Solution

July 8, 2022

Exercise 1

When using the sequence of RPY-type angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)$ defined around the fixed axes XZY, the orientation of the robot end-effector is given by the rotation matrix

$$
\begin{aligned}
\boldsymbol{R}_{XZY}(\alpha,\beta,\gamma) &= \boldsymbol{R}_Y(\gamma)\boldsymbol{R}_Z(\beta)\boldsymbol{R}_X(\alpha)\\
&= \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}\\
&= \begin{pmatrix} \cos\beta\cos\gamma & \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma & \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma \\ \sin\beta & \cos\alpha\cos\beta & -\sin\alpha\cos\beta \\ -\cos\beta\sin\gamma & \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma \end{pmatrix}.
\end{aligned}
$$

Note that this is the same orientation obtained by using the sequence of Euler angles (γ, β, α) defined around the moving axes YZ′X″.

The angular velocity $\boldsymbol{\omega}$ of the body can be obtained from the formula $\boldsymbol{S}(\boldsymbol{\omega}) = \dot{\boldsymbol{R}}_{XZY}(\boldsymbol{\phi})\boldsymbol{R}^T_{XZY}(\boldsymbol{\phi})$, where $\boldsymbol{S}$ is a skew-symmetric matrix. With the shorthand notation for trigonometric functions, taking the time derivative of $\boldsymbol{R}_{XZY}$ and post-multiplying by the transpose of the same rotation matrix yields

$$
\begin{aligned}
&\dot{\boldsymbol{R}}_{XZY}(\boldsymbol{\phi}) \cdot \boldsymbol{R}^T_{XZY}(\boldsymbol{\phi})\\
&= \begin{pmatrix} -s_\beta c_\gamma\dot\beta - c_\alpha s_\beta\dot\gamma & \begin{matrix}(c_\alpha s_\gamma + s_\alpha s_\beta c_\gamma)\dot\alpha - c_\alpha c_\beta c_\gamma\,\dot\beta \\ +\,(s_\alpha c_\gamma + c_\alpha s_\beta s_\gamma)\dot\gamma\end{matrix} & \begin{matrix}(c_\alpha s_\beta c_\gamma - s_\alpha s_\gamma)\dot\alpha + s_\alpha c_\beta c_\gamma\,\dot\beta \\ +\,(c_\alpha c_\gamma - s_\alpha s_\beta s_\gamma)\dot\gamma\end{matrix} \\ c_\beta\dot\beta & -s_\alpha c_\beta\dot\alpha - c_\alpha s_\beta\dot\beta & -c_\alpha c_\beta\dot\alpha + s_\alpha s_\beta\dot\beta \\ s_\beta s_\gamma\dot\beta - c_\beta c_\gamma\dot\gamma & \begin{matrix}(c_\alpha c_\gamma - s_\alpha s_\beta s_\gamma)\dot\alpha + c_\alpha c_\beta s_\gamma\,\dot\beta \\ +\,(c_\alpha s_\beta c_\gamma - s_\alpha s_\gamma)\dot\gamma\end{matrix} & \begin{matrix}-(s_\alpha c_\gamma + c_\alpha s_\beta s_\gamma)\dot\alpha - s_\alpha c_\beta s_\gamma\,\dot\beta \\ -\,(s_\alpha s_\beta c_\gamma + c_\alpha s_\gamma)\dot\gamma\end{matrix} \end{pmatrix}\\
&\quad\cdot \begin{pmatrix} c_\beta c_\gamma & s_\beta & -c_\beta s_\gamma \\ s_\alpha s_\gamma - c_\alpha s_\beta c_\gamma & c_\alpha c_\beta & s_\alpha c_\gamma + c_\alpha s_\beta s_\gamma \\ c_\alpha s_\gamma + s_\alpha s_\beta c_\gamma & -s_\alpha c_\beta & c_\alpha c_\beta - s_\alpha s_\beta s_\gamma \end{pmatrix} \qquad (2)\\
&= \begin{pmatrix} 0 & c_\beta s_\gamma\,\dot\alpha - c_\gamma\,\dot\beta & s_\beta\,\dot\alpha + \dot\gamma \\ -c_\beta s_\gamma\,\dot\alpha + c_\gamma\,\dot\beta & 0 & -c_\beta c_\gamma\,\dot\alpha - s_\gamma\,\dot\beta \\ -s_\beta\,\dot\alpha - \dot\gamma & c_\beta c_\gamma\,\dot\alpha + s_\gamma\,\dot\beta & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} = \boldsymbol{S}(\boldsymbol{\omega}).
\end{aligned}
$$

