Definition of Determinant:

For $A \in \mathbb{C}^{n\times n}$, determinant of A is defined to be the scalar: $\det(A) = \sum_{p} \sigma(p) \cdot a_{1p_1} a_{2p_2} \cdots a_{np_n}$

where the sum is taken over ($n!$) permutations $p = (p_1, p_2, \ldots, p_n)$ of $(1, 2, \ldots, n)$. Note that each term $a_{1p_1} a_{2p_2} \cdots a_{np_n}$ contains exactly one entry from each row and each column of A.

The sign of permutation p is defined to be $\sigma(p)$:

$$\sigma(p) = \begin{cases} +1 & \text{if } p \text{ can be restored to natural order by an even number of interchanges;} \\ -1 & \text{if } p \text{ can be restored to natural order by an odd number of interchange} \end{cases}$$

(The determinant of a non-square matrix is NOT defined) matrix is NOT defined) by an odd number of interchange.

Example: 1). $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\det(A) = \sigma(1,2)\, a_{11}a_{22} - \sigma(2,1)\cdot a_{12}a_{21}$
$= a_{11}a_{22} - a_{12}\cdot a_{21}$.

2). Triangular determinants:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}$$

(Special case: diagonal matrix)

Properties:

1. [Effects of Row operations].

Let B be the matrix obtained from $A_{n\times n}$ by one of the three elementary row operations: ( Meyer's book P463 ).

- Type(I): interchange rows $i$ and $j$;
- Type(II): Multiply row $i$ by $\alpha \neq 0$;
- Type(III): Add $\alpha$ times row $i$ to row $j$.

The value of $\det(B)$ is as follows:

$\Rightarrow \det(B) = -\det(A)$ for Type I. operations ($\det(B) = \det(E)\det(A)$)

$\Rightarrow \det(B) = \alpha\cdot\det(A)$ for Type II. operations ($\det(B) = \det(F)\cdot\det(A)$)

$\Rightarrow \det(B) = \det(A)$ for Type III —— ($\det(B) = \det(G)\cdot\det(A)$)

In other words, suppose $P_1, P_2, \ldots, P_k$ are a set of elementary matrices, A is a non-singular matrix.

$$\det(P_1P_2\cdots P_kA) = \det(P_1)\cdot\det(P_2\cdots P_kA)$$
$$= \det(P_1)\cdot\det(P_2)\cdots\det(P_3\cdots A)$$
(*) $$= \det(P_1)\cdot\det(P_2)\cdots\det(P_k)\cdot\det(A).$$

NOTE: Elementary matrices have non-zero determinants.

example. $A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \xrightarrow[\text{③}+\text{①}]{\text{②}-\text{①}\times 2} \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix} \xrightarrow{\text{③}-\text{②}} \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$

pivots.

$\therefore \det(A) = 2\times1\times4 = 8$.

2. [Invertibility & Determinants].

$A \in \mathbb{C}^{n\times n}$ is non-singular iff $\det(A) \neq 0$

($\Leftrightarrow A \in \mathbb{C}^{n\times n}$ is singular iff $\det(A) = 0$)

($\Leftarrow$ $A \in \mathbb{C}^{n\times n}$ is singular iff $\det(A)=0$)

Proof: Let $P_1, P_2, \dots, P_k$ be a sequence of elementary matrices s.t. $P_1 P_2 \cdots P_k \cdot A = E_A$, apply (*) we have:

$$\det(P_1)\cdot\det(P_2)\cdots\det(P_k)\cdot\det(A)=\det(E_A)$$

※NOTE: $E_A$ is row echelon matrix, see Meyer's book, P.48

Because elementary matrices have non-zero determinants,

$\det(A)\neq 0 \Leftrightarrow \det(E_A)\neq 0 \Leftrightarrow$ there are no zero pivots

$\Leftrightarrow$ every column in $E_A$ (and in $A$) is a basis.

