Solution

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Exercise 1

The correct (and unique) DH frame assignment for the RPR robot of Fig. 1 satisfying all requests is shown in Fig. 2. The associated DH parameters are reported in Tab. 1.

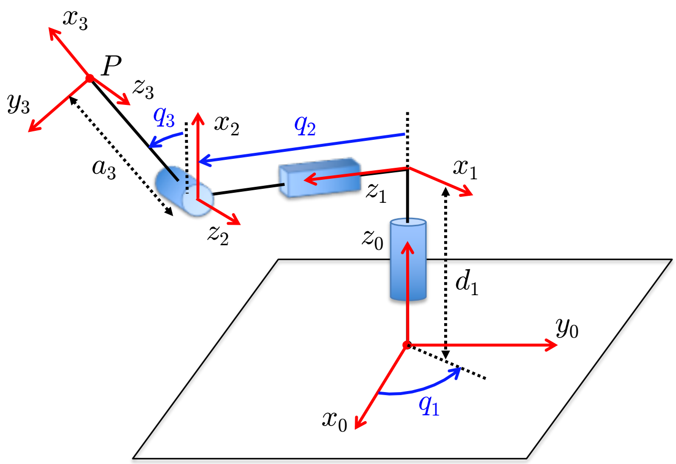

Figure 2: DH frames for the spatial RPR robot.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 > 0$	$q_1 > 0$
2	$\pi/2$	0	$q_2 > 0$	$\pi/2$
3	0	$a_3 > 0$	0	$q_3 > 0$

Table 1: DH parameters corresponding to the frames of Fig. 2.

From the associated homogeneous transformation matrices

$$\boldsymbol{A}_1(q_1) = \begin{pmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{A}_2(q_2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{A}_3(q_3) = \begin{pmatrix} c_3 & -s_3 & 0 & a_3 c_3 \\ s_3 & c_3 & 0 & a_3 s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we compute

$$\boldsymbol{p}_H = \begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = \boldsymbol{A}_1(q_1) \left(\boldsymbol{A}_2(q_2) \left(\boldsymbol{A}_3(q_3) \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix} \right) \right)$$

yielding the direct kinematics of the position of point P as

$$\boldsymbol{p} = \begin{pmatrix} s_1 (q_2 + a_3 s_3) \\ -c_1 (q_2 + a_3 s_3) \\ d_1 + a_3 c_3 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}).$$

Exercise 1b

Differentiating the direct kinematics yields the 3×3 Jacobian matrix

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}} = \begin{pmatrix} c_1\,(q_2 + a_3 s_3) & s_1 & a_3 s_1 c_3 \\ s_1\,(q_2 + a_3 s_3) & -c_1 & -a_3 c_1 c_3 \\ 0 & 0 & -a_3 s_3 \end{pmatrix}.$$

Its determinant is

$$\det \boldsymbol{J}(\boldsymbol{q}) = a_3 s_3\,(q_2 + a_3 s_3)$$

so that the singularities occur when

$$s_3 = 0 \quad (q_3 = \{0, \pi\}) \qquad \text{or} \qquad q_2 = -a_2 s_3.$$

In the first case, setting $q_3 = 0$ for illustration (and for $q_2 \neq 0$), we have

$$\boldsymbol{J}_I = \boldsymbol{J}(\boldsymbol{q})\big|_{q_3=0} = \begin{pmatrix} q_2 c_1 & s_1 & a_3 s_1 \\ q_2 s_1 & -c_1 & -a_3 c_1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \operatorname{rank}(\boldsymbol{J}_I) = 2.$$

Bases for the null space and range space of the Jacobian, and for the space of lost Cartesian mobility are

$$\mathcal{N}(\boldsymbol{J}_I) = \left\{ \begin{pmatrix} 0 \\ -a_3 \\ 1 \end{pmatrix} \right\}, \qquad \mathcal{R}(\boldsymbol{J}_I) = \left\{ \begin{pmatrix} c_1 \\ s_1 \\ 0 \end{pmatrix}, \begin{pmatrix} s_1 \\ -c_1 \\ 0 \end{pmatrix} \right\}, \qquad \mathcal{R}^{\perp}(\boldsymbol{J}_I) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

where $\mathcal{R}^{\perp}$ is the complementary subspace to $\mathcal{R}$ in $\mathbb{R}^3$. In this singular configuration, the third link is vertical so that point P is at the boundary of the reachable workspace. Thus, it cannot move along the vertical $\boldsymbol{z}_0$ direction.

