LEC. 12: EIGENANALYSIS

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AGENDA

1. ALGEBRAIC & GEOMETRIC MULTIPLICITIES

2. SIMILARITY

3. DIAGONALIZATION

   - IMPLICATIONS
   - NECESSARY & SUFFICIENT CONDITIONS
   - SOLVING LINEAR DIFFERENTIAL EQUATIONS

ALGEBRAIC & GEOMETRIC MULTIPLICITIES

Recall

$$\det(A - \lambda I) = 0$$

$$\Rightarrow c_0 + c_1\lambda + \cdots + c_n\lambda^n = 0$$

$$\Rightarrow \underbrace{(\lambda-\lambda_1)^{\alpha_1}(\lambda-\lambda_2)^{\alpha_2}\cdots(\lambda-\lambda_k)^{\alpha_k}}_{c_0+c_1\lambda+\cdots+c_n\lambda^n} = 0$$

Here $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct complex numbers

$\alpha_1, \alpha_2, \ldots, \alpha_k$ are positive integers

$$\alpha_i \geq 1, \quad i = 1, 2, \ldots, k$$

where α_i denote the number of times the root λ_i repeats.

$$\boxed{\sum_{i=1}^{k} \alpha_i = n}$$

Given an eigenvalue λ^* of A,

(i) ALGEBRAIC multiplicity of λ^*, denoted $\alpha_A(\lambda^*)$ is the number of times λ^* repeats as a root of the characteristic polynomial.

(ii) GEOMETRIC multiplicity of λ^*, denoted

$$g_A(\lambda^*) = \dim\left(\mathcal{N}(A - \lambda^* I)\right)$$

For any A,

$$g_A(\lambda^*) \leq \alpha_A(\lambda^*).$$

In particular, if an eigenvalue λ^* repeats only once, then $g_A(\lambda^*) = 1$.

SIMILARITY & DIAGONALIZATION

SIMILAR MATRICES: A matrix $A \in \mathbb{C}^{n \times n}$ is said to be <u>similar</u> to matrix $B \in \mathbb{C}^{n \times n}$, if there exists an <u>invertible</u> matrix $P \in \mathbb{C}^{n \times n}$ such that

$$PAP^{-1} = B \iff A = P^{-1}BP$$

Relation of eigenvectors of similar matrices discussed during Discussions

SIMILAR MATRICES SHARE Eigenvalues

Let λ be an eigenvalue of $PAP^{-1} = B$

$$\det(B - \lambda I) = 0 \iff \det(PAP^{-1} - \lambda I) = 0$$

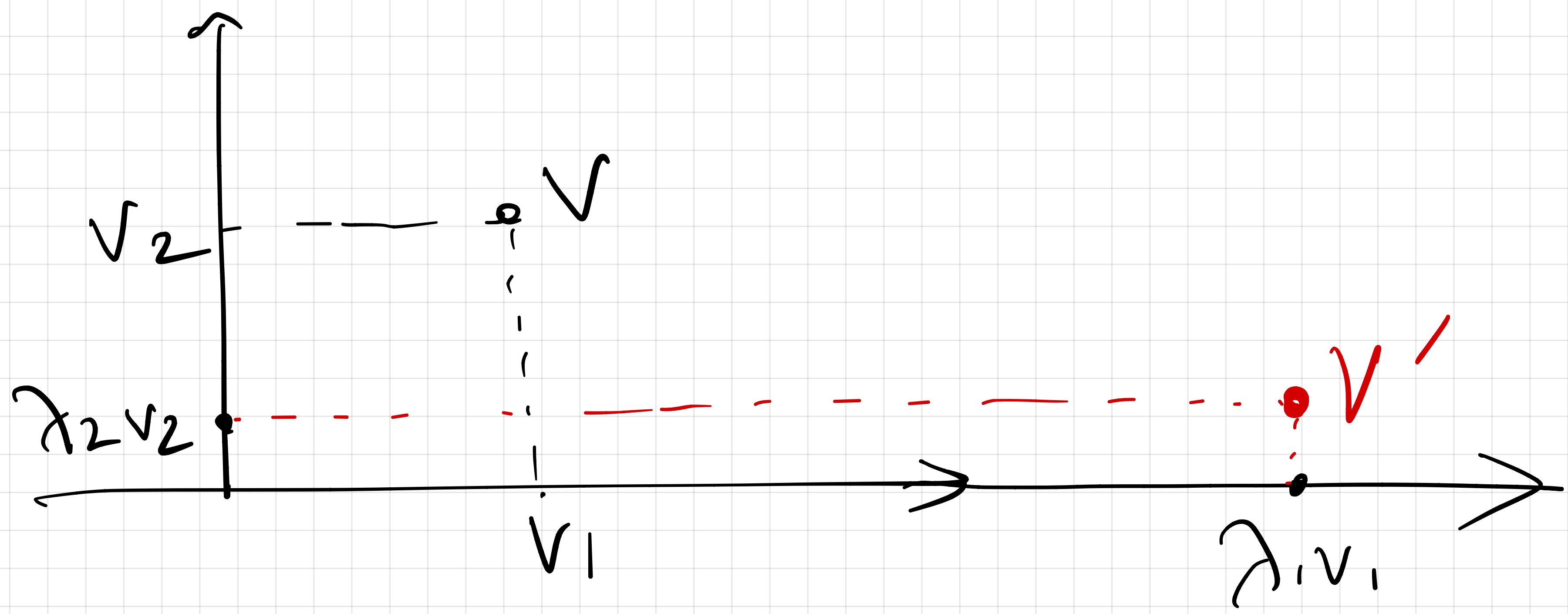
$$\iff \det\left(P(A - \lambda I)P^{-1}\right) = 0$$

$$\iff \det(P)\det(A - \lambda I)\det(P^{-1}) = 0$$

$$\iff \det(A - \lambda I) = 0.$$

DIAGONALIZATION (MOTIVATION)

$$v' = \Lambda v = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} v.$$



Diagonalization plays important role in deriving solution to linear dynamical system of equations (control theory).

