Lecture 4: Linear Independence, Basis

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AGENDA

1. Linear Independence
    - Implications
    - Example: DFT

2. BASIS
    - PROPERTIES
    - EXAMPLES
    - RANK
    - RANK NULLITY: PREVIF

LINEAR INDEPENDENCE

Let $\mathcal{V}$ be a vector space over a field $\mathbb{F}$. A set of (non-zero) vectors $v_1, v_2, v_3, \ldots, v_k \in \mathcal{V}$ are said to be <u>linearly dependent</u>, if there exist $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{F}$ (not all of them zeros) such that

$$\sum_{i=1}^{k} \alpha_i v_i = \underbrace{0}_{\text{zero of } \mathcal{V}}$$

On the other hand, $v_1, v_2, \ldots, v_k$ are <u>linearly independent</u> if

$$\sum_{i=1}^{k} \alpha_i v_i = 0 \iff \alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_k = \underbrace{0}_{0 \in \mathbb{F}.}$$

Example:

Linearly Independent.

① $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $(\mathcal{V} = \mathbb{R}^3)$

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0 \Rightarrow \begin{bmatrix} \alpha_2 \\ \alpha_3 \\ \alpha_1 + \alpha_2 + \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$$

② $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ -2.8 \\ 7.1 \end{bmatrix}$ $(\mathcal{V} = \mathbb{R}^3)$

Verify they are linearly dependent!

IMPLICATIONS OF LINEAR INDEPENDENCE

Suppose $v_1, \dots, v_k$ in $\mathbb{F}^n$ are linearly independent

Let $V = [v_1 \ v_2 \ \dots \ v_k]$ ($n \times k$ matrix)

(1) $N(V) = \{x \mid Vx = 0\}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}$, $x_i \in \mathbb{F}$.

$$= \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \;\middle|\; \sum_{i=1}^{k} x_i v_i = 0 \right\}$$

$$= \left\{ \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$$

[$N(V) = \{0\}$ is equivalent to $v_1, \dots, v_k$ being linearly independent]

(2) Let $v \in \operatorname{span}\{v_1, \dots, v_k\} = R(V)$.

Then we know there exist $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{F}$ s.t.

$$v = \sum_{i=1}^{k} \alpha_i v_i \quad (\text{since } v \in \operatorname{span}\{v_1, \dots, v_k\})$$

Does there exist another choice of scalars, $\beta_1, \beta_2, \dots, \beta_k \in \mathbb{F}$ (such that not all α_i's are equal to β_i's) such that

$$v = \sum_{i=1}^{k} \beta_i v_i$$

No!

Proof: Suppose there exist $\alpha_1, \dots, \alpha_K$ and $\beta_1, \dots, \beta_K$ s.t.

$$V = \sum_{i=1}^{K} \alpha_i v_i \quad \text{and}$$

$$V = \sum_{i=1}^{K} \beta_i v_i$$

$$\Rightarrow \sum_{i=1}^{k} \alpha_i v_i = \sum_{i=1}^{K} \beta_i v_i$$

$$\Rightarrow \sum_{i=1}^{K} (\alpha_i - \beta_i) \cdot v_i = 0$$

Since $v_1, v_2, \dots, v_K$ are linearly independent

$$\Rightarrow \alpha_i - \beta_i = 0, \quad i = 1, 2, \dots, K$$

$$\Rightarrow \alpha_i = \beta_i, \quad i = 1, 2, \dots, K$$

If $v_1, v_2, \dots, v_K$ are linearly independent then any vector $v \in \text{span}\{v_1, \dots, v_K\}$ can be represented as a linear combination of $v_1, \dots, v_K$ in EXACTLY ONE WAY!

Example of Unique Representation Property

Let $\mathcal{V} = \mathbb{C}^n$, $\mathbb{F} = \mathbb{C}$.

Consider a set of n vectors: $v_0, v_1, \ldots, v_{n-1}$ given by

$$v_i = \begin{bmatrix} 1 \\ e^{j\frac{2\pi i}{n}} \\ e^{j\frac{2\pi \cdot 2i}{n}} \\ \vdots \\ e^{j\frac{2\pi(n-1)i}{n}} \end{bmatrix}, \quad i = 0, 1, 2, \ldots, n-1$$

- It can be shown that $\{v_0, \ldots, v_{n-1}\}$ are linearly independent
- Furthermore, it can also be shown that
$$\text{span}\{v_0, v_1, \ldots, v_{n-1}\} = \mathbb{C}^n$$

$\Rightarrow$ Due to unique representation property, any vector $v \in \mathbb{C}^n$ can be represented as a linear combination of $v_0, v_1, \ldots, v_{n-1}$ in exactly one way!

i.e. $v = \sum_{i=0}^{n-1} \alpha_i v_i$,

$\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$ are known as the Discrete Fourier Transform (DFT) coefficients of the signal v.

