LEC. 8: ORTHOGONAL PROJECTION

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ECE 269

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AGENDA

1. ORTHOGONAL PROJECTION: BASICS
   - ORTHOGONALITY PRINCIPLE

ORTHOGONAL PROJECTION

BASIC SETUP

Let V be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, endowed with an inner product $\langle \cdot, \cdot \rangle$ and let $\| \ \|_{ip}$ be the norm induced by this inner product. V is finite dimensional.

Let $y \in V$, and $S \subseteq V$ be a subspace

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{F} \quad \text{(inner product)}$$

$$\| \cdot \|_{ip} : V \longrightarrow \mathbb{R}^+ \quad \text{(norm induced by inner product)}$$

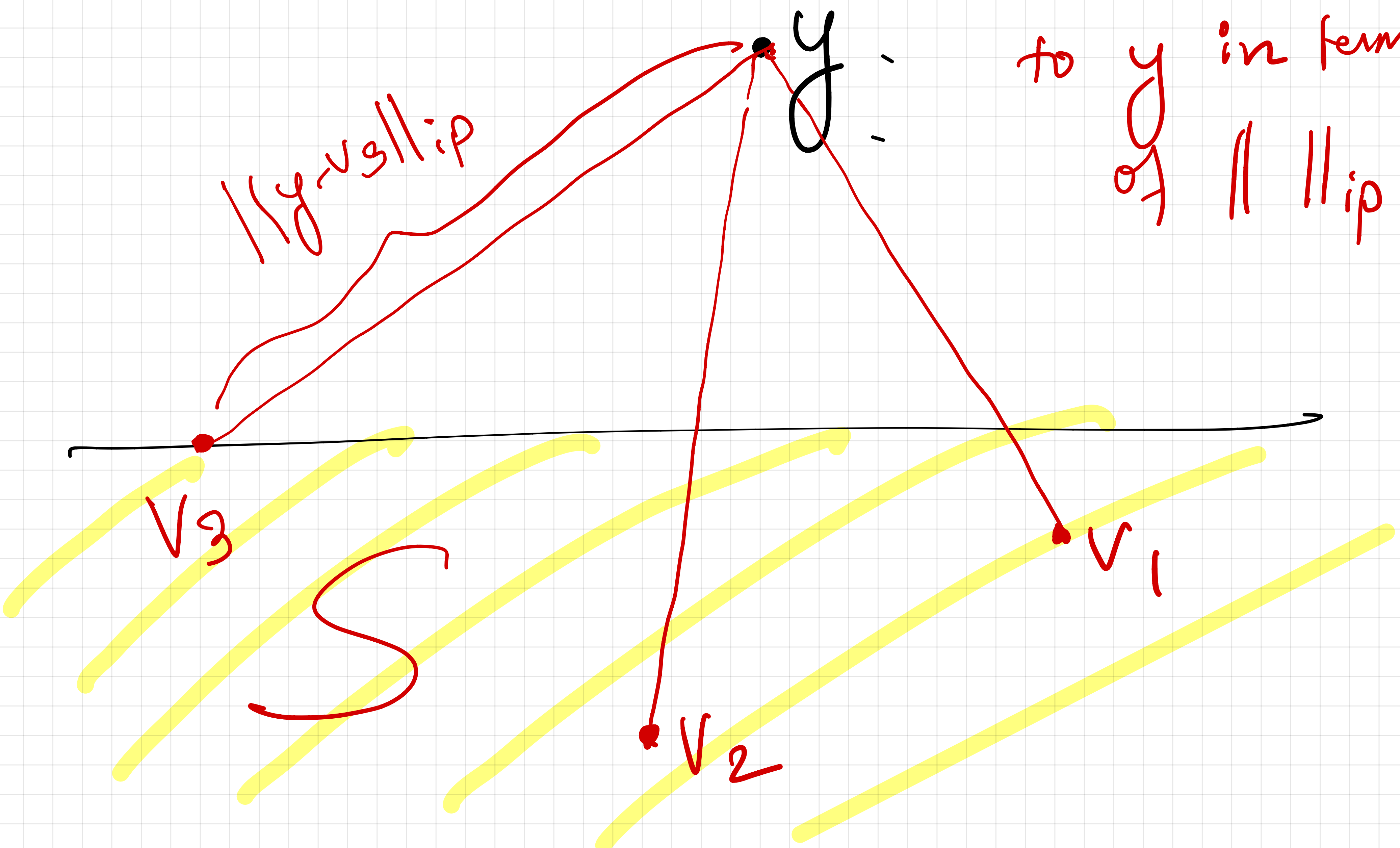
$$\| v \|_{ip} = \sqrt{\langle v, v \rangle}$$

