LECTURE 3: SUBSPACES, LINEAR INDEPENDENCE

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ECE 269
FALL 2021

AGENDA

1. SUBSPACES
   - Examples
   - Properties
   - $R(A)$ and $\mathcal{N}(A)$

2. Linear Independence
   - Example
   - Implications

Example of subspaces induced by a Matrix:

Suppose we are given a matrix $A \in \mathbb{F}^{m \times n}$ ($\mathbb{F}$ is a Field).

Then, define

(i) $\mathcal{R}(A) = \{ y \in \mathbb{F}^m, \text{ s.t. } y = Ax, \ x \in \mathbb{F}^n \}$

$\mathcal{R}(A)$ is also known as range space / column span of A.

$0 \in \mathbb{F}^m$

(ii) $\mathcal{N}(A) = \{ x \in \mathbb{F}^n, \text{ s.t. } Ax = 0 \}$.

$\mathcal{N}(A)$ is also known as the Null space of A.

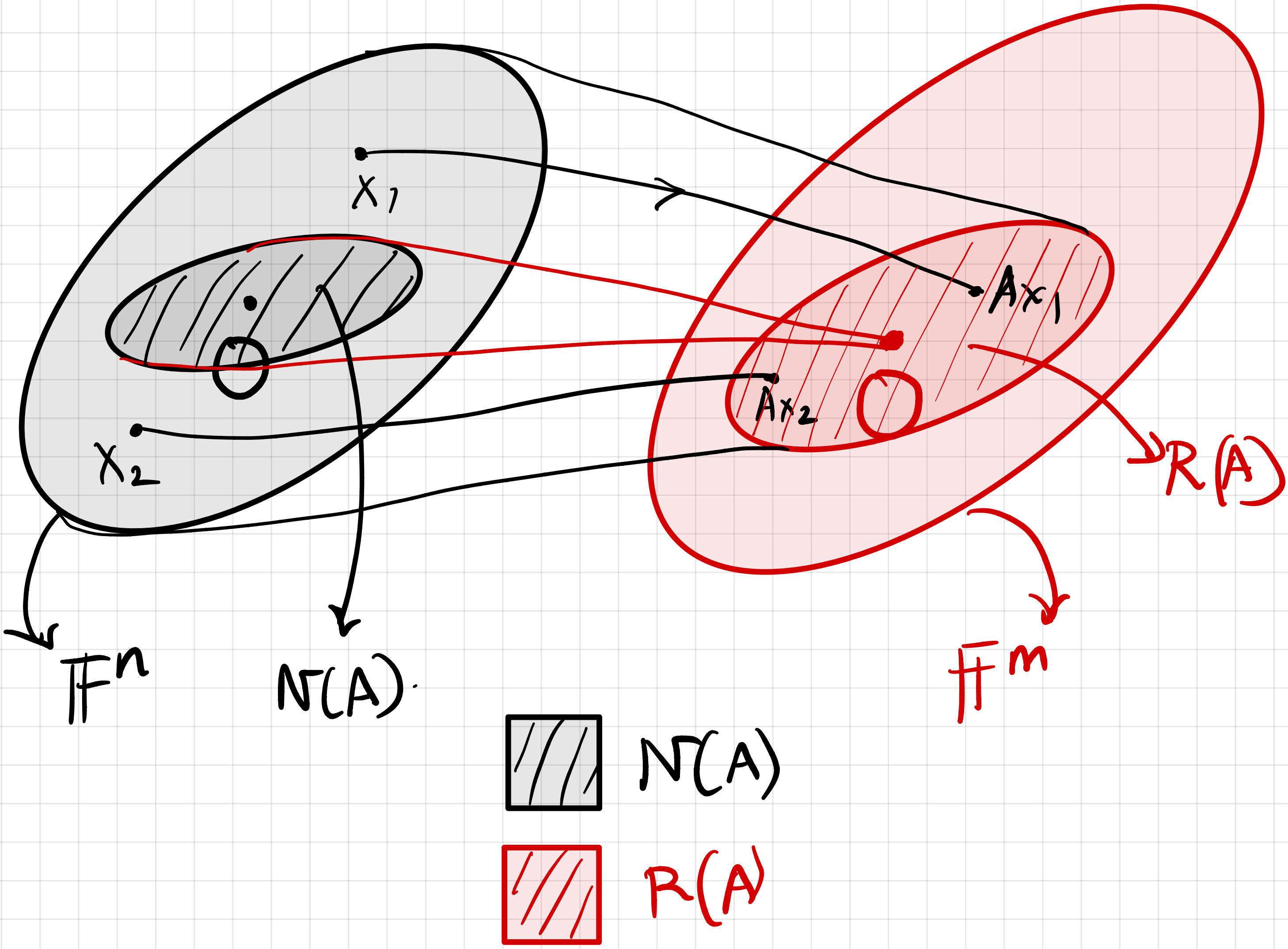
What kind of sets are $R(A)$ and $N(A)$?

