



Machine Learning

# Regularization

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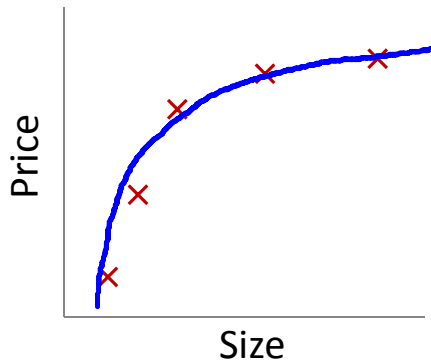
The problem of  
overfitting

## Example: Linear regression (housing prices)



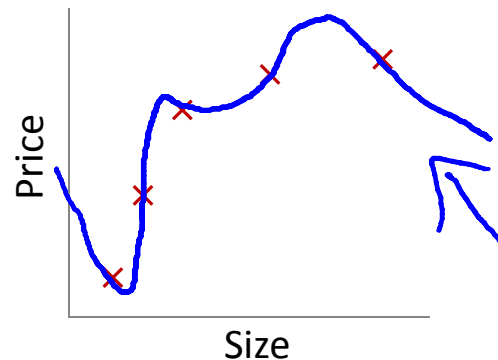
$$\rightarrow \theta_0 + \theta_1 x$$

"Underfit" "High bias"



$$\rightarrow \theta_0 + \theta_1 x + \theta_2 x^2$$

"Just right"



$$\rightarrow \theta_0 + \theta_1 x + \theta_2 x^2 + \theta_3 x^3 + \theta_4 x^4$$

"Overfit" "High variance"

**Overfitting:** If we have too many features, the learned hypothesis may fit the training set very well ( $J(\theta) = \frac{1}{2m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)})^2 \approx 0$ ), but fail to generalize to new examples (predict prices on new examples).

## Example: Logistic regression



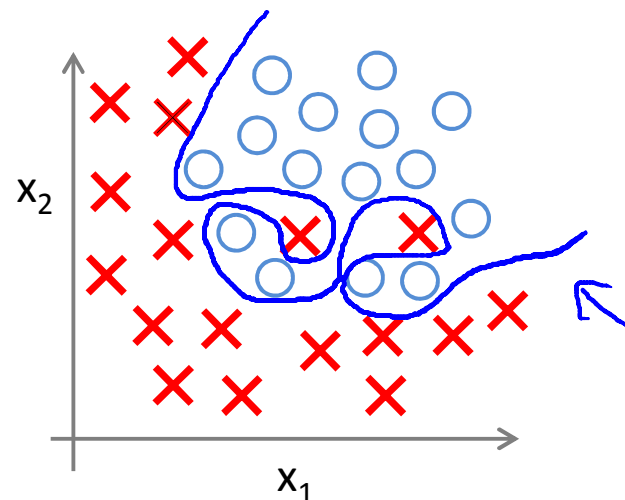
$\rightarrow h_{\theta}(x) = g(\theta_0 + \theta_1 x_1 + \theta_2 x_2)$   
 ( $g$  = sigmoid function)

↩  
 "Underfit"



$g(\theta_0 + \theta_1 x_1 + \theta_2 x_2$   
 $+ \theta_3 x_1^2 + \theta_4 x_2^2$   
 $+ \theta_5 \underline{x_1 x_2})$

↗



$g(\theta_0 + \theta_1 x_1 + \theta_2 x_1^2$   
 $+ \theta_3 \underline{x_1^2 x_2} + \theta_4 \underline{x_1^2 x_2^2}$   
 $+ \theta_5 \underline{x_1^2 x_2^3} + \theta_6 \underline{x_1^3 x_2} + \dots)$

↖  
 "Overfit"

## Addressing overfitting:

$x_1$  = size of house

$x_2$  = no. of bedrooms

$x_3$  = no. of floors

$x_4$  = age of house

$x_5$  = average income in neighborhood

$x_6$  = kitchen size

$\vdots$

$x_{100}$



# Addressing overfitting:

## Options:

1. Reduce number of features.
  - — Manually select which features to keep.
  - — Model selection algorithm (later in course).
2. Regularization.
  - — Keep all the features, but reduce magnitude/values of parameters  $\theta_j$ .
  - Works well when we have a lot of features, each of which contributes a bit to predicting  $y$ .