The above derivation is greatly simplified by using symbolic computation in MATLAB. The linear mapping $\boldsymbol{\omega} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}$ is then extracted from the elements of matrix $\boldsymbol{S}$ in (2) as

$$
\boldsymbol{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} c_\beta c_\gamma\,\dot\alpha + s_\gamma\,\dot\beta \\ s_\beta\,\dot\alpha + \dot\gamma \\ -c_\beta s_\gamma\,\dot\alpha + c_\gamma\,\dot\beta \end{pmatrix} = \begin{pmatrix} \cos\beta\cos\gamma & \sin\gamma & 0 \\ \sin\beta & 0 & 1 \\ -\cos\beta\sin\gamma & \cos\gamma & 0 \end{pmatrix}\begin{pmatrix} \dot\alpha \\ \dot\beta \\ \dot\gamma \end{pmatrix} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}.
$$

The singularities of this mapping occur when $\det \boldsymbol{T}(\boldsymbol{\phi}) = -\cos\beta = 0$, i.e., for $\beta = \pm\pi/2$.

In alternative to the above procedure, and perhaps more quickly, one can build the matrix $\boldsymbol{T}(\boldsymbol{\phi})$ by noting the individual contributions to the angular velocity $\boldsymbol{\omega}$ in the Euler interpretation of the rotation matrix $\boldsymbol{R}_{XZY}$: $\dot{\gamma}$ is a rotation around the initial (fixed) Y-axis; $\dot{\beta}$ is a rotation around the Z'-axis, i.e., the Z-axis after the rotation $\boldsymbol{R}_Y(\gamma)$; and $\dot{\alpha}$ is a rotation around the X''-axis, i.e., the X-axis after the first two rotations $\boldsymbol{R}_Y(\gamma)\boldsymbol{R}_Z(\beta)$. Thus, we have

$$\begin{aligned}\boldsymbol{\omega} = \boldsymbol{\omega}_{\dot{\gamma},Y} + \boldsymbol{\omega}_{\dot{\beta},Z'} + \boldsymbol{\omega}_{\dot{\alpha},X''} &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \dot{\gamma} + \boldsymbol{R}_Y(\gamma) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \dot{\beta} + \boldsymbol{R}_Y(\gamma)\boldsymbol{R}_Z(\beta) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \dot{\alpha} \\ &= \begin{pmatrix} \cos\beta\cos\gamma \\ \sin\beta \\ -\cos\beta\sin\gamma \end{pmatrix} \dot{\alpha} + \begin{pmatrix} \sin\gamma \\ 0 \\ \cos\gamma \end{pmatrix} \dot{\beta} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \dot{\gamma} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}},\end{aligned}$$

where, being each contribution to $\boldsymbol{\omega}$ a vector itself, the order in the sum is irrelevant.
Choose now, for instance, $\beta = \pi/2$. Thus

$$\bar{\boldsymbol{T}}(\gamma) = \boldsymbol{T}(\boldsymbol{\phi})|_{\beta=\pi/2} = \begin{pmatrix} 0 & \sin\gamma & 0 \\ 1 & 0 & 1 \\ 0 & \cos\gamma & 0 \end{pmatrix}, \qquad \operatorname{rank} \bar{\boldsymbol{T}}(\gamma) = 2.$$

In this representation singularity (for any value of α and γ), one has that angular velocities $\boldsymbol{\omega}$ of the form

$$\boldsymbol{\omega} = \rho \begin{pmatrix} \cos\gamma \\ 0 \\ -\sin\gamma \end{pmatrix} \notin \mathcal{R}\left(\bar{\boldsymbol{T}}(\gamma)\right), \qquad \forall \rho \neq 0$$

are not realizable by any possible choice of $\dot{\boldsymbol{\phi}}$. Moreover, time derivatives of $\boldsymbol{\phi}$ of the form

$$\dot{\boldsymbol{\phi}} = \sigma \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \in \mathcal{N}\left(\bar{\boldsymbol{T}}(\gamma)\right), \qquad \forall \sigma \in \mathbb{R}$$

generate zero angular velocity.

