Solution

January 23, 2023

Exercise 1

A possible conventional DH frame assignment for the TIAGo robotic arm is shown in Fig. 2. The two views are used in order to better illustrate the assignment (which is indeed consistent in the two pictures). The corresponding DH parameters are reported in Tab. 1. Note that $\boldsymbol{x}_0 \| \boldsymbol{x}_1$ ($\theta_1 = 0$).

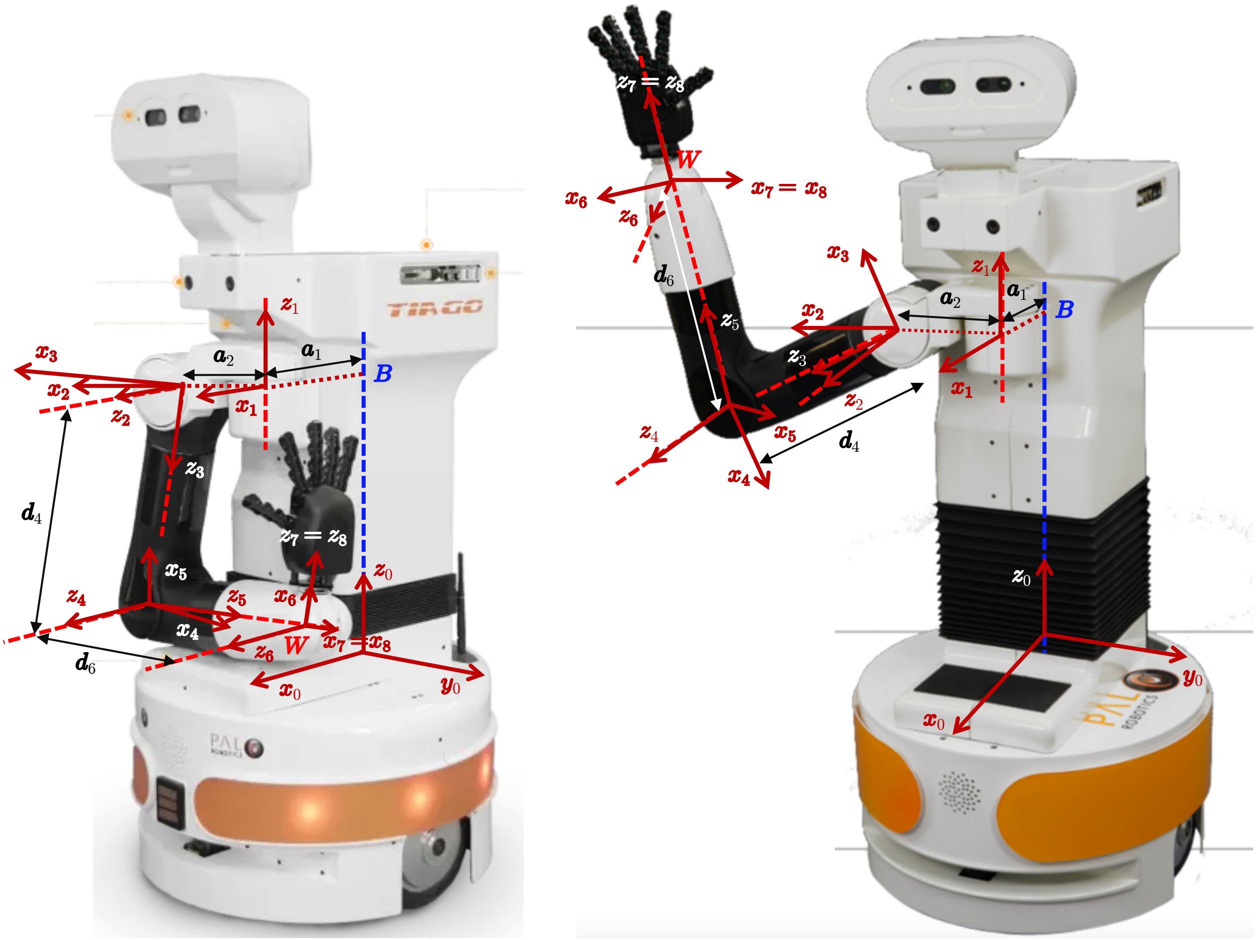

Figure 2: Two views of the DH frames assigned to the TIAGo robotic arm.

