Fields

Definition: A field is a set $\mathbb{F}$ equipped with two operations $+$ and $\cdot$, (called addition and multiplication, respectively) obeying the following rules or axioms, $\forall x, y, z \in \mathbb{F}$.

(i) $x+y \in \mathbb{F}$, $x \cdot y \in \mathbb{F}$

(ii) $x+y = y+x$ (commutativity of addition)

(iii) $(x+y)+z = x+(y+z)$ (associativity of add.)

(iv) There is a "zero" in $\mathbb{F}$ such that $x+0 = x$

(v) For each $x \in \mathbb{F}$ there is an "additive inverse" $(-x)$ $x+(-x)=0$.

(vi) $x \cdot y = y \cdot x$ (vii) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

(viii) $(x+y) \cdot z = x \cdot z + y \cdot z$ (distributivity).

(ix) There is a $1 \in \mathbb{F}$, $1 \neq 0$, s.t. $x \cdot 1 = x$

(x) If $x \neq 0$, there is an element in $\mathbb{F}$ "Multiplicative Inverse" of x (denoted x^{-1}) s.t. $x \cdot x^{-1} = 1$

Property:

① 0 is unique ② 1 is unique

③ Additive of every element is unique.

④ Multi inverse - - -- - is unique.

⑤ $a \cdot 0 = 0$ ⑥ $(-1) \cdot a = -a$

e.g. $\mathbb{C}$: Set of all complex numbers

Vector Spaces

A vector space $\mathcal{V}$, defined over a field $\mathbb{F}$, is a non-empty set (whose members are called vectors) equipped with two operations:

(i) Vector addition, $+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$

(ii) Scalar multiplication, $\cdot : \mathbb{F} \times \mathcal{V} \to \mathcal{V}$

which must satisfy the following properties

$u, v \in \mathcal{V}$

(A1): $u+v \in \mathcal{V}$

(A2) $u+v = v+u, \forall u, v \in \mathcal{V}$

(A3) $(u+v)+w = u+(v+w)$

(A4) $0+v = v$, $0 \in \mathbb{F}, v \in \mathcal{V}$

(A5) $v + v_I = 0$, $v_I \in \mathcal{V}$

$\alpha, \beta \in \mathbb{F}$

(M1) $\alpha \cdot v \in \mathcal{V}$

(M2) $\alpha \cdot (\beta \cdot v) = (\alpha \cdot \beta) \cdot v$

(M3) $1 \cdot v = v$ $1 \in \mathbb{F}$

(M4) $\alpha \cdot (v+u) = \alpha v + \alpha u$

(M5) $(\alpha+\beta) \cdot v = \alpha v + \beta v$

e.t.c.

Define: $C(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R},\ f \text{ is continuous} \}$

$f \in C(\mathbb{R})$, $g \in C(\mathbb{R})$

$(f+g)(t) = f(t) + g(t)$

$(\alpha f)(t) = \alpha \cdot f(t)$.

0 is $f_0(t) = 0, \forall t$.

$f_I(t) = -f(t), \forall t$.

$C(\mathbb{R})$ is a vector space over $\mathbb{R}$

Subspaces

A subspace $\mathcal{W}$ of a vector space $\mathcal{V}$ over a field $\mathbb{F}$, is a non-empty subset of $\mathcal{V}$ (i.e. $\mathcal{W} \subseteq \mathcal{V}$) such that

(i) If $x, y \in \mathcal{W}$, then $x+y \in \mathcal{W}$

(iii) If $x \in \mathcal{W}$, then $\alpha \cdot x \in \mathcal{W}$, $\forall \alpha \in \mathbb{F}$

Q: Is $0 \in \mathcal{V}$ always included in any subspace $\mathcal{W}$? YES.

Q: Is $\mathcal{W}$ a vectorspace? YES (All axioms can be chosen)

※ R(A), N(A)

Suppose we are given a matrix $A \in \mathbb{F}^{m \times n}$

Then: define:

(i) $R(A) = \{ y \in \mathbb{F}^m,\ \text{s.t. } y = Ax, x \in \mathbb{F}^n \}$

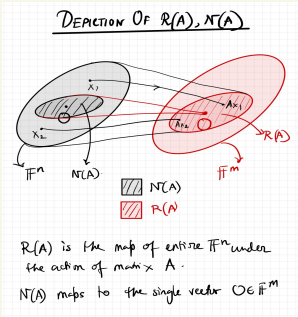
$R(A)$ is also known as Range Space/ column space of A.

