UFRJ Universidade Federal do Rio de Janeiro
COPPE Coordenação dos Programas de Pós-Graduação em Engenharia
PEE Programa de Engenharia Elétrica
ECA Engenharia de Controle e Automação

Adaptive control

Conteúdo

0 Organização do curso

Contents

0.1 Horário

COE835	:	H–312B
	:	$4^{a.}$ feira 10:00 – 12:00
	:	$6^{a.}$ feira 10:00 – 12:00
COE603	:	H–219B
	:	$3^{a.}$ feira 13:00 – 15:00
	:	$5^{a.}$ feira 12:00 – 15:00
Professor	:	Ramon R. Costa
Laboratório	:	H–345
Telefone	:	(cel) 98887-9355
e-mails	:	ramonrcosta@gmail.com

0.2 Prerequisites

- Linear algebra
- Linear systems
- Discrete control
- Nonlinear systems $\Leftarrow$!

0.3 Objetivos do curso

- Apresentar as principais técnicas de análise e projeto
- Introdução ao estado da arte
- Capacitar o aluno a ler e pesquisar no tema

0.4 Plano de aulas simplificado

(...)

0.5 Ementa *não-oficial*

- Revisão: controle 2DOF
- Exemplos simples
  - MRAC $(n^* = 1)$ direto e indireto
  - VS-MRAC
- Estimação adaptativa de parâmetros
- MRAC SISO
- LS-MRAC
- Backstepping
- MRAC MIMO
- VS-MRAC

0.6 Duração do curso

COPPE/PEE COE835 Controle adaptativo	POLI/ECA COE603 Controle adaptativo
Início : 24/set/25	Início : 18/mar/25
Término : 12/dez/25	Término : 19/jul/25
★ 12 semanas	★ 15 semanas
★ 45 horas-aula	★ 75 horas-aula

0.7 Atividades

(1) Aulas

(2) Seminários

(3) Pesquisa bibliográfica

(4) Simulações digitais

(5) Relatórios

0.8 Bibliography

[1] Eugene Lavretsky & Kevin A. Wise
Robust and Adaptive Control. With Aerospace Applications [2 Ed].
Springer, **2024**.

[2] Nhan T. Nguyen
Model-Reference Adaptive Control. A Primer.
Springer, **2018**.

[3] Hovakimyan & Cao
$\mathcal{L}_1$ *Adaptive Control Theory*.
SIAM, **2010**.

[4] Karl Johan Åström & Björn Wittenmark
Adaptive Control [2 Ed].
Dover, **2008**.

[5] Petros Ioannou & Bariş Fidan
Adaptive Control Tutorial.
SIAM, **2006**.

[6] Gang Tao
Adaptive Control Design and Analysis.
John Wiley & Sons, **2003**.

[7] Jeffrey T. Spooner, Manfredi Maggiore, Raúl Ordóñez & Kevin M. Passino
Stable Adaptive Control and Estimation for Nonlinear Systems: Neural and Fuzzy Approximator Techniques.
John Wiley & Sons, **2002**.

[8] Gang Feng & Rogelio Lozano
Adaptive Control Systems.
Newnes, **1999**.

[9] Iven Mareels & Jan Willem Polderman
Adaptive Systems. An Introduction.
Springer, **1996**.

[10] Petros A. Ioannou & Jing Sun
Robust Adaptive Control.
Prentice Hall, **1996**.

[11] Miroslav Krstić, Ioannis Kanellakopoulos & Petar Kokotović
Nonlinear and Adaptive Control Design.
John Wiley & Sons, Inc., **1995**.

[12] Edoardo Mosca
Optimal, Predictive, and Adaptive Control.
Prentice-Hall, **1994**.

[13] Jean-Jacques E. Slotine & Weiping Li
Applied Nonlinear Control.
Prentice Hall, **1991**.

[14] Robert R. Bitmead, Michel Gevers & Vincent Wertz
Adaptive Optimal Control. The Thinking Man's GPC.
Prentice Hall, **1990**.

[15] Kumpati S. Narendra & Anuradha M. Annaswamy
Stable Adaptive Systems.
Prentice Hall, **1989**.

[16] Shankar Sastry & Marc Bodson
Adaptive Control. Stability, Convergence, and Robustness.
Prentice Hall, **1989**.

[17] Yoan D. Landau
Adaptive Control.
Marcel Dekker Inc., **1979**.

0.9 Bibliografia extra

1. Gilbert Strang
   Introduction to Linear Algebra [4 Ed].
   Wellesley - Cambridge Press, **2009**.

2. Karl Johan Åström & Björn Wittenmark
   Computer Controlled Systems Theory and Design.
   Prentice-Hall Inc., **1997**.

0.10 Artigos

1. Liu Hsu & Ramon R. Costa
   Variable structure model reference adaptive control using only input and output measurements: Part 1.
   Int. Journal of Control, Vol. 49, No. 2, 399–416, **1989**.

2. Ramon R. Costa, Liu Hsu, Alvaro K. Imai & Petar Kokotović
   Lyapunov-based adaptive control of MIMO systems.
   Automatica, Vol. 39, pp. 1251-1257, **2003**.

3. Ramon R. Costa
   Lyapunov design of least-squares model-reference adaptive control.
   IFAC 2020, Berlin, Germany, July 11-17, **2020**.

4. Ramon R. Costa
   Model-reference adaptive control with high-order parameter tuners.
   ACC 2022, Atlanta, USA, June 8-10, **2022**.

5. Ramon R. Costa
Least-squares model-reference adaptive control with high-order parameter tuners.
Automatica, **2024**.

6. Ramon R. Costa
Least-squares adaptive control and observer.
IEEE-TAC (To appear), **2025**.

0.11 Avaliação

★ Baseado em trabalhos:

- Simulações digitais
- Relatórios
- Seminários

0.12 Apresentações e entrega dos relatórios

:	Trabalho	Data
1	MRAC (Caso $n = 1$, $n^* = 1$ e $n_p = 2$)	
2	Algoritmos de identificação	
3	Solução do controle 2DOF	
4	MRAC (Casos $n^* = 1$ e 2, $\forall n$)	
5	MRAC (Caso $n^* = 3$, $\forall n$)	
6	LS-MRAC (Caso $n^* = 1$)	
7	Composite M-MRAC+LS (Caso $n^* = 1$)	
8	Backstepping (Caso $n^* = 2$), Observador de ordem completa. Indireto	
9	Backstepping (Caso $n^* = 2$), Observador de ordem reduzida. Indireto	
10	Backstepping (Caso $n^* = 2$), Observador de ordem reduzida. Direto	
11	Backstepping (Caso $n^* = 3$)	
12	MIMO MRAC (Caso $n^* = 1$)	
13	MIMO MRAC (Caso $n^* = 2$)	
14	MIMO LS-MRAC (Caso $n^* = 1$)	
15	VS-MRAC (Caso $n^* = 1$)	

1 Revisão: controle 2DOF

Contents

1.1 Introduction

"Despite the vast literature on the subject, there is still a general feeling that adaptive control is a collection of unrelated technical tools and tricks."

[Ioannou & Sun:1996] (pag. 25)

1.2 Problema geral

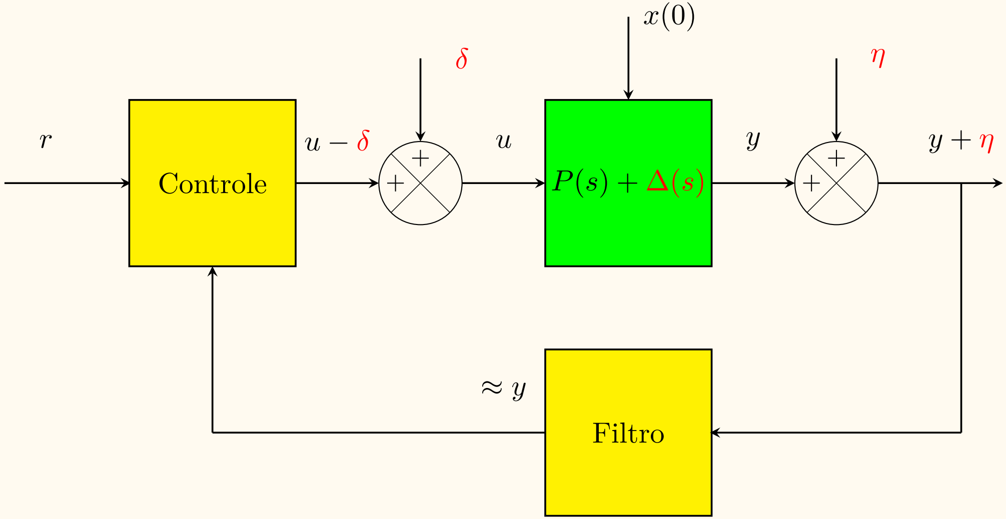
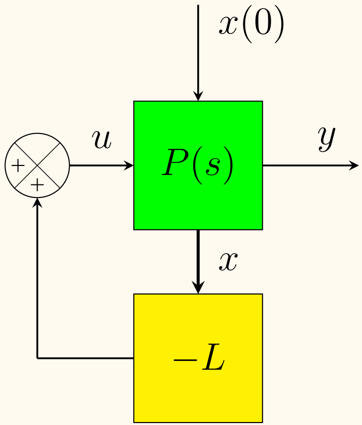


Figura 1: Diagrama de blocos com incertezas e perturbações.

- δ : Perturbação de entrada (baixa frequência)
- η : Perturbação de saída (alta frequência)
- $\Delta(s)$: Incerteza na planta

1.3 Realimentação de estado

- Solução para o problema de estabilização de sistemas lineares.

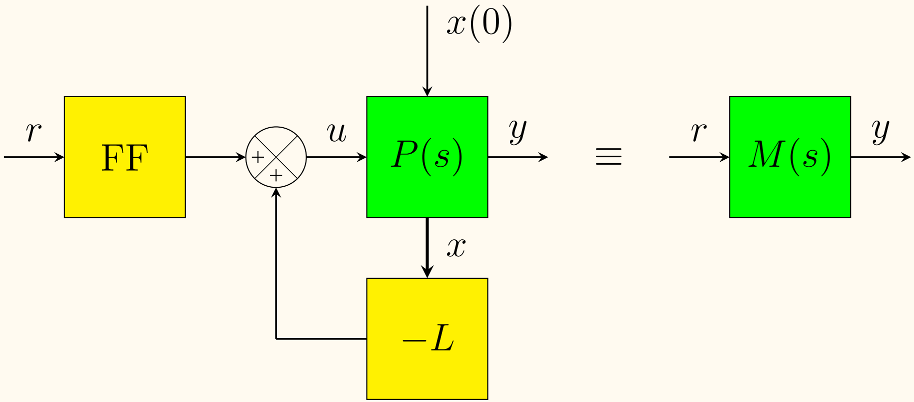


★ Problema mais simples: Não há incertezas nem perturbações.

- Vantagem:
  - Liberdade para realocar todos os polos

- Desvantagens:
  - Necessita medição do estado
  - Análise de robustez difícil

- Solução para o problema de rastreamento.



★ A realimentação de estado realoca os polos.

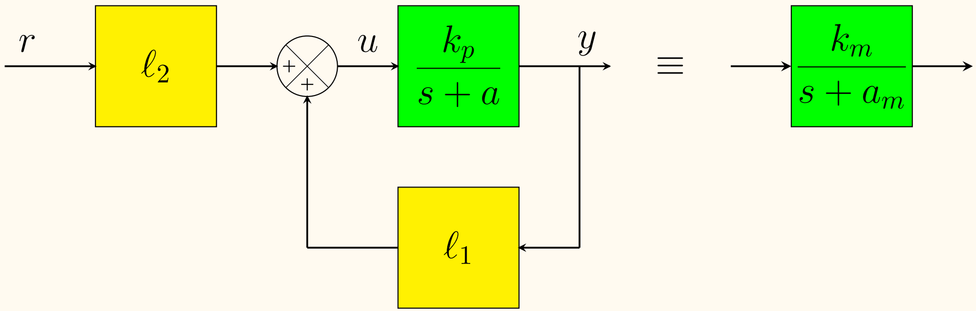
★ O bloco *feedforward* (FF) realoca os zeros (via cancelamento).

★ Condição de rastreabilidade : os zeros da planta devem ser estáveis.

Exemplo 1 Planta de 1a. ordem.

Planta : $$P(s) = \frac{k_p}{s+a}$$

Modelo : $$M(s) = \frac{k_m}{s+a_m}$$ (Dinâmica desejada)



Controle : $\boxed{u = \ell_1 y + \ell_2 r}$

Malha fechada: $(s + \underbrace{a - k_p\,\ell_1}_{a_m})y = \underbrace{k_p\,\ell_2}_{k_m}\,r$

Solução: $\boxed{\ell_1 = \dfrac{a - a_m}{k_p}}$ Feedback (FB)

$\boxed{\ell_2 = \dfrac{k_m}{k_p}}$ Feedforward (FF)

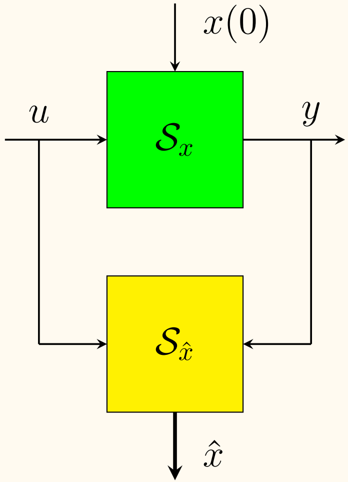
★ Note que k_p aparece no denominador!

Problema: Quando o parâmetro k_p é adaptado $\Rightarrow$ $\hat{k}_p \neq 0$!!

$\Rightarrow$ $\text{sign}(k_p)$ deve ser conhecido.

1.4 Observadores

- Estimador dinâmico de estado.



★ O observador é uma cópia da planta ($+$ uma realimentação).

Hipótese fundamental :

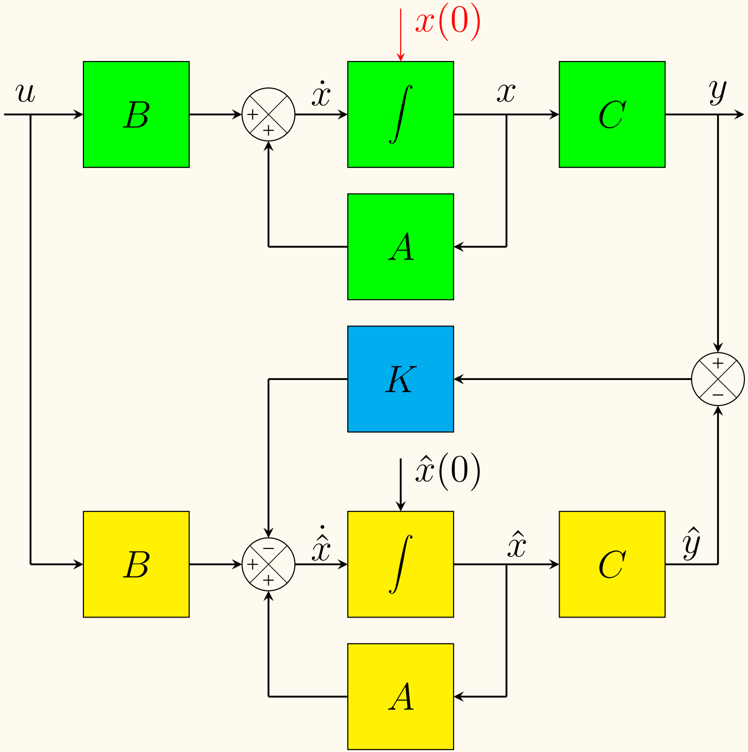
- A planta é perfeitamente conhecida.
- A planta é completamente observável.

★ Somente $x(0)$ é desconhecido !!

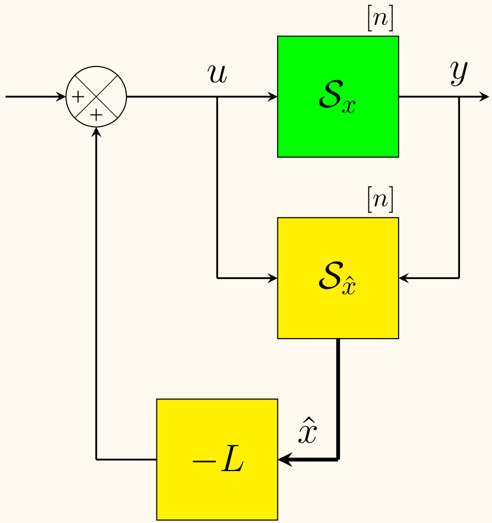
Planta : $$\boxed{\dot{x} = Ax + Bu}$$

Observador : $$\boxed{\dot{\hat{x}} = A\hat{x} + Bu + \underbrace{K\big(y - \hat{y}\big)}_{\text{Realimentação}}}$$

Diagrama de blocos : Observador de ordem completa com realimentação.



- Solução para o problema de estabilização sem medição do estado.

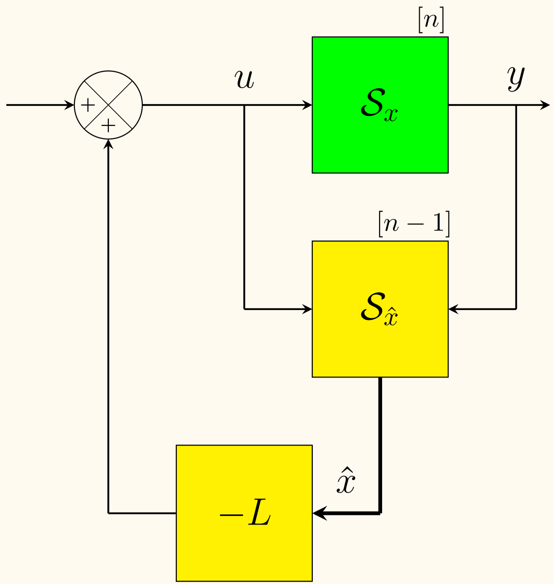


$$u = -L\hat{x}$$

★ Muito importante : Princípio da separação.

Matrizes K e L podem ser projetadas separadamente !!

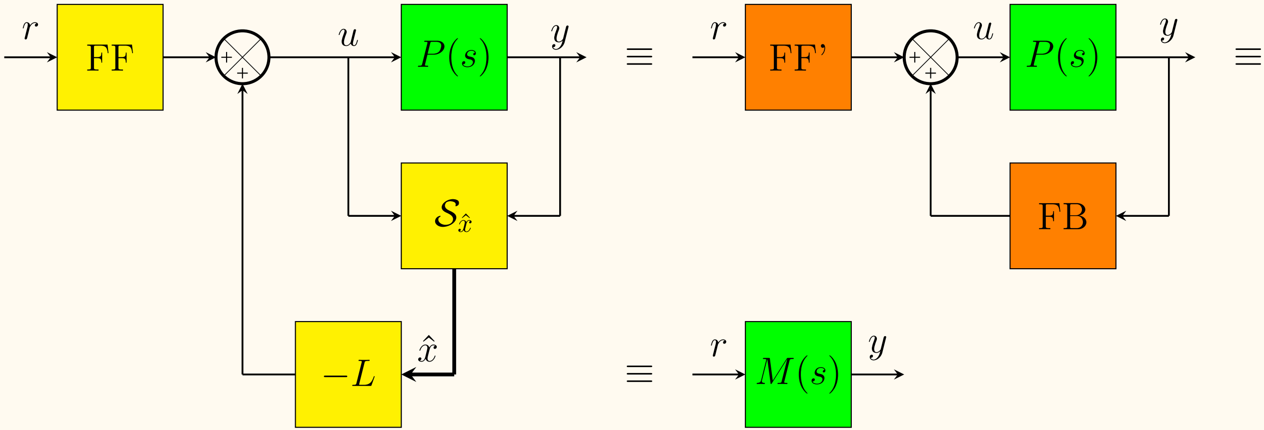
1.4.1 Observadores de ordem mínima



★ Ideia : $y = x_1$ não precisa ser estimado. É medido!

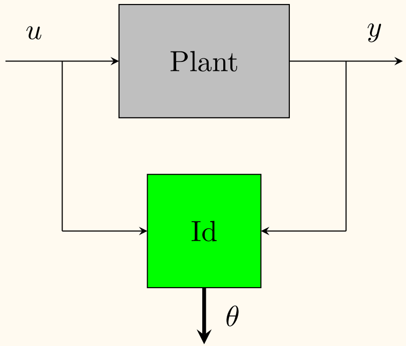
1.5 Estrutura 2DOF

- Solução geral para o problema de rastreamento sem medição do estado.



★ $M(s)$: *Modelo de referência*.
Dinâmica desejada para o sistema em malha fechada.

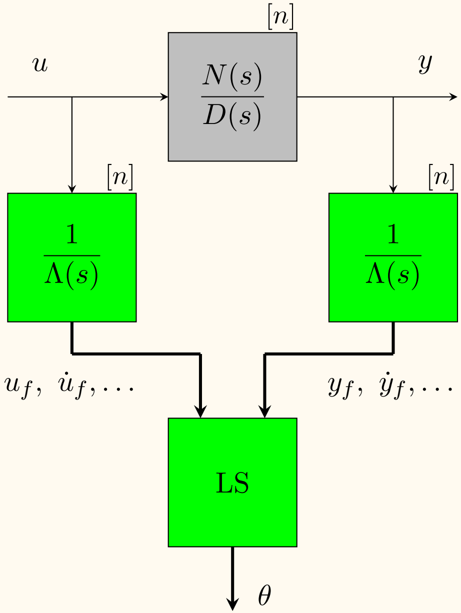
1.6 Identificação de parâmetros



- ★ θ não é realimentado na planta!
- ★ Os algoritmos utilizam predições. $\Rightarrow$ Problema não-linear.

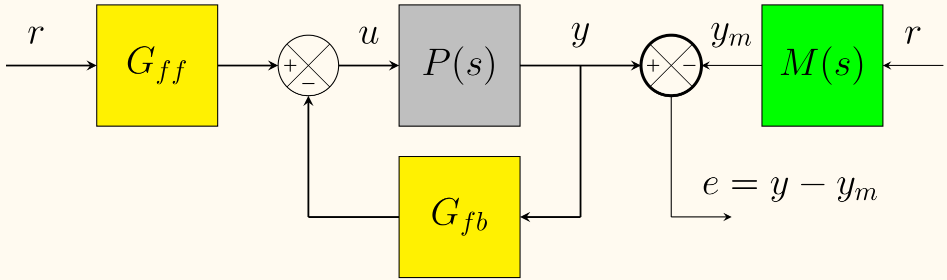
Problema: Como implementar?

Solução: Utilizar estrutura com filtros.



★ LS = *Least-Squares*.

1.7 Estrutura do MRAC



★ MRAC : *Model-reference adaptive control*

★ G_{fb} e G_{ff} são *ajustados/adaptados* tal que $e_0(t) \to 0$.

★ O problema torna-se não linear e variante no tempo.

Problema: Como implementar?

Solução: Utilizar estrutura com filtros (semelhante ao caso do identificador).

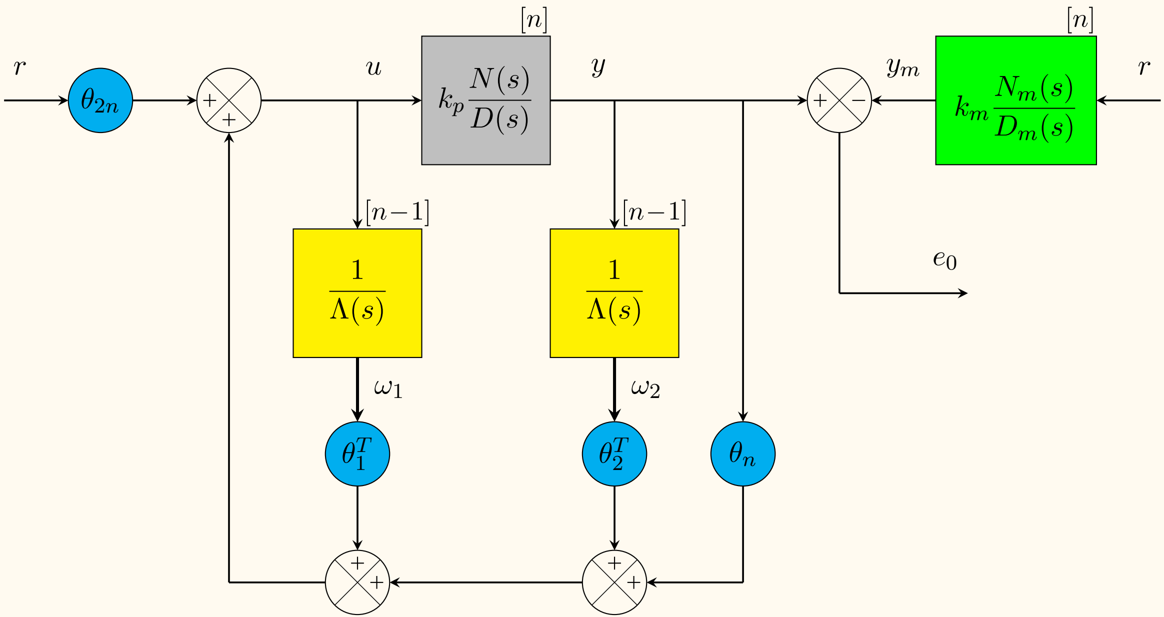


Figura 2: Estrutura do MRAC.

2 Exemplos simples

Contents

2.1 Planta com 1 parâmetro desconhecido (a_p)

Classificação do sistema:

$$n = 1 \quad \text{(ordem)}$$
$$n^* = 1 \quad \text{(grau relativo)}$$
$$n_p = 1 \quad \text{(\# de parâmetros)}$$

- Planta : $\boxed{\dot{y} = a_p\, y + u}$ $\quad y = \dfrac{1}{s - a_p} u$ $\quad$ a_p desconhecido !
- Modelo : $\boxed{\dot{y}_m = -a_m y_m + r}$ $\quad y_m = \dfrac{1}{s + a_m} r$

★ Neste caso, o problema pode ser resolvido utilizando apenas *feedback*.
Não é necessário *feedforward*.

Os seguintes algoritmos são discutidos:

MRAC direto : Os parâmetros da lei de controle são adaptados diretamente utilizando um critério de estabilidade.

Método do Gradiente : Os parâmetros do controle são projetados por otimalidade.

MRAC indireto : Os parâmetros da planta são identificados e então utilizados para gerar o controle.

VS-MRAC : É uma abordagem onde a lei de controle é baseada em relés.

VS-MRAC = Variable structure MRAC

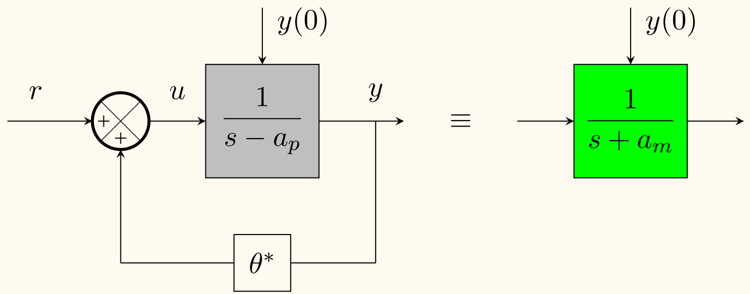
2.1.1 MRAC DIRETO

Exemplo 1 [Tao:2003], (pag. 17)

Planta : $\boxed{\dot{y} = a_p y + u}$ $\quad y = \dfrac{1}{s - a_p} u \quad$ a_p desconhecido !

Modelo : $\boxed{\dot{y}_m = -a_m y_m + r}$ $\quad y_m = \dfrac{1}{s + a_m} r$

Se a_p fosse conhecido, então



Lei de controle ideal : $$u^* = \theta^* y + r$$

Matching gain : $$\theta^* = -a_p - a_m$$ $\exists!\, \theta^*$

Quando a_p é desconhecido, usamos o controle

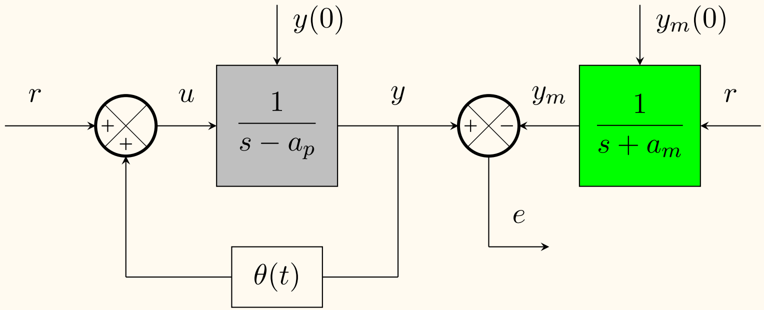
$$u = \theta(t)\, y + r$$

Certainty equivalence principle

★ $\theta(t)$ é uma estimativa de θ^*.

★ $\theta(t)$ é variante no tempo! $\Rightarrow$ O problema torna-se não linear.

Estrutura do MRAC:



Lei de controle: $$\boxed{u = \theta(t)\, y + r}$$ **Não linear !**

Erro de saída :

$$\boxed{e = y - y_m}$$

Equação do erro (dinâmica de $e(t)$):

$$\begin{aligned}
\dot{e} &= \dot{y} - \dot{y}_m \\
&= \big(a_p y + u\big) - \big(-a_m y_m + r\big) + \big(a_m y\big) - \big(a_m y\big) \\
&= -a_m \underbrace{\big(y - y_m\big)}_{e} + \underbrace{\big(a_p + a_m\big)}_{-\theta^*} y - r + u \\
&= -a_m e - \underbrace{\big(\theta^* y + r\big)}_{u^*} + u
\end{aligned}$$

Portanto,

$$\boxed{\dot{e} = -a_m e + \big[u - u^*\big]}$$

Erro paramétrico:

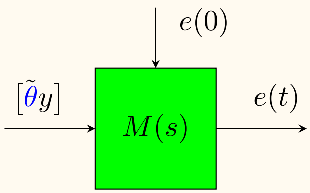
$$\boxed{\tilde{\theta} = \theta - \theta^*}$$

A equação do erro pode ser escrita como:

$$\begin{aligned} \dot{e} &= -a_m e + \left[\theta y + r - \theta^* y - r\right] \\ &= -a_m e + \tilde{\theta} y \end{aligned}$$

Ou ainda,

$$\boxed{e = M(s)\left[\tilde{\theta} y\right]}, \qquad M(s) = \frac{1}{s + a_m}$$



Erro de controle: $$\boxed{\tilde{u} = u - u^* = \tilde{\theta} y}$$ (*mismatching control*)

Outra forma para a equação do erro:

$$\boxed{e = M(s)[\tilde{u}]}$$

★ Note que, $\tilde{\theta} \equiv 0 \quad \Rightarrow \quad e(t) \to 0$ exponencialmente !

Resumo das equações do sistema (em termos de erros):

$$\begin{cases} \dot{e} = -a_m e + \tilde{\theta} y & \text{(dinâmica do erro de rastreamento)} \\ \dot{\tilde{\theta}} = \dot{\theta} = ? & \text{(dinâmica do erro paramétrico)} \end{cases}$$

Próximo passo: Achar uma lei de adaptação para $\theta(t)$ tal que $e(t) \to 0$ e também $\tilde{\theta}(t) \to 0$.

★ Fundamental: estabilidade !

Tomamos a função de Lyapunov [Parks:1966]:

$$\boxed{2V(e,\tilde{\theta}) = e^2 + \tilde{\theta}^2} \qquad \text{positiva definida} \tag{1}$$

Derivando,

$$\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial e}\dot{e} + \frac{\partial V}{\partial \tilde{\theta}}\dot{\theta} \\
&= e\dot{e} + \tilde{\theta}\dot{\theta} \\
&= e\big(-a_m e + \tilde{\theta} y\big) + \tilde{\theta}\dot{\theta} \\
&= -a_m e^2 + \tilde{\theta} e y + \tilde{\theta}\dot{\theta} \\
&= -a_m e^2 + \tilde{\theta}\big[ey + \dot{\theta}\big]
\end{aligned}$$

Escolhemos: $\boxed{\dot{\theta} = -ey}$

Resultado: $\dot{V} = -a_m e^2 \leq 0$ negativa semi-definida !

Conclusão: $\boxed{e(t)\,,\ \tilde{\theta}(t) \in \mathcal{L}_\infty}$ $\Rightarrow$ Estabilidade

Análise de convergência

- $r \in \mathcal{L}_\infty \quad \Rightarrow \quad \boxed{y_m \in \mathcal{L}_\infty}$
- $\left.\begin{array}{r} e \in \mathcal{L}_\infty \\ y_m \in \mathcal{L}_\infty \end{array}\right\} \quad \Rightarrow \quad \boxed{y \in \mathcal{L}_\infty}$
- $\dot{V} \leq 0 \quad \Rightarrow \quad \boxed{V(t) \leq V(0)}$

Integrando a expressão de $\dot{V}$, temos:

$$\int_0^t \dot{V}(\tau)d\tau = \int_0^t -a_m e^2(\tau)d\tau = -a_m \int_0^t e^2(\tau)d\tau$$

Ou melhor,

$$\underbrace{V(t) - V(0)}_{\leq 0} = -a_m \int_0^t e^2(\tau)d\tau$$

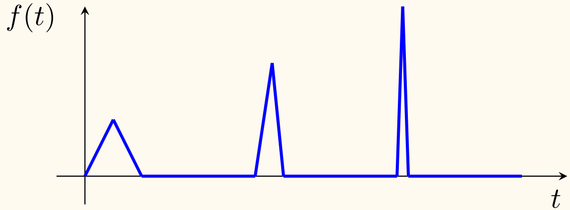
$$\underbrace{V(0) - V(t)}_{\geq 0} = a_m \int_0^t e^2(\tau)d\tau < \infty\,, \qquad \forall t\ !$$

Conclusão: $\boxed{e \in \mathcal{L}_2}$

★ Propriedade importante! Isto é *quase convergência*.

★ Aparentemente, $e \in \mathcal{L}_2 \;\Rightarrow\; e(t) \to 0$. Porém isto não é verdade.

Exemplo 2 Função soluço.



$$\int_0^\infty f^2(d\tau) < \infty \quad \text{porém} \quad f(t) \not\to 0$$

Ref.: [Slotine & Li:1991], (pag. 122)
[Chen:1999], (pag. 123)

Lema. [Tao:2003], (pag. 80)

$$\left.\begin{array}{l} f(t) \in \mathcal{L}_2 \\ \dot{f}(t) \in \mathcal{L}_\infty \end{array}\right\} \Rightarrow \boxed{\lim_{t\to\infty} f(t) = 0}$$

Outro lema mais geral ainda,

Lema. (Barbalăt) [Tao:2003], (pag. 81)

$$\left.\begin{array}{r} f(t) \text{ unif. contínua} \\ \int_0^\infty f(t)dt < \infty \end{array}\right\} \Rightarrow \boxed{\lim_{t\to\infty} f(t) = 0}$$

Lembrete :

$$\begin{bmatrix} f(t) \text{ dif.} \\ \dot{f}(t) \in \mathcal{L}_\infty \end{bmatrix} \begin{array}{c} \Rightarrow \\ \not\Leftarrow \end{array} \big[f(t) \text{ Lip.} \big] \begin{array}{c} \Rightarrow \\ \not\Leftarrow \end{array} \big[f(t) \text{ u.c.} \big] \begin{array}{c} \Rightarrow \\ \not\Leftarrow \end{array} \big[f(t) \text{ contínua} \big]$$

Note que

$$\left.\begin{array}{r} e(t) \in \mathcal{L}_\infty \\ \tilde{\theta} \in \mathcal{L}_\infty \\ y \in \mathcal{L}_\infty \end{array}\right\} \Rightarrow \quad \dot{e} = \underbrace{-a_m e}_{\mathcal{L}_\infty} + \underbrace{\tilde{\theta} y}_{\mathcal{L}_\infty} \quad \Rightarrow \quad \boxed{\dot{e}(t) \in \mathcal{L}_\infty}$$

Aplicando o lema, conclui-se que:

$$\left.\begin{array}{r} e(t) \in \mathcal{L}_2 \\ \dot{e}(t) \in \mathcal{L}_\infty \end{array}\right\} \Rightarrow \quad \boxed{\lim_{t\to\infty} e(t) = 0}$$

Fato.

$$\theta(t) \not\to \theta^*$$

Condição para convergência: Excitação persistente (PE)

$$\int_t^{t+\tau} y^2(\xi)d\xi \geq \alpha > 0$$

Variação da análise de estabilidade

Modificando a função de Lyapunov (1) para

$$2V(e,\tilde{\theta}) = e^2 + \gamma^{-1}\tilde{\theta}^2$$

sua derivada resulta

$$\begin{aligned}\dot{V} &= e\big(-a_m e + \tilde{\theta} y\big) + \gamma^{-1}\tilde{\theta}\dot{\theta} \\ &= -a_m e^2 + \tilde{\theta}\big(ey + \gamma^{-1}\dot{\theta}\big)\end{aligned}$$

e, portanto,

$$\boxed{\dot{\theta} = -\gamma e y} \qquad (\gamma > 0)$$

★ $\gamma =$ ganho de adaptação.

Summary of the algorithm

Subsystem	Equation	Order
Plant	$\dot{y} = a_p y + u$	1
Model	$\dot{y}_m = -a_m y_m + r$	1
Tracking error	$e_0 = y - y_m$	
Control law	$u = \theta\, y + r$	
Update law	$\dot{\theta} = -\gamma e_0 y$	1
	System order $=$	3

Resultados de simulação Vide: [relatorio-01.pdf]

Simulação #1 Condições iniciais nulas.

Condições iniciais . . . : $y(0) = 0$

$y_m(0) = 0$

$\theta(0) = 0$

Parâmetros . . . . : $a_p = 2$

$a_m = 1$

$\gamma = 2,\ 100$

Sinal de referência . . : $r = 1$

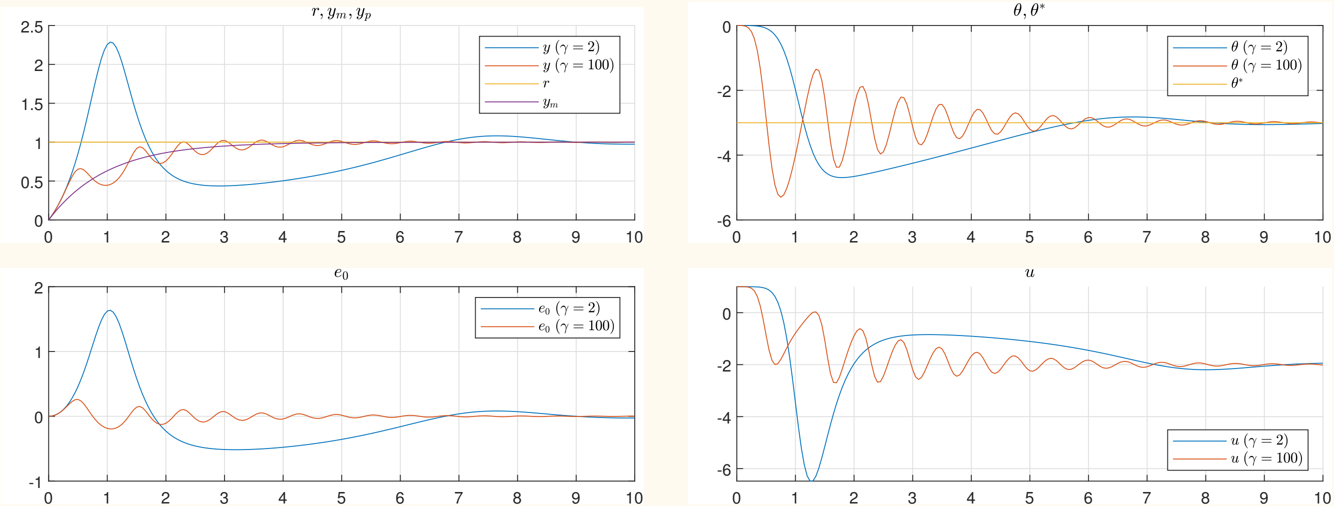


Figura 3: Resultado da simulação com algoritmo MRAC direto. (Script: `simu01.m`)

- Observa-se que o aumento de γ somente causa um aumento na frequência de oscilação de θ.
- A envoltória permanece aproximadamente a mesma.

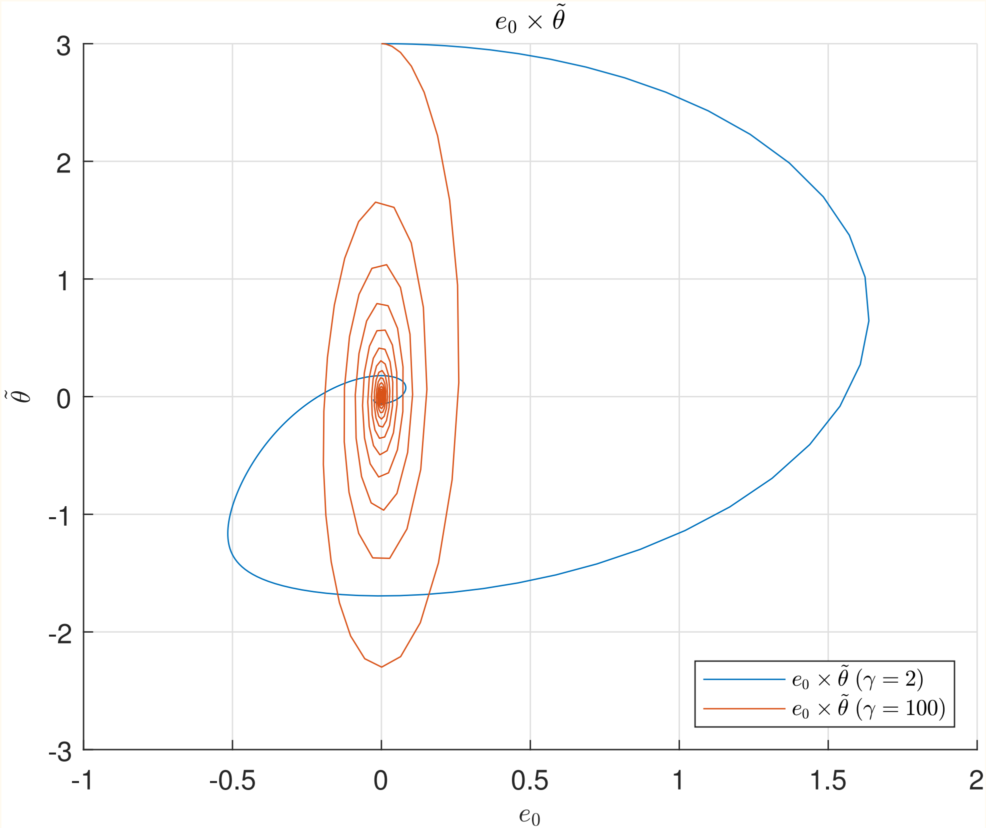


Figura 4: Resultado da simulação com algoritmo MRAC direto. (Script: `simu01.m`)

Simulação #2 Efeito de condição inicial pequena.

Condições iniciais . . . : $y(0) = \boxed{2}$

$y_m(0) = 0$

$\theta(0) = 0$

Parâmetros . . . . : $a_p = 2$

$a_m = 1$

$\gamma = 2, \ 100$

Sinal de referência . . : $r = 1$

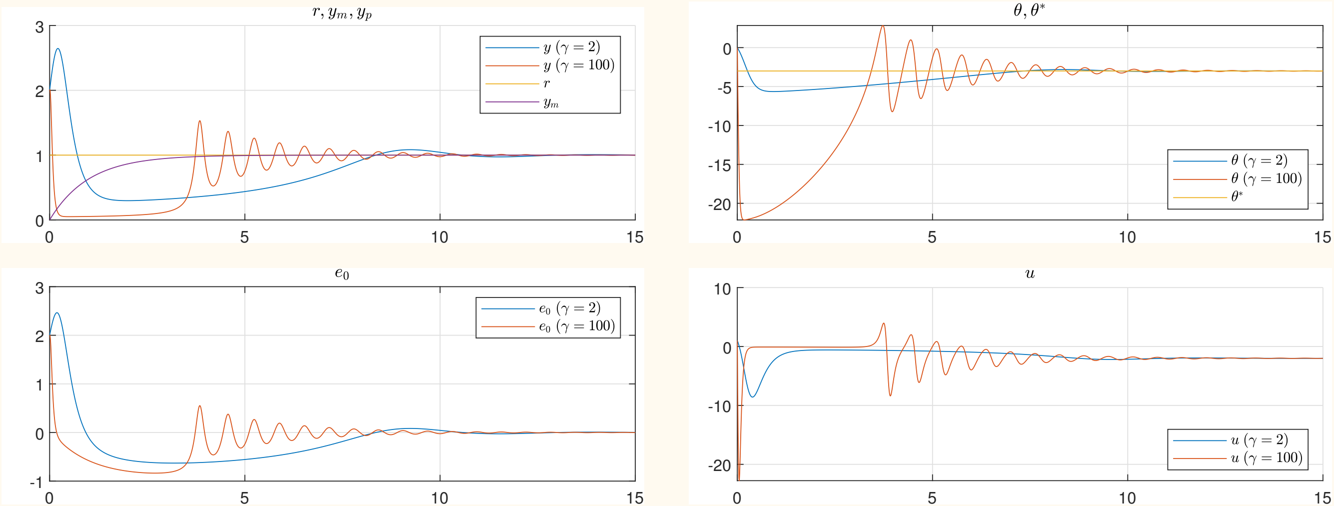


Figura 5: Resultado da simulação com algoritmo MRAC direto. (Script: `simu02.m`)

- Esta simulação mostra a não uniformidade do transitório de adaptação em relação às condições iniciais.
- O comportamento de θ para $\gamma = 2$ é completamente diferente do comportamento para $\gamma = 100$.

Simulação #3 Efeito de condição inicial grande.

