Lecture 5: Linear Maps, Matrices, Rank-Nullity.

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AGENDA

1. Linear Maps
   - Definition
   - Examples
   - $R(T)$ and $N(T)$

2. Linear Maps between FINITE Dimensional Vector Spaces.

3. Rank-Nullity Theorem : Sketch of Proof.

Linear Maps

Let $\mathcal{U}$ and $\mathcal{V}$ be vector spaces over a field $\mathbb{F}$. A function $f : \mathcal{U} \longrightarrow \mathcal{V}$ is linear if

(i) $f(\underbrace{u+v}_{\text{vector addition in } \mathcal{U}}) = f(u) \underbrace{+}_{\text{vector addition in } \mathcal{V}} f(v), \quad \forall\, u, v \in \mathcal{U}.$

(ii) $f(\underbrace{\alpha . u}_{\text{scalar multiplication in } \mathcal{U}}) = \alpha . f(u), \quad \forall\, u \in \mathcal{U}, \alpha \in \mathbb{F}.$

scalar multiplication in $\mathcal{V}$.

Question: What is $f(\underbrace{0}_{0 \in \mathcal{U}})$? $= 0 \in \mathcal{V}$.

$$f(\underbrace{0}_{0 \in \mathbb{F}} . \underbrace{0}_{0 \in \mathcal{U}}) = \underbrace{0}_{0 \in \mathbb{F}} . \underbrace{f(0)}_{\in \mathcal{V}} = 0 \ (\in \mathcal{V})$$

$$\Rightarrow f(0) = 0 \in \mathcal{V}.$$

Examples

1. $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \alpha x, \quad \alpha \in \mathbb{R}.$$

show f is linear

$$f(x) = \alpha x + \beta, \quad \alpha, \beta \in \mathbb{R}, \quad \beta \neq 0$$

Not linear ($\because f(0) \neq 0$)

2. $f: \mathbb{R}^n \to \mathbb{R}^m$

Given $A \in \mathbb{R}^{m \times n}$

define $\boxed{f(x) = Ax}$

SHOW THIS IS LINEAR BY SHOWING,

$f(x+y) = f(x) + f(y)$

& $f(\alpha x) = \alpha f(x)$

3. $f: \mathbb{R}^{n \times n} \to \mathbb{R}$

$$f(X) = \sum_{i=1}^{n} X_{ii} = \mathrm{Trace}(X)$$

Show that $f(x)$ is linear map

Range & Kernel of Linear Maps

Let $T : \mathcal{U} \to \mathcal{V}$ be a linear map. Then

$$\mathcal{R}(T) = \{ v \in \mathcal{V},\ \text{s.t. } v = T(u) \text{ for some } u \in \mathcal{U} \}$$