$R(A)$ is a subspace of $\mathbb{F}^m$

$N(A)$ is a subspace of $\mathbb{F}^n$.

(Prove this yourself by using properties of matrix-vector multiplication and the two axioms of a subspace)

DEPICTION OF $R(A)$, $N(A)$



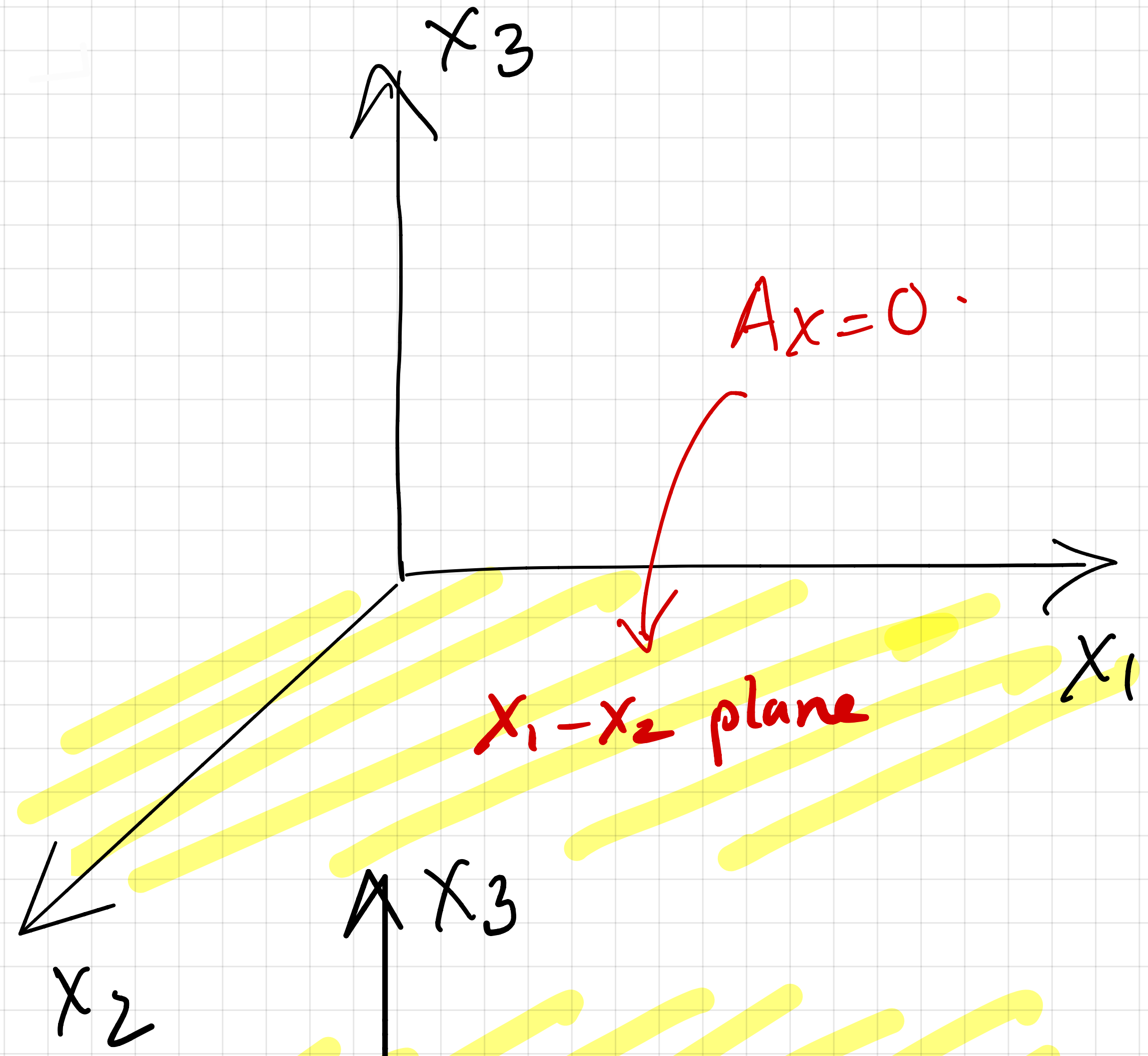
$R(A)$ is the map of entire $\mathbb{F}^n$ under the action of matrix A.