Condições iniciais . . . :

$$y(0) = 10$$
$$y_m(0) = 0$$
$$\theta(0) = 0$$

Parâmetros . . . . :

$$a_p = 2$$
$$a_m = 1$$
$$\gamma = 2,\ 100$$

Sinal de referência . . :

$$r = 1$$

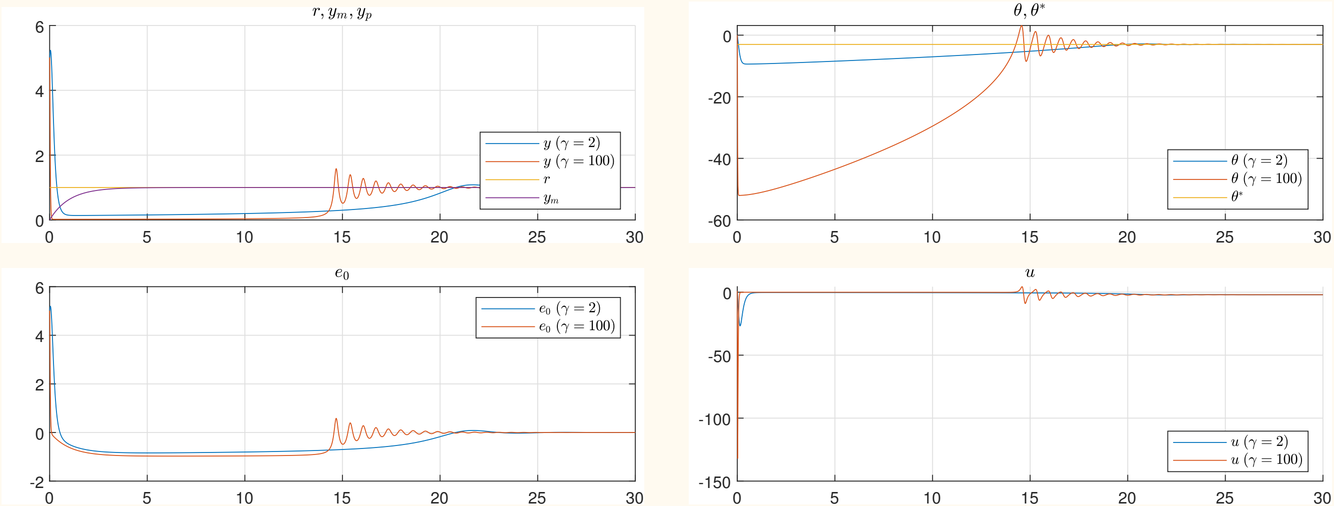


Figura 6: Resultado da simulação com algoritmo MRAC direto. (Script: `simu03.m`)

- A duração do transitório de adaptação aumenta com as condições iniciais.
- $e(0)$ grande $\Rightarrow$ $\dot{\theta}(0)$ grande $\Rightarrow$ Salto inicial de θ. (*Super-estabilização*)
- θ grande $\Rightarrow$ $y = \frac{1}{s+\theta} r = \frac{1/\theta}{s/\theta+1} r \simeq 0$. (*Atenuação de* r)

Simulação #4 Efeito de variações descontínuas no parâmetro a_p.

Condições iniciais . . . : $$y(0) = 2$$ $$y_m(0) = 0$$ $$\theta(0) = 0$$

Parâmetros . . . . : $$a_p = [2,\ 3,\ 4,\ 5,\ 6]$$ $$a_m = 1$$ $$\gamma = 10$$

Sinal de referência . . : $$r = 1$$

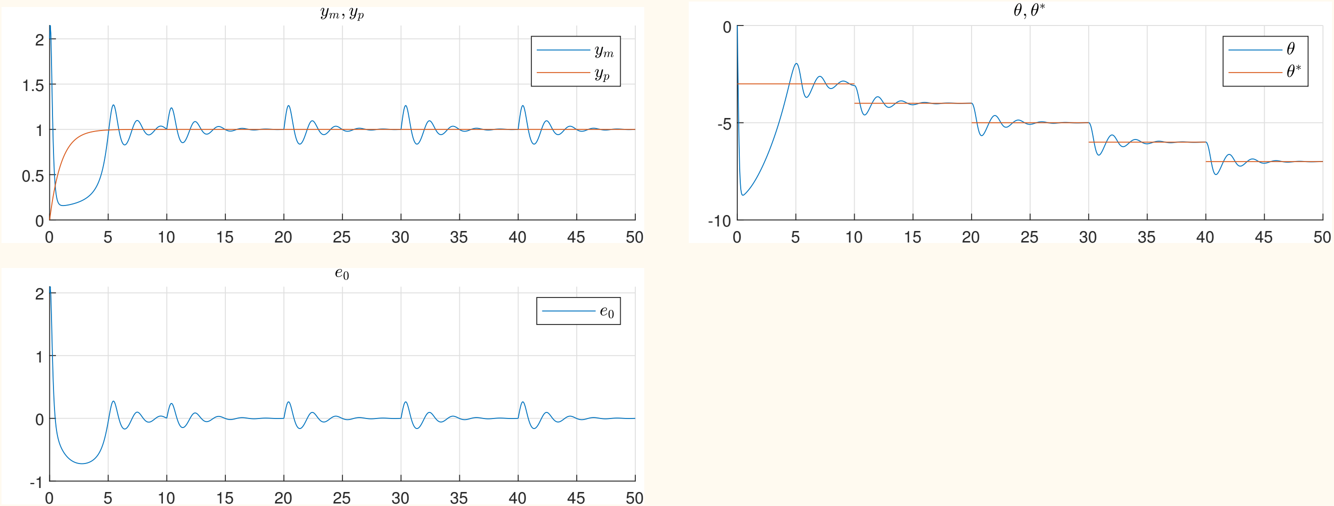


Figura 7: Resultado da simulação com algoritmo MRAC direto. (Script: `simu04.m`)

2.1.2 Método do Gradiente

Exemplo 3 Ref.: [Tao:2003], (pag. 20)

Planta : $\boxed{\dot{y} = a_p y + u}$ $P(s) = \dfrac{1}{s - a_p}$ a_p desconhecido

Modelo : $\boxed{\dot{y}_m = -a_m y_m + r}$ $M(s) = \dfrac{1}{s + a_m}$

Lei de controle: $\boxed{u = \theta y + r}$

Equação do erro: $\boxed{\dot{e} = -a_m e + \tilde{\theta} y}$

Vamos definir o seguinte sinal filtrado:

$$\zeta = M(s)y = \frac{1}{s + a_m}\,y$$

Então, podemos reescrever a equação do erro como

$$\begin{aligned} e &= M\big[\theta y - \theta^* y\big] \\ &= M\big[\theta y\big] - \theta^* M\big[y\big] \\ &= M\big[\theta y\big] - \theta^* \zeta \end{aligned}$$

Note que:
- e : sinal medido
- θ : parâmetro adaptado
- y : sinal medido
- θ^* : parâmetro desconhecido
- ζ : sinal filtrado

Idéia: Ajustar o parâmetro θ tal que o erro de estimativa de e seja pequeno.

Estimador: $\boxed{\hat{e} = M\big[\theta y\big] - \theta\zeta}$ θ = estimativa de θ^*

Erro de estima: $\boxed{\varepsilon = e - \hat{e}}$

Nota. Partindo-se da equação do erro na forma

$$e = M\big[\theta y - \theta^* y\big]$$

o estimador resulta

$$\hat{e} = M\big[\theta y - \theta y\big] = 0 \text{ !!!}$$

O erro de estima também pode ser escrito como

$$\begin{aligned}\varepsilon &= e - \hat{e} \\ &= M\big[\theta y\big] - \theta^* \zeta - M\big[\theta y\big] + \theta\zeta \quad \Rightarrow \quad \boxed{\varepsilon = \tilde{\theta}\zeta}\end{aligned}$$

Vamos agora escolher a função de Lyapunov (parcial)

$$2V\big(\tilde{\theta}\big) = \gamma^{-1}\tilde{\theta}^2, \qquad \gamma > 0$$

Derivando,

$$\dot{V} = \frac{\partial V}{\partial \tilde{\theta}}\dot{\theta} = \gamma^{-1}\tilde{\theta}\dot{\theta}$$

Considere a seguinte lei de adaptação normalizada

$$\boxed{\dot{\theta} = -\frac{\gamma \varepsilon \zeta}{m^2}}$$

com sinal normalizante

$$\boxed{m^2 = 1 + \zeta^2}$$

★ Lembrar que $\zeta = M(s)y$.

Substituindo em $\dot{V}$,

$$\begin{aligned}\dot{V} &= \gamma^{-1}\tilde{\theta}\dot{\theta} \\ &= \gamma^{-1}\tilde{\theta}\left(-\frac{\gamma\varepsilon\zeta}{m^2}\right) \\ &= -\frac{\tilde{\theta}\varepsilon\zeta}{m^2}\end{aligned}$$

Lembrando que $\varepsilon = \tilde{\theta}\zeta$,

$$\boxed{\dot{V} = -\frac{\varepsilon^2}{m^2}} \leq 0$$

Conclusão:

- $V(\tilde{\theta})$ não cresce $\Rightarrow$ $\boxed{\tilde{\theta} \in \mathcal{L}_\infty}$

- $\varepsilon = \tilde{\theta}\zeta \quad \Rightarrow \quad \dfrac{\varepsilon}{m} = \tilde{\theta}\,\underbrace{\dfrac{\zeta}{m}}_{<1} \quad \Rightarrow \quad \boxed{\dfrac{\varepsilon}{m} \in \mathcal{L}_\infty}$

- $\dot{V} = -\dfrac{\varepsilon^2}{m^2} \quad \Rightarrow \quad \displaystyle\int_0^\infty \dfrac{\varepsilon^2}{m^2} d\tau < \infty \quad \Rightarrow \quad \boxed{\dfrac{\varepsilon}{m} \in \mathcal{L}_2}$

- $\dot{\theta} = -\gamma \underbrace{\dfrac{\varepsilon}{m}}_{\mathcal{L}_\infty \bigcap \mathcal{L}_2} \underbrace{\dfrac{\zeta}{m}}_{<1} \quad \Rightarrow \quad \boxed{\dot{\theta} \in \mathcal{L}_\infty}$ e $\boxed{\dot{\theta} \in \mathcal{L}_2}$

Análise de convergência

- ★ É possível mostrar com estes resultados que $\boxed{\; y \to y_m \;}$.

- ★ A demonstração é trabalhosa !

  Ver [Tao:2003], (pag. 212) .

Nota. Aplicação do método do Gradiente.

Função custo: $$\boxed{J = \frac{\varepsilon^2}{m^2} = \frac{\tilde{\theta}^2 \zeta^2}{m^2}}$$

Derivando em relação a $\tilde{\theta}$,

$$\frac{\partial J}{\partial \tilde{\theta}} = 2\tilde{\theta}\frac{\zeta^2}{m^2} = 2\varepsilon\frac{\zeta}{m^2} \quad \text{!!}$$

Lei do gradiente: A adaptação é feita no sentido negativo de $\dfrac{\partial J}{\partial \tilde{\theta}}$.

Isto é,

$$\boxed{\dot{\theta} = -\frac{\partial J}{\partial \tilde{\theta}} = -\gamma\varepsilon\frac{\zeta}{m^2}}$$

★ Note o efeito da normalização: $J = \tilde{\theta}^2 \underbrace{\frac{\zeta^2}{m^2}}_{<1}$

Summary of the algorithm

Subsystem	Equation	Order
Plant	$\dot{y} = a_p y + u$	1
Model	$\dot{y}_m = -a_m y_m + r$	1
Tracking error	$e_0 = y - y_m$	
Filter	$\dot{\zeta} = -a_m \zeta + y$	1
Control law	$u = \theta y + r$	
Prediction	$\hat{e} = \dfrac{1}{s + a_m} [\theta y] - \theta \zeta$	1
Prediction error	$\varepsilon = e_0 - \hat{e}$	
Normalizing signal	$m^2 = 1 + \zeta^2$	
Update law	$\dot{\theta} = -\dfrac{\gamma \varepsilon \zeta}{m^2}$	1
	System order $=$	5

Resultados de simulação

Vide: [relatorio-02.pdf]

Simulação #1

Condições iniciais nulas.

Condições iniciais.....:

$$y(0) = 0$$
$$y_m(0) = 0$$
$$\theta(0) = 0$$

Parâmetros..............:

$$a_p = 2$$
$$a_m = 1$$
$$\gamma = 10,\ 100$$

Sinal de referência....: $r = 1$

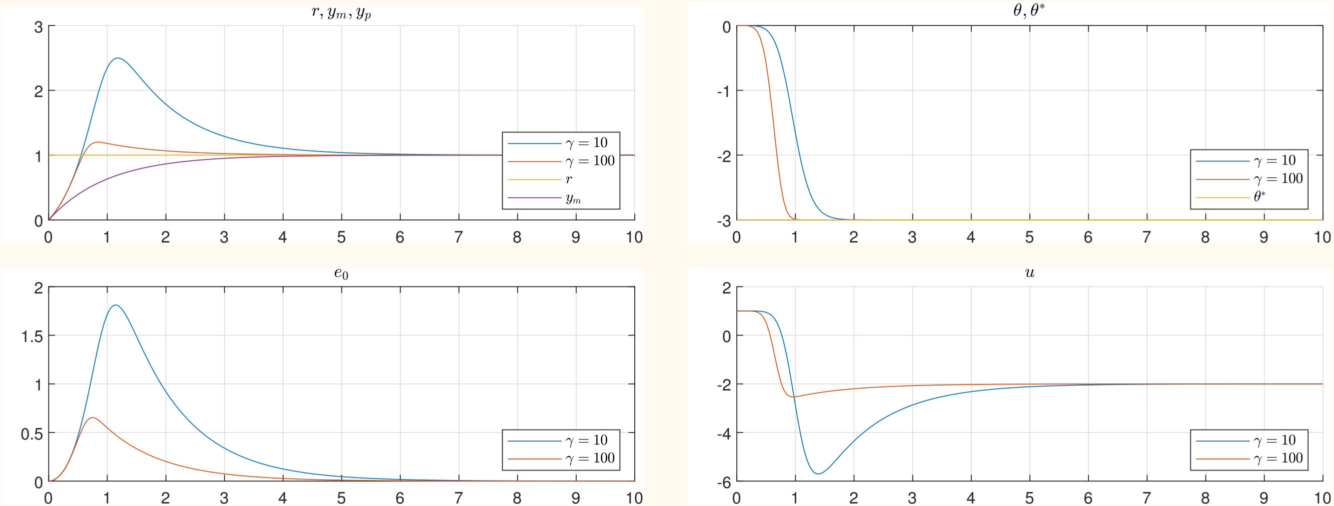


Figura 8: Resultado da simulação com algoritmo Gradiente normalizado. (Script: `simu01.m`)

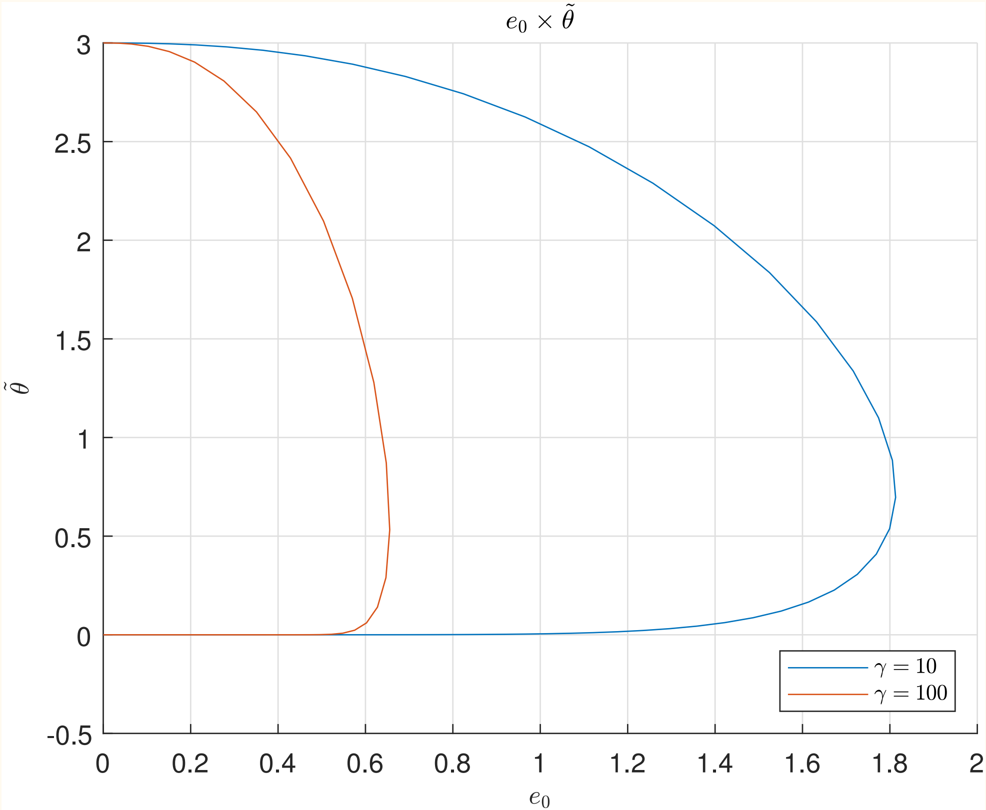


Figura 9: Resultado da simulação com algoritmo Gradiente normalizado. (Script: `simu01.m`)

Simulação #2 Efeito de condição inicial pequena.

Condições iniciais.....:
$$y(0) = \boxed{2}$$
$$y_m(0) = 0$$
$$\theta(0) = 0$$

Parâmetros...............:
$$a_p = 2$$
$$a_m = 1$$
$$\gamma = 2,\ 100$$

Sinal de referência....: $r = 1$

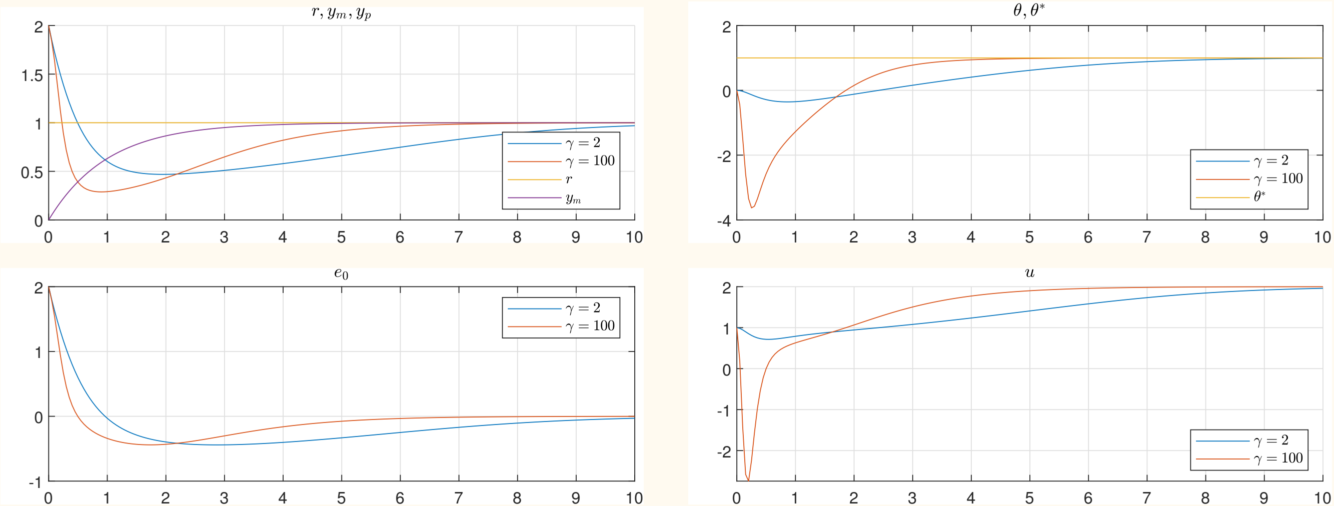


Figura 10: Resultado da simulação com algoritmo Gradiente normalizado. (Script: `simu02.m`)

Simulação #3 Efeito de condição inicial grande.

Condições iniciais.....: $y(0) = \boxed{10}$

$y_m(0) = 0$

$\theta(0) = 0$

Parâmetros..............: $a_p = 2$

$a_m = 1$

$\gamma = 1,\ 100$

Sinal de referência....: $r = 1$

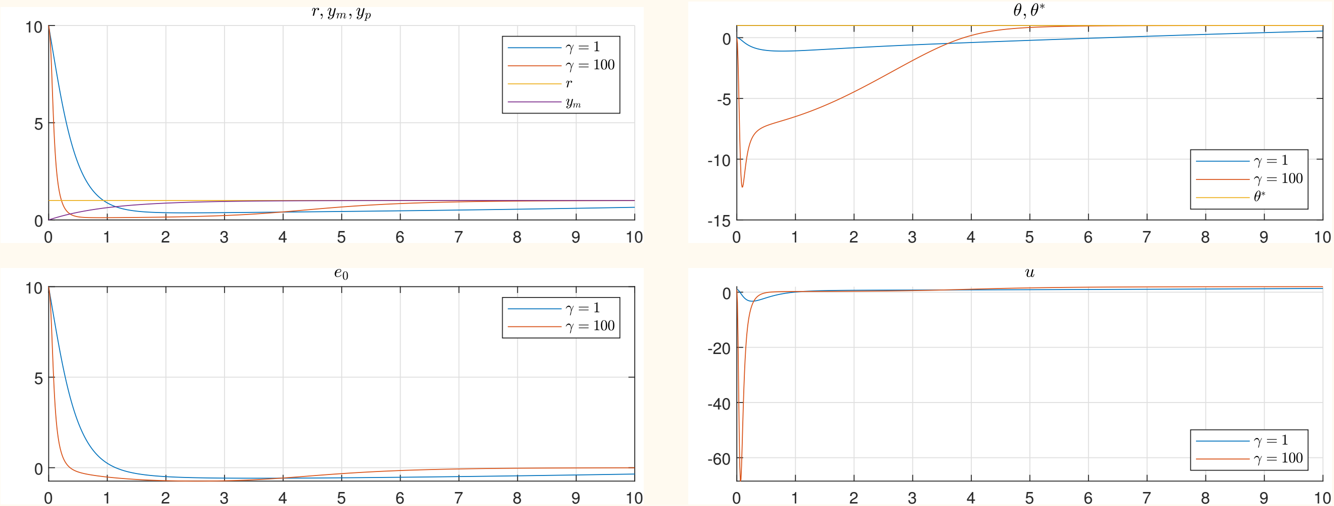


Figura 11: Resultado da simulação com algoritmo Gradiente normalizado. (Script: `simu03.m`)

2.1.3 MRAC INDIRETO

Exemplo 4 Ref.: [Tao:2003], (pag. 21)

Planta : $\boxed{\dot{y} = a_p y + u}$ $P(s) = \dfrac{1}{s - a_p}$ a_p desconhecido

Modelo : $\boxed{\dot{y}_m = -a_m y_m + r}$ $M(s) = \dfrac{1}{s + a_m}$

Lei de controle: $\boxed{u = \theta y + r}$, $\theta^* = -a_p - a_m$

Seja $\hat{a}_p$ uma estima de a_p.

Podemos calcular $\theta(t)$ como: $\boxed{\theta(t) = -\hat{a}_p - a_m}$

Para encontrar $\hat{a}_p$, introduz-se um filtro estável

$$\frac{1}{s+a_f}, \quad a_f > 0$$

A planta pode ser expressa como

$$\begin{aligned} sy &= a_p y + u + (a_f y) - (a_f y) \\ &= -a_f y + (a_f + a_p) y + u \end{aligned}$$

Portanto,

$$(s + a_f)y = (a_f + a_p)y + u$$

ou melhor,

$$y = \underbrace{(a_f + a_p)}_{\psi^*} \underbrace{\frac{1}{s+a_f} y}_{y_f} + \underbrace{\frac{1}{s+a_f} u}_{u_f} \quad \Rightarrow \quad \boxed{y = \psi^* y_f + u_f}$$

Predição de y: $\boxed{\hat{y} = \psi y_f + u_f}$

★ Lembrar que $\psi = a_f + \hat{a}_p$ é uma estima de ψ^*. $\Rightarrow$ $\boxed{\theta(t) = -\psi + a_f - a_m}$

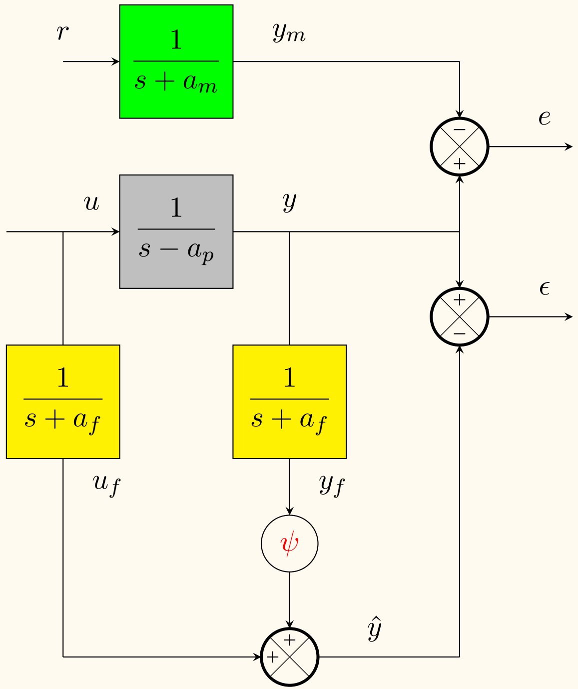
★ A análise pode ser feita com $\hat{a}_p$ ou ψ.

Erro de predição:

$$\begin{aligned} \varepsilon &= \hat{y} - y \\ &= \big(\psi\, y_f + u_f\big) - \big(\psi^* y_f + u_f\big) \quad \Rightarrow \quad \boxed{\varepsilon = \tilde{\psi}\, y_f} \end{aligned}$$

★ A equação do erro não tem dinâmica !!

Estrutura do MRAC indireto:



Lei de adaptação: $\boxed{\dot{\psi} = -\dfrac{\gamma\varepsilon\, y_f}{m^2}}$, $m^2 = 1 + y_f^2\,,\qquad \gamma > 0$

Análise de estabilidade

Função de Lyapunov: $2V(\tilde{\psi}) = \gamma^{-1}\tilde{\psi}^2$

Derivando,

$$\dot{V} = \gamma^{-1}\tilde{\psi}\dot{\psi} = \gamma^{-1}\tilde{\psi}\left(-\frac{\gamma\varepsilon\, y_f}{m^2}\right) = -\frac{\varepsilon\, y_f\, \tilde{\psi}}{m^2} = -\frac{\varepsilon^2}{m^2} \leq 0$$

Conclusão: $\boxed{\tilde{\psi} \in \mathcal{L}_\infty}$

Summary of the algorithm

Subsystem	Equation	Order
Plant	$\dot{y} = a_p y + u$	1
Model	$\dot{y}_m = -a_m y_m + r$	1
Tracking error	$e_0 = y - y_m$	
Filters	$\dot{y}_f = -a_f y_f + y$	1
	$\dot{u}_f = -a_f u_f + u$	1
Identification	$\hat{a}_p = \psi - a_f$	
Control gain	$\theta = -\hat{a}_p - a_m$	
Control law	$u = \theta y + r$	
Prediction	$\hat{y} = \psi\, y_f + u_f$	
Prediction error	$\varepsilon = \hat{y} - y$	
Normalizing signal	$m^2 = 1 + y_f^2$	
Update law	$\dot{\psi} = -\gamma \varepsilon \dfrac{y_f}{m^2}$	1
	System order $=$	5

Resultados de simulação Vide: [relatorio-03.pdf]

Simulação #1 Condições iniciais nulas.

Condições iniciais.....: $y(0) = \boxed{0}$

$$y_m(0) = 0$$

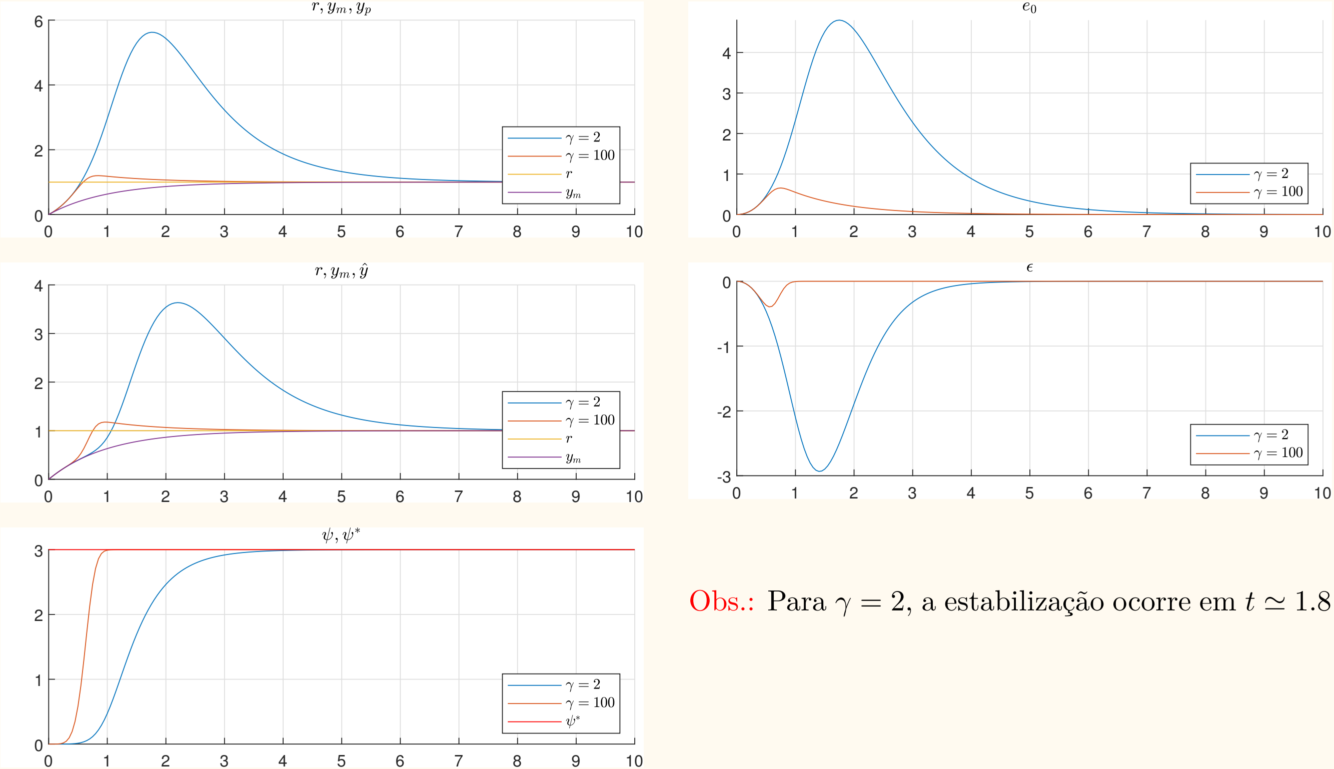
$$\psi(0) = 0$$

Parâmetros..............: $a_p = 2$

$$a_m = 1$$

$$\gamma = 2,\ 100$$

Sinal de referência....: $r = 1$



Obs.: Para $\gamma = 2$, a estabilização ocorre em $t \simeq 1.8$

Figura 12: Resultado da simulação com algoritmo MRAC indireto. (Script: `simu01.m`)

Simulação #2 Efeito de condição inicial pequena.

Condições iniciais.....: $y(0) = \boxed{2}$

$$y_m(0) = 0$$

$$\psi(0) = 0$$

Parâmetros..............: $a_p = 2$

$$a_m = 1$$

$$\gamma = 2,\ 100$$

Sinal de referência....: $r = 1$

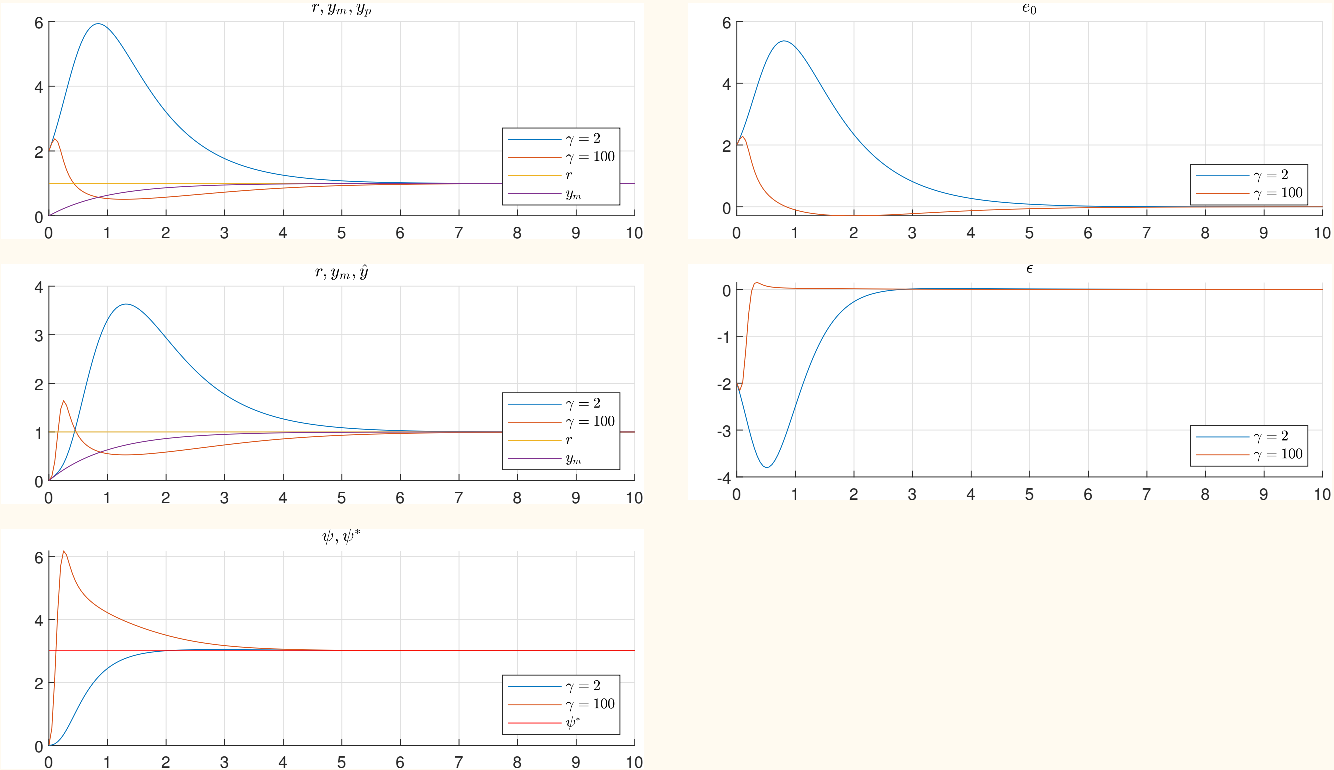


Figura 13: Resultado da simulação com algoritmo MRAC indireto. (Script: `simu02.m`)

Simulação #3 Efeito de condição inicial grande.

Condições iniciais.....: $y(0) = 10$

$y_m(0) = 0$

$\psi(0) = 0$

Parâmetros..............: $a_p = 2$

$a_m = 1$

$\gamma = 2,\ 100$

Sinal de referência....: $r = 1$

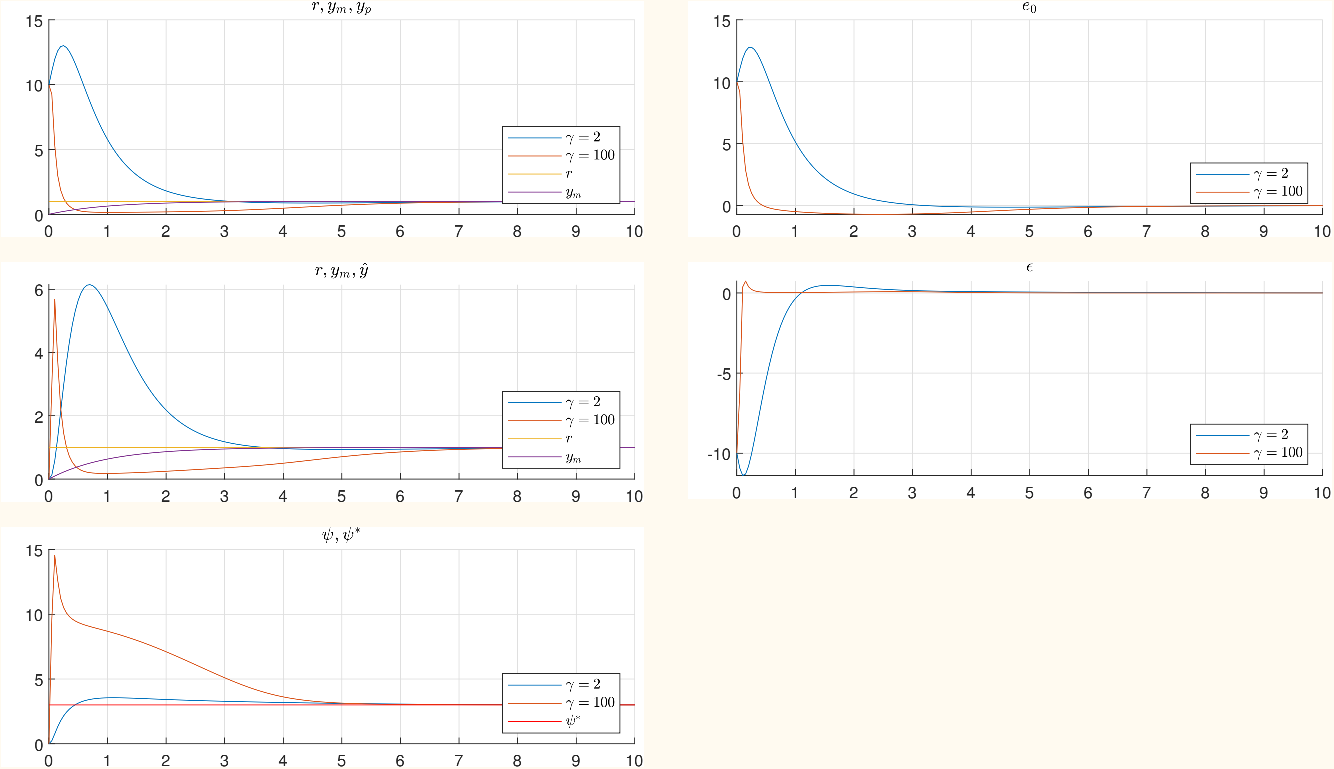


Figura 14: Resultado da simulação com algoritmo MRAC indireto. (Script: `simu03.m`)

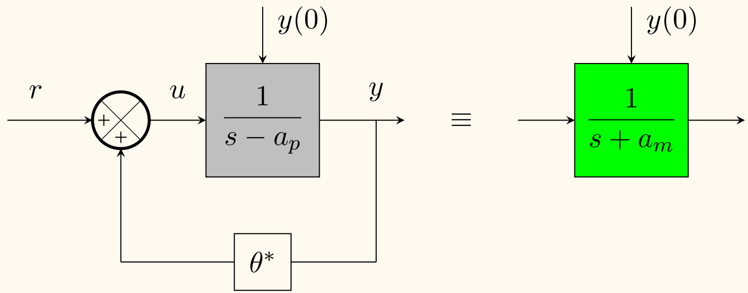
2.1.4 VS-MRAC

Ref.: [Hsu & Costa:1989]

Planta : $\boxed{\dot{y} = a_p y + u}$ $\quad y = \dfrac{1}{s - a_p} u$ $\quad$ a_p desconhecido !

Modelo : $\boxed{\dot{y}_m = -a_m y_m + r}$ $\quad y_m = \dfrac{1}{s + a_m} r$

Se a_p fosse conhecido, então



Lei de *controle ideal*: $$\boxed{u^* = \theta^* y + r}$$

Matching gain : $$\boxed{\theta^* = -a_p - a_m}$$

Controle proposto: $$\boxed{u = \theta(t)\, y + r}$$

★ $\theta(t)$ é um sinal chaveado (*switching function*).

Erro de saída: $\boxed{e = y - y_m}$

Equação do erro:

$$
\begin{aligned}
\dot{e} &= \dot{y} - \dot{y}_m \\
&= \big(a_p y + u\big) - \big(-a_m y_m + r\big) + \big(a_m y\big) - \big(a_m y\big) \\
&= -a_m\big(\underbrace{y - y_m}_{e}\big) + \big(\underbrace{a_p + a_m}_{-\theta^*}\big)y - r + u \\
&= -a_m e - \theta^* y - r + \theta y + r \\
&= -a_m e + \big[\underbrace{\theta - \theta^*}_{\tilde{\theta}}\big]y
\end{aligned}
$$

Erro paramétrico: $\boxed{\tilde{\theta} = \theta - \theta^*}$

Portanto, $\boxed{\dot{e} = -a_m e + \tilde{\theta} y}$

Resumo das equações do sistema (em termos de erros):

$$\begin{cases} \dot{e} = -a_m e + \tilde{\theta} y \\ \theta = ? \qquad \text{(switching function)} \end{cases}$$

Nova proposta: *Variable Structure - MRAC* (VS-MRAC)

★ O sinal θ não é definido por uma equação diferencial !!

Função de Lyapunov para este sistema (sem o termo $\tilde{\theta}^2$) :

$$2V(e) = e^2$$

Derivando,

$$\begin{aligned}\dot{V}(e) = e\dot{e} &= -a_m e^2 + \tilde{\theta} y e \\ &= -a_m e^2 + (\theta - \theta^*) y e \\ &= -a_m e^2 + \theta y e - \theta^* y e\end{aligned}$$

Hipótese: Conhecemos um *upper-bound* $\bar{\theta}$ para θ^*. Isto é $\boxed{\bar{\theta} > |\theta^*|}$

Isto permite escolher $\boxed{\theta = -\bar{\theta}\ \mathrm{sign}(ye)}$

Resultado:

$$
\begin{aligned}
\dot{V} &= -a_m e^2 - \bar{\theta}\, ye \ \text{sign}(ye) - \theta^* ye \\
&= -a_m e^2 - \underbrace{\bar{\theta}|ye|}_{>0} - \theta^* ye
\end{aligned}
$$

★ Note que $\bar{\theta}|ye| > |\theta^* ye| > 0$

Portanto,

$$
\begin{aligned}
\dot{V} &= -a_m e^2 - \Delta \qquad\qquad (\Delta > 0) \\
&< -a_m e^2 \qquad \text{(!!)}
\end{aligned}
$$

Conclusão: $\boxed{\dot{V} < 0}$ $\Rightarrow$ $\boxed{e(t) \to 0}$ pelo menos exponencialmente !!

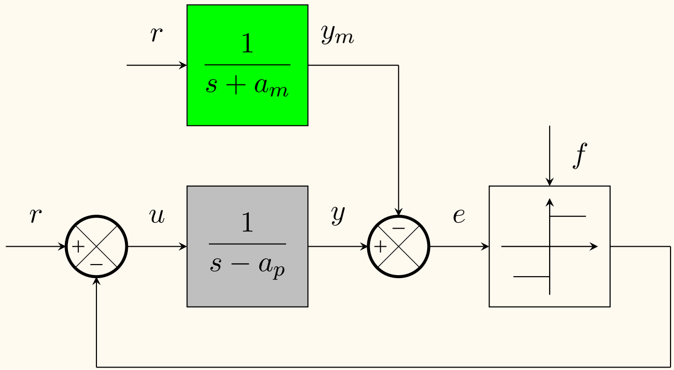
A lei de controle pode ser escrita como

$$\begin{aligned} u &= \theta\, y + r \\ &= -\bar{\theta}\ \mathrm{sign}(ey)y + r \\ &= -\underbrace{\bar{\theta}\,|y|}_{f}\ \mathrm{sign}(e) + r \end{aligned}$$

Ou melhor, $\boxed{u = -f\,\mathrm{sign}(e) + r}$

★ $f = \bar{\theta}\,|y|$ é um upper-bound para $\theta^*\,|y|$.

Estrutura do VS-MRAC:

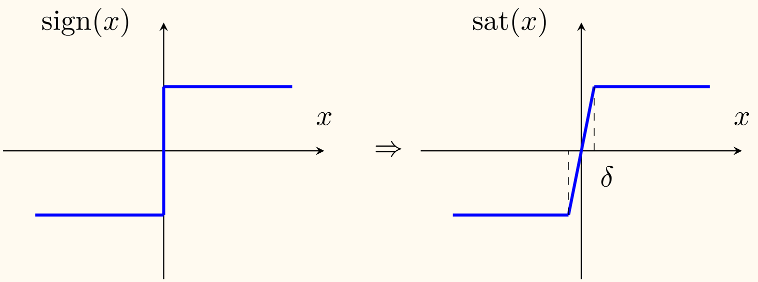


Resultados de simulação

Vide: [relatorio-04.pdf]

★ **Problema**: descontinuidade da função $\operatorname{sign}(\cdot)$.

★ Para a implementação da lei de controle, o relé é substituído por uma zona linear.



Summary of the algorithm

Subsystem	Equation	Order
Plant	$\dot{y} = a_p(t)y + u$	1
Model	$\dot{y}_m = -a_m y_m + r$	1
Tracking error	$e_0 = y - y_m$	
Filter	$\tau\dot{\theta}_{eq} = -\theta_{eq} + \theta$	1
Control law	$u = \theta_{eq}\, y + r$	
Update law	$\theta = -\bar{\theta}\ \mathrm{sat}(ye_0)$	
Linear zone	$\mathrm{sat}(x) = \begin{cases} \delta^{-1}x & \text{se } \lvert x\rvert < \delta \\ \mathrm{sign}(x) & \text{se } \lvert x\rvert \geq \delta \end{cases}$	
	System order $=$	3

★ Note that the plant is time varying!

