Lec. 14: Matrix Exponential, Cayley-Hamilton, Hermitian Matrices.

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AGENDA

1. MATRIX EXPONENTIALS
   - DIFFERENTIAL EQUATIONS
   - DIAGONALIZATION & MATRIX EXPONENTIALS.
2. CAYLEY-HAMILTON
3. HERMITIAN MATRICES.

MATRIX EXPONENTIALS

Given a matrix $A \in \mathbb{C}^{n\times n}$, consider the following infinite series

$$I + A + \frac{A^2}{2!} + \cdots + \frac{A^k}{k!} + \cdots\cdots$$

where $A^k = \underbrace{A \cdot A \cdot \cdots A}_{k \text{ times}}$.

Does the above series converge? YES.
For what values of A? Any complex $A \in \mathbb{C}^{n\times n}$

[Proof :- Can be proved via absolute convergence in the matrix Frobenius Norm, ie by showing $\sum_{p=0}^{\infty} \frac{\|A^p\|_F}{p!}$ converges.]

Since the above infinite series converges (to a matrix), the limit is denoted by e^A, ie we define

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots\cdots$$

$$= \sum_{p=0}^{\infty} \frac{A^p}{p!}$$

The above series, when $A \in \mathbb{C}^{1\times 1}$, (ie a complex scalar) yields the well know exponential series

$$e^{z} = 1 + z + \frac{z^2}{2!} + \cdots$$

which converges for all $z \in \mathbb{C}$.

Revisiting Linear Differential Equations

Let $x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$, consider

$$\boxed{\frac{dx(t)}{dt} = Ax(t)} \quad — \quad (1)$$

CLAIM: Solution to (1) is

$$x(t) = e^{At} x(0). \quad — \quad (2)$$

Proof: Verify that

$$\frac{d}{dt} x(t) = \frac{d}{dt} e^{At} x(0) = \frac{d}{dt} \left[I + tA + \frac{t^2 A^2}{2!} + \cdots \right] x(0)$$

$$= \left[A + \frac{2tA^2}{2!} + \frac{3t^2A^3}{3!} + \cdots \right] x(0)$$

$$= A \underbrace{\left[I + \frac{tA}{1!} + \frac{t^2A^2}{2!} + \cdots \right]}_{e^{tA}} x(0)$$

$$= A.e^{tA}x(0) = Ax(t)$$

$\therefore$ (2) satisfies (1).

Solution to (1) is given by (2)

regardless of whether A is diagonalizable.

Computing e^A when A is Diagonalizable

Suppose $A = P\Lambda P^{-1}$.

Then $A^k = \underbrace{(P\Lambda P^{-1})(P\Lambda P^{-1}) \cdots\cdots (P\Lambda P^{-1})}_{k \text{ times}}$

$$= P\Lambda^k P^{-1}$$

$$\Rightarrow e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots\cdots$$

$$= P\left(I + \Lambda + \frac{\Lambda^2}{2!} + \frac{\Lambda^3}{3!} + \cdots\cdots\right)P^{-1}$$

$$= P\begin{bmatrix} \sum_{p=0}^{\infty} \frac{\lambda_1^p}{p!} & & & O \\ & \sum_{p=0}^{\infty} \frac{\lambda_2^p}{p!} & & \\ & & \ddots & \\ O & & & \sum_{p=0}^{\infty} \frac{\lambda_n^p}{p!} \end{bmatrix} P^{-1}$$

$$= P\begin{bmatrix} e^{\lambda_1} & & O \\ & \ddots & \\ O & & e^{\lambda_n} \end{bmatrix} P^{-1} = Pe^{\Lambda}P^{-1}$$

We can also compute e^A in closed form when A is non-diagonalizable, using its Jordan form.

CAYLEY HAMILTON THEOREM

Let $A \in \mathbb{C}^{n \times n}$ with characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ given by

$$p(\lambda) = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots\cdots (\lambda - \lambda_k)^{n_k}$$

where $\lambda_1, \ldots, \lambda_k$ are its distinct eigenvalues with $n_1, n_2, \ldots, n_k$ being respective algebraic multiplicities.

<u>CLAIM:</u>

$$p(A) := (A - \lambda_1 I)^{n_1} (A - \lambda_2 I)^{n_2} \cdots\cdots (A - \lambda_k I)^{n_k}$$

is equal to the "zero" matrix.

ie $(A - \lambda I)^{n_1} \cdots\cdots (A - \lambda_k I)^{n_k} = 0$

<u>Proof:</u> We provide the proof for diagonalizable A. Proof for non-diagonalizable A can be derived using Jordan Form.

