



Machine Learning

Neural Networks: Learning

Cost function

Neural Network (Classification)



$$\{(x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \dots, (x^{(m)}, y^{(m)})\}$$

$L =$ total no. of layers in network

$s_l =$ no. of units (not counting bias unit) in layer l

Binary classification

$y = 0$ or 1

1 output unit

Multi-class classification (K classes)

$y \in \mathbb{R}^K$ E.g. $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$
pedestrian car motorcycle truck

K output units

Cost function

Logistic regression:

$$J(\theta) = -\frac{1}{m} \left[\sum_{i=1}^m y^{(i)} \log h_{\theta}(x^{(i)}) + (1 - y^{(i)}) \log(1 - h_{\theta}(x^{(i)})) \right] + \frac{\lambda}{2m} \sum_{j=1}^n \theta_j^2$$

Neural network:

$$h_{\Theta}(x) \in \mathbb{R}^K \quad (h_{\Theta}(x))_i = i^{th} \text{ output}$$

$$J(\Theta) = -\frac{1}{m} \left[\sum_{i=1}^m \sum_{k=1}^K y_k^{(i)} \log(h_{\Theta}(x^{(i)}))_k + (1 - y_k^{(i)}) \log(1 - (h_{\Theta}(x^{(i)}))_k) \right] \\ + \frac{\lambda}{2m} \sum_{l=1}^{L-1} \sum_{i=1}^{s_l} \sum_{j=1}^{s_{l+1}} (\Theta_{ji}^{(l)})^2$$



Machine Learning

Neural Networks: Learning

Backpropagation algorithm

Gradient computation

$$\rightarrow \underline{J(\Theta)} = -\frac{1}{m} \left[\sum_{i=1}^m \sum_{k=1}^K y_k^{(i)} \log h_{\theta}(x^{(i)})_k + (1 - y_k^{(i)}) \log(1 - h_{\theta}(x^{(i)})_k) \right] \\ + \frac{\lambda}{2m} \sum_{l=1}^{L-1} \sum_{i=1}^{s_l} \sum_{j=1}^{s_{l+1}} (\Theta_j^{(l)})^2$$

$$\rightarrow \min_{\Theta} J(\Theta)$$

Need code to compute:

$$\rightarrow - \underline{J(\Theta)}$$

$$\rightarrow - \underline{\frac{\partial}{\partial \Theta_{ij}^{(l)}} J(\Theta)} \quad \leftarrow$$

$$\Theta_{ij}^{(l)} \in \mathbb{R}$$

Gradient computation

Given one training example (x, y) :

Forward propagation:

$$\begin{aligned} &\rightarrow \underline{a^{(1)}} = \underline{x} \\ &\rightarrow z^{(2)} = \Theta^{(1)} a^{(1)} \\ &\rightarrow a^{(2)} = g(z^{(2)}) \quad (\text{add } \underline{a_0^{(2)}}) \\ &\rightarrow z^{(3)} = \Theta^{(2)} a^{(2)} \\ &\rightarrow a^{(3)} = g(z^{(3)}) \quad (\text{add } a_0^{(3)}) \\ &\rightarrow z^{(4)} = \Theta^{(3)} a^{(3)} \\ &\rightarrow \underline{a^{(4)}} = \underline{h_{\Theta}(x)} = g(z^{(4)}) \end{aligned}$$



Gradient computation: Backpropagation algorithm

Intuition: $\delta_j^{(l)}$ = "error" of node j in layer l .

For each output unit (layer $L = 4$)

$$\delta_j^{(4)} = a_j^{(4)} - y_j$$

(handwritten note: $(\text{activation})_j$ $\delta_j^{(4)} = a_j^{(4)} - y_j$)



$$\delta^{(3)} = (\Theta^{(3)})^T \delta^{(4)} \cdot g'(z^{(3)})$$

$$\delta^{(2)} = (\Theta^{(2)})^T \delta^{(3)} \cdot g'(z^{(2)})$$

(handwritten notes for $g'(z)$):
 $a^{(3)} \cdot (1 - a^{(3)})$
 $a^{(2)} \cdot (1 - a^{(2)})$

$$\frac{\partial}{\partial \Theta_{ij}^{(l)}} J(\Theta) = a_j^{(l)} \delta_i^{(l+1)}$$

(ignoring λ ; if $\lambda = 0$)

Backpropagation algorithm

→ Training set $\{(x^{(1)}, y^{(1)}), \dots, (x^{(m)}, y^{(m)})\}$

Set $\Delta_{ij}^{(l)} = 0$ (for all l, i, j).

