Hardware - Tutorial 2 - Correction
Boolean Algebra and Karnaugh Maps

Exercice 1: Boolean Algebra

1) Simplify the following expressions:

- $S_1 = (A + B)\cdot(\bar{A} + \bar{B})$
- $S_2 = A\cdot B + \bar{A}\cdot\bar{B} + \bar{A}\cdot B$
- $S_3 = (A + \bar{B})\cdot(A + B) + C\cdot(\bar{A} + B)$
- $S_4 = (A + C + D)\cdot(B + C + D)$
- $S_5 = (A\cdot\bar{B} + A\cdot B + A\cdot C)\cdot(\bar{A}\cdot\bar{B} + A\cdot B + A\cdot\bar{C})$
- $S_6 = (A + \bar{B} + C)\cdot(A + \bar{C})\cdot(\bar{A} + \bar{B})$
- $S_7 = A\cdot B\cdot C + A\cdot\bar{B}\cdot\bar{C} + \bar{A}\cdot B\cdot\bar{C} + \bar{A}\cdot B\cdot C$
- $S_8 = A\cdot B\cdot C + A\cdot\bar{B}\cdot C + A\cdot B\cdot\bar{C}\cdot D$

Solutions:

$S_1 = (A + B)\cdot(\bar{A} + \bar{B})$
$S_1 = A\cdot\bar{A} + A\cdot\bar{B} + \bar{A}\cdot B + B\cdot\bar{B}$
$S_1 = \cancel{A\cdot\bar{A}} + A\cdot\bar{B} + \bar{A}\cdot B + \cancel{B\cdot\bar{B}}$
$S_1 = A\cdot\bar{B} + \bar{A}\cdot B$
$\boxed{S_1 = A \oplus B}$

$S_2 = A\cdot B + \bar{A}\cdot\bar{B} + \bar{A}\cdot B$
$S_2 = A\cdot B + \bar{A}\cdot(B + \bar{B})$
$S_2 = A\cdot B + \bar{A}\cdot 1$
$S_2 = A\cdot B + \bar{A}$
$\boxed{S_2 = B + \bar{A}}$

$S_3 = (A + \bar{B})\cdot(A + B) + C\cdot(\bar{A} + B)$
$S_3 = A\cdot\cancel{A} + A\cdot B + A\cdot\bar{B} + \cancel{B\cdot\bar{B}} + \bar{A}\cdot C + B\cdot C$
$S_3 = A + \cancel{A\cdot B} + \cancel{A\cdot\bar{B}} + \bar{A}\cdot C + B\cdot C$
$S_3 = A + \cancel{\bar{A}}\cdot C + B\cdot C$
$S_3 = A + C + \cancel{B\cdot C}$
$\boxed{S_3 = A + C}$

$S_4 = (A + C + D)\cdot(B + C + D)$
$S_4 = A\cdot B + A\cdot C + A\cdot D + B\cdot C + C\cdot\cancel{C} + C\cdot D + B\cdot D + \cancel{C\cdot D} + D\cdot\cancel{D}$
$S_4 = A\cdot B + \cancel{A\cdot C} + \cancel{A\cdot D} + \cancel{B\cdot C} + C + \cancel{C\cdot D} + \cancel{B\cdot D} + D$
$\boxed{S_4 = A\cdot B + C + D}$

$S_5 = (A\cdot\bar{B} + A\cdot B + A\cdot C)\cdot(\bar{A}\cdot\bar{B} + A\cdot B + A\cdot\bar{C})$
$S_5 = (A\cdot\cancel{(\bar{B} + B)} + \cancel{A\cdot C})\cdot(\bar{A}\cdot\bar{B} + A\cdot B + A\cdot\bar{C})$
$S_5 = A\cdot(\bar{A}\cdot\bar{B} + A\cdot B + A\cdot\bar{C})$
$S_5 = \cancel{A\cdot\bar{A}\cdot\bar{B}} + \cancel{A}\cdot A\cdot B + \cancel{A}\cdot A\cdot\bar{C}$
$\boxed{S_5 = A\cdot B + A\cdot\bar{C}}$