Simulação #1 Efeito da zona linear δ.

Condições iniciais.....: $$y(0) = 2$$ $$y_m(0) = 0$$ $$\theta(0) = 0$$

Parâmetros..............: $$a_p = -2$$ $$a_m = 1$$ $$\bar{\theta} = 5$$ $$\tau = 0.1$$ $$\delta = 0.1,\ 0.01$$

Sinal de referência....: $$r = 1$$

★ O aumento da zona linear suaviza os sinais porém cria um erro de regime que pode ser observado em y.

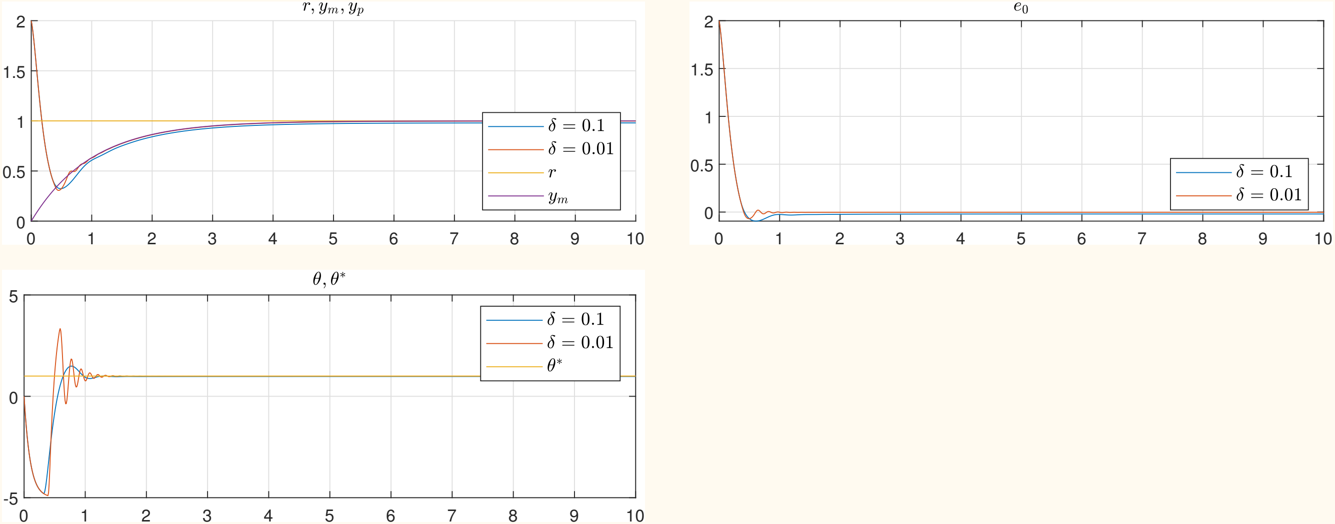


Figura 15: Resultado da simulação com algoritmo VS-MRAC. (Script: `simu01.m`)

Simulação #2 Efeito da constante de tempo τ do filtro.

Condições iniciais.....: $y(0) = 2$

$y_m(0) = 0$

$\theta(0) = 0$

Parâmetros..............: $a_p = -2$

$a_m = 1$

$\bar{\theta} = 5$

$\tau = 0.1,\ 0.01$

$\delta = 0.01$

Sinal de referência....: $r = 1$

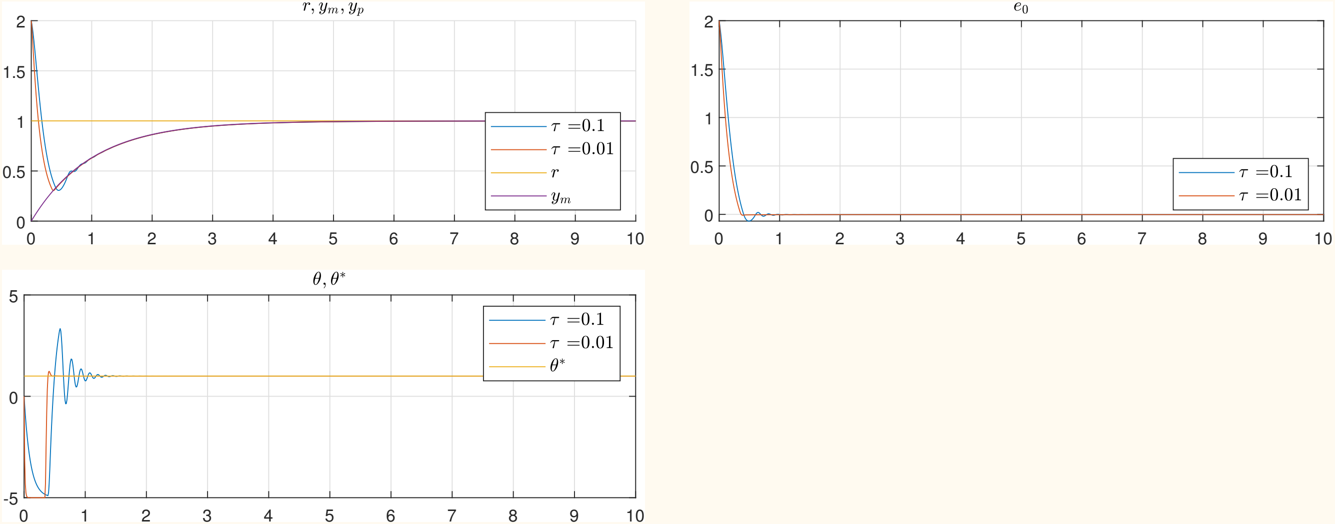


Figura 16: Resultado da simulação com algoritmo VS-MRAC. (Script: `simu02.m`)

Simulação #3 Efeito de uma constante de tempo τ muito pequena.

Condições iniciais.....: $y(0) = 10$

$y_m(0) = 0$

$\theta(0) = 0$

Parâmetros..............: $a_p = -2$

$a_m = 1$

$\bar{\theta} = 5$

$\tau = 0.1,\ 0.00001$

$\delta = 0.00001$

Sinal de referência....: $r = 1$

★ Neste caso, o *chattering* torna-se visível.

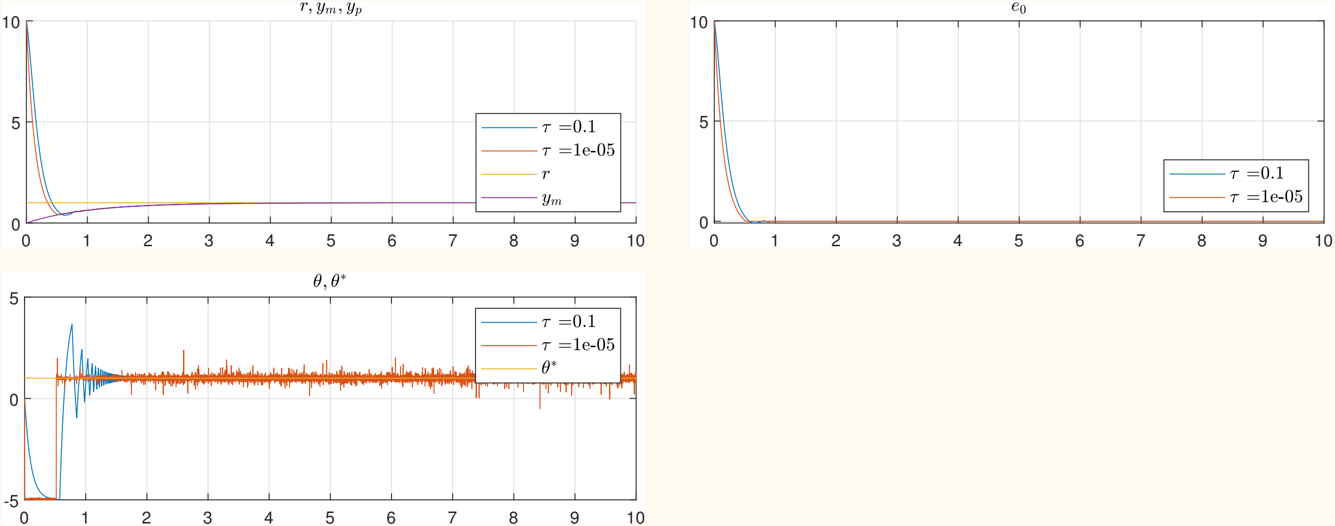


Figura 17: Resultado da simulação com algoritmo VS-MRAC. (Script: `simu03.m`)

Simulação #4 Caso em que o parâmetro da planta é variante no tempo.

Condições iniciais.....: $y(0) = 2$

$$y_m(0) = 0$$

$$\theta(0) = 0$$

Parâmetros..............: $a_p(t) = \begin{cases} -2\,, & 0 \le t \le 10 \\ -2 + \sin(2t)\,, & t > 10 \end{cases}$

$$a_m = 1$$

$$\bar{\theta} = 5$$

$$\tau = 0.1$$

$$\delta = 0.1$$

Sinal de referência....: $r = 1$

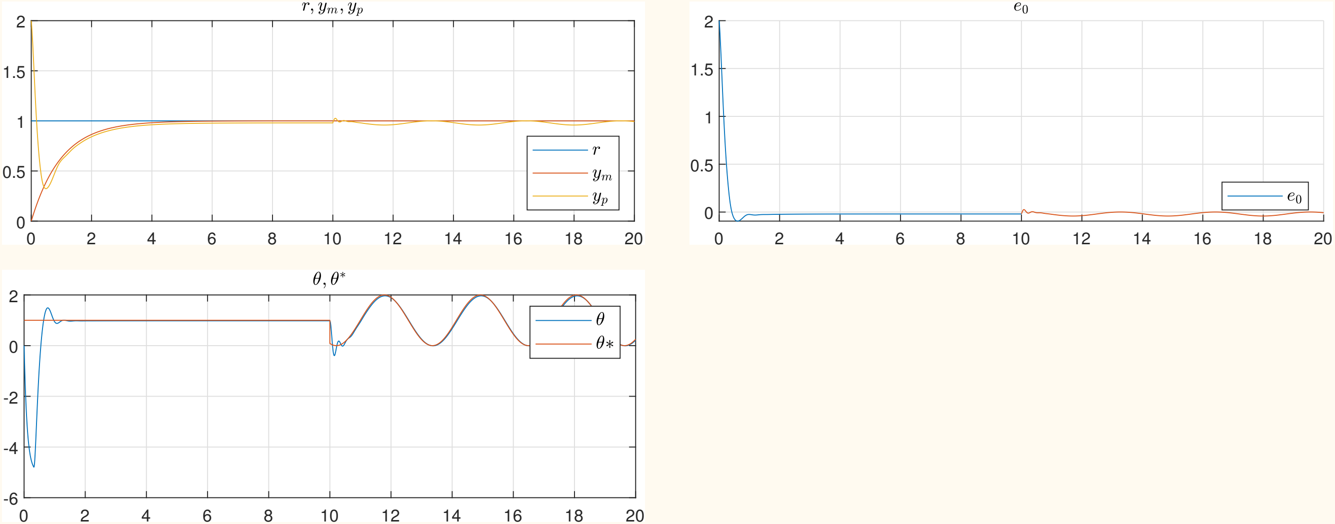


Figura 18: Resultado da simulação com algoritmo VS-MRAC. (Script: `simu04.m`)

2.1.5 Problemas & exercícios

(1) Observe o diagrama de blocos do MRAC indireto. O sinal de erro e não é utilizado na lei de controle e nem na lei de adaptação.

- **(a)** O modelo de referência é realmente necessário?
- **(b)** Explique?

Solução:

(2) Considere o algoritmo Gradiente Normalizado apresentado na seção 2.1.2. Modifique o fator de normalização para

$$m^2 = 1 + \zeta^2 + \dot{\zeta}^2$$

e mostre que, com essa modificação,

$$\frac{\dot{\varepsilon}}{m} \in \mathcal{L}_\infty \,.$$

Solução:

Relembrando...

Erro de estima : $\boxed{\varepsilon = \tilde{k}\zeta}$

Lei de adaptação : $\boxed{\dot{k} = -\dfrac{\gamma\varepsilon\zeta}{m^2}}$

Derivando a equação do erro,

$$\begin{aligned}\dot{\varepsilon} &= \dot{k}\zeta + \tilde{k}\dot{\zeta} \\ &= -\gamma\varepsilon\frac{\zeta^2}{m^2} + \tilde{k}\dot{\zeta}\end{aligned}$$

Porém,

$$\zeta = \frac{1}{s + a_m} y \quad \Rightarrow \quad \dot{\zeta} = -a_m\zeta + y$$

Portanto,

$$\begin{aligned}
\dot{\varepsilon} &= -\gamma\varepsilon\frac{\zeta^2}{m^2} + \tilde{k}\dot{\zeta} \\
&= -\gamma\varepsilon\frac{\zeta^2}{m^2} + \tilde{k}\big(-a_m\zeta + y\big) \\
&= -\gamma\varepsilon\frac{\zeta^2}{m^2} - a_m\underbrace{\tilde{k}\zeta}_{\varepsilon} + \tilde{k}y \\
&= -\left(a_m + \gamma\frac{\zeta^2}{m^2}\right)\varepsilon + \tilde{k}y
\end{aligned}$$

Dividindo por m,

$$\frac{\dot{\varepsilon}}{m} = -\underbrace{\left(a_m + \gamma\frac{\zeta^2}{m^2}\right)\frac{\varepsilon}{m}}_{\in\mathcal{L}_\infty} + \tilde{k}\underbrace{\frac{y}{m}}_{\in\mathcal{L}_\infty} \quad\Rightarrow\quad \boxed{\frac{\dot{\varepsilon}}{m} \in \mathcal{L}_\infty}$$

Detalhe : $\dot{\zeta} = -a_m\zeta + y \quad\Rightarrow\quad \dfrac{y}{m} \in \mathcal{L}_\infty$!!

2.2 Enfoque discreto

Algoritmos analisados:

- Método do Gradiente
- MRAC indireto

★ Não existe um equivalente discreto para o MRAC direto.

2.2.1 Método do Gradiente discreto

Exemplo 5 Ref.: [Tao:2003], (pag. 23)

Planta : $\boxed{y(t+1) = a_p y + u}$ $P(z) = \dfrac{1}{z - a_p}$ a_p desconhecido

Modelo : $\boxed{y_m(t+1) = -a_m y_m + r}$ $M(z) = \dfrac{1}{z + a_m}, \quad |a_m| < 1$

Parâmetro ideal: $\theta^* = -(a_p + a_m)$

Controle ideal : $\boxed{u^* = \theta^* y + r}$

Controle adaptativo: $\boxed{u = \theta y + r}$

Erro de saída: $\boxed{e = y - y_m}$

Equação do erro (dinâmica de $e(t)$):

$$
\begin{aligned}
e(t+1) &= y(t+1) - y_m(t+1) \\
&= \big(a_p\, y + u\big) - \big(-a_m y_m + r\big) + \big(a_m y\big) - \big(a_m y\big) \\
&= -a_m \underbrace{\big(y - y_m\big)}_{e} + \underbrace{\big(a_p + a_m\big)}_{-\theta^*} y - r + u \\
&= -a_m e - \theta^* y + \theta y
\end{aligned}
$$

Portanto,

$$\boxed{e(t+1) = -a_m e(t) + \big[\tilde{\theta} y\big]} \quad \text{ou} \quad \boxed{e = M(z)\big[\tilde{\theta} y\big]}$$

★ Similar ao caso contínuo.

Como no caso contínuo, define-se o sinal filtrado

$$\boxed{\zeta = M(z)y = \frac{1}{z + a_m} y}$$

Pode-se reescrever a equação do erro como

$$\begin{aligned} e &= M\big[\theta y - \theta^* y\big] \\ &= M\big[\theta y\big] - \theta^* M\big[y\big] \\ &= M\big[\theta y\big] - \theta^* \zeta \end{aligned}$$

Estimador: $\boxed{\hat{e} = M\big[\theta y\big] - \theta \zeta}$ θ $=$ estimativa de θ^*

Erro de estima: $\boxed{\varepsilon = e - \hat{e}}$

Que pode ser escrito como

$$\begin{aligned} \varepsilon &= e - \hat{e} \\ &= M\big[\theta y\big] - \theta^{*}\zeta - M\big[\theta y\big] + \theta\zeta \quad \Rightarrow \quad \boxed{\varepsilon = \tilde{\theta}\zeta} \end{aligned}$$

Função de Lyapunov: $V\big(\tilde{\theta}\big) = \gamma^{-1}\tilde{\theta}^{2}$

A variação é dada por

$$\begin{aligned} \Delta V(\tilde{\theta}) &= V\big(\tilde{\theta}(t+1)\big) - V\big(\tilde{\theta}(t)\big) \\ &= \gamma^{-1}\tilde{\theta}^{2}(t+1) - \gamma^{-1}\tilde{\theta}^{2}(t) \end{aligned}$$

Considere a seguinte lei de adaptação normalizada

$$\boxed{\theta(t+1) = \theta(t) - \frac{\gamma\varepsilon(t)\zeta(t)}{m^2(t)}}$$

com sinal normalizante

$$\boxed{m^2 = 1 + \zeta^2}$$

Portanto,

$$\begin{aligned}
\Delta V(\tilde{\theta}) &= \gamma^{-1}\tilde{\theta}^2(t+1) - \gamma^{-1}\tilde{\theta}^2(t) \\
&= \gamma^{-1}\big[\theta(t+1) - \theta^*\big]^2 - \gamma^{-1}\big[\theta(t) - \theta^*\big]^2 \\
&= \gamma^{-1}\left[\theta^2(t+1) - 2\theta(t+1)\theta^* + \theta^{*2}\right] - \gamma^{-1}\left[\theta^2(t) - 2\theta(t)\theta^* + \theta^{*2}\right]
\end{aligned}$$

Usando

$$\theta^2(t+1) = \theta^2 - 2\frac{\gamma\varepsilon\zeta}{m^2}\theta + \left(\frac{\gamma\varepsilon\zeta}{m^2}\right)^2$$

e simplificando a notação, tem-se que

$$\begin{aligned}
\Delta V(\tilde{\theta}) &= \gamma^{-1}\left[-2\frac{\gamma\varepsilon\zeta}{m^2}\theta + \left(\frac{\gamma\varepsilon\zeta}{m^2}\right)^2 - 2\left(\theta - \frac{\gamma\varepsilon\zeta}{m^2}\right)\theta^*\right] + 2\gamma^{-1}\theta\theta^* \\
&= -2\frac{\varepsilon\zeta}{m^2}\theta + \gamma^{-1}\left(\frac{\gamma\varepsilon\zeta}{m^2}\right)^2 + 2\frac{\varepsilon\zeta}{m^2}\theta^* \\
&= \gamma^{-1}\left(\frac{\gamma\varepsilon\zeta}{m^2}\right)^2 - 2\frac{\varepsilon\zeta}{m^2}\tilde{\theta} \\
&= \left(\frac{\gamma\zeta^2}{m^2} - 2\right)\frac{\varepsilon^2}{m^2}
\end{aligned}$$

Lembrando que $\dfrac{\zeta^2}{m^2} < 1$,

$$\Rightarrow \quad \boxed{\dot{V} \leq -\alpha_1 \frac{\varepsilon^2}{m^2}} \leq 0 \quad \text{para} \quad \boxed{0 < \gamma < 2}$$

★ Note que neste caso (discreto), o ganho de adaptação é limitado.

Conclusão:

- $V(\tilde{\theta})$ não cresce $\quad \Rightarrow \quad \boxed{\tilde{\theta} \in \mathcal{L}_\infty}$ e $\boxed{\theta \in \mathcal{L}_\infty}$

- $\varepsilon = \tilde{\theta}\zeta \quad \Rightarrow \quad \dfrac{\varepsilon}{m} = \tilde{\theta}\underbrace{\dfrac{\zeta}{m}}_{<1} \quad \Rightarrow \quad \boxed{\dfrac{\varepsilon}{m} \in \mathcal{L}_\infty}$

- $\Delta\theta = \theta(t+1) - \theta(t) = -\gamma \underbrace{\frac{\varepsilon}{m}}_{\mathcal{L}_\infty} \underbrace{\frac{\zeta}{m}}_{<1} \quad \Rightarrow \quad \boxed{\Delta\theta \in \mathcal{L}_\infty}$

- $\theta(t) = \theta(0) + \sum_{t=0}^{t} \Delta\theta < \infty \quad \Rightarrow$

$$\Rightarrow \quad |\theta(t)| \leq |\theta(0)| + \sum_{t=0}^{t} |\Delta\theta| < \infty, \ \forall t \quad \Rightarrow \quad \boxed{\sum_{t=0}^{\infty} |\Delta\theta| < \infty}$$

- $\sum_{t=0}^{\infty} \Delta\theta = -\gamma \sum_{t=0}^{\infty} \frac{\varepsilon}{m} \underbrace{\frac{\zeta}{m}}_{<1} < \infty \quad \Rightarrow \quad \boxed{\sum_{t=0}^{\infty} \frac{\varepsilon}{m} < \infty}$

- $\Delta V \leq 0 \quad \Rightarrow \quad V(t) \leq V(0) < \infty$

- $V(t) = V(0) + \sum \Delta V \leq V(0) < \infty \quad \Rightarrow \quad \sum_{t=0}^{\infty} |\Delta V| < \infty$

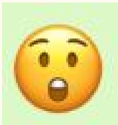
- $\Delta V \leq -\alpha_1 \dfrac{\varepsilon^2}{m^2} \quad \Rightarrow \quad \sum_{t=0}^{\infty} \dfrac{\varepsilon^2}{m^2} d\tau < \infty \quad \Rightarrow \quad \boxed{\dfrac{\varepsilon^2}{m^2} \in \mathcal{L}_\infty}$ e $\boxed{\dfrac{\varepsilon}{m} \in \mathcal{L}_2}$

★ É possível mostrar com estes resultados que $\boxed{y \to y_m}$

A demonstração é trabalhosa !

2.2.2 MRAC indireto discreto

★ Algoritmo similar ao caso contínuo.

Exercício! 

2.2.3 Problemas & exercícios

Exercício. ...

2.3 Planta com 1 parâmetro desconhecido (k_p)

A planta ainda é bastante simples: $n = 1$ (ordem)
$n^* = 1$ (grau relativo)
$n_p = 1$ (# de parâmetros)

Apenas o ganho de alta frequência k_p é desconhecido.

- Planta : $\boxed{\dot{y} = -y + k_p u}$ $\quad y = \dfrac{k_p}{s+1}u$ $\quad k_p$ desconhecido !
- Modelo : $\boxed{\dot{y}_m = -y_m + r}$ $\quad y_m = \dfrac{1}{s+1}r$

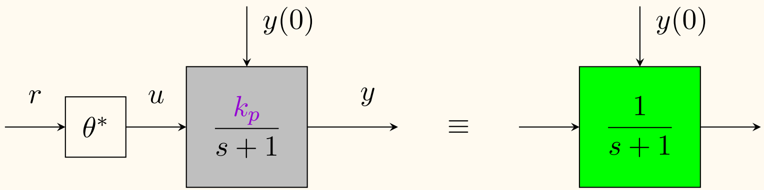
★ Neste caso, o problema pode ser resolvido utilizando apenas *feedforward*. Não é necessário *feedback*.

2.3.1 MRAC DIRETO

Planta : $\boxed{\dot{y} = -y + k_p u}$ $\quad y = \dfrac{k_p}{s+1} u$ $\quad$ k_p desconhecido !

Modelo : $\boxed{\dot{y}_m = -y_m + r}$ $\quad y_m = \dfrac{1}{s+1} r$

Se k_p fosse conhecido, então



Lei de *controle ideal*: $\boxed{u^* = \theta^* r}$ $\boxed{\theta^* = \dfrac{1}{k_p}}$

Lei de controle: $\boxed{u = \theta r}$

Erro de saída: $\boxed{e = y - y_m}$

Equação do erro:

$$\begin{aligned}
\dot{e} &= \dot{y} - \dot{y}_m \\
&= \big(-y + k_p u \big) - \big(-y_m + r \big) \\
&= -(y - y_m) + k_p \Big(u - \frac{1}{k_p} r \Big) \\
&= -e + k_p \big(u - \theta^* r \big) \\
&= -e + k_p \big(u - u^* \big)
\end{aligned}$$

Erro paramétrico: $$\boxed{\tilde{\theta} = \theta - \theta^*}$$

A equação do erro pode ser escrita como:

$$\begin{aligned}\dot{e} &= -e + k_p\big[\theta r - \theta^* r\big] \\ &= -e + k_p\tilde{\theta} r\end{aligned}$$

Ou ainda,

$$\boxed{e = M(s)k_p\big[\tilde{\theta} r\big]}$$

★ Note a presença de k_p na equação do erro.

Resumo das equações do sistema (em termos de erros):

$$\begin{cases} \dot{e} = -e + k_p \tilde{\theta} r \\ \dot{\theta} = ? \end{cases}$$

Função de Lyapunov:

$$2V(e, \tilde{\theta}) = e^2 + \gamma^{-1} \tilde{\theta}^2$$

Derivando,

$$\begin{aligned} \dot{V} &= e\dot{e} + \gamma^{-1} \tilde{\theta}\dot{\theta} \\ &= -e^2 + k_p \tilde{\theta} e r + \gamma^{-1} \tilde{\theta}\dot{\theta} \\ &= -e^2 + \gamma^{-1} \tilde{\theta} \big[\dot{\theta} + \gamma k_p e r\big] \end{aligned}$$

Qual é o problema?!

Nova função de Lyapunov:

$$2V(e,\tilde{\theta}) = e^2 + \gamma^{-1}\,|k_p|\,\tilde{\theta}^2 \;\; > 0$$

Derivando,

$$\begin{aligned}\dot{V} &= e\dot{e} + \gamma^{-1}\,|k_p|\tilde{\theta}\dot{\theta}\\ &= -e^2 + k_p\tilde{\theta}er + \gamma^{-1}\,|k_p|\tilde{\theta}\dot{\theta}\\ &= -e^2 + \gamma^{-1}\,|k_p|\tilde{\theta}\Big[\dot{\theta} + \gamma\,\mathrm{sign}(k_p)er\Big]\end{aligned}$$

Escolhemos: $\boxed{\dot{\theta} = -\gamma\ \mathrm{sign}(k_p)er}$

Resultado: $\dot{V} = -e^2 \le 0 \quad\Rightarrow\quad \boxed{e(t)\,,\ \theta(t) \in \mathcal{L}_\infty}$

Hipótese fundamental. É necessário o conhecimento de $\mathrm{sign}(k_p)$.

- ★ É interessante notar que o único mecanismo para instabilização neste caso é quando $\theta \to \infty$.
- ★ A planta é estável.

Summary of the algorithm

Subsystem	Equation	Order
Plant	$\dot{y} = -a_m y + k_p u$	1
Model	$\dot{y}_m = -a_m y_m + r$	1
Tracking error	$e_0 = y - y_m$	
Control law	$u = \theta\, r$	
Update law	$\dot{\theta} = -\gamma \ \mathrm{sign}(k_p) e r$	1
	System order =	3

Simulation results

Simulation 1

Initial conditions ......: $y(0) = 2$

$y_m(0) = 0$

$\theta(0) = 0$

Parameters ....: $a_m = 1$

$k_p = 0.5$

$\gamma = 10, 100$

Reference signal ....: $r = 1$

Matching ....: $\boxed{\theta^* = 2}$

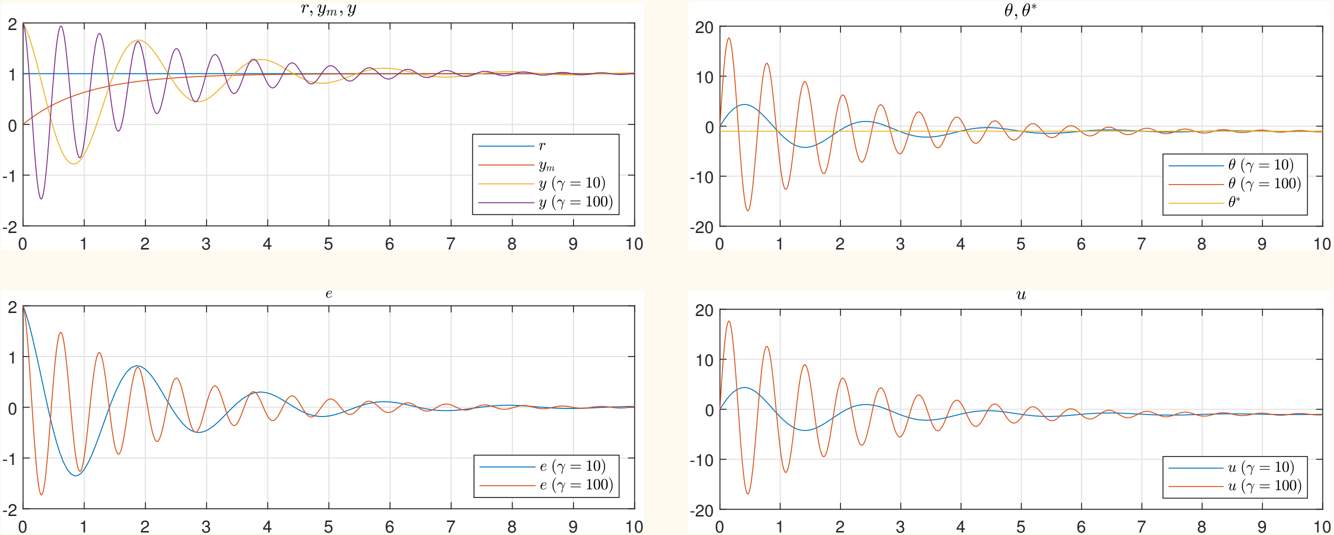


Figura 19: Direct MRAC. Case: $\gamma = 10, 100$.

*** To be checked. $\theta \not\to -2$

★ Simulations are performed with the Matlab ODE package.

2.3.2 Solução analítica

Planta : $\boxed{\dot{y} = -a_m y + k_p u}$ $\quad y = \dfrac{k_p}{s + a_m} u \,, \qquad k_p > 0$

Modelo : $\boxed{\dot{y}_m = -a_m y_m + k_m r}$ $\quad y_m = \dfrac{1}{s + a_m} r$

Controle : $\boxed{u = \theta r}$

Adaptação : $\boxed{\dot{\theta} = -\gamma e r}$

Sinal de referência : $\boxed{r = 1}$ (Simplificação !!)

★ A solução analítica mostra o efeito de γ no comportamento transitório.

Como $r = 1$, a dinâmica é regulada pelo sistema linear

$$\begin{cases} \dot{e} = -a_m e + k_p \tilde{\theta} \\ \dot{\theta} = -\gamma e \end{cases}$$

Aplicando Transformada de Laplace,

$$\begin{cases} se - e(0) = -a_m e + k_p \tilde{\theta} \\ s\tilde{\theta} - \tilde{\theta}(0) = -\gamma e \end{cases}$$

Resolvendo para e, tem-se

$$se + a_m e = e(0) + k_p \tilde{\theta} \qquad (\times s)$$

$$s^2 e + a_m se = e(0)s + k_p s\tilde{\theta}$$

$$s^2 e + a_m se = e(0)s + k_p \big(\tilde{\theta}(0) - \gamma e\big)$$

$$\big(s^2 + a_m s + k_p \gamma\big)e = e(0)s + k_p \tilde{\theta}(0)$$

Porém,

$$\begin{aligned} k_p\,\tilde{\theta}(0) &= k_p\,\big(\theta(0) - \theta^*\big) \\ &= k_p\,\theta(0) - k_p\frac{1}{k_p} \\ &= k_p\,\theta(0) - 1 \end{aligned}$$

Então,

$$\big(s^2 + a_m s + k_p\gamma\big)e = e(0)s + k_p\,\theta(0) - 1$$

Resultado: $$\boxed{e = \frac{e(0)s + k_p\,\theta(0) - 1}{s^2 + a_m s + k_p\gamma}}$$ $$\big(\theta(0) = \theta^* \;\Rightarrow\; e = 0\big)$$

- ★ O ganho γ só influencia a frequência de oscilação.
- ★ A atenuação é definida pelo parâmetro a_m do modelo.
- ★ Note que o sistema oscila mesmo para $e(0) = 0$.

2.3.3 MRAC INDIRETO

Planta : $\boxed{\dot{y} = -y + k_p u}$ $\quad y = \dfrac{k_p}{s+1} u$ $\quad$ k_p desconhecido !

Modelo : $\boxed{\dot{y}_m = -y_m + r}$ $\quad y_m = \dfrac{1}{s+1} r$

Lei de *controle ideal*: $\boxed{u^* = \theta^* r}$ $\quad \boxed{\theta^* = \dfrac{1}{k_p}}$

Lei de controle: $\boxed{u = \theta r}$

Estima de k_p : $\boxed{\hat{k}_p}$ $\quad \Rightarrow \quad \boxed{\theta = \dfrac{1}{\hat{k}_p}}$

A planta pode ser escrita como: $y = k_p \underbrace{M(s)u}_{u_f}$ (*)

Filtro: $u_f = M(s)u = \dfrac{1}{s+1}u$

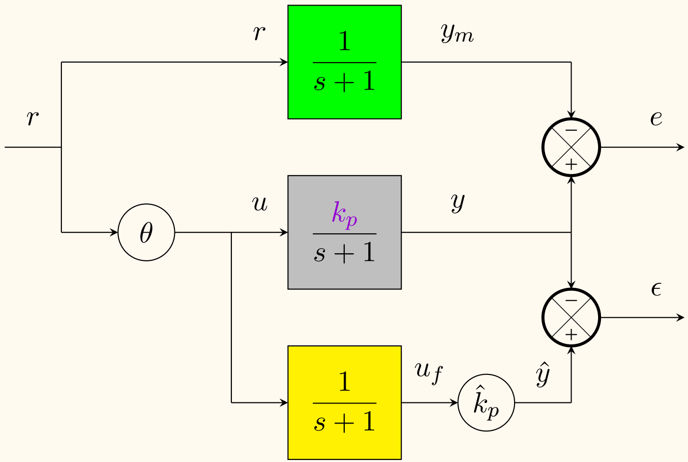
Predição: $\hat{y} = \hat{k}_p\, M(s)u \quad \Rightarrow \quad \boxed{\hat{y} = \hat{k}_p u_f}$

Erro de predição:

$$
\begin{aligned}
\varepsilon &= \hat{y} - y \\
&= \hat{k}_p u_f - k_p u_f \\
&= \tilde{k}_p u_f
\end{aligned}
$$

(*) Desprezando-se as condições iniciais.

Diagrama de blocos do MRAC indireto:



Lei de adaptação: $\boxed{\dot{\hat{k}}_p = -\gamma \varepsilon u_f}$

Análise de estabilidade

Função de Lyapunov: $2V\left(\tilde{k}_p\right) = \tilde{k}_p^2$

Derivando,

$$\dot{V} = \tilde{k}_p \dot{\hat{k}}_p = -\gamma \tilde{k}_p \varepsilon u_f = -\gamma \varepsilon^2 \leq 0$$

Conclusão: $\boxed{\tilde{k}_p \, , \; \hat{k}_p \in \mathcal{L}_\infty}$ e $\boxed{\varepsilon \in \mathcal{L}_2}$

Dúvida: Não é necessário conhecer $\mathrm{sign}(k_p)$?

★ De fato! Não é necessário conhecer $\mathrm{sign}(k_p)$ para identificação.

★ O conhecimento de $\mathrm{sign}(k_p)$ é necessário somente para estabilização.

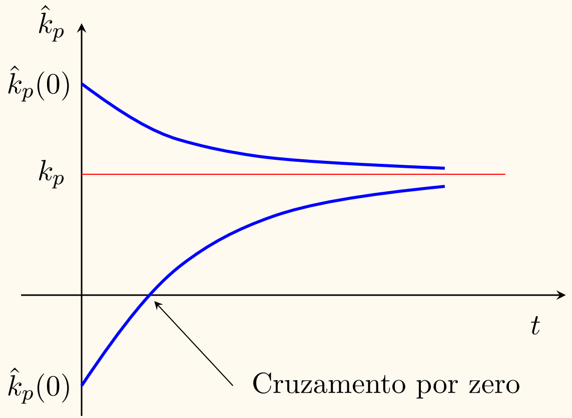
O controle é dado por : $\boxed{u = \frac{1}{\hat{k}_p} r}$

★ $\hat{k}_p(t)$ não pode cruzar por zero!

A lei de adaptação pode ser escrita como:

$$\dot{\hat{k}}_p = -\gamma \varepsilon u_f = -\underbrace{\gamma u_f^2}_{\geq 0} \tilde{k}_p$$

Portanto, $\boxed{\tilde{k}_p \to 0} \quad \Rightarrow \quad \boxed{\hat{k}_p \to k^*}$ monotonicamente.



★ O cruzamento de $\hat{k}_p$ por zero deve ser evitado.

Solução: Inicializar $\hat{k}_p$ do lado correto!

★ É necessário o conhecimento de $\mathrm{sign}(k_p)$. (!!)

Condição de identificabilidade

Referência: [Sastry & Bodson:1989], (pag. 15)

A solução do sistema linear variante no tempo

$$\dot{\tilde{k}}_p = -\gamma u_f^2 \tilde{k}_p \qquad (\dot{\tilde{k}}_p = \dot{\hat{k}}_p)$$

é dada por

$$\tilde{k}_p(t) = \tilde{k}_p(0)\, \exp\Big(-\gamma \int_0^t u_f^2(\tau) d\tau \Big)$$

Portanto, se $\boxed{\int_0^t u_f^2(\tau) d\tau \to \infty}$ então $\boxed{\tilde{k}_p(t) \to 0}$

★ Esta condição depende de u_f (sinal interno que depende de θ e r).

Pontos que serão abordados mais adiante:

- Persistent excitation (PE) : condição sobre o sinal u_f para que ocorra a identificação.
- Richness : condição sobre o sinal r para que u_f seja PE.

2.3.4 Simulações

(...)

2.3.5 Problemas & exercícios

Exercício. Considere o seguinte caso em que a planta é instável.

Planta : $\boxed{\dot{y} = y + k_p u}$ $\quad y = \dfrac{k_p}{s-1} u$ **Apenas k_p desconhecido !**

Modelo : $\boxed{\dot{y}_m = -y_m + r}$ $\quad y_m = \dfrac{1}{s+1} r$

Obtenha os algoritmos de controle e de adaptação.

2.4 Planta com 2 parâmetros desconhecidos (a_p e k_p)

A planta ainda é bastante simples:

$$
\begin{aligned}
n &= 1 \quad &&\text{(ordem)}\\
n^* &= 1 \quad &&\text{(grau relativo)}\\
n_p &= 2 \quad &&\text{(\# de parâmetros)}
\end{aligned}
$$

Planta : $\boxed{\dot{y} = a_p y + k_p u}$ $\quad y = \dfrac{k_p}{s - a_p} u$ $\quad$ a_p e k_p desconhecidos !

Modelo : $\boxed{\dot{y}_m = -a_m y_m + k_m r}$ $\quad y_m = \dfrac{k_m}{s + a_m} r$

★ Neste caso, são necessários *feedback* e *feedforward*.

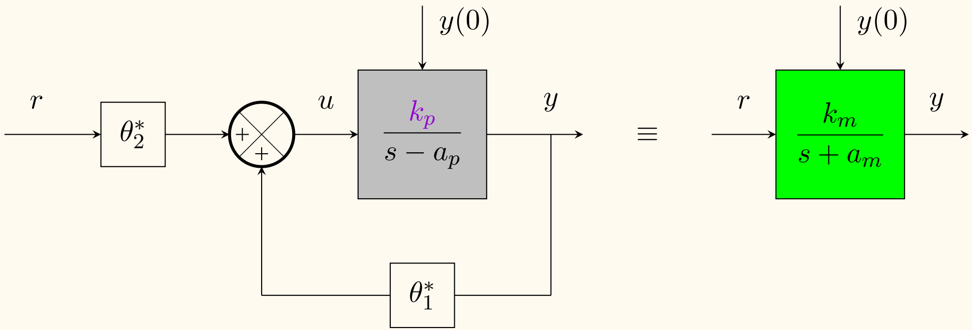
2.4.1 MRAC DIRETO

Exemplo 6 Ref.: [...]

Planta : $\boxed{\dot{y} = a_p y + k_p u}$ k_p e a_p desconhecidos !

Modelo : $\boxed{\dot{y}_m = -a_m y_m + k_m r}$

Lei de *controle ideal*: $\boxed{u^* = \theta_1^* y + \theta_2^* r}$



Verificação:

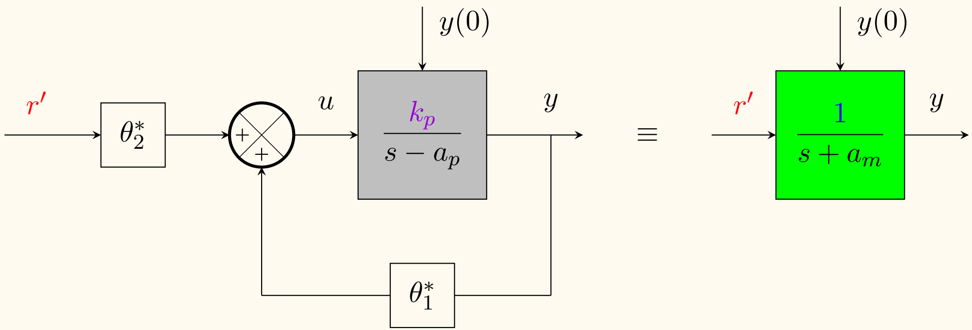
$$
\begin{aligned}
\dot{y} &= a_p y + k_p\left(\theta_1^* y + \theta_2^* r\right) \\
&= \big(\underbrace{a_p + k_p\theta_1^*}_{-a_m}\big)y + \underbrace{k_p\theta_2^*}_{k_m} r
\end{aligned}
$$

Matching gains : $\boxed{\theta_1^* = -\dfrac{a_p + a_m}{k_p}}$ $\boxed{\theta_2^* = \dfrac{k_m}{k_p}}$

Obs.: It is usual to assume $k_m = 1$.

Rewrite the reference model as

$$\dot{y}_m = -a_m y_m + \underbrace{k_m r}_{r'} = -a_m y_m + r'$$



Matching gains : $\boxed{\theta_1^* = -\dfrac{a_p + a_m}{k_p}}$ $\boxed{\theta_2^* = \dfrac{1}{k_p}}$

Forma vetorial: $u^* = \theta_1^* y + \theta_2^* r \quad \Rightarrow \quad \boxed{u^* = \theta^{*T} \omega}$

★ $\theta^* = \begin{bmatrix} \theta_1^* \\ \theta_2^* \end{bmatrix}$ e $\omega = \begin{bmatrix} y \\ r \end{bmatrix}$ (Vetores !!)

★ For convenience, the reference signal is written r.

Controle adaptativo: $u = \theta_1\, y + \theta_2 r \quad \Rightarrow \quad \boxed{u = \theta^T \omega}$

★ $\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$

Erro de saída: $$\boxed{e = y - y_m}$$

Equação do erro:

$$
\begin{aligned}
\dot{e} &= \dot{y} - \dot{y}_m \\
&= \big(a_p y + k_p u\big) - \big(-a_m y_m + r\big) + \big(a_m y\big) - \big(a_m y\big) \\
&= -a_m \underbrace{\big(y - y_m\big)}_{e} + \big(a_p + a_m\big) y + k_p u - r \\
&= -a_m e + k_p \Big[\underbrace{\frac{a_p + a_m}{k_p}}_{-\theta_1^*} y + u - \underbrace{\frac{1}{k_p}}_{\theta_2^*} r\Big] \\
&= -a_m e + k_p \Big[u - \theta_1^* y - \theta_2^* r\Big] \\
&= -a_m e + k_p \Big[u - u^*\Big]
\end{aligned}
$$

Portanto,

$$\boxed{\dot{e} = -a_m e + k_p \Big[u - u^* \Big]}$$

ou ainda,

$$\boxed{e = M(s) k_p \Big[u - u^* \Big]}$$

★ Note que $\boxed{k_p = \left(\theta_2^* \right)^{-1}}$

Controle: $$u = \theta_1 y + \theta_2 r = \begin{bmatrix}\theta_1 & \theta_2\end{bmatrix}\begin{bmatrix} y \\ r \end{bmatrix} = \theta^T \omega$$

Erro paramétrico: $$\begin{cases} \tilde{\theta}_1 = \theta_1 - \theta_1^* \\ \tilde{\theta}_2 = \theta_2 - \theta_2^* \end{cases} \Rightarrow \boxed{\tilde{\theta} = \theta - \theta^*}$$

A equação do erro pode ser escrita como:

$$\dot{e} = -a_m e + k_p[\tilde{\theta}^T \omega]$$

Ou ainda,

$$\boxed{e = M(s) k_p [\tilde{\theta}^T \omega]}$$

Resumo das equações do sistema (em termos de erros):

$$\begin{cases} \dot{e} = -a_m e + k_p\left[\tilde{\theta}_1 y + \tilde{\theta}_2 r\right] \\ \dot{\theta}_1 = ? \\ \dot{\theta}_2 = ? \end{cases}$$

Notação compacta (vetorial):

$$\begin{cases} \dot{e} = -a_m e + k_p\left[\tilde{\theta}^T \omega\right] \\ \dot{\theta} = ? \end{cases}$$

Função de Lyapunov:

$$
\begin{aligned}
2V(e,\tilde{\theta}_1,\tilde{\theta}_2) &= e^2 + \gamma_1^{-1}\,|k_p|\,\tilde{\theta}_1^2 + \gamma_2^{-1}\,|k_p|\,\tilde{\theta}_2^2 \\
&= e^2 + |k_p|\,\tilde{\theta}^T\Gamma^{-1}\tilde{\theta}
\end{aligned}
$$

Derivando,

$$
\begin{aligned}
\dot{V} &= e\dot{e} + |k_p|\left(\gamma_1^{-1}\tilde{\theta}_1\dot{\theta}_1 + \gamma_2^{-1}\tilde{\theta}_2\dot{\theta}_2\right) \\
&= -a_m e^2 + k_p\tilde{\theta}^T\omega e + |k_p|\,\tilde{\theta}^T\Gamma^{-1}\dot{\theta} \\
&= -a_m e^2 + |k_p|\,\tilde{\theta}^T\left[\,\mathrm{sign}(k_p)\omega e + \Gamma^{-1}\dot{\theta}\right]
\end{aligned}
$$

Escolhemos: $\boxed{\dot{\theta} = -\Gamma\,\mathrm{sign}(k_p)\omega e}$

Resultado: $\dot{V} = -a_m e^2 \leq 0$ negativa semi-definida !