(used to compute $\frac{\partial}{\partial \Theta_{ij}^{(l)}} J(\Theta)$)

For $i = 1$ to m ← $(\underline{x^{(i)}}, \underline{y^{(i)}})$

Set $\underline{a^{(1)}} = \underline{x^{(i)}}$

→ Perform forward propagation to compute $\underline{a^{(l)}}$ for $l = \underline{2}, \underline{3}, \dots, \underline{L}$

→ Using $\underline{y^{(i)}}$, compute $\underline{\delta^{(L)}} = \underline{a^{(L)}} - \underline{y^{(i)}}$

→ Compute $\underline{\delta^{(L-1)}}, \underline{\delta^{(L-2)}}, \dots, \underline{\delta^{(2)}}$ ~~set~~

→ $\underline{\Delta_{ij}^{(l)}} := \underline{\Delta_{ij}^{(l)}} + \underline{a_j^{(l)}} \underline{\delta_i^{(l+1)}}$ ← Δ

$\Delta^{(l)} := \Delta^{(l)} + \delta^{(l+1)} (a^{(l)})^T$

→ $\underline{D_{ij}^{(l)}} := \frac{1}{m} \underline{\Delta_{ij}^{(l)}} + \underline{\lambda \Theta_{ij}^{(l)}}$ if $\underline{j \neq 0}$

→ $\underline{D_{ij}^{(l)}} := \frac{1}{m} \underline{\Delta_{ij}^{(l)}}$ if $\underline{j = 0}$