(ii) $N(A) = \{ x \in \mathbb{F}^n,\ \text{s.t. } Ax = 0 \}$

$N(A)$ is Null space of A

$R(A)$ is a subspace of $\mathbb{F}^m$

$N(A)$ is a subspace of $\mathbb{F}^n$



Linear Independence

Let $\mathcal{V}$ be a vector space. A set of (non-zero) vectors $v_1, v_2, \dots v_k \in \mathcal{V}$ are said to be linearly dependent, if there exist $\alpha_1, \dots \alpha_k \in \mathbb{F}$ (not all of them zeros), such that $\sum_{i=1}^{k} \alpha_i v_i = 0$ ($\alpha \neq 0$)

on the other hand, $v_1, \dots v_k$ are linearly independent if $\sum_{i=1}^{k} \alpha_i v_i = 0$, $\forall \alpha_i = 0$, $0 \in \mathbb{F}$

e.g. $v_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\mathcal{V} \in \mathbb{R}^3$

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0 \Rightarrow \begin{bmatrix} 2\alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0 \Rightarrow \alpha_1, \alpha_2, \alpha_3 = 0$.

Implication of Linear Independence -

Suppose $v_1, \dots v_k$ in $\mathbb{F}^n$ are linearly independent

(let $V = [v_1 \sim v_k]$ $(n \times k$ matrix)

① $N(V) = \{ x \mid Vx = 0 \}$ $x = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$

$= \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \;\middle|\; \sum_{i=1}^{k} x_i v_i = 0 \right\}$

$= \left\{ \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$ [$N(V) = \{0\}$ is equal to $v_1 \sim v_k$ being linearly independent]

② Let $v \in \text{span}\{v_1 \sim v_k\} = R(V)$

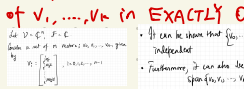
Then we know there exist $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{F}$ s.t. $v = \sum_{i=1}^{k} \alpha_i v_i$ (since $v \in \text{span}\{v_1 \sim v_k\}$)

Does there exist another choice of scalars, $\beta_1, \beta_2, \dots, \beta_k \in \mathbb{F}$ (such that not all α_i are equal to β_i) s.t. $v = \sum_{i=1}^{k} \beta_i v_i$

No! $\sum_{i=1}^{k} \alpha_i v_i = \sum_{i=1}^{k} \beta_i v_i \Rightarrow \sum_{i=1}^{k} \alpha_i v_i - \sum_{i=1}^{k} \beta_i v_i = 0$

$\Rightarrow \alpha_i - \beta_i = 0 \to \alpha_i = \beta_i$

If $v_1, v_2, \dots, v_k$ are linearly independent then any vector $v \in \text{span}\{v_1 \sim v_k\}$ can be represented as a linear combination of $v_1, \dots, v_k$ in EXACTLY ONE WAY!



Basis

Basis: Let $\mathcal{V}$ be a v.s. over $\mathbb{F}$, A set of vectors.

$v_1, v_2, \dots v_k \in \mathcal{V}$ is said to be a basis of $\mathcal{V}$ if

(i) $\text{span}\{v_1, \dots, v_k\} = \mathcal{V}$ (i.e. every vector in $\mathcal{V}$ can be represent as a linear combination of $v_1 \sim v_k$)

(ii) $\{v_1 \sim v_k\}$ is a linearly independent set.

[All bases (if finite) of $\mathcal{V}$ must have same number of elements (cardinality)]

Dimension:

The cardinality (or the number of elements) of a basis of $\mathcal{V}$ is known as the dimension of $\mathcal{V}$

$\dim(\mathcal{V}) = k$ Number of possible bases in $\mathbb{R}^n$ or $\mathbb{C}^n$ is infinite.

ex. Verify: $u_1 = \begin{bmatrix} u_{11} \\ \vdots \\ u_{n1} \end{bmatrix} = u_{11} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + u_{21} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \dots + u_{n1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

$v_1 \quad v_2 \quad \dots \quad v_n$

any $u \in \mathbb{F}^n$ can be represented as a linear combination of $v_1 \sim v_n$.

