Simple Random Walk on $\mathbb{Z}^d$

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Abstract

We show that simple symmetric random walk is recurrent for $d \leqslant 2$ and transient for $d > 2$.

Setup

$\mathbb{Z}^d$ is the set of vectors in $\mathbb{R}^d$ with integer coordinates. Each site $x \in \mathbb{Z}^d$ has $2d$ neighbors, $x \pm e_j$, $j = 1, \dots, d$, where $e_1 = (1, 0, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, etc.

Consider an i.i.d. sequence X_n in $\mathbb{Z}^d$ distributed as

$$P(X_n = e_j) = P(X_n = -e_j) = \tfrac{1}{2d}, \quad j = 1, \dots, d.$$

The *simple symmetric random walk on* $\mathbb{Z}^d$ is defined by

$$S_0 = 0, \quad S_n = S_{n-1} + X_n.$$

Theorem 1.

$$P(S_n = 0 \ i.o.) = \begin{cases} 1, & d \leqslant 2, \\ 0, & d > 2. \end{cases}$$

1 Idea of the proof

Write $Z_n = \mathbb{1}_{S_n=0}$, and let $R = \sum_{n=1}^{\infty} Z_n$ count the number of returns to the origin. Then

$$ER = \sum_{n=1}^{\infty} P(S_n = 0).$$

For $d = 1$, using Stirling's formula we have, for a positive constant c,

$$P(S_n = 0) = \binom{n}{n/2} 2^{-n} \asymp cn^{-1/2}$$

if n is even, and $P(S_n = 0) = 0$ if n is odd. For $d = 2$, in the first n steps, about $\frac{n}{2}$ are horizontal and $\frac{n}{2}$ are vertical. Moreover, if we know how many steps were made in each direction, their signs are independent, so it is plausible that

$$P(S_n = 0) \asymp \tfrac{1}{2}\ c^2\ (\tfrac{n}{2})^{-1}$$

for n even (the extra $\frac{1}{2}$ is related to whether the number of horizontal and vertical jumps are both even or both odd). In both $d = 1$ and $d = 2$, $ER = \infty$.

For $d = 3$, in the first n steps, about $\frac{n}{3}$ are in the x-direction $\frac{n}{3}$ are in the y-direction, and $\frac{n}{3}$ are in the z-direction. Again it is plausible that

$$P(S_n = 0|\text{parity}) \asymp c^3(\tfrac{n}{3})^{-3/2}$$

or 0 depending on parity. This in turn implies that $ER < \infty$, which certainly implies that $R < \infty$ a.s. The same reasoning also works for $d = 4, 5, 6, \dots$.

There are two steps that require careful justification: for $d \leqslant 2$, why $ER = \infty$ implies $P(R = \infty) = 1$; for $d > 2$, formalize the idea that about $\frac{n}{d}$ steps are made in each direction (while the sign of the steps remain independent).

2 Proof of transience

We consider $d = 3$. Higher dimensions are treated analogously.

Let $J_n \in \{1, 2, 3\}$ denote the direction of X_n, that is, $J_n = 1$ if $X_n = \pm e_1$, $J_n = 2$

if $X_n = \pm e_2$, and $J_n = 3$ if $X_n = \pm e_3$, . Take $Y_n = +1$ or -1 according to whether $X_n = +e_{J_n}$ or $-e_{J_n}$. Then Y_n is independent of J_n, and they are distributed as discrete uniforms on $\{-1, +1\}$ and $\{1, 2, 3\}$, respectively.

For $n \in \mathbb{N}$ fixed, let $N_1 = \sum_{k=1}^n \mathbb{1}_{J_k=1}$ count the number of steps given in the x-direction. Analogously for $N_2 = \sum_{k=1}^n \mathbb{1}_{J_k=2}$ and $N_3 = \sum_{k=1}^n \mathbb{1}_{J_k=3}$.