The actual values of the constant DH parameters are:

$$a_1 = \overrightarrow{BO_1} = 0.1557, \quad a_2 = \overrightarrow{O_1O_2} = 0.125, \quad d_4 = \overrightarrow{O_3O_4} = 0.3115, \quad d_6 = \overrightarrow{O_5O_6} = 0.312 \ \ [\mathrm{m}].$$

Note that the above assignment may not correspond to the one used by the manufacturer (or in URDF models). Also, very minor offsets exist at the elbow and at the shoulder of the robotic arm; these offsets have been neglected in this exercise.

i	α_i	a_i	d_i	θ_i
1	0	a_1	q_1	0
2	$-\pi/2$	a_2	0	q_2
3	$-\pi/2$	0	0	q_3
4	$-\pi/2$	0	d_4	q_4
5	$\pi/2$	0	0	q_5
6	$-\pi/2$	0	d_6	q_6
7	$-\pi/2$	0	0	q_7
8	0	0	0	q_8

Table 1: Table of DH parameters corresponding to the frames in Fig. 2.

Exercise 2

The direct kinematics of the robotic arm from RF_w to RF_8 using homogeneous transformation matrices is

$$ {}^w\boldsymbol{T}_8\left({}^w\boldsymbol{p}_0, \phi, \boldsymbol{q}\right) = {}^w\boldsymbol{T}_0\left({}^w\boldsymbol{p}_0, \phi\right)\, {}^0\boldsymbol{T}_8(\boldsymbol{q}), \tag{4} $$

with

$$ {}^w\boldsymbol{T}_0\left({}^w\boldsymbol{p}_0, \phi\right) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 & p_{0,x} \\ \sin\phi & \cos\phi & 0 & p_{0,y} \\ 0 & 0 & 1 & p_{0,z} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 1.5 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & -4.5 \\ 0 & 0 & 1 & 0.3 \\ 0 & 0 & 0 & 1 \end{pmatrix}. $$

In general, the most efficient way for computing only the position of the origin O_8 of the last DH frame expressed in the reference frame RF_0 is by the nested matrix-vector product

$$ {}^0\boldsymbol{p}_{8,hom} = \begin{pmatrix} {}^0\boldsymbol{p}_8 \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\left({}^1\boldsymbol{A}_2(q_2)\left(\cdots\left({}^7\boldsymbol{A}_8(q_8)\begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}\right)\cdots\right)\right). \tag{5} $$

In the present case, it is $W = O_8 = O_7 = O_6$. Thus, the last column of the two matrices ${}^6\boldsymbol{A}_7(q_7)$ and ${}^7\boldsymbol{A}_8(q_8)$ is simply $\begin{pmatrix} \boldsymbol{0}^T & 1 \end{pmatrix}^T$. Moreover, being the last column of matrix ${}^5\boldsymbol{A}_6(q_6)$ equal to $\begin{pmatrix} 0 & 0 & d_6 & 1 \end{pmatrix}^T$, equation (5) simplifies to

$$ {}^0\boldsymbol{p}_{W,hom} = \begin{pmatrix} {}^0\boldsymbol{p}_W \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\left({}^1\boldsymbol{A}_2(q_2)\left({}^2\boldsymbol{A}_3(q_3)\left({}^3\boldsymbol{A}_4(q_4)\left({}^4\boldsymbol{A}_5(q_5)\begin{pmatrix} 0 \\ 0 \\ d_6 \\ 1 \end{pmatrix}\right)\right)\right)\right). \tag{6} $$

The symbolic outcome of (6) can be easily obtained adapting the MATLAB code for the direct kinematics of a robot manipulator `dirkin.m` that is available on the web site of the course. Since

$$ {}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} c_2 & 0 & -s_2 & a_2c_2 \\ s_2 & 0 & c_2 & a_2s_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} c_3 & 0 & -s_3 & 0 \\ s_3 & 0 & c_3 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, $$

$$
{}^3\boldsymbol{A}_4(q_4) = \begin{pmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^4\boldsymbol{A}_5(q_5) = \begin{pmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

one obtains

$$
{}^0\boldsymbol{p}_W = \begin{pmatrix} a_1 + a_2 c_2 - d_4 c_2 s_3 + d_6 \left(s_2 s_4 s_5 - c_2 s_3 c_5 - c_3 c_4 s_5 \right) \\ a_2 s_2 - d_4 s_2 s_3 - d_6 \left(s_2 s_3 c_5 + c_2 s_4 s_5 + c_3 c_4 s_5 \right) \\ q_1 - d_4 c_3 - d_6 \left(c_3 c_5 + s_3 c_4 s_5 \right) \end{pmatrix}. \tag{7}
$$