$$\frac{\partial}{\partial \Theta_{ij}^{(l)}} J(\Theta) = D_{ij}^{(l)}$$

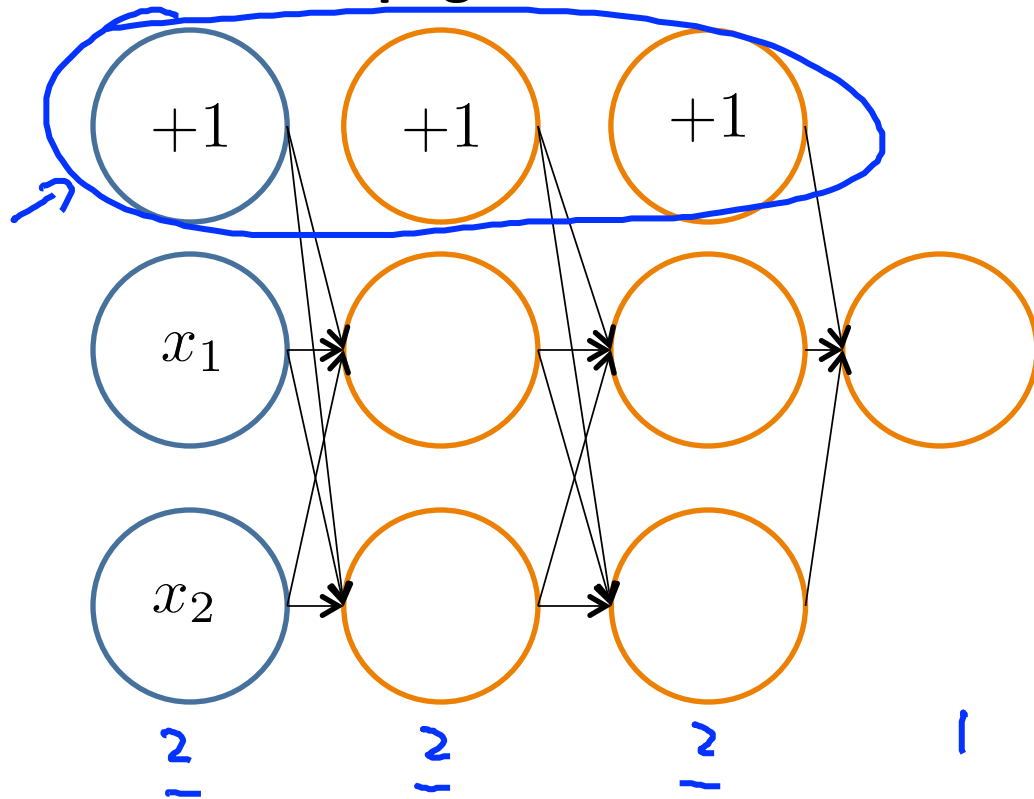


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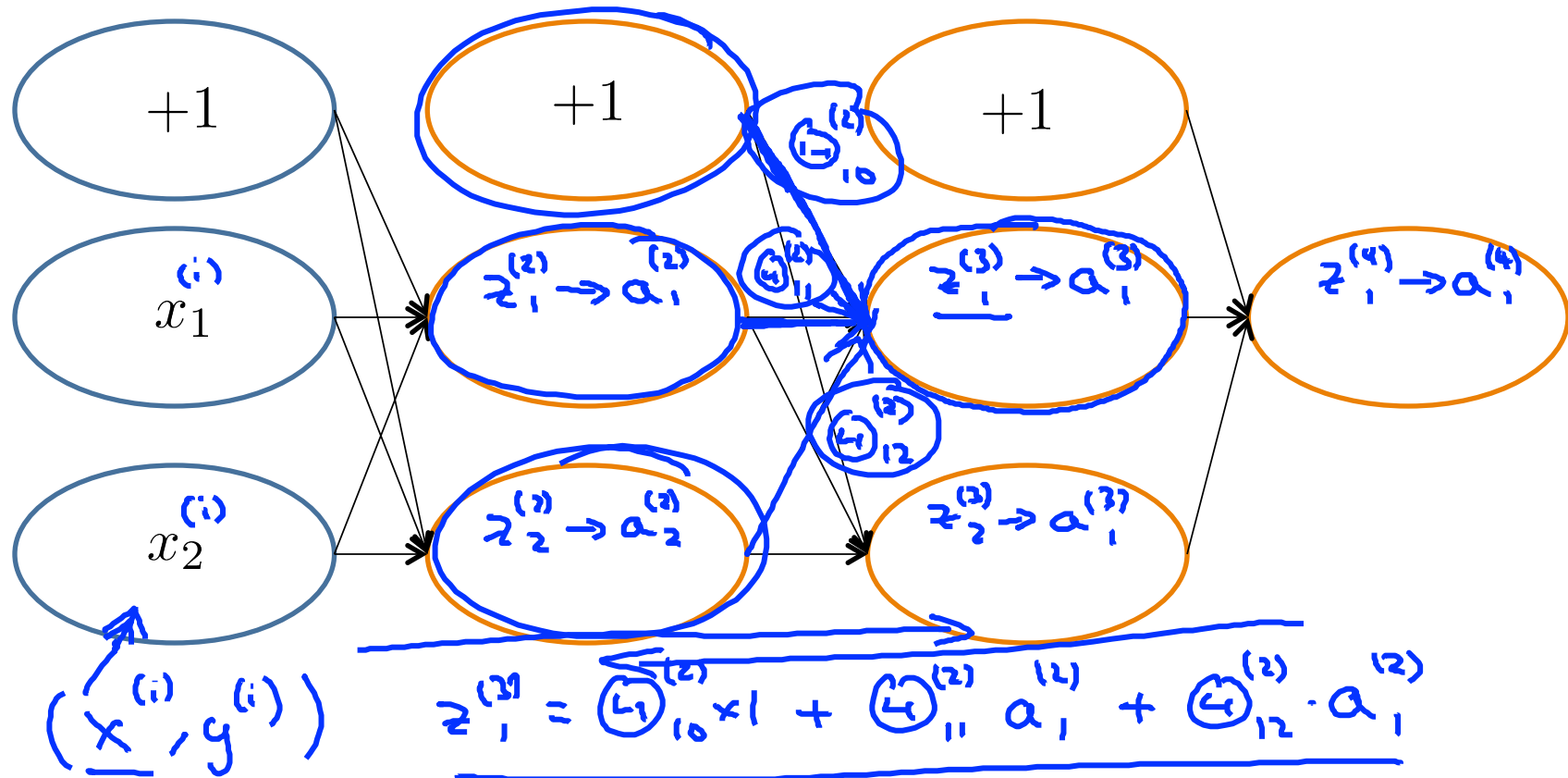
Neural Networks: Learning

