Lecture 1: Field & Vector Spaces

Prof. Piya Pal

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AGENDA

1. Fields
   - Axioms
   - Properties
   - Examples.

2. Vector space (over a field)
   - Axioms

FIELDS

Definition: A field is a set F, equipped with two operations $+$ and $\cdot$ (called addition and multiplication, respectively) obeying the following rules or axioms, $\forall\, x, y, z \in F$.

(i) $x+y \in F$, $x \cdot y \in F$

(ii) $x+y = y+x$ (commutativity of addition)

(iii) $(x+y)+z = x+(y+z)$ (associativity of addition)

(iv) There is an element in F, called "<u>zero</u>" (denoted 0) such that $x+0 = x$

(v) For each $x \in F$, there is an element in F, called its "<u>additive inverse</u>" (denoted $-x$), such that

$$x+(-x) = 0$$

(vi) $x \cdot y = y \cdot x$ (commutativity),

(vii) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ (associativity),

(viii) $(x+y) \cdot z = x \cdot z + y \cdot z$ (distributivity)

(ix) There is an element $1 \in F$, $1 \neq 0$, such that $x \cdot 1 = x$

(x) If $x \neq 0$, there is an element in F called <u>multiplicative inverse</u> of x (denoted x^{-1}) such that

$$x \cdot x^{-1} = 1$$

Example of a Field?

$\mathbb{R} = \{$set of all real numbers$\}$

$+$: Regular addition of real numbers

$\cdot$: Regular multiplication of reals.

What is 0? the real zero $0 \in \mathbb{R}$

What is 1? " " number 1, $1 \in \mathbb{R}$

What is additive inverse of $x \in \mathbb{R}$

"negative of x"

What is the multiplicative inverse of $x \in \mathbb{R}$? $(x \neq 0)$ $\dfrac{1}{x}$

Some (obvious?) Properties of Fields

① **0 is unique**: Suppose there are two zeros 0_1 and 0_2 in F.

$0_1 + x = x$, $\forall x \in F$, choose $x = 0_2 \Rightarrow 0_1 + 0_2 = 0_2$

Similarly, $0_2 + x = x$, $\forall x \in F$ choose $x = 0_1$

$\Rightarrow 0_2 + 0_1 = 0_1$

$0_2 + 0_1 = 0_1$ and $0_1 + 0_2 = 0_2$ However $0_1 + 0_2 = 0_2 + 0_1$ (commutativity of addition)

$\Rightarrow \boxed{0_1 = 0_2}$

$\Rightarrow$ There is a unique 0 in F

② **1 is unique**: Follow similar argument as above.

3 Additive inverse of every element is unique

Suppose there are two additive inverses of x, denoted x_1 and x_2. $\Rightarrow$ $x + x_1 = 0$ and $x + x_2 = 0$.

$$(x+x_1) + x_2 = 0 + x_2 = x_2$$

$$x + (x_1 + x_2) = x_2 \quad \text{(associativity)}$$

$$x + (x_2 + x_1) = x_2 \quad \text{(commutativity)}$$

$$\underbrace{(x + x_2)}_{=0} + x_1 = x_2 \quad \text{(associativity)}$$

$$0 + x_1 = x_2 \Rightarrow \boxed{x_1 = x_2}$$

4 Multiplicative inverse of every (non-zero) element is unique:

Similarly establish this fact.

⑤ $a.0 = 0$:-

(Distributivity)

$$a.0 + a.0 \overset{\downarrow}{=} a.(0+0) = a.0$$

$$\Rightarrow a.0 + a.0 = a.0$$

Let b be an additive inverse of $a.0$

(Note that b exists, by axioms of the field)

$$\Rightarrow (a.0 + a.0) + b = \underbrace{a.0 + b} = 0$$

$$\Rightarrow a.0 + (\underbrace{a.0+b}_{=0}) = 0 \overset{=0}{\Rightarrow} a.0 + 0 = 0$$

⑥ $(-1).a = -a$:-

(Formula for computing additive inverse of any field element).

-1 : additive inverse of 1

$-a$: " " of a

Exercise (Do on your own)

Examples of Fields

1. **Smallest Field ?**

Any field F must have a 0 and a 1 $(1 \neq 0)$. Can we define $+$ and $\cdot$ so that the set $\{1, 0\}$ is a field ?

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Additive inverse of 0 is 0
" " " 1 is 1

Multiplicative inverse of 1 is 1.

$\{0, 1\}$ is a field with $+$ and $\cdot$ defined as above.

2. $\mathbb{R}$: Already shown.

(3) $\mathbb{C}$: Set of all complex numbers.

$$\mathbb{C} = \{(x,y),\ x,y \in \mathbb{R}\}.$$

Using standard addition, multiplication of complex numbers, show $\mathbb{C}$ is a field.

$$+ : (x,y) + (p+q) = (x+y,\ p+q)$$

real addition

$$\cdot : (x,y)\cdot(p,q)$$

$$= (x\cdot p - y\cdot q,\ x\cdot q + y\cdot p)$$

0 of $\mathbb{C}$? $(0,0)$

1 of $\mathbb{C}$? $(1,0)$

additive inverse of (x,y)? $(-x,-y)$

multiplicative inverse of (x,y)? DIY.

Vector Spaces

A vector space $\mathcal{V}$, defined over a field F, is a non-empty set (whose members are called vectors) equipped with two operations :-

(i) Vector addition. $+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$

(ii) Scalar Multiplication. $\cdot : F \times \mathcal{V} \to \mathcal{V}$

which must satisfy the following properties

(0) Closure under Vector Addition and Scalar Multiplication:

(follows from the way these operations are defined).

If $u, v \in \mathcal{V}$, then $u+v \in \mathcal{V}$, and $\alpha \in F$, $u \in \mathcal{V}$, $\alpha \cdot u \in \mathcal{V}$

(i) $u + v = v + u, \quad \forall\, u, v \in \mathcal{V}$ (Commutativity of vector addition).

(ii) $(u+v)+w = u+(v+w), \quad \forall\, u, v, w \in \mathcal{V}$ (Associativity).

(iii) Existence of zero: There is a vector $0 \in \mathcal{V}$ such that $0 + v = v, \ \forall\, v \in \mathcal{V}$

(iv) Existence of additive inverse: Every $v \in \mathcal{V}$ has another element $v_I \in \mathcal{V}$ such that $v + v_I = 0$

(v) $\alpha \cdot (\beta \cdot v) = (\alpha \cdot \beta) \cdot v, \quad \forall\, \alpha, \beta \in F, \ v \in \mathcal{V}$ (Associativity of scalar multiplication)

(vi) $1 \cdot v = v, \quad 1 \in F, \ v \in \mathcal{V}$

(vii) $\alpha \cdot (u+v) = \alpha \cdot u + \alpha \cdot v$
$\forall\, \alpha \in F, \ u, v \in \mathcal{V}$

(viii) $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$
$\alpha, \beta \in F, \ v \in \mathcal{V}$

Distributivity.