Exercise 2

The problem has four inverse kinematics solutions, provided the desired end-effector position $\boldsymbol{p}_d$ is inside the primary (or, reachable) workspace WS_1 of the robot[1]. For this to happen, the following inequalities should *necessarily* hold for the components p_{dx}, p_{dy} and p_{dz} of the desired position $\boldsymbol{p}_d$ of the end effector:

$$|L - N| \leq \sqrt{p_{dx}^2 + p_{dy}^2 + (p_{dz} - M)^2} \leq L + N, \qquad M - N \leq p_{dz} \leq M + N, \tag{3}$$

with strict inequalities enforced for the interior of WS_1. For the given data, these inequalities are satisfied being, respectively,

$$0 < 0.4690 < 1, \qquad 0 < 0.7 < 1.$$

[1]This workspace is in fact a *solid torus*, see Fig. 3 in the solution of Exercise #1 of the exam of June 10, 2022.

Even if this check is not done (or even known!) in advance, these inequalities will appear as necessary conditions to be satisfied in order to proceed with the computation of analytic inverse kinematics solutions.

From the third equation in the direct kinematics (1), one has

$$M + N \sin q_3 = p_{dz} \quad \Rightarrow \quad s_3 = \frac{p_{dz} - M}{N} \in [-1, 1] \quad \text{—and also } c_3 = \pm\sqrt{1 - s_3^2} \in [-1, 1],$$

with the admissible interval for the trigonometric function s_3 leading to the second pair of inequalities in (3). The two solutions for q_3 are then

$$q_3^{[+]} = \text{atan2}\left\{ s_3, +\sqrt{1 - s_3^2} \right\}, \qquad q_3^{[-]} = \text{atan2}\left\{ s_3, -\sqrt{1 - s_3^2} \right\}. \tag{4}$$

Each of these will branch in two solution pairs for (q_1, q_2). In fact, the first two equations in (1) can be interpreted as the direct kinematics of a planar 2R arm with link lengths

$$l_1 = L, \qquad l_2 = N \cos q_3^{[+]} \quad \text{or} \quad l_2 = N \cos q_3^{[-]}. \tag{5}$$

For each resulting value of l_2, one can use the solution for the 2R arm, which is obtained though the standard formulas. First, evaluate

$$c_2 = \frac{p_{dx}^2 + p_{dy}^2 - (l_1^2 + l_2^2)}{2 l_1 l_2} \in [-1, 1], \qquad s_2 = \pm\sqrt{1 - c_2^2}.$$

It can be shown that the admissible interval for the trigonometric function c_2 leads to the necessity of the first pair of inequalities in (3). For each value of l_2 in (5), the two solutions for q_2 are then

$$q_2^{[+]} = \text{atan2}\left\{ \sqrt{1 - c_2^2}, c_2 \right\}, \qquad q_2^{[-]} = \text{atan2}\left\{ -\sqrt{1 - c_2^2}, c_2 \right\}. \tag{6}$$

By the property of the atan2 function, it follows that $q_2^{[-]} = -q_2^{[+]}$. Finally, for each solution pair (q_2, q_3), a 2×2 linear system of equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ in the unknowns $\boldsymbol{x} = (s_1, c_1)$ can be set up, whose solution is found provided the determinant of the system matrix

$$\det \boldsymbol{A} = l_1^2 + l_2^2 + 2 l_1 l_2 c_2 = L^2 + N^2 \cos^2 q_3 + 2LN \cos q_2 \cos q_3 \neq 0.$$

In this case, we have

$$s_1 = \frac{p_{dy}\,(l_1 + l_2 c_2) - p_{dx} l_2 s_2)}{\det \boldsymbol{A}}, \qquad c_1 = \frac{p_{dx}\,(l_1 + l_2 c_2) + p_{dy} l_2 s_2)}{\det \boldsymbol{A}},$$

and the associated (unique) solution for q_1 is

$$q_1 = \text{atan2}\{s_1, c_1\}. \tag{7}$$

Note that in this case the determinant cannot be eliminated from the denominator of the two arguments of this atan2 function; in fact, when the determinant is different from zero, its sign may change depending on the particular solution for the pair (q_2, q_3) inserted in the linear system.