Backpropagation intuition

Forward Propagation



Forward Propagation



What is backpropagation doing?

$$J(\Theta) = -\frac{1}{m} \left[\sum_{i=1}^m y^{(i)} \log(h_{\Theta}(x^{(i)})) + (1 - y^{(i)}) \log(1 - (h_{\Theta}(x^{(i)}))) \right] + \frac{\lambda}{2m} \sum_{l=1}^{L-1} \sum_{i=1}^{s_l} \sum_{j=1}^{s_{l+1}} (\Theta_{ji}^{(l)})^2$$

(x⁽ⁱ⁾, y⁽ⁱ⁾)

Focusing on a single example x⁽ⁱ⁾, y⁽ⁱ⁾, the case of 1 output unit, and ignoring regularization (λ = 0),

Note: Mistake on lecture, it is supposed to be 1-h(x).

$$\text{cost}(i) = y^{(i)} \log h_{\Theta}(x^{(i)}) + (1 - y^{(i)}) \log h_{\Theta}(x^{(i)})$$

(Think of cost(i) ≈ (h_Θ(x⁽ⁱ⁾) - y⁽ⁱ⁾)²)

I.e. how well is the network doing on example i?

Forward Propagation



$$\delta_1^{(4)} = y^{(i)} - a_1^{(4)}$$

$$\delta_2^{(3)} = w_{12}^{(3)} \delta_1^{(4)} + w_{22}^{(3)} \delta_2^{(4)}$$

$$\delta_2^{(3)} = w_{12}^{(3)} \delta_1^{(4)}$$

→ $\delta_j^{(l)}$ = "error" of cost for $a_j^{(l)}$ (unit j in layer l).

Formally, $\delta_j^{(l)} = \frac{\partial}{\partial z_j^{(l)}} \text{cost}(i)$ (for $j \geq 0$), where

$$\text{cost}(i) = y^{(i)} \log h_{\Theta}(x^{(i)}) + (1 - y^{(i)}) \log h_{\Theta}(x^{(i)})$$



Machine Learning

Neural Networks: Learning

Implementation
note: Unrolling
parameters

Advanced optimization

```
function [jVal, gradient] = costFunction(theta)  
...  
optTheta = fminunc(@costFunction, initialTheta, options)
```

Handwritten annotations: \mathbb{R}^{n+1} (twice) and \mathbb{R}^{n+1} (vectors) with arrows pointing to gradient, theta, and initialTheta respectively.

Neural Network (L=4):

→ $\Theta^{(1)}, \Theta^{(2)}, \Theta^{(3)}$ - matrices (Theta1, Theta2, Theta3)

→ $D^{(1)}$, $D^{(2)}$, $D^{(3)}$ - matrices (D1, D2, D3)

“Unroll” into vectors

Example

$$s_1 = 10, s_2 = 10, s_3 = 1$$

$$\rightarrow \Theta^{(1)} \in \mathbb{R}^{10 \times 11}, \Theta^{(2)} \in \mathbb{R}^{10 \times 11}, \Theta^{(3)} \in \mathbb{R}^{1 \times 11}$$

$$\rightarrow D^{(1)} \in \mathbb{R}^{10 \times 11}, D^{(2)} \in \mathbb{R}^{10 \times 11}, D^{(3)} \in \mathbb{R}^{1 \times 11}$$



$$\rightarrow \text{thetaVec} = [\text{Theta1}(:); \text{Theta2}(:); \text{Theta3}(:)] ;$$

$$\rightarrow \text{DVec} = [\text{D1}(:); \text{D2}(:); \text{D3}(:)] ;$$

$$\text{Theta1} = \text{reshape}(\text{thetaVec}(1:110), 10, 11) ;$$

$$\rightarrow \text{Theta2} = \text{reshape}(\text{thetaVec}(111:220), 10, 11) ;$$

$$\rightarrow \text{Theta3} = \text{reshape}(\text{thetaVec}(221:231), 1, 11) ;$$

