De Moivre - Laplace Theorem

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Theorem 1 (De Moivre 1730, Laplace 1812)**.** *Let* $(X_n)_n$ *be i.i.d. random variables distributed as Bernoulli with parameter* $p = 1 - q \in (0,1)$*, and write* $S_n = X_1 + \cdots + X_n$*. The, for all* $a < b$*,*

$$P\left(a < \frac{S_n - np}{\sqrt{npq}} \leqslant b\right) \to \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{x^2}{2}} \,\mathrm{d}x.$$

Proof

The theorem was proved by De Moivre assuming that $p = \frac{1}{2}$ and Laplace to $0 < p < 1$. In fact, it follows from a much finer approximation:

$$\frac{n!}{k!\,(n-k)!}\; p^k \; q^{n-k} \asymp \frac{1}{\sqrt{2\pi npq}} e^{-\frac{x_k^2}{2}}, \tag{$*$}$$

where

$$x_k = x_{n,k} = \frac{k - np}{\sqrt{npq}}$$

and $\asymp$ means that the ratio between both terms tends to 1 when n tends to infinity. The limit at $(*)$ is uniform if restricted to $|x_k| < M$ with any $M < \infty$ fixed.

This approximation is much finer because it says not only that the probability of fluctuation being within a certain range is well approximated by the normal but also that the probability function of each of the possible values within a fixed interval is well approximated by the Gaussian density.

To understand where this approach comes from, we first need a more palpable expression for $n!$. How likely are we to get exactly 60 if we toss a coin 120 times? The answer is easy:

$$P(S_{120} = 60) = \frac{120!}{60!\,60!} \times \frac{1}{2^{120}}.$$

This expression is simple and mathematically perfect. But how much is this probability worth? More than 15%? Less than 5%? Between 5% and 10%? A pocket calculator hangs when calculating 120!. On a computer this calculation results in 0.07268.... But what if it were 40,000 coin tosses? And if we were interested in calculating $P(S_{40.000} \leqslant 19.750)$, would we do a similar calculation 250 times to then add? The expressions with factorial are perfect for the combinatorial, but impractical to make estimates. Our help will be Stirling's formula:

$$n! \asymp n^n\, e^{-n}\, \sqrt{2\pi n}.$$

The approximation of $n!$ by the Stirling formula is very good even without taking large n. It approximates 1! by 0.92, 2! by 1.92, 4! by 23.5, and from $9! = 362,880$, which is approximated by $359,537$, the error is less Than 1%. At first glance, $n^n\, e^{-n}\, \sqrt{2\pi n}$ does not seem any nicer than $n!$. But let's see how that helps with the previous calculation. We have:

$$\frac{(2k)!}{k!\,k!} \times \frac{1}{2^{2k}} \asymp \frac{(2k)^{2k} e^{-2k} \sqrt{4\pi k}}{(k^k e^{-k} \sqrt{2\pi k})^2} \times \frac{1}{2^{2k}} = \frac{1}{\sqrt{\pi k}} =_{|k=60} 0.0728...,$$

which can be done even without calculator. More than this, we have just obtained the approximation $(*)$ in the case $p = q = \frac{1}{2}$ and $x_k = 0$.

Let's show $(*)$ for $|x_k| < M$ where M is fixed. Applying the Stirling's formula we obtain

$$\frac{n!}{k!\,(n-k)!}\, p^k\, q^{n-k} \asymp \frac{n^n e^{-n} \sqrt{2\pi n}\, p^k\, q^{n-k}}{k^k e^{-k} \sqrt{2\pi k} (n-k)^{n-k} e^{k-n} \sqrt{2\pi (n-k)}} = \frac{(\frac{np}{k})^k (\frac{nq}{n-k})^{n-k}}{\sqrt{2\pi k(n-k)/n}}.$$

Note that for $|x_k| < M$ we have

$$k = np + \sqrt{npq}\, x_k \asymp np \qquad \text{and} \qquad n-k = nq - \sqrt{npq}\, x_k \asymp nq,$$

whence

$$\frac{n!}{k!\,(n-k)!}\, p^k\, q^{n-k} \asymp \frac{\left(\frac{np}{k}\right)^k \left(\frac{nq}{n-k}\right)^{n-k}}{\sqrt{2\pi npq}} = \frac{f(n,k)}{\sqrt{2\pi npq}}.$$