Finally, using (4) and keeping only the first three components of the result yields

$$
{}^w\boldsymbol{p}_W = \begin{pmatrix} 1.5 + \dfrac{1}{\sqrt{2}} \Big(a_1 + (s_2 + c_2)(a_2 - d_4 s_3) + d_6 \left((s_2 - c_2) s_4 s_5 - (s_2 + c_2) s_3 c_5 - 2 c_3 c_4 s_5 \right) \Big) \\ -4.5 + \dfrac{1}{\sqrt{2}} \Big(a_1 + (c_2 - s_2)(a_2 - d_4 s_3) + d_6 \left((s_2 + c_2) s_4 s_5 + (s_2 - c_2) s_3 c_5 \right) \Big) \\ 0.3 + q_1 - d_4 c_3 - d_6 \left(c_3 c_5 + s_3 c_4 s_5 \right) \end{pmatrix}. \tag{8}
$$

Exercise 3

This problem is solved by the algebraic transformation method used in inverse kinematics. Expand the cosine function in (1) to get

$$
\sin\theta_1 + 2\left(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2\right) = 2,
$$

which is of the form

$$
a \sin\theta_1 + b \cos\theta_1 = c, \tag{9}
$$

with

$$
a = 1 - 2\sin\theta_2, \qquad b = 2\cos\theta_2, \qquad c = 2.
$$

The transcendental eq. (9) has already been studied in the lecture slides (InverseKinematics.pdf, slide #13). From there, we know that this equation has (one or two) real solutions if and only if

$$
a^2 + b^2 \geq c^2 \qquad \Rightarrow \qquad (1 - 2\sin\theta_2)^2 + 4\cos^2\theta_2 \geq 4,
$$

or

$$
\sin\theta_2 \leq 0.25 \qquad \Rightarrow \qquad \theta_2 \in (-\pi, 0.2526] \cup [\pi - 0.2526, \pi] \ \text{[rad]}.
$$

The range of admissible solutions for θ_2, i.e., those providing a real solution θ_1 to (1), is shown in Fig. 3.

For instance, when $\theta_2 = 0$, eq. (1) becomes

$$
\sin\theta_1 + 2\cos\theta_1 = 2,
$$

which has the two real solutions

$$
\begin{aligned}
\theta_1^+ &= 2\arctan\left(\frac{a + \sqrt{a^2 + b^2 - c^2}}{b + c} \right) = 2\arctan\left(\frac{1+1}{4} \right) = 2\arctan 0.5 = 0.9273 \ \text{[rad]}, \\
\theta_1^- &= 2\arctan\left(\frac{a - \sqrt{a^2 + b^2 - c^2}}{b + c} \right) = 2\arctan\left(\frac{1-1}{4} \right) = 2\arctan 0 = 0.
\end{aligned}
$$

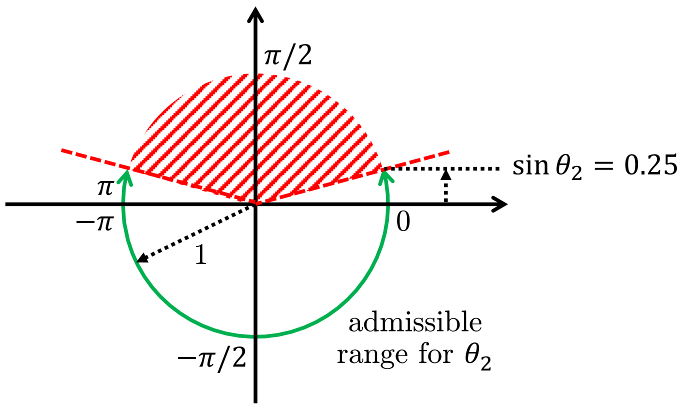

Figure 3: Admissible range for θ_2.

