Lec. 15: Hermitian & PSD matrices, SVD

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AGENDA

1. HERMITIAN MATRICES

2. POSITIVE (SEMI) DEFINITE MATRICES

3. SVD

Rayleigh Quotient

Given a Hermitian $A \in \mathbb{C}^{n \times n}$, define

$$R_A : \mathbb{C}^n \setminus \{0\} \to \mathbb{R}, \text{ as}$$

$$\boxed{R_A(x) \triangleq \frac{x^H A x}{x^H x}, \quad x \neq 0.}$$

Always real valued.

(note since $A = A^H$, $(x^H A x)^H = x^H A x \Rightarrow$ $x^H A x$ is real).

Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of A. Then

$$\boxed{\begin{aligned} \lambda_1 &= \max_{x \neq 0} R_A(x) \\ \lambda_n &= \min_{x \neq 0} R_A(x). \end{aligned}}$$

$$\boxed{\lambda_n \leq R_A(x) \leq \lambda_1}, \quad \forall x \neq 0$$

Proof: Given $x \neq 0$

$$x^H A x = x^H U \Lambda U^H x$$

Define $y = U^H x$, $y \neq 0$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\Rightarrow x^H A x = y^H \Lambda y$$

$$= \begin{bmatrix} y_1^* & y_2^* & \cdots & y_n^* \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$= \sum_{i=1}^{n} \lambda_i |y_i|^2$$

$$= \lambda_1 |y_1|^2 + \lambda_2 |y_2|^2 + \cdots + \lambda_n |y_n|^2$$

Since $\lambda_1 \geq \lambda_2 \geq \lambda_3, \cdots \geq \lambda_n$

We have

$$\lambda_2 |y_2|^2 \leq \lambda_1 |y_2|^2$$

$$\lambda_3 |y_3|^2 \leq \lambda_1 |y_3|^2$$

$$\vdots$$

$$\lambda_n |y_n|^2 \leq \lambda_1 |y_n|^2$$

$$\Rightarrow x^H A x = \lambda_1 |y_1|^2 + \lambda_2 |y_2|^2 + \cdots + \lambda_n |y_n|^2$$

$$\leq \lambda_1 |y_1|^2 + \lambda_1 |y_2|^2 + \cdots + \lambda_1 |y_n|^2$$

$$= \lambda_1 \left(|y_1|^2 + \cdots + |y_n|^2 \right)$$

$$= \lambda_1 \|y\|_2^2 = \lambda_1 \|U^H x\|_2^2$$

$$= \lambda_1 \left(\underbrace{x^H U U^H x} \right)$$

$$= \lambda_1 x^H x$$

$$\Rightarrow \quad x^H A x \leq \lambda_1 x^H x$$

$$\Rightarrow \quad R_A(x) \leq \lambda_1$$

Arguing similarly (DIY) show that

$$R_A(x) \geq \lambda_n.$$

$$\Rightarrow \boxed{\lambda_n \leq R_A(x) \leq \lambda_1}$$

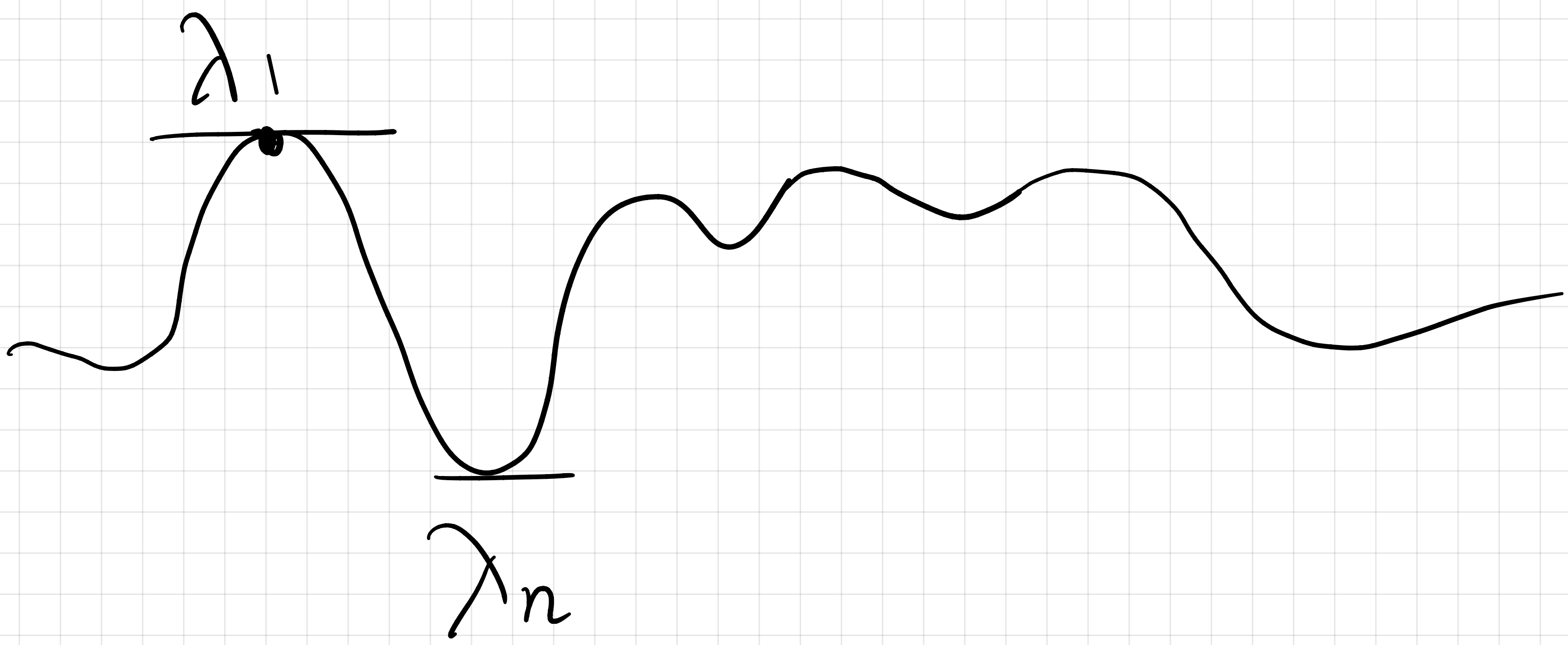
Specifically if $x = u_1$ (u_1 is eigenvector of A corresponding to λ_1)