S is a subspace of V.

Consider the Optimization Problem:

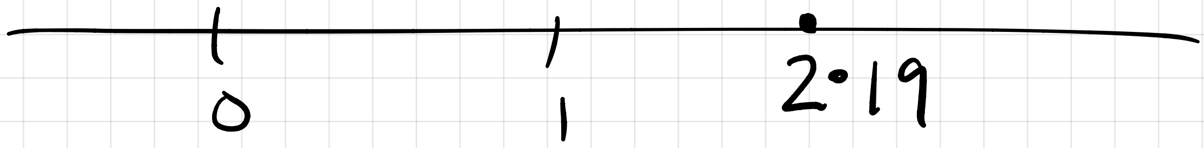
$$\boxed{\begin{aligned} &\min_{v} \|y - v\|_{ip} \\ &\text{s.t. } v \in S. \end{aligned}} \quad \text{— (P)}$$

What is it trying to find? A point in S which is "closest" to y in terms of $\| \ \|_{ip}$.



Can an optimization problem always be solved?

Does a minimizer always exist? Not always



$$\left.\begin{array}{l}\min\limits_{x} \; |2.19 - x| \\ 0 < x < 1\end{array}\right\}$$ No minimizer

$$\left.\begin{array}{l}\min\limits_{x} \; |2.19 - x| \\ 0 < x \leq 1\end{array}\right\}$$ Minimizer exists $x^* = 1$.

We will show that the Problem (P) has a minimizer (minimizer exists)

When $\mathcal{V}$ is finite-dimensional. If $\mathcal{V}$ is infinite-dimensional, existence can be again established if $\mathcal{V}$ is a complete inner-product space, also known as a Hilbert Space and S is a closed subspace of $\mathcal{V}$.

SUMMARY (WILL PROVE).

We will show that the minimizer to (P)

(i) exists (will do the proof for finite-dimensional V)

(ii) is unique.

(iii) Satisfies the orthogonality principle.

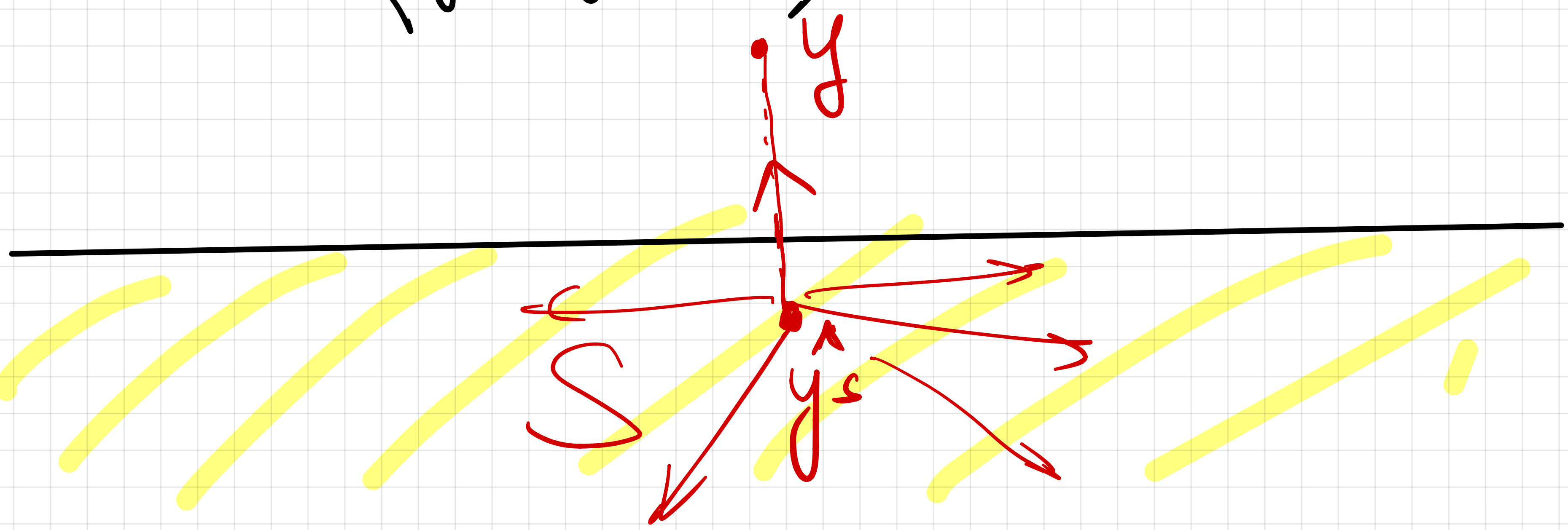
(will define next).