Conclusão: $\boxed{e(t)\,,\ \theta(t) \in \mathcal{L}_\infty}$ e $\boxed{e(t) \in \mathcal{L}_2}$

Análise de convergência

Semelhante ao caso do exemplo 1 :

$$\left.\begin{array}{l} e(t) \in \mathcal{L}_2 \\ \dot{e}(t) \in \mathcal{L}_\infty \end{array}\right\} \Rightarrow \boxed{\lim_{t\to\infty} e(t) = 0}$$

Summary of the algorithm

Subsystem	Equation	Order
Plant	$\dot{y} = a_p y + k_p u$	1
Model	$\dot{y}_m = -a_m y_m + k_m r$	1
Tracking error	$e_0 = y - y_m$	
Regressor	$\omega = [y \quad r]^T$	
Control law	$u = \theta^T \omega$	
Update law	$\dot{\theta} = -\Gamma\,\mathrm{sign}(k_p)\omega e$	2
	System order $=$	4

Simulation results

Simulation #1 Null initial conditions.

Initial conditions......:
$$y(0) = \boxed{0}$$
$$y_m(0) = 0$$
$$\theta(0) = [0 \quad 0]^T$$

Parameters..............:
$$a_p = 2 \qquad a_m = 1 \qquad \Gamma = I$$
$$k_p = 0.5 \qquad k_m = 1$$

Reference signal.......:
$$r(t) = dc + a\sin(\omega t)$$
$$dc = 1$$
$$a = 1$$
$$\omega = 1$$

Matching parameter.: $\theta^* = [-6, \ 2] \quad , \quad k^* = 0.5 \quad , \quad \|\theta^*\| = 6.32$

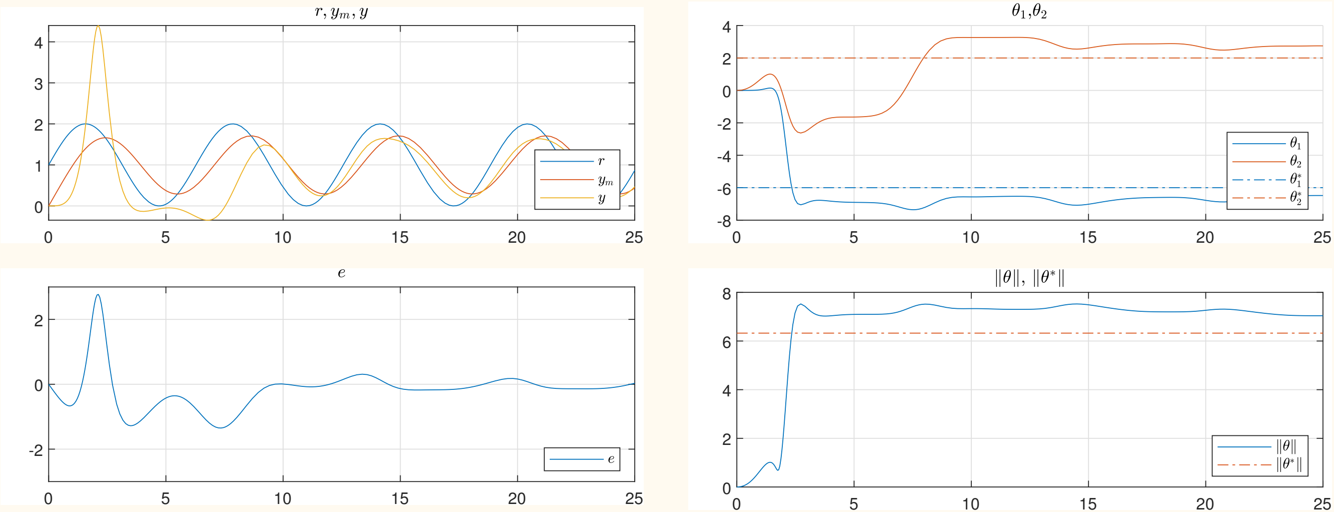


Figura 20: Resultado da simulação com algoritmo MRAC direto. (Script: `simu01.m`)

Simulação #2 Efeito da condição inicial pequena.

Initial conditions . . . . . . :

$$y(0) = 1$$
$$y_m(0) = 0$$
$$\theta(0) = [0 \quad 0]^T$$

Parameters . . . . :

$$a_p = 2 \qquad a_m = 1 \qquad \Gamma = I$$
$$k_p = 0.5 \qquad k_m = 1$$

Reference signal . . . . :

$$dc = 1$$
$$a = 1$$
$$\omega = 1$$

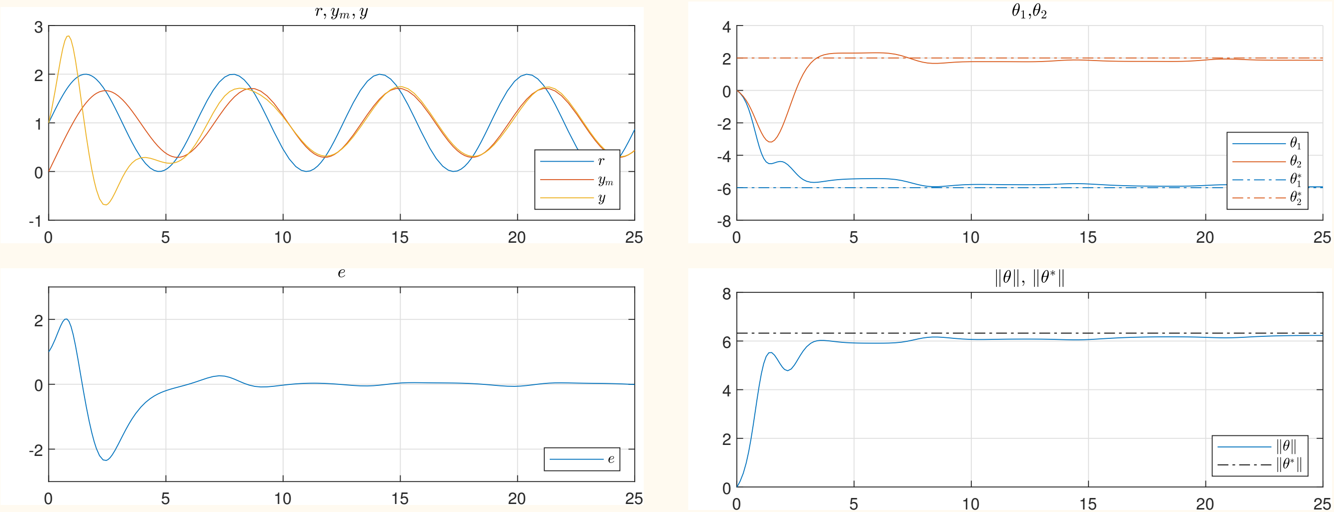


Figura 21: Resultado da simulação com algoritmo MRAC direto. (Script: `simu02.m`)

Simulação #3 Efeito da condição inicial grande.

Initial conditions......:
$$y(0) = 5$$
$$y_m(0) = 0$$
$$\theta(0) = [0 \quad 0]^T$$

Parameters..............:
$$a_p = 2 \qquad a_m = 1 \qquad \Gamma = I$$
$$k_p = 0.5 \qquad k_m = 1$$

Reference signal.......:
$$dc = 1$$
$$a = 1$$
$$\omega = 1$$

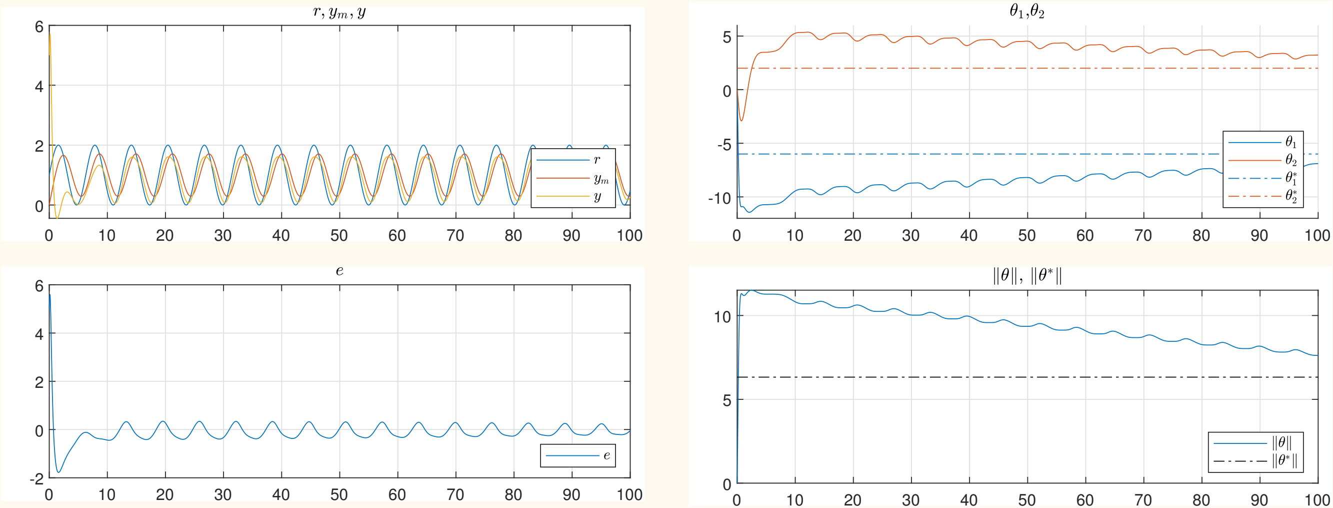


Figura 22: Resultado da simulação com algoritmo MRAC direto. (Script: `simu03.m`)

Simulação #4 Efeito do ganho de adaptação.

Initial conditions......: $y(0) = 2$

$y_m(0) = 0$

$\theta(0) = [0 \;\; 0]^T$

Parameters..............: $a_p = 2$ $\quad a_m = 1$ $\quad \Gamma = 10\,I$

$k_p = 0.5$ $\quad k_m = 1$

Reference signal.......: $dc = 1$

$a = 1$

$\omega = 1$

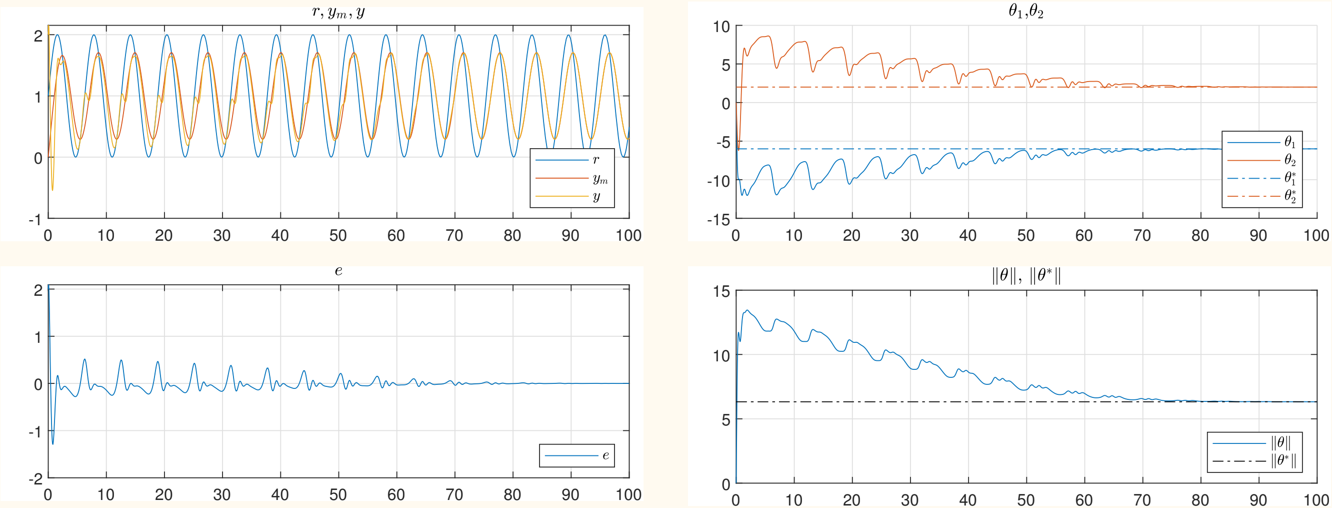


Figura 23: Resultado da simulação com algoritmo MRAC direto. (Script: `simu04.m`)

Simulação #5 Efeito do sinal DC.

Initial conditions......: $y(0) = 2$

$$y_m(0) = 0$$

$$\theta(0) = [0 \quad 0]^T$$

Parameters..............: $a_p = 2$ $\quad a_m = 1$ $\quad \Gamma = 10\,I$

$$k_p = 0.5 \qquad k_m = 1$$

Reference signal.......: $dc = \{1,\ 2,\ 3\}$

$$a = 1$$

$$\omega = 1$$

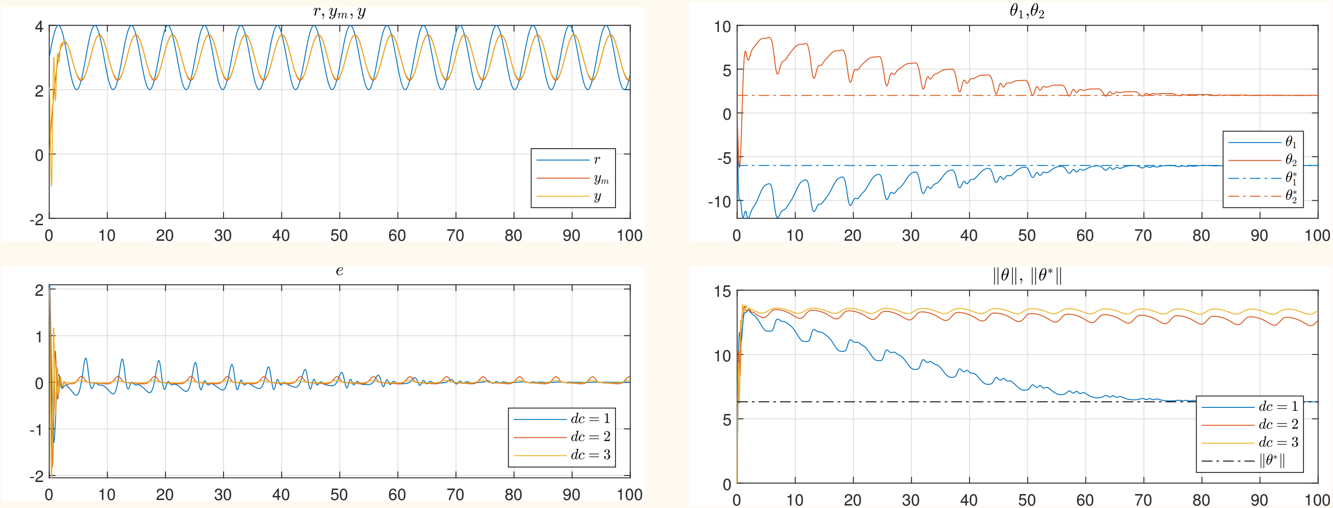


Figura 24: Resultado da simulação com algoritmo MRAC direto. (Script: `simu05.m`)

Resultados de simulações

- Mostrar convergência
- Mostrar não uniformidade em relação às condições iniciais

2.4.2 Problemas & exercícios

Exercício. ...

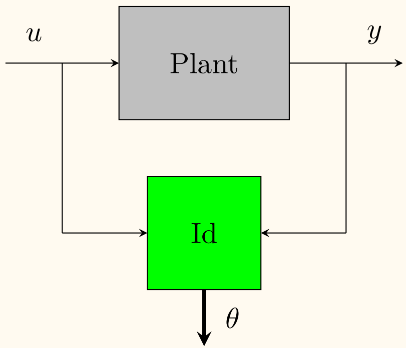
3 Identificação adaptativa de parâmetros

Contents

3.1 Introdução

Referência. [Tao:2003], (pag. 99)

Problema de identificação



- ★ θ não é realimentado na planta!
- ★ A planta pode ser instável! $\Rightarrow$ Necessidade de normalização do regressor.

3.2 Parametrização do modelo

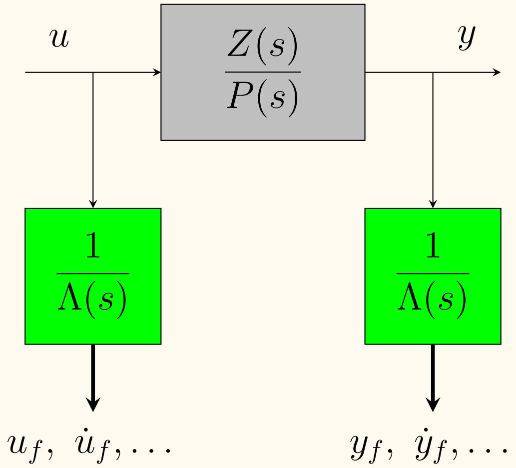
Planta : $\boxed{P(s)y = Z(s)u}$

onde : $P(s) = s^n + \cdots + p_1 s + p_0$

$Z(s) = z_m s^m + \cdots + z_1 s + z_0$

Filtros : $\boxed{\dfrac{1}{\Lambda(s)}}$ (grau n)

onde : $\Lambda(s) = s^n + \cdots + \lambda_1 s + \lambda_0$



Podemos escrever a equação da planta como:

$$\begin{aligned}\frac{P}{\Lambda}y &= \frac{Z}{\Lambda}u \\ &= \frac{Z}{\Lambda}u + \left(\frac{\Lambda}{\Lambda}y - y\right)\end{aligned}$$

ou melhor,

$$\boxed{y = \frac{Z}{\Lambda}u + \frac{\Lambda - P}{\Lambda}y} \quad \text{ou} \quad \boxed{y = Zu_f + (\Lambda - P)y_f}$$

★ $\mathrm{grau}(\Lambda - P) = n - 1$

★ O sinal $y(t)$ pode ser escrito em função de sinais filtrados.

Definimos o vetor de parâmetros

$$\theta^* = \begin{bmatrix} z_0 & z_1 & \cdots & z_m & (\lambda_0 - p_0) & (\lambda_1 - p_1) & \cdots & (\lambda_{n-1} - p_{n-1}) \end{bmatrix}^T$$

e o vetor regressor

$$\begin{aligned} \phi &= \begin{bmatrix} \frac{1}{\Lambda}u & \frac{s}{\Lambda}u & \cdots & \frac{s^m}{\Lambda}u & \frac{1}{\Lambda}y & \frac{s}{\Lambda}y & \cdots & \frac{s^{n-1}}{\Lambda}y \end{bmatrix}^T \\ &= \begin{bmatrix} u_f & \dot{u}_f & \ddot{u}_f & \cdots & y_f & \dot{y}_f & \ddot{y}_f & \cdots \end{bmatrix}^T \end{aligned}$$

★ Note que $\theta^*, \ \phi \in \mathbb{R}^{n+m+1}$.

Então,

$$\boxed{y = \theta^{*T}\phi}$$

Exemplo 7 Parametrização de um sistema de 2a. ordem.

Planta: $$\left(s^2 + p_1 s + p_0\right) y = \left(z_1 s + z_0\right) u$$

Filtros: $$\frac{1}{\Lambda(s)} = \frac{1}{s^2 + \lambda_1 s + \lambda_0} \quad \Rightarrow \quad \boxed{u_f = \frac{1}{\Lambda} u} \quad \text{e} \quad \boxed{y_f = \frac{1}{\Lambda} y}$$

Definindo

$$\theta^* = \begin{bmatrix} z_0 & z_1 & (\lambda_0 - p_0) & (\lambda_1 - p_1) \end{bmatrix}^T \in \mathbb{R}^4$$

$$\phi = \begin{bmatrix} u_f & \dot{u}_f & y_f & \dot{y}_f \end{bmatrix}^T \in \mathbb{R}^4$$

Então, $$\boxed{y = \theta^{*T} \phi}$$

3.2.1 Parametrização alternativa

- Na parametrização anterior, θ contem coeficientes de $Z(s)$ e de $\Lambda(s) - P(s)$.
- Na parametrização alternativa a seguir, definimos θ_p, que contem somente os coeficientes de $Z(s)$ e $P(s)$.

★ Um exemplo com um sistema de 2a. ordem ilustra o procedimento.

Exemplo 8 Parametrização alternativa de um sistema de 2a. ordem.

Plant: $\big(s^2 + p_1 s + p_0\big)y = \big(z_1 s + z_0\big)u$

Filter: $\dfrac{1}{\Lambda(s)} = \dfrac{1}{s^2 + \lambda_1 s + \lambda_0} \quad \Rightarrow \quad \boxed{u_f = \dfrac{1}{\Lambda}u}$ and $\boxed{y_f = \dfrac{1}{\Lambda}y}$

Applying the filter to both sides: $\big(s^2 + p_1 s + p_0\big)\dfrac{y}{\Lambda} = \big(z_1 s + z_0\big)\dfrac{u}{\Lambda}$

$$\Rightarrow \ \ddot{y}_f = \big(z_1 s + z_0\big)u_f - \big(p_1 s + p_0\big)y_f$$

Defining $\boxed{\theta_p^* = \begin{bmatrix} z_0 & z_1 & p_0 & p_1 \end{bmatrix}^T}$ $\boxed{\phi = \begin{bmatrix} u_f & \dot{u}_f & -y_f & -\dot{y}_f \end{bmatrix}^T}$

Then, $\boxed{\ddot{y}_f = \theta_p^{*T}\phi}$ Note that $\boxed{\ddot{y}_f = y - \big(\lambda_1 s + \lambda_0\big)y_f}$

3.2.2 Implementação dos filtros

★ Os filtros são idênticos.

Realização de estado:

$$\begin{cases} \dot{\omega}_1 = A\omega_1 + bu \\ \dot{\omega}_2 = A\omega_2 + by \end{cases}$$

Escolhemos

$$A = \begin{bmatrix} 0 & 1 \\ -\lambda_0 & -\lambda_1 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Portanto,

$$
\begin{aligned}
\omega_1 &= (sI - A)^{-1} b u \\
&= \begin{bmatrix} s & -1 \\ \lambda_0 & s + \lambda_1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
&= \frac{1}{\Lambda} \begin{bmatrix} s + \lambda_1 & 1 \\ -\lambda_0 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = \frac{1}{\Lambda} \begin{bmatrix} 1 \\ s \end{bmatrix} u = \begin{bmatrix} u_f \\ \dot{u}_f \end{bmatrix}
\end{aligned}
$$

Da mesma forma,

$$
\omega_2 = (sI - A)^{-1} b y = \begin{bmatrix} y_f \\ \dot{y}_f \end{bmatrix}
$$

Vetor regressor :

$$
\boxed{\phi = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}}
$$

★ Note que utiliza-se o mesmo filtro para u e y.

3.2.3 Adaptação

Seja $\theta(t)$ uma estimativa de θ^* obtida por um algoritmo adaptativo.

Estimativa de y : $\boxed{\hat{y} = \theta^T \phi}$ (Parametrização linear)

Erro de estimativa : $\boxed{\epsilon = \hat{y} - y}$

Erro paramétrico: $\boxed{\tilde{\theta} = \theta - \theta^*}$

Portanto,

$$\boxed{\epsilon = \tilde{\theta}^T \phi}$$

(Equação do erro)

3.3 Gradiente normalizado

Ideia: Atualizar a estimativa do parâmetro na direção do negativo do gradiente.

Função custo: $$\boxed{J(\theta) = \frac{\epsilon^2}{2\,m^2}}$$

Sinal normalizante: $$\boxed{m^2 = 1 + \kappa\,\phi^T\phi} \qquad \kappa > 0$$

★ Em geral, m é definido de forma a garantir que $\dfrac{\phi}{m} \in \mathcal{L}_\infty$.

Note que: $$J(\theta) = \frac{\epsilon^2}{2\,m^2} = \frac{\tilde{\theta}^T \phi \phi^T \tilde{\theta}}{2\,m^2}$$

Gradiente de J : $$\frac{\partial J}{\partial \theta} = \frac{2\phi\phi^T\tilde{\theta}}{2\,m^2} = \frac{\phi\epsilon}{m^2}$$

Outra forma de obtenção: $$\frac{\partial J}{\partial \theta} = \frac{\partial J}{\partial \epsilon}\,\frac{\partial \epsilon}{\partial \theta} = \frac{\epsilon}{m^2}\,\phi$$

Algoritmo do Gradiente: $$\boxed{\dot{\theta} = -\Gamma\,\frac{\phi\epsilon}{m^2}} \qquad \Gamma > 0$$

Lema. Estabilidade do algoritmo do Gradiente:

- $\boxed{\theta \in \mathcal{L}_\infty}$
- $\boxed{\dot{\theta},\ \frac{\epsilon}{m} \in \mathcal{L}_\infty \bigcap \mathcal{L}_2}$

Prova. [Tao:2003], (pag. 103)

Função de Lyapunov: $2V\left(\tilde{\theta}\right) = \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$

Derivando,

$$\dot{V}\left(\tilde{\theta}\right) = \tilde{\theta}^T \Gamma^{-1} \dot{\theta} = \tilde{\theta}^T \Gamma^{-1} \left(-\frac{\Gamma \phi \epsilon}{m^2}\right) = -\frac{\epsilon^2}{m^2} = -\frac{(\tilde{\theta}^T \phi)^2}{m^2} \leq 0$$

★ Porque não é definida negativa?!

Conclusões:

- $\tilde{\theta} \in \mathcal{L}_\infty$

- $\dfrac{\epsilon}{m} \in \mathcal{L}_2$

- $\dfrac{|\epsilon|}{m} \leq \|\tilde{\theta}\| \underbrace{\dfrac{\|\phi\|}{m}}_{<1} \quad \Rightarrow \quad \boxed{\dfrac{|\epsilon|}{m} \in \mathcal{L}_\infty}$

- $\dot{\theta} = -\Gamma \underbrace{\dfrac{\phi}{m}}_{<1} \underbrace{\dfrac{\epsilon}{m}}_{\mathcal{L}_2 \cap \mathcal{L}_\infty} \quad \Rightarrow \quad \boxed{\dot{\theta} \in \mathcal{L}_2 \cap \mathcal{L}_\infty}$

■

Note que

$$\dot{\theta} = -\Gamma \frac{\phi\epsilon}{m^2} = -\underbrace{\frac{\Gamma\,\phi\phi^T}{m^2}}_{A(t)}\tilde{\theta} = -A(t)\,\tilde{\theta}$$

onde

$$\boxed{A(t) = A^T(t) \geq 0} \qquad \operatorname{rank}\big(A(t)\big) = 1$$

★ Quer dizer, $\tilde{\theta} = 0$ é um ponto de equilíbrio unif. estável !!

Importante:

- $\tilde{\theta} \equiv 0 \qquad \Rightarrow \qquad \epsilon = 0$

- $\epsilon = \tilde{\theta}^T \phi \equiv 0 \qquad \not\Rightarrow \qquad \tilde{\theta} = 0$

  $\qquad\qquad\qquad \Rightarrow \qquad \tilde{\theta} \perp \phi \quad (!!)$

- $\begin{cases} \dot{\theta} \in \mathcal{L}_2 \cap \mathcal{L}_\infty \\ \text{“Se”}\ \ddot{\theta} \in \mathcal{L}_\infty \end{cases} \quad \Rightarrow \quad \lim_{t\to\infty} \dot{\theta} = 0 \qquad \not\Rightarrow \qquad \theta \to \text{constante} \quad (!!)$

★ Ver exemplos em [Slotine & Li:1991], (pag. 122) .

3.3.1 Simulações

Case 1 First order plant, one unknown parameter.

Plant .... : $P(s) = \dfrac{N(s)}{D(s)} = \dfrac{1}{\tau s + 1}$

Filter .... : $\dfrac{1}{\Lambda(s)} = \dfrac{1}{s + \lambda}$

★ As simulações utilizam o pacote ODE do Matlab.

Resumo das equações

Planta.....................: $\dot{y} = (1/\tau)(-y + u) \quad \Rightarrow \quad \tau\dot{y} = -y + u$

Parametrização 1 ....: $\underbrace{\dot{y}_f}_{Y_1} = \underbrace{(1/\tau)}_{\theta_1^*}\underbrace{(-y_f + u_f)}_{\phi_1}$

Parametrização 2 ....: $\underbrace{-y_f + u_f}_{Y_2} = \underbrace{(\tau)}_{\theta_2^*}\underbrace{(\dot{y}_f)}_{\phi_2}$

Estimativa...............: $\hat{Y}_i = \theta_i\phi_i\,, \qquad (i = 1, 2)$

Erro.......................: $\epsilon_i = \hat{Y}_i - Y_i$

Normalização...........: $m_i^2 = 1 + \kappa_i\,\phi_i^2$

Lei de adaptação .....: $\dot{\theta}_i = -\gamma_i\,\dfrac{\phi_i\epsilon_i}{m_i^2}$

Simulation 1 Zero initial conditions, small gains.

Algorithm : **Normalized Gradient**

Parameters . . . . :

$$\tau = 0.7 \qquad \gamma_1 = 1 \qquad \kappa_1 = 0$$
$$\lambda = 1 \qquad \gamma_2 = 1 \qquad \kappa_2 = 0$$

Initial conditions . . . . :

$$y(0) = 0$$
$$\theta_1(0) = 0$$
$$\theta_2(0) = 0$$

Input signal . . . . :

$$r = 1 + 2\,\mathrm{sign}\big(\sin(0.1\,\pi\,t)\big)$$

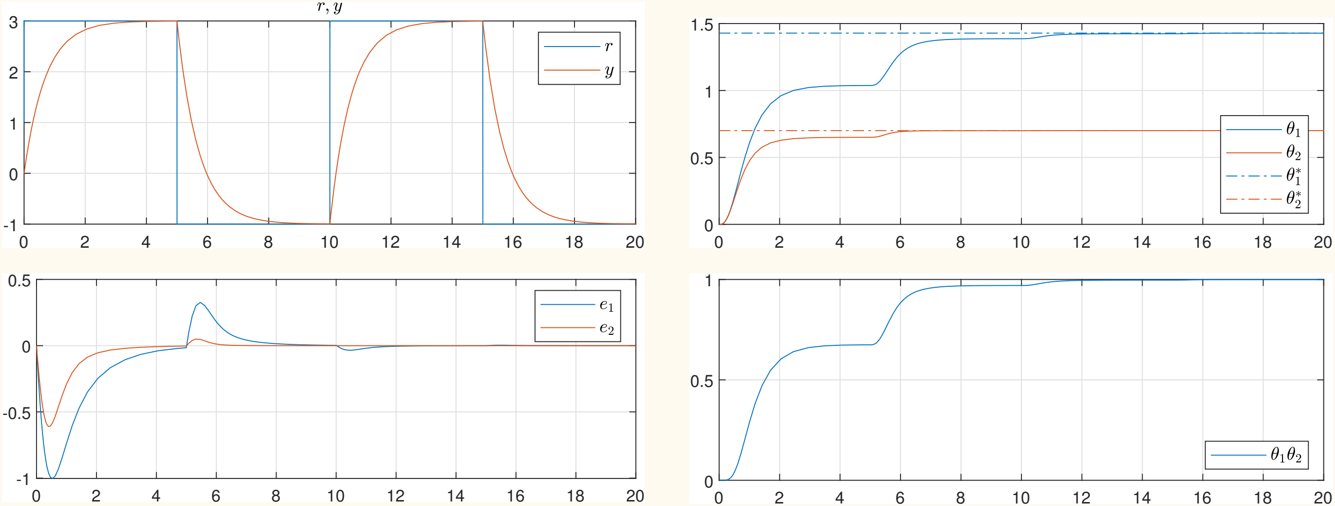


Figura 25: Normalized Gradient. Plant-11.

Simulation 2 Zero initial conditions, large gains.

Algorithm : **Normalized Gradient**

Parameters . . . . :

$$\tau = 0.7 \qquad \gamma_1 = 10 \qquad \kappa_1 = 0$$
$$\lambda = 1 \qquad \gamma_2 = 10 \qquad \kappa_2 = 0$$

Initial conditions . . . . . :

$$y(0) = 0$$
$$\theta_1(0) = 0$$
$$\theta_2(0) = 0$$

Input signal . . . . :

$$r = 1 + 2\,\mathrm{sign}\big(\sin(0.1\,\pi\,t)\big)$$

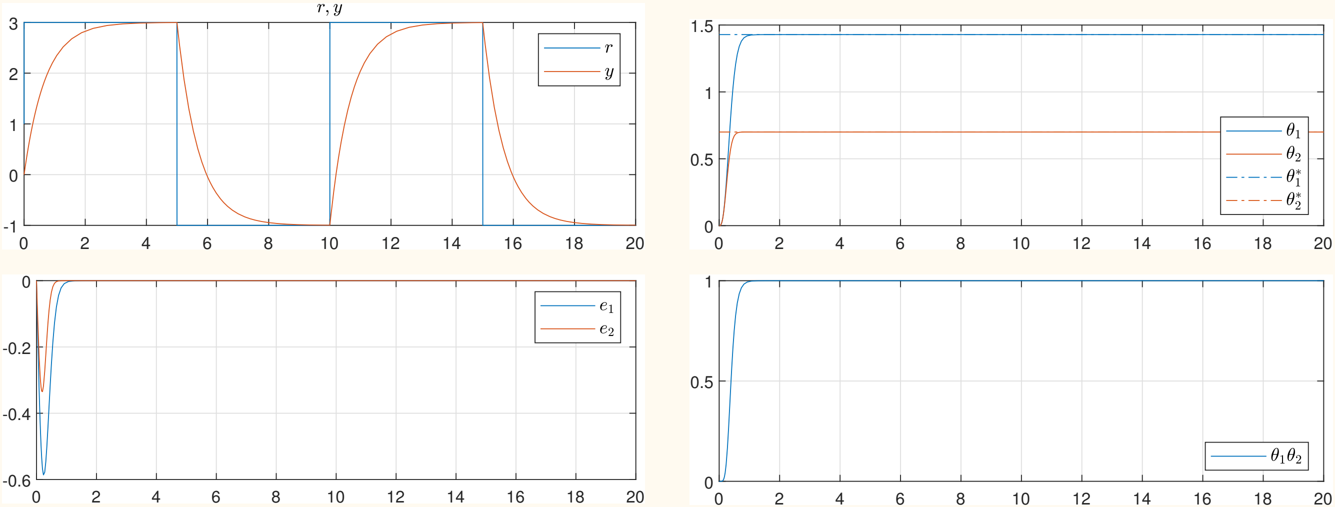


Figura 26: Normalized Gradient. Plant-11.

Simulation 3 Small initial conditions, large gains.

Algorithm : Normalized Gradient

Parameters . . . . : $\tau = 0.7$ $\gamma_1 = 10$ $\kappa_1 = 0$

$\lambda = 1$ $\gamma_2 = 10$ $\kappa_2 = 0$

Initial conditions . . . . . : $y(0) = 5$

$\theta_1(0) = 0$

$\theta_2(0) = 0$

Input signal . . . . : $r = 1 + 2\,\mathrm{sign}\big(\sin(0.1\,\pi\,t)\big)$

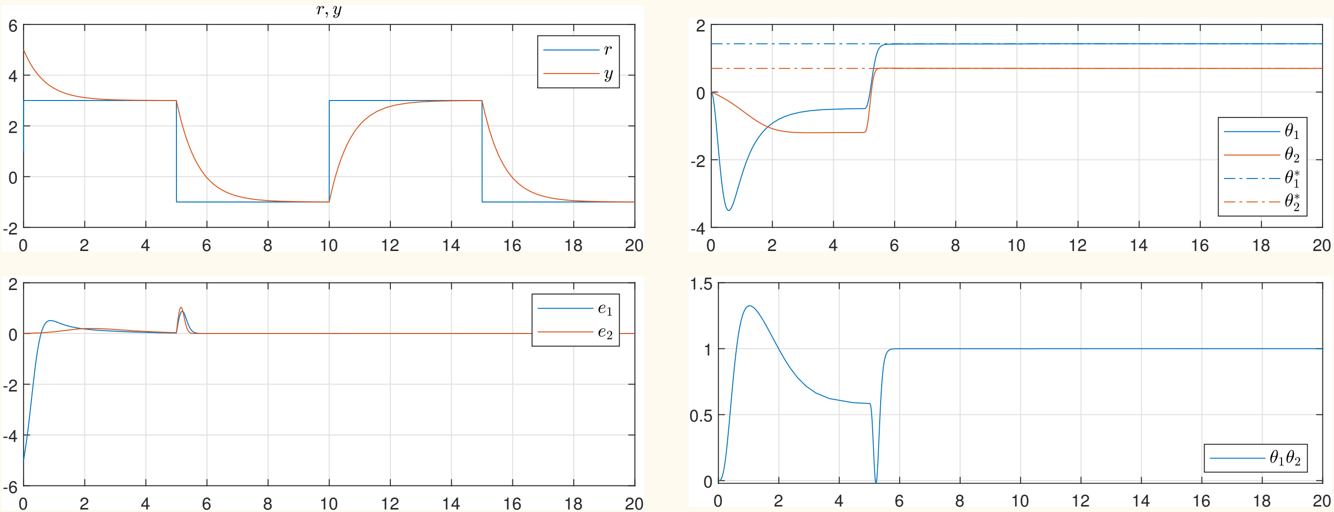


Figura 27: Normalized Gradient. Plant-11.

Simulation 4 Effect of normalization.

Algorithm : Normalized Gradient

Parameters . . . . : $\tau = 0.7$ $\gamma_1 = 10$ $\kappa_1 = 10$

$\lambda = 1$ $\gamma_2 = 10$ $\kappa_2 = 10$

Initial conditions . . . . : $y(0) = 5$

$\theta_1(0) = 0$

$\theta_2(0) = 0$

Input signal . . . . : $r = 1 + 2\,\mathrm{sign}\big(\sin(0.1\,\pi\,t)\big)$

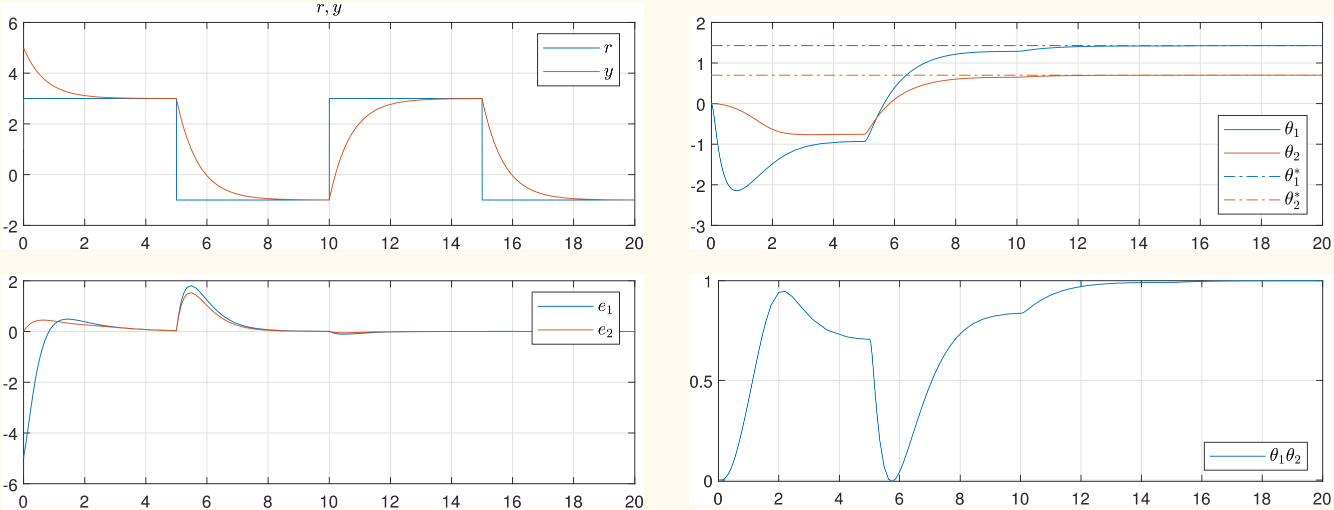


Figura 28: Normalized Gradient. Plant-11.

Case 2 First order plant, 2 unknown parameters.

Plant ....: $P(s) = \dfrac{N(s)}{D(s)} = \dfrac{b_0}{s + a_0}$

Filter ....: $\dfrac{1}{\Lambda(s)} = \dfrac{1}{s + \lambda}$

★ As simulações utilizam o pacote ODE do Matlab.

Resumo das equações

Planta.....................: $\dot{y} = -a_0 y + b_0 u$

Filtro.......................: $\Lambda(s) = s + \lambda$

Parametrização 1 ....: $\underbrace{y}_{Y_1} = b_0 u_f + (\lambda - a_0) y_f = \begin{bmatrix} b_0 & (a_0 - \lambda) \end{bmatrix} \underbrace{\begin{bmatrix} u_f \\ -y_f \end{bmatrix}}_{\phi_1}$

Parametrização 2 ....: $\underbrace{\dot{y}_f}_{Y_2} = b_0 u_f - a_0 y_f = \begin{bmatrix} b_0 & a_0 \end{bmatrix} \underbrace{\begin{bmatrix} u_f \\ -y_f \end{bmatrix}}_{\phi_2}$

Parametrização 3 ....: $\underbrace{u_f}_{Y_3} = (1/b_0) \dot{y}_f + (a_0/b_0) y_f = \begin{bmatrix} 1/b_0 & a_0/b_0 \end{bmatrix} \underbrace{\begin{bmatrix} \dot{y}_f \\ y_f \end{bmatrix}}_{\phi_3}$

Estimativa................: $\hat{Y}_i = \theta_i^T \phi_i$

Erro........................: $\epsilon_i = \hat{Y}_i - Y_i$

Normalização...........: $m_i^2 = 1 + \kappa_i \, \phi_i^T \phi_i$

Lei de adaptação.....: $\dot{\theta}_i = -\gamma_i \, \dfrac{\phi_i \epsilon_i}{m_i^2}$

Simulation 1 Zero initial conditions, small gains.

Algorithm : Normalized Gradient

Parameters . . . . : $a_0 = 2 \qquad b_0 = 3$

$\lambda = 1$

$\Gamma_i = 1\, I$

$\kappa_i = 0$

Initial conditions . . . . : $y(0) = 0$

$\theta_i(0) = 0$

Input signal . . . . : $r = 1 + 2\ \mathrm{sign}\big(\sin(0.1\,\pi\, t)\big)$

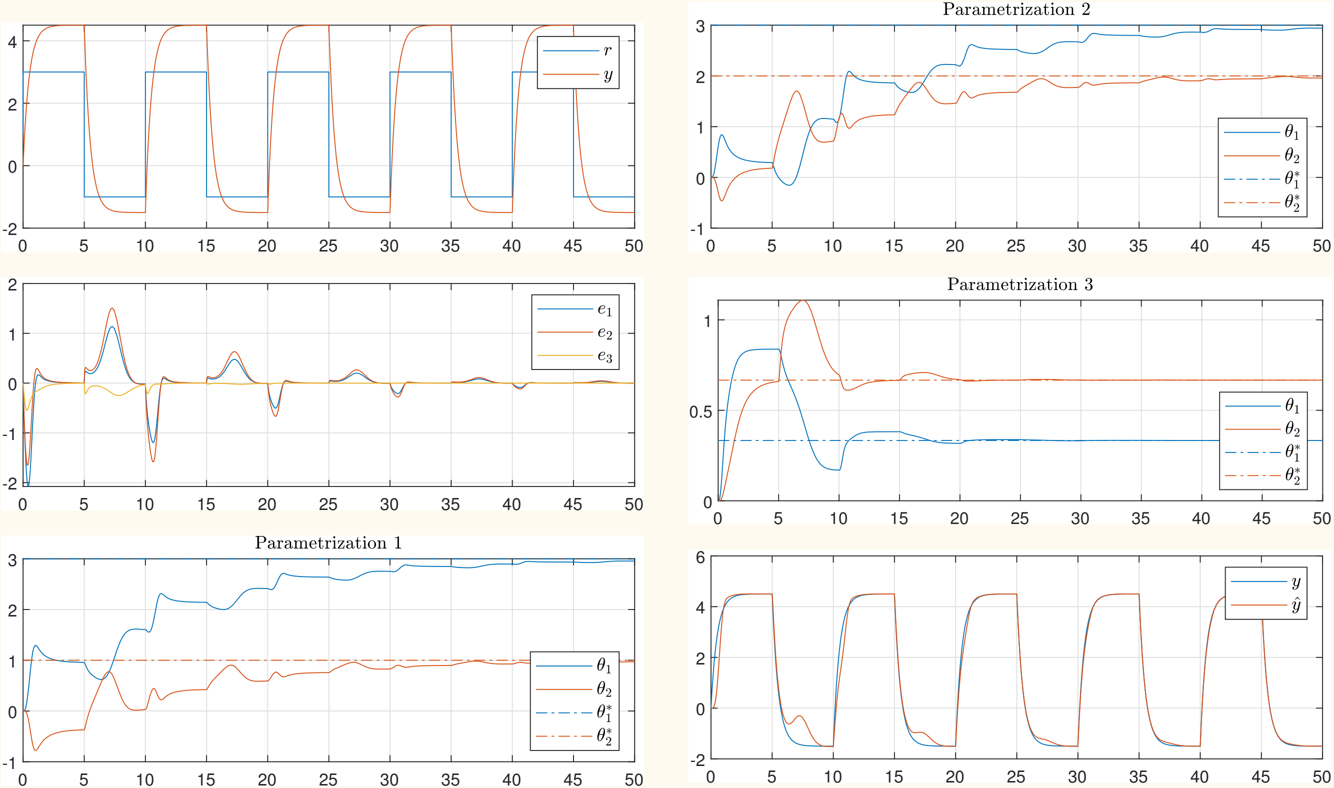


Figura 29: Normalized Gradient. Plant-12.