Machine Learning

# Regularization

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# Cost function

# Intuition



Size of house

$$\theta_0 + \theta_1 x + \theta_2 x^2$$



Size of house

$$\theta_0 + \theta_1 x + \theta_2 x^2 + \cancel{\theta_3 x^3} + \cancel{\theta_4 x^4}$$

↑
↑

Suppose we penalize and make  $\theta_3, \theta_4$  really small.

$$\rightarrow \min_{\theta} \frac{1}{2m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)})^2 + \underbrace{1000 \theta_3^2}_{\theta_3 \approx 0} + \underbrace{1000 \theta_4^2}_{\theta_4 \approx 0}$$



## Regularization.

Small values for parameters  $\theta_0, \theta_1, \dots, \theta_n$

- “Simpler” hypothesis
- Less prone to overfitting

$$\rightarrow \boxed{\theta_3, \theta_4} \approx 0$$

Housing:

- Features:  $x_1, x_2, \dots, x_{100}$
- Parameters:  $\theta_0, \theta_1, \theta_2, \dots, \theta_{100}$

$$J(\theta) = \frac{1}{2m} \left[ \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)})^2 + \lambda \sum_{j=1}^n \theta_j^2 \right]$$

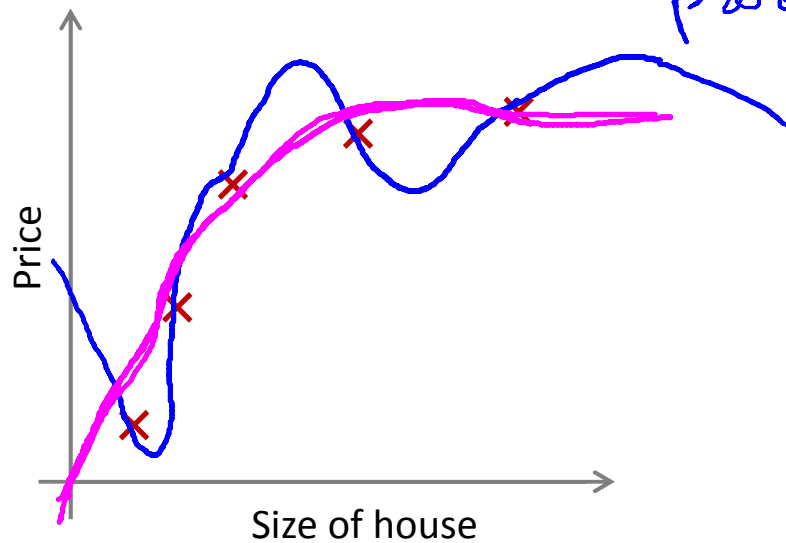
~~$\theta_0$~~   $\theta_1, \theta_2, \theta_3, \dots, \theta_{100}$   ~~$\theta_0$~~

## Regularization.

$$\rightarrow J(\theta) = \frac{1}{2m} \left[ \underbrace{\sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)})^2}_{\text{blue bracket}} + \underbrace{\lambda \sum_{j=1}^n \theta_j^2}_{\text{pink bracket}} \right]$$

$\min_{\theta} J(\theta)$

regularization parameter



In regularized linear regression, we choose  $\theta$  to minimize

$$J(\theta) = \frac{1}{2m} \left[ \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)})^2 + \lambda \sum_{j=1}^n \theta_j^2 \right]$$

What if  $\lambda$  is set to an extremely large value (perhaps far too large for our problem, say  $\lambda = 10^{10}$ )?

- Algorithm works fine; setting  $\lambda$  to be very large can't hurt it
- Algorithm fails to eliminate overfitting.
- Algorithm results in underfitting. (Fails to fit even training data well).
- Gradient descent will fail to converge.

In regularized linear regression, we choose  $\theta$  to minimize

$$J(\theta) = \frac{1}{2m} \left[ \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)})^2 + \lambda \sum_{j=1}^n \theta_j^2 \right]$$

What if  $\lambda$  is set to an extremely large value (perhaps far too large for our problem, say  $\lambda = 10^{10}$ )?



$h_{\theta}(x)$

$$\theta_0 + \cancel{\theta_1 x} + \cancel{\theta_2 x^2} + \cancel{\theta_3 x^3} + \cancel{\theta_4 x^4}$$

$\theta_1, \theta_2, \theta_3, \theta_4$

$\theta_1 \approx 0, \theta_2 \approx 0$

$\theta_3 \approx 0, \theta_4 \approx 0$

$$h_{\theta}(x) = \theta_0$$





Machine Learning

# Regularization

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Regularized linear  
regression

# Regularized linear regression

$$J(\theta) = \frac{1}{2m} \left[ \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)})^2 + \underbrace{\lambda \sum_{j=1}^n \theta_j^2} \right]$$

$$\min_{\theta} \underline{J(\theta)}$$

# Gradient descent

$$\theta_0$$

$$\theta_1, \theta_2, \dots, \theta_n$$

Repeat {

$$\frac{\partial}{\partial \theta_0} J(\theta)$$

$$\rightarrow \theta_0 := \theta_0 - \alpha \frac{1}{m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)}) x_0^{(i)}$$

$$\theta_j := \theta_j - \alpha$$

$$\frac{1}{m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)}) x_j^{(i)}$$

$$+ \frac{\lambda}{m} \theta_j$$

$$(j = \cancel{x}, 1, 2, 3, \dots, n)$$

}

$$\theta_j := \theta_j (1 - \alpha \frac{\lambda}{m}) - \alpha \frac{1}{m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)}) x_j^{(i)}$$

$$\rightarrow J(\theta)$$

$$\theta_j^2$$

$$1 - \alpha \frac{\lambda}{m} < 1$$

$$0.99$$

$$\theta_j \times 0.99$$



# Normal equation

$$\underline{X} = \begin{bmatrix} (x^{(1)})^T \\ \vdots \\ (x^{(m)})^T \end{bmatrix} \leftarrow$$