In the second singular case, we have (for $q_3 \neq 0$ or π)

$$\boldsymbol{J}_{II} = \boldsymbol{J}(\boldsymbol{q})\big|_{q_2 + a_3 s_3 = 0} = \begin{pmatrix} 0 & s_1 & a_3 s_1 c_3 \\ 0 & -c_1 & -a_3 c_1 c_3 \\ 0 & 0 & -a_3 s_3 \end{pmatrix}, \qquad \operatorname{rank}(\boldsymbol{J}_{II}) = 2.$$

Bases for the null space and range space of the Jacobian, and for the space of lost Cartesian mobility are in this case

$$\mathcal{N}(\boldsymbol{J}_I) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}, \qquad \mathcal{R}(\boldsymbol{J}_I) = \left\{ \begin{pmatrix} s_1 \\ -c_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \qquad \mathcal{R}^{\perp}(\boldsymbol{J}_I) = \left\{ \begin{pmatrix} c_1 \\ s_1 \\ 0 \end{pmatrix} \right\}.$$

In this singular configuration, point P is placed on the axis $\boldsymbol{z}_0$ and cannot move along the normal direction to the vertical plane being defined by the links 2 and 3.

Finally, at the intersection of the two singularities, e.g., for $q_2 = q_3 = 0$, we obtain

$$\boldsymbol{J}_{I+II} = \boldsymbol{J}(\boldsymbol{q})\big|_{q_2 = q_3 = 0} = \begin{pmatrix} 0 & s_1 & a_3 s_1 \\ 0 & -c_1 & -a_3 c_1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \operatorname{rank}(\boldsymbol{J}_{I+II}) = 1.$$

Bases for the null space and range space of the Jacobian, and for the space of lost Cartesian mobility are then

$$\mathcal{N}\left(\boldsymbol{J}_{I+II}\right) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -a_3 \\ 1 \end{pmatrix} \right\}, \ \mathcal{R}\left(\boldsymbol{J}_{I+II}\right) = \left\{ \begin{pmatrix} s_1 \\ -c_1 \\ 0 \end{pmatrix} \right\}, \ \mathcal{R}^{\perp}\left(\boldsymbol{J}_{I+II}\right) = \left\{ \begin{pmatrix} c_1 \\ s_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

As a result, the third link is vertical and point P is on the axis $\boldsymbol{z}_0$ at the boundary of the reachable workspace. Thus, it cannot move neither vertically nor along the normal direction to the plane defined by link 2 and 3.

Exercise 1c

Out of singularities, the required joint velocity control law is

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\boldsymbol{K}\left(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q})\right), \tag{1}$$

using a diagonal and uniform gain matrix $\boldsymbol{K} = k\boldsymbol{I}$, with $k > 0$. Note that no trajectory is being planned between the initial position $\boldsymbol{p}(0) = \boldsymbol{f}(\boldsymbol{q}(0))$ and the constant desired Cartesian position $\boldsymbol{p}_d$, so that this is a pure feedback law (of the nonlinear type). The position error $\boldsymbol{e} = \boldsymbol{p}_d - \boldsymbol{p}$ will evolve as

$$\dot{\boldsymbol{e}} = -\dot{\boldsymbol{p}} = -\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}} = -\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q})\boldsymbol{K}\boldsymbol{e} = -k\,\boldsymbol{e},$$

yielding the solution

$$e_i(t) = \exp(-kt)\,e_i(0), \qquad i = x, y, z.$$

Thus, the robot end-effector will exponentially converge to the desired position $\boldsymbol{p}_d \in \mathbb{R}^3$ in the reachable workspace, unless it encounters a singularity where the control law (1) is not defined. Moreover, starting from the initial position $\boldsymbol{p}(0)$, the error $\boldsymbol{e}(t) = \boldsymbol{p}_d - \boldsymbol{p}(t)$ will always be aligned to $\boldsymbol{e}(0) = \boldsymbol{p}_d - \boldsymbol{p}(0)$. Hence, in the absence of perturbations, $\boldsymbol{p}(t)$ remains along the straight line going through $\boldsymbol{p}(0)$ and $\boldsymbol{p}_d$.

Exercise 2

The assigned motion task has to be converted in the joint space, where the command input is defined together with the velocity bounds. Through the standard inverse kinematics of the planar 2R robot we obtain

$$\boldsymbol{q}_{in} = \boldsymbol{f}^{-1}(P_{in}) = \begin{pmatrix} 0 \\ \frac{\pi}{4} \end{pmatrix} \text{ [rad]}, \qquad \boldsymbol{q}_{fin} = \boldsymbol{f}^{-1}(P_{fin}) = \begin{pmatrix} -\frac{\pi}{4} \\ \frac{\pi}{2} \end{pmatrix} \text{ [rad]}, \tag{2}$$

where the right arm solution (with the '+' sign) has been chosen, both at the initial and final points (so as to avoid a singularity crossing). Thus, the required displacement in the joint space is

$$\Delta\boldsymbol{q} = \boldsymbol{q}_{fin} - \boldsymbol{q}_{in} = \begin{pmatrix} -\frac{\pi}{4} \\ \frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} -0.7854 \\ 0.7854 \end{pmatrix} \text{ [rad]}.$$