Simulation 2 Zero initial conditions, large gains.

Algorithm : **Normalized Gradient**

Parameters . . . . : $a_0 = 2$ $\quad b_0 = 3$

$\lambda = 1$

$\Gamma_i = 10\,I$

$\kappa_i = 0$

Initial conditions . . . . : $y(0) = 0$

$\theta_i(0) = 0$

Input signal . . . . : $r = 1 + 2\,\mathrm{sign}\big(\sin(0.1\,\pi\,t)\big)$

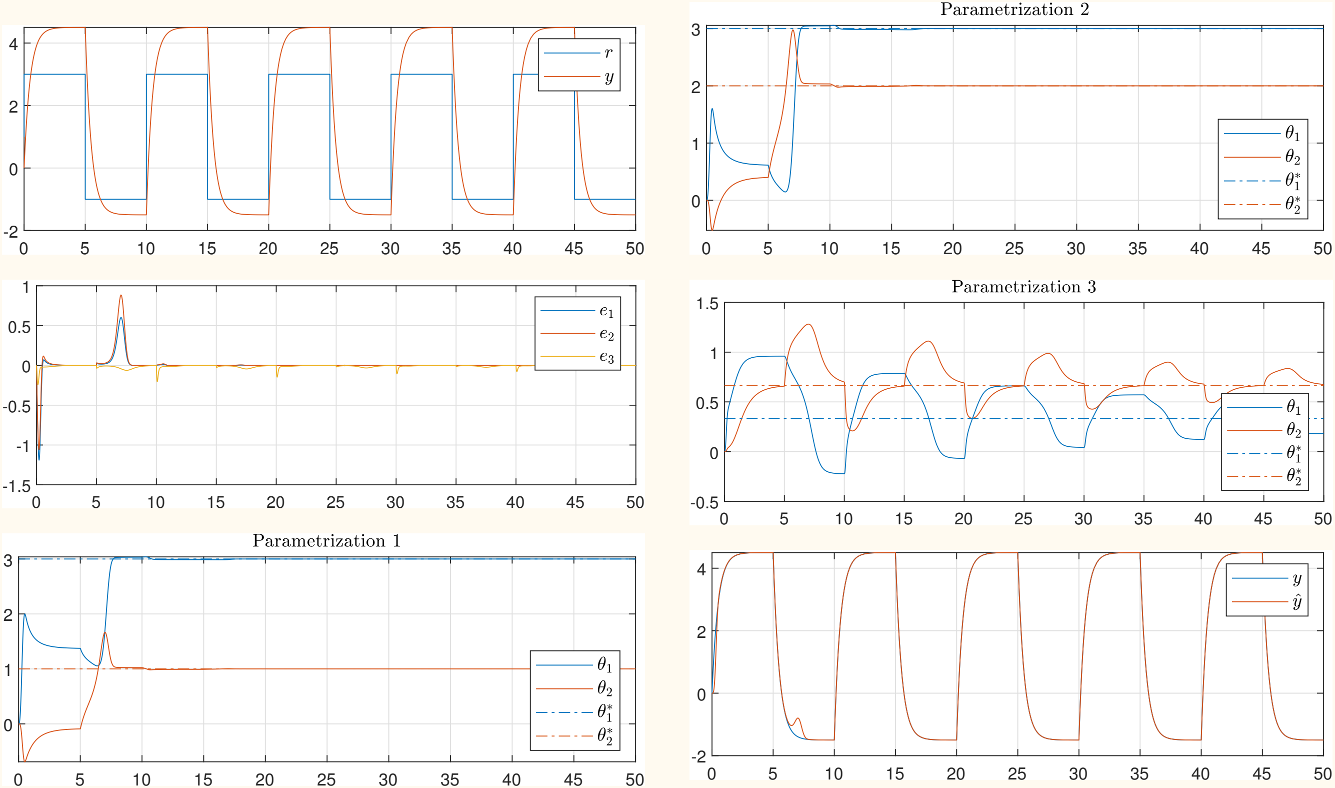


Figura 30: Normalized Gradient. Plant-12.

Simulation 3 Small initial conditions, large gains.

Algorithm : **Normalized Gradient**

Parameters . . . . : $a_0 = 2$ $\quad b_0 = 3$

$$\lambda = 1$$

$$\Gamma_i = 10\, I$$

$$\kappa_i = 0$$

Initial conditions . . . . . : $y(0) = 5$

$$\theta_i(0) = 0$$

Input signal . . . . : $r = 1 + 2\,\mathrm{sign}\big(\sin(0.1\,\pi\, t)\big)$

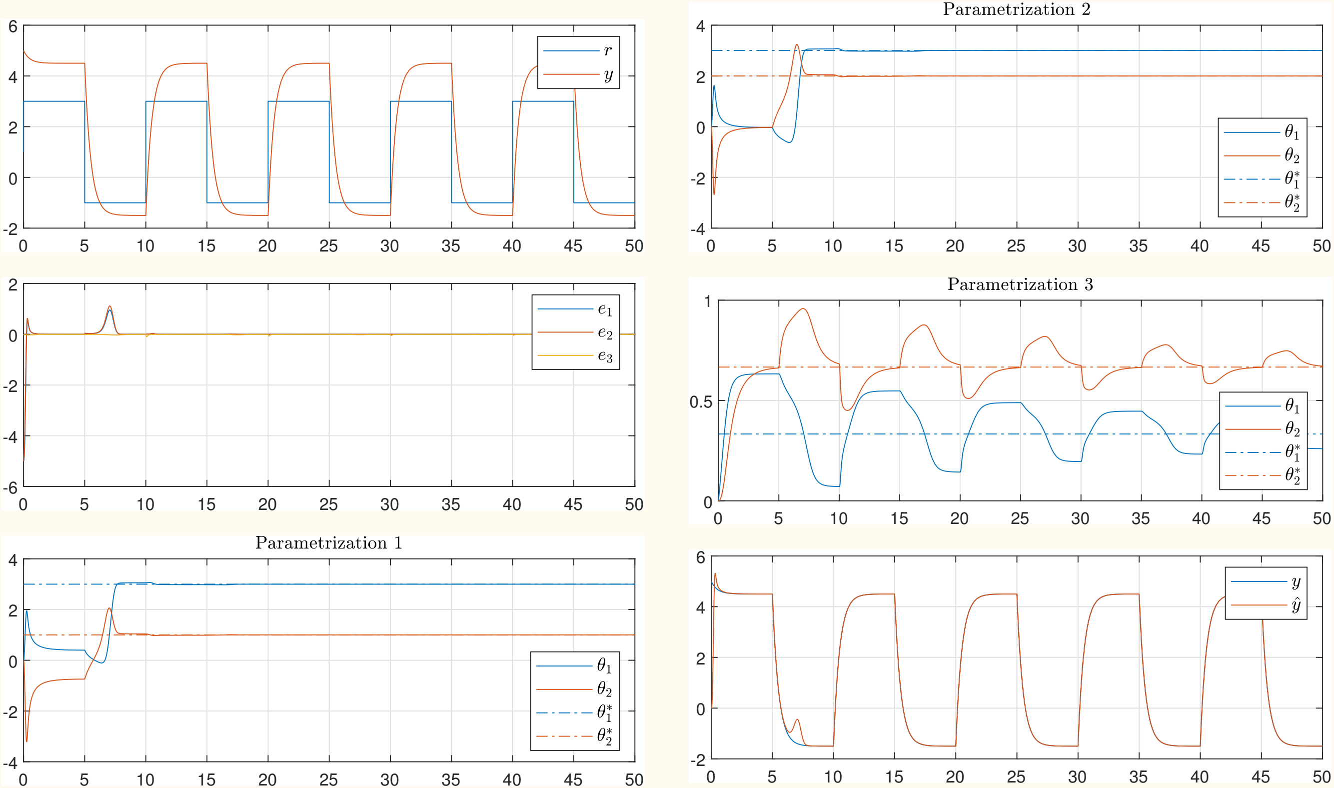


Figura 31: Normalized Gradient. Plant-12.

Simulation 4 Effect of normalization (large κ).

Algorithm : **Normalized Gradient**

Parameters . . . . :

$$a_0 = 2 \qquad b_0 = 3$$

$$\lambda = 1$$

$$\Gamma_i = 10\, I$$

$$\kappa_i = 10$$

Initial conditions . . . . :

$$y(0) = 5$$

$$\theta_i(0) = 0$$

Input signal . . . . :

$$r = 1 + 2\ \mathrm{sign}\big(\sin(0.1\,\pi\, t)\big)$$

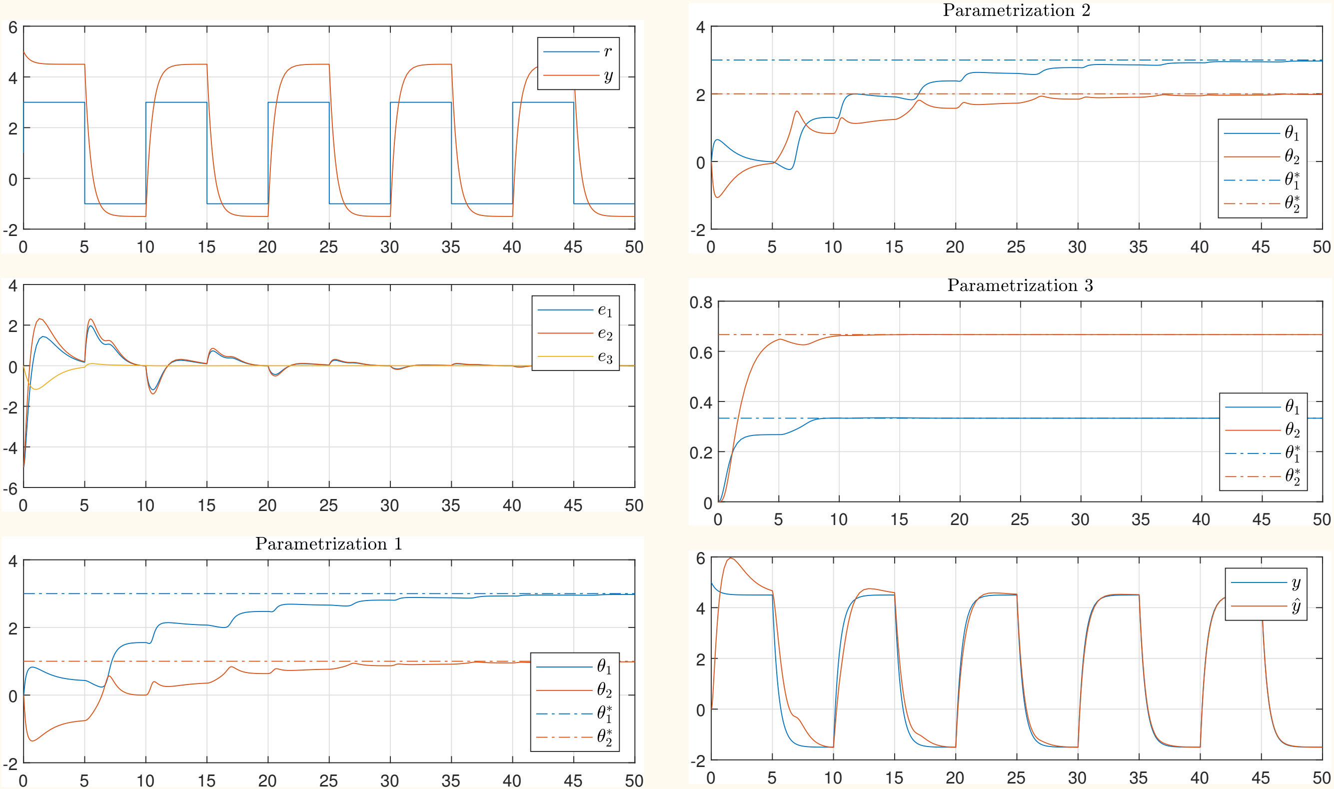


Figura 32: Normalized Gradient. Plant-12.

Simulation 5 Plant is an integrator.

Algorithm : **Normalized Gradient**

Parameters . . . . : $a_0 = \boxed{0}$ $\quad b_0 = 3$

$\lambda = 1$

$\Gamma_i = \boxed{10\, I}$

$\kappa_i = \boxed{0}$

Initial conditions . . . . . : $y(0) = \boxed{0}$

$\theta_i(0) = \boxed{0}$

Input signal . . . . : $r = 1 + 2\ \mathrm{sign}\big(\sin(0.1\,\pi\,t)\big)$

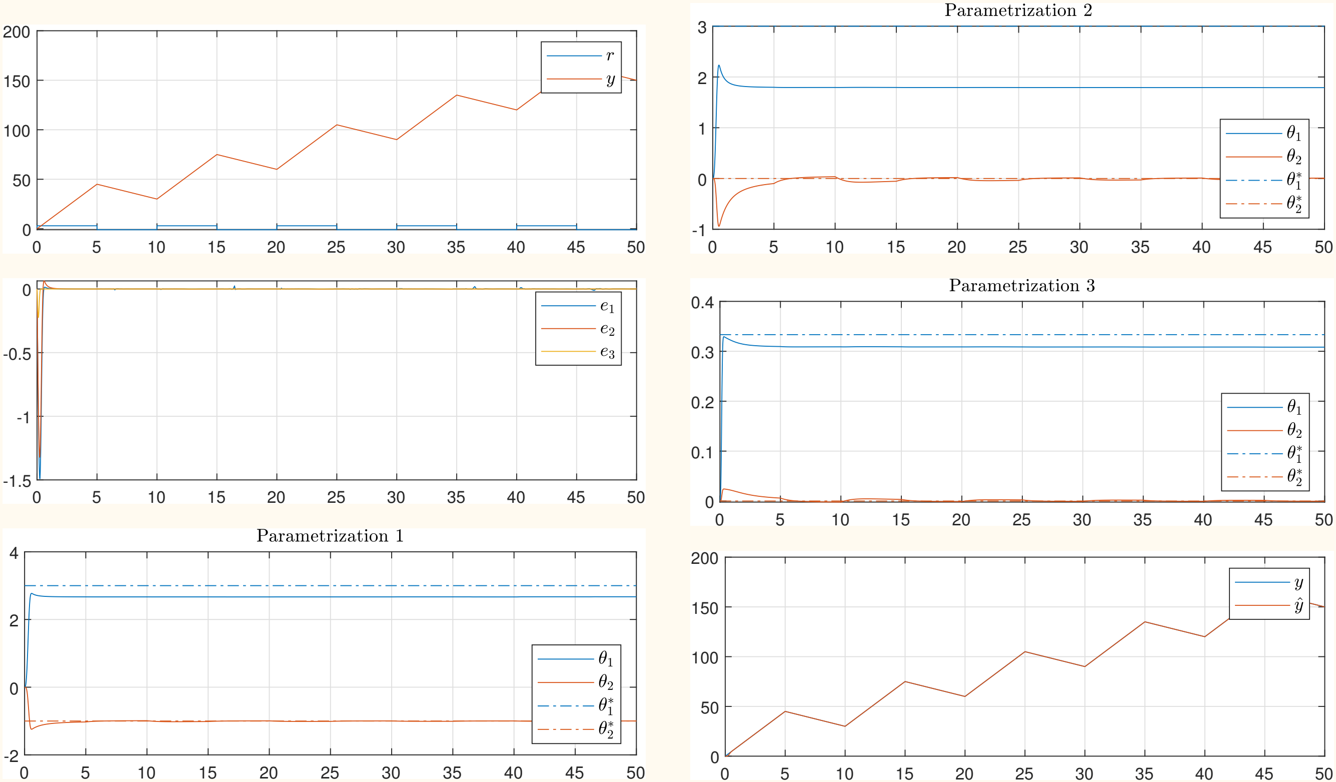


Figura 33: Normalized Gradient. Plant-12.

Case 3 2nd order plant, 4 unknown parameters.

Plant .... : $P(s) = \dfrac{N(s)}{D(s)} = \dfrac{b_1 s + b_0}{s^2 + a_1 s + a_0}$

Filter .... : $\dfrac{1}{\Lambda(s)} = \dfrac{1}{s^2 + \lambda_1 s + \lambda_0}$

★ As simulações utilizam o pacote ODE do Matlab.

Resumo das equações

Parametrização 1 ....: $\underbrace{y}_{Y_1} = b_0 u_f + b_1 \dot{u}_f + (a_0 - \lambda_0) y_f + (a_1 - \lambda_1) \dot{y}_f = \theta_1^* \phi_1$

Regressor ....: $\phi_1 = \begin{bmatrix} u_f & \dot{u}_f & -y_f & -\dot{y}_f \end{bmatrix}^T$

Parâmetros..............: $\theta_1^* = \begin{bmatrix} b_0 & b_1 & (a_0 - \lambda_0) & (a_1 - \lambda_1) \end{bmatrix}^T$

Parametrização 2 ....: $\underbrace{\ddot{y}_f}_{Y_2} = b_0 u_f + b_1 \dot{u}_f - a_0 y_f - a_1 \dot{y}_f = \theta_1^* \phi_1$

Regressor ....: $\phi_2 = \begin{bmatrix} u_f & \dot{u}_f & -y_f & -\dot{y}_f \end{bmatrix}^T$

Parâmetros..............: $\theta_2^* = \begin{bmatrix} b_0 & b_1 & a_0 & a_1 \end{bmatrix}^T$

Parametrização 3 .....: $\underbrace{u_f}_{Y_3} = \frac{1}{b_0}\ddot{y}_f - \frac{b_1}{b_0}\dot{u}_f + \frac{a_0}{b_0}y_f + \frac{a_1}{b_0}\dot{y}_f$

Regressor ....: $\phi_3 = \begin{bmatrix}\ddot{y}_f & -\dot{u}_f & y_f & \dot{y}_f\end{bmatrix}^T$

Parâmetros..............: $\theta_3^* = \begin{bmatrix}1/b_0 & b_1/b_0 & a_0/b_0 & a_1/b_0\end{bmatrix}^T$

Estimativa...............: $\hat{Y}_i = \theta_i^T \phi_i$

Erro........................: $\epsilon_i = \hat{Y}_i - Y_i$

Normalização...........: $m_i^2 = 1 + \kappa_i\, \phi_i^T \phi_i$

Lei de adaptação .....: $\dot{\theta}_i = -\gamma_i \dfrac{\phi_i \epsilon_i}{m_i^2}$

Nota. $P(s)y_f = Z(s)u_f \quad \Rightarrow \quad \ddot{y}_f + a_1\dot{y}_f + a_0 y_f = b_1\dot{u}_f + b_0 u_f$

Simulation 1 Zero initial conditions, small gains.

Algorithm : **Normalized Gradient**

Parameters . . . . : $a_1 = 2.5 \quad a_0 = 1.5 \quad b_1 = 2 \quad b_0 = 0.8$

$\lambda_0 = 1 \quad \lambda_1 = 2$

$\Gamma_i = 1\,I$

$\kappa_i = 0$

Initial conditions . . . . : $x(0) = [0 \;\; 0]^T$

$\theta_i(0) = [0 \;\; 0 \;\; 0 \;\; 0]^T$

Input signal . . . . : $r = 1 + 2\,\text{sign}\big(\sin(0.1\,\pi\,t)\big)$

Matching parameters:

$$\theta_1^* = \begin{bmatrix}0.8 & 2 & 0.5 & 0.5\end{bmatrix}$$

$$\theta_2^* = \begin{bmatrix}0.8 & 2 & 1.5 & 2.5\end{bmatrix}$$

$$\theta_3^* = \begin{bmatrix}1 & 2.5 & 1.875 & 3.125\end{bmatrix}$$

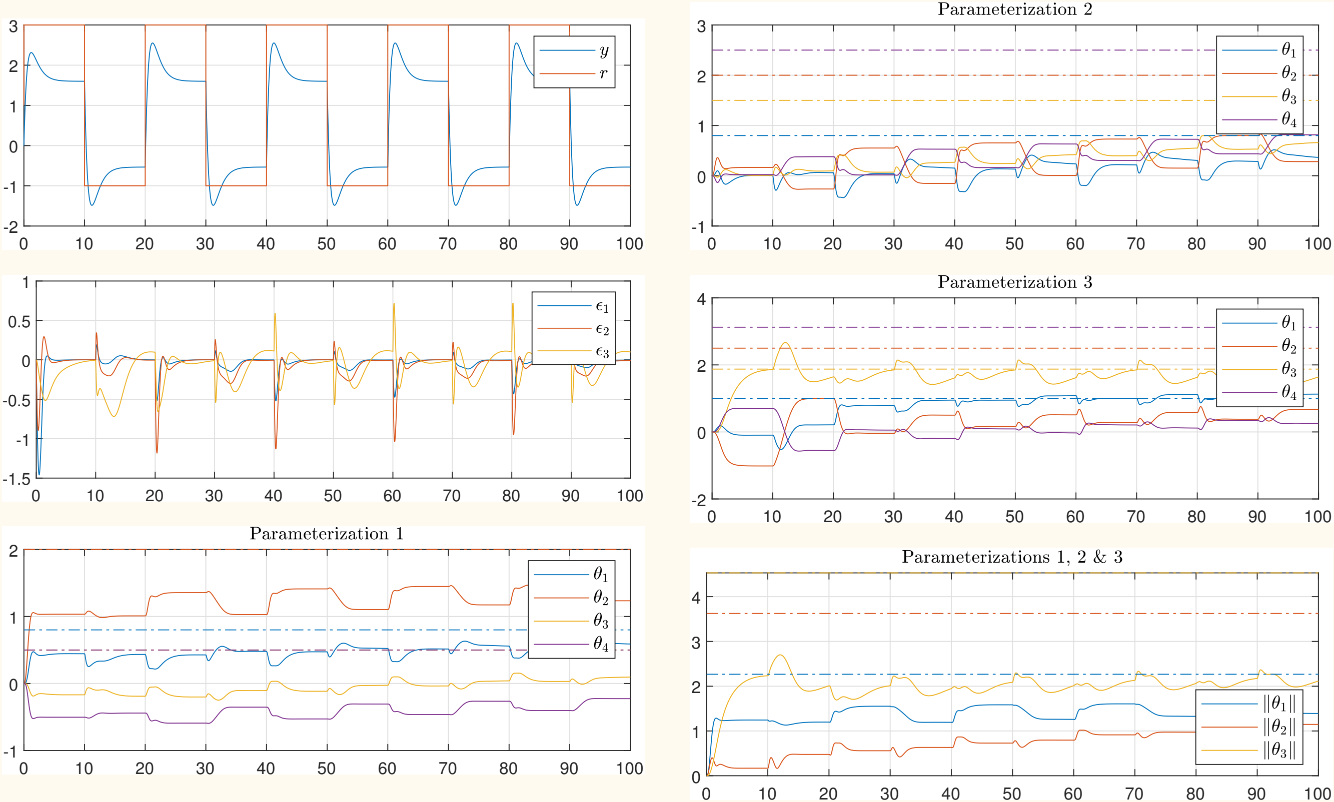


Figura 34: Normalized Gradient. Plant-24.

Simulation 2 Zero initial conditions, large gains.

Algorithm : Normalized Gradient

Parameters . . . . : $a_1 = 2.5$ $a_0 = 1.5$ $b_1 = 2$ $b_0 = 0.8$

$\lambda_0 = 1$ $\lambda_1 = 2$

$\Gamma_i = 10\, I$

$\kappa_i = 0$

Initial conditions . . . . . : $x(0) = [0 \;\; 0]^T$

$\theta_i(0) = [0 \;\; 0 \;\; 0 \;\; 0]^T$

Input signal . . . . : $r = 1 + 2\,\mathrm{sign}\big(\sin(0.1\,\pi\, t)\big)$

Matching parameters: $\theta_1^* = \begin{bmatrix} 0.8 & 2 & 0.5 & 0.5 \end{bmatrix}$

$\theta_2^* = \begin{bmatrix} 0.8 & 2 & 1.5 & 2.5 \end{bmatrix}$

$\theta_3^* = \begin{bmatrix} 1 & 2.5 & 1.875 & 3.125 \end{bmatrix}$

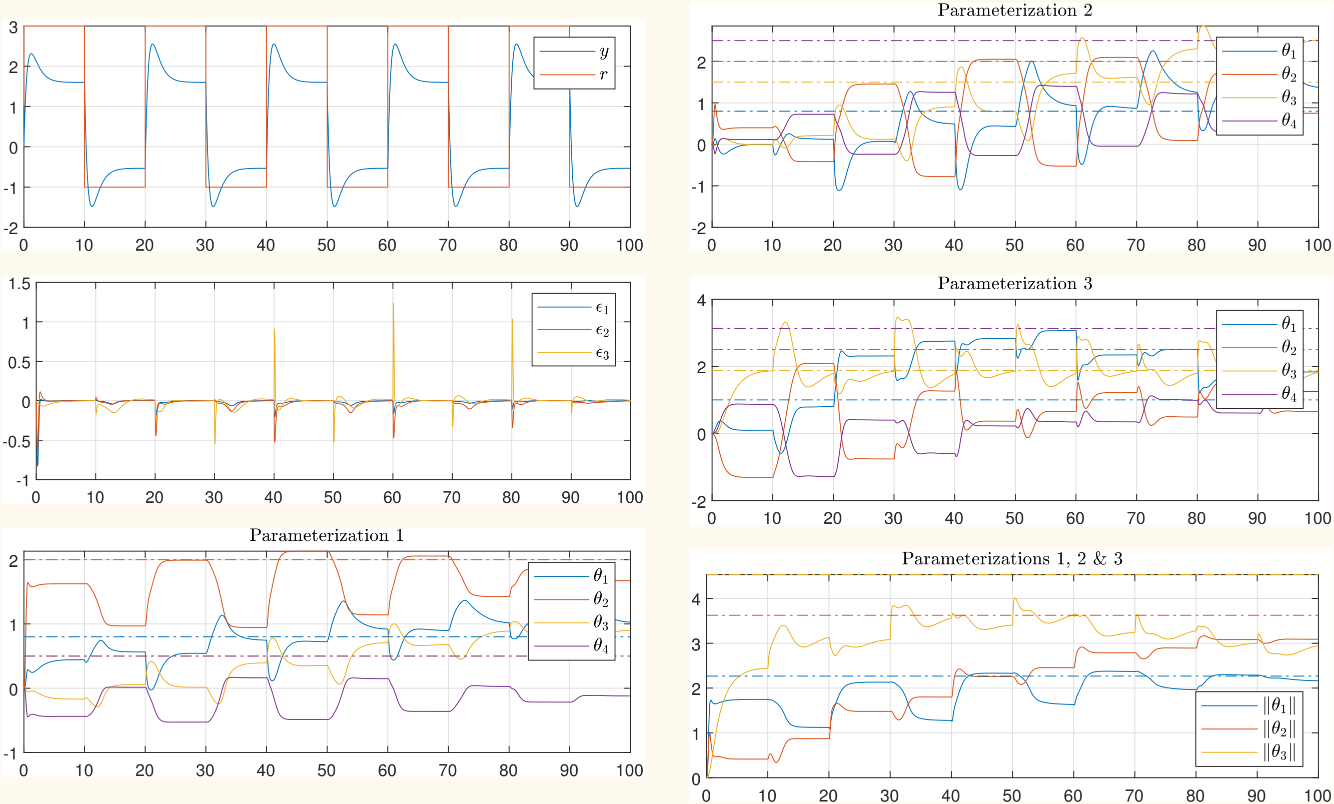


Figura 35: Normalized Gradient. Plant-24.

Simulation 3 Zero initial conditions, large gains & more excitation.

Algorithm : Normalized Gradient

Parameters . . . . :

$$a_1 = 2.5 \qquad a_0 = 1.5 \qquad b_1 = 2 \qquad b_0 = 0.8$$

$$\lambda_0 = 1 \qquad \lambda_1 = 2$$

$$\Gamma_i = \boxed{10\, I}$$

$$\kappa_i = 0$$

Initial conditions . . . . :

$$x(0) = \boxed{[0 \;\; 0]^T}$$

$$\theta_i(0) = \boxed{[0 \;\; 0 \;\; 0 \;\; 0]^T}$$

Input signal . . . . :

$$r = 1 + 2\ \mathrm{sign}\big(\sin(0.1\,\pi\, t)\big) + 2\sin(2t)$$

Matching parameters:

$$\theta_1^* = \begin{bmatrix} 0.8 & 2 & 0.5 & 0.5 \end{bmatrix}$$

$$\theta_2^* = \begin{bmatrix} 0.8 & 2 & 1.5 & 2.5 \end{bmatrix}$$

$$\theta_3^* = \begin{bmatrix} 1 & 2.5 & 1.875 & 3.125 \end{bmatrix}$$

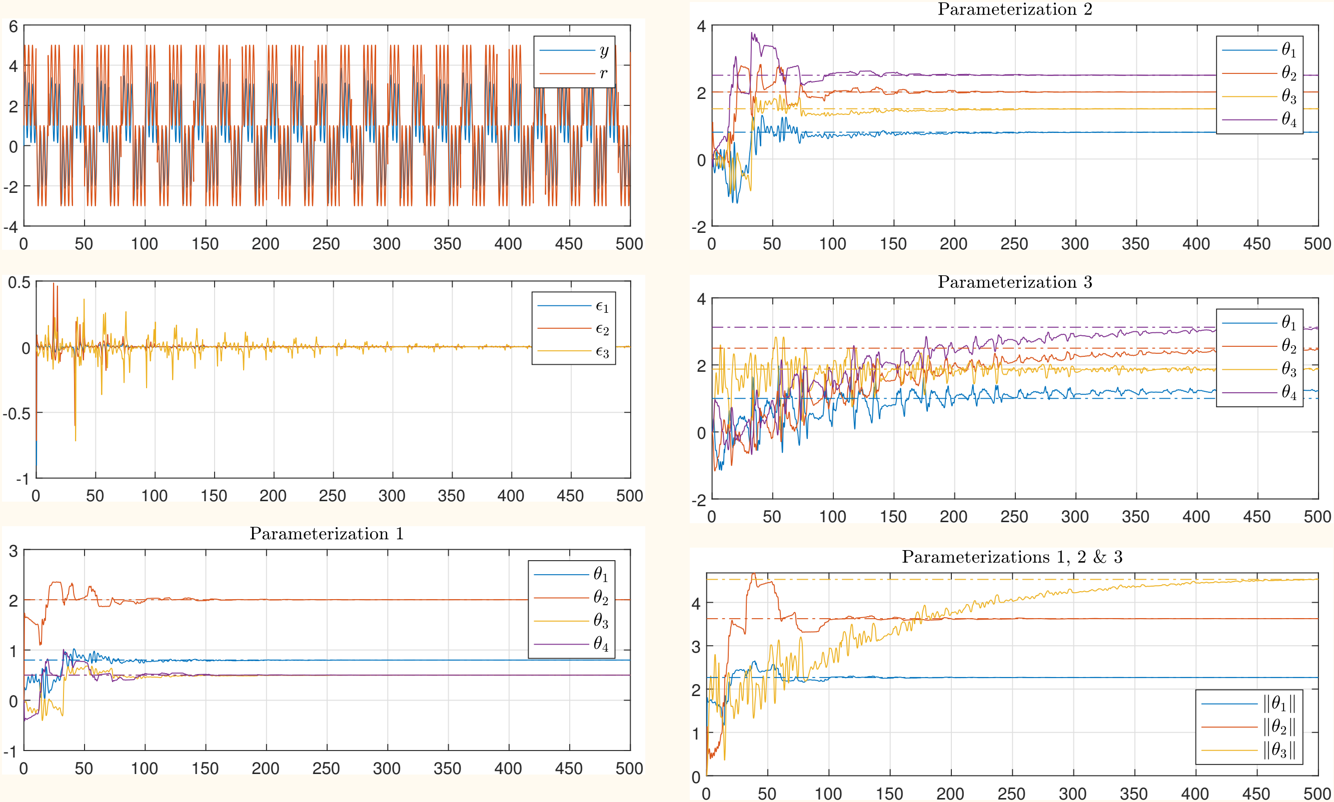


Figura 36: Normalized Gradient. Plant-24.

Simulation 4 Effect of a large initial condition.

Algorithm : **Normalized Gradient**

Parameters . . . . :

$$a_1 = 2.5 \qquad a_0 = 1.5 \qquad b_1 = 2 \qquad b_0 = 0.8$$

$$\lambda_0 = 1 \qquad \lambda_1 = 2$$

$$\Gamma_i = 10\, I$$

$$\kappa_i = 0$$

Initial conditions . . . . :

$$x(0) = [15 \;\; 0]^T$$

$$\theta_i(0) = [0 \;\; 0 \;\; 0 \;\; 0]^T$$

Input signal . . . . :

$$r = 1 + 2\,\mathrm{sign}\big(\sin(0.1\,\pi\,t)\big) + 2\sin(2t)$$

Matching parameters:

$$\theta_1^* = \begin{bmatrix} 0.8 & 2 & 0.5 & 0.5 \end{bmatrix}$$

$$\theta_2^* = \begin{bmatrix} 0.8 & 2 & 1.5 & 2.5 \end{bmatrix}$$

$$\theta_3^* = \begin{bmatrix} 1 & 2.5 & 1.875 & 3.125 \end{bmatrix}$$

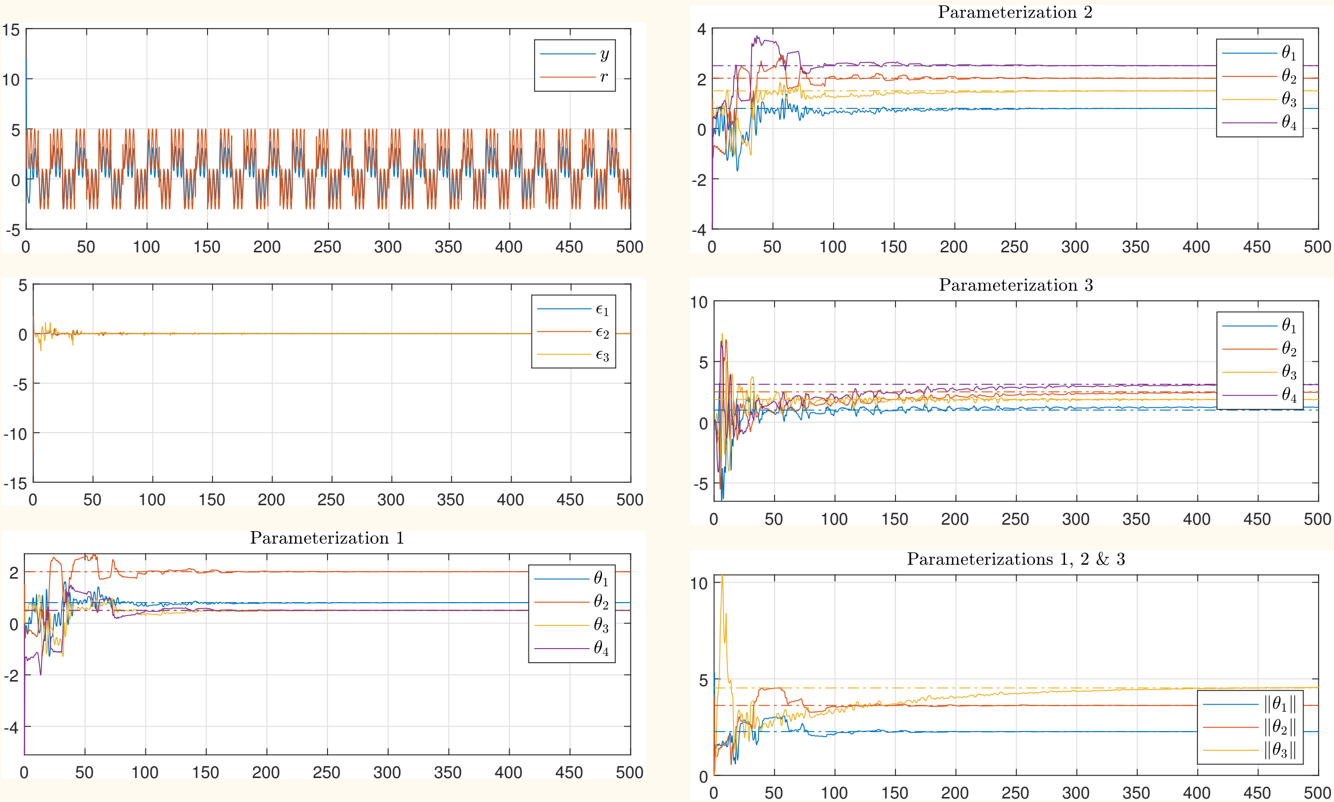


Figura 37: Normalized Gradient. Plant-24.

Simulation 5 Effect of the normalization.

Algorithm : Normalized Gradient

Parameters . . . . : $a_1 = 2.5$ $a_0 = 1.5$ $b_1 = 2$ $b_0 = 0.8$

$\lambda_0 = 1$ $\lambda_1 = 2$

$\Gamma_i = 10\, I$

$\kappa_i = 1$

Initial conditions . . . . : $x(0) = [15 \;\; 0]^T$

$\theta_i(0) = [0 \;\; 0 \;\; 0 \;\; 0]^T$

Input signal . . . . : $r = 1 + 2\,\mathrm{sign}\big(\sin(0.1\,\pi\, t)\big) + 2\sin(2t)$

Matching parameters: $\theta_1^* = \begin{bmatrix} 0.8 & 2 & 0.5 & 0.5 \end{bmatrix}$

$\theta_2^* = \begin{bmatrix} 0.8 & 2 & 1.5 & 2.5 \end{bmatrix}$

$\theta_3^* = \begin{bmatrix} 1 & 2.5 & 1.875 & 3.125 \end{bmatrix}$

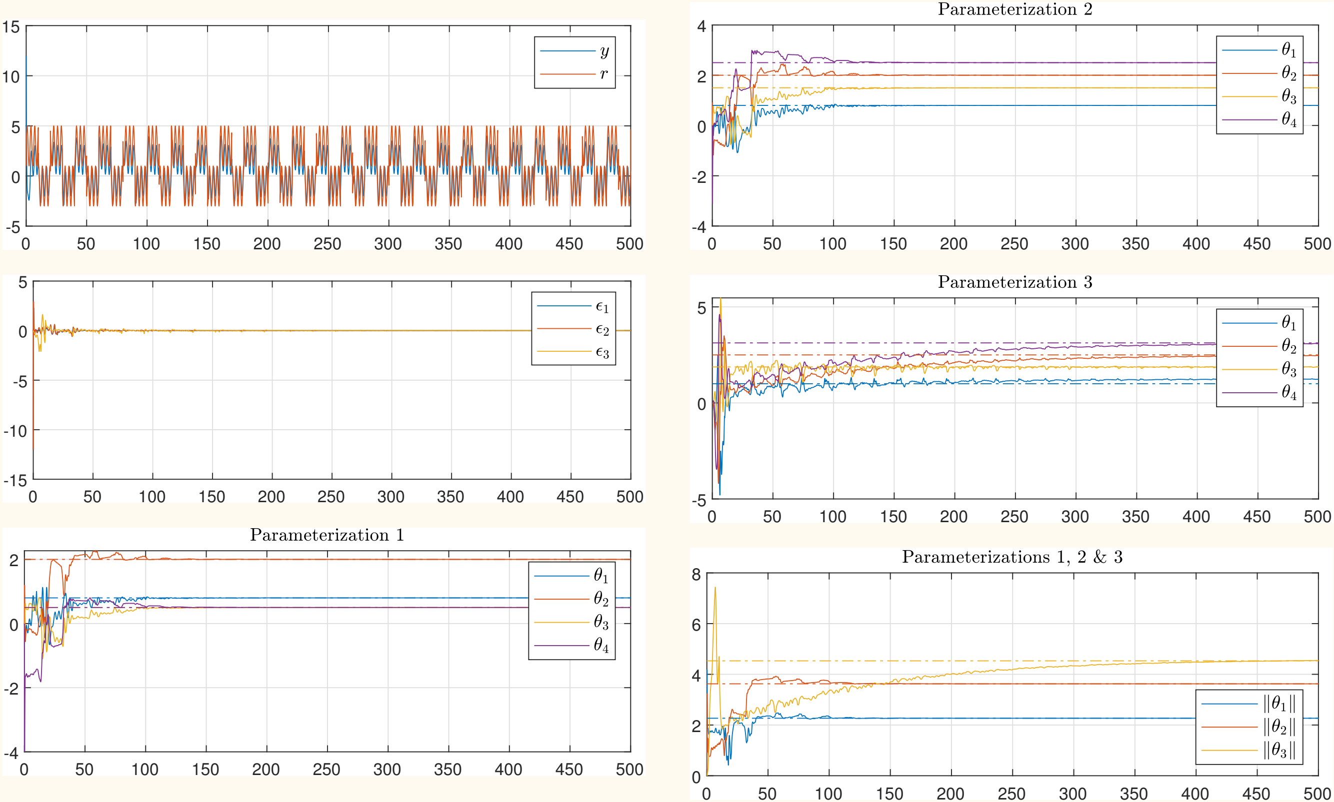


Figura 38: Normalized Gradient. Plant-24.

3.4 *Least-squares* normalizado

Algoritmo:

$$\dot{\theta} = -\frac{P\phi\epsilon}{m^2}, \qquad \theta(t_0) = \theta_0 \tag{2}$$

$$\dot{P} = -\frac{P\phi\phi^T P}{m^2}, \qquad P(t_0) = P_0 > 0 \tag{3}$$

$$m^2 = 1 + \kappa\,\phi^T P\phi, \qquad \kappa > 0$$

★ Note que P é um ganho de adaptação variante no tempo.

Lema. Estabilidade do algoritmo LS normalizado.

- $P(t) = P^T(t) > 0\,, \quad \forall t$
- $P\,,\ \dot{P}\ \in \mathcal{L}_\infty$
- $\theta\,,\ \dfrac{\epsilon}{m}\,,\ \dfrac{\epsilon}{\bar{m}}\ \in \mathcal{L}_\infty$
- $\dot{\theta}\,,\ \dfrac{\epsilon}{m}\,,\ \dfrac{\epsilon}{\bar{m}}\ \in \mathcal{L}_2$
- $\lim\limits_{t\to\infty} P(t) = P_\infty$
- $\lim\limits_{t\to\infty} \theta(t) = \theta_\infty$

$$\boxed{\bar{m}^2 = 1 + \phi^T\phi}$$

Prova. [Tao:2003], (pag. 105)

Como $P(t_0) > 0$, então para $t = t_0$

$$P(t)P^{-1}(t) = I$$

Derivando,

$$\dot{P}P^{-1} + P\frac{d}{dt}\big(P^{-1}\big) = 0$$

Portanto,

$$\begin{aligned}\frac{d}{dt}\big(P^{-1}\big) &= -P^{-1}\dot{P}P^{-1} \\ &= -P^{-1}\left(-\frac{P\phi\phi^T P}{m^2}\right)P^{-1} = \frac{\phi\phi^T}{m^2} \geq 0\end{aligned}$$

Portanto,

- P^{-1} é monotonicamente crescente $\Rightarrow$ $\boxed{P^{-1}(t) \geq P^{-1}(t_0) > 0}$

- $P(t) > 0\,, \quad \forall t$

- P^{-1} é monot. crescente $\Rightarrow$ P é monot. decrescente $\Rightarrow$ $\boxed{P(t) \in \mathcal{L}_\infty}$

- $\dot{P} = -\dfrac{P\phi\phi^T P}{m^2} = -\underbrace{Q}_{\mathcal{L}_\infty}\underbrace{\dfrac{Q\phi\phi^T Q}{1+\kappa\phi^T QQ\phi}}_{\mathcal{L}_\infty}\underbrace{Q}_{\mathcal{L}_\infty}$ $\Rightarrow$ $\boxed{\dot{P} \in \mathcal{L}_\infty}$

★ No item anterior, $P = QQ$.

Note que P pode $\to 0$ e, portanto, não se pode afirmar que $\dfrac{\phi\phi^T}{m^2} \in \mathcal{L}_\infty$.

