

# Lyapunov's Central Limit Theorem

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**Theorem 1.** *Let  $(X_n)_n$  be independent with finite third moments such that*

$$\frac{\sum_{i=1}^n E|X_i - EX_i|^3}{(\sum_{i=1}^n V X_i)^{3/2}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

*Then,*

$$Ef\left(\frac{S_n - ES_n}{\sqrt{VS_n}}\right) \rightarrow \int_{\mathbb{R}} f(x) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \text{ as } n \rightarrow \infty$$

*for any  $f$  such that  $f$ ,  $f'$ ,  $f''$  and  $f'''$  are bounded.*

It turns out, convergence for all  $f$  satisfying these conditions is equivalent to convergence for all bounded continuous  $f$ , which is equivalent to convergence in distribution.

## Proof


We can assume  $EX_i = 0$  without loss of generality. Write  $\sigma_i = \sqrt{VX_i}$ . Consider  $(Y_n)_n$  independent, and also independent of  $(X_n)_n$ , with distribution  $\mathcal{N}(0, \sigma_i^2)$ .

[We accept without proof existence of such sequence in the same probability space.]

Write

$$W_n = \frac{Y_1 + \cdots + Y_n}{\sqrt{VS_n}} \sim \mathcal{N}(0, 1) \quad \text{and} \quad Z_n = \frac{X_1 + \cdots + X_n}{\sqrt{VS_n}}.$$

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Fix  $n$ , define  $Z^0 = W_n$  and, for  $i = 1, \dots, n$ , define

$$V^i = Z^{i-1} - \frac{Y_i}{\sqrt{VS_n}} \quad \text{and} \quad Z^i = V^i + \frac{X_i}{\sqrt{VS_n}},$$

so that  $Z^n = Z_n$ . Now,

$$Ef(Z_n) - Ef(W_n) = Ef(Z^n) - Ef(Z^0) = \sum_{i=1}^n Ef(Z^i) - Ef(Z^{i-1}).$$

Using the Taylor expansion of  $f$ ,

$$\begin{aligned} f(Z^i) &= f(V_i + \frac{X_i}{\sqrt{VS_n}}) = f(V^i) + f'(V^i) \frac{X_i}{\sqrt{VS_n}} + \frac{f''(V_i)}{2} \frac{X_i^2}{VS_n} + \frac{f'''(\theta^i)}{6} \frac{X_i^3}{(VS_n)^{3/2}}, \\ f(Z^{i-1}) &= f(V_i + \frac{Y_i}{\sqrt{VS_n}}) = f(V^i) + f'(V^i) \frac{Y_i}{\sqrt{VS_n}} + \frac{f''(V_i)}{2} \frac{Y_i^2}{VS_n} + \frac{f'''(\tilde{\theta}^i)}{6} \frac{Y_i^3}{(VS_n)^{3/2}}, \end{aligned}$$

where  $\theta_i$  and  $\tilde{\theta}_i$  come from the Taylor polynomial and depend on  $V_i$ ,  $X_i$  and  $Y_i$ .

Notice that  $EX_i = EY_i$ ,  $EX_i^2 = EY_i^2$ ,  $X_i \perp V^i$  and  $Y_i \perp V^i$ , and recall that  $f$  and its first three derivatives are bounded. We want to take expectation above and subtract. Since  $f$  is bounded,  $f(V^i)$  is integrable and this first term cancels out. Also,  $f'(V^i)$  is integrable and, by independence,

$$E[f'(V^i)X_i] = Ef'(V^i) \cdot EX_i = Ef'(V^i) \cdot EY_i = E[f'(V^i)Y_i],$$

so this term also cancels out, as well as the third one for the same reason. Thus,

$$Ef(Z^i) - Ef(Z^{i-1}) = \frac{E[f'''(\theta_i)X_i^3] - E[f'''(\tilde{\theta}_i)Y_i^3]}{6(VS_n)^{3/2}},$$

so taking  $C = \sup_x |f'''(x)|$  we get

$$\begin{aligned} |Ef(Z^i) - Ef(Z^{i-1})| &\leq \frac{CE|X_i|^3 + CE|Y_i|^3}{6(VS_n)^{3/2}} \\ &= C \frac{E|X_i|^3 + \sigma_i^3 E|\mathcal{N}|^3}{6(VS_n)^{3/2}} \\ &\leq C \frac{E|X_i|^3}{(VS_n)^{3/2}}, \end{aligned}$$

where in the last inequality we used that  $E|\mathcal{N}|^3 = \frac{4}{\sqrt{2\pi}} < 2$  and that

$$\sigma_i = (E|X_i|^2)^{\frac{1}{2}} \leq (E|X_i|^3)^{\frac{1}{3}}$$

by Jensen's inequality. Finally, summing over  $i$ ,

$$|Ef(Z^i) - Ef(Z^{i-1})| \leq \sum_{i=1}^n |Ef(Z^i) - Ef(Z^{i-1})| \leq C \frac{\sum_{i=1}^n E|X_i|^3}{(\sum_{i=1}^n VX_i)^{3/2}} \rightarrow 0$$

as  $n \rightarrow \infty$  by assumption, concluding the proof of the theorem.