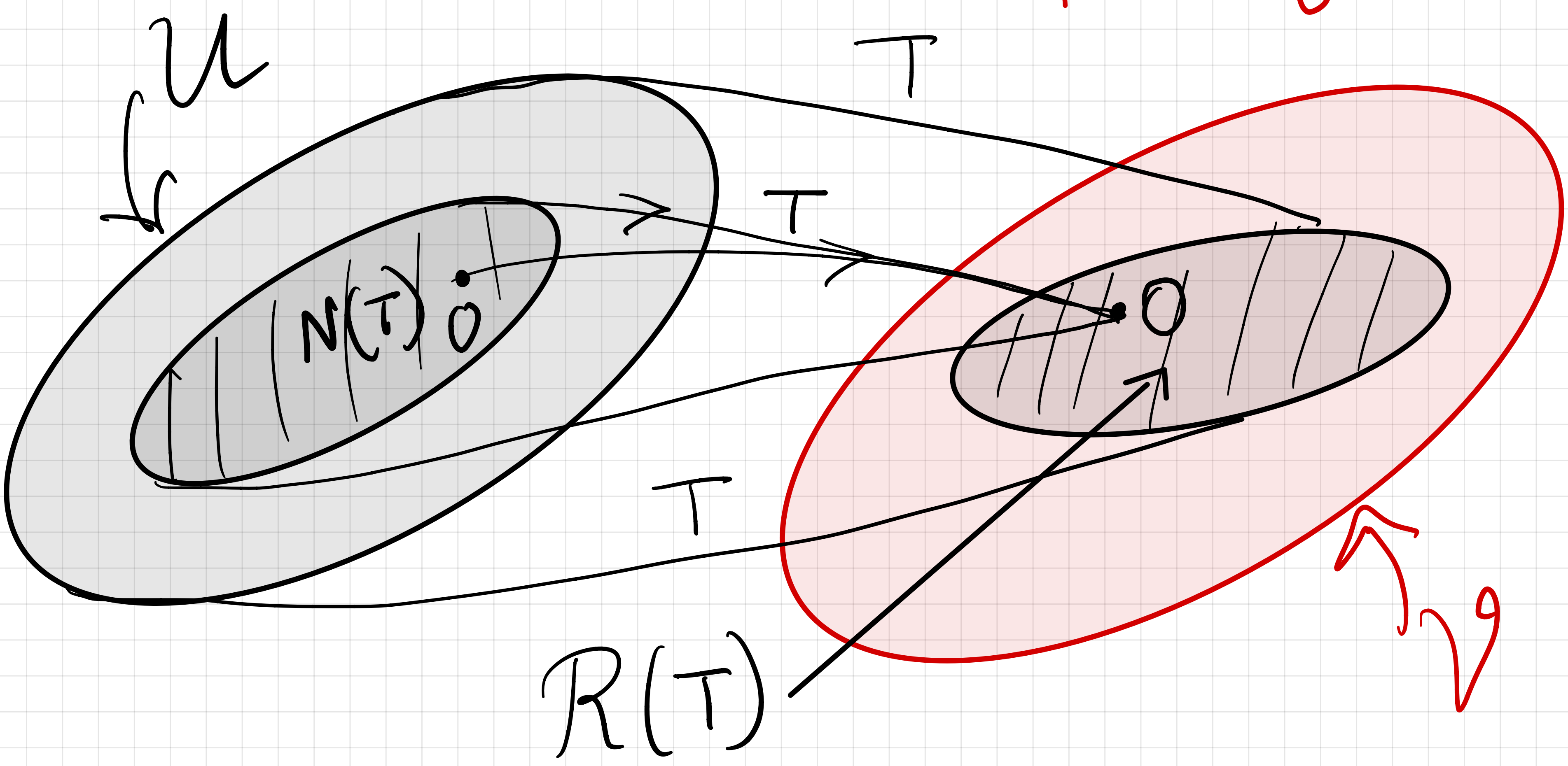
$$\mathcal{N}(T) = \{ u \in \mathcal{U},\ \text{s.t. } T(u) = 0 \}$$

(aka Kernel)

VERIFY: If T is linear (need not be representable by a matrix since $\mathcal{U}$ or $\mathcal{V}$ can be infinite dimensional)

$\mathcal{R}(T)$ is a subspace.
$\mathcal{N}(T)$ " " " .

Verify using basic definition of linearity of T.



LINEAR MAPS BETWEEN FINITE DIMENSIO-NAL VECTOR SPACES

Let $\mathcal{U}, \mathcal{V}$ be finite dimensional vector spaces over a field $\mathbb{F}$.

Say, $\dim(\mathcal{U}) = n$, $\dim(\mathcal{V}) = m$.

Consider a <u>linear</u> map $T : \mathcal{U} \to \mathcal{V}$.

We will show that T can be represented by a matrix.

Consider a basis $\{u_1, u_2, \ldots, u_n\}$ of $\mathcal{U}$.

Similarly, let $\{v_1, \ldots, v_m\}$ be a basis of $\mathcal{V}$.

Let u be any vector in $\mathcal{U}$.

and $v = T(u)$

Notice that u can be represented as

$$u = \sum_{i=1}^{n} \alpha_i u_i$$

Since $\{u_1, \ldots, u_n\}$ is a basis, $\{\alpha_1, \ldots, \alpha_n\}$ are unique for a given u.

Similary, v can be represented as

$$v = \sum_{i=1}^{m} \beta_i v_i$$

Since $\{v_1, \dots, v_m\}$ is a basis of $\mathcal{V}$, $\{\beta_1, \dots, \beta_m\}$ are unique for a given v.

We will show that $\beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix}$ and $\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$ are related by a matrix $C \in \mathbb{F}^{m \times n}$

ie $$\boxed{\beta = C\,\alpha}$$

[Note that β uniquely defines v and α uniquely defines u]

<u>Proof:</u>

$$v = T(u) = T\left(\sum_{i=1}^{n} \alpha_i u_i\right)$$

Linearity of T

$$= \sum_{i=1}^{n} \alpha_i T(u_i) \quad (1)$$

Now $T(u_1), T(u_2), \dots, T(u_n) \in \mathcal{V}$

Since $\{v_1, \dots, v_m\}$ is a basis of $\mathcal{V}$

we must have

$$T(u_1) = \sum_{j=1}^{m} \gamma_{1j} v_j \quad \text{---} \ (2)$$

$$T(u_2) = \sum_{j=1}^{m} \gamma_{2j} v_j \quad \text{---} \ (3)$$

$$\vdots$$

$$T(u_n) = \sum_{j=1}^{m} \gamma_{nj} v_j \quad \text{---} \ (n+1)$$

Substituting (2), (3), ..., (n+1) into (1)

we get

$$v = \sum_{i=1}^{n} \alpha_i T(u_i) = \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m} \gamma_{ij} v_j$$

$$= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} \gamma_{ij} \alpha_i \right) v_j$$

