Robotics 1

September 10, 2021

Exercise #1

Consider the 3-dof planar PRR robot in Fig. 1, with the joint coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$ defined therein. The second and third links have a common length $L > 0$. The robot performs three-dimensional tasks that involve the position $\boldsymbol{p} = (p_x, p_y)$ of its end-effector point $\boldsymbol{P}$ and the orientation angle α of the end-effector w.r.t. the axis $\boldsymbol{x}_0$.

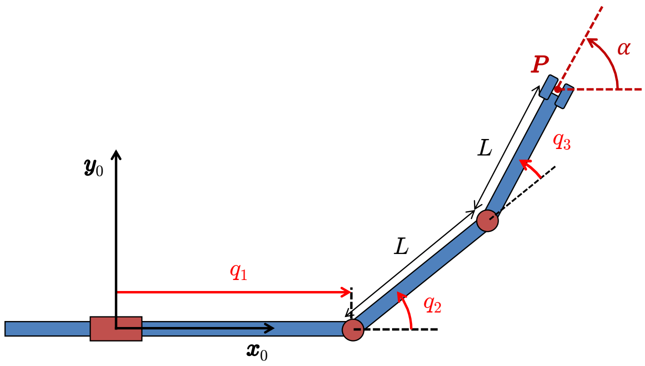

Figure 1: A planar PRR robot.

a) Determine the direct task kinematics $\boldsymbol{r} = \boldsymbol{f}(\boldsymbol{q})$ between $\boldsymbol{q} = (q_1, q_2, q_3)$ and $\boldsymbol{r} = (p_x, p_y, \alpha)$. Derive the task Jacobian $\boldsymbol{J}(\boldsymbol{q})$ of the map $\boldsymbol{f}(\boldsymbol{q})$ and find all singularities $\boldsymbol{q}_s$ of this 3×3 matrix.

b) When the robot is in a singular configuration $\boldsymbol{q}_s$ (choose one at will), determine:
   - a null-space joint velocity $\dot{\boldsymbol{q}}_0 \in \mathcal{N}\{\boldsymbol{J}(\boldsymbol{q}_s)\}$;
   - a task velocity $\dot{\boldsymbol{r}}_1 \in \mathcal{R}\{\boldsymbol{J}(\boldsymbol{q}_s)\}$ and an associated joint velocity $\dot{\boldsymbol{q}}$ that realizes it;
   - an unfeasible task velocity $\dot{\boldsymbol{r}}_2 \notin \mathcal{R}\{\boldsymbol{J}(\boldsymbol{q}_s)\}$;
   - a generalized task force $\boldsymbol{F}_0 = (F_x, F_y, M_z)$ applied at the end effector that is statically balanced by joint forces/torques $\boldsymbol{\tau} = \mathbf{0}$.

c) Find a closed-form expression for the inverse task kinematics $\boldsymbol{q} = \boldsymbol{f}^{-1}(\boldsymbol{r}_d)$, whenever at least a solution exist. Compute then the numerical value of all inverse solutions for $L = 0.5$ [m] and when $\boldsymbol{r}_d = (0.3, 0.7, \pi/3)$ [m,m,rad].

d) Draw the primary and secondary workspaces for this robot, when the prismatic joint has a finite range $q_1 \in [0, L]$ while the revolute joints have unlimited range.

Exercise #2

For the same PRR robot in Fig. 1 (with a generic value L for link lengths), determine a smooth, coordinated rest-to-rest joint trajectory $\boldsymbol{q}_d(t)$ that will move the robot in T seconds from the initial value $\boldsymbol{r}_i = (2L, 0, \pi/4)$ of the task vector to the final value $\boldsymbol{r}_f = (2L, 0, -\pi/4)$, *without* ever changing the position $\boldsymbol{p}_d = (2L, 0)$ of the point $\boldsymbol{P}$. Sketch a plot of the obtained joint trajectory $\boldsymbol{q}_d(t) = (q_{1d}(t), q_{2d}(t), q_{3d}(t))$.

[180 minutes (3 hours); open books]

Solution

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Exercise #1

The direct kinematics of the task is given by

$$\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ \alpha \end{pmatrix} = \begin{pmatrix} q_1 + L\left(\cos q_2 + \cos(q_2+q_3)\right) \\ L\left(\sin q_2 + \sin(q_2+q_3)\right) \\ q_2+q_3 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}). \tag{1}$$

The task Jacobian is thus

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} 1 & -L\left(\sin q_2 + \sin(q_2+q_3)\right) & -L\sin(q_2+q_3) \\ 0 & L\left(\cos q_2 + \cos(q_2+q_3)\right) & L\cos(q_2+q_3) \\ 0 & 1 & 1 \end{pmatrix}. \tag{2}$$

The singularities occur when

$$\det \boldsymbol{J}(\boldsymbol{q}) = L\cos q_2 = 0 \qquad \Longleftrightarrow \qquad q_2 = \pm\frac{\pi}{2}. \tag{3}$$

The condition (3) is easy to interpret in terms of loss of mobility. When the second link is orthogonal to the first one, the linear motion of the prismatic joint and the rotation of the second joint both produce linear contributions to the end-effector motion restricted to the $\boldsymbol{x}_0$ direction. If the third joint is used to impose a desired rotation of the end effector around the $\boldsymbol{z}_0$ axis, there is no remaining freedom for achieving instantaneously also a non-zero velocity along $\boldsymbol{y}_0$. The robot end effector has lost its full mobility in the task space and we are thus in a singularity.

We set now $\boldsymbol{q}_s = (*, \pi/2, q_3)$, where $*$ denotes an arbitrary value. The task Jacobian becomes

$$\boldsymbol{J}(\boldsymbol{q}_s) = \begin{pmatrix} 1 & -L\left(1+\cos q_3\right) & -L\cos q_3 \\ 0 & -L\sin q_3 & -L\sin q_3 \\ 0 & 1 & 1 \end{pmatrix}, \tag{4}$$

with $\operatorname{rank}\{\boldsymbol{J}(\boldsymbol{q}_s)\} = 2$. All joint velocities in the null space of $\boldsymbol{J}(\boldsymbol{q}_s)$ are expressed as

$$\dot{\boldsymbol{q}}_0 = \beta \begin{pmatrix} L \\ 1 \\ -1 \end{pmatrix} \in \mathcal{N}\left\{\boldsymbol{J}(\boldsymbol{q}_s)\right\}, \ \forall\beta \qquad \Longleftrightarrow \qquad \boldsymbol{J}(\boldsymbol{q}_s)\dot{\boldsymbol{q}}_0 = \boldsymbol{0}.$$

Thus, null-space motions always involve all three joints. A basis for the two-dimensional range space of $\boldsymbol{J}(\boldsymbol{q}_s)$ is

$$\mathcal{R}\left\{\boldsymbol{J}(\boldsymbol{q}_s)\right\} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -L\sin q_3 \\ 1 \end{pmatrix}\right\}. \tag{5}$$

The complementary space to $\mathcal{R}\left\{\boldsymbol{J}(\boldsymbol{q}_s)\right\}$ in $\mathbb{R}^3$ is the one-dimensional subspace

$$\mathcal{R}\left\{\boldsymbol{J}(\boldsymbol{q}_s)\right\}^{\perp} = \mathcal{N}\left\{\boldsymbol{J}^T(\boldsymbol{q}_s)\right\} = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \\ L\sin q_3 \end{pmatrix}\right\}. \tag{6}$$