Learning Algorithm

- Have initial parameters $\Theta^{(1)}, \Theta^{(2)}, \Theta^{(3)}$.
- Unroll to get `initialTheta` to pass to
- `fminunc(@costFunction, initialTheta, options)`

```
function [jval, gradientVec] = costFunction(thetaVec)
```

- From thetaVec, get $\Theta^{(1)}, \Theta^{(2)}, \Theta^{(3)}$. *reshape*
- Use forward prop/back prop to compute $D^{(1)}, D^{(2)}, D^{(3)}$ $J(\Theta)$
and $D^{(1)}, D^{(2)}, D^{(3)}$
Unroll _____ to get gradientVec.



Machine Learning

Neural Networks: Learning

Gradient checking

Numerical estimation of gradients



$$\frac{d}{d\Theta} J(\Theta) \approx$$

$$\frac{J(\Theta + \epsilon) - J(\Theta - \epsilon)}{2\epsilon}$$

$\epsilon = 10^{-4}$

~~$$\frac{J(\Theta + \epsilon) - J(\Theta)}{\epsilon}$$~~

Implement: gradApprox = (J(theta + EPSILON) - J(theta - EPSILON)) / (2*EPSILON)

Parameter vector θ

→ $\theta \in \mathbb{R}^n$ (E.g. θ is “unrolled” version of $\underline{\Theta^{(1)}}$, $\underline{\Theta^{(2)}}$, $\underline{\Theta^{(3)}}$)

→ $\theta = [\theta_1, \theta_2, \theta_3, \dots, \theta_n]$

→ $\frac{\partial}{\partial \theta_1} J(\theta) \approx \frac{J(\theta_1 + \epsilon, \theta_2, \theta_3, \dots, \theta_n) - J(\theta_1 - \epsilon, \theta_2, \theta_3, \dots, \theta_n)}{2\epsilon}$

→ $\frac{\partial}{\partial \theta_2} J(\theta) \approx \frac{J(\theta_1, \theta_2 + \epsilon, \theta_3, \dots, \theta_n) - J(\theta_1, \theta_2 - \epsilon, \theta_3, \dots, \theta_n)}{2\epsilon}$

⋮

→ $\frac{\partial}{\partial \theta_n} J(\theta) \approx \frac{J(\theta_1, \theta_2, \theta_3, \dots, \theta_n + \epsilon) - J(\theta_1, \theta_2, \theta_3, \dots, \theta_n - \epsilon)}{2\epsilon}$

```

for i = 1:n, ←
    thetaPlus = theta;
    thetaPlus(i) = thetaPlus(i) + EPSILON;
    thetaMinus = theta;
    thetaMinus(i) = thetaMinus(i) - EPSILON;
    gradApprox(i) = (J(thetaPlus) - J(thetaMinus))
                    / (2*EPSILON);
end;

```

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_i + \epsilon \\ \vdots \\ \theta_n \end{bmatrix} \rightarrow \theta_i - \epsilon$$


$$\frac{\partial}{\partial \theta_i} J(\theta).$$

Check that gradApprox \approx DVec ←

↑
From back prop.

Implementation Note:

- - Implement backprop to compute DVec (unrolled $D^{(1)}$, $D^{(2)}$, $D^{(3)}$).

- - Implement numerical gradient check to compute gradApprox.
- - Make sure they give similar values.
- - Turn off gradient checking. Using backprop code for learning.


Important:

- - Be sure to disable your gradient checking code before training your classifier. If you run numerical gradient computation on every iteration of gradient descent (or in the inner loop of `costFunction(...)`) your code will be very slow.