Facts About BASES

① All vector space have bases

② All bases of a (finite dim) vector space have the same cardinality

③ If $\dim(\mathcal{V}) = P$

(i) Any set of P linearly independent vectors in $\mathcal{V}$ form a basis of $\mathcal{V}$

(ii) Any set of $k \leq P$ linear. In. vectors in $\mathcal{V}$ is contained in some basis of $\mathcal{V}$

(iii) Any set of $k > P$ vectors in $\mathcal{V}$ must be L.D.

(iv) Any spanning set of $\mathcal{V}$ contains a basis

(v) Any spanning set of $\mathcal{V}$ with exactly P elements is a basis of $\mathcal{V}$

Additional Properties of Subspaces

$\mathcal{U}, \mathcal{W}$ are subspaces of a vector space $\mathcal{V}$

Define following subset of $\mathcal{V}$

(i) $\mathcal{U}+\mathcal{W} = \{ u+w,\ u \in \mathcal{U}, w \in \mathcal{W} \}$ ($u+w$ in $\mathcal{V}$)

(ii) $\mathcal{U} \cap \mathcal{W} = \{ u,\ u \in \mathcal{U},\ u \in \mathcal{W} \}$ (Intersection)

(iii) $\mathcal{U} \cup \mathcal{W} = \{ u,\ \text{either } u \in \mathcal{U} \text{ or } u \in \mathcal{W} \}$ (Union)

(i) $\mathcal{U}+\mathcal{W}$ is a subspace of $\mathcal{V}$

(ii) $\mathcal{U} \cap \mathcal{W}$ is a subspace of $\mathcal{V}$

(iii) $\mathcal{U} \cup \mathcal{W}$ is Not Always A subspace.

Span: Given a set of vectors $\{v_1 \sim v_k\} \in \mathcal{V}$ their span is define as

$\text{Span}\{v_1 \sim v_k\} = \{ \sum_{i=1}^{k} \alpha_i v_i,\ \alpha_i \in \mathbb{F} \}$

$\text{span}\{v_1 \sim v_k\}$ is a subspace of $\mathcal{V}$.

R(A): Given $A \in \mathbb{F}^{m \times n}$, $A = [a_1 \sim a_n]$

$R(A) = \{ Ax,\ x \in \mathbb{F}^n \}$

$= \{ \sum_{i=1}^{n} a_i x_i,\ x_i \in \mathbb{F} \}$

$= \text{span}\{a_1 \sim a_n\}$

Dimension And Rank.

Given $A \in \mathbb{F}^{m\times n}$, $A = [a_1 \cdots a_n]$, let $\dim(R(A)) = 2$.

Since $R(A) = \text{span}\{a_1, a_2, \cdots a_n\}$, s.t.

- there exist "2" columns of A which form a basis for R(A).
- The above is equivalent to saying that there exist 2 linearly independent colums of A and other colums of A can be represented as a linear combination of these 2 colums.

"Rank(A) is the maximum number of linearly independent colums in A"

Therefore, if rank(A) = r, this means there are r linearly independent columns in A, AND any other colums of A can be represented as a linear combination of these columns $\Rightarrow \boxed{\dim(R(A)) = \text{RANK}(A)}$

Linear Maps

Let $\mathcal{U}$ and $\mathcal{V}$ be v.s. over $\mathbb{F}$.

A function $f: \mathcal{U} \to \mathcal{V}$ is linear if

(i) $f(\underbrace{u+v}_{\text{in } \mathcal{U}}) = \underbrace{f(u)}_{\text{in } \mathcal{V}} + f(v)$, $\forall\ u, v \in \mathcal{U}$

(ii) $f(\underbrace{\alpha \cdot u}_{\text{in } \mathcal{U}}) = \alpha \underbrace{f(u)}_{\text{in } \mathcal{V}}$, $\forall\ u \in \mathcal{U}, \alpha \in \mathbb{F}$

$f(0) = 0$ (0 in $\mathcal{U}$ ↕ 0 in $\mathcal{V}$)

E.x.

