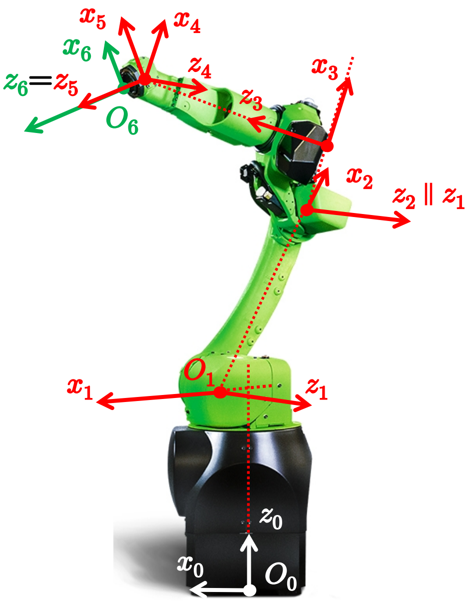
Solution

September 9, 2022

Exercise 1

A possible DH frame assignment for the Fanuc CR15ia robot is shown in Fig. 2, in the front and back views. The associated DH parameters are reported in Tab. 1.



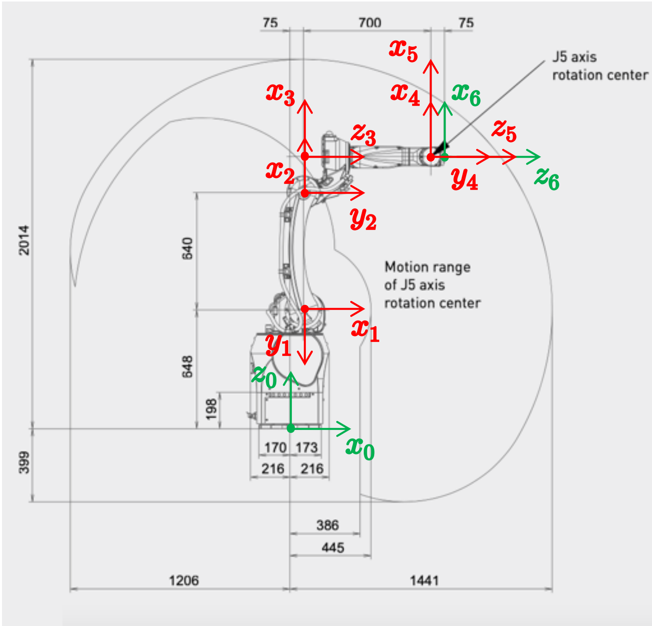


Figure 2: DH frames for the Fanuc CR15ia robot: front view (left) and back view (right).

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	75	648	q_1
2	0	640	0	q_2
3	$-\pi/2$	a_3	0	q_3
4	$\pi/2$	0	700	q_4
5	$-\pi/2$	0	0	q_5
6	0	0	75	q_6

Table 1: Parameters associated to the DH frames of Fig. 2. Lengths are in [mm].

Parameter a_3 is the only one not directly given in the data sheet. By geometric reasoning one has

$$a_3 = \sqrt{\left(2014 - (648 + 640)\right)^2 - 700^2} \simeq 192.55 \text{ [mm]},$$

which is best evaluated when the forearm is pointing upward and reaches the top of the workspace.

When the robot is in the configuration shown in the back view, the values of the joint variables are:

$$q_1 = 0, \qquad q_2 = -\frac{\pi}{2}\ \text{[rad]}, \qquad q_3 = q_4 = q_5 = q_6 = 0.$$

On the other hand, one can approximately guess the joint values (for convenience, expressed in degree) also when the robot is in the configuration shown in the front view:

$$q_1 = 15^\circ, \quad q_2 = -110^\circ, \quad q_3 = 5^\circ, \quad q_4 = 0^\circ, \quad q_5 = 40^\circ, \quad q_1 = 0^\circ.$$

Exercise 2

1. When the robot is in a singularity, there is always an infinite number of inverse solutions.

   False A planar 2R robot is singular at the outer boundary, with only one inverse solution.

2. A 6-dof Cartesian robot with a spherical wrist has two inverse solutions, out of singularities.

   True For such PPP-3R robot, these are the two orientation solutions of the spherical wrist.

3. If a closed-form inverse solution is not known in advance, a numerical method cannot provide one.

   False This is exactly one of the main reasons for using a numerical method for inversion.

4. A 6R industrial robot may have sixteen inverse solutions in its workspace, out of singularities.

   True This maximum number of solutions has been actually reached by a 6R robot.

5. A planar manipulator with $n \geq 3$ revolute joints has up to n inverse solutions for a positioning task.

   False The robot is redundant for the task and can have an infinity of inverse solutions.

6. At workspace boundaries, there is never an analytic solution to the inverse kinematics.

   False For a stretched planar 2R robot: $q_1 = \text{atan2}\{p_y, p_x\}$, $q_2 = 0$.

7. A 3R robot with twist angles α_i different from 0, $\pm\pi/2$, or $\pm\pi$ has no closed-form inverse solution.

   False Though more complex, closed-form inverse solutions can be found in other 3R cases.

