ECE 269: Linear Algebra and Applications
Fall 2021

Practice Problems

Try to solve the following problems on your own. We will post solutions on Thursday Nov. 18.

1) **Problem 1: Orthogonal Projection Matrices.** Let $\mathcal{M}$ and $\mathcal{N}$ be subspaces of $\mathbb{C}^n$, and consider the associated orthogonal projectors $\mathbf{P}_{\mathcal{M}}$ and $\mathbf{P}_{\mathcal{N}}$.
   a) Prove that $\mathbf{P}_{\mathcal{M}}\mathbf{P}_{\mathcal{N}} = \mathbf{0}$ if and only if $\mathcal{M} \perp \mathcal{N}$.
   b) Is it true that $\mathbf{P}_{\mathcal{M}}\mathbf{P}_{\mathcal{N}} = \mathbf{0}$ if and only if $\mathbf{P}_{\mathcal{N}}\mathbf{P}_{\mathcal{M}} = \mathbf{0}$? Justify
   c) Show $R(\mathbf{P}_{\mathcal{M}} + \mathbf{P}_{\mathcal{N}}) = R(\mathbf{P}_{\mathcal{M}}) + R(\mathbf{P}_{\mathcal{N}})$

2) **Problem 2: Orthonormal Basis Expansion and Parseval's Theorem.** Suppose we are given a set of orthonormal basis vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N\}$ of an inner product vector space $\mathcal{U}$
   a) Let $\mathbf{x} \in \mathcal{U}$, we can find a unique representation of $\mathbf{x} = \sum_{i=1}^{N} \alpha_i \mathbf{u}_i$. Prove

$$||\mathbf{x}||_2^2 = \sum_{i=1}^{N} |\alpha_i|^2 \tag{1}$$

   (Note: This is known as Parseval's identity)
   b) Suppose you have a subset of orthonormal vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_s\}$ (where $s < N$) from the given basis. Show that any vector $\mathbf{v} \in \mathcal{U}$ satisfies

$$||\mathbf{v}||_2^2 \geq \sum_{i=1}^{s} |\langle \mathbf{v}, \mathbf{u}_i \rangle|^2 \tag{2}$$

3) **Problem 3: Range Space perpendicular to Null Spaces.**
   Let $\mathbf{A} \in \mathbb{C}^{m \times n}$ satisfy $\mathbf{A}^H\mathbf{A} = \mathbf{A}\mathbf{A}^H$. Show that $R(\mathbf{A}) \perp N(\mathbf{A})$, i.e, show that for all $\mathbf{x} \in R(\mathbf{A})$, $\mathbf{y} \in N(\mathbf{A})$, $\mathbf{x}^H\mathbf{y} = 0$

4) **Problem 4: Householder Reflections.** A Householder matrix is defined as

$$\mathbf{Q} = \mathbf{I} - \mathbf{2uu^T}$$

   for a unit vector $\mathbf{u} \in \mathbb{R}^n$
   a) Show that $\mathbf{Q}$ is orthogonal.
   b) Show that $\mathbf{Qu} = -\mathbf{u}$ and that $\mathbf{Qv} = \mathbf{v}$ for every $\mathbf{v} \perp \mathbf{u}$. Thus, the linear transformation $\mathbf{y} = \mathbf{Qx}$ reflects $\mathbf{x}$ through the hyperplane with normal vector $\mathbf{u}$.
   c) Given $\mathbf{y}$, find $\mathbf{x}$ such that $\mathbf{y} = \mathbf{Qx}$.
   d) Given nonzero vectors $\mathbf{x}$ and $\mathbf{y}$, find a unit vector u such that $(\mathbf{I} - \mathbf{2uu^T})\mathbf{x} \in \text{span}(\mathbf{y})$, in terms of $\mathbf{x}$ and $\mathbf{y}$.

5) **Problem 5: System Identification.** Consider a system whose input $x(n)$ and output $y(n)$ are related by:

$$y(n) = \sum_{k=0}^{L-1} h(k)x(n-k), \quad n = 0, 1, 2, \ldots \tag{3}$$

Here $h(n)$ is called the impulse response of the system. Suppose you are given an input signal $\overline{x}(n)$ (non-zero for all n) and are able to observe a noisy version ($\overline{y}(n)$) of the output of the system, contaminated with noise $w(n)$, i.e., you observe

$$\overline{y}(n) = \sum_{k=0}^{L-1} h(k)\overline{x}(n-k) + w(n), \quad n = 0, 1, 2, \ldots \tag{4}$$

Using the idea of orthogonal projection, describe a method to estimate the impulse response $h(n)$ using $\overline{y}(n)$ and $\overline{x}(n)$. In the absence of noise, under what conditions can you exactly identify $h(n)$? Justify your answer.

6) **Problem 6: Variations of Orthogonal Projection**

a) Given $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, consider the following problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} ||\mathbf{x} + \mathbf{c}||_2 \tag{5}$$

$$s.t.\ \mathbf{A}\mathbf{x} = \mathbf{b} \tag{6}$$

Cast it as an orthogonal projection problem. Identify the subspace you are projecting on? What is the point being projected?

b) Given $\mathbf{x}_0 \in \mathbb{R}^n$, $\mathbf{a} \in \mathbb{R}^n, b \in \mathbb{R}$, solve the following problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} ||\mathbf{x}_0 - \mathbf{x}||_2 \tag{7}$$

$$s.t\ \mathbf{a}^T\mathbf{x} = b \tag{8}$$

Derive the solution in closed form.