Consider, $IF = \frac{1}{Mx-Ny} = \frac{1}{x^2y^2sinxy+xycosxy+xycosxy-x^2y^2sinxy} = \frac{1}{2xycosxy}$

Multiplying the given equation by the IF we have,

$M = \frac{ytanxy}{2} + \frac{1}{2x}$, $N = \frac{xtanxy}{2} - \frac{1}{2y}$

$$\int Mdx + \int N(y)dy = C$$

$$\int \frac{1}{y} - \frac{2}{x}dx + 1\int \frac{xtanxy}{2} - \frac{1}{2y}\,dy = C$$

$$\frac{1}{2}\frac{ylog(secxy)}{y} + logx - \frac{1}{2}logy = c$$

$$\log(secxy) + logx - logy = 2logc$$

$$log\frac{xsecxy}{y} = 2logc$$

$$\frac{xsecxy}{y} = k$$

....

3. Solve $\left(\boldsymbol{xy^2 + e^{1/x^3}}\right)\boldsymbol{dx} - \boldsymbol{x^2y}\ \mathbf{dy} = \mathbf{0}$

Solution :

$M = xy^2 + e^{1/x^3}, N = -\,x^2y$

Consider, $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = \frac{2xy-(-2xy)}{x^2y} = \frac{-4}{x}$ (function of x only)

Then, $IF = e^{\int -\frac{y}{x}dx} = e^{-4logx} = x^{-4}$

Multiplying the given equation by the IF we have,

$M = \frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^{-4}}$, $N = \frac{y}{x^2}$

$$\int M dx + \int N(y) dy = C$$

$$\int \left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^{-4}}\right) dx + \int 0\, dy = C$$

$$\frac{-y^2x^{-2}}{2} + \frac{1}{3}\int e^{x^{-3}}(-3x^{-4})\, dx = c$$

$$\frac{e^{x^{-3}}}{3} - \frac{y^2}{2x^2} = c$$

$$\frac{1}{3e^{x^3}} - \frac{y^2}{2x^2} = c$$

....

4. Solve $(\mathbf{1 + (x + y)tany})\frac{\mathbf{dx}}{\mathbf{dy}} + \mathbf{1 = 0}$

Solution :

$M = 1, N = 1 + (x + y)tany$

Consider, $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{0 - tany}{1} = -tany = g(y)$ (function of y only)

Then, $IF = e^{-\int g(y)dy} = e^{\int tanydy} = e^{\log(secy)} = secy$

Multiplying the given equation by the IF we have,

$M = secy$, $N = secy + (x + y)tanysecy$

$$\int M dx + \int N(y) dy = C$$

$$\int secydx + \int secy + ytanysecydy = C$$

$$xsecy + \log(secy + tany) + ysecy - \log\,(secytany) = c$$

$$xsecy + ysecy = c$$

....

5. $(\mathbf{2xlogx - xy})\mathbf{dy + 2ydx = 0}$

Solution :

$M = 2y, N = 2xlogx - xy$

Consider, $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2-(2logx+2-y)}{2xlogx-xy} = \frac{-(2logx-y)}{x(2logx-y)} = \frac{-1}{x}$ = f(x) (function of x only)

Then, $IF = e^{\int \frac{-1}{x}dx} = e^{-logx} = \frac{1}{x}$

Multiplying the given equation by the IF we have,

$M = \frac{2y}{x}, N = 2logx - y$

$$\int Mdx + \int N(y)dy = C$$

$$\int \frac{2y}{x}dx - \int ydy = c$$

$$2ylogx - \frac{y^2}{2} = c$$

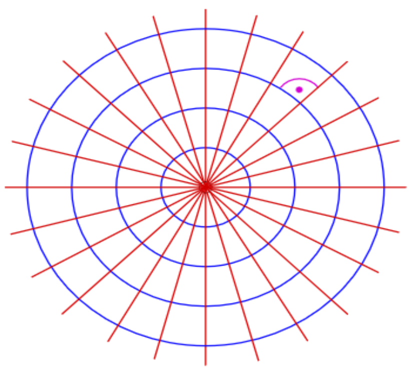
....

Orthogonal Trajectories:

Definition : *Two families of curves are said to be orthogonal trajectories of each other if every member of one family cuts every other member of the other family orthogonally.*

Example :

1. Family of *straight lines* *y=mx* & the family of *circles* $x^2 + y^2 = a^2$ are orthogonal trajectories of each other.



