check $\mathbb{C}$ is a field.

Let $(x,y), (x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{C}$

(i) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in \mathbb{C}$

as $x_1 + x_2 \in \mathbb{R}$ & $y_1 + y_2 \in \mathbb{R}$.

$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2) \in \mathbb{C}$

since $x_1 x_2 - y_1 y_2 \in \mathbb{R}$

$x_1 y_2 + y_1 x_2 \in \mathbb{R}$.

(ii) Commutativity of +

$X + Y = Y + X$

$X, Y, Z \in \mathbb{C}$

(iii) Associativity of +

↳ check $(X+Y)+Z = X+(Y+Z)$

(iv) $(x, y) + (0,0) = (x+0, y+0) = (x, y)$

thus $(0,0)$ is zero of $\mathbb{C}$

v) $(x, y) + (-x, -y) = (x+(-x), y+(-y))$

$= (0,0)$

Thus $(-x, -y)$ are AI of (x, y) in $\mathbb{C}$

vi) $(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$

$= (x_2x_1 - y_2y_1, x_2y_1 + x_1y_2)$

$= (x_2, y_2) \cdot (x_1, y_1)$

(vii) Associativity over $(\cdot)$ check.

(viii) Distributivity

$((x_1, y_1) + (x_2, y_2)) \cdot (x_3, y_3)$

$= (x_1 + x_2, y_1 + y_2) \cdot (x_3, y_3)$

$= (((x_1 + x_2) \cdot x_3 - (y_1 + y_2) y_3), (x_1 + x_2) \cdot y_3 + x_3 (y_1 + y_2))$

$= (x_1x_3 + x_2x_3 - y_1y_3 - y_2y_3, x_1y_3 + x_2y_3 + x_3y_1 + x_3y_2)$

$= (x_1x_3 - y_1y_3, x_1y_3 + x_3y_1) + (x_2x_3 - y_2y_3, x_2y_3 + x_3y_2)$

$= (x_1, y_1) \cdot (x_3, y_3) + (x_2, y_2) \cdot (x_3, y_3)$

ix) Multiplicative identity

$$(x,y)\cdot(1,0) = (x\cdot 1 - y\cdot 0,\ x\cdot 0 + 1\cdot y)$$

$$= (x,y)$$

x) $(x,y) \neq 0$ Let $(x',y') = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$

$$(x,y)\cdot(x',y') = (x,y)\cdot\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$$

$$= (1,0).$$

→ Prove $a \cdot 0 = 0$ and 1 is unique

→ $a \cdot 0 = a(0+0) = a \cdot 0 + a \cdot 0 \cdot \quad a, 0 \in \mathbb{F}$

Let $a \cdot 0$ be $u \in \mathbb{F}$

$u = u + u$

Let $-u$ be AI of u on $\mathbb{F}$. (AI : Additive inverse)

$(u + (-u)) = u + (u + (-u))$

$0 = u \Rightarrow a \cdot 0 = u$

→ 1 is unique

Let $\tilde{1}, \hat{1}$ be ones of $\mathbb{F}$. $\tilde{1}, \hat{1} \in \mathbb{F}$ & $\tilde{1} \neq \hat{1}$

the $y, x \in \mathbb{F}$ $\quad \tilde{1} \cdot y = y$ & $\hat{1} \cdot x = x$

choose $x = \tilde{1}$ and $y = \hat{1}$

$\hat{1} = \tilde{1} \cdot \hat{1} = \hat{1} \cdot \tilde{1} = \tilde{1}$

commutativity over .

Thus $\hat{1} = \tilde{1}$ $\Rightarrow$ 1 is unique .

① $0 \cdot v = \underline{0}$ $\quad 0 \in \mathbb{F} \quad v \in V \quad \underline{0} \in V$

Let $0 \cdot v = u \in V$

$$u = 0 \cdot v = (0+0) \cdot v = 0 \cdot v + 0 \cdot v = u + u$$

u has an additive inverse $-u \in V$

$$u + (-u) = u + u + (-u)$$

$$\underline{0} = u \Rightarrow 0 \cdot v = \underline{0}.$$

② $\alpha \cdot \underline{0} = \underline{0}$ $\quad \alpha \in \mathbb{F}$ & $\underline{0} \in V$

$$\alpha \cdot \underline{0} = \alpha(\underline{0} + \underline{0}) = \alpha \cdot \underline{0} + \alpha \cdot \underline{0}$$

Same argument as above

$\Rightarrow \alpha \cdot \underline{0} = \underline{0}$

where $\underline{0} \in V$ and $0 \in \mathbb{F}$.

