Methods to find a Jordan basis

Note: we use (a, b, c) to denote the column vector $[\ a\ b\ c\ ]^T$.

Quick and Dirty methods

- General method. For each eigenvalue λ:
  - Find the eigenspace $E(\lambda, T)$ by solving $Tu = \lambda u$.
  - Find a basis $\mathcal{A}$ to the eigenspace $E(\lambda, T)$.
  - For each v in $\mathcal{A}$:
    - Find one v' which solves $Tv' = \lambda v' + v$, if possible.
    - Find one v'' which solves $Tv'' = \lambda v'' + v'$, if possible.
    - Find v''', v'''', etc., until the equation has no solutions.

  The result is always an L.I. family, but may not be spanning.

- Method indicated for the case of a unique λ:
  - Pick a v at random on $G(\lambda, T)$, write $u = v$.
  - Let $u' = Tu - \lambda u$, $u'' = Tu' - \lambda u'$, $u''' = Tu'' - \lambda u''$, etc., until it gives $\mathbf{0}$.
  - If fewer than n vectors have been found, find $v', v'', v''', \dots$ as above.
  - Pick random v outside the span of previous vectors, and repeat the process.

  The result is always a spanning family, but may not be L.I.

Comments

Typically, these methods fail if and only if there is an eigenvalue λ whose Jordan blocks have different sizes. Exceptions in both directions are unlikely or impossible.

The first method will fail if the basis $\mathcal{A}$ does not have vectors v that belong to $\mathsf{range}(T - \lambda I)^k$ with k as large as possible. Then the chain $v, v', v'', \dots$ will not be long enough.

An example is $T(x, y, z) = (x, y + z, z)$, so $\lambda = 1$ and $\mathcal{A} = \{(1, 1, 0), (1, 2, 0)\}$ for $E(1, T)$. The basis $\mathcal{A}$ does not have a vector in $\mathsf{range}(T - I)$. So the equation $Tv' = v' + v$ has no solutions, and the method falls short of producing 3 vectors.

The second method will fail if the threads $u_1 \mapsto u_1' \mapsto \cdots$, $u_2 \mapsto u_2' \mapsto \cdots$, etc become linearly dependent instead of reaching $\mathbf{0}$.

An example is $T(x, y, z) = (x, y + z, z)$ with $u_1 = (1, 2, 3)$, $u_1' = (0, 3, 0)$, $u_1'' = \mathbf{0}$ and $u_2 = (1, 1, 1), u_2' = (0, 1, 0), u_2'' = \mathbf{0}$, so u_2' is a multiple of u_1'.

Guaranteed method

- Find all the eigenvalues.
- For each eigenvalue λ:
  - Let $N = T - \lambda I$.
  - Compute $N^2, N^3, \ldots, N^n$.
  - Find the generalized eigenspace $G = G(\lambda, T)$ of solutions u to $N^n u = \mathbf{0}$.
  - Find a temporary basis for G.
  - Let $U_0 = G$, $U_n = \{\mathbf{0}\}$ and $\mathcal{B}_n = \emptyset$. Then $\mathcal{B}_n$ is a Jordan basis for U_n.
  - For $k = n-1, \ldots, 1, 0$:
    * Find $U_k = \mathsf{range}(N_{|_G})^k$ by applying N^k to the temporary basis of G.
    * From the previous step we have a Jordan basis $\mathcal{B}_{k+1}$ to $T_{|_{U_{k+1}}}$ given by $N^{d_1}v_1, \ldots, N^2 v_1, Nv_1, v_1, \ldots, N^{d_m}v_m, \ldots, N^2 v_m, Nv_m, v_m$, with the property that $N^{d_j+1}v_j = \mathbf{0}$ for all j.
    * For $j = 1, \ldots, m$, find one u_j such that $Nu_j = v_j$.
      Let $\tilde{\mathcal{B}}_k = N^{d_1}v_1, \ldots, N^2 v_1, Nv_1, v_1, u_1 \ldots, N^{d_m}v_m, \ldots, N^2 v_m, Nv_m, v_m, u_m$
      Then $\tilde{\mathcal{B}}_k$ is a Jordan basis for $T_{|_{\mathsf{span}\,\tilde{\mathcal{B}}_k}}$.
    * Find $\mathcal{A}_k$ be such that $\tilde{\mathcal{B}}_k \cup \mathcal{A}_k$ is a basis of U_k.
    * For each $w \in \mathcal{A}_k$:
      · Find $x \in \mathsf{span}\,\tilde{\mathcal{B}}_k$ such that $Nx = Nw$.
      · Let $u = w - x$, so $Nu = \mathbf{0}$.
    * Let $\tilde{\mathcal{A}}_k$ be the set of vectors obtained above, so $\#\tilde{\mathcal{A}}_k = \#\mathcal{A}_k$.
    * Let $\mathcal{B}_k = \tilde{\mathcal{B}}_k, \tilde{\mathcal{A}}_k$. Then $\mathcal{B}_k$ is a Jordan basis for $T_{|_{U_k}}$.
  - In the end, $\mathcal{B}_0$ is a Jordan basis for $T_{|_G}$.
- Recollecting the Jordan bases for each $T_{|_{G(\lambda,T)}}$ produces a Jordan basis for T.

