Ziming Qi

ECE269

HW2

A13418961.

1. **Problem 1: Affine functions.** A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *affine* if for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and any $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta = 1$, we have

$$f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}).$$

Note that without the restriction $\alpha + \beta = 1$, this would be the definition of linearity.

(a) Suppose that $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Show that the function $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ is affine.

(b) Prove the converse, namely, show that any affine function f can be represented uniquely as $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ for some $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$.

(a) For $\alpha, \beta \in \mathbb{R}$ $\alpha+\beta = 1$.

$$f(\alpha x+\beta y) = A(\alpha x + \beta y) + b$$
$$= \alpha A x + \beta A y + b$$

$$\alpha f(x) + \beta f(y) = \alpha A x + \alpha b + \beta A y + \beta b$$
$$= \alpha A x + \beta A y + (\alpha+\beta) b$$

$\because \alpha + \beta = 1$

$\therefore \alpha f(x) + \beta f(y) = \alpha A x + \beta A y + b = f(\alpha x + \beta y)$

$\therefore f(x) = Ax + b$ is affine.

(b) Suppose linear map function $T: R^n \to R^m$

$T(x) = Ax$ $\quad T(\alpha x+\beta y) = A(\alpha x + \beta y)$ $\quad A \in \mathbb{R}^{m\times n}$
$$= \alpha A x + \beta A y.$$
$T(0) = 0$
$$= \alpha T(x) + \beta T(y)$$

Suppose an affine function f: $f(x) = A_1 x + b_1$

$f(0) = b_1$

$g(x) = f(x) - f(0) = A_1 x$. proof $g(x)$ is linear can proof A_1 is unique represent $g(x)$.

so.

$$
\begin{aligned}
g(\alpha x) &= f(\alpha x) - f(0) \\
&= f(\alpha x + 0) - f(0) \longrightarrow \beta = 1-\alpha \\
&= f(\alpha x + (1-\alpha)0) - f(0) \\
&= \alpha f(x) + (1-\alpha) f(0) - f(0) \\
&= \alpha f(x) + f(0) - \alpha f(0) - f(0) \\
&= \alpha f(x) - \alpha f(0)
\end{aligned}
$$

$$
\begin{aligned}
\alpha g(x) &= \alpha (f(x) - f(0)) \\
&= \alpha f(x) - \alpha f(0) \\
&= g(\alpha x).
\end{aligned}
$$

So $g(x)$ closed the multiplication

$$
\begin{aligned}
g(x_1 + x_2) &= 2 \cdot g\left(\tfrac{1}{2} x_1 + \tfrac{1}{2} x_2\right) \quad \because g(\alpha x) = \alpha (g x). \\
&= 2 \cdot \left[f\left(\tfrac{1}{2} x_1 + \tfrac{1}{2} x_2\right) - f(0) \right] \\
&= 2 \cdot \left[\tfrac{1}{2} f(x_1) + \tfrac{1}{2} f(x_2) - f(0) \right] \\
&= f(x_1) + f(x_2) - 2 f(0).
\end{aligned}
$$

$$
\begin{aligned}
g(x_1) + g(x_2) &= f(x_1) - f(0) + f(x_2) - f(0) \\
&= f(x_1) + f(x_2) - 2 f(0) \\
&= g(x_1 + x_2)
\end{aligned}
$$

$\therefore$ $g(x)$ is a linear function.

$\therefore$ $g(x) = A_1 x$ where A_1 is unique, $= A$.

$g(x) = f(x) - f(0)$

$f(x) = g(x) + f(0)$.

$\therefore$ $f(x) = Ax + f(0)$.

$b_1 = f(0)$ $\because$ $f(0)$ is unique, so b_1 is unique

$\therefore$ $f(x) = Ax + b$. for any affine function.

2. **Problem 2: Linear Maps and Differentiation of polynomials.** Let $\mathcal{P}_n$ be the vector space consisting of all polynomials of degree $\leq n$ with real coefficients.

(a) Consider the transformation $T : \mathcal{P}_n \to \mathcal{P}_n$ defined by

$$T(p(x)) = \frac{dp(x)}{dx}.$$

For example, $T(1 + 3x + x^2) = 3 + 2x$. Show that T is linear.

