ECE269: Linear Algebra and Applications
Fall 2021
Homework # 2 Solutions

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1. **Affine Functions.** A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *affine* if for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and any $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta = 1$, we have

$$f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y})$$

Note that without the restriction $\alpha + \beta = 1$, this would be the definition of linearity.

(a) Suppose that $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Show that the function $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ is affine.

(b) Prove the converse, namely, show that any affine function f can be represented uniquely as $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ for some $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$.

Solution:

(a) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta = 1$

$$\begin{aligned}
f(\alpha\mathbf{x} + \beta\mathbf{y}) &= \mathbf{A}(\alpha\mathbf{x} + \beta\mathbf{y}) + \mathbf{b} \\
&= \mathbf{A}(\alpha\mathbf{x}) + \mathbf{A}(\beta\mathbf{y}) + (\alpha + \beta)\mathbf{b} \\
&= \alpha\mathbf{A}\mathbf{x} + \beta\mathbf{A}\mathbf{y} + \alpha\mathbf{b} + \beta\mathbf{b} \\
&= \alpha(\mathbf{A}\mathbf{x} + \mathbf{b}) + \beta(\mathbf{A}\mathbf{y} + \mathbf{b}) \\
&= \alpha f(\mathbf{x}) + \beta f(\mathbf{y})
\end{aligned}$$

(b) Define $g(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{0})$. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$

$$\begin{aligned}
g(\alpha\mathbf{x}) &= f(\alpha\mathbf{x}) - f(\mathbf{0}) \\
&= f(\alpha\mathbf{x} + (1-\alpha)\mathbf{0}) - f(\mathbf{0}) \\
&= \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{0}) - f(\mathbf{0}) \\
&= \alpha(f(\mathbf{x}) - f(\mathbf{0})) \\
&= \alpha g(\mathbf{x}) \\
g(\mathbf{x} + \mathbf{y}) &= g\left(2\frac{\mathbf{x} + \mathbf{y}}{2}\right) \\
&= 2g\left(\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}\right) \qquad [\because g(\alpha\mathbf{x}) = \alpha g(\mathbf{x})] \\
&= 2\left(f\left(\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}\right) - f(\mathbf{0})\right) \\
&= 2\left(\frac{1}{2}f(\mathbf{x}) + \frac{1}{2}f(\mathbf{y}) - f(\mathbf{0})\right) \\
&= (f(\mathbf{x}) - f(\mathbf{0})) + (f(\mathbf{y}) - f(\mathbf{0})) \\
&= g(\mathbf{x}) + g(\mathbf{y})
\end{aligned}$$

Hence $g : \mathbb{R}^n \to \mathbb{R}^m$ is a linear function. Hence $g(\mathbf{x})$ can be uniquely written as $\mathbf{A}\mathbf{x}$ for some $\mathbf{A} \in \mathbb{R}^{m\times n}$. With $\mathbf{b} = f(\mathbf{0})$, $f(\mathbf{x}) = g(\mathbf{x}) + f(\mathbf{0}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ for some unique $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$.

2. Linear Maps and Differentiation of polynomials. Let $\mathcal{P}_n$ be the vector space consisting of all polynomial of degree $\leq n$ with real coefficients.

(a) Consider the transformation $T : \mathcal{P}_n \to \mathcal{P}_n$ defined by

$$T(p(x)) = \frac{dp(x)}{dx}.$$

For example, $T(1 + 3x + x^2) = 3 + 2x$. Show that T is linear.

(b) Using $\{1, x, \ldots, x^n\}$ as a basis, represent the transformation in part (a) by a matrix $\mathbf{A} \in \mathbb{R}^{(n+1)\times(n+1)}$. Find the rank of $\mathbf{A}$.

