Solution

June 10, 2022

Exercise 1

A D-H frame assignment for the spatial 3R robot is shown in Fig. 2, with the associated table of D-H parameters given in Tab. 1.

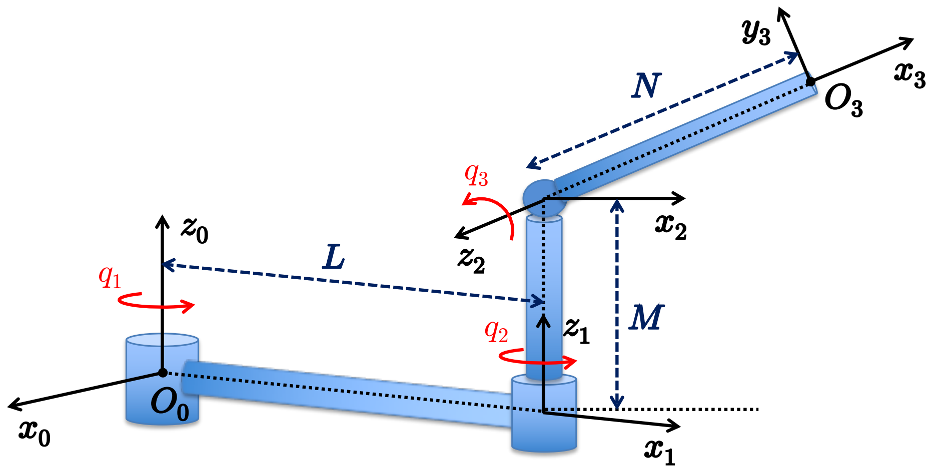

Figure 2: A D-H frame assignment for the spatial 3R robot of Fig. 1.

i	α_i	a_i	d_i	θ_i
1	0	L	0	$q_1 > 0$
2	$\pi/2$	0	M	$q_2 > 0$
3	0	N	0	$q_3 > 0$

Table 1: Table of D-H parameters associated to Fig. 2.

Based on Tab. 1, one can evaluate the D-H homogeneous transformation matrices ${}^{i-1}\boldsymbol{A}_i(q_i)$, for $i = 1, 2, 3$. An efficient symbolic computation for obtaining the end-effector position $\boldsymbol{p} = \boldsymbol{p}_3(\boldsymbol{q})$ makes use of recursive matrix-vector products in homogeneous coordinates as

$$\begin{pmatrix} \boldsymbol{p}_3(\boldsymbol{q}) \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\left({}^1\boldsymbol{A}_2(q_2)\left({}^2\boldsymbol{A}_3(q_3)\begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}\right)\right) = \begin{pmatrix} L\cos q_1 + N\cos(q_1+q_2)\cos q_3 \\ L\sin q_1 + N\sin(q_1+q_2)\cos q_3 \\ M + N\sin q_3 \\ 1 \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix}. \tag{2}$$

It is easy to verify that the following inequalities on the components of the position of the end effector should *necessarily* hold

$$|L - N| \le \sqrt{p_x^2 + p_y^2 + (p_z - M)^2} \le L + N, \qquad M - N \le p_z \le M + N$$

in order for $\boldsymbol{p}$ to belong to the primary (or reachable) workspace WS_1 of the robot, namely the set of all points in $\mathbb{R}^3$ that can be reached by the end-effector position. These inequalities are also helpful for sketching WS_1. As shown in Fig. 3, the workspace is in fact a solid torus parallel to the (x_0, y_0) plane, with center at $(0, 0, M)$, inner radius $R_{in} = |L - N|$ and outer radius $R_{out} = L + N$. Any vertical section of the 3D object with a plane passing though the origin is a circle of radius $r = N$.

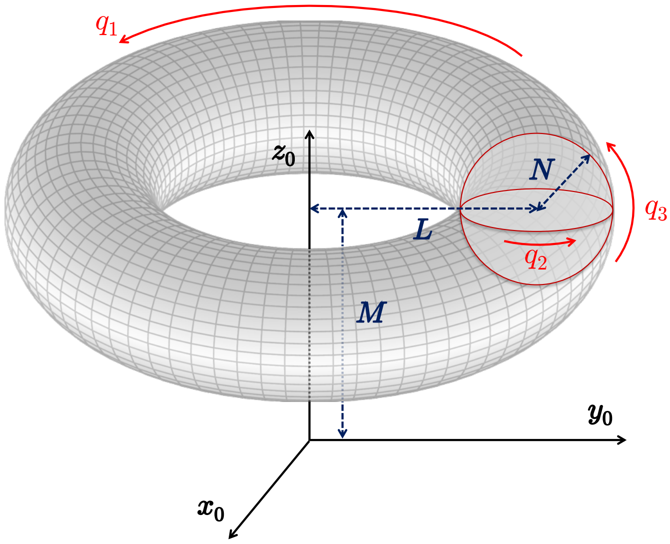

Figure 3: The primary workspace of the spatial 3R robot of Fig. 1.