$S_6 = (A + \bar{B} + C)\cdot(A + \bar{C})\cdot(\bar{A} + \bar{B})$
$S_6 = (A\cdot\cancel{A} + \cancel{A\cdot\bar{C}} + \cancel{A\cdot\bar{B}} + \bar{B}\cdot\bar{C} + \cancel{A\cdot C} + \cancel{C\cdot\bar{C}})\cdot(\bar{A} + \bar{B})$
$S_6 = (A + \bar{B}\cdot\bar{C})\cdot(\bar{A} + \bar{B})$
$S_6 = \cancel{A\cdot\bar{A}} + A\cdot\bar{B} + \bar{A}\cdot\bar{B}\cdot\bar{C} + \cancel{\bar{B}}\cdot\bar{B}\cdot\bar{C}$
$S_6 = A\cdot\bar{B} + \cancel{\bar{A}\cdot\bar{B}\cdot\bar{C}} + \bar{B}\cdot\bar{C}$
$\boxed{S_6 = A\cdot\bar{B} + \bar{B}\cdot\bar{C}}$

$S_7 = A\cdot B\cdot C + A\cdot\bar{B}\cdot\bar{C} + \bar{A}\cdot B\cdot\bar{C} + \bar{A}\cdot B\cdot C$
$S_7 = B\cdot(A\cdot C + \bar{A}\cdot\bar{C} + \bar{A}\cdot C) + A\cdot\bar{B}\cdot\bar{C}$
$S_7 = B\cdot(A\cdot C + \bar{A}) + A\cdot\bar{B}\cdot\bar{C}$
$S_7 = B\cdot(\bar{A} + C) + A\cdot\bar{B}\cdot\bar{C}$
$S_7 = B\cdot(\overline{\overline{\bar{A} + C}}) + A\cdot\bar{B}\cdot\bar{C}$
$S_7 = B\cdot(\overline{A\cdot\bar{C}}) + \bar{B}\cdot(A\cdot\bar{C})$
$\boxed{S_7 = B \oplus A\cdot\bar{C}}$

$S_8 = A\cdot B\cdot C + A\cdot\bar{B}\cdot C + A\cdot B\cdot\bar{C}\cdot D$
$S_8 = A\cdot C\cdot\cancel{(B + \bar{B})} + A\cdot B\cdot\bar{C}\cdot D$
$S_8 = A\cdot C + A\cdot B\cdot\bar{C}\cdot D$
$S_8 = A\cdot(C + B\cdot\bar{C}\cdot D)$
$S_8 = A\cdot(C + B\cdot D)$
$\boxed{S_8 = A\cdot C + A\cdot B\cdot D}$

2) Calculate and simplify the complement of S1, S5 and S6.

Solutions:

$\overline{S_1} = \overline{A\cdot\bar{B} + \bar{A}\cdot B}$
$\overline{S_1} = \overline{A\cdot\bar{B}}\cdot\overline{\bar{A}\cdot B}$
$\overline{S_1} = (\bar{A} + B)\cdot(A + \bar{B})$
$\overline{S_1} = \bar{A}\cdot\bar{B} + A\cdot B$
$\boxed{\overline{S_1} = \overline{A \oplus B} = \bar{A} \oplus B = A \oplus \bar{B}}$

$\overline{S_5} = \overline{A\cdot B + A\cdot\bar{C}}$
$\overline{S_5} = \overline{A\cdot(B + \bar{C})}$
$\overline{S_5} = \bar{A} + (\overline{B + \bar{C}})$
$\boxed{\overline{S_5} = \bar{A} + \bar{B}\cdot C}$

$\overline{S_6} = \overline{A\cdot\bar{B} + \bar{B}\cdot\bar{C}}$
$\overline{S_6} = \overline{\bar{B}\cdot(A\cdot\bar{C}}$
$\overline{S_6} = B + (\overline{A + \bar{C}})$
$\boxed{\overline{S_6} = B + \bar{A}\cdot C}$

3) Design the NOT, AND and OR gates by using only NAND gantes, then only NOR gates.