Orthogonality principle will also lead to a computationally efficient way to solve (P)

Orthogonal Projection

Given a point $y \in \mathcal{V}$ and a subspace $S \subseteq \mathcal{V}$, a vector $\hat{y}_S \in S$ is called an orthogonal projection of y onto S if

$$\langle y - \hat{y}_S, v \rangle = 0, \quad \forall v \in S$$



The existence and uniqueness of an orthogonal projection can be established for an infinite dimensional vector space $\mathcal{V}$, if $\mathcal{V}$ is a Hilbert space (a complete inner product space) and S is a closed subspace of $\mathcal{V}$.

We will establish existence & uniqueness for finite dimensional $\mathcal{V}$.

First, suppose that a genie told you that for every $y \in \mathcal{V}$, and subspace $S \subset \mathcal{V}$, its orthogonal projection $\hat{y}_S$ exists.

How does it help us solve (P) ?

SOLUTION TO (P)

Consider any vector $\hat{y} \in S$

$$\|y-\hat{y}\|_{ip}^2 = \|y-\hat{y}_S+\hat{y}_S-\hat{y}\|_{ip}^2$$

$$= \|y-\hat{y}_S\|_{ip}^2 + \|\hat{y}_S-\hat{y}\|_{ip}^2 + \langle y-\hat{y}_S, \hat{y}_S-\hat{y}\rangle + \langle \hat{y}_S-\hat{y}, y-\hat{y}_S\rangle$$

Notice that $\hat{y}_S \in S$ and $\hat{y} \in S$

$\Rightarrow \hat{y}_S-\hat{y} \in S$ (since S is a subspace).

$\Rightarrow \langle y-\hat{y}_S, \hat{y}_S-\hat{y}\rangle = 0$ (property of $\hat{y}_S$)

$\therefore \langle \hat{y}_S-\hat{y}, y-\hat{y}_S\rangle = 0$

$\therefore$ We obtain in above expression:-

$$\|y-\hat{y}\|_{ip}^2 = \|y-\hat{y}_S\|_{ip}^2+\|\hat{y}_S-\hat{y}\|_{ip}^2 \geq \|y-\hat{y}_S\|_{ip}^2$$

(Since $\|\hat{y}_S-\hat{y}\|_{ip}^2 \geq 0$)

Therefore, for any $\hat{y} \in S$, we have

$$\|y - \hat{y}\|_{ip}^2 \geq \|y - \hat{y}_s\|_{ip}^2$$

$\therefore$ $\|y - \hat{y}_s\|_{ip}^2$ is a lower bound on $\|y - \hat{y}\|_{ip}^2$ and it is attained when $\hat{y} = \hat{y}_s$.

$\therefore$ $\hat{y}_s$ minimizes (P).