Solution:

(a) Let $p(x), q(x) \in \mathcal{P}_n$ and $\alpha, \beta \in \mathbb{R}$

$$\begin{aligned} T(\alpha p(x) + \beta q(x)) &= \frac{d}{dx}(\alpha p(x) + \beta q(x)) \\ &= \frac{d}{dx}(\alpha p(x)) + \frac{d}{dx}(\beta q(x)) \\ &= \alpha \frac{dp(x)}{dx} + \beta \frac{dq(x)}{dx} \\ &= \alpha T(p(x)) + \beta T(q(x)) \end{aligned}$$

Hence T is a linear function.

(b) Any vector $p(x)$ in $\mathcal{P}_n$ can be represented as $\sum_{i=0}^{n} \alpha_i x^i$ for $\alpha_i \in \mathbb{R}$ for $0 \leq i \leq n$. A linear map is completely characterized by its action on the basis vectors and can be represented as a matrix for the basis.

$T(1) = 0, T(x^i) = ix^{i-1}$ for all $1 \leq i \leq n$. Hence

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 2 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & n \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}_{(n+1)\times(n+1)}$$

The last n columns of $\mathbf{A}$ denoted $c_1, c_2, \ldots c_n$ are linearly independent as $c_i = i * e_i$ where e_i are the vectors of the standard basis. As column c_0 is the zero vector, the number of linearly independent columns of $\mathbf{A}$ is n. Hence the rank of $\mathbf{A}$ is n.

3. Matrix Rank. Show the following identities about rank.

(a) If $\mathbf{A} \in \mathbb{F}^{m\times n}$,$\mathbf{B} \in \mathbb{F}^{n\times k}$, then

$$\text{rank}(\mathbf{B}) \leq \text{rank}(\mathbf{AB}) + \dim(\mathcal{N}(\mathbf{A}))$$

(b) If $\mathbf{A} \in \mathbb{F}^{m\times n}$,$\mathbf{B} \in \mathbb{F}^{m\times n}$, then

$$\text{rank}(\mathbf{A} + \mathbf{B}) \leq \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B})$$

(c) Suppose $\mathbf{A}, \mathbf{B} \in \mathbb{F}^{m\times m}$. Then, show that if $\mathbf{AB} = \mathbf{0}$, then

$$\text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B}) \leq m$$

(d) Suppose $\mathbf{A} \in \mathbb{F}^{m\times m}$. Then, show that $\mathbf{A}^2 = \mathbf{A}$ if and only if

$$\text{rank}(\mathbf{A}) + \text{rank}(\mathbf{A} - \mathbf{I}) = m$$

Solution:

(a) $\mathbf{A} \in \mathbb{F}^{m\times n}, \mathbf{B} \in \mathbb{F}^{n\times k}$
The range space of $\mathbf{B}$ denoted $\mathcal{R}(\mathbf{B})$ is a subspace of $\mathbb{F}^n$.
Restrict the linear map $\mathbf{A}$ to $\mathcal{R}(\mathbf{B})$ i.e. consider new linear map $f : \mathcal{R}(\mathbf{B}) \to \mathbb{F}^m$ given by $f(\mathbf{x}) = \mathbf{Ax}$ for all vectors $\mathbf{x}$ in $\mathcal{R}(\mathbf{B})$.

Domain of f is $\mathcal{R}(\mathbf{B})$ and hence dimension of domain is rank($\mathbf{B}$).

$$\mathcal{R}(f) = \{\mathbf{Ay} : \mathbf{y} \in \mathcal{R}(\mathbf{B})\} = \{\mathbf{ABx} : \mathbf{x} \in \mathbb{F}^k\} = \mathcal{R}(\mathbf{AB})$$

$$\mathcal{N}(f) = \{\mathbf{y} \in \mathcal{R}(\mathbf{B}) : \mathbf{Ay} = 0\} = \mathcal{N}(\mathbf{A}) \cap \mathcal{R}(\mathbf{B})$$

We can apply Rank-Nullity Theorem on this new map f.