Differentiating the first three components in (2), we obtain $\boldsymbol{v} = \dot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$ with the Jacobian

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -Ls_1 - Ns_{12}c_3 & -Ns_{12}c_3 & -Nc_{12}s_3 \\ Lc_1 + Nc_{12}c_3 & Nc_{12}c_3 & -Ns_{12}s_3 \\ 0 & 0 & Nc_3 \end{pmatrix}, \tag{3}$$

where the compact notation for trigonometric functions has been used (e.g., $s_{12} = \sin(q_1 + q_2)$). The determinant of $\boldsymbol{J}(\boldsymbol{q})$ is

$$\det \boldsymbol{J}(\boldsymbol{q}) = L\, N^2 s_2\, c_3^2,$$

which is independent from q_1 as it should be. Therefore, singularities occur when:

- $s_2 = 0 \iff q_2 = 0$ or $q_2 = \pi$: the three links live in the vertical plane (x_1, z_0).

  The rank of the Jacobian is then always $\rho(\boldsymbol{J}) = 2$, for all q_3. This can be seen also more clearly expressing the Jacobian in the rotated frame RF_1. For instance, when $q_2 = 0$ it is

$$\boldsymbol{J}(\boldsymbol{q})\big|_{q_2=0} = \begin{pmatrix} -s_1(L + Nc_3) & -Ns_1c_3 & -Nc_1s_3 \\ c_1(L + Nc_3) & Nc_1c_3 & -Ns_1s_3 \\ 0 & 0 & Nc_3 \end{pmatrix},$$

$${}^1\boldsymbol{J}(\boldsymbol{q})\big|_{q_2=0} = {}^0\boldsymbol{R}_1^T(q_1)\boldsymbol{J}(\boldsymbol{q})\Big|_{q_2=0} = \begin{pmatrix} 0 & 0 & -Ns_3 \\ L + Nc_3 & Nc_3 & 0 \\ 0 & 0 & Nc_3 \end{pmatrix}.$$

- $c_3 = 0 \iff q_3 = \pi/2$ or $q_3 = -\pi/2$: the third link is straight vertical. In this case, $\rho(\boldsymbol{J}) = 2$, for all $q_2 \neq \pm\pi/2$. For instance, when $q_3 = \pi/2$ it is

$$\boldsymbol{J}(\boldsymbol{q})\big|_{q_3=\pi/2} = \begin{pmatrix} -Ls_1 & 0 & -Nc_{12} \\ Lc_1 & 0 & -Ns_{12} \\ 0 & 0 & 0 \end{pmatrix}, \qquad {}^1\boldsymbol{J}(\boldsymbol{q})\big|_{q_3=\pi/2} = \begin{pmatrix} 0 & 0 & -Nc_2 \\ L & 0 & -Ns_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

- In particular[1], when $c_3 = 0$ and $c_2 = 0$, the rank drops further to $\rho(\boldsymbol{J}) = 1$. For instance, when $q_2 = q_3 = \pi/2$ it is

$$\boldsymbol{J}(\boldsymbol{q})\big|_{q_2=q_3=\pi/2} = \begin{pmatrix} -Ls_1 & 0 & Ns_1 \\ Lc_1 & 0 & -Nc_1 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \mathcal{R}\left\{ \boldsymbol{J}(\boldsymbol{q})\big|_{q_2=q_3=\pi/2} \right\} = \text{span}\left\{ \begin{pmatrix} s_1 \\ -c_1 \\ 0 \end{pmatrix} \right\}.$$

Consider now this last case, with $\boldsymbol{q}_s$ such that $q_2 = q_3 = \pi/2$. In this singularity, any admissible end-effector velocity $\boldsymbol{v}_s$, as well as the infinite set of joint velocities $\dot{\boldsymbol{q}}_s$ that will realize them, will be of the form

$$\boldsymbol{v}_s = \alpha \begin{pmatrix} s_1 \\ -c_1 \\ 0 \end{pmatrix}, \quad \forall \alpha \qquad \Rightarrow \qquad \dot{\boldsymbol{q}}_s = \begin{pmatrix} \beta \\ 0 \\ \gamma \end{pmatrix}, \quad \text{with } \gamma N - \beta L = \alpha.$$