Let's study $\log f(n,k)$. Rewriting every term we have

$$\frac{np}{k} = 1 - \frac{\sqrt{npq}\, x_k}{k} \qquad \text{and} \qquad \frac{nq}{n-k} = 1 + \frac{\sqrt{npq}\, x_k}{n-k}.$$

Making the Taylor expansion of $\log(1+x)$ we have

$$\log(1+x) = x - \frac{x^2}{2} + r(x), \qquad \text{where } \frac{r(x)}{x^2} \to 0 \text{ as } x \to 0.$$

Thus,

$$\log f(n,k) = k\left(-\frac{\sqrt{npq}\, x_k}{k} - \frac{npqx_k^2}{2k^2} + r\left(\tfrac{\sqrt{npq}\, x_k}{k}\right)\right) + \\ + (n-k)\left(\frac{\sqrt{npq}\, x_k}{n-k} - \frac{npqx_k^2}{2(n-k)^2} + r\left(\tfrac{\sqrt{npq}\, x_k}{n-k}\right)\right).$$

Notice that the first terms are canceled, and as $n \to \infty$,

$$\log f(n,k) \asymp -\frac{npqx_k^2}{2k} - \frac{npqx_k^2}{2(n-k)} = -\frac{n^2pqx_k^2}{2k(n-k)} \asymp -\frac{n^2pqx_k^2}{2npnq} = -\frac{x_k^2}{2},$$

from which we get

$$f(n,k) \asymp e^{-\frac{x_k^2}{2}}$$

uniformly for $|x_k| < M$, concluding the proof of $(*)$.

Summing up on the possible values of S_n we have

$$P\left(a < \tfrac{S_n - np}{\sqrt{npq}} \leqslant b\right) = \sum_{a < x_k \leqslant b} P(S_n = k) = \sum_{a < x_k \leqslant b} \frac{n!}{k!\,(n-k)!}\, p^k\, q^{n-k},$$

Where the sums are about k with the condition on x_k, which is given by $x_k = \frac{k-np}{\sqrt{npq}}$. Noting that

$$x_{k+1} - x_k = \frac{1}{\sqrt{npq}},$$

and substituting $(*)$, we get

$$P\left(a < \tfrac{S_n - np}{\sqrt{npq}} \leqslant b\right) \asymp \sum_{a < x_k \leqslant b} \frac{e^{-\frac{x_k^2}{2}}}{\sqrt{2\pi npq}} = \frac{1}{\sqrt{2\pi}} \sum_{a < x_k \leqslant b} e^{-\frac{x_k^2}{2}} \cdot [x_{k+1} - x_k].$$

Finally, we observe that the summation above is a Riemann sum that approaches the integral $\frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{x^2}{2}}\, \mathrm{d}x$. This concludes de proof of Theorem 1.

Stirling's formula

In this section we prove the following.

Theorem 2 (Stirling Formula)**.** $n! \asymp n^n\, e^{-n} \sqrt{2\pi n}$.

To understand how the formula appears, note that

$$\log n! = \log 1 + \log 2 + \cdots + \log n = \sum_{k=1}^{n} \log k$$

is an upper approximation for

$$\int_0^n \log x \, \mathrm{d}x = n \log n - n = \log(n^n e^{-n}).$$

With this argument of approximation of sum by integral, it can be shown that

$$\log n! = n \log n - n + r(n) \qquad \text{with } \frac{r(n)}{n} \to 0,$$

Which is sufficient in many applications, but we want a finer approximation. In fact, we want to asymptotically approximate $n!$ and not just $\log n!$.

Assuming a polynomial correction, let's try to approximate $n!$ by a multiple of $n^n e^{-n} n^\alpha$ with $\alpha \in \mathbb{R}$. Taking

$$c_n = \log\left(\frac{n^n e^{-n} n^\alpha}{n!}\right),$$

we get

$$\begin{aligned} c_{n+1} - c_n = \log\Big[(n+1)\left(\tfrac{n+1}{n}\right)^n \tfrac{e^{-n-1}}{e^{-n}} \left(\tfrac{n+1}{n}\right)^\alpha \tfrac{n!}{(n+1)!}\Big] = \\ = \big[n\log(1+\tfrac{1}{n}) - 1\big] + \alpha\log(1+\tfrac{1}{n}). \end{aligned}$$

Making the Taylor expansion of $\log(1+x)$ para $x \in [0,1]$ we have

$$\log(1+x) = x - \frac{x^2}{2} + r(x)$$

where $r(x)$ equals $\frac{2}{(1+\tilde{x})^3}\frac{x^3}{6}$ for some $\tilde{x} \in [0,x]$ and satisfies $0 \leqslant r(x) \leqslant \frac{x^3}{3}$.