$m \times (n+1)$

$$\underset{\uparrow}{y} = \begin{bmatrix} y^{(1)} \\ \vdots \\ y^{(m)} \end{bmatrix} \quad \mathbb{R}^m$$

$$\rightarrow \min_{\theta} \underline{J(\theta)}$$

$$\Rightarrow \Theta = \left( X^T X + \lambda \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{(n+1) \times (n+1)} \right)^{-1} X^T y$$

$\frac{\partial}{\partial \theta_j} J(\theta) \stackrel{\text{set}}{=} 0$

$\in \mathbb{R}^n \quad n=2$

## Non-invertibility (optional/advanced).

Suppose  $m \leq n$ ,  $\leftarrow$   
(#examples) (#features)

$$\theta = \underbrace{(X^T X)^{-1}}_{\text{non-invertible / singular}} X^T y$$

pinv

inv  
 $\nearrow$

If  $\lambda > 0$ ,

$$\theta = \left( X^T X + \lambda \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{bmatrix} \right)^{-1} X^T y$$

invertible.





Machine Learning

# Regularization

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Regularized  
logistic regression

# Regularized logistic regression.



$$h_\theta(x) = g(\theta_0 + \theta_1 x_1 + \theta_2 x_1^2 + \theta_3 x_1^2 x_2 + \theta_4 x_1^2 x_2^2 + \theta_5 x_1^2 x_2^3 + \dots)$$

Handwritten blue arrows point to  $\theta_1$  and  $\theta_2$ . A handwritten pink arrow points to the  $\theta_5$  term.

Cost function:

$$\rightarrow J(\theta) = - \left[ \frac{1}{m} \sum_{i=1}^m y^{(i)} \log h_\theta(x^{(i)}) + (1 - y^{(i)}) \log(1 - h_\theta(x^{(i)})) \right] + \frac{\lambda}{2m} \sum_{j=1}^n \theta_j^2$$

The term  $\theta_1, \theta_2, \dots, \theta_n$  is enclosed in a pink box.

# Gradient descent

Repeat {

$$\rightarrow \theta_0 := \theta_0 - \alpha \frac{1}{m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)}) x_0^{(i)}$$

$$\rightarrow \theta_j := \theta_j - \alpha \left[ \underbrace{\frac{1}{m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)}) x_j^{(i)}}_{\substack{(j = \cancel{0}, 1, 2, 3, \dots, n) \\ \theta_1, \dots, \theta_n}} + \frac{\lambda}{n} \theta_j \right] \leftarrow$$

}

$$\frac{\partial}{\partial \theta_j} J(\theta)$$

$$\underline{h_{\theta}(x)} = \frac{1}{1 + e^{-\theta^T x}}$$

# Advanced optimization

→ `function [jVal, gradient] = costFunction(theta)`

`jVal = [code to compute  $J(\theta)$ ];`

$$\rightarrow J(\theta) = \left[ -\frac{1}{m} \sum_{i=1}^m y^{(i)} \log(h_{\theta}(x^{(i)})) + (1 - y^{(i)}) \log(1 - h_{\theta}(x^{(i)})) \right] + \left[ \frac{\lambda}{2m} \sum_{j=1}^n \theta_j^2 \right]$$

→ `gradient(1) = [code to compute  $\frac{\partial}{\partial \theta_0} J(\theta)$ ];`

$$\frac{1}{m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)}) x_0^{(i)} \leftarrow$$

→ `gradient(2) = [code to compute  $\frac{\partial}{\partial \theta_1} J(\theta)$ ];`

$$\left( \frac{1}{m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)}) x_1^{(i)} \right) - \frac{\lambda}{m} \theta_1 \leftarrow$$

→ `gradient(3) = [code to compute  $\frac{\partial}{\partial \theta_2} J(\theta)$ ];`

$$\vdots \left( \frac{1}{m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)}) x_2^{(i)} \right) - \frac{\lambda}{m} \theta_2$$

`gradient(n+1) = [code to compute  $\frac{\partial}{\partial \theta_n} J(\theta)$ ];`

minimize (cost function)  $\theta = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_n \end{bmatrix}$   $\theta_0 \leftarrow \text{theta}(1)$   
 $\theta_1 \leftarrow \text{theta}(2)$   
 $\theta_n \leftarrow \text{theta}(n+1)$

$J(\theta)$

