Lec. 13: Eigenanalysis (Contd.)

[Asynchronous Lecture]

Prof. Piya Pal
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AGENDA

① DIAGONALIZATION

- SUFFICIENT CONDITION (DISTINCT EIGENVALUES)
- NECESSARY & SUFFICIENT CONDITION

② NON DIAGONALIZABLE MATRICES & JORDAN FORM

- KEY POINTS

RECALL

A matrix $A \in \mathbb{C}^{n \times n}$ is diagonalizable if there exists an invertible matrix $P \in \mathbb{C}^{n \times n}$ and a diagonal matrix $\Lambda \in \mathbb{C}^{n \times n}$ such that

$$\boxed{A = P \Lambda P^{-1}}$$

(i.e, A is "similar" to a diagonal matrix)

- **FACT!** $A \in \mathbb{C}^{n \times n}$ is diagonalizable if and only if A has "n" linearly independent eigenvectors.

Q: CAN WE DEVELOP CONDITIONS FOR DIAGONALIZABILITY THAT INVOLVE EIGENVALUES ?

A SUFFICIENT CONDITION FOR DIAGONALIZABILITY

(DISTINCT EIGENVALUES)

<u>Fact</u>: Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be a set of distinct eigenvalues of A (i.e. $\lambda_i \neq \lambda_j$, $i \neq j$) and let $(\lambda_1, v_1), (\lambda_2, v_2), \ldots, (\lambda_k, v_k)$ be a set of corresponding eigenpairs of A.

Then $v_1, v_2, \ldots, v_k$ are linearly independent.

<u>Proof</u>: Suppose $v_1, v_2, \ldots, v_k$ are linearly dependent.

$$V = [v_1 \cdots v_k]$$

Suppose $\text{rank}(V) = r < k$

$\Rightarrow$ There exists a set of "r" linearly independent columns of V.

Without loss of generality, let those "r" columns be $v_1, v_2, \ldots, v_r$,

$\therefore$ Every column v_j, $j \geq r+1$ is a linear combination of $v_1, \ldots, v_r$

$$v_{r+1} = \sum_{i=1}^{r} \alpha_i v_i$$

$$(A-\lambda_{r+1}I)v_{r+1} = (A-\lambda_{r+1}I)\sum_{i=1}^{r}\alpha_i v_i$$

$$\Rightarrow \underbrace{Av_{r+1} - \lambda_{r+1}v_{r+1}}_{=0} = \sum_{i=1}^{r}\alpha_i \underbrace{Av_i}_{\lambda_i v_i} - \lambda_{r+1}\sum_{i=1}^{r}\alpha_i v_i$$

(since $Av_{r+1} = \lambda_{r+1}v_{r+1}$

$$\Rightarrow \quad 0 = \sum_{i=1}^{r}\alpha_i\lambda_i v_i - \lambda_{r+1}\sum_{i=1}^{r}\alpha_i v_i$$

$$0 = \sum_{i=1}^{r}\alpha_i(\lambda_i - \lambda_{r+1})v_i$$

Since $v_1, \dots, v_r$ are linearly independent, we must have

$$\alpha_i(\lambda_i - \lambda_{r+1}) = 0, \quad i = 1, 2, \dots, r$$

Since $\lambda_1, \lambda_2, \dots, \lambda_r, \lambda_{r+1}, \dots, \lambda_k$ are all distinct

$$\lambda_i - \lambda_{r+1} \neq 0, \quad i = 1, 2, \dots, r$$

$\Rightarrow$ We must have

$$\alpha_i = 0, \quad i = 1, 2, \dots, r$$

$\Rightarrow$ This violates that v_{r+1} is a (non-zero) eigenvector

$\Rightarrow$ v_{r+1} cannot be expressed as a linear combination of $v_1, \dots, v_r$

$\Rightarrow \{v_1, \dots, v_r, v_{r+1}\}$ are a set of $r+1$ linearly independent vectors.

Repeat above argument to show $\{v_1, \dots, v_{r+2}\}$ are linearly independent & keep proceeding until you show $\{v_1, \dots, v_k\}$ are linearly independent.

FACT: (Sufficient Condition for Diagonalizability)

If $A \in \mathbb{C}^{n \times n}$ has n distinct eigenvalues, then it is diagonalizable.