Continuing the development of $c_{n+1} - c_n$, we get

$$\begin{aligned} c_{n+1} - c_n &= \big[n\big(\tfrac{1}{n} - \tfrac{1}{2n^2} + r(\tfrac{1}{n})\big) - 1\big] + \alpha\left(\tfrac{1}{n} - \tfrac{1}{2n^2} + r(\tfrac{1}{n})\right) \\ &= \tfrac{\alpha}{n} - \tfrac{1}{2n} + n\, r(\tfrac{1}{n}) - \tfrac{\alpha}{2n^2} + \alpha\, r(\tfrac{1}{n}) \\ &= n\, r(\tfrac{1}{n}) + \tfrac{1}{2} r(\tfrac{1}{n}) - \tfrac{1}{4n^2} \end{aligned}$$

if we choose $\alpha = \frac{1}{2}$ to cancel the terms of order $\frac{1}{n}$.

Finally, combining the latest identity and the Taylor expansion we have

$$|c_{n+1} - c_n| \leqslant \tfrac{1}{2n^2},$$

Which is summable, so $c_n \to c$ for some $c \in \mathbb{R}$. Therefore,

$$\frac{n!}{n^n e^{-n}\sqrt{n}} \to e^{-c} = \sqrt{2\lambda}$$

for some $\lambda > 0$, that is

$$n! \asymp n^n e^{-n}\sqrt{2\lambda n}.$$

It remains to show that the constant is given $\lambda = \pi$.

Finding the Constant

Stirling's formula was first proved by De Moivre, and Stirling found the value of the constant. Let's prove that $\lambda = \pi$ in two different ways.

Using the proof of De Moivre's theorem The first proof assumes that the reader saw the demonstration of De Moivre-Laplace's theorem in the previous section. By Chebyshev's inequality,

$$1 - \tfrac{1}{m^2} \leqslant P\left(-m \leqslant \tfrac{S_n - np}{npq} \leqslant +m\right) \leqslant 1.$$

Now notice that the proof of De Moivre-Laplace's theorem works assuming Stirling's formula with an unknown constant λ replacing π. Thus, taking $n \to \infty$,

$$1 - \tfrac{1}{m^2} \leqslant \int_{-m}^{m} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\lambda}} \, \mathrm{d}x \leqslant 1.$$

Taking $m \to \infty$ we get

$$\int_{\mathbb{R}} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\lambda}} \, \mathrm{d}x = 1,$$

and therefore $\lambda = \pi$.

Using Wallis product Wallis product is given by

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \left(\frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \right) = \lim_{n\to\infty} \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1}.$$

Taking the square root and using $\frac{2n}{2n+1} \to 1$ we obtain

$$\sqrt{\frac{\pi}{2}} = \lim_{n\to\infty} \frac{2 \cdot 4 \cdot 6 \cdots (2n-2)}{3 \cdot 5 \cdot 7 \cdots (2n-1)} \cdot \sqrt{2n}.$$

Multiplying by the numerator we arrive at

$$\begin{aligned} \sqrt{\frac{\pi}{2}} &= \lim_{n\to\infty} \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots (2n-2) \cdot (2n-2)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdots (2n-2) \cdot (2n-1)} \cdot \frac{2n \cdot 2n}{2n} \cdot \frac{\sqrt{2n}}{2n} \\ &= \lim_{n\to\infty} \frac{2^{2n} \left(1^2 \cdot 2^2 \cdot 3^2 \cdots n^2\right)}{(2n)!} \cdot \frac{1}{\sqrt{2n}} = \lim_{n\to\infty} \frac{2^{2n} (n!)^2}{(2n)! \sqrt{2n}}. \end{aligned}$$

Finally, substituting Stirling's formula we get

$$\sqrt{\frac{\pi}{2}} = \lim_{n\to\infty} \frac{2^{2n} n^{2n} e^{-2n} 2\lambda n}{(2n)^{2n} e^{-2n} \sqrt{4\lambda n}\sqrt{2n}} = \sqrt{\frac{\lambda}{2}},$$

and therefore $\lambda = \pi$.