Inner product

Let $F = \mathbb{R}$ or $\mathbb{C}$, A vector space V over $\mathbb{F}$ is said to be an inner-product space if it is equipped with a function called inner product $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$

(i) $\langle v+u, \omega \rangle = \langle v, w \rangle + \langle u, w \rangle \quad \forall u, v, \omega \in V$

(ii) $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle \quad \forall u, v \in V, \alpha \in \mathbb{F}$

(iii) $\langle u, v \rangle = \overline{\langle v, u \rangle} \quad \forall u, v \in V$

(iv) $\langle u, u \rangle \geq 0 \quad \forall u \in V$ moreover

$\langle u, u \rangle = 0$ if and only if $u = 0$

→ Cauchy Schwarz inequality

For all vectors u and v in V over field $\mathbb{F}$ ($\mathbb{C}$ or $\mathbb{R}$) associated with an inner product the following is true,

$$|\langle u,v\rangle|^2 \leq \langle u,u\rangle \cdot \langle v,v\rangle$$

consider the definition of a norm induced by an inner product,

$$\langle u,u\rangle = \|u\|^2$$

$$\boxed{|\langle u,v\rangle|^2 \leq \|u\|^2 \|v\|^2}$$

$$\Rightarrow \boxed{|\langle u,v\rangle| \leq \|u\| \|v\|}$$

Cauchy Schwarz inequality

First let's show, $\lambda \in \mathbb{F}$ $u, v_1, v_2 \in V$

(i) $\langle u, \lambda v \rangle = \overline{\langle \lambda v, u \rangle} = \overline{\lambda \langle v, u \rangle}$

$= \bar{\lambda} \, \overline{\langle v, u \rangle}$

$= \bar{\lambda} \langle u, v \rangle$

(ii) $\langle u, v_1 + v_2 \rangle = \overline{\langle v_1 + v_2, u \rangle}$

$= \overline{\langle v_1, u \rangle} + \overline{\langle v_2, u \rangle}$

$= \langle u, v_1 \rangle + \langle u, v_2 \rangle$

Proof of cauchy schwarz inequality:

$\langle u - \lambda v, u - \lambda v \rangle \geq 0$

$\langle u, u - \lambda v \rangle + \langle -\lambda v, u - \lambda v \rangle \geq 0$

$\langle u, u \rangle + \langle u, -\lambda v \rangle + \langle -\lambda v, u \rangle + \langle -\lambda v, -\lambda v \rangle$

≥ 0

$\|u\|^2 - \bar{\lambda}\langle u,v\rangle - \lambda\langle v,u\rangle + \|\lambda v\|^2 \geq 0$

Let $\lambda = \dfrac{\langle u,v\rangle}{\|v\|^2}$ Since it is true for all $\lambda \in F$

$$\|u\|^2 - \frac{\overline{\langle u,v\rangle}\langle u,v\rangle}{\|v\|^2} - \frac{\langle u,v\rangle\langle v,u\rangle}{\|v\|^2}$$

$$+ \langle \lambda v, \lambda v\rangle \geq 0$$

$$\boxed{\|\lambda v\|_2^2 = \langle \lambda v, \lambda v\rangle = \lambda\bar{\lambda}\langle v,v\rangle = |\lambda|^2\langle v,v\rangle}$$

$$\|u\|^2 - 2\frac{\langle u,v\rangle\overline{\langle u,v\rangle}}{\|v\|^2} + \frac{|\langle u,v\rangle|^2}{\|v\|^4} \times \|v\|^2 \geq 0$$

$$\|u\|^2 - 2\frac{|\langle u,v\rangle|^2}{\|v\|^2} + \frac{|\langle u,v\rangle|^2}{\|v\|^2} \geq 0$$

$$\|u\|^2 - \frac{|\langle u,v\rangle|^2}{\|v\|^2} \geq 0 \Rightarrow |\langle u,v\rangle|^2 \leq \|v\|^2\|u\|^2$$

Thus,

$$\boxed{|\langle u,v\rangle| \leq \|v\|\|u\|}$$

Note: The choice of λ is such that $v \neq 0$
if u or $v = 0$ then cauchy schwarz inequality is trivial to show.