$$\frac{dx(t)}{dt} = \alpha x(t). \quad \text{——— } (1)$$

$\rightarrow$ solution gives us the trajectory $x(t)$

Generalize to system of first order linear differential equations:

$$\frac{dx_1(t)}{dt} = \alpha_{11} x_1(t) + \alpha_{12} x_2(t) + \cdots + \alpha_{1n} x_n(t)$$

2 — $\vdots \qquad \vdots \qquad \vdots$

$$\frac{dx_n(t)}{dt} = \alpha_{n1} x_1(t) + \alpha_{n2} x_2(t) + \cdots + \alpha_{nn} x(t)$$

Suppose you know how to solve the scalar system 1

Can you solve 2 using the knowledge from solving 1 & linear algebra?

DIAGONALIZATION

A matrix $A \in \mathbb{C}^{n \times n}$ is said to be diagonalizable if there exists a non-singular (i.e. invertible) matrix $P \in \mathbb{C}^{n \times n}$ and a diagonal matrix $\Lambda \in \mathbb{C}^{n \times n}$ s.t.

$$P^{-1} A P = \Lambda$$

ie A is diagonalizable if and only if it is similar to a diagonal matrix.

CAUTION: Not all square matrices are diagonalizable.

Consequences Of Diagonalizability.

Suppose $A \in \mathbb{C}^{n \times n}$ is diagonalizable.

$\Rightarrow \quad P^{-1} A P = \Lambda \qquad P = [p_1 \; p_2 \; \cdots \; p_n]$

$AP = P\Lambda$

$Ap_i = \lambda_i p_i \quad , \; i = 1, 2, \ldots, n$

(i) It must hold that p_i is an eigenvector of A and λ_i is its corresponding eigenvalue

(ii) Also since P^{-1} exists, the vectors $p_1, \ldots, p_n$ must be linearly independent.

$\Rightarrow$ If A is diagonalizable, then A has a set of "n" linearly independent eigenvectors.

On the other hand, suppose $A \in \mathbb{C}^{n\times n}$ is a matrix with n linearly independent eigenvectors $v_1, v_2, \dots, v_n$ with corresponding eigenvalues (possibly repeated) $\lambda_1, \lambda_2, \dots, \lambda_n$.

$$Av_1 = \lambda_1 v_1$$
$$Av_2 = \lambda_2 v_2$$
$$\vdots \qquad \vdots$$
$$Av_n = \lambda_n v_n$$

$$\Rightarrow A\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

Since $v_1, \dots, v_n$ are linearly independent,

$V = [v_1 \cdots, v_n]$ is invertible.

$$\left(\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\right)$$

$$\Rightarrow AV = V\Lambda$$

$$\Rightarrow V^{-1}AV = \Lambda$$

$\Rightarrow$ $\boxed{A \text{ is diagonalizable}}$

A matrix $A \in \mathbb{C}^{n \times n}$ is diagonalizable if and only if A has a set of n linearly independent eigenvectors.

Necessary & sufficient condition for A to be diagonalizable.

Revisiting Differential Equations

Recall the system of differential equations (2)

Define $x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ such that $A_{ij} = \alpha_{ij}$

Then (2) can be re-written as

$$\boxed{\frac{dx(t)}{dt} = Ax(t)} \quad \text{--- (3)}$$

How to solve (3) ? Diagonalizable A helps!

Say $A = P \Lambda P^{-1}$, $\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

(i.e. A is diagonalizable)

Do variable transformation :-

$$y(t) = P^{-1} x(t).$$

$$\Rightarrow \frac{dy(t)}{dt} = \frac{d}{dt}\left(P^{-1}x(t)\right) = P^{-1}\frac{dx(t)}{dt}$$ (Linearity of Derivative)

$\Rightarrow$ (3) can be re-cast as

$$\underbrace{P^{-1}\frac{dx(t)}{dt}}_{\frac{dy(t)}{dt}} = \Lambda \underbrace{P^{-1}x(t)}_{y(t)}$$

$$\Rightarrow \quad \frac{dy(t)}{dt} = \Lambda y(t)$$

$$\Rightarrow \quad \underbrace{\frac{d}{dt} y_i(t) = \lambda_i y_i(t), \quad i = 1, 2, \dots, n}$$

"n" scalar differential equations, which can be solved independently.

$\therefore \quad y_i(t) = y_i(0) \cdot e^{\lambda_i t}$ (solution of scalar differential equation).

$\therefore \quad x(t) = P\, y(t) \longrightarrow$ Final solution