Thus, for a given α, there will be infinite possible solutions $\dot{\boldsymbol{q}}_s$. For instance, for $\alpha = 1$, the joint velocity solution with minimum norm[2] and a generic second solution are

$$\dot{\boldsymbol{q}}_{s,1} = \boldsymbol{J}^{\#}(\boldsymbol{q})\Big|_{q_2=q_3=\pi/2} \boldsymbol{v}_s = \frac{1}{L^2+N^2} \begin{pmatrix} -L \\ 0 \\ N \end{pmatrix}, \qquad \dot{\boldsymbol{q}}_{s,2} = \frac{1}{L} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

Exercise 2

Differentiating eq. (1) once and twice w.r.t. time gives

$$\dot{\boldsymbol{p}} = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \begin{pmatrix} -q_2 \sin q_1 & \cos q_1 \\ q_2 \cos q_1 & \sin q_1 \end{pmatrix} \dot{\boldsymbol{q}}$$

and

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \begin{pmatrix} -q_2 \cos q_1 \, \dot{q}_1^2 - 2 \sin q_1 \, \dot{q}_1 \dot{q}_2 \\ -q_2 \sin q_1 \, \dot{q}_1^2 + 2 \cos q_1 \, \dot{q}_1 \dot{q}_2 \end{pmatrix}.$$

Therefore, in order to obtain $\ddot{\boldsymbol{p}} = \boldsymbol{0}$ out of a singular configuration ($q_2 \neq 0$), the *unique* choice for the joint acceleration is

$$\ddot{\boldsymbol{q}} = -\boldsymbol{J}^{-1}(\boldsymbol{q})\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = -\frac{1}{q_2} \begin{pmatrix} -2\dot{q}_1\dot{q}_2 \\ q_2^2 \dot{q}_1^2 \end{pmatrix}.$$

We note also that it will never be possible to obtain $\ddot{\boldsymbol{p}} = \boldsymbol{0}$ in a singularity, when the product $\dot{q}_1\dot{q}_2 \neq 0$ (i.e., in the generic case for $\dot{\boldsymbol{q}} \neq \boldsymbol{0}$).

[1]The further exploration of what happens in the singularity $c_3 = 0$ is also suggested by the fact that this factor appears as squared in the symbolic expression of the determinant of the Jacobian.

[2]The pseudoinverse can be computed symbolically with MATLAB in this simple case.

Exercise 3

The mapping between Cartesian forces $\boldsymbol{F} \in \mathbb{R}^2$ applied at the end effector of the RP robot and balancing joint torques $\boldsymbol{\tau} \in \mathbb{R}^2$ guaranteeing static equilibrium is given by

$$\boldsymbol{\tau} = -\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F} = -\begin{pmatrix} -q_2 \sin q_1 & q_2 \cos q_1 \\ \cos q_1 & \sin q_1 \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix}, \tag{4}$$

thus being linear at a given configuration $\boldsymbol{q}$. Vice versa, balancing joint torques map into Cartesian forces as

$$\boldsymbol{F} = -\boldsymbol{J}^{-T}(\boldsymbol{q})\boldsymbol{\tau} = \frac{1}{q_2}\begin{pmatrix} \sin q_1 & -q_2 \cos q_1 \\ -\cos q_1 & -q_2 \sin q_1 \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}.$$

This mapping will transform the rectangular region of feasible joint torques (whose vertices are given by the four combinations of signs in $\boldsymbol{\tau} = (\pm\tau_{max,1}, \pm\tau_{max,2})$ into a polytope (here, a convex polygon) of admissible Cartesian forces $\boldsymbol{F} = (F_x, F_y)$ that can be applied at the robot end effector and effectively balanced. At $\boldsymbol{q} = (\pi/3, 1.5)$ [rad,m], the inverse of the Jacobian transpose is

$$\bar{\boldsymbol{J}}^{-T} = \boldsymbol{J}^{-T}(\boldsymbol{q})\Big|_{\boldsymbol{q}=(\pi/3,1.5)} = \begin{pmatrix} -0.5774 & 0.5000 \\ 0.3333 & 0.8660 \end{pmatrix},$$

and the four vertices of this Cartesian region are computed as

$$\boldsymbol{F}_{++} = -\bar{\boldsymbol{J}}^{-T}\begin{pmatrix} 10 \\ 5 \end{pmatrix} = \begin{pmatrix} 3.2735 \\ -7.6635 \end{pmatrix} \qquad \boldsymbol{F}_{+-} = -\bar{\boldsymbol{J}}^{-T}\begin{pmatrix} 10 \\ -5 \end{pmatrix} = \begin{pmatrix} 8.2735 \\ 0.9968 \end{pmatrix}$$