(Since by above proof, "n" distinct eigenvalues $\Rightarrow$ n linearly independent eigenvectors).

NECESSARY & SUFFICIENT CONDITION FOR DIAGONALIZABILITY

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of $A \in \mathbb{C}^{n\times n}$. Then A is diagonalizable if and only if

$$\boxed{\underbrace{g_A(\lambda_i)}_{\text{geometric multiplicity of } \lambda_i} = \underbrace{\alpha_A(\lambda_i)}_{\text{algebraic multiplicity of } \lambda_i}}, \quad i = 1, 2, \ldots, k.$$

PROOF: We first consider the forward direction, ie, we show that

if $g_A(\lambda_i) = \alpha_A(\lambda_i)$ $i = 1, 2, \ldots, k$

then A is diagonalizable.

[DIY: Use the previous fact that eigenvectors corresponding to distinct eigenvalues are linearly independent.

Note that $\dim(N(A - \lambda_i I)) = g_A(\lambda_i) = \alpha_A(\lambda_i)$. Hence we can construct a basis of $N(A - \lambda_i I)$ of size $\alpha_A(\lambda_i)$. This basis consists of $\alpha_A(\lambda_i)$ eigenvectors (corresponding to λ_i) which are linearly independent. Stack all these "k" bases (each of size $\alpha_A(\lambda_i)$, $i = 1, 2, \ldots, k$) together to get a total of $\sum_{i=1}^{k} \alpha_A(\lambda_i) = n$ eigenvectors of A, which will be linearly independent.]

We now establish the "only if" part:–

We assume that A is diagonalizable and show that we must have
$$g_A(\lambda_i) = \alpha_A(\lambda_i), \quad i = 1, 2, \ldots, k.$$

Proof. Since A is diagonalizable, there exists P such that

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & & & & & & \\ & \lambda_1 & & & & & O & \\ & & \ddots & & & & & \\ & & & \lambda_1 & & & & \\ & & & & \lambda_2 & & & \\ & & & & & \ddots & & \\ & & & & & & \lambda_2 & \\ & O & & & & & & \ddots \\ & & & & & & & & \lambda_k \\ & & & & & & & & & \ddots \\ & & & & & & & & & & \lambda_k \end{bmatrix}$$

repeats $\alpha_A(\lambda_1)$ times.

repeats $\alpha_A(\lambda_k)$ times

Recall
$$g_A(\lambda_1) = \dim \mathcal{N}(A - \lambda_1 I) = n - \operatorname{rank}(A - \lambda_1 I)$$

Now,
$$\operatorname{rank}(A - \lambda_1 I) = \operatorname{rank}\Big(\underbrace{P \begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & O & \\ & & \lambda_1 & & & \\ & & & \ddots & & \\ & O & & & \lambda_k & \\ & & & & & \ddots & \\ & & & & & & \lambda_k \end{bmatrix} P^{-1}}_{A} - \lambda_1 \underset{\sim}{I}\Big)$$

$I = PP^{-1}$

$$= \operatorname{rank}\left(P \begin{bmatrix} \lambda_1 - \lambda_1 & & & & & & \\ & \ddots & & & & O & \\ & & \lambda_1 - \lambda_1 & & & & \\ & & & \lambda_2 - \lambda_1 & & & \\ & & & & \ddots & & \\ & & & & & \lambda_2 - \lambda_1 & \\ & O & & & & & \lambda_k - \lambda_1 \\ & & & & & & & \ddots \\ & & & & & & & & \lambda_k - \lambda_1 \end{bmatrix} P^{-1}\right)$$

Since P is full rank (ie rank n),

$$\operatorname{rank}(A-\lambda_1 I) = \operatorname{rank}\begin{pmatrix}\begin{bmatrix}\lambda_1-\lambda_1 & & & & & & \\ & \ddots & & & & 0 & \\ & & \lambda_1-\lambda_1 & & & & \\ & & & \lambda_2-\lambda_1 & & & \\ & & & & \ddots & & \\ & 0 & & & & \lambda_2-\lambda_1 & \\ & & & & & & \ddots \\ & & & & & & & \lambda_k-\lambda_1\end{bmatrix}\end{pmatrix}$$

$$= \operatorname{rank}\begin{pmatrix}\begin{bmatrix}0 & & & & & & & \\ & 0 & & & & & 0 & \\ & & \ddots & & & & & \\ & & & 0 & & & & \\ & & & & \lambda_2-\lambda_1 & & & \\ & & & & & \ddots & & \\ & & & & & & \lambda_2-\lambda_1 & \\ & 0 & & & & & & \lambda_k-\lambda_1 \\ & & & & & & & & \ddots \\ & & & & & & & & & \lambda_k-\lambda_1\end{bmatrix}\end{pmatrix}$$