Solutions:

NAND gates only:

$\bar{A} = \overline{A\cdot A}$

$A\cdot B = \overline{\overline{A\cdot B}}$
$A\cdot B = \overline{(\overline{A\cdot B})\cdot(\overline{A\cdot B})}$

$A + B = \overline{\overline{A + B}}$
$A + B = \overline{\bar{A}\cdot\bar{B}}$
$A + B = \overline{(\overline{A\cdot A})\cdot(\overline{B\cdot B})}$

NOR gates only:

$\bar{A} = \overline{A + A}$

$A\cdot B = \overline{\overline{A\cdot B}}$
$A\cdot B = \overline{\bar{A} + \bar{B}}$
$A\cdot B = \overline{(\overline{A + A}) + (\overline{B + B})}$

$A + B = \overline{\overline{A + B}}$
$A + B = \overline{(\overline{A + B}) + (\overline{A + B})}$

Exercice 2: Equality

Demonstrate the following equalities:

- $\overline{A{\cdot}C + B{\cdot}\bar{C}} = \bar{A}{\cdot}C + \bar{B}{\cdot}\bar{C}$
- $(A + B){\cdot}(\bar{A} + C){\cdot}(B + C) = (A + B){\cdot}(\bar{A} + C)$

Solution:

Let's demonstrate that $\overline{A{\cdot}C + B{\cdot}\bar{C}} = \bar{A}{\cdot}C + \bar{B}{\cdot}\bar{C}$ by changing the left part of the equation into the right part:
$\overline{A{\cdot}C + B{\cdot}\bar{C}} = \overline{A{\cdot}C}{\cdot}\overline{B{\cdot}\bar{C}}$
$\overline{A{\cdot}C + B{\cdot}\bar{C}} = (\bar{A} + \bar{C}){\cdot}(\bar{B} + C)$
$\overline{A{\cdot}C + B{\cdot}\bar{C}} = \bar{A}{\cdot}\bar{B} + \bar{A}{\cdot}C + \bar{B}{\cdot}\bar{C} + \cancel{C{\cdot}\bar{C}}$
$\overline{A{\cdot}C + B{\cdot}\bar{C}} = \cancel{\bar{A}{\cdot}\bar{B}{\cdot}C} + \cancel{\bar{A}{\cdot}\bar{B}{\cdot}\bar{C}} + \bar{A}{\cdot}C + \bar{B}{\cdot}\bar{C}$
$\boxed{\overline{A{\cdot}C + B{\cdot}\bar{C}} = \bar{A}{\cdot}C + \bar{B}{\cdot}\bar{C}}$

Now let's demonstrate that $(A + B){\cdot}(\bar{A} + C){\cdot}(B + C) = (A + B){\cdot}(\bar{A} + C)$ by developping each side of the equation:
$(A + B){\cdot}(\bar{A} + C){\cdot}(B + C) = (A + B){\cdot}(\bar{A} + C)$
$(\cancel{A{\cdot}\bar{A}} + A{\cdot}C + \bar{A}{\cdot}B + B{\cdot}C){\cdot}(B + C) = \cancel{\bar{A}{\cdot}A} + A{\cdot}C + \bar{A}{\cdot}B + B{\cdot}C$
$A{\cdot}B{\cdot}C + A{\cdot}C{\cdot}\cancel{C} + \bar{A}{\cdot}B{\cdot}\cancel{B} + \bar{A}{\cdot}B{\cdot}C + \cancel{B}{\cdot}B{\cdot}C + B{\cdot}C{\cdot}\cancel{C} = A{\cdot}C + \bar{A}{\cdot}B + B{\cdot}C$
$\cancel{A{\cdot}B{\cdot}C} + A{\cdot}C + \bar{A}{\cdot}B + B{\cdot}C = A{\cdot}C + \bar{A}{\cdot}B + B{\cdot}C$
$A{\cdot}C + \bar{A}{\cdot}B + B{\cdot}C = A{\cdot}C + \bar{A}{\cdot}B + B{\cdot}C$

Exercice 3: Problem

Given 3 binary variables A, B and C, design an expression S that is true only if the number of true variables is odd.