Summarizing, we have found the following four inverse kinematics solutions (all distinct in the regular case):

$$\begin{pmatrix} q_1^{[+,+,+]} & q_2^{[+,+]} & q_3^{[+]} \end{pmatrix}^T, \quad \begin{pmatrix} q_1^{[+,-,+]} & q_2^{[+.-]} & q_3^{[+]} \end{pmatrix}^T,$$
$$\begin{pmatrix} q_1^{[-,+,-]} & q_2^{[-.+]} & q_3^{[-]} \end{pmatrix}^T, \quad \begin{pmatrix} q_1^{[-,-,-]} & q_2^{[-.+]} & q_3^{[-]} \end{pmatrix}^T.$$

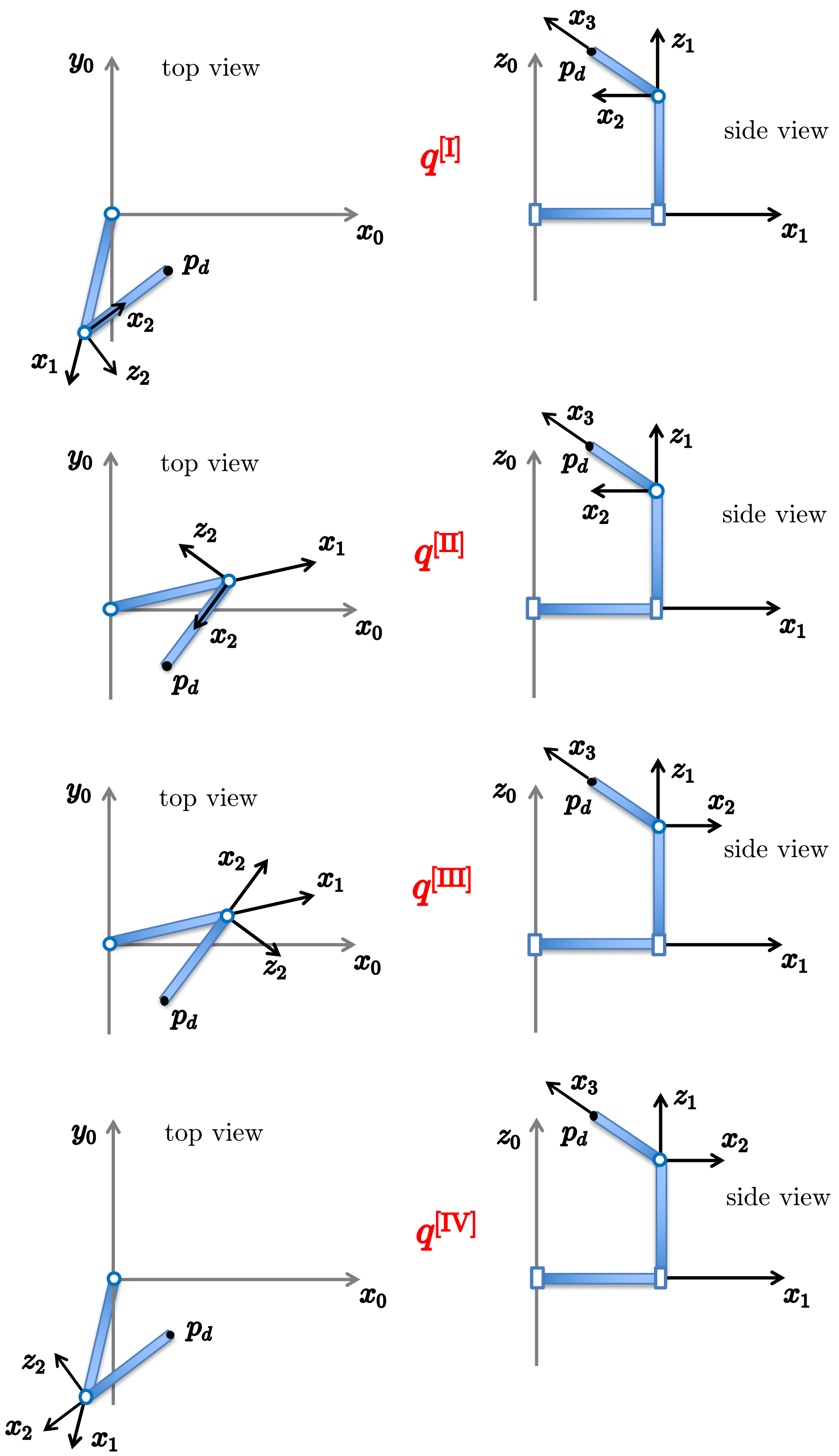

Figure 2: The four inverse kinematics solutions for the spatial 3R robot (top and side views).

