

Algebraic topology - Homework 8

December 25, 2024

(★) = not for submission, but make sure you understand how to do it

(★★) = not for submission, a bonus question which I find interesting

- (1) **The Eckman-Hilton argument.** A unitary binary operation on a set X is a function $\otimes: X \times X \rightarrow X$ such that there exists a unit $e \in X$ satisfying $x \otimes e = e \otimes x = x$ for all $x \in X$.

- (a) Suppose X has two unitary binary operations $\otimes, \boxtimes: X \times X \rightarrow X$, which satisfy

$$(a \otimes b) \boxtimes (c \otimes d) = (a \boxtimes c) \otimes (b \boxtimes d)$$

for every $a, b, c, d \in X$. Prove that \otimes and \boxtimes are the same, and that they are in fact associative and commutative.

- (b) Let G be a topological group with unit $e \in G$, prove that $\pi_1(G, e)$ is Abelian. Deduce that $\pi_1(S^1) \simeq \mathbb{Z}$.
- (c) (★) Prove that $\pi_1(G, g) \simeq \pi_1(G, e)$ for every $g \in G$.
- (d) For every $n \geq 0$, determine whether \mathbb{R}^2 with n points removed has a topological group structure.

- (2) **Gluing cells.**

- (a) Suppose a group G has a presentation with finitely many generators and relations. Find a pointed space X such that Wirtinger presentation $\pi_1(X) \simeq G$.
- (b) Let X be the space formed from the cube I^3 by identifying each face with its opposite face after applying a right-handed clockwise quarter turn. Prove that $\pi_1(X)$ is isomorphic to the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$.

- (3) **Complements of knots.** Solve exercise 22 in [Hatcher's Algebraic Topology](#), page 55.