Given that $N_1 = n_1$, $N_2 = n_2$ and $N_3 = n_3$, the conditional probability of $S_n = 0$ is given by

$$P\big(S_n = 0 \big| (N_1, N_2, N_3) = (n_1, n_2, n_3)\big) = \tbinom{n_1}{n_1/2} 2^{-n_1} \tbinom{n_2}{n_2/2} 2^{-n_2} \tbinom{n_3}{n_3/2} 2^{-n_3} \asymp \tfrac{c^3}{\sqrt{n_1 n_2 n_3}}$$

if n_1, n_2, n_3 are all even, and 0 if some of them is odd.

Therefore,

$$\begin{aligned}
P(S_n = 0) &\asymp \sum_{\substack{n_1,n_2,n_3 \\ \text{all even}}} \frac{c^3}{\sqrt{n_1 n_2 n_3}} \cdot P\big((N_1, N_2, N_3) = (n_1, n_2, n_3)\big) \\
&\leqslant \sum_{\substack{n_1,n_2,n_3 \\ \text{all greater than } \frac{n}{4}}} \frac{c^3}{\sqrt{n_1 n_2 n_3}} \cdot P\big((N_1, N_2, N_3) = (n_1, n_2, n_3)\big) + \\
&\quad + \sum_{\substack{n_1,n_2,n_3 \\ \text{some smaller than } \frac{n}{4}}} \frac{c^3}{\sqrt{n_1 n_2 n_3}} \cdot P\big((N_1, N_2, N_3) = (n_1, n_2, n_3)\big) \\
&\leqslant \frac{c^3}{(\frac{n}{4})^{3/2}} \sum_{\substack{n_1,n_2,n_3 \\ \text{all greater than } \frac{n}{4}}} P\big((N_1, N_2, N_3) = (n_1, n_2, n_3)\big) + \\
&\quad + \sum_{\substack{n_1,n_2,n_3 \\ \text{some smaller than } \frac{n}{4}}} P\big((N_1, N_2, N_3) = (n_1, n_2, n_3)\big) \\
&= \frac{8c^3}{n^{3/2}} P(\text{all } N_1, N_2, N_3 \text{ are greater than } \tfrac{n}{4}) + \\
&\qquad + P(\text{some of } N_1, N_2, N_3 \text{ is smaller than } \tfrac{n}{4}) \\
&\leqslant \frac{8c^3}{n^{3/2}} + 3 \cdot \frac{E(N_1 - \frac{n}{3})^4}{(\frac{n}{12})^4} \\
&\leqslant (8c^3) n^{-3/2} + (3 \cdot 3 \cdot 12^4) n^{-2}.
\end{aligned}$$

In the last inequality we used the same estimate as in the proof of Cantelli's Law of Large Numbers, where the centered fourth moment of an i.i.d. sum is less than $3n^2$ times the centered fourth moment of each variable.

From the above estimate, we see that $P(S_n = 0)$ is summable over n, thus R is integrable, and therefore finite almost surely. This concludes the proof that the simple symmetric random walk in dimension 3 is a.s. transient.

3 Proof of recurrence

Lemma 2. *For every $k \in \mathbb{N}$, $P(R \geqslant k) = P(R \geqslant 1)^k$*

Proof. When the event "$R \geqslant k$" occurs, define $T_1 < \cdots < T_k$ as the first k times that the walk returns, that is

$$S_0 = S_{T_1} = \cdots = S_{T_k} = 0, \quad S_n \neq 0 \text{ for } T_j < n < T_{j+1}.$$

Then

$$P(R \geqslant k) = \sum_{t_1 < t_2 < \cdots < t_k} P(T_1 = t_1, T_2 = t_2, \ldots, T_k = t_k).$$

For the latter probability, notice that every possible path until time t_k has the same probability $\frac{1}{(2d)^{t_k}}$. Therefore,

$$\frac{\#\{(x_1, \ldots, x_{t_k}) : s_{t_1} = \cdots = s_{t_k} = 0, s_n \neq 0 \text{ for } t_j < n < t_{j+1}\}}{(2d)^{t_k}} = \frac{\#A_{t_1,\ldots,t_k}}{(2d)^{t_k}},$$

where, given a deterministic string $x_1, \ldots, x_{t_k}$, we denote the corresponding path by $s_n = x_1 + \cdots + x_n$.