$$\boldsymbol{F}_{--} = -\bar{\boldsymbol{J}}^{-T}\begin{pmatrix} -10 \\ -5 \end{pmatrix} = \begin{pmatrix} -3.2735 \\ 7.6635 \end{pmatrix} \qquad \boldsymbol{F}_{-+} = -\bar{\boldsymbol{J}}^{-T}\begin{pmatrix} -10 \\ +5 \end{pmatrix} = \begin{pmatrix} -8.2735 \\ -0.9968 \end{pmatrix}.$$

The resulting admissible region is shown (in blue) in Fig. 4 (try to verify the correspondence between the vertices).

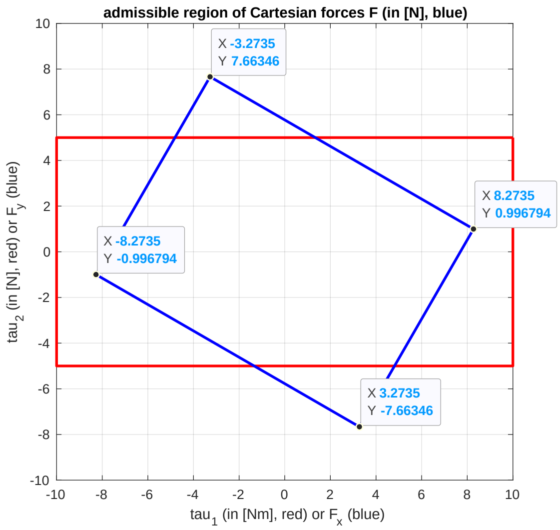

Figure 4: The set of feasible joint torques (rectangle in red) and the region of associated admissibile Cartesian forces (skewed rectangle in blue) that can be statically balanced by the RP robot.

For an additional check, take one Cartesian force that belongs to the blue region and is close to a boundary, and compute the balancing torque by (4) to verify its feasibility. For instance, with

$$\boldsymbol{F} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} \text{ [N]} \qquad \Rightarrow \qquad \boldsymbol{\tau} = -\left.\boldsymbol{J}^T(\boldsymbol{q})\right|_{\boldsymbol{q}=(\pi/3,1.5)} \boldsymbol{F} = \begin{pmatrix} -9.6962 \\ -3.1962 \end{pmatrix} \text{ [Nm,N]},$$

the obtained $\boldsymbol{\tau}$ is feasible.

Exercise 4

To address the problem one applies the following Cartesian kinematic control,

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\Big(\dot{\boldsymbol{p}}_d + \boldsymbol{K}_P\left(\boldsymbol{p}_d - \boldsymbol{p}(\boldsymbol{q})\right)\Big), \qquad \text{with } \boldsymbol{K}_P = k_P \cdot \boldsymbol{I}_{2\times 2} > 0, \tag{5}$$

where the common scalar gain k_p i s used in both Cartesian directions because of the requested uniformity of error behavior. For the given 2R planar robot and motion task, we have

$$\boldsymbol{p}(\boldsymbol{q}) = \begin{pmatrix} c_1 + c_{12} \\ s_1 + s_{12} \end{pmatrix}, \qquad \boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -(s_1 + s_{12}) & -s_{12} \\ c_1 + c_{12} & c_{12} \end{pmatrix},$$

$$\boldsymbol{p}_d(t) = \boldsymbol{P}_1 + v_d t\big(\boldsymbol{P}_2 - \boldsymbol{P}_1\big) = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix} + 0.5\,t \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \dot{\boldsymbol{p}}_d = v_d\big(\boldsymbol{P}_2 - \boldsymbol{P}_1\big) = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}.$$

The initial position of the end effector $\boldsymbol{P}_0 = (0.5, 0.5)$ [m] corresponds to an initial Cartesian error at $t = 0$ that is non-zero only along the x-direction

$$\boldsymbol{e}_p(0) = \boldsymbol{p}_d(0) - \boldsymbol{p}(\boldsymbol{q}(0)) = \boldsymbol{P}_1 - \boldsymbol{P}_0 = \begin{pmatrix} 0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} e_{p,x}(0) \\ e_{p,y}(0) \end{pmatrix}.$$