$$\begin{aligned}\text{rank}(\mathbf{B}) &= \dim(\mathcal{R}(f)) + \dim(\mathcal{N}(f))\\ &= \text{rank}(\mathbf{AB}) + \dim(\mathcal{N}(\mathbf{A}) \cap \mathcal{R}(\mathbf{B}))\\ &\leq \text{rank}(\mathbf{AB}) + \dim(\mathcal{N}(\mathbf{A}))\end{aligned}$$

(b) $\mathbf{A} \in \mathbb{F}^{m\times n}, \mathbf{B} \in \mathbb{F}^{m\times n}$

$$\begin{aligned}\mathcal{R}(\mathbf{A} + \mathbf{B}) &= \{(\mathbf{A} + \mathbf{B})\mathbf{x} : \mathbf{x} \in \mathbb{F}^n\}\\ &= \{\mathbf{Ax} + \mathbf{Bx} : \mathbf{x} \in \mathbb{F}^n\}\\ &\subseteq \{\mathbf{Ax} : \mathbf{x} \in \mathbb{F}^n\} + \{\mathbf{Bx} : \mathbf{x} \in \mathbb{F}^n\}\\ &= \mathcal{R}(\mathbf{A}) + \mathcal{R}(\mathbf{B})\end{aligned}$$

Hence rank($\mathbf{A} + \mathbf{B}$) $\leq \dim(\mathcal{R}(\mathbf{A}) + \mathcal{R}(\mathbf{B}))$

Let $\mathcal{B}_1$, $\mathcal{B}_2$ be basis for $\mathcal{R}(\mathbf{A})$ and $\mathcal{R}(\mathbf{B})$ respectively. Then $\mathcal{B}_1 \cup \mathcal{B}_2$ spans $\mathcal{R}(\mathbf{A}) + \mathcal{R}(\mathbf{B})$. Hence $\dim(\mathcal{R}(\mathbf{A}) + \mathcal{R}(\mathbf{B})) \leq \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B})$.

Hence rank($\mathbf{A} + \mathbf{B}$) $\leq$ rank($\mathbf{A}$) + rank($\mathbf{B}$)

(c) Let $\mathbf{A}, \mathbf{B} \in \mathbb{F}^{m\times m}$. From part (a),

$$\begin{aligned}\text{rank}(\mathbf{B}) &\leq \text{rank}(\mathbf{AB}) + \dim(\mathcal{N}(\mathbf{A}))\\ &= \dim(\mathcal{N}(\mathbf{A}))\\ &= m - \text{rank}(\mathbf{A}) \quad [\because \text{Rank Nullity for } \mathbf{A}]\end{aligned}$$

Hence $\text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B}) \leq m$

(d) Let $\mathbf{A} \in \mathbb{F}^{m\times m}$
Need to show $\mathbf{A}^2 = \mathbf{A} \iff \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{A} - \mathbf{I}) = m$

($\implies$) Given $\mathbf{A}^2 = \mathbf{A}$
From part (b),

$$m = \text{rank}(\mathbf{I}) = \text{rank}(\mathbf{A} + (\mathbf{I} - \mathbf{A})) \leq \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{I} - \mathbf{A})$$

giving $\text{rank}(\mathbf{A}) + \text{rank}(\mathbf{I} - \mathbf{A}) \geq m$.
From part (c),

$$\text{rank}(\mathbf{A}) + \text{rank}(\mathbf{I} - \mathbf{A}) \leq m$$

Hence $m = \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{I} - \mathbf{A})$.
As $\text{rank}(\mathbf{I} - \mathbf{A}) = \text{rank}(\mathbf{A} - \mathbf{I})$, we get $m = \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{A} - \mathbf{I})$

