Lecture 2: Vector Spaces & subspaces

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AGENDA

1. VECTOR SPACES
   - EXAMPLES
   - PROPERTIES

2. SUBSPACES
   - DEFINITION

Examples of Vector Spaces

① Let $\mathbb{F}$ be a field. Define

$$\mathbb{F}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad x_i \in \mathbb{F} \right\}$$

Vector addition: $x, y \in \mathbb{F}^n$

Define:

$$x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

field addition

Scalar multiplication: $\alpha \in \mathbb{F}$, $x \in \mathbb{F}^n$

$$\alpha \cdot x = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

Field multiplication

$\boxed{\mathbb{F}^n \text{ is a vector space with these operations}}$

zero: $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow 0 \in \mathbb{F}$. repeated n times

$$x_I = \begin{bmatrix} -x_1 \\ \vdots \\ -x_n \end{bmatrix},$$ $-x_i$ is the additive inverse of $x_i \in \mathbb{F}$.

$\boxed{\mathbb{F}^n \text{ is a Vector Space over } \mathbb{F}}$

Special cases of $\mathbb{F}^n$

(i) $\mathbb{F} = \mathbb{R}$

$\mathbb{F}^n = \mathbb{R}^n$

n-dimensional real Euclidean space.

(ii) $\mathbb{F} = \mathbb{C}$

$\mathbb{F}^n = \mathbb{C}^n$.

$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \neq 0$

(2) Let $\mathbb{F}$ be a field. Define

$$\mathbb{F}^{m\times n} = \left\{ \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ \vdots & & & \vdots \\ x_{m1} & \cdots & & x_{mn} \end{bmatrix}, x_{ij} \in \mathbb{F} \right\}$$

Vector Addition: $X, Y \in \mathbb{F}^{m\times n}$

$$X + Y = \begin{bmatrix} x_{11}+y_{11} & \cdots & x_{1n}+y_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1}+y_{m1} & \cdots & x_{mn}+y_{mn} \end{bmatrix}$$

Scalar Multiplication: $\alpha \in \mathbb{F}$, $X \in \mathbb{F}^{m\times n}$

$$\alpha . X = \begin{bmatrix} \alpha x_{11} & \alpha x_{12} & \cdots & \alpha x_{1n} \\ \vdots & & & \\ \alpha x_{m1} & \cdots & & \alpha x_{mn} \end{bmatrix}$$

Zero:

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & & 0 \end{bmatrix}$$

X_I:

$$\begin{bmatrix} -x_{11} & \cdots & -x_{1n} \\ \vdots & & \\ -x_{m1} & \cdots & -x_{mn} \end{bmatrix}$$

$-x_{ij}$ is the additive inverse of x_{ij} in $\mathbb{F}$.

Special Cases

(i) $\mathbb{F} = \mathbb{R}$

$\mathbb{F}^{m\times n} = \mathbb{R}^{m\times n}$ (real-valued $m\times n$ matrices)

(ii) $\mathbb{F} = \mathbb{C}$

$\mathbb{F}^{m\times n} = \mathbb{C}^{m\times n}$ (complex-valued $m\times n$ matrices)

$\boxed{\mathbb{F}^{m\times n} \text{ is a Vector Space over } \mathbb{F}}$

③ <u>Smallest Vectorspace?</u> Just the vector 0.

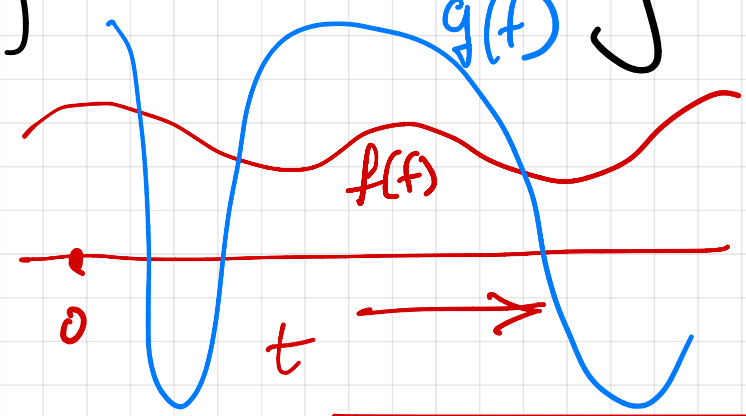
$\{0\}$ → It is a vector space by itself.

Will turn out to be a very important vector space!