Working Procedure:

For finding the Orthogonal Trajectory of Cartesian family of Curves:

Step 1 : *Form the differential equation for the given family of curves* $F(x, y, c) = 0$ *in the form* $f(x, y, dy/dx) = 0$.

Step 2 : *Replace* $\frac{dy}{dx}$ *by* $-\frac{dx}{dy}$ to obtain the *differential equation of the required orthogonal family of curves*.

Step 3 : *Solving this differential equation, the orthogonal family of curves can be obtained.*

Working Procedure:

For finding the Orthogonal Trajectory of Polar family of Curves:

Step 1 : *Form the differential equation for the given family of curves* $F(r, \theta, c) = 0$ *in the form* $f(r, \theta, dr/d\theta) = 0$.

Step 2 : *Replace* $\frac{dr}{d\theta}$ *by* $-r^2\frac{d\theta}{dr}$ *to obtain the differential equation of the required orthogonal family of curves.*

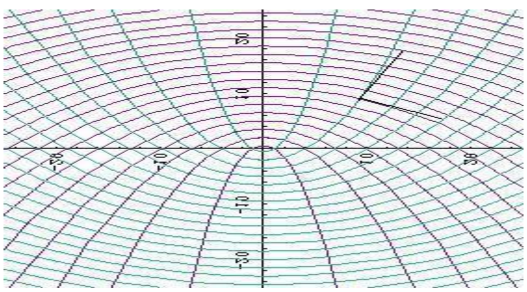
Step 3 : *Solving this differential equation, the orthogonal family of curves can be obtained.*

Self Orthogonality:

Definition : *A given family of curves is said to be self- Orthogonal if its family of Orthogonal Trajectories is the same as the given family of curves.*

Example :

The family of curves $x^2 = 4c(y + c)$ *is self- orthogonal.*



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Problems:

1. **Find the Orthogonal Trajectories of the family of circles $x^2 + y^2 = c^2$**

Answer :

Consider $\boldsymbol{x^2 + y^2 = c^2}$
Differentiating with respect to x, we have

$$2x + 2y\frac{dy}{dx} = 0$$

Therefore, *DE of the given family* : $\boldsymbol{\frac{dy}{dx} = -\frac{x}{y}}$

Replace $\frac{dy}{dx}$ *by* $-\frac{dx}{dy}$ to obtain the *differential equation of the required orthogonal family of curves.*

Hence, DE of the Orthogonal family : $\frac{dy}{y} = \frac{dx}{x}$

Solving this we obtain, $\boldsymbol{y = cx}$ ***which is the required solution.***

………

2. Find the Orthogonal Trajectories of the curves $\boldsymbol{r^2 = a^2 cos2\theta}$

Answer :

Consider

$$\boldsymbol{r^2 = a^2 cos2\theta}$$

Differentiating with respect to θ, *we have*

$$2r\frac{dr}{d\theta} = -a^2(2sin2\theta)$$

Therefore, DE of the given family : $\frac{1}{r}\frac{dr}{d\theta} = -tan2\theta$

Replace $\frac{dr}{d\theta}$ *by* $-r^2\frac{d\theta}{dr}$ *to obtain the differential equation of the required orthogonal family of curves*

Hence, DE of the Orthogonal family : $\frac{dr}{r} - cot\theta d\theta = 0$

Solving this we obtain, $r^2cosec2\theta = c^2$ *or* $r^2 = c^2sin2\theta$ *which is the required solution.*

……

3. **Prove that the system of confocal and coaxial parabolas** $\boldsymbol{y^2 = 4a(x + a)}$ **is self orthogonal. (*a* is a parameter)**

Answer :

Consider $y^2 = 4a(x + a)$

Differentiating with respect to x, we have

DE of the given family : $y = 2x\frac{dy}{dx} + y\left(\frac{dy}{dx}\right)^2$ ……………(1)

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ we obtain $y = -2x\frac{dx}{dy} + y\left(\frac{dx}{dy}\right)^2$

Dividing throughout by $\left(\frac{dx}{dy}\right)^2$ we have,

$y = 2x\frac{dy}{dx} + y\left(\frac{dy}{dx}\right)^2$ ……………(2) which is the DE of the Orthogonal family .