On the other hand, eq. (1) has a single solution when $\sin\theta_2 = 0.25$. In particular, for $\theta_2 = 0.2526$ the equation becomes

$$0.5\sin\theta_1 + 1.9365\cos\theta_1 = 2,$$

with the single solution

$$\theta_1 = 0.2499 \text{ [rad]}.$$

Similarly, for $\theta_2 = \pi - 0.2526$ the equation becomes

$$0.5\sin\theta_1 - 1.9365\cos\theta_1 = 2,$$

with the single solution

$$\theta_1 = \pi - 0.2499 = 2.8917 \text{ [rad]}.$$

Exercise 4

The robot is a planar RPRP arm. The task vector $\boldsymbol{r}$ contains the (x, y) position of the end-effector and the angle ϕ of the last link w.r.t. the $\boldsymbol{x}$-axis. The analytic 3×4 Jacobian associated to the task function in (2) is

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}} = \begin{pmatrix} -q_2 s_1 - q_4 s_{13} & c_1 & -q_4 s_{13} & c_{13} \\ q_2 c_1 + q_4 c_{13} & s_1 & q_4 c_{13} & s_{13} \\ 1 & 0 & 1 & 0 \end{pmatrix}. \tag{10}$$

Its singular configurations (corresponding to a loss of rank) are determined by computing

$$\det\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right) = 2q_2^2 + 2s_3^2 - q_2^2 s_3^2. \tag{11}$$

This determinant is zero if and only if $q_2 = 0$ **and** $s_3 = 0$ ($q_3 = 0$ or π) simultaneously, because cancelation between addends in (11) is ruled out. In fact, let $a = s_3^2 \in (0, 1]$; setting the determinant to zero would correspond to a value $q_2^2 = -2a/(2 - a) < 0$, which is impossible.

An alternative method for finding the singularities of the Jacobian would be to check the four minors obtained by deleting one of its columns. Let $\boldsymbol{J}_{-i}(\boldsymbol{q})$ be the 3×3 matrix obtained by deleting column i from $\boldsymbol{J}(\boldsymbol{q})$, for $i = 1, 2, 3, 4$. Then

$$\det \boldsymbol{J}_{-1}(\boldsymbol{q}) = -s_3, \qquad \det \boldsymbol{J}_{-2}(\boldsymbol{q}) = q_2 c_3, \qquad \det \boldsymbol{J}_{-3}(\boldsymbol{q}) = s_3, \qquad \det \boldsymbol{J}_{-4}(\boldsymbol{q}) = -q_2.$$

All minors should vanish at the same time, and this happens again if and only if $q_2 = 0$ and $s_3 = 0$.

In a generic regular configuration, the null space of $\boldsymbol{J}(\boldsymbol{q})$ is one-dimensional, i.e., it is generated by a single vector $\dot{\boldsymbol{q}}_0$ (scaled with any factor $\alpha \in \mathbb{R}$):

$$\dot{\boldsymbol{q}}_0 \in \mathcal{N}(\boldsymbol{J}(\boldsymbol{q})) = \text{span}\left\{ \begin{pmatrix} -s_3 \\ -q_2 c_3 \\ s_3 \\ q_2 \end{pmatrix} \right\} \qquad \Rightarrow \qquad \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}_0 = \boldsymbol{0}.$$

In a singular configuration $\boldsymbol{q}_s$, the Jacobian becomes[2]

$$\boldsymbol{J}(\boldsymbol{q}_s) = \boldsymbol{J}(\boldsymbol{q})\big|_{q_2=q_3=0} = \begin{pmatrix} -q_4 s_1 & c_1 & -q_4 s_1 & c_1 \\ q_4 c_1 & s_1 & q_4 c_1 & s_1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \tag{12}$$

which has rank equal to 2: the first two columns are independent for all $\boldsymbol{q}$, the other two columns are simply duplications. Thus, the null space of $\boldsymbol{J}(\boldsymbol{q}_s)$ is two-dimensional and a basis for all $\dot{\boldsymbol{q}} \in \mathbb{R}^4$ such that $\boldsymbol{J}(\boldsymbol{q}_s)\dot{\boldsymbol{q}} = \boldsymbol{0}$ is