Solution:

In order to extract a boolean expression from a given problem, we first need to fill a truth table of that problem. Then we can extract the terms of the expression from the truth table by simply finding the expression of each line where S is true.

A	B	C	S	
0	0	0	0	
0	0	1	1	$\rightarrow \bar{A}{\cdot}\bar{B}{\cdot}C$
0	1	0	1	$\rightarrow \bar{A}{\cdot}B{\cdot}\bar{C}$
0	1	1	0	
1	0	0	1	$\rightarrow A{\cdot}\bar{B}{\cdot}\bar{C}$
1	0	1	0	
1	1	0	0	
1	1	1	1	$\rightarrow A{\cdot}B{\cdot}C$

Finaly we have to compute and simplify the full expression:

$S = \bar{A}{\cdot}\bar{B}{\cdot}C + \bar{A}{\cdot}B{\cdot}\bar{C} + A{\cdot}\bar{B}{\cdot}\bar{C} + A{\cdot}B{\cdot}C$
$S = \bar{A}{\cdot}(\bar{B}{\cdot}C + B{\cdot}\bar{C}) + \bar{A}{\cdot}(\bar{B}{\cdot}\bar{C} + B{\cdot}C)$
$S = \bar{A}{\cdot}(B \oplus C) + A{\cdot}(\overline{B \oplus C})$
$\boxed{S = A \oplus B \oplus C}$

Exerice 4: 3 variables Karnaugh maps

Given a number N encoded in 3 bits binary (C, B, A) with C the most significant bit, find the most simplified boolean expression of $S = f(N)$ using Karnaugh maps for each of the following:

- $S_1 = 1$ when $N >= 3$
- $S_2 = 1$ when $2 < N <= 6$
- $S_3 = 1$ when $N = 1, 3, 5$
- $S_4 = 1$ when $N = 1, 3, 5$ and S is undefined when $N = 0$ or 7

Solution:

To solve theses problems, we will first fill a truth table for each S, then fill and solve some Karnaugh Maps and finally extract the expressions from the Karnaugh Maps:

C	B	A	S_1	S_2	S_3	S_4
0	**0**	**0**	0	0	0	Ø
0	**0**	**1**	0	0	1	1
0	**1**	**0**	0	0	0	0
0	**1**	**1**	1	1	1	1
1	**0**	**0**	1	1	0	0
1	**0**	**1**	1	1	1	1
1	**1**	**0**	1	1	0	0
1	**1**	**1**	1	0	0	Ø

S_1		**BA**			
		00	01	11	10
C	0	0	0	1	0
	1	1	1	1	1

S_2		**BA**			
		00	01	11	10
C	0	0	0	1	0
	1	1	1	0	1

S_3		**BA**			
		00	01	11	10
C	0	0	1	1	0
	1	0	1	0	0

S_4		**BA**			
		00	01	11	10
C	0	Ø	1	1	0
	1	0	1	Ø	0

$S_1 = B{\cdot}A + C$

$S_2 = \bar{C}{\cdot}B{\cdot}A + C{\cdot}\bar{B} + C{\cdot}\bar{A}$

$S_2 = \bar{C}{\cdot}(B{\cdot}A) + C{\cdot}(\bar{B} + \bar{A})$

$S_2 = \bar{C}{\cdot}(B{\cdot}A) + C{\cdot}\overline{(B{\cdot}A)}$

$S_2 = C \oplus B{\cdot}A$

$S_3 = \bar{C}{\cdot}A + \bar{B}{\cdot}A$

$S_4 = A$

Exercice 5: Problem

We want to design a circuit that can perform the two's complement of a 3 bits binary number.
This circuit has 3 inputs (C, B, A) and 3 outputs (C', B', A') with C and C' their most significant bits.