$\Leftrightarrow$ $A$ is non-singular

3. [Product Rules]

$$\det(AB)=\det(A)\cdot\det(B) \text{ for all } A, B \in \mathbb{C}^{n\times n}$$

Proof: 1). If $A$ is singular, then $AB$ also singular, (because $\text{rank}(AB) \leq \text{rank}(A)$)

Consequently, $\det(AB)=0=\det(A)\det(B)$

2) If $A$ is non-singular, then $A$ can be written as a product of elementary matrices, $A=P_1P_2\cdots P_k$

then by using (*)

$$\det(AB)=\underline{\det(P_1P_2\cdots P_kB)=\det(P_1)\dots\det(P_k)\cdot\det(B)} \quad (**)$$

$$=\det(P_1P_2\cdots P_k)\cdot\det(B)=\det(A)\cdot\det(B).$$

$\Rightarrow$ Why $\text{rank}(AB)\leq\text{rank}(A)$ with $A, B\in\mathbb{C}^{n\times n}$?

Proof: Let $C=AB \Rightarrow C\cdot x = AB\cdot x$ ) let $z=Bx$

$$= [a_1\ a_2 \cdots a_n]\cdot\begin{bmatrix} z_1\\ z_2\\ \vdots\\ z_n\end{bmatrix}$$

Clearly $R(C)$ can be represented by linear combination of $\{a_1, a_2, \dots, a_n\}$,

thus $R(C)=R(AB)\subseteq \text{span}\{a_1, \dots, a_n\}=R(A)$.

$\therefore$ $\text{rank}(AB)\leq\text{rank}(A)$.

$\Rightarrow$ Follow-up from student's question: why (**) holds without the knowledge of product rule?

without the knowledge of product rule ?

Proof : when $P$ is an elementary matrix, $\det(PA)=\det(P)\cdot\det(A)$. This holds without the knowledge of product rule (see Meyer's book P.465)

then for (**) :

$$\det(P_1 P_2 \cdots P_k B) = \det(P_1)\cdot\det(P_2 \cdots P_k B)$$
$$= \det(P_1)\cdot\det(P_2)\det(P_3 \cdots P_k B)$$
$$\vdots$$
$$= \det(P_1)\cdot\det(P_2)\cdots\det(P_k)\cdot\det(B).$$

$\checkmark$

Prob. 2. We introduced similar matrices in class.

$A, B \in \mathbb{C}^{m\times m}$, A and B are called "similar" if there exists an invertible matrix $P$, s.t. $P^{-1}AP = B$

$\Rightarrow$ We have shown that $A, B$ share the same eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$

$\Rightarrow$ Now, what's the relation between eigenvectors of A and B?

Proof: suppose for any $\lambda_i \in \{\lambda_1, \lambda_2, \dots, \lambda_m\}$, we have

$A \cdot x_i = \lambda_i \cdot x_i$, where $x_i$ is the corresponding eigenvector

$$\Rightarrow A \cdot (P \cdot P^{-1}) \cdot x_i = \lambda_i \cdot x_i$$

$$\Rightarrow \underbrace{P^{-1} \cdot A \cdot P}\cdot P^{-1} \cdot x_i = \lambda_i \cdot P^{-1} \cdot x_i$$

$$\Rightarrow B \cdot (P^{-1} \cdot x_i) = \lambda_i \cdot (P^{-1} \cdot x_i).$$

Clearly, for every eigenpair $(\lambda_i, x_i)$ of A, it corresponds to the eigenpair $(\lambda_i, P^{-1} \cdot x_i)$ of B.

Prob. 3. Show that normal equation $A^T A x = A^T b$ is equivalent to $Ax = P_{R(A)} \cdot b$

Proof: $A^T \cdot A \cdot x = A^T b$

$$\Leftrightarrow A^T (Ax - b) = 0$$

$$\Leftrightarrow (Ax - b) \in N(P_{R(A)})$$

$$\Leftrightarrow P_{R(A)} \cdot (Ax - b) = 0$$

$$\Leftrightarrow Ax = P_{R(A)} \cdot b$$

Because $A^T \cdot (Ax - b) = 0$

$$\Downarrow$$

$$(Ax - b) \in N(A^T).$$

It is true that $N(A^T) = R(A)^\perp$

(HW#3 problem! convince yourself).

also $N(P_{R(A)}) = R(A)^\perp$

$\Longleftrightarrow \quad Ax = P_{R(A)} \cdot b$

Furthermore,

suppose $\hat{x}$ is a least square solution for a possibly inconsistent linear system $Ax = b$,

then the following three statements are equivalent:

i). $\|A\hat{x} - b\|_2 = \min\limits_{x \in \mathbb{R}^n} \|Ax - b\|_2$

ii). $A\hat{x} = P_{R(A)} \cdot b$

iii). $A^T \cdot A \cdot \hat{x} = A^T \cdot b$