① $f: \mathbb{R} \to \mathbb{R}$: $f(x) = 2x$, $2 \in \mathbb{R}$

$f(x) = 2x + \beta$, $2, \beta \in \mathbb{R}$, $\beta \neq 0$ not linear.

② $f: \mathbb{R}^n \to \mathbb{R}^m$

Given $A \in \mathbb{R}^{m\times n}$

define $\boxed{f(x) = Ax}$ ← show this is linear by showing, $f(x+y) = f(x) + f(y)$, $f(\alpha x) = \alpha f(x)$

③ $f: \mathbb{R}^{n\times n} \to \mathbb{R}$

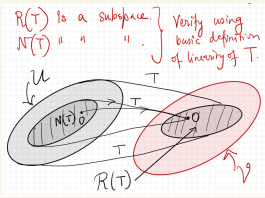
$f(x) = \sum_{i=1}^{n} x_{ii} = \text{Trace}(X)$

Range & Kernal of Linear Maps

Let $T: \mathcal{U} \to \mathcal{V}$, be a linear map. Then

$R(T) = \{v \in \mathcal{V}, \text{ s.t. } v = T(u) \text{ for some } u \in \mathcal{U}\}$

$N(T) = \{u \in \mathcal{U}, \text{ s.t. } T(u) = 0\}$ (a.k.a kernal)



Rank-Nullity Theorem

Let $\mathcal{U}, \mathcal{V}$ be vector spaces over a field $\mathbb{F}$. Let $T: \mathcal{U} \to \mathcal{V}$ be a linear map. Suppose $\mathcal{U}$ is finite dimensional, Then.

$$\dim(R(T)) + \dim(N(T)) = \dim(\mathcal{U}).$$

Proof:

$R(T) = \{v \in \mathcal{V}, \text{ s.t. } v = T(u) \text{ for some } u \in \mathcal{U}\}$

$N(T) = \{u \in \mathcal{U}, \text{ s.t. } T(u) = 0\}$

$N(T) \subseteq \mathcal{U}$. But $\dim(\mathcal{U}) = n$ (n is a finite integer)

$\dim(N(T)) \leq n$.

Let $k = \dim(N(T)) \leq n$.

Let $\{u_1, u_2, u_3, \cdots, u_k\}$ be a basis for $N(T)$

$\Rightarrow \{u_1, u_2, \cdots, u_k\}$ is a linearly independent set in $\mathcal{U}$.

$\Rightarrow$ Recall: Any linearly independent set in a vector space is contained in some basis of the vector space.

$\Rightarrow$ there exist vectors $u_{k+1}, u_{k+2}, \cdots, u_n \in \mathcal{U}$ s.t. $\{u_1, u_2, \cdots, u_k, u_{k+1}, \cdots u_n\}$ is a basis for $\mathcal{U}$.

Consider the vectors

$T(u_{k+1}) = v_1$

$T(u_{k+2}) = v_2$

$\vdots$

$T(u_n) = v_{n-k}$

Of course $v_1, v_2, \cdots v_{n-k} \in R(T)$.

If we can show that $v_1, v_2, \cdots, v_{n-k}$ is a "basis" for $R(T)$, then we will have

$\dim(R(T)) = n-k$

$\Rightarrow \dim(R(T)) + k = n$

$\Rightarrow \dim(R(T)) + \dim(N(T)) = n$.

Goal is therefore to show that $v_1, v_2, \cdots, v_{n-k}$ is a basis of $R(T)$

$\Leftrightarrow$ (i) $\{v_1, \cdots, v_{n-k}\}$ spans $R(T)$ and

(ii) $\{v_1, \cdots, v_{n-k}\}$ are linearly independent.