Comment

This method is guaranteed because is based on the proof of existence of Jordan bases found in Axler's Linear Algebra Done Right.

In the previous example, $U_1 = \mathsf{span}(0, 11, 0)$. We can take $\mathcal{A}_1 = \{(0, -7, 0)\}$, then $\tilde{\mathcal{A}}_1 = \mathcal{A}_1$ and $\mathcal{B}_1 = \mathcal{A}_1$. By solving $Nu = (0, -7, 0)$ we can take $u = (5, 8, -7)$ and $\tilde{\mathcal{B}}_0 = \{(0, -7, 0), (5, 8, -7)\}$. In order to extend $\tilde{\mathcal{B}}_0$ to a basis of $U_0 = \mathbb{C}^3$ we can take $\mathcal{A}_0 = \{(3, -2, 7)\}$. For $w = (3, -2, 7)$, we have $Nw = (0, 7, 0)$. Solving for $Nx = Nw$, the only solution $x \in \mathsf{span}(5, 8, -7)$ is $x = (-5, -8, 7)$, hence $u = w - x = (8, 6, 0)$ and $\tilde{\mathcal{A}}_0 = \{(8, 6, 0)\}$. Finally, $\mathcal{B}_0 = \tilde{\mathcal{B}}_0 \cup \tilde{\mathcal{A}}_0 = \{(0, -7, 0), (5, 8, -7), (8, 6, 0)\}$ is a Jordan basis.

Examples

Example 1.

$$[T] = A = \begin{bmatrix} -4 & 9 \\ -4 & 8 \end{bmatrix}.$$

First with the Quick and Dirty method.

Compute eigenvalues: $\mathsf{det}(A - \lambda I) = 0$... get $\lambda = 2$.
Pick a random vector: $u = (5, 3)$.
Take $u' = Au - 2u$. Multiplying... $u' = (-3, -2)$.
Quick and Dirty method succeeded!
We already know what the Jordan form is and how to write the basis. Let's double-check:

$$Q = \begin{bmatrix} -3 & 5 \\ -2 & 3 \end{bmatrix} \implies Q^{-1}AQ = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

Let's see with the Guaranteed Method.

Compute eigenvalues: $\mathsf{det}(A - \lambda I) = 0$... get $\lambda = 2$.
Take $N = A - 2I$. Multiplying... $N^2 = \mathbf{0}$, so $G(2, T) = \mathbb{C}^2$, take the canonical basis.
We now compute U_1.
Multiplying... $y = Ne_1 = (-6, -4)$ and $y' = Ne_2 = (9, 6)$.
Perform row reduction on $[y, y']$... we see that $\mathcal{B}_1 = \{y\}$ is a basis for $U_1 = \mathsf{range}\, N$.
We now compute U_2.
Multiplying... $Ny = \mathbf{0}$, so $U_2 = \{\mathbf{0}\}$.
We now build the basis from top down:
For $k = 2$, $\mathcal{B}_2 = \emptyset$.
For $k = 1$:
U_1 is one-dimensional, so take $\mathcal{B}_1 = \{y\}$.
For $k = 0$:
Solving $Nx = y$ we get a solution $x = (4, 2)$. So $\mathcal{B}_0 = \{y, x\}$.
We already know what the Jordan form is and how to write the basis. Let's double-check:

$$Q = \begin{bmatrix} -6 & 4 \\ -4 & 2 \end{bmatrix} \implies Q^{-1}AQ = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