(b) Using $\{1, x, \ldots, x^n\}$ as a basis, represent the transformation in part (a) by a matrix $\mathbf{A} \in \mathbb{R}^{(n+1)\times(n+1)}$. Find the rank of $\mathbf{A}$.

(a) $u_1(x) \in P_n \quad u_2(x) \in P_n$

$$T(u_1(x) + u_2(x)) = \frac{d(u_1(x) + u_2(x))}{dx}$$

$\because$ differetiation is linear

$$\therefore \frac{d(u_1(x) + u_2(x))}{dx} = \frac{du_1(x) + du_2(x)}{dx} = \frac{du_1(x)}{dx} + \frac{du_2(x)}{dx}$$

$$T(u_1x) + Tu_2(x) = \frac{du_1(x)}{dx} + \frac{du_2(x)}{dx} = T(u_1(x) + u_2(x))$$

$\therefore$ $T: P_n \to P_n$ enclosed addition.

$$T(\alpha u_1(x)) = \frac{d\alpha u_1(x)}{dx} = \frac{\alpha d(u_1(x))}{dx} = \alpha \frac{d(u_1(x))}{dx} = \alpha T(u_1(x))$$

$\therefore$ $T: P_n \to P_n$ enclosed multiplication.

$\therefore$ T is a linear map

(b), let $\{1, \cdots, x^n\}$ is a basis of P_n

$u(x) \in P_n$

$$u(x) = \sum_{i=0}^{n} a_i x^i$$

where $a = [a_0 \sim a_n]^T$

$$T(u(x)) = \frac{d\,u(x)}{dx} = \sum_{i=0}^{n} \beta_i x^i$$

$$\beta = [0, a_0, \cdots, na_n]^T$$

$\beta = A\alpha.$

so A has 0s in column 1 so $\dim(N(A)) = 1$.

Therefore $\text{Rank}(A) + \dim(N(A)) = n+1$

$\text{Rank}(A) + 1 = n+1$

$\text{Rank}(A) = n$.

3. **Problem 3: Matrix Rank Inequalities.**
Show the following identities about rank.

(a) If $\mathbf{A} \in \mathbb{F}^{m \times n}$,$\mathbf{B} \in \mathbb{F}^{n \times k}$ then

$$\operatorname{rank}(\mathbf{B}) \leq \operatorname{rank}(\mathbf{AB}) + \dim(\operatorname{null}(\mathbf{A}))$$

(b) If $\mathbf{A} \in \mathbb{F}^{m \times n}$,$\mathbf{B} \in \mathbb{F}^{m \times n}$ then

$$\operatorname{rank}(\mathbf{A}+\mathbf{B}) \leq \operatorname{rank}(\mathbf{A}) + \operatorname{rank}(\mathbf{B})$$

(c) Suppose $\mathbf{A}, \mathbf{B} \in \mathbb{F}^{m \times m}$. Then show that if $\mathbf{AB} = \mathbf{0}$ then

$$\operatorname{rank}(\mathbf{A}) + \operatorname{rank}(\mathbf{B}) \leq m$$

(d) Suppose $\mathbf{A} \in \mathbb{F}^{m \times m}$. Then show $\mathbf{A}^2 = \mathbf{A}$ if and only if

$$\operatorname{rank}(\mathbf{A}) + \operatorname{rank}(\mathbf{A}-\mathbf{I}) = m$$

where $\mathbf{I} \in \mathbb{F}^{m \times m}$ is the identity matrix.

(a) $AB \in \mathbb{F}^{m \times k}$.

$\dim(N(A)) = \dim(A) - \operatorname{rank}(A). = n - \operatorname{rank}(A) \leq n.$

$\operatorname{rank}(AB) \leq \min(m, k).$

$\operatorname{rank}(AB) + \dim(N(A)) \leq \text{minium}(m, k) + n$

$\leq \min(m+n, k+n)$

$\operatorname{rank}(B) \leq \min(n, k).$ $\because$ $n < m+n$, $k < k+n$

$\therefore \operatorname{rank}(B) \leq \min(m+n, k+n)$

$\leq \operatorname{rank}(AB) + \dim(N(A)).$

(b). $A+B \in \mathbb{F}^{mn}$.