④ Define

$C(\mathbb{R}) = \{$continuous functions whose domain is $\mathbb{R}\}$

$= \{f : \mathbb{R} \to \mathbb{R},\ f \text{ is continuous}\}.$



$f \in C(\mathbb{R}),\ g \in C(\mathbb{R})$

Define

$(f+g)(t) = f(t) \underset{\text{addition of reals}}{+} g(t)$

$(\alpha f)(t) = \alpha \cdot f(t)$ $\alpha \in \mathbb{R}$

multiplication of reals

$\boxed{C(\mathbb{R}) \text{ is a vector space over } \mathbb{R}}$

0 function is $f_0(t) = 0,\ \forall t$ $f_I(t) = -f(t),\ \forall t$

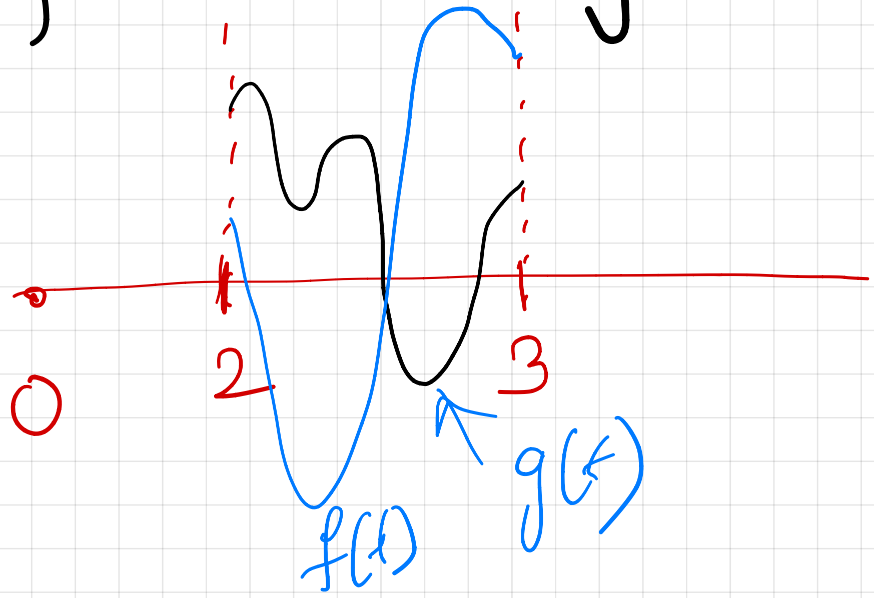
5. Let I be an interval of $\mathbb{R}$ (e.g. $I = [2,3]$)

Define

$$C(I) = \{\text{all continuous functions whose domain is } I\}$$

$$= \{f : I \to \mathbb{R},\ f \text{ is continuous}\}$$

Under same definitions of addition, scalar multiplication



Θ is $f(t) = 0$, $2 \le t \le 3$

f_I : $f_I(t) = -f(t)$, $2 \le t \le 3$

$C(I)$ is a vector space over $\mathbb{R}$ the field

Properties of Vector spaces (DIY)

1. $0 \cdot v = 0 \quad , \forall \, v \in \mathcal{V}$

zero of field F 0 of $\mathcal{V}$

2. $\alpha \cdot 0 = 0 \quad , \quad \forall \, \alpha \in F$

3. $\underbrace{(-1)}_{\text{additive inverse of } 1 \in F} \cdot v = \underbrace{-v}_{\text{additive inverse of } v \text{ in } \mathcal{V}} \quad , \quad \forall \, v \in \mathcal{V}$

4. $0 \in \mathcal{V}$ is unique.

5. For every $x \in \mathcal{V}$, its additive inverse is unique.

6. If $\alpha \cdot v = 0$, $(\alpha \in F, v \in \mathcal{V})$ then either $\alpha = 0$ (zero of field) or $v = 0$ (zero of $\mathcal{V}$)

SUBSPACES

A subspace $\mathcal{W}$ of a vector space $\mathcal{V}$ over a field $\mathbb{F}$, is a non-empty subset of $\mathcal{V}$ (ie $\mathcal{W} \subseteq \mathcal{V}$) such that

(i) If $x, y \in \mathcal{W}$, then $x+y \in \mathcal{W}$

(ii) If $x \in \mathcal{W}$, then $\alpha . x \in \mathcal{W}$, $\forall\ \alpha \in \mathbb{F}$

Question :-

(i) Is $0 \in \mathcal{V}$ always included in any subspace $\mathcal{W}$? YES

(ii) Is $\mathcal{W}$ a vector space ? YES

(All 8 axioms of vector space can be shown to be true for $\mathcal{W}$ by simply using the above two properties and the fact that $\mathcal{W} \subseteq \mathcal{V}$).