$N(A)$ maps to the single vector $0 \in \mathbb{F}^m$

Visualization of the set $Ax+v$ in $\mathbb{R}^3$

① $A = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, $v = 0$

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, Ax = 0 \right\} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, x_3 = 0 \right\}$$

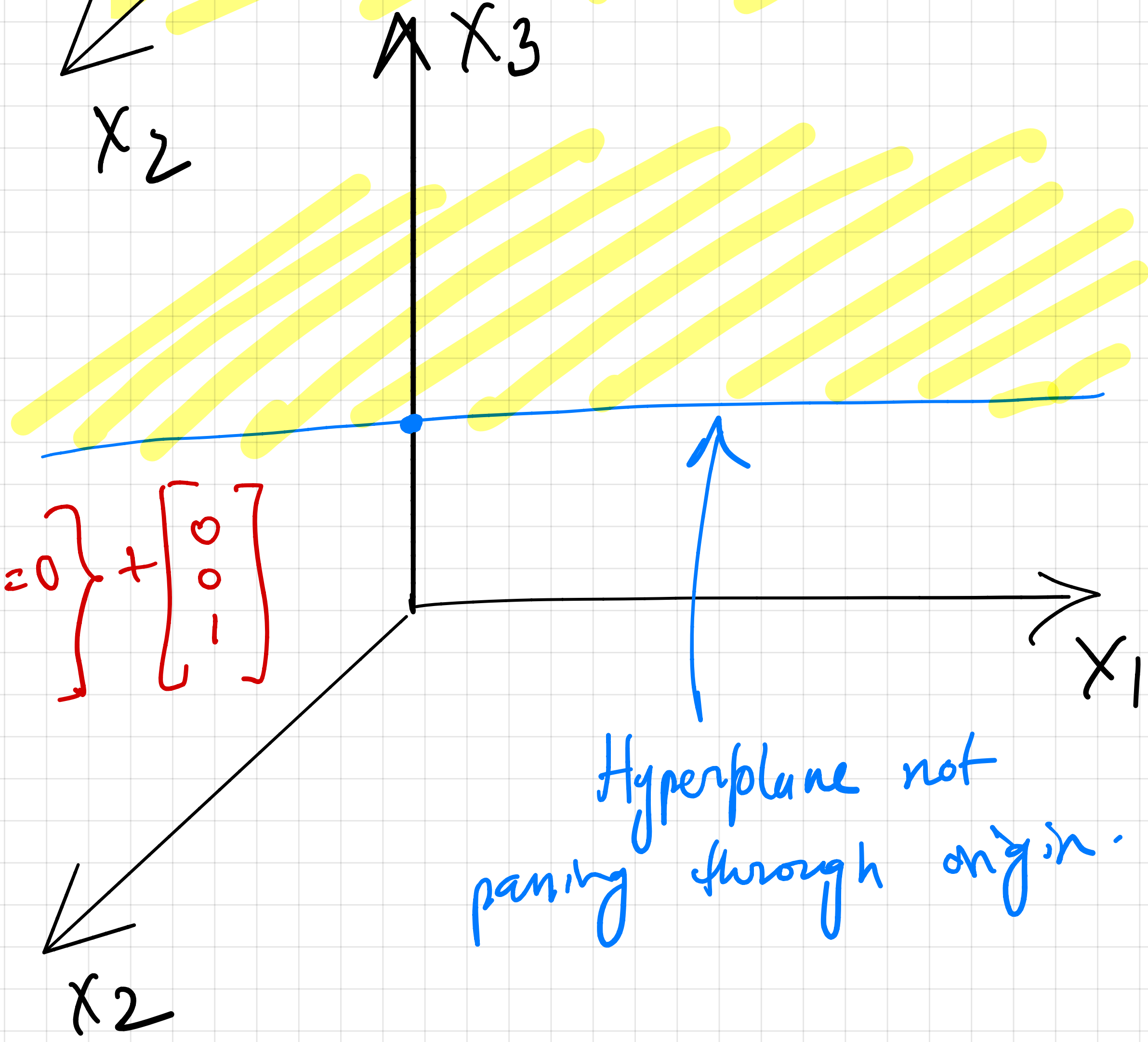


② $A = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$,

$v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$H_2 = N(A) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, x_3 = 0 \right\} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