($\impliedby$) Given $\text{rank}(\mathbf{A}) + \text{rank}(\mathbf{A} - \mathbf{I}) = m$
Let $\mathbf{x} \in \mathcal{N}(\mathbf{A})$, then $(\mathbf{A} - \mathbf{I})(-\mathbf{x}) = -\mathbf{A}\mathbf{x} + \mathbf{I}\mathbf{x} = \mathbf{x}$.
Hence $\mathcal{N}(\mathbf{A}) \subseteq \mathcal{R}(\mathbf{A} - \mathbf{I})$.
As $\dim(\mathcal{N}(\mathbf{A})) = m - \text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A} - \mathbf{I})$ we get $\mathcal{N}(\mathbf{A}) = \mathcal{R}(\mathbf{A} - \mathbf{I})$.

$$\mathcal{R}(\mathbf{A}(\mathbf{A} - \mathbf{I})) = \{\mathbf{A}(\mathbf{A} - \mathbf{I})\mathbf{x} : \mathbf{x} \in \mathbb{F}^m\} = \{\mathbf{A}\mathbf{y} : \mathbf{y} \in \mathcal{R}(\mathbf{A} - \mathbf{I})\} = \{0\}$$

Hence $\mathbf{A}(\mathbf{A} - \mathbf{I}) = \mathbf{0}$ i.e. $\mathbf{A}^2 = \mathbf{A}$.

4. Solution of Linear System of Equations. Consider the system of linear equations

$$\mathbf{y} = \mathbf{ABx}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times m}$, $m \leq n$. For each of the following cases, find conditions (in terms of null spaces and range spaces of $\mathbf{A}$ and $\mathbf{B}$) under which there can be a unique solution, no solution, or infinite number of solutions.

(a) $\text{rank}(\mathbf{A}) = n$, and $\text{rank}(\mathbf{B}) = m$.

(b) $\text{rank}(\mathbf{A}) = n$, and $\text{rank}(\mathbf{B}) < m$.

(c) $\text{rank}(\mathbf{A}) < n$, and $\text{rank}(\mathbf{B}) = m$.

Solution: Note that $\mathbf{AB}$ is a n by m matrix where $m \leq n$. So $R(\mathbf{AB})$ may not span the entire $\mathbb{R}^n$ where $\mathbf{y}$ lives. Hence,there may exist a solution or no solution according to given $\mathbf{y}$. There exists a solution if $y \in \mathbf{AB}$ and does not exist if $y \notin \mathbf{AB}$. In the case where a solution exists, we need to check $N(\mathbf{AB})$. If $N(\mathbf{AB}) = \{\mathbf{0}\}$, then we have a unique solution. Otherwise we have infinite number solutions. We need to check $N(\mathbf{AB})$ in each case.

(a)

$$\mathbf{ABx} = \mathbf{0} \Rightarrow$$

Since $rank(\mathbf{A}) = n, N(\mathbf{A}) = \{\mathbf{0}\}$,

$$\mathbf{Bx} = \mathbf{0} \Rightarrow$$

Since $rank(\mathbf{B}) = m, N(\mathbf{B}) = \{\mathbf{0}\}$,

$$\mathbf{x} = \mathbf{0} \Rightarrow$$

$$N(\mathbf{AB}) = \{\mathbf{0}\}$$

Hence

$$if\ y \notin R(\mathbf{AB}) \Rightarrow there\ exists\ no\ solution$$

$$if\ y \in R(\mathbf{AB}) \Rightarrow there\ exists\ a\ unique\ solution$$

(b)

$$\mathbf{ABx} = \mathbf{0} \Rightarrow$$

Since $rank(\mathbf{A}) = n, N(\mathbf{A}) = \{\mathbf{0}\}$,

$$\mathbf{Bx} = \mathbf{0} \Rightarrow$$

Since $rank(\mathbf{B}) < m, dim(N(\mathbf{B})) > m - m = 0$,

$$\exists \mathbf{x} \neq \mathbf{0}\ such\ that\ \mathbf{ABx} = \mathbf{0} \Rightarrow$$