Norm induced by inner product $\langle u,v\rangle$

In order to prove $\|u\| = \sqrt{\langle u,u\rangle}$ a valid norm we have to prove $\forall\, u,v \in V$ over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$)

(a) $\|u\| \geq 0$ equality iff $u=0$

(b) $\|\alpha u\| = |\alpha|\,\|u\|$ $\quad \forall\, \alpha \in \mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$)

(c) $\|u+v\| \leq \|u\| + \|v\|$

(a) we know $\langle u,u\rangle \geq 0$, $\forall\, u \in V$

and $\langle u,u\rangle = 0$ iff $u=0$.

using this property of ~~norm~~ inner product, it is easy to show,

$\|u\| = \sqrt{\langle u,u\rangle} \geq 0$ and $\|u\| = 0$

only iff $\langle u,u\rangle = 0 \iff \boxed{u=0}$

(b) $\|\alpha u\| = \sqrt{\langle \alpha u, \alpha u\rangle} = \sqrt{\alpha\bar{\alpha}\langle u,u\rangle}$

$= \sqrt{|\alpha|^2\langle u,u\rangle}$

$= |\alpha|\sqrt{\langle u,u\rangle}$

$= |\alpha|\,\|u\|$

(c) $\|u+v\|^2 = \langle u+v, u+v\rangle$

$= \langle u, u+v\rangle + \langle v, u+v\rangle$

$= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle$

$= \|u\|^2 + 2\,\mathrm{Re}\{\langle u,v\rangle\} + \|v\|^2$

$\boxed{\alpha + \bar{\alpha} = 2\,\mathrm{Re}\{\alpha\}}$

$$\boxed{|\alpha| \geq Re\{\alpha\}}$$

$$|\alpha| = \sqrt{Re\{\alpha\}^2 + Im(\alpha)^2}$$

$$\|u+v\|^2 \leq \|u\|^2 + \|v\|^2 + 2|\langle u,v\rangle|$$

from cachy schwartz inequality

$$|\langle u,v\rangle| \leq \|u\|\,\|v\|$$

$$\|u+v\|^2 \leq \|u\|^2 + \|v\|^2 + 2\|u\|\,\|v\|$$

$$\|u+v\|^2 \leq (\|u\| + \|v\|)^2$$

$$\Rightarrow \|u+v\| \leq \|u\| + \|v\| \quad \text{since } \|\cdot\| \geq 0.$$

Thus proving any norm induced by an inner product satisfied all the properties of a norm.

Ex:

(1) $\|u\|_2 = \sqrt{u^H u}$. - l_2 norm.

Show columns of a DFT matrix are orthogonal and forms a basis of C^n.

$$v_i = \begin{bmatrix} 1 \\ e^{j2\pi i/n} \\ e^{j2\pi 2i/n} \\ \vdots \\ e^{j2\pi(n-1)i/n} \end{bmatrix} \quad i^{th} \text{ column.}$$

we check if

$\langle v_l, v_m \rangle = 0$

$$\langle v_l, v_m \rangle = \sum_{k=0}^{n-1} e^{\frac{j2\pi}{n}[l-m]k}$$

Let $e^{j2\pi/n} = \omega$

$$= \sum_{k=0}^{n-1} \left((\omega)^{l-m}\right)^k$$

$$= 1 + \left(\omega^{(l-m)}\right) + \left(\omega^{(l-m)}\right)^2 + \dots + \left((\omega)^{(l-m)}\right)^{n-1}$$

If $l \neq m$,

$$\langle v_l, v_m \rangle = \frac{\left((\omega)^{(l-m)}\right)^n - 1}{(\omega)^{l-m} - 1} \quad \left(\text{since, } \omega^{l-m} \neq 1\right)$$

where, $$\left((\omega)^{l-m}\right)^n = e^{\frac{j2\pi}{n} \times (l-m) \times n}$$

$$= e^{j2\pi(l-m)}$$

where $(l-m)$ is an integer

$$= \cos(2\pi(l-m)) + j\sin 2\pi(l-m)$$

$$= 1.$$

$$(\omega)^{l-m} = e^{j\frac{2\pi}{n}(l-m)} \quad (\neq 1)$$

$$\langle v_l, v_m \rangle = \frac{1-1}{(\omega)^{l-m} - 1} = 0 \quad \text{if } l \neq m$$

Thus, $\{v_0, v_1, v_2, \dots, v_{n-1}\}$ forms an orthogonal basis since we have "n" such linearly independent vectors that are orthogonal to each other and dimension of C^n is "n"

thus $\{v_0, v_1, \ldots, v_{n-1}\}$ is an orthogonal basis of C^n.