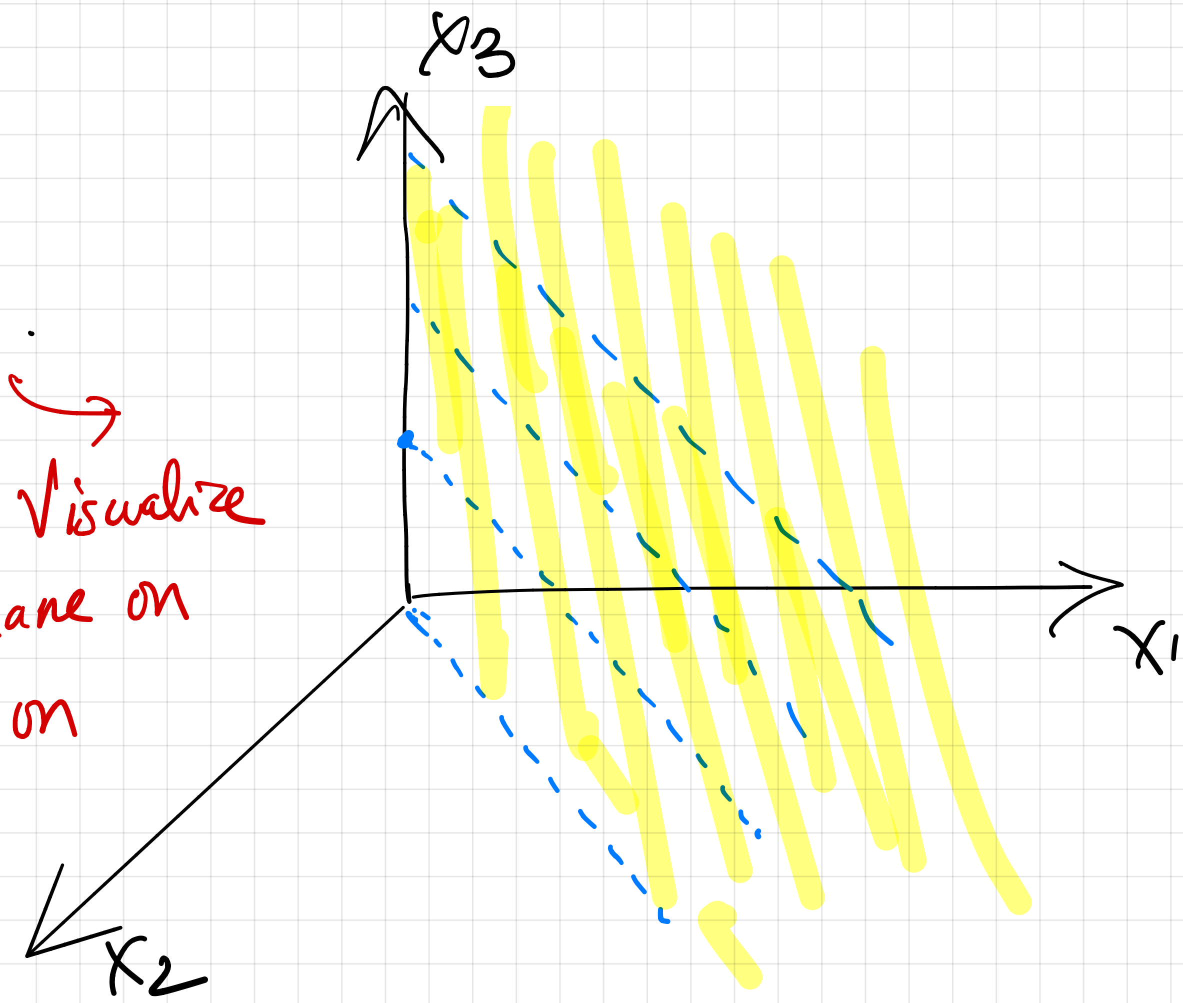
$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} \right\}.$$



③ $A = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$, $v = 0$

$$H_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, x_1 - x_2 = 0 \right\}.$$

Visualize this plane on the figure on RHS.



