LEC. 9 : ORTHOGONAL PROJECTION

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AGENDA

ORTHOGONAL PROJECTION: BASICS

- EXISTENCE
- UNIQUENESS
- PROJECTION MATRIX

Orthogonal Projection

Basic Setup

Let $\mathcal{V}$ be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, endowed with an inner product $\langle \cdot, \cdot \rangle$ and let $\| \ \|_{ip}$ be the norm induced by this inner product. $\mathcal{V}$ is finite dimensional.

Let $y \in \mathcal{V}$, and $S \subseteq \mathcal{V}$ be a subspace

$$\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{F} \quad \text{(inner product)}$$

$$\| \cdot \|_{ip} : \mathcal{V} \longrightarrow \mathbb{R}^+ \quad \text{(norm induced by inner product)}$$

$$\| v \|_{ip} = \sqrt{\langle v, v \rangle}$$

S is a subspace of $\mathcal{V}$.

For the remaining lecture, we assume that $\mathcal{V}$ is finite-dimensional:

Given $b \in \mathcal{V}$, solve

$$\min_{\hat{b} \in S} \|b - \hat{b}\|_{ip} \quad \text{——} \quad (OP)$$

LAST TIME: We showed that if there exists a vector $\hat{b}_S \in S$ satisfying

$$\langle b - \hat{b}_S, v \rangle = 0, \quad \forall\, v \in S$$

then $\hat{b}_S$ solves (OP).

THIS TIME: We show that $\hat{b}_S$ indeed exists.

Existence of $\hat{b}_S$

Let $\dim(S)=k$ and $v_1, v_2, \ldots, v_k$ be a basis for S.

$\Leftrightarrow$ $\left\{\begin{array}{l}\text{there exists a vector } \hat{b}_S \in S, \text{ such that } \langle b-\hat{b}_S, v\rangle = 0 \quad \forall\, v \in S\end{array}\right.$

$\Leftrightarrow$ $\left\{\begin{array}{l}\text{there exists a vector } \hat{b}_S \in S, \text{ such that } \begin{array}{c}\langle b-\hat{b}_S, v_1\rangle = 0 \\ \vdots \\ \langle b-\hat{b}_S, v_k\rangle = 0\end{array}\end{array}\right.$

$\Leftrightarrow$ $\left\{\begin{array}{l}\text{there exists a vector } \hat{b}_S \in S, \text{ such that } \begin{array}{c}\langle b, v_1\rangle = \langle \hat{b}_S, v_1\rangle \\ \vdots \\ \langle b, v_k\rangle = \langle \hat{b}_S, v_k\rangle\end{array}\end{array}\right.$

$\Leftrightarrow$ $\left\{\begin{array}{l}\text{there exist } x_1, x_2, \ldots, x_k \in \mathbb{F}, \text{ such that } \hat{b}_S = \sum_{i=1}^{k} v_i x_i \text{ and } \begin{array}{c}\langle b, v_1\rangle = \langle \hat{b}_S, v_1\rangle \\ \vdots \\ \langle b, v_k\rangle = \langle \hat{b}_S, v_k\rangle\end{array}\end{array}\right.$

$\Longleftrightarrow$ there exist $x_1, x_2, \ldots, x_k \in \mathbb{F}$ s.t.

$$\begin{cases} \langle b, v_1 \rangle = \left\langle \sum_{i=1}^{k} x_i v_i, v_1 \right\rangle \\ \langle b, v_2 \rangle = \left\langle \sum_{i=1}^{k} x_i v_i, v_2 \right\rangle \\ \vdots \\ \langle b, v_k \rangle = \left\langle \sum_{i=1}^{k} x_i v_i, v_k \right\rangle \end{cases}$$

$\Longleftrightarrow$ there exist $x_1, x_2, \ldots, x_k \in \mathbb{F}$ s.t.

$$\begin{cases} \langle b, v_1 \rangle = \sum_{i=1}^{k} \langle v_i, v_1 \rangle x_i \\ \vdots \\ \langle b, v_k \rangle = \sum_{i=1}^{k} \langle v_i, v_k \rangle x_i \end{cases}$$