In order to perform the required rest-to-rest motion in minimum time and in a coordinated way, first we compute separately the minimum-time motion for each joint (i.e., for $i = 1, 2$): joint i

will have a symmetric bang-bang acceleration profile $[A_{max,i}, -A_{max,i}]$, where the sign of $A_{max,i}$ depends on the sign of Δq_i, its intensity is defined so as to reach the maximum velocity $\pm V_{max,i}$ at the trajectory midpoint $t = T_i/2$, and the total motion time T_i will be such to complete the displacement Δq_i. Thus,

$$A_{max,i} = \frac{V_{max,i}^2}{\Delta q_i}, \qquad T_i = \sqrt{\frac{4\Delta q_i}{A_{max,i}}} > 0.$$

With the given data, it is

$$A_{max,1} = -5.0930 \text{ [rad/s}^2\text{]}, \quad T_1 = 0.7854 \text{ [s]}, \quad A_{max,2} = 2.8648 \text{ [rad/s}^2\text{]}, \quad T_2 = 1.0472 \text{ [s]}.$$

However, since the motion has to be coordinated, the total motion time will be dictated by the slowest joint:

$$T = \max\{T_1, T_2\} \qquad \Rightarrow \qquad T = T_2 = 1.0472 \text{ [s]}.$$

Accordingly, the faster joint should be slowed down and its actual peak velocity V_i and constant acceleration A_i in the first motion half recomputed based on the total motion time T. In the present case, joint 1 will be slowed down with

$$V_1 = \frac{2\Delta q_1}{T} = -1.5 \text{ [rad/s]}, \qquad A_1 = \frac{V_1^2}{\Delta q_1} = -2.8648 \text{ [rad/s}^2\text{]},$$

whereas it is still $A_2 = A_{max,2} = 2.8648$ [rad/s^2] and $V_2 = V_{max,2} = 1.5$ [rad/s] for joint 2. Note that, after the scaling, we have opposite values for the two joints ($A_1 = -A_2$ and $V_1 = -V_2$) simply because in this case the displacement are opposite ($\Delta_1 = -\Delta_2$). Figure 3 shows the coordinated, time-optimal profiles of $\ddot{q}_1(t)$ and $\ddot{q}_2(t)$.

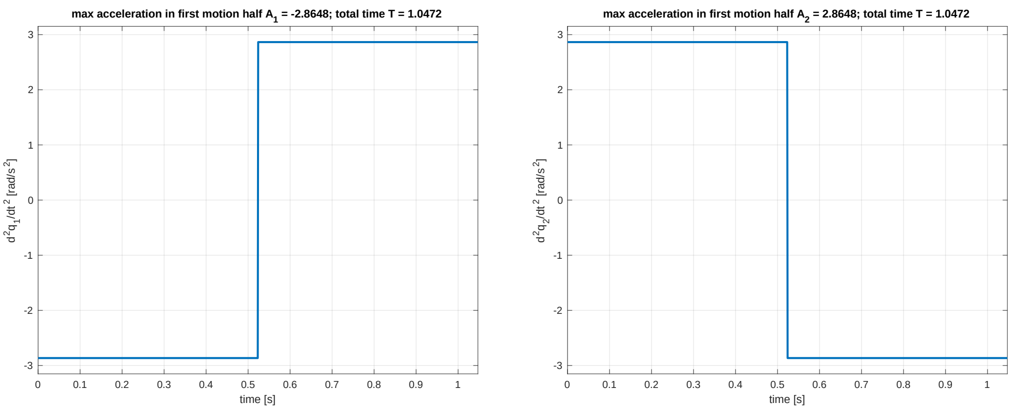

Figure 3: Final acceleration profiles of the two joints.

Note finally that choosing instead the left arm solution in place of (2),

$$\boldsymbol{q}_{in}^{-} = \boldsymbol{f}^{-1}(P_{in}) = \begin{pmatrix} 0.5110 \\ -\dfrac{\pi}{4} \end{pmatrix} \text{ [rad]}, \qquad \boldsymbol{q}_{fin}^{-} = \boldsymbol{f}^{-1}(P_{fin}) = \begin{pmatrix} 0.1419 \\ -\dfrac{\pi}{2} \end{pmatrix} \text{ [rad]}, \tag{3}$$

would have lead to different acceleration profiles, but still to the same coordinated motion time $T = 1.0472$ [s] in this particular case. This is due to the fact that the limiting joint is the second, with a displacement $\Delta q_2 = -\pi/4$ which is the same (in absolute value) as before.

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