PROPERTIES OF SUBSPACES

Given a subspace $\mathcal{U}$ of a vector space $\mathcal{V}$:-

(1) $\mathcal{U} = \{0\}$ is always a subspace

(2) $\mathcal{U}$ is itself a vector space

(can be verified by the fact that $\mathcal{U}$ is closed under vector addition and scalar multiplication, AND being a subset of $\mathcal{V}$, $\mathcal{U}$ inherits all properties of vector addition and scalar multiplication of $\mathcal{V}$).

Verify: given $v \in \mathcal{U}$, is the additive inverse v_I of v also in $\mathcal{U}$?

Yes: since $v_I = (-1) \cdot v$ and $\mathcal{U}$ is closed under scalar multiplication.

ADDITIONAL PROPERTIES OF SUBSPACES

Let $\mathcal{U}, \mathcal{W}$ be subspaces of a vector space $\mathcal{V}$ over a field $\mathbb{F}$. Define the following subsets of $\mathcal{V}$.

vector addition in $\mathcal{V}$

(i) $\underbrace{\mathcal{U}+\mathcal{W}}_{\text{Sum.}} = \{ u+w \,,\; u \in \mathcal{U},\; w \in \mathcal{W} \}$

(ii) $\underbrace{\mathcal{U}\cap\mathcal{W}}_{\text{Intersection}} = \{ u,\; u \in \mathcal{U},\; u \in \mathcal{W} \}$.

(iii) $\underbrace{\mathcal{U}\cup\mathcal{W}}_{\text{Union.}} = \{ u,\; \text{either } u \in \mathcal{U} \text{ or } u \in \mathcal{W} \}$

Is each of the above three sets a subspace?

(i) $\mathcal{U}+\mathcal{W}$ is a subspace of $\mathcal{V}$ (VERIFY)

(ii) $\mathcal{U}\cap\mathcal{W}$ " " " " $\mathcal{V}$

(iii) $\mathcal{U}\cup\mathcal{W}$ is <u>NOT ALWAYS</u> A SUBSPACE.

SPAN

Given a set of vectors $v_1, v_2, v_3, \ldots, v_K \in \mathcal{V}$ in a vector space $\mathcal{V}$ over a field $\mathbb{F}$, their span is defined as

$$\text{span}\{v_1, v_2, \ldots, v_K\} = \left\{ \sum_{i=1}^{K} \alpha_i \cdot v_i, \ \alpha_i \in \mathbb{F} \right\}$$

Is $\text{span}\{v_1, \ldots, v_K\}$ a subspace of $\mathcal{V}$?

YES (VERIFY).

Range Space Revisited

Given a matrix $A \in \mathbb{F}^{m \times n}$, $A = [a_1\, a_2 \cdots a_n]$

$$R(A) = \{Ax, \ x \in \mathbb{F}^n\}$$

$$= \left\{ \sum_{i=1}^{n} a_i x_i, \ x_i \in \mathbb{F} \right\}$$

$$= \text{span}\{a_1, a_2, \ldots, a_n\}$$

LINEAR INDEPENDENCE

Let $\mathcal{V}$ be a vector space over a field $\mathbb{F}$. A set of (non-zero) vectors $v_1, v_2, v_3, -, v_k \in \mathcal{V}$ are said to be linearly dependent, if there exist $\alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{F}$ (not all of them zeros)

such that $$\sum_{i=1}^{k} \alpha_i v_i = \underbrace{0}_{\text{zero of } \mathcal{V}}$$

On the other hand, $v_1, v_2, \cdots, v_k$ are linearly independent if

$$\sum_{i=1}^{k} \alpha_i v_i = 0 \iff \alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_k = \underbrace{0}_{0 \in \mathbb{F}.}$$

Example:

Linearly Independent.

① $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\quad (\mathcal{V} = \mathbb{R}^3)$

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0 \Rightarrow \begin{bmatrix} \alpha_2 \\ \alpha_3 \\ \alpha_1 + \alpha_2 + \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$

② $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ -2.8 \\ 7.1 \end{bmatrix}$ $\quad (\mathcal{V} = \mathbb{R}^3)$

Verify they are linearly dependent!