Plugging the numerical data in eqs. (4)–(7), we obtain (in [rad])

$$\boldsymbol{q}^{[I]} = \begin{pmatrix} -1.8110 & 2.2281 & 0.4115 \end{pmatrix}^T, \quad \boldsymbol{q}^{[II]} = \begin{pmatrix} 0.2402 & -2.2281 & 0.4115 \end{pmatrix}^T,$$
$$\boldsymbol{q}^{[III]} = \begin{pmatrix} 0.2402 & 0.9135 & 2.7301 \end{pmatrix}^T, \quad \boldsymbol{q}^{[IV]} = \begin{pmatrix} -1.8110 & -0.9135 & 2.7301 \end{pmatrix}^T,$$

or (in degrees)

$$\boldsymbol{q}^{[I]} = \begin{pmatrix} -103.7^\circ & 127.66^\circ & 23.58^\circ \end{pmatrix}^T, \quad \boldsymbol{q}^{[II]} = \begin{pmatrix} 13.77^\circ & -127.66^\circ & 23.58^\circ \end{pmatrix}^T,$$
$$\boldsymbol{q}^{[III]} = \begin{pmatrix} 13.77^\circ & 52.34^\circ & 156.42^\circ \end{pmatrix}^T, \quad \boldsymbol{q}^{[IV]} = \begin{pmatrix} -103.77^\circ & -52.34^\circ & 156.42^\circ \end{pmatrix}^T,$$

Indeed, evaluation of (1) with these solutions returns always the desired $\boldsymbol{p}_d$ (it is worth to do this check!). For each of these four inverse kinematics solutions, Figure 2 sketches two views of the resulting configuration of the spatial 3R robot.

Exercise 3

One should set up a simple code that implements the following Newton iteration

$$\boldsymbol{q}^{\{k\}} = \boldsymbol{q}^{\{k-1\}} + \boldsymbol{J}^{-1}(\boldsymbol{q}^{\{k-1\}}) \left(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q}^{\{k-1\}})\right), \qquad k = 1, 2, \ldots, \tag{8}$$

starting from an initial guess $\boldsymbol{q}^{\{0\}}$, until convergence is achieved or some other stopping criterion is reached. To obtain a reliable code, it is important to include a limit on the maximum number of iterations, signifying that no convergence is being achieved, and a warning (with exit) when a singularity of the Jacobian is being met.
For the implementation of (8), we need the direct kinematics function $\boldsymbol{f}(\boldsymbol{q})$ in (1) and the associated (analytic) Jacobian

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -Ls_1 - Ns_{12}c_3 & -Ns_{12}c_3 & -Nc_{12}s_3 \\ Lc_1 + Nc_{12}c_3 & Nc_{12}c_3 & -Ns_{12}s_3 \\ 0 & 0 & Nc_3 \end{pmatrix}, \tag{9}$$

with $\det \boldsymbol{J}(\boldsymbol{q}) = LN^2 s_2 c_3^2$. As long as this determinant is non-zero (or is far enough from it), iteration (8) is well defined. Nonetheless, convergence can be guaranteed *only* when starting sufficiently close to a solution (although the convergence rate is then the fastest possible, namely quadratic).
In the present case, for $L = M = N = 0.5$ [m] and $\boldsymbol{p}_d = (0.3, -0.3, 0.7)$ [m], when starting from $\boldsymbol{q}^{\{0\}} = (-\pi/4, \pi/4, \pi/4)$ [rad], the method converges within the error tolerance $\epsilon = 10^{-3}$ [m] in $k^* = 5$ iteration, generating