Função de Lyapunov: $V\big(\tilde{\theta}\big) = \tilde{\theta}^T P^{-1} \tilde{\theta}$

Derivando,

$$
\begin{aligned}
\dot{V} &= \dot{\theta}^T P^{-1} \tilde{\theta} + \tilde{\theta}^T \dot{P}^{-1} \tilde{\theta} + \tilde{\theta}^T P^{-1} \dot{\theta} \\
&= 2\tilde{\theta}^T P^{-1} \dot{\theta} + \tilde{\theta}^T \big(- P^{-1} \dot{P} P^{-1} \big) \tilde{\theta} \\
&= 2\tilde{\theta}^T P^{-1} \left(-\frac{P\phi\epsilon}{m^2} \right) - \tilde{\theta}^T P^{-1} \left(-\frac{P\phi\phi^T P}{m^2} \right) P^{-1} \tilde{\theta} \\
&= -2\frac{\epsilon^2}{m^2} + \frac{\epsilon^2}{m^2} \\
&= -\frac{\epsilon^2}{m^2} \leq 0
\end{aligned}
$$

Portanto,

- $\boxed{\tilde{\theta} \in \mathcal{L}_\infty}$

- $\boxed{\dfrac{\epsilon}{m} \in \mathcal{L}_2}$

- $P \in \mathcal{L}_\infty \quad \Rightarrow \quad \boxed{\dfrac{m}{\bar{m}} \in \mathcal{L}_\infty} \qquad (\ P \to 0 \ \Rightarrow \ m \to 1\ )$

- $\dfrac{\epsilon}{\bar{m}} = \underbrace{\tilde{\theta}^T}_{\mathcal{L}_\infty} \underbrace{\dfrac{\phi}{\bar{m}}}_{\mathcal{L}_\infty} \quad \Rightarrow \quad \boxed{\dfrac{\epsilon}{\bar{m}} \in \mathcal{L}_\infty}$

- $\dfrac{\epsilon}{\bar{m}} = \underbrace{\dfrac{\epsilon}{m}}_{\mathcal{L}_2} \underbrace{\dfrac{m}{\bar{m}}}_{\mathcal{L}_\infty} \quad \Rightarrow \quad \boxed{\dfrac{\epsilon}{\bar{m}} \in \mathcal{L}_2}$

- $\underbrace{\dfrac{\epsilon}{\bar{m}}}_{\mathcal{L}_\infty} = \dfrac{\epsilon}{m} \underbrace{\dfrac{m}{\bar{m}}}_{\mathcal{L}_\infty} \quad \Rightarrow \quad \boxed{\dfrac{\epsilon}{m} \in \mathcal{L}_\infty}$

- $\dot{\theta} = -\dfrac{P\phi\epsilon}{m^2} = -\underbrace{Q}_{\mathcal{L}_\infty} \underbrace{\dfrac{Q\phi}{m}}_{\mathcal{L}_\infty} \underbrace{\dfrac{\epsilon}{m}}_{\mathcal{L}_2 \cap \mathcal{L}_\infty} \quad \Rightarrow \quad \boxed{\dot{\theta} \in \mathcal{L}_2 \cap \mathcal{L}_\infty}$

★ Note que pode ocorrer $P \to 0$ e $\phi \to \infty$.

Portanto, pode ocorrer $m \to 1$ e $\bar{m} \to \infty$.

★ $\dfrac{\phi}{\bar{m}} \in \mathcal{L}_\infty$. Porém, $\dfrac{\phi}{m} \notin \mathcal{L}_\infty$.

Para provar que $P \to P_\infty$, integramos a expressão de P:

$$P(t) = \underbrace{P_0}_{>0} - \underbrace{\int_{t_0}^{t} \frac{P\phi\phi^T P}{m^2} d\tau}_{\mathcal{I}} > 0$$

Então, para todo z,

$$z^T P z = z^T P_0 z - z^T \mathcal{I} z > 0$$

Portanto,

- $z^T \mathcal{I} z > 0$ e monotonicamente crescente.
- $\underbrace{z^T P z}_{>0} = z^T P_0 z - \underbrace{z^T \mathcal{I} z}_{>0} > 0 \qquad \Rightarrow \quad z^T \mathcal{I} z \leq z^T P_0 z.$

Fato. Se uma função $f(t)$ é monotonicamente crescente e limitada, então

$$\lim_{t\to\infty} f(t) = f_\infty < \infty\,.$$

Portanto,

$$\lim_{t\to\infty} z^T P z = z^T P_0 z - \underbrace{\lim_{t\to\infty} z^T \mathcal{I} z}_{f_\infty} < \infty$$

- Para uma escolha adequada de z, tem-se que $\boxed{\lim_{t\to\infty} P(t) = P_\infty}$

★ Exemplo de escolha de z: $z = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T$.

- Agora vem a parte mais elaborada da prova: mostrar que $\theta \to \theta_\infty$.

O algoritmo LS minimiza a função custo

$$J(\theta) = \frac{1}{2}\int_{t_0}^{t} \frac{\varepsilon^2(\tau)}{m^2(\tau)} d\tau + \frac{1}{2}\big(\theta - \theta_0\big) P_0^{-1} \big(\theta - \theta_0\big)$$

onde $\varepsilon(\tau) = \theta^T(t)\phi(\tau) - y(\tau)$.

Para encontrar o mínimo, fazemos

$$\frac{\partial J}{\partial \theta} = \frac{\partial J}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial \theta} = \int_{t_0}^{t} \frac{\varepsilon(\tau)}{m^2(\tau)} \phi(\tau) d\tau + P_0^{-1}\big(\theta - \theta_0\big) = 0$$

Resolvendo para θ, obtém-se

$$P_0^{-1}\theta = P_0^{-1}\theta_0 - \int_{t_0}^{t} \frac{\phi^T\theta - y}{m^2} \phi d\tau$$

Portanto,

$$P_0^{-1}\theta(t) = P_0^{-1}\theta_0 - \int_{t_0}^{t} \frac{\phi\phi^T}{m^2}\theta(t)d\tau + \int_{t_0}^{t} \frac{y\phi}{m^2}d\tau$$

★ Lembrar que $\varepsilon(\tau) = \theta^T(t)\phi(\tau) - y(\tau)$.

Então,

$$P_0^{-1}\theta(t) + \int_{t_0}^{t} \frac{\phi\phi^T}{m^2}d\tau\theta(t) = P_0^{-1}\theta_0 + \int_{t_0}^{t} \frac{y\phi}{m^2}d\tau$$

$$\underbrace{\left(P_0^{-1} + \int_{t_0}^{t} \frac{\phi\phi^T}{m^2}d\tau\right)}_{P^{-1}}\theta(t) = P_0^{-1}\theta_0 + \int_{t_0}^{t} \frac{y\phi}{m^2}d\tau$$

$$\Rightarrow \quad \boxed{\theta(t) = P\left(P_0^{-1}\theta_0 + \int_{t_0}^{t} \frac{y\phi}{m^2}d\tau\right)} \tag{4}$$

Lembrando que $y = \phi^T \theta^*$, podemos escrever

$$\underbrace{\theta(t) - \theta^*}_{\tilde{\theta}} = P\left(P_0^{-1}\theta_0 + \int_{t_0}^{t} \frac{\phi\phi^T\theta^*}{m^2} d\tau\right) - \theta^*$$

Porém,

$$P^{-1} = P_0^{-1} + \int_{t_0}^{t} \frac{\phi\phi^T}{m^2} d\tau \quad \Rightarrow \quad P^{-1} - P_0^{-1} = \int_{t_0}^{t} \frac{\phi\phi^T}{m^2} d\tau$$

Portanto,

$$\begin{aligned}\tilde{\theta} &= PP_0^{-1}\theta_0 + P\big(P^{-1} - P_0^{-1}\big)\theta^* - \theta^* \\ &= PP_0^{-1}\theta_0 - PP_0^{-1}\theta^* \\ &= PP_0^{-1}\underbrace{\big(\theta_0 - \theta^*\big)}_{\tilde{\theta}(0)}\end{aligned}$$

Resumindo: $\boxed{\tilde{\theta} = P P_0^{-1} \tilde{\theta}(0)}$ $\Rightarrow$ $\boxed{P^{-1}\tilde{\theta} = P_0^{-1}\tilde{\theta}(0)}$ = constante (!!)

★ Esta é uma relação surpreendente!

Portanto,

$$\lim_{t\to\infty} \theta(t) = \theta^* + \underbrace{\lim_{t\to\infty} P(t)}_{P_\infty} P_0^{-1}\tilde{\theta}(0) = \underbrace{\theta^* + P_\infty P_0^{-1}\tilde{\theta}(0)}_{\theta_\infty}$$

- $\boxed{\lim_{t\to\infty} \theta(t) = \theta_\infty}$

■

3.4.1 Simulações

Case 1 2nd order plant, 4 unknown parameters.

Plant ....: $P(s) = \dfrac{N(s)}{D(s)} = \dfrac{b_1 s + b_0}{s^2 + a_1 s + a_0}$

Filter ....: $\dfrac{1}{\Lambda(s)} = \dfrac{1}{s^2 + \lambda_1 s + \lambda_0}$

★ As simulações utilizam o pacote ODE do Matlab.

Resumo das equações

Parametrização 1....: $\underbrace{y}_{Y} = b_0 u_f + b_1 \dot{u}_f + (a_0 - \lambda_0) y_f + (a_1 - \lambda_1)\dot{y}_f = \theta^* \phi$

Regressor..................: $\phi = \begin{bmatrix} u_f & \dot{u}_f & y_f & \dot{y}_f \end{bmatrix}^T$

Parâmetros..............: $\theta^* = \begin{bmatrix} b_0 & b_1 & (a_0 - \lambda_0) & (a_1 - \lambda_1) \end{bmatrix}^T$

Estimativa...............: $\hat{Y} = \theta^T \phi$

Erro.........................: $\epsilon = \hat{Y} - Y$

Sinal normalizante....: $m^2 = 1 + \kappa\, \phi^T P \phi, \quad \kappa > 0$

Lei de adaptação.....: $\dot{\theta} = -\dfrac{P\phi\,\epsilon}{m^2}$

Matriz de covariância: $\dot{P} = -\dfrac{P\phi\phi^T P}{m^2}, \quad P(0) > 0$

Simulation 1 Zero initial conditions, small gains.

Algorithm : **Normalized Least-squares**

Parameters . . . . :

$$P(s) = \frac{0.5s + 2}{s^2 + 3s + 1.25}$$

$$\Lambda(s) = s^2 + 3s + 2$$

$$\kappa = 0$$

Initial conditions . . . . :

$$x(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$$

$$\theta(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$P(0) = 1\, I$$

Input signal . . . . :

$$r = 3 + 5\,\mathrm{sign}\big(\sin(0.1\,\pi\, t)\big)$$

Matching parameters:

$$\theta^* = \begin{bmatrix} 2 & 0.5 & -0.75 & 0 \end{bmatrix}$$

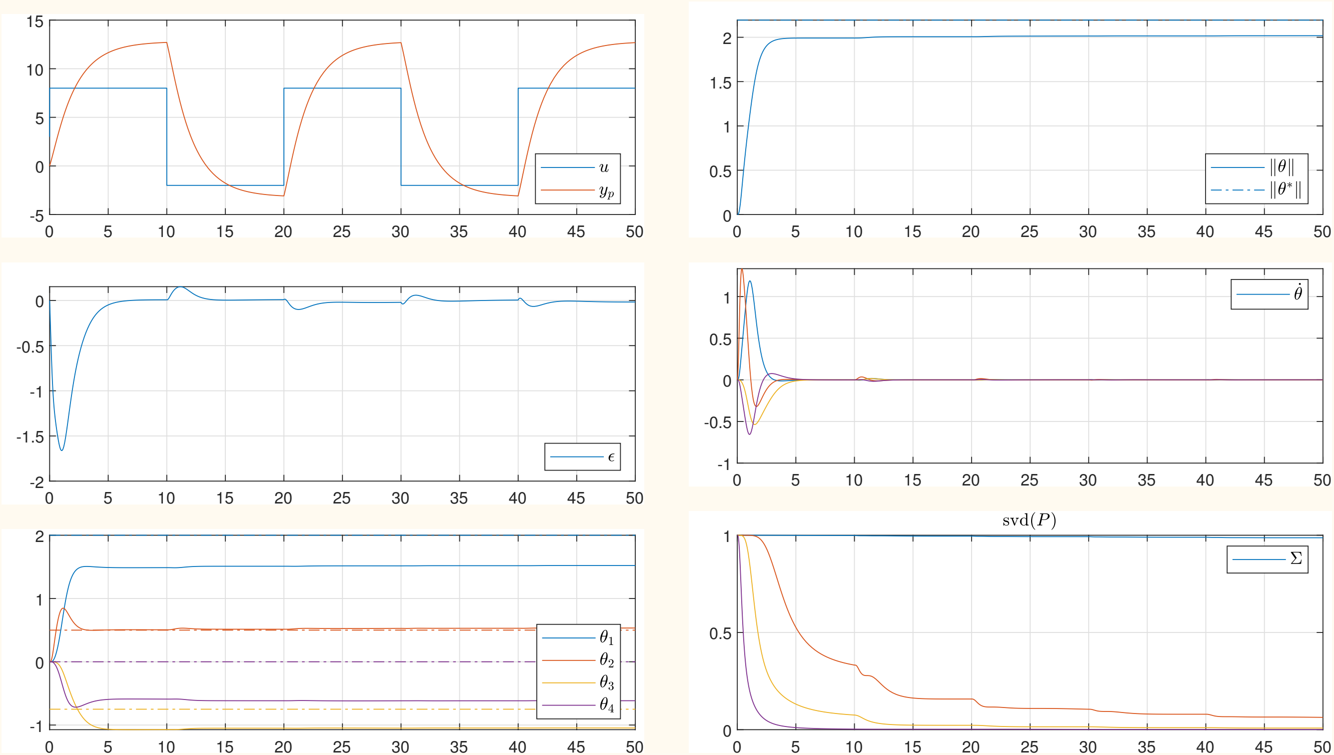


Figura 39: Normalized Least-squares. Plant-24.

Simulation 2 Zero initial conditions, large gains.

Algorithm : **Normalized Least-squares**

Parameters . . . . :

$$P(s) = \frac{0.5s + 2}{s^2 + 3s + 1.25}$$

$$\Lambda(s) = s^2 + 3s + 2$$

$$\kappa = 0$$

Initial conditions . . . . :

$$x(0) = \boxed{[0 \;\; 0]^T}$$

$$\theta(0) = \boxed{[0 \;\; 0 \;\; 0 \;\; 0]^T}$$

$$P(0) = \boxed{10\, I}$$

Input signal . . . . :

$$r = 3 + 5\,\mathrm{sign}\big(\sin(0.1\,\pi\,t)\big)$$

Matching parameters:

$$\theta^* = \begin{bmatrix} 2 & 0.5 & -0.75 & 0 \end{bmatrix}$$

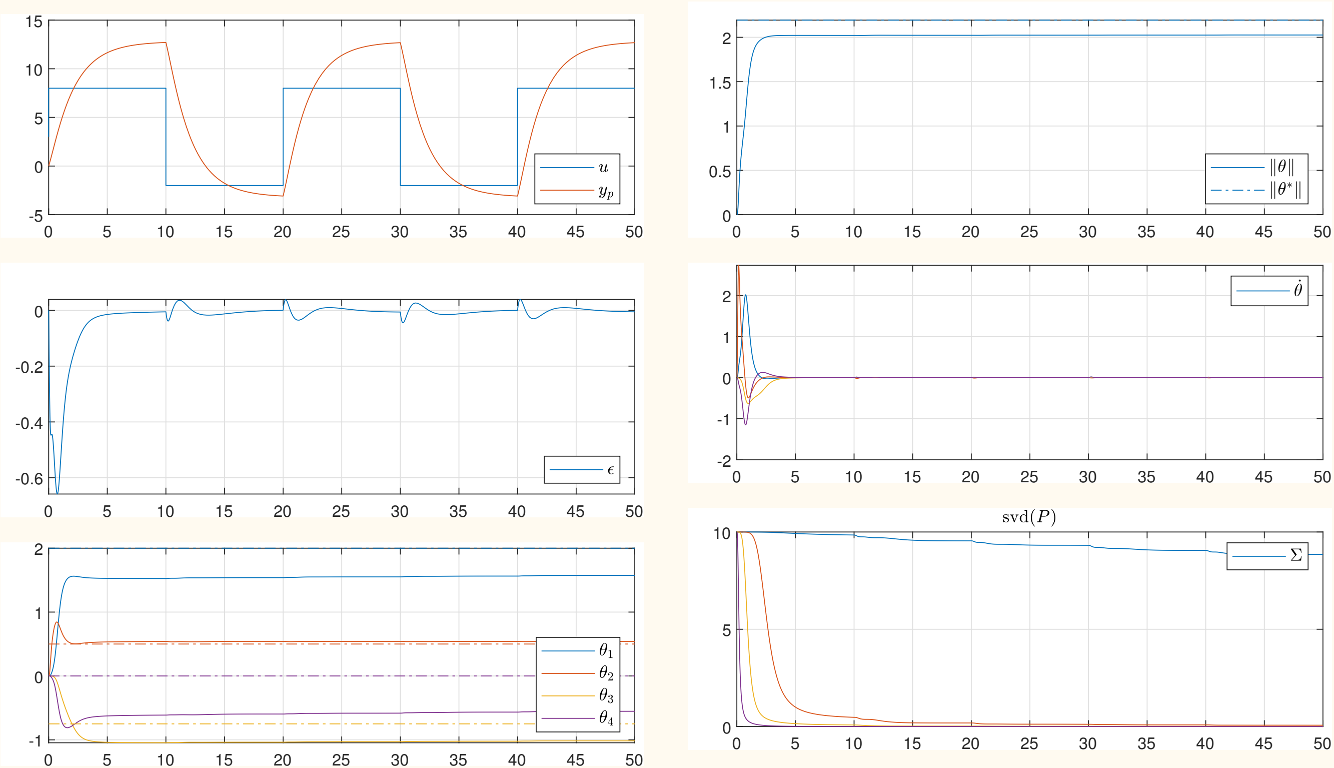


Figura 40: Normalized Least-squares. Plant-24.

Simulation 3 Effect of excitation frequencies.

Algorithm : **Normalized Least-squares**

Parameters . . . . :

$$P(s) = \frac{0.5s + 2}{s^2 + 3s + 1.25}$$

$$\Lambda(s) = s^2 + 3s + 2$$

$$\kappa = 0$$

Initial conditions . . . . :

$$x(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$$

$$\theta(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$P(0) = 10\, I$$

Input signal . . . . :

$$r = 3 + 5\,\mathrm{sign}\big(\sin(0.1\,\pi\, t)\big) + \sin(t) + 2\sin(2.3t)$$

Matching parameters:

$$\theta^* = \begin{bmatrix} 2 & 0.5 & -0.75 & 0 \end{bmatrix}$$

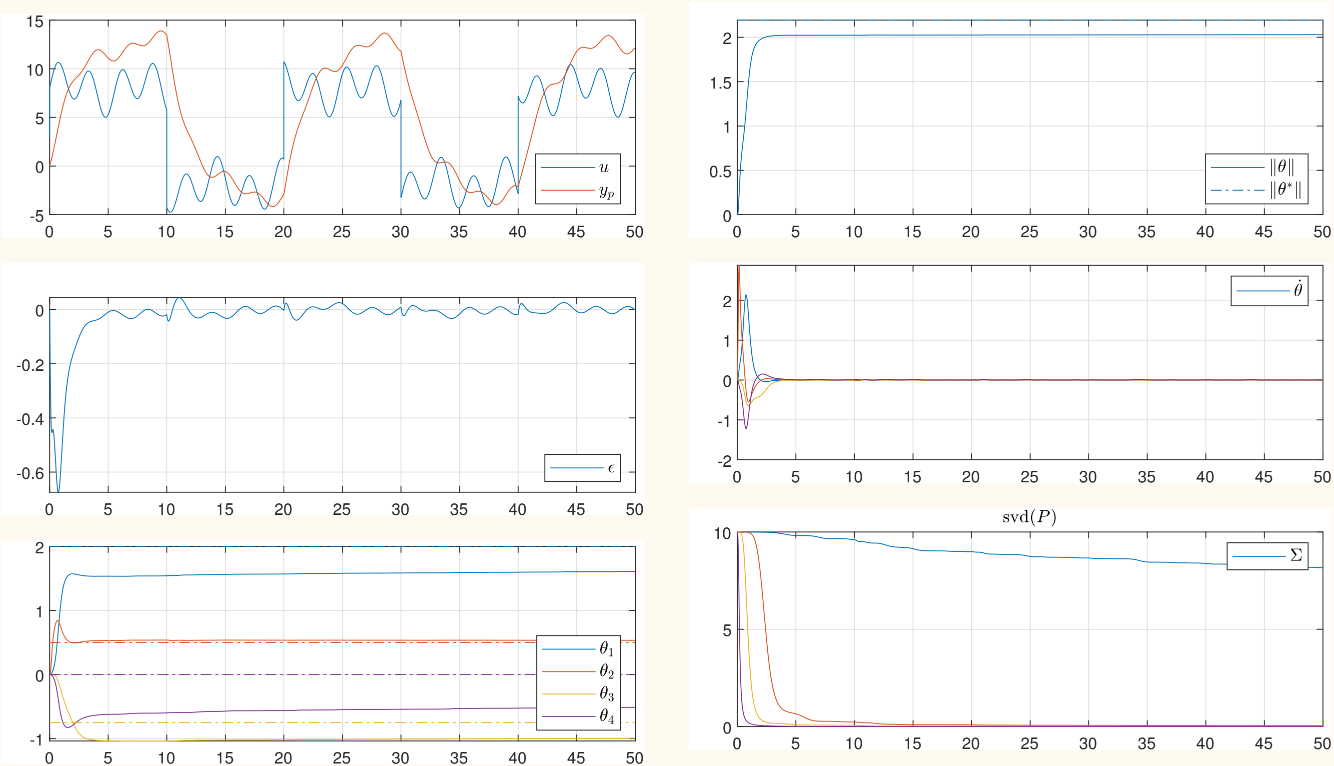


Figura 41: Normalized Least-squares. Plant-24.

Simulation 4 Effect of large initial condition.

Algorithm : **Normalized Least-squares**

Parameters . . . . :

$$P(s) = \frac{0.5s + 2}{s^2 + 3s + 1.25}$$

$$\Lambda(s) = s^2 + 3s + 2$$

$$\kappa = 0$$

Initial conditions . . . . :

$$x(0) = \boxed{[10 \quad 0]^T}$$

$$\theta(0) = \boxed{[0 \quad 0 \quad 0 \quad 0]^T}$$

$$P(0) = \boxed{10\, I}$$

Input signal . . . . :

$$r = 3 + 5\ \mathrm{sign}\big(\sin(0.1\,\pi\, t)\big) + \sin(t) + 2\sin(2.3t)$$

Matching parameters:

$$\theta^* = \begin{bmatrix} 2 & 0.5 & -0.75 & 0 \end{bmatrix}$$

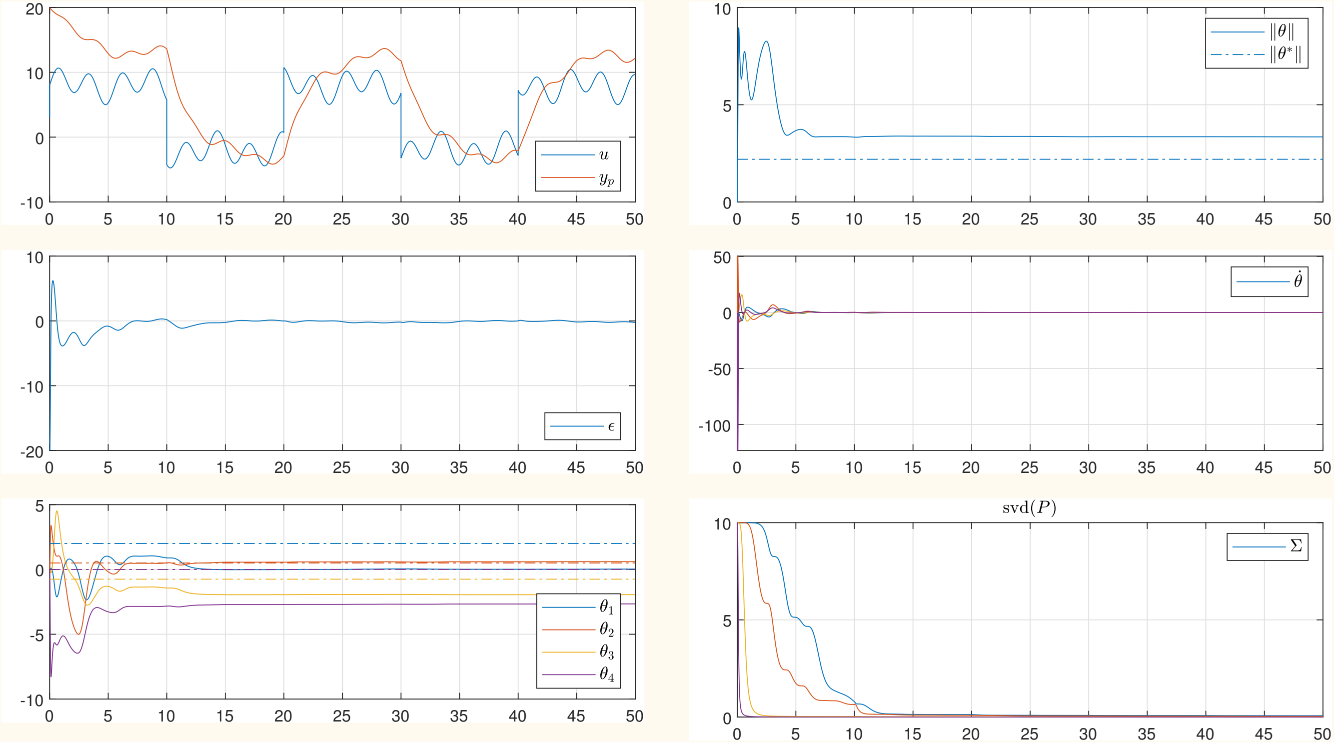


Figura 42: Normalized Least-squares. Plant-24.

Simulation 5 Effect of plant complex poles.

Algorithm : Normalized Least-squares

Parameters . . . . :

$$P(s) = \frac{0.5s+2}{s^2+s+1.25} = \frac{0.5s+2}{(s+0.5+i)(s+0.5-i)}$$

$$\Lambda(s) = s^2+3s+2$$

$$\kappa = 0$$

Initial conditions . . . . :

$$x(0) = \boxed{[0 \quad 0]^T}$$

$$\theta(0) = \boxed{[0 \quad 0 \quad 0 \quad 0]^T}$$

$$P(0) = \boxed{1\,I}$$

Input signal . . . . :

$$r = 3 + 5\,\mathrm{sign}\big(\sin(0.1\,\pi\,t)\big)$$

Matching parameters:

$$\theta^* = \begin{bmatrix}2 & 0.5 & -0.75 & -2\end{bmatrix}$$

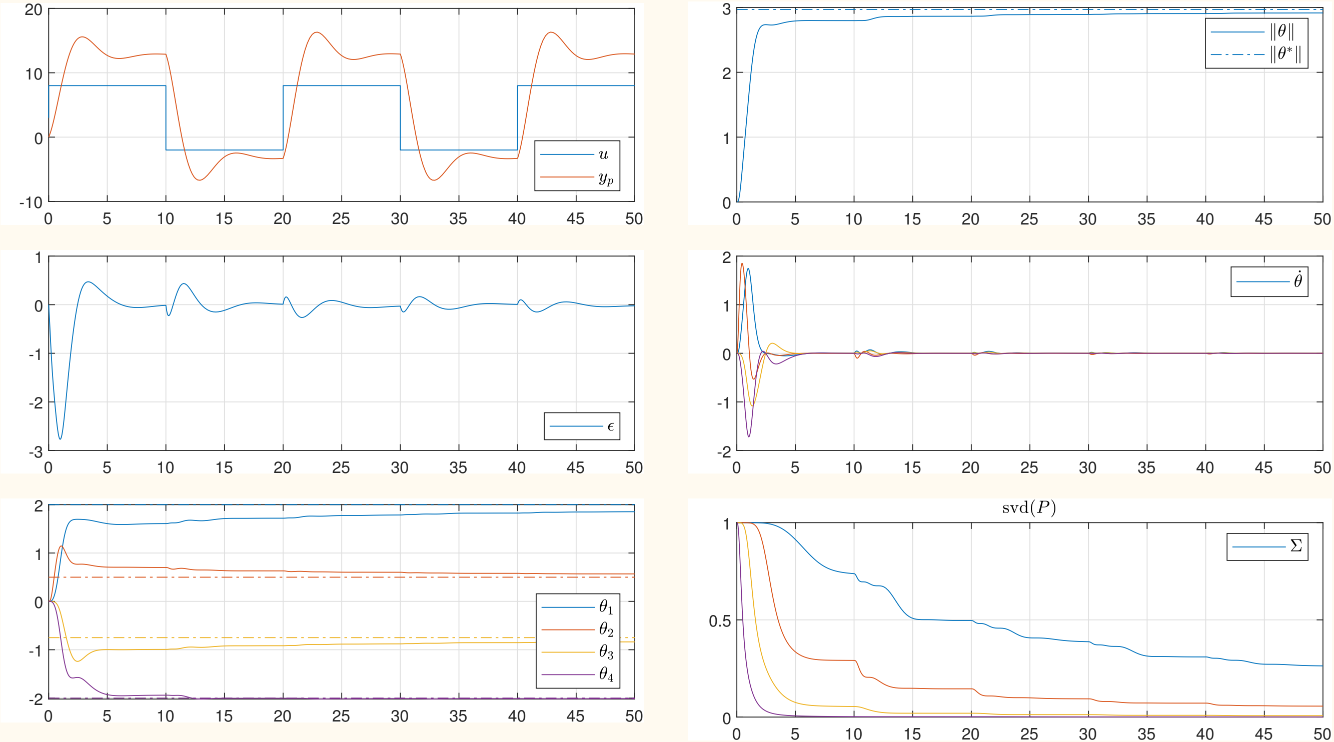


Figura 43: Normalized Least-squares. Plant-24.

Simulation 6 Effect of large gains.

Algorithm : **Normalized Least-squares**

Parameters . . . . :

$$P(s) = \frac{0.5s + 2}{s^2 + s + 1.25} = \frac{0.5s + 2}{(s + 0.5 + i)(s + 0.5 - i)}$$

$$\Lambda(s) = s^2 + 3s + 2$$

$$\kappa = 0$$

Initial conditions . . . . . :

$$x(0) = \boxed{[0 \;\; 0]^T}$$

$$\theta(0) = \boxed{[0 \;\; 0 \;\; 0 \;\; 0]^T}$$

$$P(0) = \boxed{10\, I}$$

Input signal . . . . :

$$r = 3 + 5\ \mathrm{sign}\big(\sin(0.1\,\pi\, t)\big)$$

Matching parameters:

$$\theta^* = \begin{bmatrix} 2 & 0.5 & -0.75 & -2 \end{bmatrix}$$

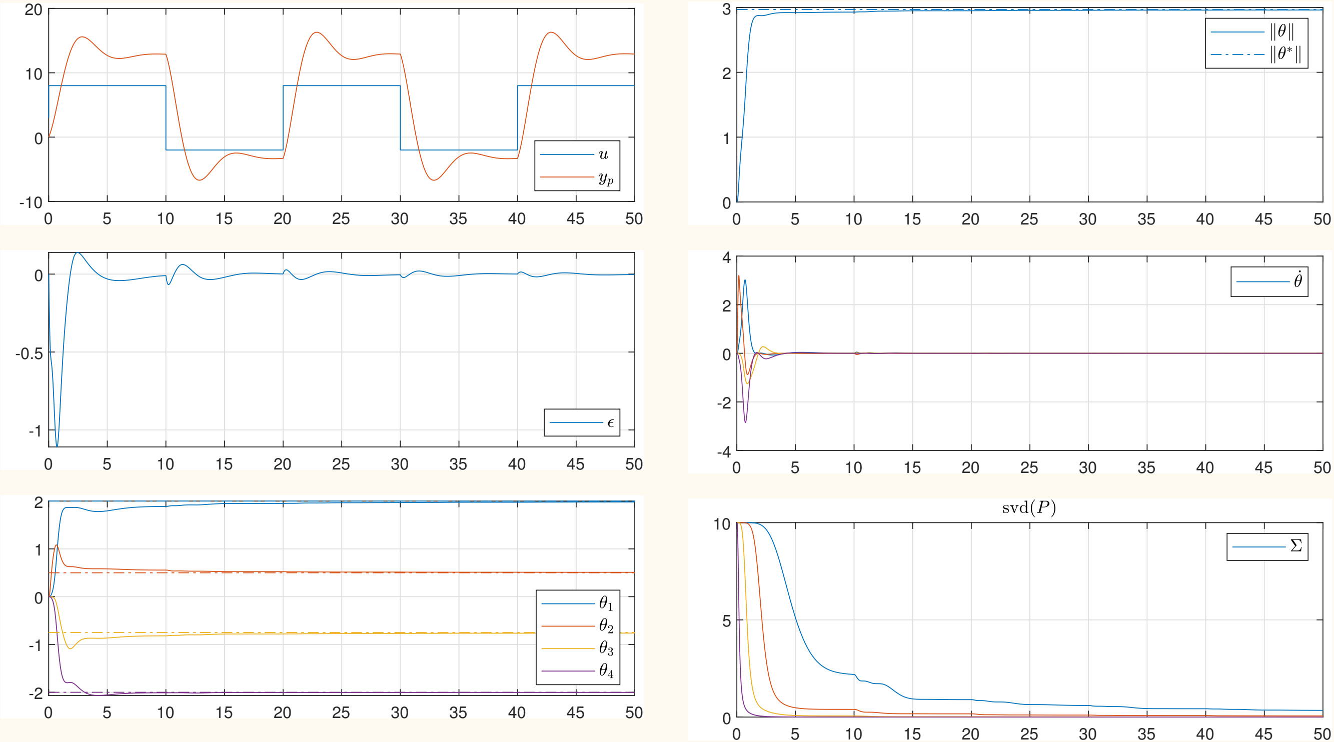


Figura 44: Normalized Least-squares. Plant-24.

3.4.2 Problemas & exercícios

(1) Mostre que a equação (4),

$$\theta(t) = P\left(P_0^{-1}\theta_0 + \int_{t_0}^{t} \frac{y\phi}{m^2} d\tau\right) ,$$

satisfaz a equação diferencial (2),

$$\dot{\theta} = -\frac{P\phi\epsilon}{m^2} , \quad \theta(t_0) = \theta_0 .$$

Solução:

3.5 Parameter convergence

Ref.: [Tao:2003], (pag. 108)

(...)

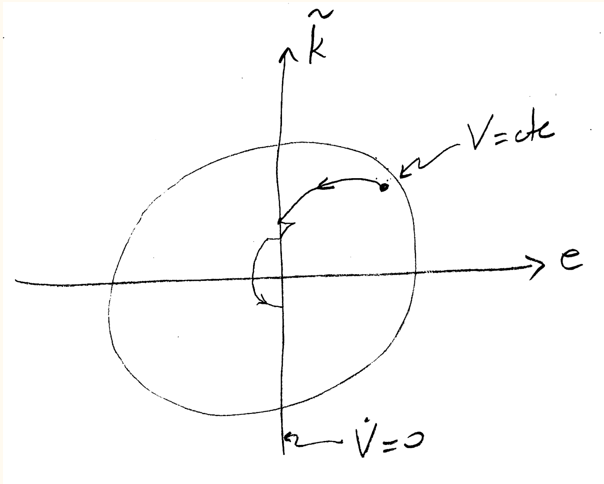


Figura 45: Interpretation: Condition for parameter convergence.

3.5.1 Improving convergence

Ref.: Adaptive observers with exponential rate of convergence, G. Kreisselmeier, 1977.

Idea: Use filters to generate multiple parameterizations.

Consider a 1st order plant with 2 unknown parameters

$$P(s) = \frac{N(s)}{D(s)} = \frac{b_0}{s + a_0}$$

and 2nd order filters

$$\frac{1}{\Lambda(s)} = \frac{1}{s^2 + \lambda_1 s + \lambda_0}$$

Filters implementation:

$$\begin{cases} \dot{\omega}_1 = A\omega_1 + bu \\ \dot{\omega}_2 = A\omega_2 + by \end{cases}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -\lambda_0 & -\lambda_1 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Filters variables:

$$\omega_1 = \begin{bmatrix} u_f \\ \dot{u}_f \end{bmatrix} \qquad \text{and} \qquad \omega_2 = \begin{bmatrix} y_f \\ \dot{y}_f \end{bmatrix}$$

We can write 2 parameterizations

Parameterization 1 . . : $\underbrace{\dot{y}_f}_{Y_1} = b_0 u_f - a_0 y_f = \underbrace{\begin{bmatrix} b_0 & a_0 \end{bmatrix}}_{\theta^*} \underbrace{\begin{bmatrix} u_f \\ -y_f \end{bmatrix}}_{\phi_1}$

Regressor 1 . . . . : $\phi_1 = \begin{bmatrix} u_f & -y_f \end{bmatrix}^T$

Parameters . . . . : $\theta^* = \begin{bmatrix} b_0 & a_0 \end{bmatrix}^T$

Parameterization 2 . . : $\underbrace{\ddot{y}_f}_{Y_2} = b_0 \dot{u}_f - a_0 \dot{y}_f = \underbrace{\begin{bmatrix} b_0 & a_0 \end{bmatrix}}_{\theta^*} \underbrace{\begin{bmatrix} \dot{u}_f \\ -\dot{y}_f \end{bmatrix}}_{\phi_2}$

Regressor 2 . . . . : $\phi_2 = \begin{bmatrix} \dot{u}_f & -\dot{y}_f \end{bmatrix}^T$

Parameters . . . . : $\theta^* = \begin{bmatrix} b_0 & a_0 \end{bmatrix}^T$

★ Notice that the parameters are the same!

Estimates.................: $\hat{Y}_1 = \theta^T \phi_1$

$$\hat{Y}_2 = \theta^T \phi_2$$

Errors.......................: $\epsilon_1 = \hat{Y}_1 - Y_1 = \tilde{\theta}^T \phi_1$

$$\epsilon_2 = \hat{Y}_2 - Y_2 = \tilde{\theta}^T \phi_2$$

Normalization..........: $m_1^2 = 1 + \kappa_1 \, \phi_1^T \phi_1$

$$m_2^2 = 1 + \kappa_2 \, \phi_2^T \phi_2$$

Gradient update law..: $\boxed{\dot{\theta} = -\gamma_1 \dfrac{\phi_1 \epsilon_1}{m_1^2} - \gamma_2 \dfrac{\phi_2 \epsilon_2}{m_2^2}}$

Using the Lyapunov function

$$2V\left(\tilde{\theta}\right) = \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$$

we get

$$\begin{aligned}
\dot{V}\left(\tilde{\theta}\right) &= \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}} \\
&= \tilde{\theta}^T \left(-\gamma_1 \frac{\phi_1 \epsilon_1}{m_1^2} - \gamma_2 \frac{\phi_2 \epsilon_2}{m_2^2} \right) \\
&= -\tilde{\theta}^T \underbrace{\left(\gamma_1 \frac{\phi_1 \phi_1^T}{m_1^2} + \gamma_2 \frac{\phi_2 \phi_2^T}{m_2^2} \right)}_{A(t)} \tilde{\theta} \quad \leq 0
\end{aligned}$$

★ Note that now $\operatorname{rank}(A) \leq 2$!

3.5.2 Simulations

Case 1 First order plant, 2 unknown parameters.

Plant .... : $P(s) = \dfrac{N(s)}{D(s)} = \dfrac{b_0}{s + a_0}$

Filter .... : $\dfrac{k}{\Lambda(s)} = \dfrac{\lambda_1 \, \lambda_2}{(s - \lambda_1)(s - \lambda_2)}$

Summary of the equations

Plant ....: $\dot{y} = -a_0 y + b_0 u$

Filter ....: $$\frac{k}{\Lambda(s)} = \frac{\lambda_1 \lambda_2}{(s-\lambda_1)(s-\lambda_2)}$$

Parameterization 1 . . : $$\underbrace{\dot{y}_f}_{Y_1} = b_0 u_f - a_0 y_f = \begin{bmatrix} b_0 & a_0 \end{bmatrix} \underbrace{\begin{bmatrix} u_f \\ -y_f \end{bmatrix}}_{\phi_1}$$

Regressor 1 ....: $\phi_1 = \begin{bmatrix} u_f & -y_f \end{bmatrix}^T$

Parameterization 2 . . : $$\underbrace{\ddot{y}_f}_{Y_2} = b_0 \dot{u}_f - a_0 \dot{y}_f = \begin{bmatrix} b_0 & a_0 \end{bmatrix} \underbrace{\begin{bmatrix} \dot{u}_f \\ -\dot{y}_f \end{bmatrix}}_{\phi_2}$$

Regressor 2 ....: $\phi_2 = \begin{bmatrix} \dot{u}_f & -\dot{y}_f \end{bmatrix}^T$

Estimates................: $\hat{Y}_i = \theta^T \phi_i$

Errors......................: $\epsilon_i = \hat{Y}_i - Y_i$

Normalization..........: $m_i^2 = 1 + \kappa_i \, \phi_i^T \phi_i$

Update law..............: $$\boxed{\dot{\theta} = -\gamma_1 \frac{\phi_1 \epsilon_1}{m_1^2} - \gamma_2 \frac{\phi_2 \epsilon_2}{m_2^2}}$$

Simulation 1 Zero initial conditions.

Algorithm : **Normalized Gradient**

Parameters . . . . :

$$a_0 = 3.3 \qquad b_0 = 5$$
$$\lambda_1 = -1 \qquad \lambda_2 = -2$$
$$\gamma_1 = 20 \qquad \gamma_2 = 20$$
$$\kappa_1 = 0 \qquad \kappa_2 = 0$$

Initial conditions . . . . :

$$y(0) = 0$$
$$\theta_i(0) = 0$$

Input signal . . . . :

$$r = 1.5 + 1\ \mathrm{sign}\big(\sin(0.2\,\pi\,t)\big)$$

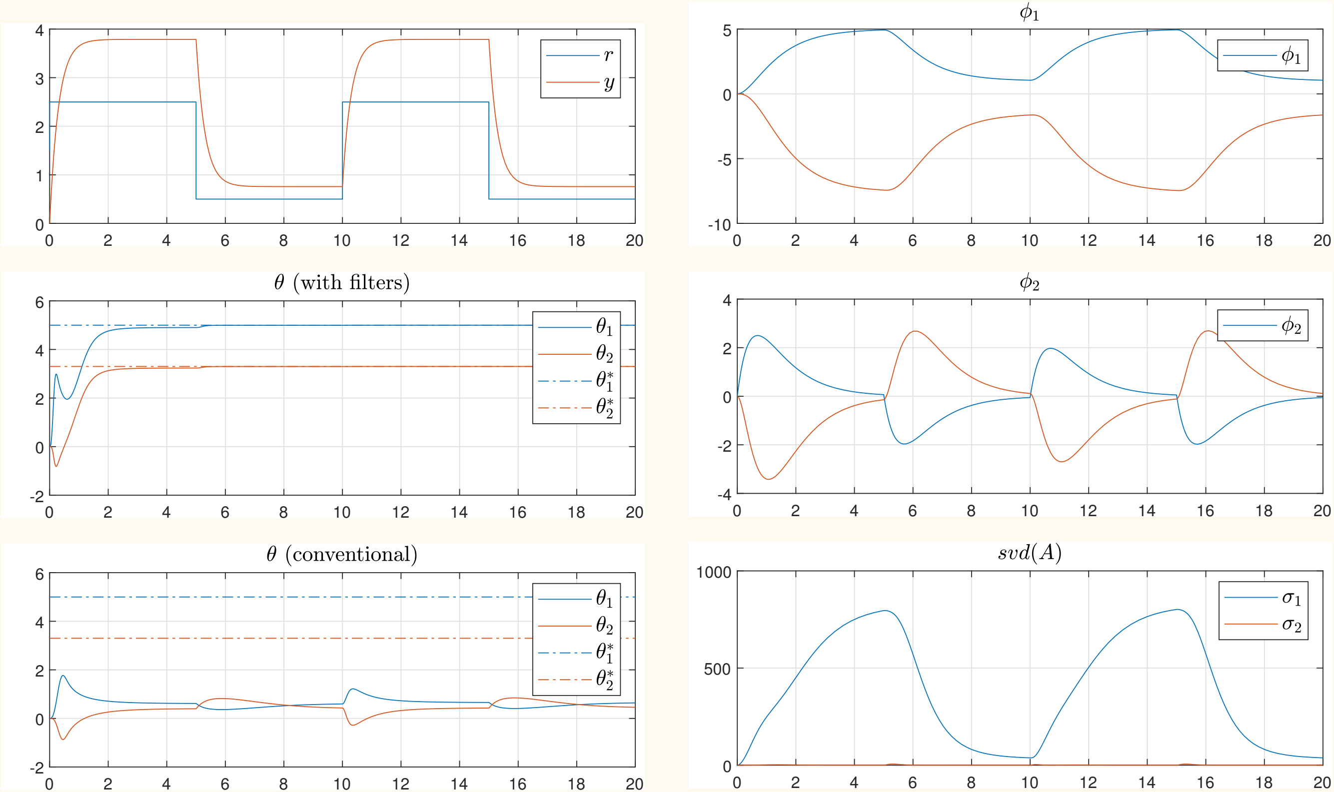


Figura 46: Normalized Gradient with filters. Plant-12.

Simulation 2 Zero initial conditions. New filter.