$$\Rightarrow \frac{x^H A x}{x^H x} = \frac{\lambda_1 u_1^H u_1}{u_1^H u_1}$$

$$= \lambda_1$$

Similarly if $x = u_n$ (u_n is eigenvector of A corresponding to λ_n)

$$\frac{x^H A x}{x^H x} = \frac{\lambda_n u_n^H u_n}{u_n^H u_n} = \lambda_n$$



Positive Semidefinite (PSD) Matrices

A Hermitian $A \in \mathbb{C}^{n \times n}$ is PSD if

$$x^H A x \geq 0 \quad \text{for all } x \in \mathbb{C}^n$$

A Hermitian $A \in \mathbb{C}^{n \times n}$ is positive definite (PD)

$$\text{if } \quad x^H A x > 0, \quad \forall \ x \neq 0.$$

Consequences:

① **Eigenvalues:** Eigenvalues of A are non negative if A is PSD.

Let λ be an eigenvalue of A and v be a corresponding eigenvector $(v \neq 0)$

$$\Rightarrow Av = \lambda v \Rightarrow v^H A v = \lambda \|v\|_2^2$$

Since $v^H A v \geq 0 \Rightarrow \lambda \geq 0 \quad (\because \|v\|_2^2 > 0)$

Additionally, if A is PD, then

$$v^H A v > 0 \Rightarrow \lambda > 0$$

$\Rightarrow$ All eigenvalues of A are positive.

(In particular, A is non-singular since it has no zero eigenvalues).

Since A is Hermitian

$$A = U \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} U^H$$

where $\lambda_i \geq 0$ if A is PSD.

(ii) Factorization of PSD Matrices:

$$A = U \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} U^H$$

$$= \underbrace{U \begin{bmatrix} \sqrt{\lambda_1} & & & 0 \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ 0 & & & \sqrt{\lambda_n} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \sqrt{\lambda_1} & & & 0 \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ 0 & & & \sqrt{\lambda_n} \end{bmatrix} U^H}_{B^H}$$

$$= B B^H.$$

Let $\Lambda^{1/2} = \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix}$

Also,

$$A = U \Lambda^{1/2} \underbrace{U^H U}_{I} \Lambda^{1/2} U^H = C C$$

$C = U \Lambda^{1/2} U^H.$

APPLICATIONS OF PSD MATRICES

COVARIANCE MATRIX OF A RANDOM VECTOR:

Let $X \in \mathbb{C}^n$ be a random vector.

- $E(X) = \mu$
- $E\left[(X-\mu)(X-\mu)^H\right] = \Sigma$ [covariance matrix]

Applications in:

(i) Estimation Theory:

- Signal/Image Processing
- Communication

(ii) Probabilistic Graphical Models

(iii) Dimension Reduction / Feature Extraction.

. . . . .

[WILL REVIST AGAIN]

Covariance matrix is PSD

$$y = x - \mu \qquad (\mu = E(x))$$

$$\Sigma = E(yy^H)$$

$$v^H \Sigma v = v^H E(yy^H) v$$

$$= E(v^H y \underbrace{y^H v}_{z \text{ (scalar)}})$$

$$= E(|z|^2) \geq 0.$$

$$\Rightarrow \quad v^H \Sigma v \geq 0$$

$\Rightarrow$ Σ is PSD.

$R(A)$ and $N(A)$ of Hermitian A

Suppose A is Hermitian, ie $A = A^H$, $A \in \mathbb{C}^{n \times n}$

Then

$N(A)$ is orthogonal complement of $R(A)$:

ie, For every $x \in N(A)$, $y \in R(A)$, $x^H y = 0$

AND $\dim(N(A)) + \dim(R(A)) = n$

Proof: Let $x \in N(A) \Rightarrow Ax = 0 \Rightarrow A^H x = 0$
$\Rightarrow x \perp R(A)$

$\Rightarrow N(A) \perp R(A)$

and $\dim(R(A)) + \dim(N(A)) = n$ (Rank-Nullity)

$\therefore$ Every vector $x \in \mathbb{C}^n$ can be decomposed as

$x = x_1 + x_2$, x_1 is orth. projection of x onto $R(A)$ and $x_2 \in N(A)$

Singular Value Decomposition

Any $A \in \mathbb{C}^{m \times n}$ permits the following decomposition, (known as SVD) :-

$$\boxed{A = \underbrace{U}_{m \times m} \Sigma V^H}$$

(i) $U \in \mathbb{C}^{m \times m}$ is unitary, ie. $U^H U = U U^H = I_{m \times m}$

(ii) $V \in \mathbb{C}^{n \times n}$ is unitary, ie $V^H V = V V^H = I_{n \times n}$.

(iii) $\Sigma \in \mathbb{R}^{m \times n}$ is of the form

$$\Sigma = \begin{bmatrix} \begin{matrix} \sigma_1 & \cdots & 0 \\ 0 & \ddots & \sigma_r \end{matrix} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \cdots \geq \sigma_r > 0$$

σ_i are called singular values of A.