$$\boldsymbol{q}^{\{0\}} = \begin{pmatrix} -0.7854 \\ 0.7854 \\ 0.7854 \end{pmatrix} \to \boldsymbol{q}^{\{1\}} = \begin{pmatrix} -2.3712 \\ 4.1084 \\ 0.3511 \end{pmatrix} \to \boldsymbol{q}^{\{2\}} = \begin{pmatrix} -1.1056 \\ 2.2074 \\ 0.4108 \end{pmatrix} \to$$
$$\to \boldsymbol{q}^{\{3\}} = \begin{pmatrix} -1.8344 \\ 2.4611 \\ 0.4115 \end{pmatrix} \to \boldsymbol{q}^{\{4\}} = \begin{pmatrix} -1.8426 \\ 2.2346 \\ 0.4115 \end{pmatrix} \to \boldsymbol{q}^{\{5\}} = \begin{pmatrix} -1.8110 \\ 2.2286 \\ 0.4115 \end{pmatrix},$$

with a final error norm $\left\|\boldsymbol{e}^{\{5\}}\right\| = 1.97 \cdot 10^{-4} < \epsilon = 10^{-3}$. The evolution of this norm is shown in Fig. 3. Note that the method initially increases the norm, but then starts converging fast when

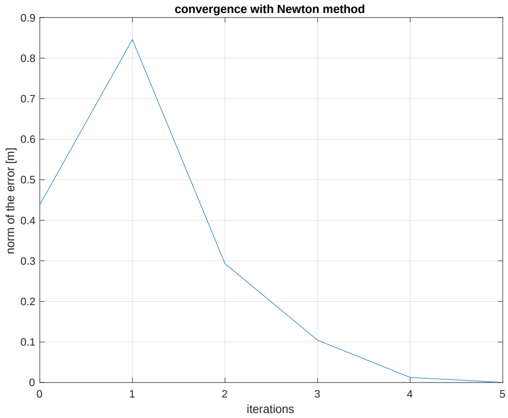

Figure 3: Evolution of the error norm $\left\|\boldsymbol{e}^{\{k\}}\right\|$ during the iterations of Newton method.

closer to a solution. In particular, in the last two iterations, the error norm decreases by one order, and eventually by two orders of magnitude: from 0.104391 at $k = 3$, to 0.012584 at $k = 4$, stopping with $0.000197 < \epsilon = 10^{-3}$ at $k^* = 5$.
The obtained solution corresponds to the analytic solution $\boldsymbol{q}^{[I]}$ found in Exercise #2. To obtain a more accurate numeric value, one should set a tighter error tolerance: with $\epsilon = 10^{-4}$, Newton method runs for two more iterations and stops at $k^* = 7$, returning $\boldsymbol{q}^{[I]}$ with the same four-digit accuracy on all joint variables.

Finally, in order to obtain another inverse kinematics solution, the method has to be restarted from another initial guess, hopefully closer to a different solution. For instance, when starting with $\boldsymbol{q}^{\{0\}} = (\pi/10, \pi/3, 3\pi/4)$ [rad], the method will converge to $\boldsymbol{q}^{[III]}$ in $k^* = 3$ iterations (with the original tolerance $\epsilon = 10^{-3}$) or in $k^* = 6$ iterations (with the tighter tolerance $\epsilon = 10^{-4}$).

Exercise 4

The stated trajectory planning problem is solved by using a single quintic polynomial for each joint, with suitable boundary conditions. For the generic joint i, with $i = 1, 2, 3$, we consider the desired smooth trajectory written as

$$q_i(t) = q_{s,i} + \Delta q_i \cdot q_{n,i}(\tau), \qquad \tau = \frac{t}{T}, \tag{10}$$

with $\Delta q_i = q_{g,i} - q_{s,i}$ and the doubly normalized quintic polynomial

$$q_{n,i}(\tau) = a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3 + a_4\tau^4 + a_5\tau^5, \qquad \tau \in [0, 1],$$

ranging in values between 0 and 1. The first and second time derivatives of (10) are

$$\dot{q}_i(t) = \frac{\Delta q_i}{T} \cdot q'_{n,i}(\tau) = \frac{\Delta q_i}{T} \cdot \left(a_1 + 2a_2\tau + 3a_3\tau^2 + 4a_4\tau^3 + 5a_5\tau^4\right) \tag{11}$$

