LEC. 7: INNER PRODUCTS, NORM, ORTHOGONALITY

Prof. Piya Pal
ECE 269
FALL 2021

AGENDA

1. Inner Products, Orthogonality
2. Norms
3. INTRODUCTION TO PROJECTION

INNER PRODUCT

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A vector space $\mathcal{V}$ over $\mathbb{F}$ is said to be an inner-product space if it is equipped with a function called inner product

$$\langle \cdot , \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{F} \quad (\mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}).$$

which satisfies the following properties.

(i) $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle, \ \forall u, v, w \in \mathcal{V}.$

(ii) $\langle \alpha u, v \rangle = \alpha . \langle u, v \rangle, \ \forall u, v \in \mathcal{V}, \alpha \in \mathbb{F}.$

multiplication of real or complex numbers

(iii) $\langle u, v \rangle = \overline{\langle v, u \rangle}$ $\forall u, v \in \mathcal{V}$ ($\bar{\alpha}$ represents complex conjugate of α)

(iv) $\langle u, u \rangle \geq 0, \ \forall u \in \mathcal{U}$. Moreover, $\langle u, u \rangle = 0$ if and only if $u = 0$

Example:

(i) $\mathcal{V} = \mathbb{C}^n$

Define

$$\langle u, v \rangle = v^H u \quad \left(= \sum_{i=1}^{n} v_i^* u_i \right)$$

Verify it is an inner product w.r.t. $\mathbb{C}$

(ii) $\mathcal{V} = \mathbb{R}^n$

Define

$$\langle u, v \rangle = v^T u \quad \left(= \sum_{i=1}^{n} u_i v_i \right)$$

$(= u^T v)$

Verify it is an inner product w.r.t. $\mathbb{R}$

(iii) $\mathcal{V} = \mathbb{C}^{m \times n}$,

Can be shown that this defines an inner product in $\mathbb{C}^{m\times n}$

Define

$$\langle U, V \rangle = \mathrm{Trace}\big(\overbrace{V^H U}^{n \times n}\big)$$

$V, U \in \mathbb{C}^{m \times n}$

Here $V^H \in \mathbb{C}^{n \times m}$, defined as

$$[V^H]_{ij} = \overline{V}_{ji} \quad \text{(transposed \& conjugated)}$$

Given a square matrix A $(n \times n)$

$$\mathrm{Trace}(A) = \sum_{i=1}^{n} A_{ii} \quad \text{(sum of diagonal elements of } A\text{)}$$

NORM

Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. A vector space $\mathcal{V}$ over $\mathbb{F}$ equipped with a norm is called a normed vector space. A norm is a function

$$\| \ \| : \mathcal{V} \to \mathbb{R}^+$$

such that

(i) $\|\alpha . v\| = |\alpha| \, \|v\|, \quad \forall \alpha \in \mathbb{F}, \ v \in \mathcal{V}$

(ii) $\|v\| \geq 0, \forall v \in \mathcal{V}$, and $\|v\| = 0$ iff $v = 0$

(iii) $\|u + v\| \leq \|u\| + \|v\|, \ \forall u, v \in \mathcal{V}$

(Triangle Inequality)

A vector space $\mathcal{V}$ equipped with a norm is called a normed vector space.

Normed Vector Spaces

A vector space $\mathcal{V}$ equipped with a norm, is called a normed vector space.

$R^+ = \{x \in \mathbb{R}, x \geq 0\}$

Examples

(1) $\mathcal{V} = \mathbb{R}^n$ or $\mathbb{C}^n$: Define $\| \ \|_p : \mathcal{V} \to R^+$ as

$$\|v\|_p = \left(\sum_{i=1}^{n} |v_i|^p\right)^{\frac{1}{p}}, \quad v \in \mathcal{V}$$

It can be shown that $\| \ \|_p$ is a norm! for $p \geq 1$

(non-trivial to verify triangle inequality except when $p=2$)

examples:

(i) l_1 norm : $\|v\|_1 = \sum_{i=1}^{n} |v_i|$

(ii) l_2 norm ; $\|v\|_2 = \left(\sum_{i=1}^{n} |v_i|^2\right)^{\frac{1}{2}}$

[can be related to the inner product]

$$\|v\|_2 = \sqrt{\sum_{i=1}^{n} |v_i|^2} = \sqrt{\sum_{i=1}^{n} \bar{v}_i v_i} = \sqrt{\langle v, v\rangle}$$

inner product in $\mathbb{C}^n$

Norm Induced By Inner Product.