Since equations (1) & (2) are the same, the given family of parabolas are self Orthogonal.

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First order Non-linear differential equations

A differential equation of first order and higher degree is of the form

$$\boldsymbol{f(x, y, y') = 0} \text{ or } \boldsymbol{f(x, y, p) = 0} \text{…………(1)}$$

where $p = y' = \frac{dy}{dx}$.

Eq (1) is a non linear first order Differential Equation.

Example : $\boldsymbol{p^4 - (x + 2y)p^3 + (x + y + 2xy)p^2 - 2xyp = 0}$

is a first order, 4^{th} degree, non linear DE.

In general, a first order non linear DE of nth degree is of the form,

$p^n + a_1p^{n-1} + a_2p^{n-2} + \cdots \dots \dots \dots \dots \dots \dots \dots + a_{n-1}p + a_n = 0$

where the coefficients $a_1, a_2, \dots \dots \dots \dots \dots \dots \dots \dots \dots, a_n$ *are functions x and y.*

Solutions to such non linear equations can be obtained by reducing to differential equations of first order and first degree by,

i. Solving for p ii. Solving for y iii. Solving for x

Working procedure for Equations solvable for *p*:

Step 1. Given an nth degree non linear DE, express it as a nth degree polynomial in p.

Step 2. Resolve the polynomial into n linear real factors in the form $(\mathbf{p}-\mathbf{b_1})(\mathbf{p}-\mathbf{b_2})\ldots\ldots\ldots\ (\mathbf{p}-\mathbf{b_n})=0$,

where $\mathrm{b_1}, \mathrm{b_2}, \ldots\ldots\ldots . \mathrm{b_n}$ are functions of x and y

Step 3. Equate the n factors in the LHS to zero which reduces to n differential equations of first order and first degree given by, $\frac{\mathbf{dy}}{\mathbf{dx}}=\mathbf{b_1}(\mathbf{x},\mathbf{y})$, $\frac{\mathbf{dy}}{\mathbf{dx}}=\mathbf{b_2}(\mathbf{x},\mathbf{y})$, ... $\frac{\mathbf{dy}}{\mathbf{dx}}=\mathbf{b_n}(\mathbf{x},\mathbf{y})$

Step 4. Solve the n differential equations to obtain the solutions of the form, $\mathbf{f_1}(\mathbf{x},\mathbf{y},\mathbf{c})=\mathbf{0}$, $\mathbf{f_2}(\mathbf{x},\mathbf{y},\mathbf{c})=\mathbf{0}$.......... .... $\mathbf{f_n}(\mathbf{x},\mathbf{y},\mathbf{c})=\mathbf{0}$

Step 5. The general solution is then given by,

$\mathbf{f_1}(\mathbf{x},\mathbf{y},\mathbf{c}).\mathbf{f_2}(\mathbf{x},\mathbf{y},\mathbf{c})\ldots\ldots\ldots\ldots\ \mathbf{f_n}(\mathbf{x},\mathbf{y},\mathbf{c})=\mathbf{0}$

Note: The general or complete solution of a differential equation is that in which the number of arbitrary constants is equal to the order of the differential equation.

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Problems:

1. **Solve :** $\boldsymbol{p^2+2xp-3x^2=0}$

Answer : Solving the equation for *p* we have,

$$p=\frac{-2x\pm\sqrt{4x^2+12x^2}}{2}=-x\pm 2x=x,\ -3x$$

$p=x$; $p=-3x$

$\frac{dy}{dx}=x$; $\frac{dy}{dx}=-3x$

$dy-xdx=0$; $dy+3xdx=0$

$y-\frac{x^2}{2}=c$; $y+\frac{3x^2}{2}=c$

$$y-\frac{x^2}{2}-c=0\quad ;\quad y+\frac{3x^2}{2}-c=0$$

General Solution : $\left(y-\frac{x^2}{2}-c\right)\left(y+\frac{3x^2}{2}-c\right)=0$