$\alpha_A(\lambda_1)$ zeros

remaining diagonal terms are all non-zero -

since $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct

$$= n - \alpha_A(\lambda_1)$$

$$\therefore \operatorname{rank}(A-\lambda_1 I) = n - \alpha_A(\lambda_1)$$

But

$$g_A(\lambda_1) = n - \operatorname{rank}(A-\lambda_1 I) = n-(n-\alpha_A(\lambda_1))$$

$$= \alpha_A(\lambda_1)$$

Repeat the same argument for $\lambda_2, \lambda_3, \ldots \lambda_k$

to establish

$$g_A(\lambda_i) = \alpha_A(\lambda_i), \quad i=1,2,\ldots,k.$$

Non-Diagonalizable Matrices

Jordan Canonical Form

Consider any matrix $A \in \mathbb{C}^{n\times n}$ (need not be diagonalizable).

Then there exists a non-singular matrix $T \in \mathbb{C}^{n\times n}$ such that

$$T^{-1}AT = \begin{bmatrix} J_1 & & & & 0 \\ & J_2 & & & \\ & & J_3 & & \\ & & & \ddots & \\ 0 & & & & J_q \end{bmatrix} \triangleq J$$

where $J_i \in \mathbb{C}^{n_i \times n_i}$, $i = 1, 2, \dots, q$.

$$J_i = \begin{array}{cc} n_i & \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix} \\ & n_i \end{array}$$

(upper bidiagonal)

- Each J_i is called a Jordan block (of size n_i)

- The same eigenvalue λ_i can appear in more than one Jordan block.

For example, we can have $J_1 \in \mathbb{C}^{3\times 3}$ (i.e $n_1 = 3$), $J_2 \in \mathbb{C}^{2\times 2}$ (i.e $n_2 = 2$) both containing eigenvalue λ_1

ie $J_1 = \begin{bmatrix} \lambda_1 & 1 & 0 \\ & \lambda_1 & 1 \\ 0 & & \lambda_1 \end{bmatrix}$, $J_2 = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$

- However, we always have $\boxed{\sum_{i=1}^{q} n_i = n}$

- SPECIAL CASE: If A is diagonalizable, then $n_i = 1$, $i = 1, 2, \dots, q$ and we automatically have $q = n$. We have "n" Jordan blocks of size 1×1, ie $J_i = \lambda_i$

- ALGEBRAIC MULTIPLICITY: $\alpha_A(\bar{\lambda})$ is the total numbes of time $\bar{\lambda}$ appears across all Jordan blocks

$$\alpha_A(\bar{\lambda}) = \sum_{i;\, \lambda_i = \bar{\lambda}} n_i$$

- **GEOMETRIC MULTIPLICITY:**

If $\bar{\lambda}$ is an eigenvalue of A, then

$g_A(\bar{\lambda})$ = # of Jordan blocks that contain $\bar{\lambda}$.

(Think about it yourself)

[Hint: $g_A(\bar{\lambda}) = \dim(\mathcal{N}(A - \bar{\lambda}I))$.

Show that $\dim(\mathcal{N}(A - \bar{\lambda}I))$

$$= \dim(\mathcal{N}(J - \bar{\lambda}I))$$

Then, examine the structure of $J - \bar{\lambda}I$ and argue what its rank will be.]

An Important Observation On Jordan Blocks

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix} \in \mathbb{C}^{n_i \times n_i}$$

$$(J_i - \lambda_i I) = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix} - \lambda_i I$$

$$= \begin{bmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}$$

$$(J_i - \lambda_i I)^2 = \begin{bmatrix} 0 & 0 & 1 & & 0 \\ & 0 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \ddots & 0 \\ 0 & & & & 0 \end{bmatrix}$$

$$\boxed{(J_i - \lambda_i I)^{n_i} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} = 0}$$

$i = 1, 2, \ldots, q.$