and

$$\ddot{q}_i(t) = \frac{\Delta q_i}{T^2} \cdot q''_{n,i}(\tau) = \frac{\Delta q_i}{T^2} \cdot \left(2a_2 + 6a_3\tau + 12a_4\tau^2 + 20a_5\tau^3\right), \tag{12}$$

where $'$ is used to denote a derivative with respect to τ. The boundary conditions to be imposed are

$$
\begin{array}{lllll}
q_i(0) = q_{s,i} & \Rightarrow & q_{n,i}(0) = 0 & \Rightarrow & a_0 = 0; \\
q_i(T) = q_{g,i} & \Rightarrow & q_{n,i}(1) = 1 & \Rightarrow & a_0 + a_1 + a_2 + a_3 + a_4 + a_5 = 1; \\
\dot{q}_i(0) = \dot{q}_{s,i} \neq 0 & \Rightarrow & q'_{n,i}(0) = \dot{q}_{s,i} \dfrac{T}{\Delta q_i} & \Rightarrow & a_1 = \dot{q}_{s,i} \dfrac{T}{\Delta q_i}; \\
\dot{q}_i(T) = 0 & \Rightarrow & q'_{n,i}(0) = 0 & \Rightarrow & a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 = 0; \\
\ddot{q}_i(0) = 0 & \Rightarrow & q''_{n,i}(0) = 0 & \Rightarrow & a_2 = 0; \\
\ddot{q}_i(T) = 0 & \Rightarrow & q''_{n,i}(1) = 0 & \Rightarrow & 2a_2 + 6a_3 + 12a_4 + 20a_5 = 0,
\end{array}
\tag{13}
$$

where the scalar component $\dot{q}_{s,i}$ of the initial velocity $\dot{\boldsymbol{q}}_s$ at the start configuration is determined by inversion of the differential kinematics

$$
\dot{\boldsymbol{q}}_s = \dot{\boldsymbol{q}}(0) = \boldsymbol{J}^{-1}(\boldsymbol{q}_s)\dot{\boldsymbol{p}}(0).
$$

Being

$$
a_0 = a_2 = 0, \qquad a_1 = \dot{q}_{s,i} \frac{T}{\Delta q_i},
$$

we need to solve the remaining three linear equations in (13) as

$$
\begin{pmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 6 & 12 & 20 \end{pmatrix}
\begin{pmatrix} a_3 \\ a_4 \\ a_5 \end{pmatrix}
=
\begin{pmatrix} 1 - \dot{q}_{s,i}\, T/\Delta q_i \\ -\dot{q}_{s,i}\, T/\Delta q_i \\ 0 \end{pmatrix}
$$

or

$$
\begin{pmatrix} a_3 \\ a_4 \\ a_5 \end{pmatrix}
=
\begin{pmatrix} 10 & -4 & 0.5 \\ -15 & 7 & -1 \\ 6 & -3 & 0.5 \end{pmatrix}
\begin{pmatrix} 1 - \dot{q}_{s,i}\, T/\Delta q_i \\ -\dot{q}_{s,i}\, T/\Delta q_i \\ 0 \end{pmatrix}
=
\begin{pmatrix} 10 - 6\dot{q}_{s,i}\, T/\Delta q_i \\ -15 + 8\dot{q}_{s,i}\, T/\Delta q_i \\ 6 - 3\dot{q}_{s,i}\, T/\Delta q_i \end{pmatrix}.
$$

Next, we compute the required data for the problem at hand. For the initial joint velocity at $\boldsymbol{q}_s = (-\pi/4, \pi/4, \pi/4)$, we obtain

$$
\dot{\boldsymbol{q}}_s = \boldsymbol{J}^{-1}(\boldsymbol{q}_s)\, \dot{\boldsymbol{p}}(0) =
\begin{pmatrix} 0.3536 & 0 & -0.3536 \\ 0.7071 & 0.3536 & 0 \\ 0 & 0 & 0.3536 \end{pmatrix}^{-1}
\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}
=
\begin{pmatrix} 2.8284 \\ -8.4853 \\ 0 \end{pmatrix}
=
\begin{pmatrix} \dot{q}_{s,1} \\ \dot{q}_{s,2} \\ \dot{q}_{s,3} \end{pmatrix}.
$$