Basis

Let $\mathcal{V}$ be a vector space over a field $\mathbb{F}$.
A set of vectors $v_1, v_2, \ldots, v_k \in \mathcal{V}$ is said to be a **basis** of $\mathcal{V}$ if

(i) $\text{span}\{v_1, \ldots, v_k\} = \mathcal{V}$. (i.e. every vector in $\mathcal{V}$ can be represented as a linear combination of $v_1, \ldots, v_k$)

AND

(ii) $\{v_1, \ldots, v_k\}$ is a linearly independent set.

FACT

[All bases (if finite) of $\mathcal{V}$ must have the same number of elements (cardinality)]

Dimension

The cardinality (or the number of elements) of a basis of $\mathcal{V}$ is known as the dimension of $\mathcal{V}$. (denoted $\dim(\mathcal{V})$)

In the above example, $k = \dim(\mathcal{V})$

V can have multiple bases

eg. $V = \mathbb{R}^2$

One basis : $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Another basis : $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

another basis : $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

⋮

Verify, in each case, that the vectors are linearly independent AND they span all of $\mathbb{R}^2$.

$$\dim(\mathbb{R}^2) = 2$$

Number of possible Bases in $\mathbb{R}^n$ or $\mathbb{C}^n$ is infinite.

Basis of $\mathbb{F}^n$

More generally, consider $V = \mathbb{F}^n$, field $= \mathbb{F}$

Consider the following vectors in V :-

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; \ldots, \; v_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; \ldots, \; v_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$1 \in F$

$0 \in F$

ith position

- **Fact 1:** $\{v_1, v_2, \ldots, v_n\}$ are linearly independent.

(Verify: $\sum_{i=1}^{n} \alpha_i v_i = 0 \iff \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \iff \alpha_i = 0$, $i = 1, 2, \ldots, n$)

- **Fact 2:** Any $u \in \mathbb{F}^n$ can be represented as a linear combination of $v_1, \ldots, v_n$

(Verify: $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = u_1 \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + u_2 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + u_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$)

$\Rightarrow$ $\boxed{\{v_1, \ldots, v_n\} \text{ is a basis of } \mathbb{F}^n \text{ and therefore, } \dim(\mathbb{F}^n) = n.}$

Basis of $\mathbb{F}^{m\times n}$?

Let $V = \mathbb{F}^{m\times n}$, Field $= \mathbb{F}$.

Question: What is a Basis of $\mathbb{F}^{m\times n}$?

Question: Using the Basis you found, argue that $\dim(\mathbb{F}^{m\times n}) = mn$.

(DIY).

Symmetric Matrices

Let $V = \mathbb{F}^{m \times m}$, field $= \mathbb{F}$.

Define the following subset of $\mathbb{F}^{m \times m}$:-

$$\mathbb{S} := \left\{ X \in \mathbb{F}^{m \times m},\ \text{s.t. } X_{ij} = X_{ji},\ i,j = 1,2,\dots m \right\}$$

- Verify that $\mathbb{S}$ is a SUBSPACE.
- Prove that $\boxed{\dim(\mathbb{S}) = \frac{m(m+1)}{2}}$

(Hint: Construct a basis for $\mathbb{S}$).

FACTS ABOUT BASES

1. All vector spaces have bases
[A deep fact, not easy to show for infinite dimensional vector spaces]

2. All bases of a (finite dimensional) vector space have the same cardinality.

3. If $\dim(\mathcal{V}) = p$

(i) Any set of p linearly independent vectors in $\mathcal{V}$ form a basis of $\mathcal{V}$.

(ii) Any set of $k \leq p$ linearly independent vectors in $\mathcal{V}$ is contained in some basis of $\mathcal{V}$.

(iii) Any set of $k > p$ vectors in $\mathcal{V}$ must be linearly dependent.

(iv) Any spanning set of $\mathcal{V}$ contains a basis.

(v) Any spanning set of $\mathcal{V}$ with exactly p elements is a basis of $\mathcal{V}$

Dimension And Rank

Given $A \in \mathbb{F}^{m \times n}$, $A = [a_1 \; a_2 \; \cdots \; a_n]$

Let $\dim(R(A)) = \alpha$

Since $R(A) = \text{span}\{a_1, a_2, \ldots, a_n\}$, this means that

- there exist "α" columns of A which form a basis for $R(A)$
- The above is equivalent to saying that there exist α linearly independent columns of A and other column of A can be represented as a linear combination of these α columns.

Recall the definition of rank of a matrix:

"Rank(A) is the maximum number of linearly independent columns in A"

Therefore, if $\text{rank}(A) = r$, this means that there are "r" linearly independent columns in A, AND any other column of A can be represented as a linear combination of these columns

$\Rightarrow$ $\boxed{\dim(R(A)) \text{ \& } \text{rank}(A) \text{ are same}}$

What is $\dim(\mathcal{N}(A))$?

Surprisingly, once we know $\operatorname{rank}(A)$, we can also determine $\dim(\mathcal{N}(A))$. The following beautiful result summarizes the relation between the two (to be proved in next lecture):—

RANK-NULLITY THEOREM

Given a matrix $A \in \mathbb{F}^{m\times n}$, the following holds:

$$\boxed{\operatorname{rank}(A) + \dim(\mathcal{N}(A)) = n}$$