1) Write down the truth table of each of theses outputs.

Solution:

C	B	A	C'	B'	A'
0	**0**	**0**	0	0	0
0	**0**	**1**	1	1	1
0	**1**	**0**	1	1	0
0	**1**	**1**	1	0	1
1	**0**	**0**	1	0	0
1	**0**	**1**	0	1	1
1	**1**	**0**	0	1	0
1	**1**	**1**	0	0	1

2) Find their most simplified expression using Karnaugh maps.

Solution:

C'		**BA**			
		00	01	11	10
C	0	0	1	1	1
	1	1	0	0	0

B'		**BA**			
		00	01	11	10
C	0	0	1	0	1
	1	0	1	0	1

A'		**BA**			
		00	01	11	10
C	0	0	1	1	0
	1	0	1	1	0

$C' = C{\cdot}\bar{B}{\cdot}\bar{A} + \bar{C}{\cdot}A + \bar{C}{\cdot}B$

$C' = C{\cdot}(\bar{B}{\cdot}\bar{A}) + \bar{C}{\cdot}(B + A)$

$C' = C{\cdot}\overline{(B + A)} + \bar{C}{\cdot}(B + A)$

$C' = C \oplus (B + A)$

$B' = \bar{B}{\cdot}A + B{\cdot}\bar{A}$

$B' = B \oplus A$

$A' = A$

Exercice 6: 4 variables Karnaugh maps

Given a number N encoded in 4 bits binary (D, C, B, A) with D the most significant bit, find the most simplified boolean expression of $S = f(N)$ using Karnaugh maps for each of the following:

- $S_1 = 1$ when $N >= 10$
- $S_2 = 1$ when $N = 0, 4, 8, 10, 12$ or 14
- $S_3 = 1$ when $N = 0, 2, 5, 7, 8, 10, 13$ or 15
- $S_4 = 1$ when $N = 2, 10, 11$ or 14
- $S_5 = 1$ when $N = 2, 10, 11$ or 14 and S is undefined when $N = 6, 9, 13$ or 15

Solution:

Let's fill directly the Karnaugh Maps for each expression:

S_1		**BA**			
		00	01	11	10
DC	00	0	0	0	0
	01	0	0	0	0
	11	1	1	1	1
	10	0	0	1	1

S_2		**BA**			
		00	01	11	10
DC	00	1	0	0	0
	01	1	0	0	0
	11	1	0	0	1
	10	1	0	0	1

S_3		**BA**			
		00	01	11	10
DC	00	1	0	0	1
	01	0	1	1	0
	11	0	1	1	0
	10	1	0	0	1

S_4		**BA**			
		00	01	11	10
DC	00	0	0	0	1
	01	0	0	0	0
	11	0	0	0	1
	10	0	0	1	1

S_5		**BA**			
		00	01	11	10
DC	00	0	0	0	1
	01	0	0	0	Ø
	11	0	Ø	Ø	1
	10	0	Ø	1	1

Then we can extract the boolean expression of each of the maps:

$S_1 = D{\cdot}C + D{\cdot}B$

$S_2 = \bar{B}{\cdot}\bar{A} + D{\cdot}\bar{A}$

$S_3 = \bar{C}{\cdot}\bar{A} + C{\cdot}A$

$S_3 = \overline{C \oplus A}$

$S_4 = D{\cdot}B{\cdot}\bar{A} + \bar{C}{\cdot}B{\cdot}\bar{A} + D{\cdot}\bar{C}{\cdot}B$

$S_5 = D{\cdot}A + B{\cdot}\bar{A}$