Moreover,

$$
T = 2, \qquad \boldsymbol{q}_s = \begin{pmatrix} -0.7854 \\ 0.7854 \\ 0.7854 \end{pmatrix} = \begin{pmatrix} q_{s,1} \\ q_{s,2} \\ q_{s,3} \end{pmatrix}, \qquad
\boldsymbol{\Delta q} = \boldsymbol{q}_g - \boldsymbol{q}_s = \begin{pmatrix} 0.7854 \\ -0.7854 \\ 0 \end{pmatrix} = \begin{pmatrix} \Delta q_1 \\ \Delta q_2 \\ \Delta q_3 \end{pmatrix}.
$$

Evaluating then the coefficients a_i's in the solution, we obtain the following quintic joint trajectories, written using the doubly normalized polynomials $q_{n,1}(\tau)$ and $q_{n,2}(\tau)$ (($q_{n3}(\tau)$ is also present, but actually irrelevant):

$$
\begin{aligned}
q_1(t) &= -0.7854 + 0.7854 \left(7.2025\, \tau - 33.2152\, \tau^3 + 42.6202\, \tau^4 - 15.6076\, \tau^5\right) \\
q_2(t) &= 0.7854 - 0.7854 \left(21.6076\, \tau - 119.6455\, \tau^3 + 157.8607\, \tau^4 - 58.8228\, \tau^5\right) \\
q_3(t) &= 0.7854 + 0 \left(10\, \tau^3 - 15\, \tau^4 + 6\, \tau^5\right) = 0.7854 \qquad \text{(no motion needed for this joint!)}
\end{aligned}
\tag{14}
$$

The plots of the joint trajectories, with their velocity and acceleration as given by (11) and (12), are reported in Fig. 4.

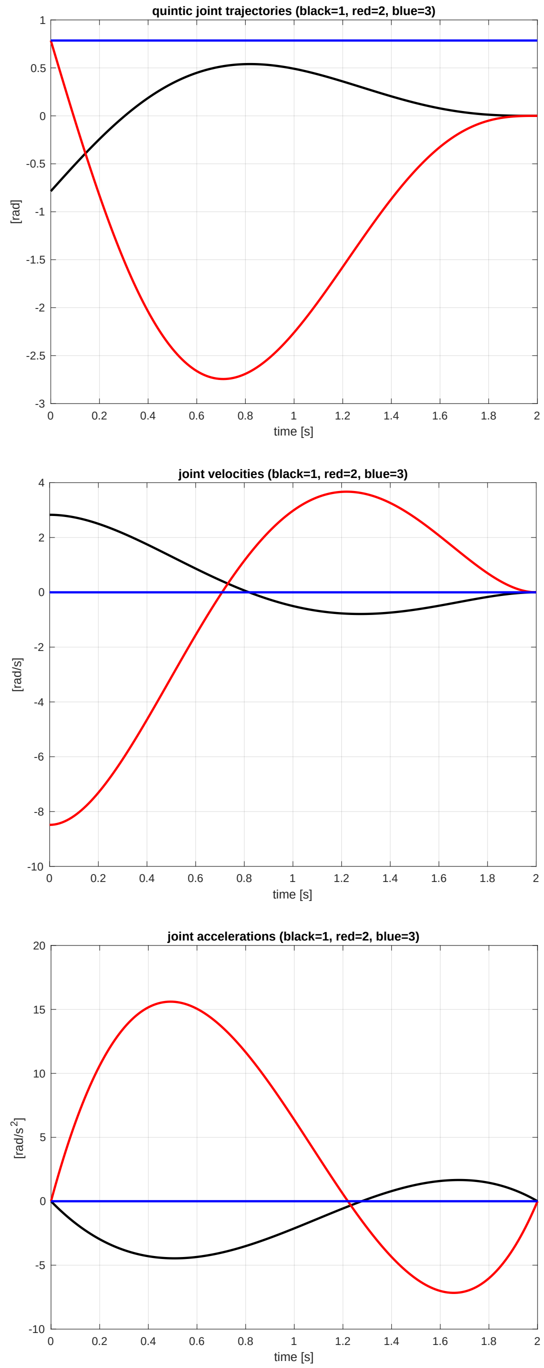

Figure 4: Solution of the trajectory planning problem: joint position (top), velocity (center) and acceleration (bottom).

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