$\therefore \operatorname{rank}(A+B) \leq \min(m, n)$

$\operatorname{rank}(A) \leq \min(m, n)$
$\operatorname{rank}(B) \leq \min(m, n).$

$\therefore$ $\operatorname{rank}(A) + \operatorname{rank}(B) \geq \min(m, n)$

$\therefore$ $\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$

(c). $\text{rank}(A) + \dim(N(A)) = m$.

$\text{rank}(B) + \dim(N(B)) = m$.

$N(B) = \{x \in \mathbb{F}^m, Bx = 0\}$.

$\because AB = 0$,

$\therefore \text{rank}(A) \leq \dim(N(B))$,

$\therefore \text{rank}(A) + \text{rank}(B) \leq m$.

(d), $A^2 = A$

$A^2 - A = 0$.

$A \cdot A - A = 0$.

$A \cdot (A - I) = 0$.

$\text{rank}(A) + \dim(N(A)) = m$.

If $A = 0$. $\text{rank}(A) = 0$ $\quad A - I = -I \quad \text{rank}(-I) = m$

If $A - I = 0$ $\quad A = I \quad \text{rank}(I) = m$

$\therefore \text{rank}(A) + \text{rank}(A - I) = m$.

4. **Problem 4: Solution of Linear System of Equations.** Consider the system of linear equations

$$\mathbf{y} = \mathbf{A}\mathbf{B}\mathbf{x}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times m}$, $m \le n$. For each of the following cases, find conditions (in terms of null spaces and range spaces of $\mathbf{A}$ and $\mathbf{B}$) under which there can be a unique solution, no solution, or infinite number of solutions.

(a) $\text{rank}(\mathbf{A}) = n$, and $\text{rank}(\mathbf{B}) = m$.

(b) $\text{rank}(\mathbf{A}) = n$, and $\text{rank}(\mathbf{B}) < m$.

(c) $\text{rank}(\mathbf{A}) < n$, and $\text{rank}(\mathbf{B}) = m$.

(a). $C = AB$

$y = Cx$

$$C = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & & \vdots \\ a_{31} & & \vdots \\ \vdots & & \vdots \\ a_{n1} & & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$\begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ \vdots \\ a_{n2} \end{bmatrix} x_2 + \cdots + \begin{bmatrix} a_{1m} \\ \vdots \\ a_{nm} \end{bmatrix} x_m = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = Y \in \mathbb{R}^n$$

① For unique solution, we need $x_1 \sim x_m$ has unique value to map unique vector $y_1 \sim y_n$

let $\{a_1, a_2, \cdots a_m\}$ is a basis of $\mathbb{R}^m$,

then for every y in $\mathbb{R}^m$, we can have $y = a_1x_1 + a_2x_2 \cdots a_mx_m$.

Thus matrix C should include a basis of $\mathbb{R}^n$, for a unique solution

So the $\text{Rank}(C) = m$, $\dim(N(C)) = 0$ $\qquad Y = \{y_1, y_2, \cdots, y_{n-m}, \cdots y_n\}$.

$\text{Rank}(A) = n$ $\qquad \text{Rank}(B) = m$

$\dim(R(A)) = n$ $\qquad \dim(R(B)) = m$

$\dim(N(A)) = 0$ $\qquad \dim(N(B)) = 0$.

Thus, unique solution need to depend on y_{n-m} which has no relation to $x_1 \sim x_m$

y_{n-m} need to be same value for a y in $y_{1 \sim m}$, such that $x_1 \sim x_m$ has a linear combination for $y_{n-m} \sim y_n$.

②. Because the $R(A)$, $N(A)$, $R(B)$, $N(B)$ Are stay same at (a) condition,

No solution For a no solution system, $y_{n-m} \sim y_n$ needs to be different $\mathbb{R}$, such that $x_1 \sim x_m$ has no linear combination to fit $y_{n-m} \sim y_n$.

③ Infinite solution. $R(A)$, $N(A)$, $R(B)$, $N(B)$ stay same, and $N(AB)=0$, so there is no situation for an infinite solution. Because there need be a linearly dependent $a_1 \sim a_n$ to have a infinite solution.