(i) To show $\{v_1, \cdots, v_{n-k}\}$ span $R(T)$

Easy to show that $\text{span}\{v_1, \cdots, v_{n-k}\} \subseteq R(T)$ (easy ? since $v_1, \cdots, v_{n-k} \in R(T) \Rightarrow R(T)$ is a subspace)

We now to show that $R(T) \subseteq \text{span}\{v_1, \cdots, v_{n-k}\}$ (we show that any vector $v \in R(T)$ can be expressed as a linear combination of $v_1, \cdots, v_{n-k}$)

Let $v \in R(T)$

$\Rightarrow$ there exists $u \in \mathcal{U}$ s.t. $v = T(u)$.

Now, $\{u_1, \cdots, u_k, u_{k+1}, \cdots, u_n\}$ is a basis of $\mathcal{U}$

$\Rightarrow u = \sum_{i=1}^{n} \alpha_i u_i$ $(\alpha_i \in \mathbb{F})$

$\Rightarrow v = T(u) = T\left(\sum_{i=1}^{n} \alpha_i u_i\right)$ (linearity of T)

$= \sum_{i=1}^{n} \alpha_i T(u_i)$

$= \sum_{i=1}^{k} \alpha_i T(u_i) + \sum_{i=k+1}^{n} \alpha_i T(u_i)$

$= \sum_{i=k+1}^{n} \alpha_i T(u_i) = \sum_{i=1}^{n-k} \alpha_{i+k} v_i$ ($\sum_{i=1}^{k} \alpha_i T(u_i) = 0$)

$\Rightarrow v$ is a linear combination of $v_1, \cdots v_{n-k}$

$\Rightarrow v \in \text{span}\{v_1, v_2, \cdots v_{n-k}\}$

$\Rightarrow \{v_1, v_2, \cdots, v_{n-k}\}$ spans $R(T)$

(ii) To show $\{v_1, v_2, \cdots, v_{n-k}\}$ are linearly independent.

Suppose $\sum_{i=1}^{n-k} \alpha_i v_i = 0$ for some $\alpha_1, \cdots \alpha_{n-k} \in \mathbb{F}$

$\Rightarrow \sum_{i=1}^{n-k} \alpha_i v_i = 0 \Rightarrow \sum_{i=1}^{n-k} \alpha_i T(u_{k+i}) = 0$ (linearity of T)

$\Rightarrow T\left(\sum_{i=1}^{n-k} \alpha_i (u_{k+i})\right) = 0$

$\Rightarrow \sum_{i=1}^{n-k} \alpha_i u_{k+i} \in N(T)$

Since $\{u_1, u_2, \cdots u_k\}$ is a basis of $N(T)$

$\Rightarrow$ there exist $\beta_1, \beta_2, \cdots \beta_k$ s.t.

$\sum_{i=1}^{n-k} \alpha_i u_{k+i} = \sum_{i=1}^{k} \beta_i u_i$

$\Rightarrow \sum_{i=1}^{k} \beta_i u_i - \sum_{i=1}^{n-k} \alpha_i u_{k+i} = 0$

Linear combination of $\{u_1, u_2, \cdots u_k, u_{k+1}, \cdots u_n\}$

But $\{u_1, u_2, \cdots, u_k, u_{k+1}, u_{k+2}, \cdots u_n\}$ are linearly independent.

$\Rightarrow \beta_1 = \beta_2 = \cdots = \beta_k = \alpha_{k+1} = \alpha_{k+2} = \cdots = \alpha_n = 0$

Since $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-k} = 0$, the vectors $\{v_1, v_2, \cdots, v_{n-k}\}$ must be linearly independent.

Linear Maps Between Finite Dimensional Vector Spaces

Let $\mathcal{U}, \mathcal{V}$ be finite dimensional vector spaces over a field $\mathbb{F}$.

Say, $\dim(\mathcal{U}) = n$, $\dim(\mathcal{V}) = m$.

Consider a linear map $T: \mathcal{U} \to \mathcal{V}$.

We will show that T can be represented by a matrix.

Consider a basis $\{u_1, u_2, \cdots u_n\}$ of $\mathcal{U}$.

Similarly, let $\{v_1, \cdots, v_m\}$ be a basis of $\mathcal{V}$.

Let u be any vector in $\mathcal{U}$.

and $v = T(u)$

Notice that u can be represented as

$u = \sum_{i=1}^{n} \alpha_i u_i$

Since $\{u_1, \cdots u_n\}$ is a basis, $\{\alpha_1, \cdots, \alpha_n\}$ are unique for a given u.