Algorithm : **Normalized Gradient**

Parameters . . . . :

$$a_0 = 3.3 \qquad b_0 = 5$$

$$\lambda_1 = \boxed{-1 - i} \qquad \lambda_2 = \boxed{-1 + i}$$

$$\gamma_1 = 20\, I \qquad \gamma_2 = 20$$

$$\kappa_1 = 0 \qquad \kappa_2 = 0$$

Initial conditions . . . . :

$$y(0) = \boxed{0}$$

$$\theta_i(0) = \boxed{0}$$

Input signal . . . . :

$$r = 1.5 + 1\ \mathrm{sign}\big(\sin(0.2\,\pi\,t)\big)$$

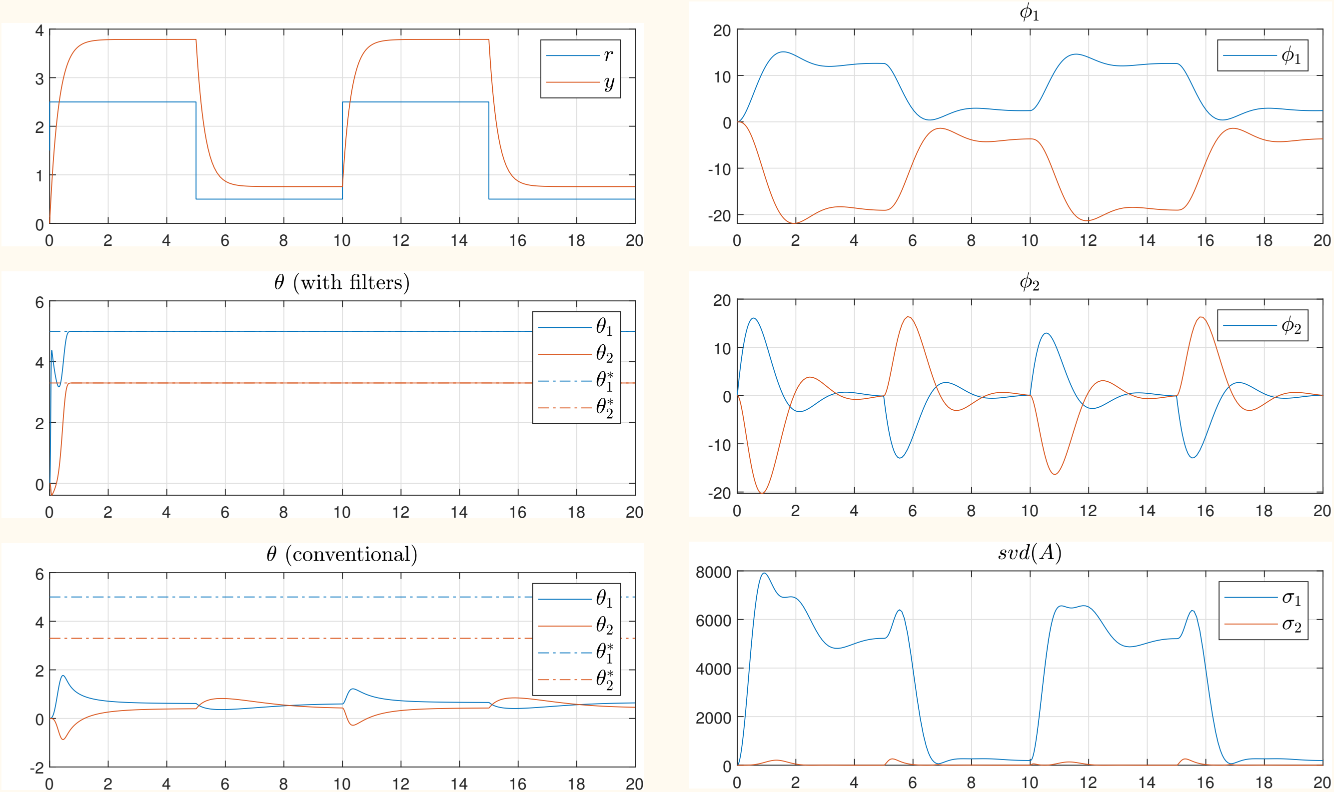


Figura 47: Normalized Gradient with filters. Plant-12.

Simulation 3 Zero initial conditions. New reference.

Algorithm : **Normalized Gradient**

Parameters . . . . :

$$a_0 = 3.3 \qquad b_0 = 5$$
$$\lambda_1 = -1 - i \qquad \lambda_2 = -1 + i$$
$$\gamma_1 = 20 \qquad \gamma_2 = 20$$
$$\kappa_1 = 0 \qquad \kappa_2 = 0$$

Initial conditions . . . . :

$$y(0) = 0$$
$$\theta_i(0) = 0$$

Input signal . . . . :

$$r = 1 + 2\,\mathrm{sign}\big(\sin(0.2\,\pi\,t)\big)$$

★ Better convergence of the conventional algorithm! Why?

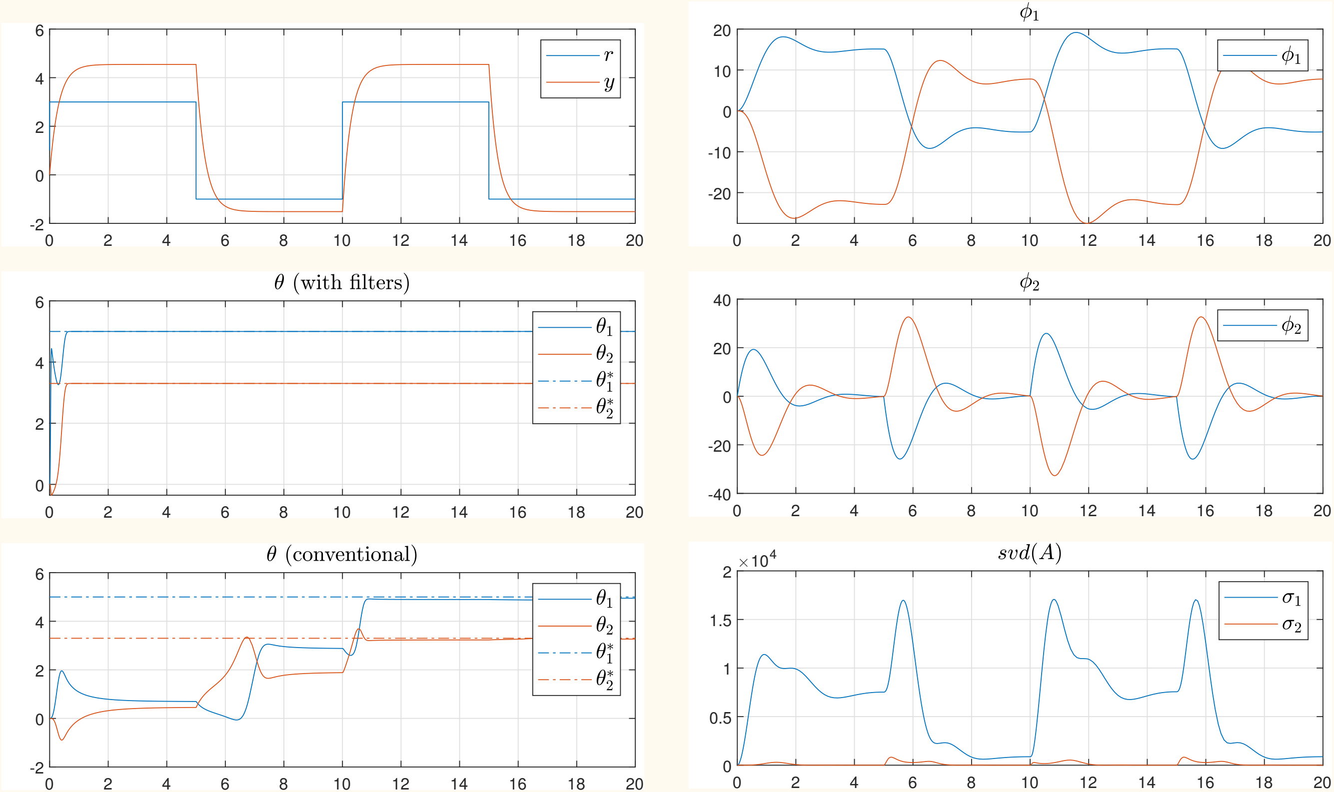


Figura 48: Normalized Gradient with filters. Plant-12.

Case 2 2nd order plant, 4 unknown parameters.

Plant ....: $P(s) = \dfrac{N(s)}{D(s)} = \dfrac{k_p(s - z_1)}{(s - p_1)(s - p_2)}$

Filter ....: $\dfrac{k}{\Lambda(s)} = \dfrac{\lambda_1\,\lambda_2\,\lambda_3\,\lambda_4\,\lambda_5}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)(s-\lambda_4)(s-\lambda_5)}$

Summary of the equations

Regressor 1 . . . . : $\phi_1 = \begin{bmatrix} u_f & \dot{u}_f & -y_f & -\dot{y}_f \end{bmatrix}^T$

Regressor 2 . . . . : $\phi_2 = \begin{bmatrix} \dot{u}_f & \ddot{u}_f & -\dot{y}_f & -\ddot{y}_f \end{bmatrix}^T$

Regressor 3 . . . . : $\phi_3 = \begin{bmatrix} \ddot{u}_f & u_f^{(3)} & -\ddot{y}_f & -y_f^{(3)} \end{bmatrix}^T$

Regressor 4 . . . . : $\phi_4 = \begin{bmatrix} u_f^{(3)} & u_f^{(4)} & -y_f^{(3)} & -y_f^{(4)} \end{bmatrix}^T$

Estimates . . . . : $\hat{Y}_i = \theta^T \phi_i$

Errors . . . . : $\epsilon_i = \hat{Y}_i - Y_i$

Normalization . . . . : $m_i^2 = 1 + \kappa_i \, \phi_i^T \phi_i$

Update law . . . . : $$\boxed{\dot{\theta} = -\sum_{i=1}^{4} \gamma_i \frac{\phi_i \epsilon_i}{m_i^2}}$$

Simulation 1 Zero initial conditions.

Algorithm : **Normalized Gradient**

Parameters . . . . :

$$p_1 = -1.3 \qquad p_2 = -2.1 \qquad z_1 = -4 \qquad k_p = 2$$

$$\lambda = \{-0.5 + i\ ,\quad -0.5 - i\ ,\quad -1\ ,\quad -1 + 2i\ ,\quad -1 - 2i\}$$

$$\gamma_i = 100$$

$$\kappa_i = 0$$

Initial conditions . . . . :

$$y(0) = 0$$

$$\theta_i(0) = 0$$

Input signal . . . . :

$$r = 1.5 + 1\ \mathrm{sign}\big(\sin(0.2\,\pi\, t)\big)$$

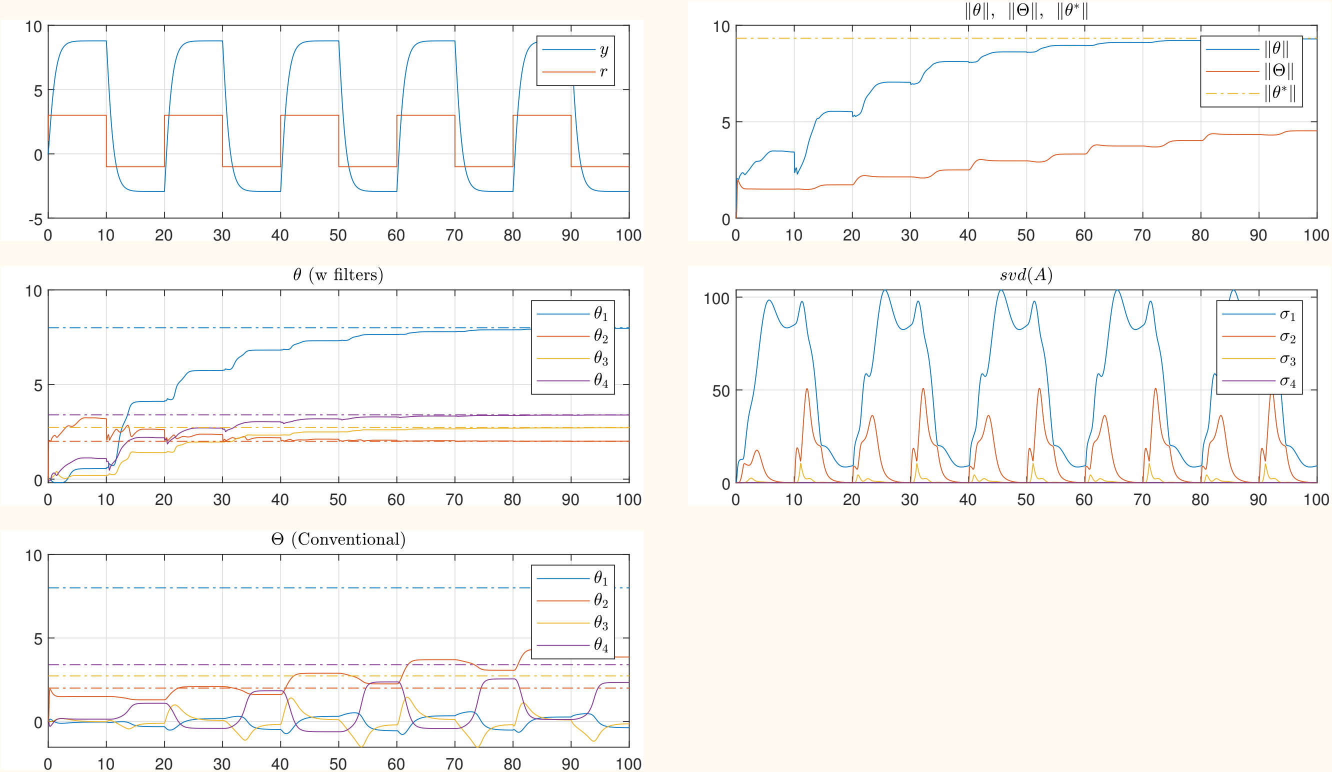


Figura 49: Normalized Gradient with filters. Plant-214.

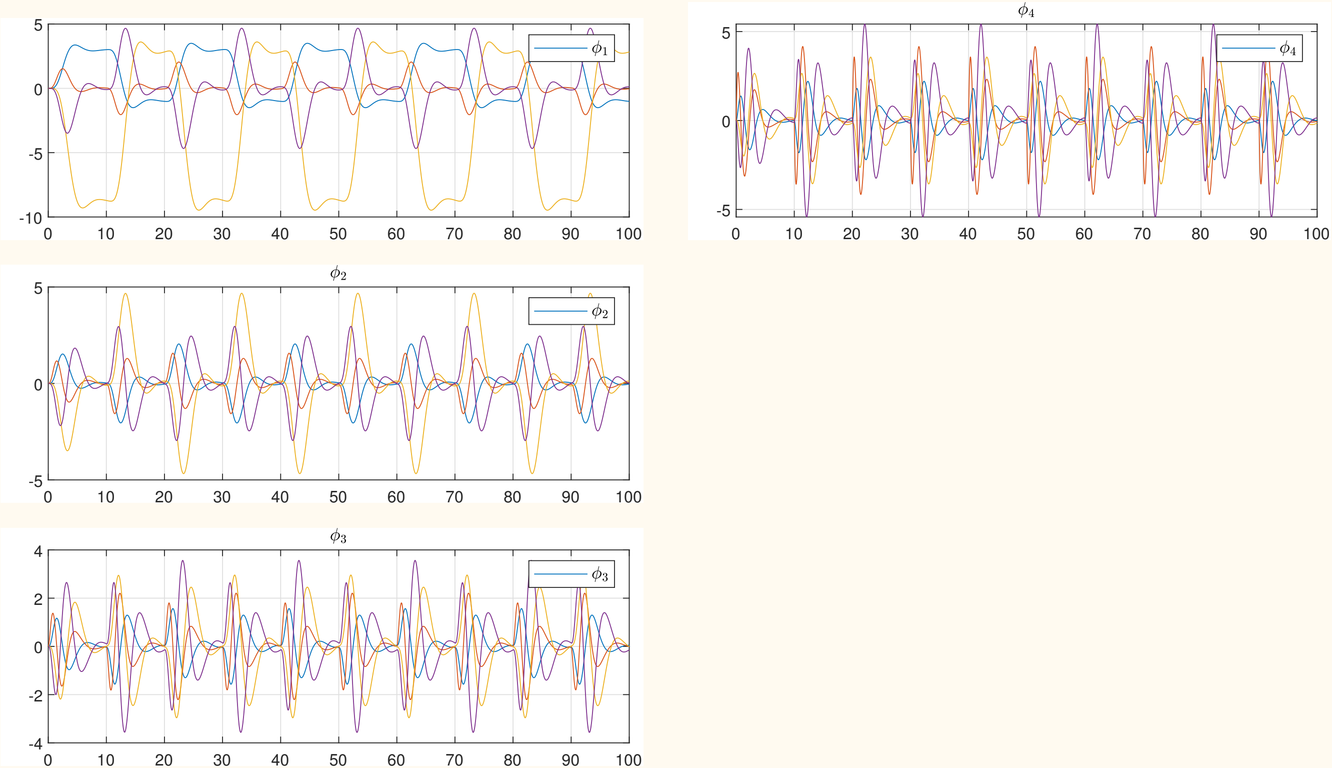


Figura 50: Normalized Gradient with filters. Plant-214.

Simulation 2 Zero initial conditions. New plant.

Algorithm : **Normalized Gradient**

Parameters . . . . : $p_1 = -3 \qquad p_2 = -5 \qquad z_1 = -4 \qquad k_p = -5$

$$\lambda = \{-0.5 + i \ , \ \ -0.5 - i \ , \ \ -1 \ , \ \ -1 + 2i \ , \ \ -1 - 2i\}$$

$$\gamma_i = 1e5$$

$$\kappa_i = 0$$

Initial conditions . . . . : $y(0) = 0$

$$\theta_i(0) = 0$$

Input signal . . . . : $r = 1.5 + 1 \ \mathrm{sign}\big(\sin(0.2\,\pi\,t)\big)$

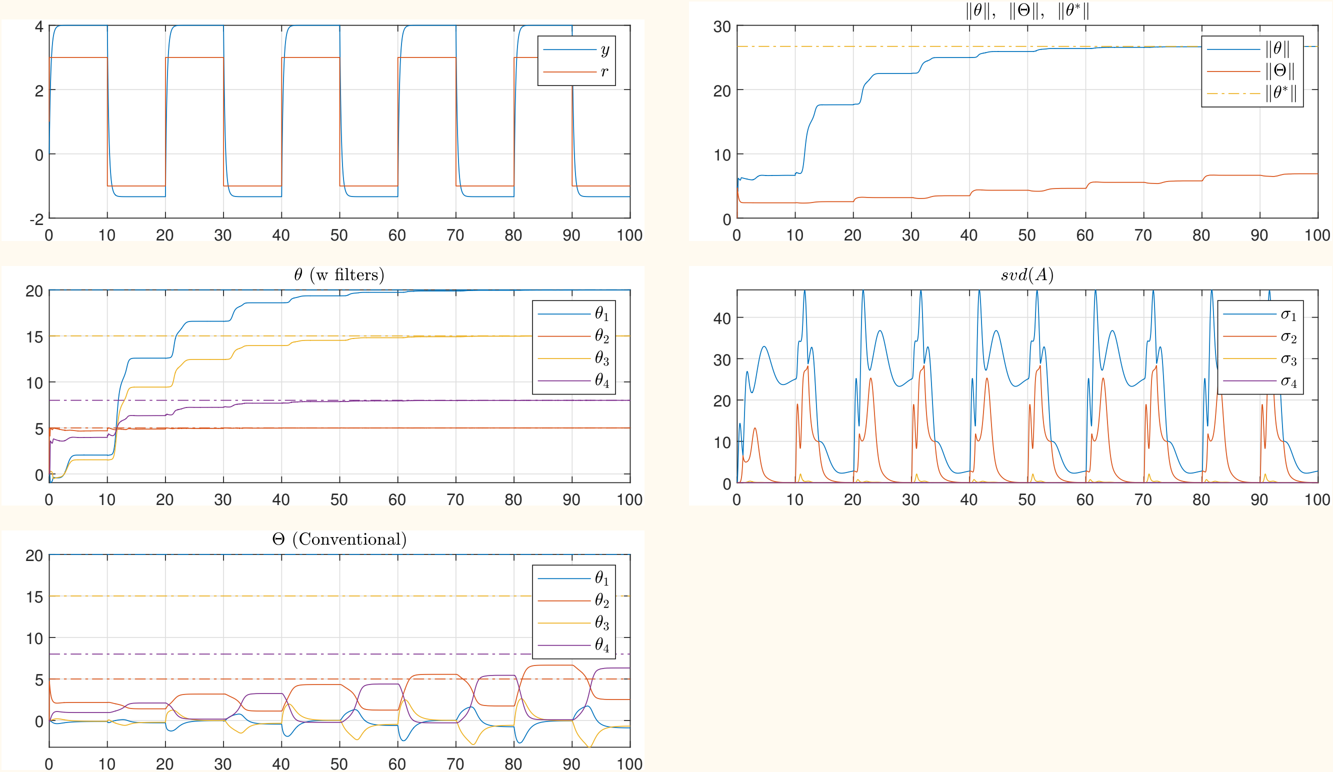


Figura 51: Normalized Gradient with filters. Plant-214.

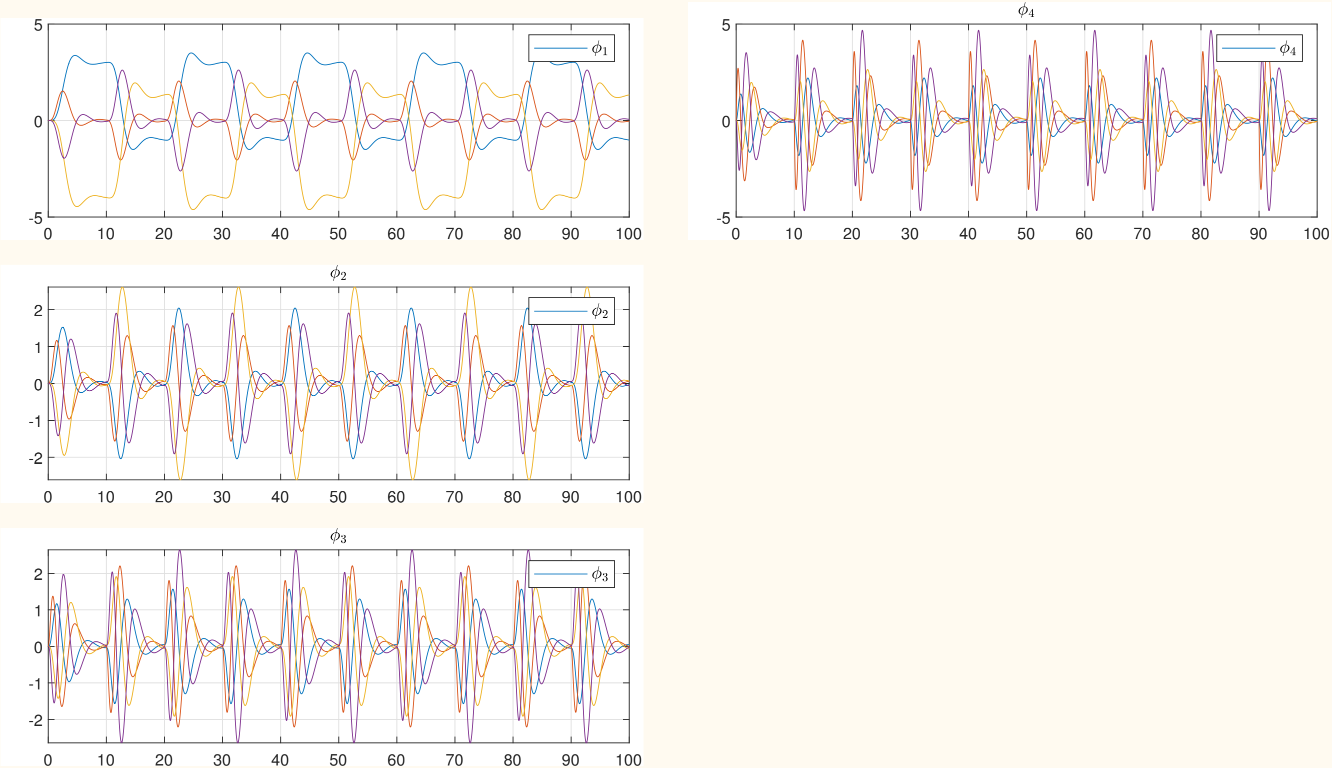


Figura 52: Normalized Gradient with filters. Plant-214.

3.6 Adaptação robusta

Ref.: [Tao:2003], (pag. 128)

Problema: Incerteza na planta.

★ Muitas modificações foram propostas para garantir as propriedades dos algoritmos.

Planta com incerteza : $$y = \theta^{*T}\phi - \delta(t)$$

Hipótese : $$|\delta(t)| \leq \delta_1(t)\|\phi(t)\| + \delta_2(t)$$

Lei de adaptação do gradiente modificada: $$\dot{\theta} = -\Gamma\frac{\phi\epsilon}{m^2} + f(t)$$

Normalização : $$m^2 = 1 + \kappa\phi^T\phi$$

★ $f(t)$ = modificação.

3.6.1 Modificação zona-morta

Algoritmo :

$$\boxed{\dot{\theta} = -\Gamma \frac{\phi \epsilon}{m^2} + f(t)}$$

$$\boxed{f(t) = \Gamma \frac{\phi f_0(\epsilon)}{m^2}}$$

$$\boxed{f_0(\epsilon) = \begin{cases} \epsilon & \text{se} \quad \frac{|\epsilon|}{m} < d_1 + \frac{d_2}{m} \\ 0 & \text{se} \quad \frac{|\epsilon|}{m} \geq d_1 + \frac{d_2}{m} \end{cases}}$$

3.6.2 Modificação σ

Algoritmo :

$$\boxed{\dot{\theta} = -\Gamma \frac{\phi \epsilon}{m^2} + f(t)}$$

$$\boxed{f(t) = -\sigma \Gamma \theta}$$

3.6.3 Modificação σ com switching

Algoritmo :

$$\dot{\theta} = -\Gamma \frac{\phi \epsilon}{m^2} + f(t)$$

$$f(t) = -\sigma(t)\Gamma\theta$$

$$\sigma(t) = \begin{cases} 0 & \text{se} \quad \|\theta\| < M_\theta \\ \sigma_0\left(\frac{\|\theta\|}{M_\theta} - 1\right) & \text{se} \quad M_\theta \leq \|\theta\| < 2M_\theta \\ \sigma_0 & \text{se} \quad \|\theta\| \geq 2M_\theta \end{cases}$$

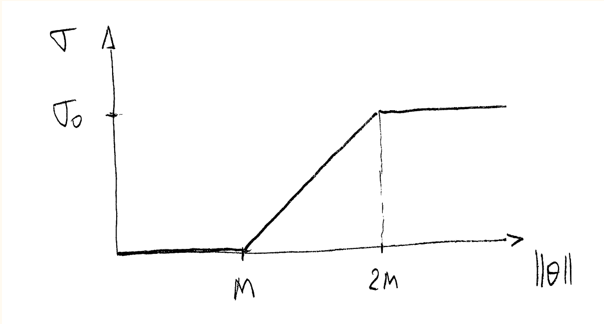


Figura 53: σ function.

3.6.4 Projeção

Algoritmo :

$$\dot{\theta} = -\Gamma \frac{\phi \epsilon}{m^2} + f(t)$$

$$f(t) = -\sigma(t)\theta$$

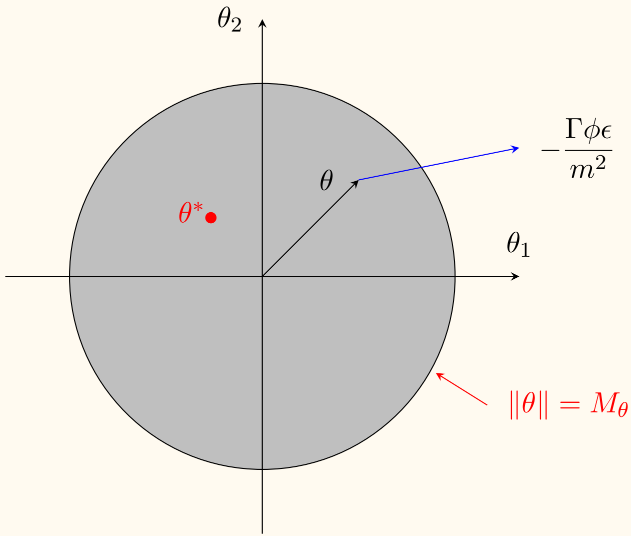
$$\sigma(t) = \begin{cases} \sigma_0 & \text{se} \quad \|\theta\| < M_\theta \quad \text{ou} \quad \sigma_{eq} < \sigma_0 \\ \sigma_{eq} & \text{se} \quad \|\theta\| \geq M_\theta \quad \text{e} \quad \sigma_{eq} \geq \sigma_0 \end{cases}$$

$$\sigma_{eq} = -\frac{\theta^T \Gamma \phi \epsilon}{\|\theta\|^2 m^2}$$

Interpretação

No caso em que $\sigma_0 = 0$, se $\|\theta\| < M_\theta$, então

$$f(t) = 0 \quad \Rightarrow \quad \boxed{\dot{\theta} = -\Gamma \frac{\phi\epsilon}{m^2}}$$



Se $\|\theta\| = M_\theta$,

$$f(t) = -\sigma\theta \quad \Rightarrow \quad \boxed{\dot{\theta} = -\Gamma\frac{\phi\epsilon}{m^2} - \sigma\theta}$$

Podemos determinar o valor de σ para que $\dot{\theta}$ seja perpendicular ao vetor θ,

$$\theta^T \perp \dot{\theta} \quad \Rightarrow \quad \theta^T\dot{\theta} = 0$$

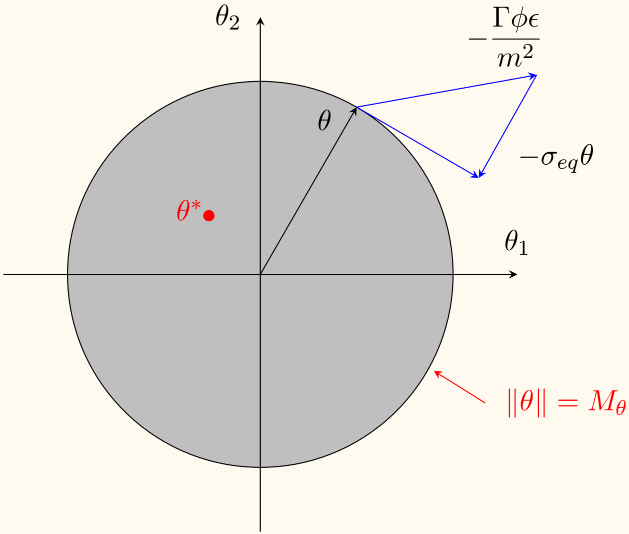
Portanto,

$$\theta^T\dot{\theta} = -\theta^T\Gamma\frac{\phi\epsilon}{m^2} - \sigma\theta^T\theta = 0 \quad \Rightarrow \quad \sigma = -\frac{\theta^T\Gamma\phi\epsilon}{\|\theta\|^2\,m^2}$$

Por conveniência, vamos chamar este valor particular de σ_{eq},

$$\boxed{\sigma_{eq} = -\frac{\theta^T\Gamma\phi\epsilon}{\|\theta\|^2\,m^2}}$$

A figura abaixo mostra a interpretação geométrica dessa expressão.



Note que os vetores θ e $\sigma_{eq}\theta$ são colineares.

Portanto,

$$\sigma_{eq} > 0 \quad \Rightarrow \quad -\Gamma\frac{\phi\epsilon}{m^2} \text{ aponta para fora da bola.}$$

$$\sigma_{eq} < 0 \quad \Rightarrow \quad -\Gamma\frac{\phi\epsilon}{m^2} \text{ aponta para dentro da bola.}$$

4 MRAC

Contents

4.1 Solução do controle 2DOF

★ Revisão...

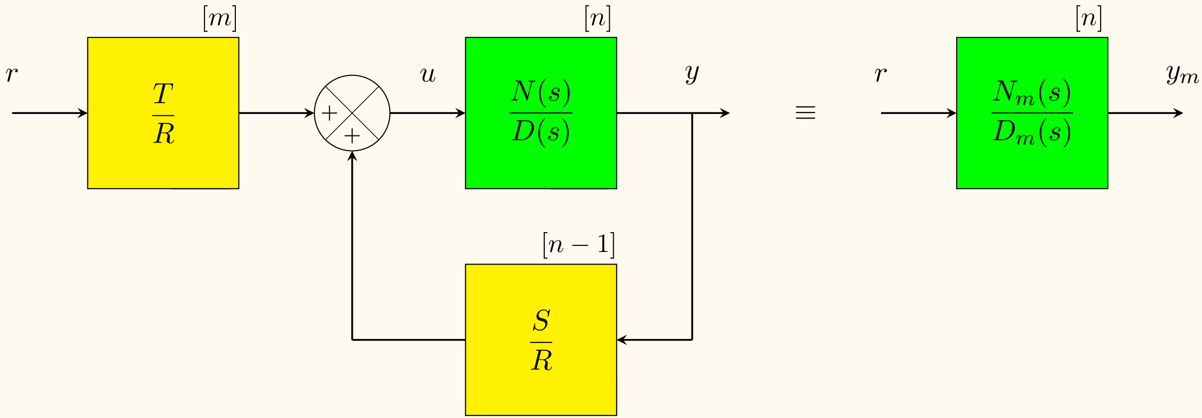


Figura 54: Estrutura do controlador 2DOF.

Lei de controle linear: $\boxed{Ru = Tr + Sy} \quad \Rightarrow \quad u = \frac{T}{R} r + \frac{S}{R} y$

Feedforward (para o caso $n^* = 1$): $\dfrac{T}{R} = \dfrac{N_m}{N} \quad \Rightarrow \quad R = N$

Problema: Como implementar?

- ★ $N(s)$ é desconhecido e será identificado.
- ★ $N(s)$ não pode aparecer no denominador!

Solução: Utilizar estrutura com filtros (semelhante ao caso do identificador).

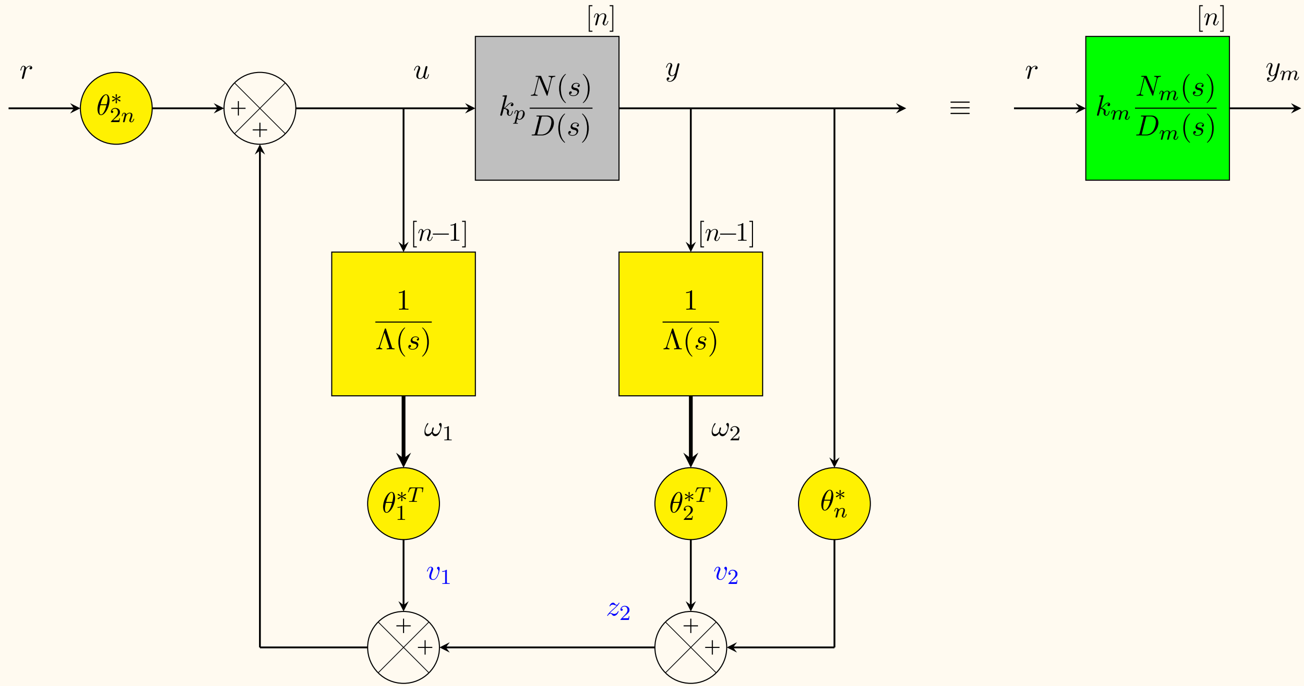


Figura 55: Estrutura do MRAC.

Podemos verificar que esta estrutura é equivalente à 2DOF.

Filtro 1: $$\begin{cases} \dot{\omega}_1 = A_f\,\omega_1 + b_f\,u \\ v_1 = \theta_1^{*T}\omega_1 \end{cases} \Rightarrow \boxed{v_1 = \frac{F}{\Lambda}\,u}$$

- $\boxed{\mathrm{grau}(F) = n - 2}$

★ Note que não há ramo direto no filtro de u (geraria um loop algébrico).

Filtro 2: $$\begin{cases} \dot{\omega}_2 = A_f\,\omega_2 + b_f\,y \\ z_2 = \theta_2^{*T}\omega_2 + \theta_n^{*}\,y \end{cases} \Rightarrow \boxed{z_2 = \frac{G}{\Lambda}\,y}$$

- $\boxed{\mathrm{grau}(G) = n - 1}$

★ Note que há um ramo direto.

Lei de controle: $$\boxed{u = \theta^{*T}\omega}$$

Parâmetros: $$\boxed{\theta^{*T} = \begin{bmatrix}\theta_1^{*T} & \theta_n^* & \theta_2^{*T} & \theta_{2n}\end{bmatrix}}$$ $\theta^* \in \mathbb{R}^{2n}$

Regressor: $$\boxed{\omega^T = \begin{bmatrix}\omega_1^T & y & \omega_2^T & r\end{bmatrix}}$$ $\omega \in \mathbb{R}^{2n}$

★ A ordem é arbitrária!

Podemos simplificar o diagrama de blocos.

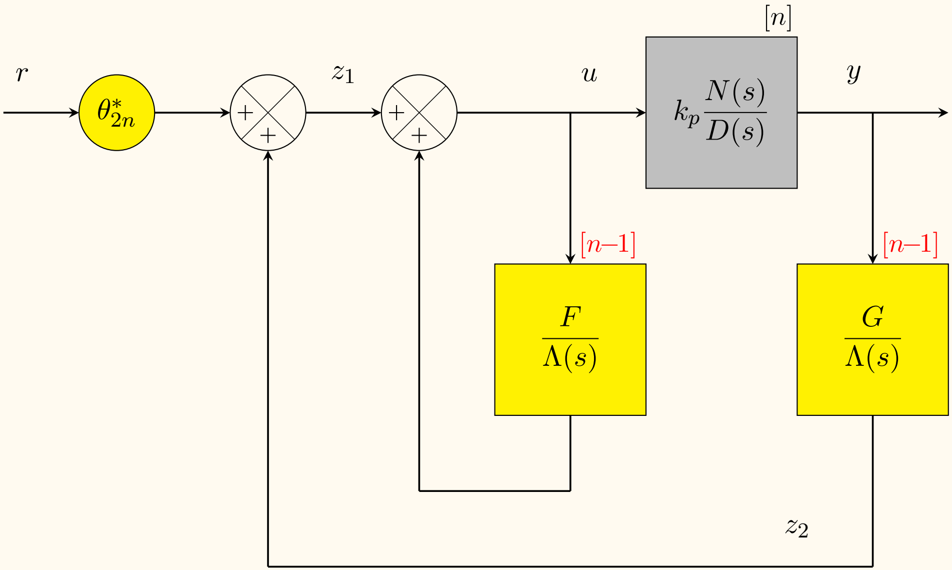


Figura 56: Estrutura do MRAC.

Note que: $$u = z_1 + \frac{F}{\Lambda} u \quad \Rightarrow \quad (\Lambda - F)u = \Lambda\, z_1$$

Portanto, $$\boxed{u = \frac{\Lambda}{\Lambda - F} z_1}$$

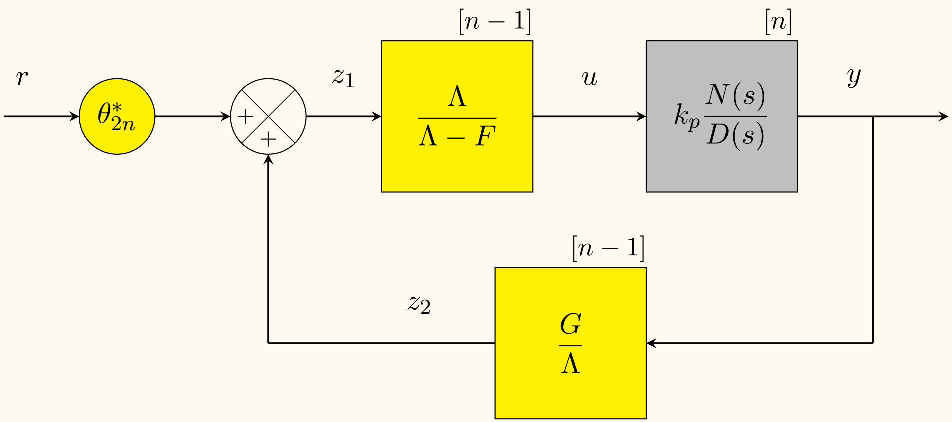


Figura 57: Estrutura do MRAC.

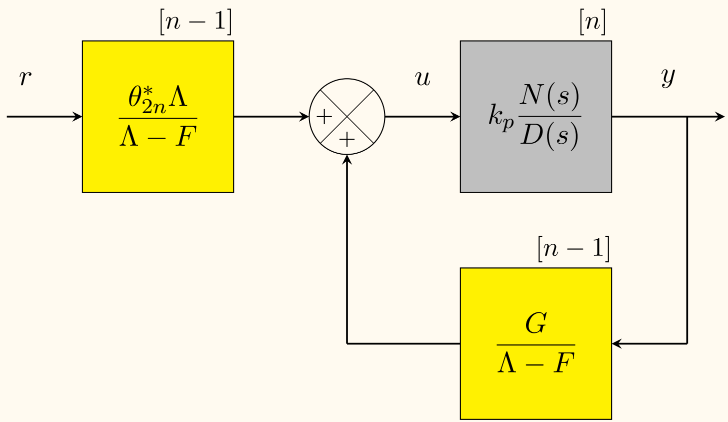


Figura 58: Estrutura do MRAC.

Malha fechada: $$y = k_p \frac{N}{D}\left(\frac{\theta_{2n}^* \Lambda}{\Lambda - F}\, r + \frac{G}{\Lambda - F}\, y\right) \quad \Rightarrow$$

$$\Rightarrow \quad D\big(\Lambda - F\big)\, y = \theta_{2n}^* k_p \Lambda N\, r + k_p N G\, y \quad \Rightarrow \quad \boxed{y = \frac{\theta_{2n}^*\, k_p\, \Lambda N}{\big(\Lambda - F\big) D - k_p N G}\, r}$$

Matching antes dos cancelamentos:

$$\boxed{\frac{\theta_{2n}^{*}\, k_p\, \Lambda N}{(\Lambda - F)D - k_p N G} = \frac{k_m\, N_m N A_0}{D_m N A_0}}$$

★ A_0: polinômio do observador.

★ Cancelamentos necessários: N e A_0.

Condições necessárias para a solução:

- $\boxed{\theta_{2n}^{*} = \frac{k_m}{k_p}} \quad \Rightarrow \quad \theta_{2n}^{*} k_p = k_m$

- $\boxed{\Lambda = N_m A_0}$ (Para o *matching* do numerador.)

- $\begin{cases} \mathrm{grau}(\Lambda) = n - 1 \\ \mathrm{grau}(N_m) = m \end{cases} \quad \Rightarrow \quad \boxed{\mathrm{grau}(A_0) = n - m - 1 = n^{*} - 1}$

Para o cancelamento de N é necessário que

- $\boxed{(\Lambda - F) = NH}$ (N deve ser fator de $\Lambda - F$)

- $\begin{cases} \text{grau}(\Lambda) = n-1 \\ \text{grau}(F) = n-2 \\ \text{grau}(N) = m \end{cases} \Rightarrow \boxed{\text{grau}(H) = n - m - 1 = n^* - 1}$

Hipótese: $P(s)$ é controlável e observável.

★ $N(s)$ e $D(s)$ são primos!

Após o cancelamento de N, tem-se a equação

$$\boxed{HD - k_p G = D_m A_0}$$ (Equação *Diophantina.*)

- $\text{grau}(HD) = \text{grau}(H) + \text{grau}(D) = (n - m - 1) + n = 2n - m - 1$
- $\text{grau}(G) = n - 1$
- $\text{grau}(D_m A_0) = \text{grau}(D_m) + \text{grau}(A_0) = n + (n - m - 1) = 2n - m - 1$

★ $H(s)$ é mônico.

Hipótese: Grau relativo de $P(s)$ $=$ Grau relativo de $M(s)$.

- $$\boxed{\text{grau}(D) - \text{grau}(N) = \text{grau}(D_m) - \text{grau}(N_m) = n^*}$$

★ O grau relativo é invariante.

Resumo do algoritmo

(1) Determinar $\mathrm{grau}(A_0)$ e $\mathrm{grau}(G)$.

(2) Achar a solução $H(s)$ e $G(s)$ da equação *Diophantina*

$$H(s)D(s) - k_p G(s) = D_m(s)A_0(s)$$

(3) Obter θ_2^* e θ_n^* a partir de $G(s)$.

(4) Calcular $F(s)$ usando a relação $F = \Lambda(s) - N(s)H(s)$

(5) Obter θ_1^* a partir de $F(s)$.

(6) $\theta_{2n}^* = \dfrac{k_m}{k_p}$.