………

2. Solve : $p^3+2xp^2-y^2p^2-2xy^2p=0$

Answer : Solving the equation for *p* we have,

$$p^2(p+2x)-y^2p(p+2x)=0$$
$$(p^2-y^2p)(p+2x)=0$$
$$p(p-y^2)(p+2x)=0$$

$$p=0 \quad ; \quad p-y^2=0 \quad ; \quad p+2x=0$$
$$\frac{dy}{dx}=0 \quad ; \quad \frac{dy}{dx}-y^2=0 \quad ; \quad \frac{dy}{dx}+2x=0$$
$$dy=0 \quad ; \quad \frac{dy}{y^2}-dx=0 \quad ; \quad dy+2xdx=0$$
$$y=c \quad ; \quad -\frac{1}{y}-x=c \quad ; \quad y+x^2=c$$
$$y-c=0 \quad ; \quad (x+c)y+1=0 \quad ; \quad y+x^2-c=0$$

General Solution : $(y-c)[(x+c)y+1](y+x^2-c)=0$

………

<u>Working procedure for Equations solvable for y:</u>

Step 1. Rewrite the given differential equation $\mathrm{f(x,y,p)}=0$ in the form

$\mathrm{y}=\mathrm{F(x,p)}$…………(1)

Step 2. Differentiate (1) w.r.t ‘x’ to obtain the equation of the form,

$\mathrm{p}=\emptyset\left(\mathrm{x},\mathrm{p},\frac{\mathrm{dp}}{\mathrm{dx}}\right)$………….(2) which is a first order and first degree differential equation in the variable p.

Step 3. Solve the differential equation (2) . The solution is of the form

$\mathrm{G(x,p,c)}=0$……………(3)

Step 4. Eliminating p from equations (1) and (3), the required solution of the DE (1).

NOTE :

1. Whenever it is not possible to eliminate p from equations (1) & (3), the solution of the DE (1) is given by the parametric equations $\mathrm{x} = \mathrm{x}(\mathrm{p}, \mathrm{c})$

   $\& \, \mathrm{y} = \mathrm{y}(\mathrm{p}, \mathrm{c})$.

2. When the factor which does not contain dp/dx is equated to zero and solved, we obtain another solution called the singular solution of the given differential equation. Observe that the singular solution does not contain any arbitrary constant.

....

Problems:

1. **Find the general solution of $3x^4p^2 - xp - y = 0$**

Answer : *The given equation can be written as,*

$y = 3x^4p^2 - xp$*..........(1)*

Differentiating w.r.t 'x',

$$\frac{dy}{dx} = 12x^3p^2 + 6x^4p\frac{dp}{dx} - x\frac{dp}{dx} - p$$
$$p = 12x^3p^2 + 6x^4p\frac{dp}{dx} - x\frac{dp}{dx} - p$$
$$2p = 12x^3p^2 + 6x^4p\frac{dp}{dx} - x\frac{dp}{dx}$$
$$2p - 12x^3p^2 = 6x^4p\frac{dp}{dx} - x\frac{dp}{dx}$$
$$2p(1 - 6x^3p) = -x(1 - 6x^3p)\frac{dp}{dx}$$
$$\left(2p + x\frac{dp}{dx}\right)(1 - 6x^3p) = 0$$

$\frac{dp}{2p} = -\frac{dx}{x}$ *(By equating the first factor to zero)*

$$\frac{dp}{2p} + \frac{dx}{x} = 0$$
$$\frac{1}{2}logp + logx = logk$$

$px^2 = c$ *or* $p = \frac{c}{x^2}$

Substituting in (1)

$$3x^4\left(\frac{c}{x^2}\right)^2 - x\left(\frac{c}{x^2}\right) - y = 0$$

$$3c^2 - \frac{c}{x} - y = 0$$
$$3c^2x - c - xy = 0$$
$$c(3cx - 1) = xy$$

....

2. **Solve :** $\boldsymbol{y + px = p^2x^4}$

Answer : The given equation can be written as,

$y = x^4p^2 - xp$..................(1)

Differentiating w.r.t ‘x’,

$$\frac{dy}{dx} = 4x^3p^2 + 2p\frac{dp}{dx} - x\frac{dp}{dx} - p$$
$$p = 4x^3p^2 + 2x^4p\frac{dp}{dx} - x\frac{dp}{dx} - p$$
$$2p = 4x^3p^2 + 2x^4p\frac{dp}{dx} - x\frac{dp}{dx}$$
$$2p - 4x^3p^2 = 2px^4\frac{dp}{dx} - x\frac{dp}{dx}$$
$$2p(1 - 2x^3p) = -x(1 - 2x^3p)\frac{dp}{dx}$$
$$\left(2p + x\frac{dp}{dx}\right)(1 - 2x^3p) = 0$$

