ECE 269: Linear Algebra and Applications
Fall 2021

Homework # 2
Due: Tuesday, October 26 11:59pm,
via Gradescope

Collaboration Policy: This homework set allows limited collaboration. You are expected to try to solve the problems on your own. You may discuss a problem with other students to clarify any doubts, but you must fully understand the solution that you turn in and write it up entirely on your own. Blindly copying results from any resource (such as your friend, or the internet) will be considered a violation of academic integrity. *

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1. **Problem 1: Affine functions.** A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *affine* if for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and any $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta = 1$, we have

$$f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}).$$

Note that without the restriction $\alpha + \beta = 1$, this would be the definition of linearity.

(a) Suppose that $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Show that the function $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ is affine.

(b) Prove the converse, namely, show that any affine function f can be represented uniquely as $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ for some $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$.

2. **Problem 2: Linear Maps and Differentiation of polynomials.** Let $\mathcal{P}_n$ be the vector space consisting of all polynomials of degree $\leq n$ with real coefficients.

(a) Consider the transformation $T : \mathcal{P}_n \to \mathcal{P}_n$ defined by

$$T(p(x)) = \frac{dp(x)}{dx}.$$

For example, $T(1 + 3x + x^2) = 3 + 2x$. Show that T is linear.

(b) Using $\{1, x, \dots, x^n\}$ as a basis, represent the transformation in part (a) by a matrix $\mathbf{A} \in \mathbb{R}^{(n+1)\times(n+1)}$. Find the rank of $\mathbf{A}$.

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3. **Problem 3: Matrix Rank Inequalities.**
   Show the following identities about rank.

   (a) If $\mathbf{A} \in \mathbb{F}^{m\times n}$,$\mathbf{B} \in \mathbb{F}^{n\times k}$ then

   $$\text{rank}(\mathbf{B}) \leq \text{rank}(\mathbf{AB}) + \dim(\text{null}(\mathbf{A}))$$

   (b) If $\mathbf{A} \in \mathbb{F}^{m\times n}$,$\mathbf{B} \in \mathbb{F}^{m\times n}$ then

   $$\text{rank}(\mathbf{A}+\mathbf{B}) \leq \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B})$$

   (c) Suppose $\mathbf{A}, \mathbf{B} \in \mathbb{F}^{m\times m}$. Then show that if $\mathbf{AB} = \mathbf{0}$ then

   $$\text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B}) \leq m$$

   (d) Suppose $\mathbf{A} \in \mathbb{F}^{m\times m}$. Then show $\mathbf{A}^2 = \mathbf{A}$ if and only if

   $$\text{rank}(\mathbf{A}) + \text{rank}(\mathbf{A}-\mathbf{I}) = m$$

   where $\mathbf{I} \in \mathbb{F}^{m\times m}$ is the identity matrix.

4. **Problem 4: Solution of Linear System of Equations.** Consider the system of linear equations

   $$\mathbf{y} = \mathbf{ABx}$$

   where $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times m}$, $m \leq n$. For each of the following cases, find conditions (in terms of null spaces and range spaces of $\mathbf{A}$ and $\mathbf{B}$) under which there can be a unique solution, no solution, or infinite number of solutions.

   (a) rank($\mathbf{A}$) $=$ n, and rank($\mathbf{B}$) $=$ m.

   (b) rank($\mathbf{A}$) $=$ n, and rank($\mathbf{B}$) $<$ m.

   (c) rank($\mathbf{A}$) $<$ n, and rank($\mathbf{B}$) $=$ m.

5. **Problem 5: Infinite Dimensional Vector Spaces.** Recall that $C^0([0,1])$ is defined as the set of all continuous functions $f : [0,1] \to \mathbb{R}$, is a vector space over $\mathbb{R}$. Let $S = \{1, (x+1), (x+2)^2, (x+3)^3, \ldots, (x+i)^i, \ldots\}$.

   (a) Is there a vector in S which can be represented as a finite linear combination of other vectors in S?

   (b) Can any vector in $C^0([0,1])$ be represented as a finite linear combination of vectors in S?

   [Finite linear combination is a linear combination with finite number of terms]