8. The number of inverse solutions under joint limits is always strictly less than that without limits.

   False Not always, though this is often the case.

9. A 6R spatial robot without spherical wrist or spherical shoulder has no closed-form inverse solution.

   False Another sufficient condition is having three parallel joint axes, as in the UR10 robot.

10. A 3-dof gantry-type robot has only one inverse kinematic solution in its workspace.

    True This is a PPP robot and there is a unique solution, say, $q_1 = p_x$, $q_2 = p_y$, $q_3 = p_z$.

Exercise 3

Differentiating the given task kinematics[1] gives $\dot{\boldsymbol{r}} = (\partial \boldsymbol{f}(\boldsymbol{q})/\partial \boldsymbol{q})\,\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$, with the task Jacobian

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -q_2 \sin q_1 - L\sin(q_1+q_3) & \cos q_1 & -L\sin(q_1+q_3) \\ q_2\cos q_1 + L\cos(q_1+q_3) & \sin q_1 & L\cos(q_1+q_3) \\ 1 & 0 & 1 \end{pmatrix}.$$

Its determinant is

$$\det \boldsymbol{J}(\boldsymbol{q}) = -q_2,$$

so that the only singularity occurs when $q_2 = 0$. Substituting this value in the Jacobian yields

$$\boldsymbol{J}_s = \boldsymbol{J}(\boldsymbol{q})|_{q_2=0} = \begin{pmatrix} -L\sin(q_1+q_3) & \cos q_1 & L\sin(q_1+q_3) \\ L\cos(q_1+q_3) & \sin q_1 & L\cos(q_1+q_3) \\ 1 & 0 & 1 \end{pmatrix},$$

having rank 2. Thus, all task velocities that can be realized in a singularity by any possible choice of joint velocities $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ span a two-dimensional subspace, namely $\mathcal{R}(\boldsymbol{J}_s)$, and are of the form

$$\dot{\boldsymbol{r}} = \begin{pmatrix} -L\sin(q_1+q_3) \\ L\cos(q_1+q_3) \\ 1 \end{pmatrix} \alpha + \begin{pmatrix} \cos q_1 \\ \sin q_1 \\ 0 \end{pmatrix} \beta, \qquad \text{with } \alpha = \dot{q}_1 + \dot{q}_3,\ \beta = \dot{q}_2.$$

Differentiating further $\dot{\boldsymbol{r}}$, we obtain the task acceleration

$$\ddot{\boldsymbol{r}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}),$$

where the term $\boldsymbol{h}$ is quadratic in $\dot{\boldsymbol{q}}$ and is given by

$$\boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -2\sin q_1\,\dot{q}_1\dot{q}_2 - q_2\cos q_1\,\dot{q}_1^2 - L\cos(q_1+q_3)\,(\dot{q}_1+\dot{q}_3)^2 \\ 2\cos q_1\,\dot{q}_1\dot{q}_2 - q_2\sin q_1\,\dot{q}_1^2 - L\sin(q_1+q_3)\,(\dot{q}_1+\dot{q}_3)^2 \\ 0 \end{pmatrix}.$$

Suppose now that the robot is at rest ($\dot{\boldsymbol{q}} = \boldsymbol{0}$), so that $\boldsymbol{h} = \boldsymbol{0}$. Then, we can obtain $\ddot{\boldsymbol{r}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \boldsymbol{0}$ for a joint acceleration $\ddot{\boldsymbol{q}} \neq \boldsymbol{0}$ if and only if the task Jacobian $\boldsymbol{J}$ is singular, i.e., it is $\boldsymbol{J}_s$. In this case, any non-zero acceleration $\ddot{\boldsymbol{q}}$ that lies in the null space of $\boldsymbol{J}_s$ solves the requested problem:

$$\ddot{\boldsymbol{q}}_0 \in \mathcal{N}\{\boldsymbol{J}_s\} = \gamma \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \ \forall \gamma \qquad \Rightarrow \qquad \boldsymbol{J}_s \ddot{\boldsymbol{q}}_0 = \boldsymbol{0}.$$

Note that the same acceleration applied at a generic nonsingular configuration and with zero joint velocity would produce instead

$$\ddot{\boldsymbol{r}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}}_0 = \gamma \begin{pmatrix} q_2 \sin q_1 \\ -q_2\cos q_1 \\ 0 \end{pmatrix} \neq \boldsymbol{0}.$$

[1] The robot is a planar RPR arm with the third link of length L, while the task is the position and orientation of its end-effector. All requested derivations are done analytically, so this information is of limited use.