Since $\{v_1, \cdots, v_m\}$ is a basis of $\mathcal{V}$, $\{\beta_1, \cdots \beta_m\}$ are unique for a given v.

We will show that $\beta = \begin{bmatrix}\beta_1 \\ \vdots \\ \beta_m\end{bmatrix}$ and $\alpha = \begin{bmatrix}\alpha_1 \\ \vdots \\ \alpha_n\end{bmatrix}$ are related by a matrix $C \in \mathbb{F}^{m\times n}$

ie $\boxed{\beta = C\alpha}$

[Note that β uniquely defines v and α uniquely defines u]

Proof:

$v = T(u) = T\left(\sum_{i=1}^{n} \alpha_i u_i\right)$ (linearity of T)

$= \sum_{i=1}^{n} \alpha_i T(u_i)$ — (1)

Now $T(u_1), T(u_2), \cdots, T(u_n) \in \mathcal{V}$

Since $\{v_1, v_2, \cdots, v_m\}$ is a basis of $\mathcal{V}$,

$T(u_1) = \sum_{j=1}^{m} \gamma_{1j} v_j$ — (2)

$T(u_2) = \sum_{j=1}^{m} \gamma_{2j} v_j$ — (3)

$\vdots$

$T(u_n) = \sum_{j=1}^{m} \gamma_{nj} v_j$ — (n+1)

Substituting (2), (3), ..., (n+1) into (1)

we get

$v = \sum_{i=1}^{n} \alpha_i T(u_i) = \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m} \gamma_{ij} v_j$

$= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} \gamma_{ij} \alpha_i\right) v_j$

However

$v = \sum_{j=1}^{m} \beta_j v_j$

Since basis expansion coefficients are unique, we have

$\beta_j = \sum_{i=1}^{n} \alpha_i \gamma_{ij}$, $j = 1, 2, \cdots, m$

$\Rightarrow \begin{bmatrix}\beta_1 \\ \beta_2 \\ \vdots \\ \beta_m\end{bmatrix} = \begin{bmatrix}\sum_{i=1}^{n} \gamma_{i1}\alpha_i \\ \sum_{i=1}^{n} \gamma_{i2}\alpha_i \\ \vdots \\ \sum_{i=1}^{n} \gamma_{im}\alpha_i\end{bmatrix}$ ($\in \mathbb{F}^m$)

$= \begin{bmatrix}\gamma_{11} & \gamma_{12} & \cdots & \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2n} \\ \vdots & \vdots & & \vdots \\ \gamma_{m1} & \gamma_{m2} & \cdots & \gamma_{mn}\end{bmatrix} \begin{bmatrix}\alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n\end{bmatrix}$ ($\in \mathbb{F}^n$)

Call this C

$\Rightarrow \boxed{\beta = C\alpha}$

Hence $T: \mathcal{U} \to \mathcal{V}$ is represented by an $m \times n$ matrix $C \in \mathbb{F}^{m\times n}$, where $\dim(\mathcal{U}) = n$, $\dim(\mathcal{V}) = m$.

This means that if you need to compute $T(u)$ for any u in $\mathcal{U}$, you can equivalently do so using "matrix vector multiplication" with C, as follows:-

① Obtain the basis expansion coefficients $\alpha_1, \alpha_2, \cdots \alpha_n$ for u, with respect to the basis $\{u_1, u_2, \cdots, u_n\}$.

Note $\alpha_1, \cdots \alpha_n$ are uniquely identifies "u".

② Apply the matrix C on $\alpha = \begin{bmatrix}\alpha_1 \\ \vdots \\ \alpha_n\end{bmatrix}$ to obtain $\beta \in \mathbb{F}^m$ as follows:-

$\beta = C\alpha$

③ Using $\beta_1, \beta_2, \cdots \beta_m$ construct (or Synthesize) v as:

$v = \sum_{i=1}^{m} \beta_i v_i$, $\{v_1, v_2, \cdots v_m\}$ is the basis of $\mathcal{V}$.

④ v is the desired $T(u)$.