Let $\mathcal{V}$ be equipped with inner product $\langle \cdot , \cdot \rangle$.

Define a function

$$\| \ \|_{ip} : \mathcal{V} \to \mathbb{R}^+ \text{ as}$$

$$\| v \|_{ip} = \sqrt{\langle v, v \rangle}, \quad v \in \mathcal{V}$$

Can be shown that:

$\boxed{\| \ \|_{ip} \text{ is a norm.}}$

[Using Cauchy-Schwartz inequality triangle inequality can be established]

Examples:

also called Euclidean Norm.

① $\mathbb{C}^n$: $\langle u, v \rangle = v^H u$.

$$\Rightarrow \quad \|u\|_{ip} = \sqrt{u^H u} = \underbrace{\left(\sum_{i=1}^{n} |u_i|^2 \right)^{\frac{1}{2}}}_{l_2\text{-norm.}}$$

② $l_2 = \left\{ \{x_n\}_{n \geq 1}, \ \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\}$

Can be shown that following is an inner product:

$$\langle u, v \rangle = \sum_{i=1}^{\infty} u_i \overline{v_i}$$

$$\|u\|_{ip} = \left(\sum_{i=1}^{\infty} |u_i|^2 \right)^{\frac{1}{2}}$$

CAUCHY-SCHWARZ INEQUALITY

Let $\mathcal{V}$ be an inner product space. Then, for any $u, v \in \mathcal{V}$, we have

$$\boxed{|\langle u, v\rangle| \leq \|u\|_{ip} \|v\|_{ip}}$$

where $\|u\|_{ip} = \sqrt{\langle u, u\rangle}$.

TA's will prove in discussion. And, using CS inequality, show that $\| \ \|_{ip}$ is a norm.

Notion of Angle:

$\mathcal{V} = \mathbb{R}^n$,

$\langle u, v\rangle = v^T u = u^T v$, $\quad \|v\|_{ip} = \left(\sum_{i=1}^{n} |v_i|^2\right)^{1/2}$

$$-1 \leq \underbrace{\frac{\langle u, v\rangle}{\|u\|_2 \|v\|_2}}_{\triangleq \cos(\theta)} \leq 1$$

θ is called the "angle" between vectors u, v.

Inner product (via CS inequality) leads to a notion of "angle" between vectors u, v.

(i) $u = v \Rightarrow \cos\theta = \dfrac{\langle u, u\rangle}{\|u\|_{ip}\|u\|_{ip}}$

$$= \frac{\|u\|_{ip}^2}{\|u\|_{ip}^2}$$

$$= 1$$

$\Rightarrow \theta = 0$

(ii) $u = -v$, $\cos\theta = -\dfrac{\langle u, u\rangle}{\|u\|_{ip}\|u\|_{ip}}$

$$= -1$$

$\Rightarrow \theta = 180^\circ$

ORTHOGONALITY

A set of vectors $v_1, v_2, \ldots, v_k$ are orthogonal if

$$\langle v_i, v_j \rangle = 0, \quad \text{whenever } i \neq j$$

[Any subset of $\{v_1, \ldots, v_k\}$ will also be orthogonal]

Example:

(i) $\mathcal{V} = \mathbb{C}^n$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

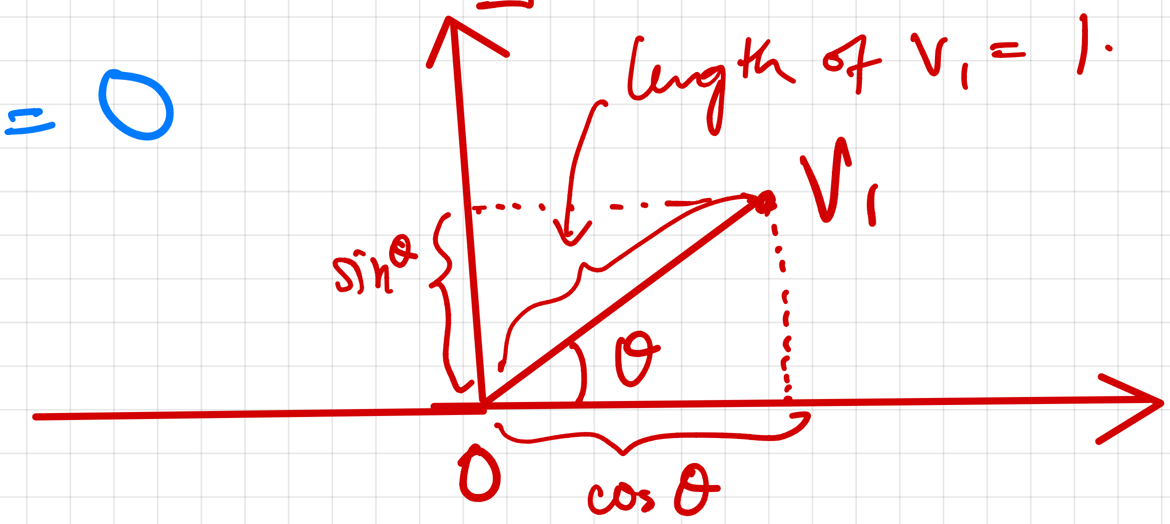
are orthogonal.