$\frac{dp}{2p} = -\frac{dx}{x}$ (Equating the first factor to zero)

$$\frac{dp}{2p} + \frac{dx}{x} = 0$$
$$\frac{1}{2}logp + logx = logk$$

$px^2 = c$ or $p = \frac{c}{x^2}$

Substituting in (1)

$$y = x^4\left(\frac{c}{x^2}\right)-^2 x\left(\frac{c}{x^2}\right)$$

$$y + \frac{c}{x} = c^2$$
$$xy + c = c^2x$$

....

Working procedure for Equations solvable for *x*:

Step 1. Rewrite the given differential equation $f(x, y, p) = 0$ in the form
x= $F(y, p)$............(1)
Step 2. Differentiate (1) w.r.t 'y' to obtain the equation of the form,
$\frac{1}{p} = \emptyset\left(y, p, \frac{dp}{dy}\right)$.............(2) which is a first order and first degree differential equation in the variable p.
Step 3. Solve the differential equation (2) . The solution is of the form
$G(y, p, c) = 0$...............(3)
Step 4. Eliminating p from equations (1) and (3), the required solution of the DE (1).
NOTE :
Whenever it is not possible to eliminate p from equations (1) & (3), the solution of the DE (1) is given by the parametric equations $x = x(p, c)$ & $y = y(p, c)$

Problems:

1. Solve $yp^2 - 2xp + y = 0$.........................(1)

Answer : Solving the given equation for *'x'*

$$x = \frac{yp^2 + y}{2p} = \frac{yp}{2} + \frac{y}{2p}$$

Differentiating w.r.t *'y'*,

$$\frac{dx}{dy} = \frac{1}{2}\left\{p + y\frac{dp}{dy} + \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy}\right\}$$

$$2\frac{1}{p} = p + \frac{dp}{dy} + \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy}$$

$$\frac{1}{p} - p = \left(1 - \frac{y}{p^2}\right)\frac{dp}{dy}$$

$$\frac{1}{p} - p = -\frac{y}{p}\left(\frac{1}{p} - p\right)\frac{dp}{dy}$$

$$\left(1+\frac{y}{p}\frac{dp}{dy}\right)\left(\frac{1}{p}-p\right)=0$$

Consider $\left(1+\frac{y}{p}\frac{dp}{dy}\right)=0$

$$\frac{dp}{p}+\frac{dy}{y}=0$$

$$logp+logy=logc$$

$$py=c$$

$$p=\frac{c}{y}$$

Substituting in (1)

$$\mathrm{y}\left(\frac{\mathrm{c}^2}{\mathrm{y}^2}\right)-2\mathrm{x}\left(\frac{\mathrm{c}}{\mathrm{y}}\right)+\mathrm{y}=0$$

$$\mathrm{c}^2-2\mathrm{cx}+\mathrm{y}^2=0$$

$$\mathrm{y}^2=2\mathrm{cx}-\mathrm{c}^2$$

………

2. Solve $y=2px+y^2p^3$……………………(1)

Answer : Solving the given equation for *'x'*

$$x=\frac{y(1-yp^3)}{2p}$$

Differentiating (1) w.r.t *'y'*,

$$1=2p.\frac{1}{p}+2x\frac{dp}{dy}+3y^2p^2\frac{dp}{dy}+2yp^3$$

$$-(1+2yp^3)=(2x+3y^2p^2)\frac{dp}{dy}$$

$$-(1+2yp^3)=\left(\frac{y(1-yp^3)}{p}+3y^2p^2\right)\frac{dp}{dy}$$

$$-(1+2yp^3) = \left(\frac{y - y^2p^3 + 3y^2p^3}{p}\right)\frac{dp}{dy}$$

$$-(1+2yp^3) = \frac{y(1+2yp^3)}{p}\frac{dp}{dy}$$

$$(1+2yp^3) + \frac{y(1+2yp^3)}{p}\frac{dp}{dy} = 0$$

$$(1+2yp^3)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$$

Consider

$$1 + \frac{y}{p}\frac{dp}{dy} = 0$$

$\frac{y}{p}\frac{dp}{dy} = -1$ *or* $\frac{dp}{p} + \frac{dy}{y} = 0$

$logp + logy = logc$ *or* $py = c$ *or* $p = c/y$

Substituting in (1)

$$y = 2\left(\frac{c}{y}\right)x + y^2\left(\frac{c}{y}\right)^3$$

$$y^2 = 2cx + c^3$$

………

Applications of differential equations –Newton's Law of Cooling

Newton's law of cooling:

According to this law, the temperature of a body changes at a rate which is proportional to the difference in temperature between that of the surrounding medium and that of the body itself.