(b). $\text{Rank}(A)=n$ $\quad \text{Rank}(B)<m.$ $\quad \text{Rank}(AB)<m.$

① For unique solution, let $\{a_1, a_2, \cdots a_n\} \subset \mathbb{R}^m$ so $\{a_1 \sim a_n\}$ is no a basis of $\mathbb{R}^m$, because $\text{Rank}(B)<m$, s.t. y can not represent by $a_1x_1 \sim a_nx_n$

So there is no situation of unique solution.

② For no solution, $\text{Rank}(B)=0$ $\quad \text{Rank}(AB)=0$ $\quad N(AB)=m$., such that
$\dim(R(B))=0$ $\quad \dim(N(B))=m$
$ABx=0$ $\forall x \in \mathbb{R}^m$, but y can be any other $\mathbb{R} \neq 0$, so, system has no equation.

③ For infinite solution, $\text{Rank}(B)=m-p$ $(p \geq 1)$ $\quad N(AB) \geq 1$, such that.

$ABx=0$ in some situation, if $y=0$ in that situation, x can be any $\mathbb{R}$

so system has infinite solution when $\dim(R(B))=m-p$ $(1 \leq p < m)$

c.) $\text{Rank}(A)<n$ $\quad \text{Rank}(B)=m$ $\quad \text{Rank}(AB)=\min(n,m)$.

① unique solution, $\text{Rank}(AB)=m$, $\dim(R(A))=k$ $(m \leq k < n)$ $\dim(R(B))=m$
$\dim(N(A))=n-k$ $\quad \dim(N(B))=0$
such that, it is the same situation as (a)①. where there is a basis $\{a_1 \sim a_m\} \in \mathbb{R}^n$ for x can present every $y \in \mathbb{R}^m$.

②, no solution, $\text{Rank}(AB)=0$ $\quad \text{Rank}(A)=0$ $\dim(R(A))=0$ $\dim(N(A))=n$.
$\text{Rank}(B)=m$ $\dim(R(B))=m$ $\dim(N(B))=0$.

s.t. $ABx=0$, $y \neq 0$ for some y, and system has no solution.

③ infinite solution, $1 \leq \text{Rank}(AB) < m$. $1 \leq \text{Rank}(A) < m$ $1 \leq \dim(R(A)) < m$, $\dim(N(A)) > 1$
$\text{Rank}(B)=m$, $\dim(R(B))=m$, $\dim(N(B))=0$.

s.t. $ABx=0$ in $1 \sim m-1$ situations, if $y=0$ in that situation, system has no solution.

5. **Problem 5: Infinite Dimensional Vector Spaces.** Recall that $C^0([0,1])$ is defined as the set of all continuous functions $f : [0,1] \to \mathbb{R}$, is a vector space over $\mathbb{R}$. Let $S = \{1, (x+1), (x+2)^2, (x+3)^3, \ldots, (x+i)^i, \ldots\}$.

   (a) Is there a vector in S which can be represented as a finite linear combination of other vectors in S?

   (b) Can any vector in $C^0([0,1])$ be represented as a finite linear combination of vectors in S?

   [Finite linear combination is a linear combination with finite number of terms]

(a). let a vector $(x+k)^k \in S$.

Suppose $(x+k)^k = \alpha_1 + \alpha_2 (x+2)^2 + \cdots + \alpha_{k-1} (x+k-1)^{k-1}$

by binomial thereom $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

$$(x+k)^k = \sum_{i=0}^{k} \binom{k}{i} x^i k^{k-i}$$

in combination $\alpha_1 + \alpha_2 (x+2)^2 + , \cdots + \alpha_{k-1} (x+k-1)^{k-1}$

there is not a element that can contains. kx^k

therefore, there is no vector in S can be represented as a finite combination of other vectors

(b). Suppose we got function 2^x, if we want binomial function to represent 2^x, we need they have same derivative after they got derivative to 0, however, $\frac{d\,2^x}{dx} = 2^x \ln 2$ the deriviate of 2^x never goes to 0, and $\frac{d(\alpha_1 + \alpha_2(x+1) \cdots \alpha_n(x+n)^n}{dx}$ will ultimately goes to 0 at end.

therefore, finite binomial function, could not represent non-polinomial function.