Place v_2 in the figure (H.W.)

(ii) $v_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, $v_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ $(\mathcal{V} = \mathbb{R}^2)$

Can show $v_1^T v_2 = 0$

$$\|v_1\|_2 = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$



(iii) Let $A \in \mathbb{C}^{m\times n}$, where $a_1, a_2, \dots, a_n$ are orthogonal.

Define $A^H \in \mathbb{C}^{n\times m}$ as the Hermitian of A (conjugate transpose)

ie $\quad [A^H]_{ij} = \overline{A_{ji}}, \quad \begin{matrix} i = 1, 2, \dots, m \\ j = 1, 2, \dots, n \end{matrix}$

Then, we have

$$\underbrace{A^H}_{n\times m}\underbrace{A}_{m\times n} = \begin{bmatrix} a_1^H \\ a_2^H \\ \vdots \\ a_n^H \end{bmatrix} \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

$$= \begin{bmatrix} a_1^H a_1 & a_1^H a_2 & a_1^H a_3 & \cdots & a_1^H a_n \\ a_2^H a_1 & a_2^H a_2 & a_2^H a_3 & \cdots & a_2^H a_n \\ \vdots & & & & \\ a_n^H a_1 & a_n^H a_2 & \cdots & \cdots & a_n^H a_n \end{bmatrix}$$

$$= \begin{bmatrix} \|a_1\|_2^2 & & & 0 \\ & \|a_2\|_2^2 & & \\ & & \ddots & \\ 0 & & & \|a_n\|_2^2 \end{bmatrix}$$

If $\|a_i\|_2 = 1$, $i = 1, 2, \ldots, n$.

then $A^H A = I_{n \times n}$ (However, we may not have $AA^H = I$)

Such a matrix $A \in \mathbb{C}^{m \times n}$ is said to have orthonormal columns

Unitary matrix:

A is square.

A matrix $A \in \mathbb{C}^{n \times n}$ is unitary if it has orthogonal columns with unit norm.

ie $A = [a_1 \; a_2 \; \cdots \; a_n]$, $a_i \in \mathbb{C}^n$

$$\langle a_i, a_j \rangle = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

The above can be written in a compact way as

$$\boxed{A^H A = I}$$

$\Rightarrow$ A^H is "the" inverse of $A \Rightarrow \boxed{AA^H = A^H A = I}$

Equivalent Definition

A matrix $A \in \mathbb{C}^{n \times n}$ is unitary iff

$$A^H A = A A^H = I_{n \times n}.$$

IMPLICATIONS OF ORTHOGONALITY.

(1) If $v_1, v_2, \ldots, v_K$ are non-zero vectors which are orthogonal, then $v_1, v_2, \ldots, v_K$ are also linearly independent.

Proof:

Suppose
$$\sum_{i=1}^{K} \alpha_i v_i = 0, \quad \alpha_i \in \mathbb{F}$$

verify that $\langle 0, v\rangle = 0$

$$\Rightarrow \left\langle \sum_{i=1}^{K} \alpha_i v_i, v_1 \right\rangle = \langle 0, v_1 \rangle = 0$$

$$\Rightarrow \sum_{i=1}^{K} \alpha_i \underbrace{\langle v_i, v_1 \rangle}_{=0,\ i \neq 1} = 0$$

$$\Rightarrow \alpha_1 \|v_1\|_{ip}^2 = 0$$

Since $v_1 \neq 0 \Rightarrow \alpha_1 = 0$. Similarly show that $\alpha_i = 0$, $i = 1, 2, \ldots, K$.