Example 2.

$$[T] = A = \begin{bmatrix} -2 & 2 & 1 \\ -7 & 4 & 2 \\ 5 & 0 & 0 \end{bmatrix}.$$

First with the Quick and Dirty method.

Compute eigenvalues: $\mathsf{det}(A - \lambda I) = 0$... get $\lambda = 1$ and 0.

For $\lambda = 0$:

Solve $Ax = \mathbf{0}$... get $u = (0, 1, -2)$.

Solve $Ax = u$... no solutions (echelon form has a pivot at the last column).

For $\lambda = 1$:

Solve $Ax = x$... get $v = (1, -1, 5)$.

Solve $Ax = x + v$... get $v' = (1, 2, 0)$.

Solve $Ax = x + v'$... no solutions (echelon form has a pivot at the last column).

Vectors $v, v' \in G(1, T)$ are L.I. because they belong to the same thread $v' \overset{N}{\mapsto} v \overset{N}{\mapsto} \mathbf{0}$. Vectors u, v, v' are L.I. because u belongs to $G(0, T)$.

Quick and Dirty method succeeded!

We already know what the Jordan form is and how to write the basis. Let's double-check:

$$Q = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 2 \\ -2 & 5 & 0 \end{bmatrix} \implies Q^{-1}AQ = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right].$$

Let's see with the Guaranteed Method.

Compute eigenvalues: $\mathsf{det}(A - \lambda I) = 0$... get $\lambda = 1$ and 0.

For $\lambda = 1$:

Take $N = A - I$. Multiplying...

$$N^3 = \begin{bmatrix} 0 & 0 & 0 \\ -10 & 5 & 3 \\ 20 & -10 & -6 \end{bmatrix}$$

Solving $N^3 x = \mathbf{0}$... $U_0 = G(1, T) = \mathsf{span}\{y, y'\}$ with $y = (6, 0, 20)$ and $y' = (5, 10, 0)$.

We now compute U_1.

Multiplying... $Ny = (2, -2, 10)$ and $Ny' = (5, -5, 25)$.

Doing row reduction on $[Ny, Ny']$... we get only one pivot, and it is at the first column.

Hence, $z = (2, -2, 10)$ is a basis for U_1.

We now compute U_2.

Multiplying... $Nz = \mathbf{0}$, so $U_2 = U_3 = \{\mathbf{0}\}$.

We now build the basis from top down:

For $k = 3$, $\mathcal{B}_3 = \emptyset$.

For $k = 2$, $\mathcal{B}_2 = \emptyset$.

For $k = 1$:

We can take $\mathcal{A}_1 = \{w\}$ with $w = z$.

No need to multiply since we know $Nw \in U_2 = \{\mathbf{0}\}$, so we take $\mathcal{B}_1 = \tilde{\mathcal{A}}_1 = \{(2, -2, 10)\}$.

For $k = 0$:

Solving $Nx = z$... get $v = (2, 4, 0)$ as solution.

So we take $\tilde{\mathcal{B}}_0 = \{z, v\}$.

Since $\mathsf{dim}\, U_0 = 2$, we take $\mathcal{A}_0 = \emptyset$, $\tilde{\mathcal{A}}_0 = \emptyset$, and $\mathcal{B}_0 = \{(2, -2, 10), (2, 4, 0)\}$.

For $\lambda = 0$:

Take $N = A$. Multiplying...

$$N^3 = \begin{bmatrix} -8 & 6 & 3 \\ -1 & 0 & 0 \\ -25 & 20 & 10 \end{bmatrix}.$$

Solving $N^3 x = \mathbf{0}$... $U_0 = G(0, T) = \mathsf{span}(x)$ with $x = (0, 1, -2)$.