$$\mathcal{N}(\boldsymbol{J}(\boldsymbol{q}_s)) = \text{span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Looking at the structure of the singular matrix in (12), the complementary space to the range space $\mathcal{R}(\boldsymbol{J}(\boldsymbol{q}_s))$ along which no task velocity can be realized at $\boldsymbol{q}_s$ is given by the single direction

$$\dot{\boldsymbol{r}}^{\perp} = \begin{pmatrix} s_1 \\ -c_1 \\ q_4 \end{pmatrix} \in \mathcal{R}^{\perp}(\boldsymbol{J}(\boldsymbol{q}_s)) = \mathcal{N}\Big(\boldsymbol{J}^T(\boldsymbol{q}_s)\Big),$$

where the latter equality between subspaces follows from the decomposition of the three-dimensional task space. In fact, all generalized task forces that can be statically balanced at $\boldsymbol{q}_s$ by a zero joint torque have the form

$$\boldsymbol{f} = \alpha \begin{pmatrix} s_1 \\ -c_1 \\ q_4 \end{pmatrix}, \quad \forall \alpha \in \mathbb{R} \qquad \Rightarrow \qquad \boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q}_s)\boldsymbol{f} = \boldsymbol{0}.$$

Note that the unit of measure for the scalar α is in this case [N]. On the other hand, when the Jacobian has full rank, i.e., has the form (10), the subspace $\mathcal{N}(\boldsymbol{J}^T(\boldsymbol{q}))$ contains only the null vector; so, there is no $\boldsymbol{f}_0 \neq \boldsymbol{0}$ in the null space of $\boldsymbol{J}^T(\boldsymbol{q})$ in the regular case.

[2] A similar analysis holds for the singularity $q_2 = 0$, $q_3 = \pi$.

Exercise 5

A parametrized description of the assigned helical path is given by [3]

$$\boldsymbol{p} = \boldsymbol{p}(s) = C + \begin{pmatrix} r \sin s \\ h\, s \\ r \cos s \end{pmatrix} = \begin{pmatrix} r \sin s \\ h\, s \\ r\,(1+\cos s) \end{pmatrix}, \qquad s \in [0, L], \tag{13}$$

so that $\boldsymbol{p}(0) = \boldsymbol{p}_0 = (0, 0, r)$. Each of the two full turns is obtained by a variation of 2π for s. Thus, the upper limit of the interval for s is $L = 4\pi$. Given this path, we need to specify a rest-to-rest timing law $s = s(t)$, with $t \in [0, T]$, that will trace it in minimum time T under the bounds (3).

From (13), by time differentiation (and use of the chain rule) we get

$$\dot{\boldsymbol{p}} = \boldsymbol{p}'\, \dot{s} = \begin{pmatrix} r \cos s \\ h \\ -r \sin s \end{pmatrix} \dot{s}, \qquad \|\boldsymbol{p}'\| = \sqrt{r^2 + h^2}, \tag{14}$$

and

$$\ddot{\boldsymbol{p}} = \boldsymbol{p}'\, \ddot{s} + \boldsymbol{p}''\, \dot{s}^2 = \begin{pmatrix} r \cos s \\ h \\ -r \sin s \end{pmatrix} \ddot{s} - \begin{pmatrix} r \sin s \\ 0 \\ r \cos s \end{pmatrix} \dot{s}^2, \tag{15}$$

where a prime ($'$) denotes differentiation with respect to the parameter s. From (14), we obtain the tangent axis $\boldsymbol{t}$ of the Frenet frame associated to the path,

$$\boldsymbol{t} = \frac{\boldsymbol{p}'}{\|\boldsymbol{p}'\|} = \frac{1}{\sqrt{r^2 + h^2}} \begin{pmatrix} r \cos s \\ h \\ -r \sin s \end{pmatrix}. \tag{16}$$