③ $(-1) \cdot v = -v$ $\quad$ where $-v$ is the A.I of $v \in V$

$$\underline{0} = 0 \cdot v = (1 + (-1)) \cdot v = 1 \cdot v + (-1) \cdot v$$

Let $1 \in \mathbb{F}$ and -1 be its A.I. (additive inverse)

$-1 \in \mathbb{F}$

we know $1 \cdot v = v$.

$\underline{0} = v + (-1) \cdot v$ $\quad$ So $(-1) \cdot v$ is AI of v.

Now, add $-v$ on both sides

$$\underline{0} + (-v) = (v + (-1) \cdot v) + (-v)$$

$$\Rightarrow -v = (v + (-v)) + (-1) \cdot v$$

$\Rightarrow -v = (\underline{0}) + (-1)\cdot v \Rightarrow \boxed{(-1)\cdot v = -v.}$ //

4) $\underline{0} \in V$ is unique

Let $\underline{\tilde{0}}$ and $\underline{\hat{0}}$ be two $\underline{0}$'s of V

$$\underline{\tilde{0}} = \underline{\tilde{0}} + \underline{\hat{0}} = \underline{\hat{0}} + \underline{\tilde{0}} = \underline{\hat{0}}$$

$\Rightarrow \underline{\tilde{0}} = \underline{\hat{0}}$.

5) $-v$ is unique

Let v_I and v_I' be two AI of v.

Since $\underline{0}$ is unique,

$$\underline{0} = v + v_I \quad \text{and} \quad \underline{0} = v + v_I'$$

$\underline{0} = v + v_I$

add v_I' on both sides,

$0 + v_I' = v + v_I + v_I'$ (use associative and commutative property over '+').

$\Rightarrow v_I' = (v + v_I') + v_I$

$\Rightarrow v_I' = \underline{0} + v_I$

$\Rightarrow \underline{v_I' = v_I}$ hence AI is unique.

(6) $\alpha \cdot v = \underline{0} \Rightarrow \alpha = 0$ or $v = \underline{0}$

(•) Let us suppose $\alpha \neq 0 \Rightarrow \alpha$ has an inverse in $\mathbb{F}$. Say α^{-1}.

LHS
$\alpha^{-1} \cdot (\alpha \cdot v) = (\alpha^{-1} \cdot \alpha) v = 1 \cdot v = v$

RHS

$\alpha^{-1} \cdot \underline{0} = \underline{0}$ Proven in 2.

$\Rightarrow v = \underline{0}$

If $\alpha \neq 0$ then $v = \underline{0}$

however if $\alpha = 0$ then from 1 we saw

$\boxed{0 \cdot v = \underline{0}}$

→ <u>vector spaces rules</u> : Let V be a vector space over F

(i) Closure under vector addition and scalar multiplication.

(ii) $u + v = v + u \quad \forall\, u, v \in V$

(iii) $(u+v)+w = u + (v+w) \;,\; \forall\, u, v, w \in V$

iv) <u>Existance of zero</u> : There is a vector $\underline{0} \in V$ such that $\underline{0} + v = v \quad \forall\, v \in V$.

(v) <u>Existance of Additive inverse (AI)</u>

Every $v \in V$ has another element $v_I \in V$ such that $v + v_I = 0$.

vi) $\alpha \cdot (\beta \cdot v) = (\alpha \cdot \beta) \cdot v \;,\; \forall\, \alpha, \beta \in F \;,\; v \in V$

(associativity of scalar multiplication)

vii) $1 \cdot v = v$, $1 \in \mathbb{F}$ and $v \in V$

(viii) $\alpha \cdot (u+v) = \alpha \cdot u + \alpha v$

viii) $(\alpha+\beta) \cdot v = \alpha \cdot v + \beta \cdot v$.

$\alpha, \beta \in F$, $v \in V$.

Let's check if $\mathbb{C}$ is a vector space over $\mathbb{R}$.

$\alpha \in \mathbb{R}$, $(x, y) \in \mathbb{C}$, $x, y \in \mathbb{R}$

$+ :$ $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$\cdot :$ $\alpha \cdot (x, y) = (\alpha \cdot x, \alpha \cdot y)$ (Exercise)