LEC. 10: LEAST SQUARES, SPARSITY (BASICS)

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AGENDA

1. ORTHOGONAL PROJECTION & LEAST SQUARES
   - SUMMARY
   - $\mathbb{C}^m$ and PROJECTION MATRIX
2. SPARSE SOLUTIONS TO LINEAR EQUATIONS
   - BASICS

Summary of Orthogonal Projection

Let $\mathcal{V}$ be a finite-dimensional inner-product space with inner product

$$\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{F} \quad (\mathbb{R} \text{ or } \mathbb{C})$$

$$\|v\|_{ip} = \sqrt{\langle v, v \rangle}, \qquad v \in \mathcal{V}$$

Given a vector $b \in \mathcal{V}$ and a subspace S of $\mathcal{V}$, solve

$$\min \|b - \hat{b}\|_{ip}, \quad \hat{b} \in S \text{ --- (OP)}$$

Fact:

The solution to the above problem exists, is unique and is given by a vector $\hat{b}_s \in S$ with the following property:-

$$\langle b - \hat{b}_s, v \rangle = 0, \quad \forall v \in S$$

How to find $\hat{b}_s$?

1. Let $\dim(S) = k$, let $\{v_1, v_2, \dots, v_k\}$ be a basis of S
2. Solve the Normal Equations : $\bar{b} = \bar{A}x$, $b \in \mathbb{F}^k$, $\bar{A} \in \mathbb{F}^{k \times k}$

   (See Lec. 9 for definition of $\bar{b}$ and $\bar{A}$)
3. $\hat{b}_s = \sum_{i=1}^{k} v_i x_i$, where $x = [x_1 \cdots x_k]^T$ is solution to Normal Equations.

APPLICATION

Let $\mathcal{V} = \mathbb{C}^m$, $S = R(Q)$ where $Q \in \mathbb{C}^{m\times n}$

Standard inner product of $\mathbb{C}^m$: $\langle x, y \rangle = y^H x$.

$$\|y\|_{ip} = \sqrt{\sum_{i=1}^{m} |y|_i^2}$$

Solve:

$$\min_{x\in\mathbb{C}^n} \|y - Qx\|_2 \quad \text{——— (LS)}$$

Note that this is equivalent to solving.

$$\min_{\hat{y}} \|y - \hat{y}\|_2 \quad \text{——— (OP)}$$
$$\text{s.t. } \hat{y} \in S, \quad S = R(Q)$$

Indeed, given the solution $\hat{y}_s$ to (OP), one can find a solution $\bar{x}$ to (LS) by solving

$$\hat{y}_s = Qx$$

To solve (OP):

1. Let $r = \text{rank}(Q)$, let $q_1, q_2, \ldots, q_r$ be "r" linearly independent columns of Q. Let $Q_r = [q_1\ q_2 \cdots q_r] \in \mathbb{C}^{m\times r}$
2. Convince yourself that Normal equations become

$$Q_r^H b = Q_r^H Q_r z$$

3. Solve for z as $\bar{z} = (Q_r^H Q_r)^{-1} Q_r^H b$

4. $\hat{y}_s$ is given by $\hat{y}_s = Q_r \bar{z}$

Hence, putting it all together,

$$\hat{y}_s = Q_r (Q_r^H Q_r)^{-1} Q_r^H \, b$$

$$= P_{R(Q)} \, b, \qquad \boxed{P_{R(Q)} = Q_r (Q_r^H Q_r)^{-1} Q_r^H}$$

Here $P_{R(Q)} \in \mathbb{C}^{m \times m}$ is called the orthogonal projection matrix onto $R(Q)$.

It takes a vector b and produces $P_{R(Q)} b$, which is nothing but the orthogonal projection of b onto $R(Q)$.

<u>Note</u>: In H.W. # 3, we provide a different (but equivalent) definition of a projection matrix. It will be your job to show that they are the same.

REVISITING $b = Ax$: ROLE OF SPARSITY.

Solve for x:

$$b = Ax$$

$\boxed{\text{Is } b \in \mathcal{R}(A)\text{ ?}}$

- YES → $\boxed{\text{Is } \mathcal{N}(A) = \{0\}\text{ ?}}$
  - YES → Unique Solution
  - NO → More than one solution
- NO → No Solution

(Can still do Least Squares to find $\bar{x}$ that minimizes $\|b - Ax\|_2$)

REGIME OF INTEREST IN COMPRESSED SENSING

Sparsity: Motivating Example.

1. Mr. X generates a vector $x_0 \in \mathbb{C}^n$.

2. He compresses it using a matrix $A \in \mathbb{C}^{m \times n}$ $(m \ll n)$ to produce

$$y_0 = Ax_0 \, , \quad y_0 \in \mathbb{C}^m$$

Since $m \ll n$, y_0 is a "compressed sketch" of x_0.

3. Mr X provides Ms. Y with y_0 and A.

Can Ms. Y recover x_0 ?

A Difficult Problem!

Since $m < n$, $\mathcal{N}(A) \neq \{0\}$ (think rank-nullity theorem)

$\Rightarrow$ $y_0 = Ax$ has "infinite solutions"

Impossible to tell which of these is x_0, UNLESS Ms. Y knows "something else" about x_0.

Typically in many scenarios x_0 turns out to be "sparse" (i.e. it has many "zero" entries).

Define:

$$S_x = \{ i \in [1, n], \text{ s.t. } x_i \neq 0 \}$$

S_x is called the support set of x.

$\|S_x\|_0$ = cardinality of S_x, also called "sparsity" of x_0.

x_0 has "sparsity" s if $\|S_x\|_0 = s$.

$\Leftrightarrow$ x_0 has "s" non-zero entries.