The main observation is that the strings in the above set can be obtained by the concatenation of substrings corresponding to each return time. More precisely, writing

$$A_t = \{(x_1, \ldots, x_t) : s_t = 0, s_n \neq 0 \text{ for } 0 < n < t\},$$

we have

$$\#A_{t_1,\ldots,t_k} = \#A_{t_1} \times \#A_{t_2-t_1} \times \#A_{t_3-t_2} \cdots \times \#A_{t_k-t_{k-1}}.$$

To conclude, we write

$$
\begin{aligned}
P(R \geqslant k) &= \sum_{r_1, r_2, \ldots, r_k > 0} P(T_1 = r_1, T_2 = r_1 + r_2, \ldots, T_k = r_1 + r_2 + \cdots + r_k) \\
&= \sum_{r_1, r_2, \ldots, r_k > 0} \frac{\# A_{r_1, r_1 + r_2, \ldots, r_1 + r_2 + \cdots + r_k}}{(2d)^{r_1 + r_2 + \cdots + r_k}} \\
&= \sum_{r_1, r_2, \ldots, r_k > 0} \frac{\# A_{r_1}}{(2d)^{r_1}} \cdot \frac{\# A_{r_2}}{(2d)^{r_2}} \cdots \frac{\# A_{r_k}}{(2d)^{r_k}} \\
&= \sum_{r_1 > 0} \frac{\# A_{r_1}}{(2d)^{r_1}} \cdot \sum_{r_2 > 0} \frac{\# A_{r_2}}{(2d)^{r_2}} \times \cdots \times \sum_{r_k > 0} \frac{\# A_{r_k}}{(2d)^{r_k}} \\
&= P(R \geqslant 1)^k.
\end{aligned}
$$

□

Corollary 3. $P(R = \infty) = \begin{cases} 1, & \sum_n P(S_n = 0) = \infty, \\ 0, & \sum_n P(S_n = 0) < \infty. \end{cases}$

Proof. Let $\rho = P(R \geqslant 1)$. There are only two possibilities:

If $\rho = 1$, then $P(R \geqslant k) = 1$ for every k, so $P(R = \infty) = 1$, and $ER = \infty$.

If $\rho < 1$, then $P(R \geqslant k) = \rho^k \to 0$ as $k \to \infty$, so $P(R = \infty) = 0$ and $ER = \sum_{n=1}^{\infty} P(R \geqslant n) = \sum_{n=1}^{\infty} \rho^n = \frac{\rho}{1-\rho} < \infty$.

Recalling that that $ER = \sum_n P(S_n = 0)$, this proves the corollary. □

We now prove recurrence for dimensions $d = 1$ and $d = 2$ by showing that $\sum_n P(S_n = 0) = \infty$. For $d = 1$, $P(S_n = 0) \asymp cn^{-1/2}$ for even n, which is non-summable, concluding the proof.

For $d = 2$, recall the definition of N_1 and N_2 from the beginning of the previous section. We claim that for n even, $P(N_1, N_2 \text{ are both even}) = \frac{1}{2}$. Indeed, consider the parity of the same counting at step $n-2$: if both are odd, then one step in each direction will make them even at step n; if both are even, then two steps in the same direction will keep them even at step n. These events occur with probability $\frac{1}{2}$ each, thus proving the claim.

So for n even we have

$$
\begin{aligned}
P(S_n = 0) &\asymp \sum_{\substack{n_1, n_2 \\ \text{both even}}} \frac{c^2}{\sqrt{n_1 n_2}} \cdot P\big((N_1, N_2) = (n_1, n_2)\big) \\
&\leqslant \frac{c^2}{\sqrt{\frac{n}{2}\frac{n}{2}}} \sum_{\substack{n_1, n_2 \\ \text{both even}}} P\big((N_1, N_2) = (n_1, n_2)\big) \\
&= \frac{c^2}{\sqrt{\frac{n}{2}\frac{n}{2}}} \, P(N_1, N_2 \text{ are both even}) \\
&= c^2 \, n^{-1},
\end{aligned}
$$

which is non-summable, concluding the proof for $d = 2$.