Since $G(0, T)$ is one-dimensional, the guaranteed method will not do much here.

Compute range by multiplying... $Nx = \mathbf{0}$.

Hence $U_3 = U_2 = U_1 = \{\mathbf{0}\}$ and $\mathcal{B}_3 = \mathcal{B}_2 = \mathcal{B}_1 = \emptyset$.

So $\tilde{\mathcal{B}}_0 = \emptyset$ as a basis for U_0 we can take $\mathcal{A}_0 = \{w\}$ with $w = (0, 1, -2)$.

Multiplying... $Nw = \mathbf{0}$, so we take $x = \mathbf{0}$ and $u = w$. So $\mathcal{B}_0 = \{(0, 1, -2)\}$.

Finished.

We already know what the Jordan form is and how to write the basis. Let's double-check:

$$Q = \begin{bmatrix} 2 & 2 & 0 \\ -2 & 4 & 1 \\ 10 & 0 & -2 \end{bmatrix} \implies Q^{-1}AQ = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right].$$

Example 3.

$$[T] = A = \begin{bmatrix} -1 & -1 & 3 \\ 0 & 2 & 0 \\ -3 & -1 & 5 \end{bmatrix}.$$

First with the Quick and Dirty method.

Compute eigenvalues: $\mathsf{det}(A - \lambda I) = 0$... get $\lambda = 2$.

Pick a random vector: $v = (1, 5, 3)$.

Multiply... $y = Tv - \lambda v = (1, 0, 1)$.

Multiply... $Ty - \lambda y = \mathbf{0}$.

Solve $Tx = \lambda x + v$... no solutions (echelon form has a pivot at the last column).

To pick a vector outside the span, perform row reduction on $[v, y, I]_{3\times5}$... there are pivots on the first three rows, so we can take $z = (1, 0, 0)$.

Solve $Tx = \lambda x + z$... no solutions (echelon form has a pivot at the last column).

Multiply... $w = Tz - \lambda z = (-3, 0, -3)$.

Multiply again... $Tw - \lambda w = \mathbf{0}$.

We got four vectors, so **the method failed**.

Let's see with the Guaranteed Method.

Compute eigenvalues: $\mathsf{det}(A - \lambda I) = 0$... get $\lambda = 2$.

Take $N = A - 2I$. Multiplying... $N^3 = \mathbf{0}$, so $U_0 = \mathbb{C}^3$, take the canonical basis.

We now compute U_1.

Multiplying... $y_1 = Ne_1 = (-3, 0, -3)$, $y_2 = Ne_2 = (-1, 0, -1)$, $y_3 = Ne_3 = (3, 0, 3)$.

Performing row reduction on $[y_1, y_2, y_3]$... we get pivot only at the first column, so $\{y_1\}$ is a basis for U_1.

We now compute U_2.

$Ny_1 = \mathbf{0}$, so $U_3 = U_2 = \{\mathbf{0}\}$.

We now build the basis from top down:

For $k = 3$, $\mathcal{B}_3 = \emptyset$.

For $k = 2$, $\mathcal{B}_2 = \emptyset$.

For $k = 1$: $\mathcal{B}_1 = \{y_1\}$.

For $k = 0$:

Solve $Nx = y_1$... get a solution $z = (2, 0, 1)$.

Take $\tilde{\mathcal{B}}_0 = \{y_1, z\}$.

Since $\{y_1, z, e_1, e_2, e_3\}$ span U_0, we perform row reduction on this 3×5 matrix... get pivots on columns 1 and 2 (as expected) as well as 4. So take $w = e_2$.

Multiplying... $Nw = (-1, 0, -1)$.

Solving for $Nx = (-1, 0, -1)$ with $x \in \mathsf{span}(z)$... we get $x = \frac{1}{3}z$. To avoid fractions, we take $u = 3(w - x) = (-2, 3, -1)$.

So $\mathcal{B}_0 = \{y_1, z, u\}$.