Moreover, from (5) it follows that $\dot{\boldsymbol{e}}_p = -\boldsymbol{K}_P\, \boldsymbol{e}_p$ and so

$$\boldsymbol{e}_p(t) = \exp\left(-\boldsymbol{K}_p t\right) \boldsymbol{e}_p(0) \qquad \Rightarrow \qquad \begin{cases} e_{p,x}(t) = \exp\left(-k_P t\right) e_{p,x}(0) \\ e_{p,y}(t) = 0, \end{cases} \qquad \forall t \geq 0.$$

The initial configuration of the robot at time $t = 0$ is found by the standard inverse kinematics of a 2R robot (choosing the elbow down solution[3]:

$$\boldsymbol{q}(0) = \text{invkin}\left(\boldsymbol{P}_{in}\right) = \begin{pmatrix} -0.4240 \\ 2.4189 \end{pmatrix} \text{ [rad]}.$$

Plugging all the above information in (5) yields at time $t = 0$

$$\dot{\boldsymbol{q}}(0) = \boldsymbol{J}^{-1}(\boldsymbol{q}(0))\left(\begin{pmatrix} 0 \\ 0.5 \end{pmatrix} + \begin{pmatrix} 0.5\,k_P \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -0.5 & -0.9114 \\ 0.5 & -0.4114 \end{pmatrix}^{-1} \begin{pmatrix} 0.5\,k_P \\ 0.5 \end{pmatrix}$$

$$= \begin{pmatrix} -0.6220 & 1.3780 \\ -0.7559 & -0.7559 \end{pmatrix} \begin{pmatrix} 0.5\,k_P \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.6890 \\ -0.3780 \end{pmatrix} + k_P \begin{pmatrix} -0.3110 \\ -0.3780 \end{pmatrix}.$$

[3] The choice of the elbow up solution would lead exactly to the same final result in this case, although passing through different numerical values in intermediate passages.

Therefore, the largest (positive) proportional control gain that can be used to speed up the decrease of the transient error along the x-direction while satisfying the joint velocity bounds on $\dot{\boldsymbol{q}}(0)$,

$$-V_{max,1} = -3 \leq 0.6890 - 0.3110\, k_P \leq 3 = V_{max,1},$$

$$-V_{max,2} = -2 \leq 0.3780 - 0.3780\, k_P \leq 2 = V_{max,2},$$

is computed as follows:

$$k_P^* = \min\left\{\frac{V_{max,1} + 0.6890}{0.3110}, \frac{V_{max,2} + 0.3780}{0.3780}\right\} = \min\{11.8610, 4.2915\} = 4.2915.$$

This choice will saturate the initial velocity of joint 2 to its largest negative value $\dot{q}_2(0) = -V_{max,2} = -2$ rad/s. The solution is in fact

$$\dot{\boldsymbol{q}}(0) = \begin{pmatrix} -0.6458 \\ -2 \end{pmatrix} \text{ [rad/s]} \qquad \Rightarrow \qquad \boldsymbol{v}(0) = \boldsymbol{J}(\boldsymbol{q}(0))\dot{\boldsymbol{q}}(0) = \begin{pmatrix} 2.1458 \\ 0.5000 \end{pmatrix} \text{ [rad/s]},$$

with the end-effector velocity pointing up and toward the path. See also the sketch of the initial situation in Fig. 5.

The time constant of the exponential decrease of the tracking error is $\tau_P = 1/k_P^* = 0.233$ [s]. This means that the error will be practically zero (i.e., reduced to less than 5% of its initial value) in about $3\tau_p \simeq 0.7$ [s], namely when the nominal trajectory is still at 1/3 of its total travel time ($T = \|\boldsymbol{P}_2 - \boldsymbol{P}_1\|/v_d = 2$ [s]).

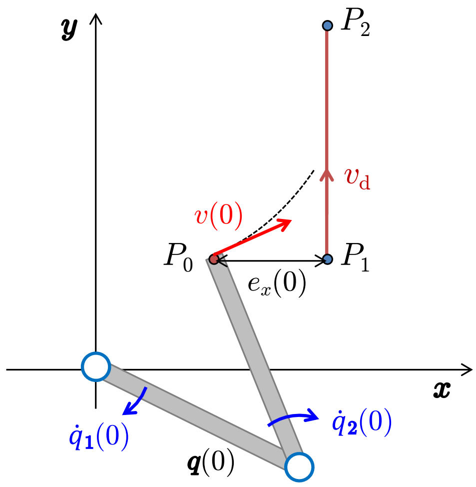

Figure 5: The 2R robot in the initial configuration, recovering the tracking error w.r.t. the desired trajectory.

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