$\Longleftrightarrow$ the following system of equations has a solution :-

$$\boxed{\bar{b} = \bar{A} x}$$

where

$$\bar{b} = \underset{(k \times 1)}{\begin{bmatrix} \langle b, v_1 \rangle \\ \vdots \\ \langle b, v_k \rangle \end{bmatrix}}, \quad \underset{(k \times k)}{\bar{A}} = \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_2, v_1 \rangle & \cdots & \langle v_k, v_1 \rangle \\ \langle v_1, v_2 \rangle & \langle v_2, v_2 \rangle & \cdots & \langle v_k, v_2 \rangle \\ \vdots & & & \vdots \\ \langle v_1, v_k \rangle & \langle v_2, v_k \rangle & \cdots & \langle v_k, v_k \rangle \end{bmatrix}$$

Does $\bar{b} = \bar{A}x$ always have solution?

YES. See Discussion Notes for proof.

SKETCH OF PROOF;

Show that $\text{rank}(\bar{A}) = k$.

In that case, columns of $\bar{A}$ will be a basis for $\mathbb{F}^k$ and any $\bar{b} \in \mathbb{F}^k$ will belong to $R(\bar{A})$.

To show that $\text{rank}(\bar{A}) = k$, show that $N(\bar{A}) = \{0\}$

Proof: Let $z \in N(\bar{A}) \Rightarrow \bar{A}z = 0$

$$\Rightarrow z^H \bar{A} z = 0, \quad z = \begin{bmatrix} z_1 \\ \vdots \\ z_k \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} z_1^* & z_2^* & \cdots & z_k^* \end{bmatrix} \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_2, v_1 \rangle & \cdots & \langle v_k, v_1 \rangle \\ \langle v_1, v_2 \rangle & \langle v_2, v_2 \rangle & \cdots & \langle v_k, v_2 \rangle \\ \vdots & & & \\ \langle v_1, v_k \rangle & \langle v_2, v_k \rangle & \cdots & \langle v_k, v_k \rangle \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_k \end{bmatrix} = 0$$

$$\Leftrightarrow \sum_{j=1}^{k} z_j^* \underbrace{\sum_{i=1}^{k} \langle v_i, v_j \rangle z_i}_{} = 0$$

express this as the $\|u\|_{ip}$ of a suitable vector u, and then argue that $z = 0$

Hence the system of equations

$$\bar{b} = \bar{A}x$$

ALWAYS HAS SOLUTION.

This system of equations are called the NORMAL EQUATIONS.

If x^* solves normal equations, then $\hat{b}_s$ can be found as

$$\boxed{\hat{b}_s = V x^*,} \qquad V = [v_1, v_2 \cdots v_k].$$

$\hat{b}_s$ will henceforth be known as the orthogonal projection of b onto S.

Among all points in S, $\hat{b}_s$ has the shortest distance to b, measured with respect to the norm induced by the inner product.

Is Orthogonal Projection Of A Given Vector Unique?

YES.

Given b,

Proof: Suppose there exist two vectors $\hat{b}_s$ and $\hat{\hat{b}}_s$ in S such that

$$\hat{b}_s \in S, \quad \langle b-\hat{b}_s, v\rangle = 0, \quad \forall\ v \in S$$

AND $$\hat{\hat{b}}_s \in S, \quad \langle b-\hat{\hat{b}}_s, v\rangle = 0, \quad \forall\ v \in S$$

Then, $$\|\hat{b}_s - \hat{\hat{b}}_s\|_{ip}^2 = \langle \hat{b}_s - \hat{\hat{b}}_s, \hat{b}_s - \hat{\hat{b}}_s\rangle$$

$$= \langle \hat{b}_s - b + b - \hat{\hat{b}}_s, \hat{b}_s - \hat{\hat{b}}_s\rangle$$

$$= \langle \hat{b}_s - b, \hat{b}_s - \hat{\hat{b}}_s\rangle + \langle b-\hat{\hat{b}}_s, \hat{b}_s - \hat{\hat{b}}_s\rangle$$

Since $\hat{b}_s, \hat{\hat{b}}_s \in S$ and S is a subspace,

we must have $\hat{b}_s - \hat{\hat{b}}_s \in S$

$$\Rightarrow \langle b-\hat{b}_s, \hat{b}_s - \hat{\hat{b}}_s\rangle = 0$$

$$\Rightarrow \|\hat{b}_s - \hat{\hat{b}}_s\|_{ip}^2 = 0$$

$$\Rightarrow \hat{b}_s = \hat{\hat{b}}_s .$$