$$N(\mathbf{AB}) \neq \{\mathbf{0}\}$$

Hence

$$if\ y \notin R(\mathbf{AB}) \Rightarrow there\ exists\ no\ solution$$

$$if\ y \in R(\mathbf{AB}) \Rightarrow there\ exists\ infinite\ solutions$$

(c) If $\mathbf{Bx} = \mathbf{0}$ then $\mathbf{x} = \mathbf{0}$. However, since $dim(N(\mathbf{A})) > n - n = 0$, there exists $\mathbf{z} \neq \mathbf{0}$ such that $\mathbf{Az} = \mathbf{0}$. This implies $N(\mathbf{AB}) \neq \{\mathbf{0}\}$ if and only if there exists $\mathbf{x}$ such that $\mathbf{Bx} = \mathbf{z}$ for $\mathbf{z} \in N(\mathbf{A})$. In other words, this means

$$N(\mathbf{AB}) \neq \{\mathbf{0}\} \Leftrightarrow N(\mathbf{A}) \cap R(\mathbf{B}) \neq \{\mathbf{0}\}$$

Note that $\mathbf{0}$ is always in $N(\mathbf{A}) \cap R(\mathbf{B})$ Hence we have

$$if\ y \notin R(\mathbf{AB}) \Rightarrow\ no\ solution$$

$$if\ y \in R(\mathbf{AB})\ and\ N(\mathbf{A}) \cap R(\mathbf{B}) = \{\mathbf{0}\} \Rightarrow\ unique\ solution$$

$$if\ y \in R(\mathbf{AB})\ and\ N(\mathbf{A}) \cap R(\mathbf{B}) \neq \{\mathbf{0}\} \Rightarrow\ infinite\ solutions$$

5. Infinite Dimensional Vector Spaces. $C^0([0,1])$, the set of all continuous functions $f : [0,1] \to \mathbb{R}$ is a vector space over $\mathbb{R}$. Let $S = \{1, (x+1), (x+2)^2, (x+3)^3, \dots, (x+i)^i, \dots\}$.

(a) Is there a vector in S which can be represented as a finite linear combination of other vectors in S?

(b) Can every vector in $C^0([0,1])$ be represented as a finite linear combination of vectors in S?

Solution:

(a) Let $(x+i_0)^{i_0} = \sum_{j=1}^{k} \alpha_j (x+i_j)^{i_j}$ for some non zero α_j and $i_1 < i_2 < \cdots < i_k$ and $i_0 \neq i_j$ for $1 \leq j \leq k$

Suppose $i_0 < i_k$, we can rewrite this as

$$(x+i_k)^{i_k} = \alpha_k^{-1}(x+i_0)^{i_0} + \sum_{j=1}^{k-1} -\alpha_k^{-1}\alpha_j (x+i_j)^{i_j}$$

resulting in the LHS having the highest degree vector.

Hence wlog, $i_0 > i_k$. Then LHS has a term x^{i_0} which does not occur on the RHS as $i_j < i_0$ for all $1 \leq j \leq k$. As the coefficient of x^{i_0} on LHS is 1 and on the RHS is 0, this leads to a contradiction. Hence no vector in S can be written as a finite linear combination of the other vectors in S.

(b) Finite linear combinations of vectors in S always results in polynomials of finite degree. However, $C^0([0,1])$ contains other functions such as $e^x, |x-0.5|$ which are continuous but are not equal to any finite degree polynomial and hence cannot be represented as finite linear combinations of vectors in S.

Suppose $e^x = \sum_{j=1}^{k} \alpha_j (x+i_j)^{i_j}$ for some non zero α_j and $i_1 < i_2 < \cdots < i_k$. Taking the l^{th} derivative with respect to x where $l = i_k + 1$,

$$\frac{d^l}{dx^l} e^x = e^x \neq 0 = \frac{d^l}{dx^l} \sum_{j=1}^{k} \alpha_j (x+i_j)^{i_j}$$

where the last equality follows from taking the fact that the l^{th} derivative of a degree $\leq l-1$ polynomial is 0.

Hence e^x cannot be represented as a finite linear combination of vectors in S.