Differentiating $\boldsymbol{t}$ with respect to the parameter s gives

$$\boldsymbol{t}' = -\frac{1}{\sqrt{r^2 + h^2}} \begin{pmatrix} r \sin s \\ 0 \\ r \cos s \end{pmatrix}, \qquad \|\boldsymbol{t}'\| = \frac{r}{\sqrt{r^2 + h^2}}, \tag{17}$$

so that the normal axis $\boldsymbol{n}$ of the Frenet frame associated to the path is

$$\boldsymbol{n} = \frac{\boldsymbol{t}'}{\|\boldsymbol{t}'\|} = -\begin{pmatrix} \sin s \\ 0 \\ \cos s \end{pmatrix}. \tag{18}$$

For checking the bounds on the components of the acceleration (15), we need to evaluate

$$\ddot{\boldsymbol{p}}^T \boldsymbol{t} = \left(\boldsymbol{p}'^T \boldsymbol{t}\right) \ddot{s} + \left(\boldsymbol{p}''^T \boldsymbol{t}\right) \dot{s}^2 \qquad \text{and} \qquad \ddot{\boldsymbol{p}}^T \boldsymbol{n} = \left(\boldsymbol{p}'^T \boldsymbol{n}\right) \ddot{s} + \left(\boldsymbol{p}''^T \boldsymbol{n}\right) \dot{s}^2. \tag{19}$$

[3]Equation (13) is not the only possible parametrization of this helix. For instance, one can also define

$$\boldsymbol{p}(s) = \begin{pmatrix} r \sin 2\pi s \\ 2\pi h\, s \\ r\,(1 + \cos 2\pi s) \end{pmatrix}, \qquad s \in [0, 2],$$

with the parameter s being scaled down by 2π. The new final value $s = 2$ of the interval of definition corresponds again to two full turns along the helix. Indeed, although the expressions are slightly different, all the following results on trajectory planning remain the same.

Being

$$\boldsymbol{p}'^T\boldsymbol{t} = \sqrt{r^2+h^2}, \qquad \boldsymbol{p}''^T\boldsymbol{t} = 0, \qquad \boldsymbol{p}'^T\boldsymbol{n} = 0, \qquad \boldsymbol{p}''^T\boldsymbol{n} = r,$$

from (14) and (19), we have the bounds

$$\|\dot{\boldsymbol{p}}\| = \sqrt{r^2+h^2}\,|\dot{s}| \leq V, \qquad \left|\ddot{\boldsymbol{p}}^T\boldsymbol{t}\right| = \sqrt{r^2+h^2}\,|\ddot{s}| \leq A, \qquad \left|\ddot{\boldsymbol{p}}^T\boldsymbol{n}\right| = r\,\dot{s}^2 \leq A,$$

or

$$|\dot{s}| \leq v_{max} = \min\left\{\frac{V}{\sqrt{r^2+h^2}}, \sqrt{\frac{A}{r}}\right\}, \qquad |\ddot{s}| \leq a_{max} = \frac{A}{\sqrt{r^2+h^2}}. \tag{20}$$

Using the numerical data, it is

$$v_{max} = \min\left\{\frac{2}{\sqrt{0.25}}, \sqrt{\frac{4.5}{0.4}}\right\} = \sqrt{\frac{4.5}{0.4}} = 3.3541 \text{ m/s}, \qquad a_{max} = 9 \text{ m/s}^2.$$

The minimum-time rest-to-rest motion for moving the scalar path parameter s between $s = 0$ and $s = L = 4\pi$ under the bounds $v_{max} > 0$ for the speed and $a_{max} > 0$ for the acceleration is a bang-coast-bang (or bang-bang) acceleration profile. In the present case, there will be a rather long coast phase since, by replacing the problem data,

$$12.5664 \approx 4\pi = L > \frac{v_{max}^2}{a_{max}} = \frac{11.25}{9} = 1.25.$$

Therefore, the minimum time is

$$T^* = \frac{L\,a_{max} + v_{max}^2}{a_{max}v_{max}} = 4.1192 \text{ s},$$

with acceleration/deceleration phases lasting $T_s = v_{max}/a_{max} = 0.3727$ s. The time-optimal profile of the parameter s and of its first two derivatives is shown in Fig. 4.