However

$$v = \sum_{j=1}^{m} \beta_j v_j$$

Since basis expansion coefficients are unique, we have

$$\beta_j = \sum_{i=1}^{n} \alpha_i \gamma_{ij} \quad , \quad j = 1, 2, \dots, m$$

$$\Rightarrow \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix}}_{\in \mathbb{F}^m} = \begin{bmatrix} \sum_{i=1}^{n} \gamma_{i1} \alpha_i \\ \sum_{i=1}^{n} \gamma_{i2} \alpha_i \\ \vdots \\ \sum_{i=1}^{m} \gamma_{im} \alpha_i \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \gamma_{11} & \gamma_{21} & \cdots & \gamma_{n1} \\ \gamma_{12} & \gamma_{22} & \cdots & \gamma_{n2} \\ \vdots & \vdots & & \vdots \\ \gamma_{1m} & \gamma_{2m} & \cdots & \gamma_{nm} \end{bmatrix}}_{\text{Call this } C} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}}_{\in \mathbb{F}^n}$$

$$\Rightarrow \boxed{\beta = C\alpha}$$

Hence $T: \mathcal{U} \to \mathcal{V}$ is represented by an $m \times n$ matrix $C \in \mathbb{F}^{m \times n}$, where $\dim(\mathcal{U}) = n$, $\dim(\mathcal{V}) = m$.

This means that if you need to compute $T(u)$ for any u in $\mathcal{U}$, you can equivalently do so using "matrix vector multiplication" with C, as follows:-

(1) Obtain the basis expansion coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$ for u, with respect to the basis $\{u_1, u_2, \ldots, u_n\}$.

Note $\alpha_1, \ldots, \alpha_n$ uniquely identifies "u".

(2) Apply the matrix C on $\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$ to obtain $\beta \in \mathbb{F}^m$ as follows:-

$$\beta = C\alpha.$$

(3) Using $\beta_1, \beta_2, \ldots, \beta_m$ construct (or Synthesize) v as:

$$v = \sum_{i=1}^{m} \beta_i v_i \quad , \{v_1, v_2, \ldots, v_m\} \text{ is the basis of } \mathcal{V}.$$

(4) v is the desired $T(u)$.

RANK-NULLITY THEOREM

Let $\mathcal{U}, \mathcal{V}$ be vector spaces over a field $\mathbb{F}$. Let $T : \mathcal{U} \to \mathcal{V}$ be a linear map. Suppose $\mathcal{U}$ is finite dimensional, Then.

$$\dim(R(T)) + \dim(\mathcal{N}(T)) = \dim(\mathcal{U}).$$

Proof:

$$R(T) = \{v \in \mathcal{V}, \text{ s.t. } v = T(u) \text{ for some } u \in \mathcal{U}\}$$

$$\mathcal{N}(T) = \{u \in \mathcal{U}, \text{ s.t. } T(u) = 0\}$$

$N(T) \subseteq \mathcal{U}$. Let $\dim(\mathcal{U}) = n$ (n is a finite integer)

$$\dim(N(T)) \leq n.$$

Let $k = \dim(N(T)) \leq n$.

Let $\{u_1, v_2, v_3, \ldots, u_k\}$ be a basis for $N(T)$

$\Rightarrow \{u_1, u_2, \ldots, u_k\}$ is a linearly independent set in $\mathcal{U}$.

$\Rightarrow$ Recall: Any linearly independent set in a vector space is contained in some basis of the vector space.

$\Rightarrow$ There exist vectors $u_{k+1}, u_{k+2}, \ldots, u_n \in \mathcal{U}$ s.t. $\{u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_n\}$ is a basis for $\mathcal{U}$.

Consider the vectors

$$T(u_{k+1}) = v_1$$
$$T(u_{k+2}) = v_2$$
$$\vdots \qquad \vdots$$
$$T(u_n) = v_{n-k}$$

Of course $v_1, v_2, \ldots, v_{n-k} \in R(T)$.

If we can show that $v_1, v_2, \ldots, v_{n-k}$ is a "basis" for $R(T)$, then we will have

$$\boxed{\dim(R(T)) = n-k}$$

$$\Rightarrow \dim(R(T)) + k = n \qquad \dim(\mathcal{U})$$

$$\Rightarrow \dim(R(T)) + \dim(N(T)) = n.$$

[We will establish this in next lecture.....]