When the task Jacobian is nonsingular, the unique joint acceleration $\ddot{\boldsymbol{q}}$ that produces $\ddot{\boldsymbol{r}} = \mathbf{0}$ is given by

$$\ddot{\boldsymbol{q}} = -\boldsymbol{J}^{-1}(\boldsymbol{q})\,\boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{1}$$

Since $\boldsymbol{q} = (\pi/2, 1, 0)$ is a regular configuration, plugging these values of joint position into (1), together with $L = 1$ and $\dot{\boldsymbol{q}} = (1, -1, -1)$, leads to

$$\ddot{\boldsymbol{q}} = -\begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}.$$

Exercise 4

The case of rest-to-rest motion is standard. Since

$$L = |q_f - q_i| = \frac{\pi}{2} = 1.570 > 1 = \frac{V^2}{A}$$

there will be a coast phase at maximum (negative) velocity $\dot{q} = -V = -2$ [m/s] during motion. Applying then the known formulas for bang-coast-bang acceleration profiles, we have

$$T_s = \frac{V}{A} = 0.5 \text{ [s]}, \qquad T_0 = \frac{LA + V^2}{AV} = \frac{2\pi + 4}{8} = \frac{\pi}{4} + 0.5 = 1.285 \text{ [s]}.$$

Thus, the cruise speed is held for $T - 2T_s = 0.285$ [s]. The resulting position, velocity and acceleration profiles are shown in Fig. 3. Note the negative trapezoidal velocity profile, since the position is being reduced (rotated clockwise!) from $q_i = \pi/2$ to $q_f = 0$.

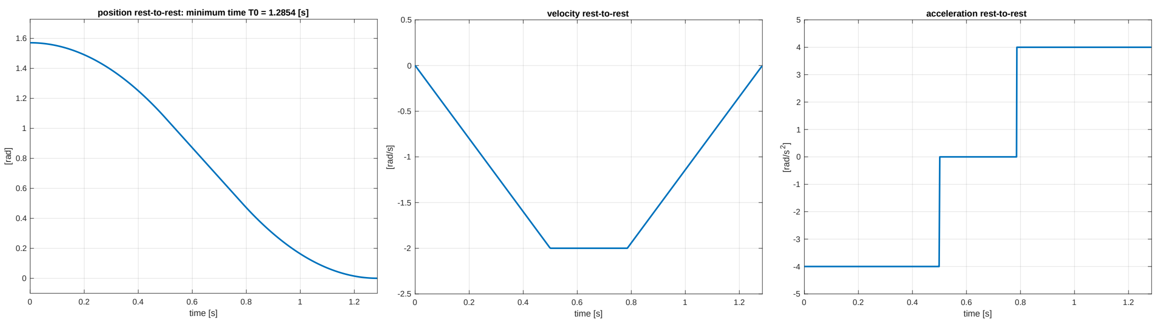


Figure 3: Motion profiles for the rest-to-rest case.

When the initial velocity is $\dot{q}_i = 1.5$ [rad/s] (state-to-rest case), the joint is moving initially in the wrong direction: thus, it needs to reverse its motion, i.e., first decelerate and stop and then move back to q_f. However, while the joint is being brought to a first stop in a time T_d, the position has progressed from $q_i = \pi/2$ to a larger positive value $q_d > q_i$. Applying the maximum negative acceleration $\ddot{q} = -A$ to stop the motion in the shortest possible time, these two quantities are then computed as

$$T_d = \frac{\dot{q}_i}{A} = 0.375 \text{ [s]}, \qquad q_d = q_i + \frac{1}{2}\dot{q}_i T_d = q_i + \frac{\dot{q}_i^2}{2A} = \frac{\pi}{2} + 0.281 = 1.852 \text{ [rad]}.$$

At this point, the remaining part of the motion is similar to the rest-to-rest case, but with the longer displacement to travel

$$L_d = |q_f - q_d| = 1.852 > L.$$

The joint will first continue with the same negative acceleration $\ddot{q} = -A$, until reaching the cruise velocity $\dot{q} = -V$ and so on. Therefore, the total minimum time in this case will be

$$T_1 = T_d + \frac{L_d A + V^2}{AV} = 0.375 + 1.426 = 1.801 \text{ [s]}.$$

The resulting position, velocity and acceleration profiles are shown in Fig. 4. Note that the overall motion is no longer symmetric.

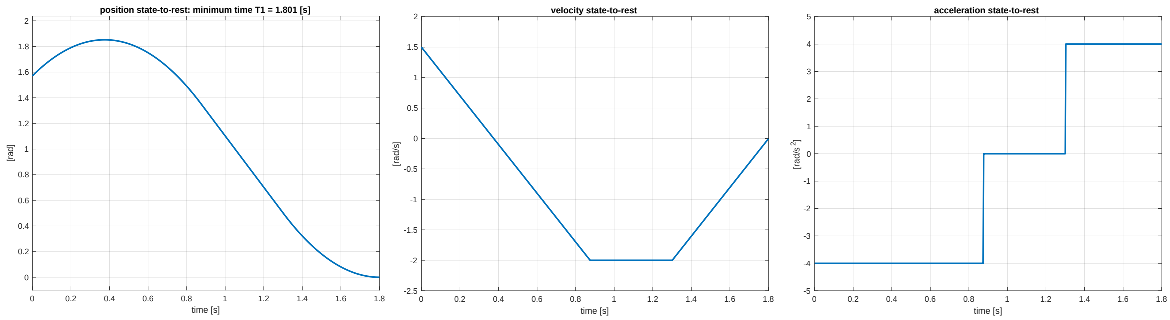


Figure 4: Motion profiles for the state-to-rest case.

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