Lyapunov's Central Limit Theorem

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Theorem 1. *Let $(X_n)_n$ be independent with finite third moments such that*

$$\frac{\sum_{i=1}^{n} E|X_i - EX_i|^3}{\left(\sum_{i=1}^{n} VX_i\right)^{3/2}} \to 0 \text{ as } n \to \infty.$$

Then,

$$Ef\left(\frac{S_n - ES_n}{\sqrt{VS_n}}\right) \to \int_{\mathbb{R}} f(x) \frac{e^{-x^2/2}}{\sqrt{2\pi}} \mathrm{d}x \text{ as } n \to \infty$$

for any f such that f, f', f'' and f''' are bounded.

It turns out, convergence for all f satisfying these conditions is equivalent to convergence for all bounded continuous f, which is equivalent to convergence in distribution.

Proof

We can assume $EX_i = 0$ without loss of generality. Write $\sigma_i = \sqrt{VX_i}$. Consider $(Y_n)_n$ independent, and also independent of $(X_n)_n$, with distribution $\mathcal{N}(0, \sigma_i^2)$.
[We accept without proof existence of such sequence in the same probability space.]

Write

$$W_n = \frac{Y_1 + \cdots + Y_n}{\sqrt{VS_n}} \sim \mathcal{N}(0,1) \quad \text{and} \quad Z_n = \frac{X_1 + \cdots + X_n}{\sqrt{VS_n}}.$$

Fix n, define $Z^0 = W_n$ and, for $i = 1, \dots, n$, define

$$V^i = Z^{i-1} - \frac{Y_i}{\sqrt{VS_n}} \quad \text{and} \quad Z^i = V^i + \frac{X_i}{\sqrt{VS_n}},$$

so that $Z^n = Z_n$. Now,

$$Ef(Z_n) - Ef(W_n) = Ef(Z^n) - Ef(Z^0) = \sum_{i=1}^{n} Ef(Z^i) - Ef(Z^{i-1}).$$

Using the Taylor expansion of f,

$$\begin{aligned} f(Z^i) &= f(V_i + \frac{X_i}{\sqrt{VS_n}}) = f(V^i) + f'(V^i)\frac{X_i}{\sqrt{VS_n}} + \frac{f''(V_i)}{2}\frac{X_i^2}{VS_n} + \frac{f'''(\theta^i)}{6}\frac{X_i^3}{(VS_n)^{3/2}}, \\ f(Z^{i-1}) &= f(V_i + \frac{Y_i}{\sqrt{VS_n}}) = f(V^i) + f'(V^i)\frac{Y_i}{\sqrt{VS_n}} + \frac{f''(V_i)}{2}\frac{Y_i^2}{VS_n} + \frac{f'''(\tilde{\theta}^i)}{6}\frac{Y_i^3}{(VS_n)^{3/2}}, \end{aligned}$$

where θ_i and $\tilde{\theta}_i$ come from the Taylor polynomial and depend on V_i, X_i and Y_i.

Notice that $EX_i = EY_i$, $EX_i^2 = EY_i^2$, $X_i \perp\!\!\!\perp V^i$ and $Y_i \perp\!\!\!\perp V^i$, and recall that f and its first three derivatives are bounded. We want to take expectation above and subtract. Since f is bounded, $f(V^i)$ is integrable and this first term cancels out. Also, $f'(V^i)$ is integrable and, by independence,

$$E[f'(V^i)X_i] = Ef'(V^i) \cdot EX_i = Ef'(V^i) \cdot EY_i = E[f'(V^i)Y_i],$$

so this term also cancels out, as well as the third one for the same reason. Thus,

$$Ef(Z^i) - Ef(Z^{i-1}) = \frac{E[f'''(\theta_i)X_i^3] - E[f'''(\tilde{\theta}_i)Y_i^3]}{6(VS_n)^{3/2}},$$

so taking $C = \sup_x |f'''(x)|$ we get

$$\begin{aligned} \left|Ef(Z^i) - Ef(Z^{i-1})\right| &\leqslant \frac{CE|X_i|^3 + CE|Y_i|^3}{6(VS_n)^{3/2}} \\ &= C\frac{E|X_i|^3 + \sigma_i^3 E|\mathcal{N}|^3}{6(VS_n)^{3/2}} \\ &\leqslant C\frac{E|X_i|^3}{(VS_n)^{3/2}}, \end{aligned}$$

where in the last inequality we used that $E|\mathcal{N}|^3 = \frac{4}{\sqrt{2\pi}} < 2$ and that

$$\sigma_i = (E|X_i|^2)^{\frac{1}{2}} \leqslant (E|X_i|^3)^{\frac{1}{3}}$$

by Jensen's inequality. Finally, summing over i,

$$\left|Ef(Z^i) - Ef(Z^{i-1})\right| \leqslant \sum_{i=1}^{n} \left|Ef(Z^i) - Ef(Z^{i-1})\right| \leqslant C\frac{\sum_{i=1}^{n} E|X_i|^3}{\left(\sum_{i=1}^{n} VX_i\right)^{3/2}} \to 0$$

as $n \to \infty$ by assumption, concluding the proof of the theorem.