Machine Learning

Neural Networks: Learning

Random initialization

Initial value of Θ

For gradient descent and advanced optimization method, need initial value for Θ .

```
optTheta = fminunc(@costFunction,  
    initialTheta, options)
```

Consider gradient descent

Set initialTheta = zeros(n,1) ?

Zero initialization



$$\rightarrow \Theta_{ij}^{(l)} = 0 \text{ for all } i, j, l.$$

$$a_1^{(2)} = a_2^{(2)} \quad \text{Also} \quad \delta_1^{(2)} = \delta_2^{(2)}$$

$$\frac{\partial}{\partial \Theta_{0,1}^{(1)}} J(\Theta) = \frac{\partial}{\partial \Theta_{0,1}^{(1)}} J(\Theta)$$

$$\Theta_{0,1}^{(1)} = \Theta_{0,2}^{(1)}$$

After each update, parameters corresponding to inputs going into each of two hidden units are identical.

$$a_1^{(2)} = a_2^{(2)}$$

Random initialization: Symmetry breaking

→ Initialize each $\Theta_{ij}^{(l)}$ to a random value in $[-\epsilon, \epsilon]$
(i.e. $-\epsilon \leq \Theta_{ij}^{(l)} \leq \epsilon$)

E.g.

Random 10x11 matrix (betw. 0 and 1)

→ Theta1 = rand(10, 11) * (2 * INIT_EPSILON)
- INIT_EPSILON; [-ε, ε]

→ Theta2 = rand(1, 11) * (2 * INIT_EPSILON)
- INIT_EPSILON;



Machine Learning

Neural Networks: Learning

Putting it together

Training a neural network

Pick a network architecture (connectivity pattern between neurons)



→ No. of input units: Dimension of features $\underline{x^{(i)}}$

→ No. output units: Number of classes

Reasonable default: 1 hidden layer, or if >1 hidden layer, have same no. of hidden units in every layer (usually the more the better)

$y \in \{1, 2, 3, \dots, 10\}$
 ~~$y = 5$~~

$y = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$

Training a neural network

- 1. Randomly initialize weights
- 2. Implement forward propagation to get $h_{\Theta}(x^{(i)})$ for any $x^{(i)}$
- 3. Implement code to compute cost function $J(\Theta)$
- 4. Implement backprop to compute partial derivatives $\frac{\partial}{\partial \Theta_{jk}^{(l)}} J(\Theta)$

→ for $i = 1:m$ { $(x^{(1)}, y^{(1)})$ $(x^{(2)}, y^{(2)})$, ..., $(x^{(m)}, y^{(m)})$

→ Perform forward propagation and backpropagation using example $(x^{(i)}, y^{(i)})$

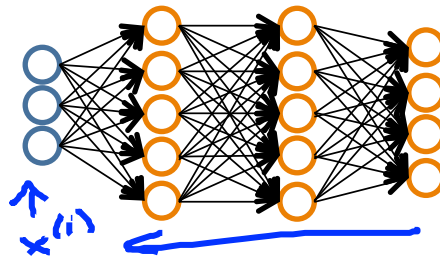
(Get activations $a^{(l)}$ and delta terms $\delta^{(l)}$ for $l = 2, \dots, L$).

→ $\Delta^{(2)} := \Delta^{(2)} + \delta^{(L)} (a^{(2)})^T$

...

}

compute $\frac{\partial}{\partial \Theta_{jk}^{(l)}} J(\Theta)$.



Training a neural network

- 5. Use gradient checking to compare $\frac{\partial}{\partial \Theta_{ik}^{(l)}} J(\Theta)$ computed using backpropagation vs. using numerical estimate of gradient of $J(\Theta)$.
- Then disable gradient checking code.
- 6. Use gradient descent or advanced optimization method with backpropagation to try to minimize $J(\Theta)$ as a function of parameters Θ

$\frac{\partial}{\partial \Theta_{ik}^{(l)}} J(\Theta)$

$J(\Theta)$ — non-convex.





Machine Learning

Neural Networks: Learning

Backpropagation
example: Autonomous
driving (optional)