We already know what the Jordan form is and how to write the basis. Let's double-check:

$$Q = \begin{bmatrix} -3 & 2 & -2 \\ 0 & 0 & 3 \\ -3 & 1 & -1 \end{bmatrix} \implies Q^{-1}AQ = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 2 & 0 \\ \hline 0 & 0 & 2 \end{array}\right].$$

Simpler method

Produce some threads by picking vectors at random, then apply the *stretch and reduce* algorithm. If necessary, add new threads to get a family that spans $G(\lambda, T)$.

See our other handout entitled *Finding a Jordan basis for a nilpotent operator*.

Example 4. Let us revisit the example where Quick and Dirty failed.

We got 4 vectors. They form 2 closed threads $\mathcal{A}_1, \mathcal{A}_2$:

$$(1,5,3) \mapsto (1,0,1) \mapsto \mathbf{0}, \quad (1,0,0) \mapsto (-3,0,-3) \mapsto \mathbf{0}.$$

Subtracting $-3\mathcal{A}_1$ from $\mathcal{A}_2$ gives

$$(1,5,3) \mapsto (1,0,1) \mapsto \mathbf{0}, \quad (4,15,9) \mapsto \mathbf{0} \mapsto \mathbf{0}.$$

The threads are closed and the tips are L.I. So regardless of how we got here, we found a Jordan basis!

We already know what the Jordan form is and how to write the basis. Let's double-check:

$$Q = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 5 & 15 \\ 1 & 3 & 9 \end{bmatrix} \implies Q^{-1}AQ = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 2 & 0 \\ \hline 0 & 0 & 2 \end{array}\right].$$

Example 5.

$$[T] = A = \begin{bmatrix} 1 & 18 & -8 & -2 & -9 \\ -4 & 1 & 1 & -4 & 1 \\ -3 & -7 & 5 & -2 & 4 \\ -2 & -17 & 8 & 1 & 9 \\ -5 & 7 & -2 & -6 & -1 \end{bmatrix}$$

Eigenvalues are given: 1 and 2.

Start with $\lambda = 2$,

$$N = \begin{bmatrix} -1 & 18 & -8 & -2 & -9 \\ -4 & -1 & 1 & -4 & 1 \\ -3 & -7 & 3 & -2 & 4 \\ -2 & -17 & 8 & -1 & 9 \\ -5 & 7 & -2 & -6 & -3 \end{bmatrix}.$$

Solving $N^5 x = \mathbf{0}$... we get $\{(1,0,0,-1,0),(0,0,1,0,-1)\}$ as a basis $G(2,T)$. We take the simple thread $(1,0,0,-1,0) \mapsto (1,0,-1,-1,1) \mapsto \mathbf{0}$ and we got a Jordan basis for T restricted to $G(2,T)$.

Now with $\lambda = 1$

$$N = \begin{bmatrix} 0 & 18 & -8 & -2 & -9 \\ -4 & 0 & 1 & -4 & 1 \\ -3 & -7 & 4 & -2 & 4 \\ -2 & -17 & 8 & 0 & 9 \\ -5 & 7 & -2 & -6 & -2 \end{bmatrix}.$$

Solving $N^5 x = \mathbf{0}$... we get $\{(1,1,0,0,2),(11,-1,0,-8,0),(3,7,16,0,0)\}$ as a basis $G(1,T)$. Computing each thread, we get first $(1,1,0,0,2) \mapsto (0,-2,-2,-1,-2) \mapsto \mathbf{0}$, and then

$$(11,-1,0,-8,0) \mapsto (-2,-12,-10,-5,-14) \mapsto (0,4,4,2,4) \mapsto \mathbf{0}.$$

We can ignore the first thread, and the second thread alone provides a Jordan basis!

We already know what the Jordan form is and how to write the basis. Let's double-check:

$$Q = \begin{bmatrix} 1 & 1 & 0 & -2 & 11 \\ 0 & 0 & 4 & -12 & -1 \\ -1 & 0 & 4 & -10 & 0 \\ -1 & -1 & 2 & -5 & -8 \\ 1 & 0 & 4 & -14 & 0 \end{bmatrix} \implies Q^{-1}AQ = \left[\begin{array}{cc|ccc} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].$$