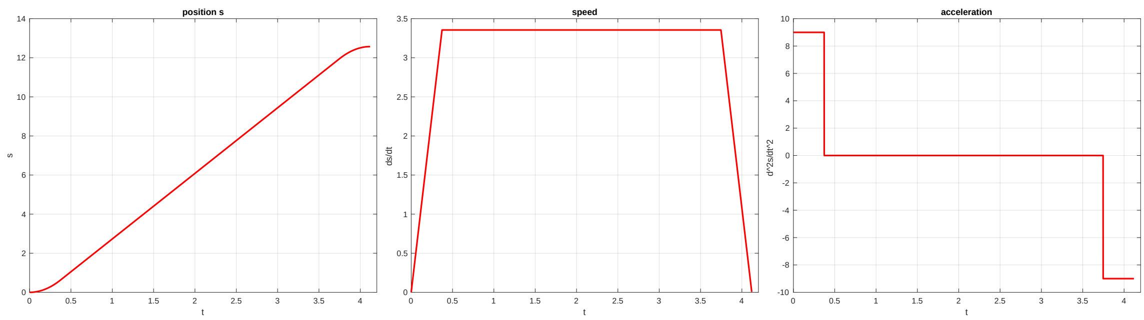

Figure 4: The optimal timing law $s(t)$, with speed and acceleration profiles.

As for the suitable positioning of the base of the spatial 3R manipulator, consider the bounding box (a parallelepiped) containing the complete helical path shown in Fig. 5. The box size is $(\Delta x, \Delta y, \Delta z) = (2r \times 4\pi h \times 2r) = (0.8 \times 3.77 \times 0.8)$ [m], with one of the four large faces lying on the plane $z = 0$ and one of the two (vertical) bases lying on the plane $y = 0$. The shoulder of the robot is at the same level of the top face of the box.

The robot base will be conveniently placed at the midpoint P_b along one of the long sides of the box, e.g., with $(x_b, y_b) = (r, 2\pi h) = (0.4, 1.885)$ [m] (and $z_b = 0$). In this way, the outreach of the second and third robot links ($L_t = L_2 + L_3 = 3$ m) starting from the shoulder point $P_s = (r, 2\pi h, L_1) = (0.4, 1.885, 0.8)$ [m] will cover the entire bounding box. In fact, even the farthest vertex $P_v = (-r, 4\pi h, 0) = (-0.4, 3.77, 0)$ [m] of the box will have a distance from P_s that is reachable, being $\|P_v - P_s\| = 2.198 < 3 = L_t$. The robot will certainly not encounter any kinematic singularity during its motion: the forearm is never outstretched or folded and the base joint axis is out of the box. Moreover, the path will never interfere with the base link of the robot, which is outside the box.

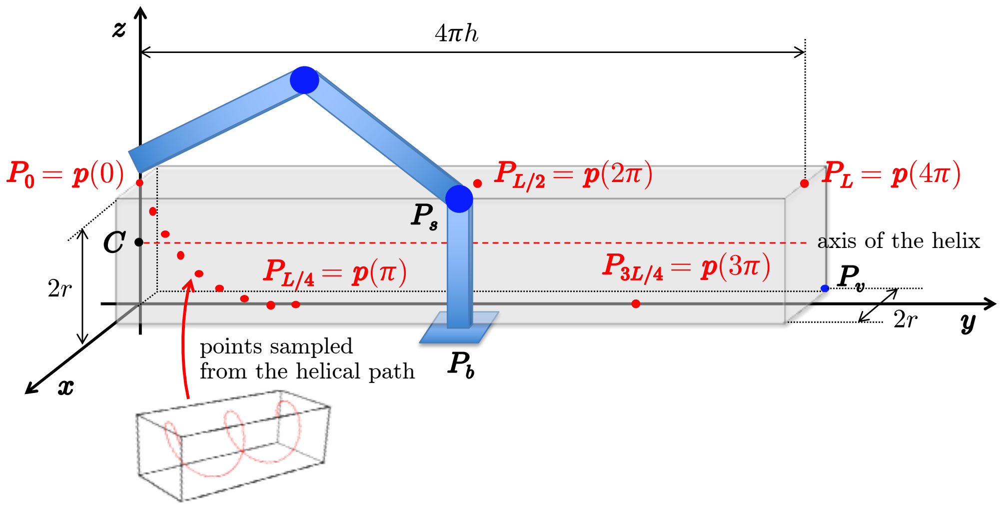

Figure 5: A box containing the helical path and a good placement of the 3R manipulator.

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