Consider $(A-\lambda_i I)^{n_i}$

$$= (P\Lambda P^{-1} - \lambda_i I)^{n_i} = P(\Lambda - \lambda_i I)^{n_i} P^{-1}$$

$$= P \begin{bmatrix} \underbrace{\begin{matrix} \lambda_1-\lambda_i & & \\ & \ddots & \\ & & \lambda_1-\lambda_i \end{matrix}}_{n_1 \text{ times}} & & & & & \text{\Large 0} \\ & \underbrace{\begin{matrix} \lambda_2-\lambda_i & & \\ & \ddots & \\ & & \lambda_2-\lambda_i \end{matrix}}_{n_2 \text{ times}} & & & & \\ & & \ddots & & & \\ & & & \underbrace{\begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix}}_{n_i \text{ zeros}} & & \\ & & & & \ddots & \\ \text{\Large 0} & & & & & \underbrace{\begin{matrix} \lambda_k-\lambda_i & & \\ & \ddots & \\ & & \lambda_k-\lambda_i \end{matrix}}_{n_k \text{ times}} \end{bmatrix}^{n_i} P^{-1}$$

$$\therefore (\Lambda - \lambda_1 I)^{n_1} \cdots (\Lambda - \lambda_k I)^{n_k}$$

$$= 0$$

HERMITIAN MATRICES

A matrix $A \in \mathbb{C}^{n\times n}$ is said to be Hermitian if $\underbrace{A^H}_{}$ $= A$ $\left(\text{ie } A_{ij} = A_{ji}^* \quad \forall\, i,j\right)$

A transposed & conjugated

Diagonal entries $A_{11}, A_{22}, \ldots, A_{nn}$ of A are real

EIGENVALUES:

The eigenvalues of A are real. $(A = A^H)$

Proof: Let λ be an eigenvalue of A & v be a corresponding eigenvector.

$$Av = \lambda v \Rightarrow v^H A v = \lambda v^H v \quad \text{—} ①$$

$$\Rightarrow v^H A^H = \lambda^* v^H \Rightarrow v^H A^H v = \lambda^* v^H v \quad \text{—} ②$$

Since $A^H = A$, LHS of ① & ② are same

$$\Rightarrow \lambda v^H v = \lambda^* v^H v \Rightarrow \boxed{\lambda = \lambda^*} \quad (\because v \neq 0)$$

EIGENVECTORS CORRESPONDING TO DISTINCT EIGENVALUES:

Suppose λ and λ' are two distinct eigenvalues of A (ie $\lambda \neq \lambda'$). Let v_1, v_2 be the eigenvectors of A $(A = A^H)$ corresponding to λ and λ'.

ie $\quad Av_1 = \lambda v_1 \qquad Av_2 = \lambda' v_2$

Then $v_1^H v_2 = 0$. $\Rightarrow$ eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof.

$$Av_1 = \lambda v_1$$

$$\Rightarrow \quad v_2^H A v_1 = \lambda v_2^H v_1 \quad \text{—— (1)}$$

Similarly $\quad Av_2 = \lambda' v_2$

$$\Rightarrow \quad v_2^H A^H = \lambda' v_2^H \quad (\lambda' \text{ is real})$$

$$\Rightarrow \quad v_2^H A^H v_1 = \lambda' v_2^H v_1 \quad \text{--- } 2$$

LHS of 1 & 2 are same, since $A = A^H$

$$\Rightarrow \quad \lambda v_2^H v_1 = \lambda' v_2^H v_1$$

$$\Rightarrow \quad (\lambda - \lambda') v_2^H v_1 = 0$$

since $\lambda \neq \lambda'$ $\Rightarrow$ $\boxed{v_2^H v_1 = 0}$

Diagonalization of Hermitian Matrices

If $A \in \mathbb{C}^{n\times n}$ is Hermitian, then there exists unitary $U \in \mathbb{C}^{n\times n}$, such that

$$\boxed{A = U \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} U^H .}$$

ie A is diagonalized by a unitary U.

In this case λ_i, $i=1,2,\ldots,n$ are eigenvalues of A and $u_1, \ldots, u_n$ are the corresponding eigenvectors ($U = [u_1 \, u_2 \cdots u_n]$).

Moreover since λ_i are real, we can always order them as

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n .$$

Rayleigh Quotient

Given a Hermitian $A \in \mathbb{C}^{n \times n}$, define

$$R_A : \mathbb{C}^n \setminus \{0\} \to \mathbb{R}, \text{ as}$$

$$\boxed{R_A(x) \triangleq \frac{x^H A x}{x^H x}, \quad x \neq 0.}$$

Always real valued.

(note since $A = A^H$, $(x^H A x)^H = x^H A x \Rightarrow x^H A x$ is real).

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of A. Then

$$\boxed{\begin{aligned} \lambda_1 &= \max_{x \neq 0} R_A(x) \\ \lambda_n &= \min_{x \neq 0} R_A(x). \end{aligned}}$$