If θ_0 is the temperature of the surroundings and θ that of the body at any time t, then $\frac{d\theta}{dt} = -k(\theta - \theta_0)$

where k is the constant of proportionality.

Note: the negative sign indicates the cooling of the body with the increase of the time.

Every geometrical or physical problem when translated into mathematical symbols gives rise to a differential equation.

The study of a differential equation consists of three phases:

- Formulation of differential equation from the given physical situation, called modelling.
- Solutions of this differential equation ,evaluating the arbitrary constants from the given conditions and
- Physical interpretation of the solution

Method of solving the problem of newton's law of cooling:

Let t_1 ˚c be the initial temperature of the body and
t_2 ˚c be the constant temperature of the medium. Further
T ˚c be the temperature of the body at any time t.
Then by Newton's law of cooling,
$\frac{dT}{dt} = -k(T - t_2)$ with the condition T(0)= t_1
$\int \frac{dT}{T-t_2} = \int -k\, dt + \alpha$
$\log(T - t_2) = -kt + \alpha$
Or $T - t_2 = e^{-kt+\alpha}$
$T - t_2 = c\, e^{-kt}$ where $c = e^{\alpha}$= constant
Applying the initial condition,
$T = t_1$ when t=0, we have
$t_1 - t_2 = ce^0$ Or $t_1 - t_2 = c$
Therefore,
$T - t_2 = (t_1 - t_2)e^{-kt}$

$$T = t_2 + (t_1 - t_2)e^{-kt}$$

Problems:

1. Water at temperature 10˚c takes 5 min to warm up to 20 ˚c in a room at temperature 40 ˚c

A) find the temperature after 20 min; after ½ hr

B) when will the temperature be 25 ˚c

Answer:

Let t_1 ˚c be the initial temperature of the water and

t_2 ˚c be the room temperature.

Further

T ˚c be the temperature of the water at any time t.

Then by Newton's law of cooling,

$$T = t_2 + (t_1 - t_2)e^{-kt} \text{........(1)}$$

Given:

$t_1 = 10\,, t_2 = 40\ \,, T = 20\ \ and\ t = 5\ min$

Substituting all these values in (1) we get $k = \frac{-1}{5}\ 0r\ 0.08109.$

a) Find T when t= 20 min

Substituting $t_1 = 10\,, t_2 = 40\ \,, k = 0.08109\ \ and\ t = 20\ min$ in (1), we have T= 34.073

Find T when t= 30 min

Substituting $t_1 = 10\,, t_2 = 40\ \,, k = 0.08109\ \ and\ t = 30\ min$ in (1), we have T= 37.36

b) Find t when T= 25

Substituting $t_1 = 10\,, t_2 = 40\ \,, k = 0.08109\ \ and\ T = 25$ in (1), we have t= 8.5478.

………

2. A copper ball is heated to a temperature of 100 ˚c. Then at time t=0 it is placed in water which is maintained at a temperature of 30 ˚c. At the end of 3 minutes temperature of the ball is reduced to 70 ˚c. Find the time at which the temperature of the ball drops to 31 ˚c.

Answer: Let t_1 ˚c be the initial temperature of the copper and t_2 ˚c be the temperature of the medium. Further let T ˚c be the temperature of the copper at any time t.

Then by Newton's law of cooling,

$$T = t_2 + (t_1 - t_2)e^{-kt} \text{ ....(1)}$$

Given:

$t_1 = 100\,, t_2 = 30\ \ , T = 70\ \ and\ t = 3\ min$

Substituting all these values in (1) we get $k = \ 0.1865$.

To find t when T= 31

Substituting $t_1 = 100\,, t_2 = 30\ \ , k = 0.1865\ \ and\ T = 31$ in (1), we have t=22.73 min.

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