ORTHOGONAL BASIS

Suppose S is a subspace with $\dim(S) = n$

Suppose $\{v_1, \dots, v_n\}$ is an orthogonal set of vectors in S, ie $v_i \in S, i = 1, \dots, n$

and $\langle v_i, v_j \rangle = 0, \; i \neq j$

Then $\{v_1, \dots, v_n\}$ is a linearly independent set in S and therefore must be a basis of S $(\dim(S) = n)$.

$\{v_1, \dots, v_n\}$ is called an orthogonal basis of S.

Representing Any Vector in S using An Orthogonal Basis

Suppose S is a subspace and $\{v_1, \ldots, v_n\}$ is an orthogonal basis of S.

Let $v \in S$.

$$\Rightarrow \quad v = \sum_{i=1}^{n} \alpha_i v_i \,, \quad \text{for some } \alpha_i \in \mathbb{F}.$$

How to determine the coefficients $(\alpha_1, \ldots, \alpha_n)$ of representation ?

Notice that

$$\langle v, v_1 \rangle = \left\langle \sum_{i=1}^{n} \alpha_i v_i, v_1 \right\rangle$$

$$= \sum_{i=1}^{n} \alpha_i \langle v_i, v_1 \rangle$$

$$= \alpha_1 \|v_1\|_{ip}^2$$

$$\Rightarrow \quad \alpha_1 = \frac{\langle v, v_1 \rangle}{\|v_1\|_{ip}^2}$$

Similarly

$$\alpha_2 = \frac{\langle v, v_2 \rangle}{\|v_2\|_{ip}^2}$$

$$\vdots$$

$$\alpha_n = \frac{\langle v, v_n \rangle}{\|v_n\|_{ip}^2}$$

Elegant Formula for computing the coefficients of representation.

Example: Signal Representation in $\mathbb{C}^N$

Finite length (ie length-N) complex valued signals can be represented as vectors in $\mathbb{C}^N$.

SIGNAL REPRESENTATION OVER A BASIS OF $\mathbb{C}^N$ $\equiv$ FINDING COEFFICIENTS OF BASIS REPRESENTATION, DESCRIBED IN TERMS OF A SUITABLE "TRANSFORM" (LINKED TO CHOICE OF BASIS)

Let $v \in \mathbb{C}^N$, $\quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}$

Let $\{b_1, b_2 \ldots, b_N\}$ be a basis of $\mathbb{C}^N$.

$$v = \sum_{i=1}^{N} \alpha_i b_i$$

$$v_1, \ldots, v_N \xleftrightarrow[\text{TRANSFORM}]{\text{MAP/}} \alpha_1, \ldots, \alpha_N$$

(e.g. pixel values of an image) (Determined by choice of basis) (e.g. frequency components of an image)

Example: DFT Basis

Consider $w_0, w_1, \dots, w_{N-1} \in \mathbb{C}^N$, defined as

$$w_n = \begin{bmatrix} 1 \\ e^{j\frac{2\pi n}{N}} \\ e^{j\frac{2\pi.(2.n)}{N}} \\ \vdots \\ e^{j\frac{2\pi(m.n)}{N}} \\ \vdots \\ e^{j\frac{2\pi(N-1)n}{N}} \end{bmatrix}, \quad n = 0, 1, \dots, N-1.$$

① Show that $\{w_0, w_1, \dots, w_{N-1}\}$ is an orthogonal set of vectors.

This will imply that $\{w_0, \dots, w_{N-1}\}$ is an orthogonal <u>BASIS</u> of $\mathbb{C}^N$.

② Let $v \in \mathbb{C}^N$

$$v = \sum_{i=0}^{N-1} \alpha_i w_i$$

Use the formula

$$\boxed{\alpha_i = \frac{\langle v, w_i \rangle}{\|w_i\|_2^2}}$$

in $\mathbb{C}^n$, $\left(\langle v, w_i \rangle = w_i^H v \right)$

And convince yourselves that

α_i = i^{th} DFT coefficient of the signal v.

This will represent the Discrete Fourier Transform of v.

Inverse DFT:

$$v = \sum_{i=0}^{N-1} \alpha_i w_i = \sum_{i=0}^{N-1} \frac{\langle v, w_i \rangle}{\|w_i\|_2^2} w_i$$

Same idea Applies to any other orthogonal Basis of $\mathbb{C}^N$.