NUMERICAL METHODS FOR PDEs

Deliverable 5: EE and mesh adaptivity

The Laplace equation is considered to be solved using the finite element method in a 2D square domain $\Omega = [0,1]^2$,

$$\begin{aligned} \Delta u &= -f \quad \text{for all} \quad (x,y) \in \Omega, \\ u &= u_D \quad \text{for all} \quad (x,y) \in \partial\Omega. \end{aligned} \tag{1}$$

for the source term f corresponds to the analytical solution $u(x,y) = g(x)g(y)$, where $g(z) = (4z(1-z))^{10}$. Starting from the already provided Matlab code (`Deliverable 2`) solving the problem (1) using FEM with P1 elements, we aim to guarantee the quality of the FE solution. Taking into account that, herein, the exact solution is know, first:

a) It is desired to obtain a mesh with a squared error density below $\tau = 10^{-2}$ (for the $\mathcal{H}^1$-seminorm). How many elements and nodes are necessary to attain such error by performing uniform refinement on all the mesh?

Available in `Atenea`, the code `main_exampleAdaptation`, provides an example to use the function `refineListElements` that, given a mesh and a set of elements to refine, generates a new mesh with those elements refined. Using the provided tool:

b) Propose and implement an algorithm that refines a mesh to ensure that the squared error density is below a given tolerance $\tau = 10^{-2}$. Write and describe your algorithm, and detail the adapted mesh (plot it and also give the number of nodes and elements). Analyze this result with respect to the one in *a)*.

Recover your codes from `Deliverable 2` to analyze the convergence for uniform refinement. For uniform and locally adapted meshes, an indicator for the mesh size is $h_{ind} = \frac{1}{N^{1/d}}$ [Arnold et al., 2000], being N the number of nodes of the mesh, and d the dimension.

c) Analyze the convergence of the $\mathcal{H}^1$-seminorm of the uniform and adaptive approaches. Show the convergence plot and analyze the behavior of both approaches.

d) Taking into account the results observed in `a)-c)` is in your opinion worth it increasing the complexity of a simulation process enabling adaptation or not? When stepping to 3D problems, do the possible advantages of adaptation become more or less significant?

In general, the exact solution of the PDE is not known. An adaptivity process based on the Zienkiewick & Zhu (ZZ) error estimator [Zienkiewicz and Zhu, 1987] is now considered.

e) Using as error indicator the ZZ-estimator, modify the previous adaptive procedure to solve the PDE imposing that the estimated squared error density is below 10^{-2}. Comment the modification performed in your previous adaptive framework. Depict and give the number of nodes and elements of the generated mesh. Analyze this result with respect to the one obtained adapting with respect to the exact error in `b)`.

f) Perform an analogous convergence analysis to the one in `c)`, but using the adaptive process developed in `e)`. Analyze the convergence of the exact $\mathcal{H}^1$-seminorm (taking into account the exact solution), and also of the estimator of the $\mathcal{H}^1$-seminorm. What advantages/disadvantages do you observe when using the adaptive procedure proposed in `e)` with respect to using the adaptive procedure proposed in `b)`?

Implementation, hints and comments

- Encapsulate your code to solve the Laplace equation in a function
  `[u]=solveLaplaceOnMesh(mesh)`
  that given a mesh `mesh` computes the corresponding FEM solution of the target problema

- In an analogous manner, encapsulate your code your adaptive process to attain a desired tolerance
  `[mesh,u]=solveLaplaceAndAdaptMesh(mesh,`τ`)`
  that given an mesh `mesh` iteratively solves the PDE, and refines the mesh to attain the desired tolerance τ.

- When performing adaptive mesh refinement, avoid demanding to overrefine the mesh from one step to the other. A good practice may be to, for instance, demand that given a mesh with a know squared elemental error density, obtain a new one such that no element exceeds the previous maximum divided by a factor (for instance, 2). Otherwise, in 2D excessively large meshes could be obtained from one step to another.

- Questions `a)-d)`, could be analogously performed in terms of the $\mathcal{L}^2$ or $\mathcal{H}^1$-norms, as already done in `Deliverable 2`. Herein, $\mathcal{H}^1$-seminorm is selected to be compared with the ZZ-estimator.

- If desired, the session can be tested with the source term and analytical solution from `Deliverable 2`, but lower the required tolerance to avoid having to deal with large meshes.

References

[Arnold et al., 2000] Arnold, D. N., Mukherjee, A., and Pouly, L. (2000). Locally adapted tetrahedral meshes using bisection. *SIAM Journal on Scientific Computing*, 22(2):431–448.

[Zienkiewicz and Zhu, 1987] Zienkiewicz, O. C. and Zhu, J. Z. (1987). A simple error estimator and adaptive procedure for practical engineerng analysis. *International journal for numerical methods in engineering*, 24(2):337–357.