4.1.1 Examples

Example 9

Classificação do sistema: $n = 2$ (ordem)
$n^* = 1$ (grau relativo)
$n_p = 4$ (# de parâmetros)

Planta......................: $$y = \frac{k_p(s+b)}{s^2 + a_1 s + a_0} u$$

Modelo...................: $$y_m = \frac{k_m(s+b_m)}{s^2 + a_{1m} s + a_{0m}} r$$

- $\text{grau}(A_0) = n^* - 1 = 0 \quad \Rightarrow \quad \boxed{A_0(s) = 1}$

- $\Lambda(s) = N_m(s)A_0(s) \quad \Rightarrow \quad \boxed{\Lambda(s) = s + b_m}$

- $\text{grau}(H) = n^* - 1 = 0 \quad \Rightarrow \quad \boxed{H(s) = 1}$

- $\text{grau}(G) = n - 1 = 1 \quad \Rightarrow \quad \boxed{G(s) = g_1 s + g_0}$

- Equação *Diophantina:* $\quad HD - k_p G = D_m A_0$

$$(1)\left(s^2 + a_1 s + a_0\right) - k_p\left(g_1 s + g_0\right) = \left(s^2 + a_{1m} s + a_{0m}\right)(1)$$

$$s^2 + \underbrace{\left(a_1 - k_p g_1\right)}_{a_{1m}} s + \underbrace{\left(a_0 - k_p g_0\right)}_{a_{0m}} = s^2 + a_{1m} s + a_{0m}$$

- Forma matricial:

$$\begin{bmatrix} a_1 & -k_p & 0 \\ a_0 & 0 & -k_p \end{bmatrix} \begin{bmatrix} 1 \\ g_1 \\ g_0 \end{bmatrix} = \begin{bmatrix} a_{1m} \\ a_{0m} \end{bmatrix}$$

- Solução da equação *Diophantina:*

$$\begin{cases} a_1 - k_p g_1 = a_{1m} \\ a_0 - k_p g_0 = a_{0m} \end{cases} \Rightarrow \boxed{g_1 = \frac{a_1 - a_{1m}}{k_p}} \quad \boxed{g_0 = \frac{a_0 - a_{0m}}{k_p}}$$

- $\dfrac{G}{\Lambda} = \dfrac{\theta_2^*}{(s+b_m)} + \theta_n^* = \dfrac{\theta_2^* + \theta_n^*(s+b_m)}{s+b_m} \quad \Rightarrow \quad G = \underbrace{\theta_n^*}_{g_1} s + \underbrace{(\theta_2^* + \theta_n^* b_m)}_{g_0}$

- $\theta_n^* = g_1 \quad \Rightarrow \quad \boxed{\theta_n^* = \dfrac{a_1 - a_{1m}}{k_p}}$

- $g_0 = \theta_2^* + \theta_n^* b_m \quad \Rightarrow \quad \boxed{\theta_2^* = \dfrac{(a_0 - a_{0m}) - b_m(a_1 - a_{1m})}{k_p}}$

- $\text{grau}(F) = n - 2 = 0$
- $F = \Lambda - NH \quad \Rightarrow \quad F(s) = (s + b_m) - (s + b) \quad \Rightarrow \quad \boxed{F(s) = b_m - b}$
- $\boxed{\theta_1^* = b_m - b}$
- $\boxed{\theta_{2n}^* = \dfrac{k_m}{k_p}}$

Example 10

Classificação do sistema:

$$\begin{aligned} n &= 2 \quad \text{(ordem)} \\ n^* &= 2 \quad \text{(grau relativo)} \\ n_p &= 4 \quad \text{(\# de parâmetros)} \end{aligned}$$

Planta.......................: $y = \dfrac{k_p}{s^2 + a_1 s + a_0} u$

Modelo ....: $y_m = \dfrac{k_m}{s^2 + a_{1m} s + a_{0m}} r$

- $\text{grau}(A_0) = n^* - 1 = 1 \quad \Rightarrow \quad \boxed{A_0(s) = s + \lambda_0}$

★ $\text{grau}(\Lambda) = n - 1 = 1$

- $\Lambda = N_m A_0 \quad \Rightarrow \quad \boxed{\Lambda(s) = (s + \lambda_0)}$

- $\text{grau}(H) = n^* - 1 = 1 \quad \Rightarrow \quad \boxed{H(s) = s + h_0}$

- $\text{grau}(G) = n - 1 = 1 \quad \Rightarrow \quad \boxed{G(s) = g_1 s + g_0}$

- Equação *Diophantina:* $$\boxed{HD - k_p G = D_m A_0}$$

$$(s+h_0)\left(s^2+a_1 s+a_0\right) - k_p\left(g_1 s+g_0\right) = \left(s^2+a_{1m}s+a_{0m}\right)(s+\lambda_0)$$

$$s^3+(a_1+h_0)s^2+(a_0+h_0a_1-k_pg_1)s+(h_0a_0-k_pg_0) = s^3+\left(\lambda_0+a_{1m}\right)s^2+\left(a_{1m}\lambda_0+a_{0m}\right)s+a_{0m}\lambda_0$$

- Sistema de equações lineares (3 equações & 3 incógnitas):

$$\begin{cases} a_1+h_0=\lambda_0+a_{1m} \\ a_0+h_0a_1-k_pg_1=a_{1m}\lambda_0+a_{0m} \\ h_0a_0-k_pg_0=a_{0m}\lambda_0 \end{cases}$$

A equação *Diophantina* pode ser escrita na forma:

$$(s+h_0)D - k_pG = D_mA_0 \quad \Rightarrow \quad h_0D - k_pG = -sD + D_mA_0$$

Resultado:

$$h_0s^2+(h_0a_1-k_pg_1)s+(h_0a_0-k_pg_0) = \left(\lambda_0+a_{1m}-a_1\right)s^2+\left(a_{1m}\lambda_0+a_{0m}-a_0\right)s+a_{0m}\lambda_0$$

★ Note que o termo s^3 é cancelado.

- Forma matricial:

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 \\ a_1 & -k_p & 0 \\ a_0 & 0 & -k_p \end{bmatrix} \begin{bmatrix} h_0 \\ g_1 \\ g_0 \end{bmatrix} &= \begin{bmatrix} \lambda_0 + a_{1m} - a_1 \\ a_{1m}\lambda_0 + a_{0m} - a_0 \\ a_{0m}\lambda_0 \end{bmatrix} \\ &= \begin{bmatrix} a_{1m} & 1 & a_1 \\ a_{0m} & a_{1m} & a_0 \\ 0 & a_{0m} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \lambda_0 \\ -1 \end{bmatrix} \end{aligned}$$

★ Note que existe uma regra de formação.

- Solução da equação *Diophantina:*

$$\begin{cases} h_0 = \lambda_0 + a_{1m} - a_1 \\ g_1 = \left[a_0 + a_1 h_0 - a_{1m}\lambda_0 - a_{0m}\right]/k_p \\ g_0 = \left[a_0 h_0 - a_{0m}\lambda_0\right]/k_p \end{cases}$$

ou melhor,

$$\begin{cases} h_0 = \lambda_0 - \left(a_1 - a_{1m}\right) \\ g_1 = \left[\left(a_0 - a_{0m}\right) + \left(\lambda_0 - a_1\right)\left(a_1 - a_{1m}\right)\right]/k_p \\ g_0 = \left[\lambda_0\left(a_0 - a_{0m}\right) - a_0\left(a_1 - a_{1m}\right)\right]/k_p \end{cases}$$

- $$\frac{G}{\Lambda} = \frac{\theta_2^*}{(s+\lambda_0)} + \theta_n^* = \frac{\theta_2^* + \theta_n^*(s+\lambda_0)}{s+\lambda_0} \quad \Rightarrow \quad G = \underbrace{\theta_n^*}_{g_1} s + \underbrace{(\theta_2^* + \theta_n^*\lambda_0)}_{g_0}$$

- $$\theta_n^* = g_1 \quad \Rightarrow \quad \boxed{\theta_n^* = \frac{\big(a_0 - a_{0m}\big) + \big(\lambda_0 - a_1\big)\big(a_1 - a_{1m}\big)}{k_p}}$$

- $$g_0 = \theta_2^* + \theta_n^*\lambda_0 \quad \Rightarrow \quad \boxed{\theta_2^* = \frac{-\big(a_0 + \lambda_0(\lambda_0 - a_1)\big)(a_1 - a_{1m})}{k_p}}$$

- $\text{grau}(F) = n - 2 = 0$

- $F = \Lambda - NH \quad \Rightarrow \quad F(s) = (s + \lambda_0) - (s + h_0) \quad \Rightarrow \quad \boxed{F(s) = \lambda_0 - h_0}$

- $\theta_1^* = \lambda_0 - h_0 \quad \Rightarrow \quad \boxed{\theta_1^* = a_1 - a_{1m}}$

- $\boxed{\theta_{2n}^* = \dfrac{k_m}{k_p}}$

Example 11

Classificação do sistema: $n = 3$ (ordem)
$n^* = 2$ (grau relativo)
$n_p = 5$ (# de parâmetros)

Planta......................: $y = \dfrac{k_p(s + b_0)}{s^3 + a_2 s^2 + a_1 s + a_0} u$

Modelo....................: $y_m = \dfrac{k_m}{s^2 + a_{1m} s + a_{0m}} r$

- $\mathrm{grau}(A_0) = n^* - 1 = 1 \quad \Rightarrow \quad \boxed{A_0(s) = s + \lambda_0}$

- $\Lambda = N_m A_0 = s + \lambda_0 \quad \Rightarrow \quad \mathrm{grau}(\Lambda) = 1 \quad$ (?)

★ Porém, sabemos que $\mathrm{grau}(\Lambda) = n - 1 = 2$

$\Rightarrow$ Aumentar o modelo para: $M(s) = \dfrac{k_m(s + \lambda_1)}{(s^2 + a_{1m}s + a_{0m})(s + \lambda_1)}$ (!)

- $\Lambda = N_m A_0 \quad \Rightarrow \quad \boxed{\Lambda(s) = (s + \lambda_0)(s + \lambda_1)}$

- $\mathrm{grau}(H) = n^* - 1 = 1 \quad \Rightarrow \quad \boxed{H(s) = s + h_0}$

- $\mathrm{grau}(G) = n - 1 = 2 \quad \Rightarrow \quad \boxed{G(s) = g_2 s^2 + g_1 s + g_0}$

- Equação *Diophantina:*

$$\boxed{HD - k_p G = D_m A_0}$$

$$\begin{aligned}(s+h_0)\left(s^3+a_2s^2+a_1s+a_0\right)-k_p\left(g_2s^2+g_1s+g_0\right)=&\\ =\left(s^2+a_{1m}s+a_{0m}\right)(s+\lambda_1)(s+\lambda_0)&\end{aligned}$$

$$\begin{aligned}s^4+(h_0+a_2)s^3+(a_1+h_0a_2-k_pg_2)s^2+(a_0+h_0a_1-k_pg_1)s+(h_0a_0-k_pg_0)=&\\ =s^4+(\lambda_0+\lambda_1+a_{1m})s^3+\left(\lambda_0\lambda_1+a_{1m}(\lambda_0+\lambda_1)+a_{0m}\right)s^2+&\\ +\left(a_{1m}\lambda_0\lambda_1+a_{0m}(\lambda_0+\lambda_1)\right)s+a_{0m}\lambda_0\lambda_1&\end{aligned}$$

- Sistema de equações lineares:

$$\begin{cases} h_0+a_2=\lambda_0+\lambda_1+a_{1m}\\ a_1+h_0a_2-k_pg_2=\lambda_0\lambda_1+a_{1m}(\lambda_0+\lambda_1)+a_{0m}\\ a_0+h_0a_1-k_pg_1=a_{1m}\lambda_0\lambda_1+a_{0m}(\lambda_0+\lambda_1)\\ h_0a_0-k_pg_0=a_{0m}\lambda_0\lambda_1\end{cases}$$

A equação *Diophantina* pode ser escrita na forma:

$$(s+h_0)D - k_pG = D_mA_0 \quad \Rightarrow \quad h_0D - k_pG = -sD + D_mA_0$$

★ Note que o termo s^4 é cancelado.

- Forma matricial:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ a_2 & -k_p & 0 & 0 \\ a_1 & 0 & -k_p & 0 \\ a_0 & 0 & 0 & -k_p \end{bmatrix} \begin{bmatrix} h_0 \\ g_2 \\ g_1 \\ g_0 \end{bmatrix} = \begin{bmatrix} \lambda_0 + \lambda_1 + a_{1m} - a_2 \\ \lambda_0\lambda_1 + a_{1m}(\lambda_0 + \lambda_1) + a_{0m} - a_1 \\ a_{1m}\lambda_0\lambda_1 + a_{0m}(\lambda_0 + \lambda_1) - a_0 \\ a_{0m}\lambda_0\lambda_1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{1m} & 1 & 0 & a_2 \\ a_{0m} & a_{1m} & 1 & a_1 \\ 0 & a_{0m} & a_{1m} & a_0 \\ 0 & 0 & a_{0m} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \lambda_0 + \lambda_1 \\ \lambda_0\lambda_1 \\ -1 \end{bmatrix}$$

★ Note que existe uma regra de formação.

- Solução da equação *Diophantina:*

$$\begin{cases} h_0 = \lambda_0 + \lambda_1 + a_{1m} - a_2 \\ g_2 = \left[a_1 + a_2 h_0 - \lambda_0\lambda_1 - a_{1m}(\lambda_0 + \lambda_1) - a_{0m}\right]/k_p \\ g_1 = \left[a_0 + a_1 h_0 - a_{1m}\lambda_0\lambda_1 - a_{0m}(\lambda_0 + \lambda_1)\right]/k_p \\ g_0 = \left[a_0 h_0 - a_{0m}\lambda_0\lambda_1\right]/k_p \end{cases}$$

Example 12

Classificação do sistema:

$$\begin{aligned} n &= 4 && \text{(ordem)} \\ n^* &= 1 && \text{(grau relativo)} \\ n_p &= 8 && \text{(\# de parâmetros)} \end{aligned}$$

Plant ....: $y = \dfrac{k_p(s^3 + b_2 s^2 + b_1 s + b_0)}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} u$

Model ....: $y_m = \dfrac{k_m(s^3 + b_{2m} s^2 + b_{1m} s + b_{0m})}{s^4 + a_{3m} s^3 + a_{2m} s^2 + a_{1m} s + a_{0m}} r$

★ The model here is the *augmented model*.

Solution

$$\theta_1^* = \begin{bmatrix} b_{0m} - b_0 \\ b_{1m} - b_1 \\ b_{2m} - b_2 \end{bmatrix}, \qquad \theta_2^* = \begin{bmatrix} (a_0 - a_{0m})/k_p - \theta_n^*\, b_{0m} \\ (a_1 - a_{1m})/k_p - \theta_n^*\, b_{1m} \\ (a_2 - a_{2m})/k_p - \theta_n^*\, b_{2m} \end{bmatrix},$$

$$\theta_n^* = (a_3 - a_{3m})/k_p, \qquad \theta_{2n}^* = k_m/k_p.$$

★ Note that the plant parameter can be easily retrieved.

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_{0m} \\ b_{1m} \\ b_{2m} \end{bmatrix} - \theta_1^*, \qquad k_p = k_m/\theta_{2n}^*,$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = k_p\theta_2^* + k_p\theta_n^* \begin{bmatrix} b_{0m} \\ b_{1m} \\ b_{2m} \end{bmatrix} + \begin{bmatrix} a_{0m} \\ a_{1m} \\ a_{2m} \end{bmatrix}, \qquad a_3 = k_p\,\theta_n^* + a_{3m}.$$

Matlab code for $n^* = 1$ and any n

Vector notation:

$$A_p = \begin{bmatrix} a_{n-1} & \dots & a_2 & a_1 & a_0 \end{bmatrix}$$

$$B_p = \begin{bmatrix} b_{n-1} & \dots & b_2 & b_1 & b_0 \end{bmatrix}$$

$$A_m = \begin{bmatrix} a_{m_{n-1}} & \dots & a_{m_2} & a_{m_1} & a_{m_0} \end{bmatrix}$$

$$B_m = \begin{bmatrix} b_{m_{n-1}} & \dots & b_{m_2} & b_{m_1} & b_{m_0} \end{bmatrix}$$

```
ks = kp/km;

for i=1:n-1
  k = n + 1 - i;
  thetas(i) = Bm(k) - Bp(k);
end

thetas(n) = (Ap(1) - Am(1))/kp;

for i=1:n-1
  k = n + 1 - i;
  thetas(n+i) = (Ap(k) - Am(k))/kp - thetas(n)*Bm(k);
end

thetas(2*n) = km/kp;

thetas
```

Verificação usando Matlab

Example 13 4nd order plant.

Classificação do sistema:
$$\begin{aligned} n &= 4 && \text{(ordem)} \\ n^* &= 2 && \text{(grau relativo)} \\ n_p &= 8 && \text{(\# de parâmetros)} \end{aligned}$$

Planta.....................: $P(s) = \dfrac{0.3\,(s+2)^2}{s^4}$

Modelo...................: $M(s) = \dfrac{1}{(s+1)^2}$

Filtro.......................: $\dfrac{1}{\Lambda(s)} = \dfrac{1}{(s+1)^3}$

Matching................: $\theta^{*T} = \begin{bmatrix} -19 & -21 & -6 & -30 & 27 & 75 & 60 & 3 \end{bmatrix}$

$$\|\theta^*\| = 119.37$$

Script:

```
syms s
P = (1/3)*(s+2)^2/(s^4)
S = [1 ; s ; s^2]

theta1 = [-19 -21 -6]
thetan = -30
theta2 = [27 75 60]
theta2n = 3

%Filters
Lambda = (s+1)^3
F = (theta1*S)
G = (theta2*S) + thetan*Lambda

%Feedforward & feedback
Hfb = G/(Lambda - F)
Hff = theta2n*Lambda/(Lambda - F)

%Closed-loop transfer function
M = P*Hff/(1 - P*Hfb)

M = simplify(M)
pretty(M)
```

Result:

```
>> verification

M =

1/(s + 1)^2

    1
--------
       2
(s + 1)
```

4.1.2 Matching equation

Referência. [Tao:2003], (pag. 197)

A referência utiliza uma notação diferente:

$$\boxed{\omega_1 = \frac{a(s)}{\Lambda} u} \qquad \boxed{\omega_2 = \frac{a(s)}{\Lambda} y}$$

$$\boxed{a(s) = \begin{bmatrix} 1 & s & s^2 & \cdots & s^{n-2} \end{bmatrix}}$$

★ Note que com essa notação $F(s)$ e $G(s)$ são escritas como

$$F(s) = \frac{\theta_1^T a(s)}{\Lambda(s)} \quad \text{e} \quad G(s) = \frac{\theta_2^T a(s)}{\Lambda(s)} + \theta_n$$

Portanto,

$$
\begin{aligned}
u &= \theta_1^T\omega_1 + \theta_2^T\omega_2 + \theta_{20}y + \theta_3 r \\
&= \frac{\theta_1^T a}{\Lambda}u + \frac{\theta_2^T a}{\Lambda}y + \theta_{20}y + \theta_3 r
\end{aligned}
$$

$$
\Rightarrow \quad \big(\Lambda - \theta_1^T a\big)u = \theta_2^T a y + \Lambda\theta_{20}y + \Lambda\theta_3 r
$$

Da equação da planta, tem-se

$$
Py = k_p Z u
$$

$$
\begin{aligned}
\Rightarrow \quad \big(\Lambda - \theta_1^T a\big)Py &= k_p Z\big(\Lambda - \theta_1^T a\big)u \\
&= k_p Z\big(\theta_2^T a y + \Lambda\theta_{20}y + \Lambda\theta_3 r\big)
\end{aligned}
$$

Ou melhor,

$$\left[\left(\Lambda - \theta_1^T a\right)P - k_p Z\left(\theta_2^T a + \theta_{20}\Lambda\right)\right]y = k_p Z \Lambda \theta_3 r$$

$$\Lambda P y + \left[-\theta_1^T a P - k_p Z\left(\theta_2^T a + \theta_{20}\Lambda\right)\right]y = k_p Z \Lambda \theta_3 r$$

$$\left[\theta_1^T a P + k_p Z\left(\theta_2^T a + \theta_{20}\Lambda\right)\right]y = \Lambda P y - k_p Z \Lambda \theta_3 r$$

Como $\boxed{P_m y_m = r}$

$$\left[\theta_1^T a P + k_p Z\left(\theta_2^T a + \theta_{20}\Lambda\right)\right]y = \Lambda P y - k_p Z \Lambda \theta_3 P_m y_m$$

Quando $\theta = \theta^*$, temos que $y = y_m$, e assim

$$\boxed{\theta_1^{*T} a P + k_p Z\left(\theta_2^{*T} a + \theta_{20}^* \Lambda\right) = \Lambda P - k_p Z \Lambda \theta_3^* P_m}$$

Matching equation

Escolhemos: $$\boxed{\theta_3^* = \frac{1}{k_p}}$$

Dessa forma, somente o lado direito contém θ^* :

$$\boxed{\theta_1^{*T} a P + k_p Z\left(\theta_2^{*T} a + \theta_{20}^* \Lambda\right) = \Lambda P - Z \Lambda P_m}$$

Se: $\boxed{P_m y_m = k_m Z_m r}$

$$\left[\theta_1^T a P + k_p Z\left(\theta_2^T a + \theta_{20}\Lambda\right)\right] k_m Z_m y = k_m Z_m \Lambda P y - k_p Z \Lambda \theta_3 k_m Z_m r$$

$$\Rightarrow \quad \left[\theta_1^{*T} a P + k_p Z\left(\theta_2^{*T} a + \theta_{20}^{*}\Lambda\right)\right] Z_m = \Lambda P Z_m - \frac{k_p}{k_m}\theta_3^{*} Z \Lambda P_m$$

Podemos simplificar fazendo: $\boxed{\Lambda = Z_m A_0}$

Resultado: $\boxed{\theta_1^{*T} a P + k_p Z\left(\theta_2^{*T} a + \theta_{20}^{*}\Lambda\right) = \Lambda P - \frac{k_p}{k_m}\theta_3^{*} Z A_0 P_m}$

★ Note que $Z(s)$ não foi cancelado.

4.2 Caso $n^* = 1$

Referência. [Tao:2003], (pag. 195)

Plant: $\boxed{y = P(s)u}$ $\boxed{P(s) = k_p \dfrac{N_p(s)}{D_p(s)}}$

Reference model: $\boxed{y_m = M(s)r}$ $\boxed{M(s) = k_m \dfrac{N_m(s)}{D_m(s)}}$

Output error: $\boxed{e_0 = y - y_m}$

★ Without loss of generality, $k_m = 1$, thus

$$\boxed{M(s) = \frac{N_m(s)}{D_m(s)}}$$

Basic assumptions

Prior available information regarding $P(s)$:

(1) The order of the plant n is known ([1]).

(2) The relative degree n^* is known.

(3) $N_p(s)$ is Hurwitz, i.e., $P(s)$ is minimum phase.

(4) $\text{sign}(k_p)$ is known.

★ ([1]) Otherwise unicity of the solution cannot be assured.

★ Unicity is not mandatory for stability !

Objective: Find a control law $u(t)$ such that

- The closed-loop system is stable and
- $\boxed{\lim_{t\to\infty} e_0 = 0}$ for arbitrary initial conditions.

★ The reference signal $r(t)$ is piece–wise continuous and uniformly bounded.

Structure of the controller

State variable filters (SVF's):

$$\boxed{\begin{aligned} \dot{\omega}_1 &= A_f\,\omega_1 + b_f u \\ \dot{\omega}_2 &= A_f\,\omega_2 + b_f y \end{aligned}} \qquad \begin{aligned} \omega_1 &\in \mathbb{R}^{n-1} \\ \omega_2 &\in \mathbb{R}^{n-1} \end{aligned}$$

★ Λ is chosen such that $N_m(s)$ is a factor of $\det(sI - \Lambda)$.

Regressor vector:

$$\boxed{\omega^T = [\omega_1^T \;\; y \;\; \omega_2^T \;\; r]} \quad , \quad \omega \in \mathbb{R}^{2n}$$

Control law:

$$\boxed{u = \theta^T \omega} \quad , \quad \theta^T = [\theta_1^T \;\; \theta_n \;\; \theta_2^T \;\; \theta_{2n}] \in \mathbb{R}^{2n}$$

Fact: There exists a unique $\theta^{*T} = [\theta_1^{*T} \quad \theta_n^* \quad \theta_2^{*T} \quad \theta_{2n}^*]$ such that

$$\boxed{y = P(s)u^* = P(s)\theta^{*T}\omega = M(s)r}$$ (Matching!)

Parameter error: $$\boxed{\tilde{\theta} = \theta - \theta^*}$$

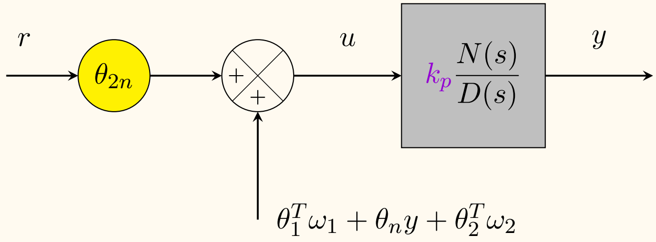
Control mismatch: $$\boxed{\tilde{u} = \tilde{\theta}^T\omega}$$

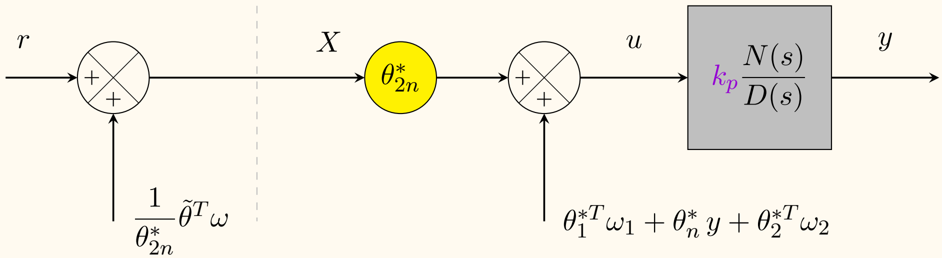
4.2.1 Error equation

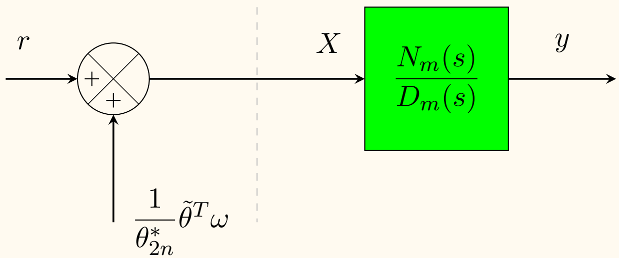
Ref.: [Slotine & Li:1991], (pag. 345)

Simple derivation of the error equation:

$$
\begin{aligned}
u &= \theta^T \omega - \theta^{*T} \omega + \theta^{*T} \omega \\
&= \tilde{\theta}^T \omega + \theta_1^{*T} \omega_1 + \theta_n^* \, y + \theta_2^{*T} \omega_2 + \theta_{2n}^* \, r \\
&= \theta_{2n}^* \underbrace{\left[r + \frac{1}{\theta_{2n}^*} \tilde{\theta}^T \omega \right]}_{\text{external signal } X} + \theta_1^{*T} \omega_1 + \theta_n^* \, y + \theta_2^{*T} \omega_2
\end{aligned}
$$


r
θ_{2n}
u
$k_p \dfrac{N(s)}{D(s)}$
y
$\theta_1^T \omega_1 + \theta_n y + \theta_2^T \omega_2$


r
X
θ_{2n}^*
u
$k_p \dfrac{N(s)}{D(s)}$
y
$\dfrac{1}{\theta_{2n}^*} \tilde{\theta}^T \omega$
$\theta_1^{*T} \omega_1 + \theta_n^* y + \theta_2^{*T} \omega_2$


r
X
$\dfrac{N_m(s)}{D_m(s)}$
y
$\dfrac{1}{\theta_{2n}^*} \tilde{\theta}^T \omega$

From the figure, one has

$$y = P(s)\big[u\big] = M(s)\big[X\big] \quad \Rightarrow \quad \boxed{y = M(s)\Big[r + \frac{1}{\theta_{2n}^*}\tilde{\theta}^T\omega\Big]}$$

★ Recall that $\boxed{\dfrac{1}{\theta_{2n}^*} = k_p}$

Error equation:

$$\begin{aligned} e_0 &= y - y_m \\ &= M(s)\big[r + k_p\,\tilde{\theta}^T\omega\big] - M(s)\big[r\big] \quad \Rightarrow \quad \boxed{e_0 = k_p\,M(s)\big[\tilde{\theta}^T\omega\big]} \end{aligned}$$

The same error equation is obtained in the case of a more general control law

$$\boxed{u = \theta^T \omega + v}$$

★ This is the case, for example, when $n^* = 2$.

The derivation is similar:

$$\begin{aligned} u &= \theta^T \omega - \theta^{*T} \omega + \theta^{*T} \omega + v \\ &= \tilde{\theta}^T \omega + \theta_1^{*T} \omega_1 + \theta_n^* y + \theta_2^{*T} \omega_2 + \theta_{2n}^* r + v \\ &= \theta_{2n}^* \underbrace{\left[r + k_p \tilde{\theta}^T \omega + k_p v \right]}_{\text{external signal } X} + \theta_1^{*T} \omega_1 + \theta_n^* y + \theta_2^{*T} \omega_2 \end{aligned}$$

This gives

$$y = P(s)\big[\,u\,\big] = M(s)\big[X\big] = M(s)\big[r + k_p\,\tilde{\theta}^T\omega + k_p v\big]$$

Error equation:

$$\begin{aligned}
e_0 = y - y_m &= M(s)\big[r + k_p\,\tilde{\theta}^T\omega + k_p v\big] - M(s)\big[\,r\,\big] \\
&= k_p\,M(s)\big[(\theta - \theta^*)^T\omega + v\big] \\
&= k_p\,M(s)\big[u - \theta^{*T}\omega\big]
\end{aligned}$$

Error equation: $$\boxed{e_0 = k_p\, M(s)\big[u - \theta^{*T}\omega\big] + \varepsilon}$$

★ ε is an exponentially decaying term due to the initial conditions.

State space representation: $$\begin{cases} \dot{e} = Ae + k_p\, B[u - \theta^{*T}\omega] \\ e_0 = Ce \end{cases}$$

★ The error vector $e \in \mathbb{R}^{3n-2}$.

★ $\{A, B, C\}$ is a non–minimal realization of $M(s)$.

★ The initial condition $e(0)$ is responsible for the term ε!

Question: Non–minimal realization? $e \in \mathbb{R}^{3n-2}$?

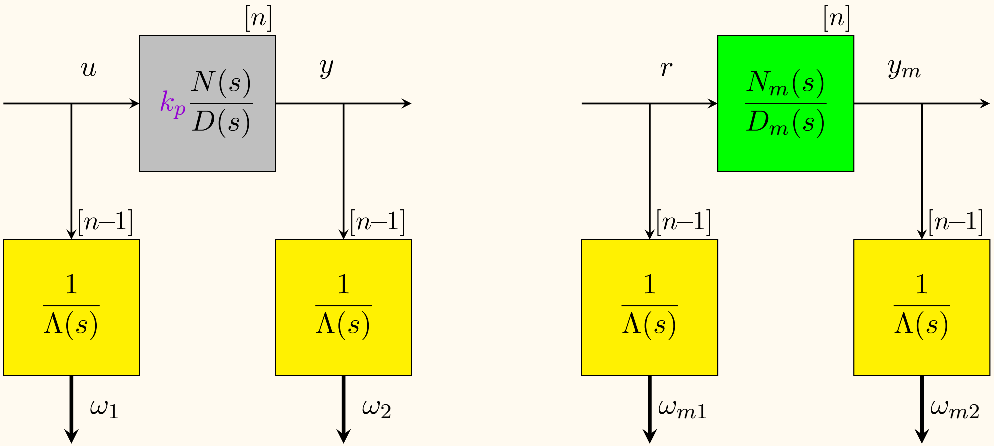


Figura 59: Non–minimal realization.

4.2.2 Meyer–Kalman–Yakubovich Lemma

Hypothesis: $M(s)$ is strictly positive real (SPR).

★ This is **fundamental** for the stability analysis (and implementation).

Lemma (Meyer–Kalman–Yakubovich)

If the system $\{A, B, C\}$ is SPR then

$$\exists \begin{cases} P = P^T > 0 \\ Q = Q^T > 0 \end{cases} \quad \text{such that} \quad \boxed{\begin{aligned} A^T P + PA &= -2Q \\ PB &= C^T \end{aligned}}$$

Ref.: [Tao:2003], (pag. 77)
Ref.: [Ioannou & Sun:1996], (pag. 126)

Sufficient condition for an SPR transfer function: Alternate poles and zeros !

Example 14 3rd order model

$$M(s) = \left(\frac{15}{8}\right) \frac{(s+2)(s+4)}{(s+1)(s+3)(s+5)}$$

Example 15 Step response of an SPR transfer function

Model: $$M(s) = \frac{k_m(s+z)}{(s+1)(s+2)}, \qquad k_m = \frac{2}{z}$$

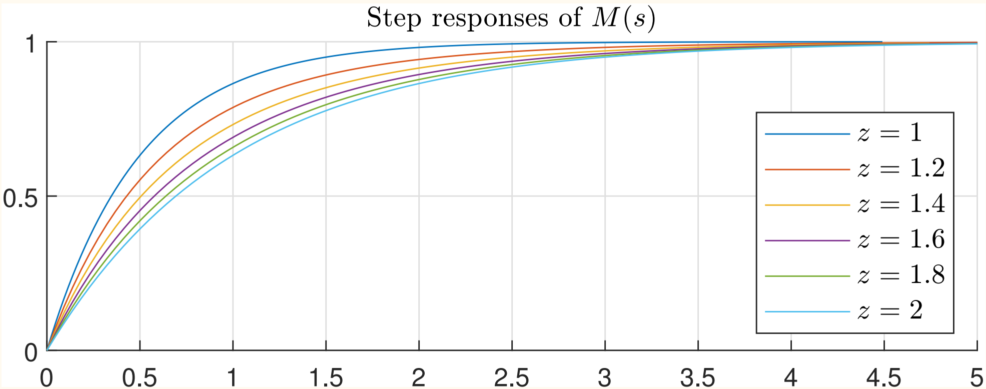


Figura 60: Step responses for $z = \{1,\ 1.2,\ 1.4,\ 1.6,\ 1.8,\ 2\}$.

4.2.3 Lyapunov design

Control law: $$\boxed{u = \theta^T \omega}$$

Set of equations: $$\begin{cases} \dot{e} = Ae + k_p B[\tilde{\theta}^T \omega]\,, & e_0 = Ce \\ \dot{\theta} = ? \end{cases}$$

Lyapunov function: $$\boxed{2V(e, \tilde{\theta}) = e^T P e + |k_p|\, \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}} \qquad \Gamma = \Gamma^T > 0$$

Time derivative:

$$
\begin{aligned}
2\dot{V} &= \dot{e}^T P e + e^T P \dot{e} + |k_p|\big(\dot{\theta}^T \Gamma^{-1} \tilde{\theta} + \tilde{\theta}^T \Gamma^{-1} \dot{\theta}\big) \\
&= \big(Ae + k_p B \tilde{\theta}^T \omega\big)^T P e + e^T P \big(Ae + k_p B \tilde{\theta}^T \omega\big) + 2|k_p| \tilde{\theta}^T \Gamma^{-1} \dot{\theta} \\
&= e^T A^T P e + k_p\, \tilde{\theta}^T \omega B^T P e + e^T P A e + k_p\, e^T P B \tilde{\theta}^T \omega + 2|k_p| \tilde{\theta}^T \Gamma^{-1} \dot{\theta} \\
&= e^T \big(\underbrace{A^T P + P A}_{-2Q}\big) e + 2\, k_p\, \tilde{\theta}^T \omega B^T P e + 2|k_p| \tilde{\theta}^T \Gamma^{-1} \dot{\theta} \\
&= -2 e^T Q e + 2 k_p\, \tilde{\theta}^T \omega B^T P e + 2|k_p| \tilde{\theta}^T \Gamma^{-1} \dot{\theta}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\dot{V} &= -e^T Q e + k_p\, \tilde{\theta}^T \omega B^T P e + |k_p| \tilde{\theta}^T \Gamma^{-1} \dot{\theta} \\
&= -e^T Q e + |k_p| \tilde{\theta}^T \Big[\operatorname{sign}(k_p)\, \omega \underbrace{B^T P e}_{Ce = e_0} + \Gamma^{-1} \dot{\theta}\Big]
\end{aligned}
$$

We choose the update law: $$\boxed{\dot{\theta} = -\,\mathrm{sign}(k_p)\,\Gamma\,\omega e_0}$$

Result: $$\boxed{\dot{V} = -e^T Q e \leq 0}$$

- $V(t)$ is monotone non–increasing, bounded above by $V(0)$ and below by 0.
- $e,\ \tilde{\theta} \in \mathcal{L}_\infty$
- Integranting $\dot{V} \quad \Rightarrow \quad e \in \mathcal{L}_2$
- $r(t) \in \mathcal{L}_\infty \quad \Rightarrow \quad \dot{e} \in \mathcal{L}_\infty$
- $\begin{cases} e \in \mathcal{L}_2 \\ \dot{e} \in \mathcal{L}_\infty \end{cases} \quad \Rightarrow \quad \boxed{e \to 0}$

- Convergence of $\tilde{\theta} \to 0$ requires the *persistent excitation* condition

$$\boxed{\int_t^{t+T} \omega(\tau)\omega^T(\tau)\, d\tau \geq \alpha I} \quad T,\ \alpha > 0\,, \quad \forall t \geq t_0$$

★ Note that $\omega(\cdot)$ is an internal signal of the system.

4.2.4 Auxiliary error

For convenience, define the auxiliary error $\boxed{e_a = \mathrm{sign}(k_p)e_0}$

Then, the error equation can be rewritten as

$$\underbrace{\mathrm{sign}(k_p)e_0}_{e_a} = \underbrace{|k_p|\, M(s)}_{\text{SPR}} \left[u - \theta^{*T}\omega\right]$$

★ $\boxed{M(s) \text{ is SPR}}$ $\Rightarrow$ $\boxed{|k_p|\, M(s) \text{ is SPR !}}$

State space representation: $\begin{cases} \dot{e} = A_m e + B_m\big[u - \theta^{*T}\omega\big] \\ e_a = C_m e \end{cases}$

where

$$\boxed{\left|k_p\right| M(s) = C_m\big(sI - A_m\big)^{-1} B_m}$$

★ The unknown term $\left|k_p\right|$ is absorbed by the matrices $\{A_m, B_m, C_m\}$.

Since the system $\{A_m, B_m, C_m\}$ is SPR then

$$\exists \begin{cases} P = P^T > 0 \\ Q = Q^T > 0 \end{cases} \quad \text{s.t.} \quad \begin{cases} A_m^T P + P A_m = -2Q \\ P B_m = C_m^T \end{cases}$$

4.2.5 Lyapunov design with e_a

Set of equations:
$$\begin{cases} \dot{e} = A_m e + B_m \left[\tilde{\theta}^T \omega\right], & e_a = C_m e \\ \dot{\theta} = ? \end{cases}$$

Lyapunov function:
$$\boxed{2V(e, \tilde{\theta}) = e^T P e + \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}} \qquad \Gamma = \Gamma^T > 0$$

★ k_p disappears from the analysis!

Time derivative:

$$
\begin{aligned}
2\dot{V} &= \dot{e}^T P e + e^T P \dot{e} + \left(\dot{\theta}^T \Gamma^{-1} \tilde{\theta} + \tilde{\theta}^T \Gamma^{-1} \dot{\theta}\right) \\
&= \left(A_m e + B_m \tilde{\theta}^T \omega\right)^T P e + e^T P \left(A_m e + B_m \tilde{\theta}^T \omega\right) + 2\tilde{\theta}^T \Gamma^{-1} \dot{\theta} \\
&= e^T A_m^T P e + \tilde{\theta}^T \omega B_m^T P e + e^T P A_m e + e^T P B_m \tilde{\theta}^T \omega + 2\tilde{\theta}^T \Gamma^{-1} \dot{\theta} \\
&= e^T \big(\underbrace{A_m^T P + P A_m}_{-2Q}\big) e + 2\, \tilde{\theta}^T \omega B_m^T P e + 2\tilde{\theta}^T \Gamma^{-1} \dot{\theta} \\
&= -2 e^T Q e + 2\tilde{\theta}^T \omega B_m^T P e + 2\tilde{\theta}^T \Gamma^{-1} \dot{\theta}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\dot{V} &= -e^T Q e + \tilde{\theta}^T \omega B_m^T P e + \tilde{\theta}^T \Gamma^{-1} \dot{\theta} \\
&= -e^T Q e + \tilde{\theta}^T \Big[\omega \underbrace{B_m^T P e}_{C_m^T e = e_a} + \Gamma^{-1} \dot{\theta}\Big]
\end{aligned}
$$

We choose the update law: $$\boxed{\dot{\theta} = -\Gamma\,\omega e_a}$$

Result: $$\boxed{\dot{V} = -e^T Q e \leq 0}$$

★ In fact, same result!

Resumo do algoritmo

Subsistema	Equação	Ordem
Planta	$y = P(s)\,u$	n
Modelo	$y_m = M(s)\,r$	n
Erro	$e_a = \operatorname{sign}(k_p)\,(y - y_m)$	
Controle	$u = \theta^T \omega$	
Λ-Filters	$\dot{\omega}_1 = A_f\,\omega_1 + b_f\,u$	$n-1$
	$\dot{\omega}_2 = A_f\,\omega_2 + b_f\,y$	$n-1$
Regressor	$\omega^T = \begin{bmatrix}\omega_1^T & y & \omega_2^T & r\end{bmatrix}$	
Adaptação	$\dot{\theta} = -\Gamma\omega\,e_a$	$2n$

Ordem total do sistema: $\boxed{N = 6n - 2}$

4.2.6 Simulações

Example 16 2nd order plant.

Classificação do sistema:

$$\begin{aligned} n &= 2 && \text{(ordem)} \\ n^* &= 1 && \text{(grau relativo)} \\ n_p &= 4 && \text{(\# de parâmetros)} \end{aligned}$$

Planta.....................: $P(s) = \dfrac{s+1}{s^2 - 2s}$

Modelo...................: $M(s) = \dfrac{1.5(s+2)}{(s+1)(s+3)}$

Filtro......................: $\dfrac{1}{\Lambda(s)} = \dfrac{1}{s+2}$

Matching................: $\theta^{*T} = \begin{bmatrix} 1 & -6 & 9 & 1.5 \end{bmatrix}$

$$\|\theta^*\| = 10.9659$$

Simulation #1 Condição inicial pequena & $\theta(0) \approx \theta^*$.

Condições iniciais.....: $y(0) = \boxed{1}$

$y_m(0) = 0$

$\theta(0) = \boxed{0.95\,\theta^*}$

Parâmetros...............: $\Gamma = 10\,I$

Sinal de referência....: $r = 2 + 5\,\sin(t)$

★ Esta simulação mostra que a parte linear do sistema está correta !

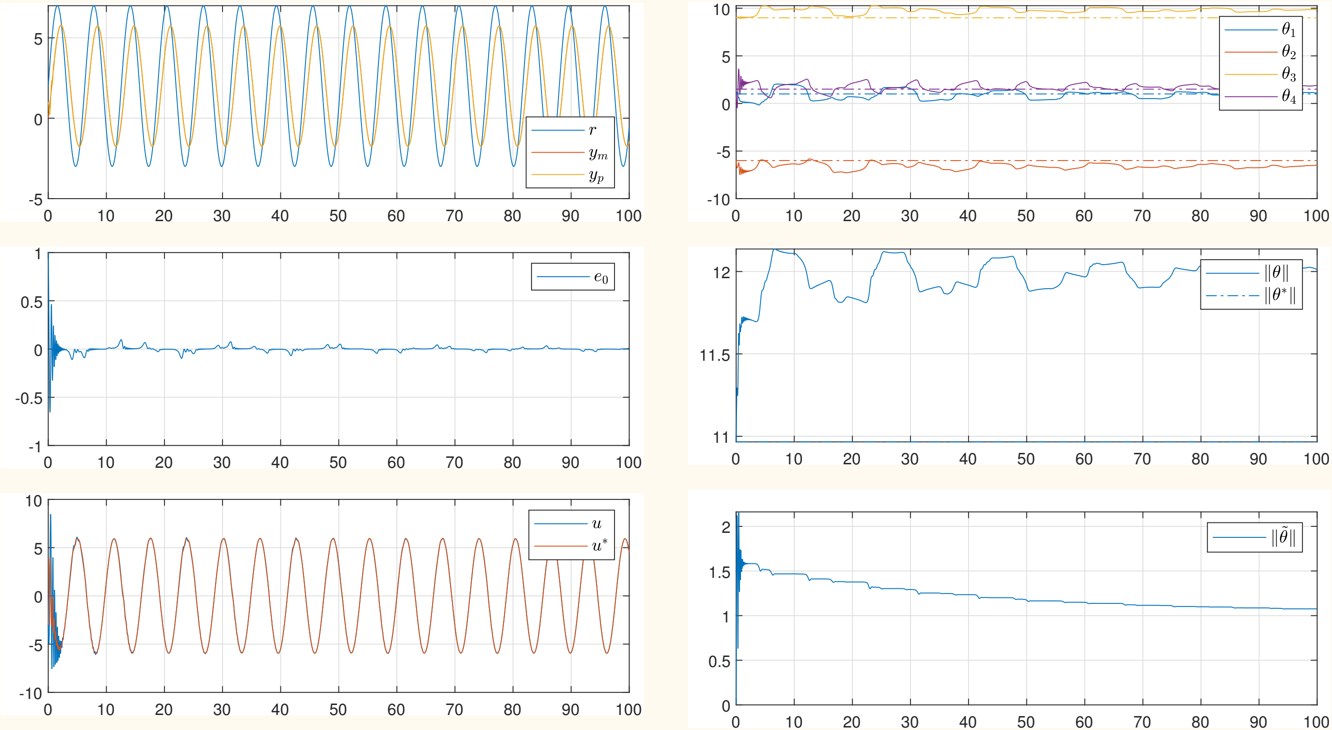


Figura 61: Resultado da simulação com algoritmo MRAC direto. (Script: `fig01.m`)

Simulation #2 Condição inicial pequena.

Condições iniciais.....: $y(0) = \boxed{1}$

$$y_m(0) = 0$$

$$\theta^T(0) = \boxed{\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}}$$

Parâmetros..............: $\Gamma = 10\, I$

Sinal de referência....: $r = 2 + 5\,\sin(t)$

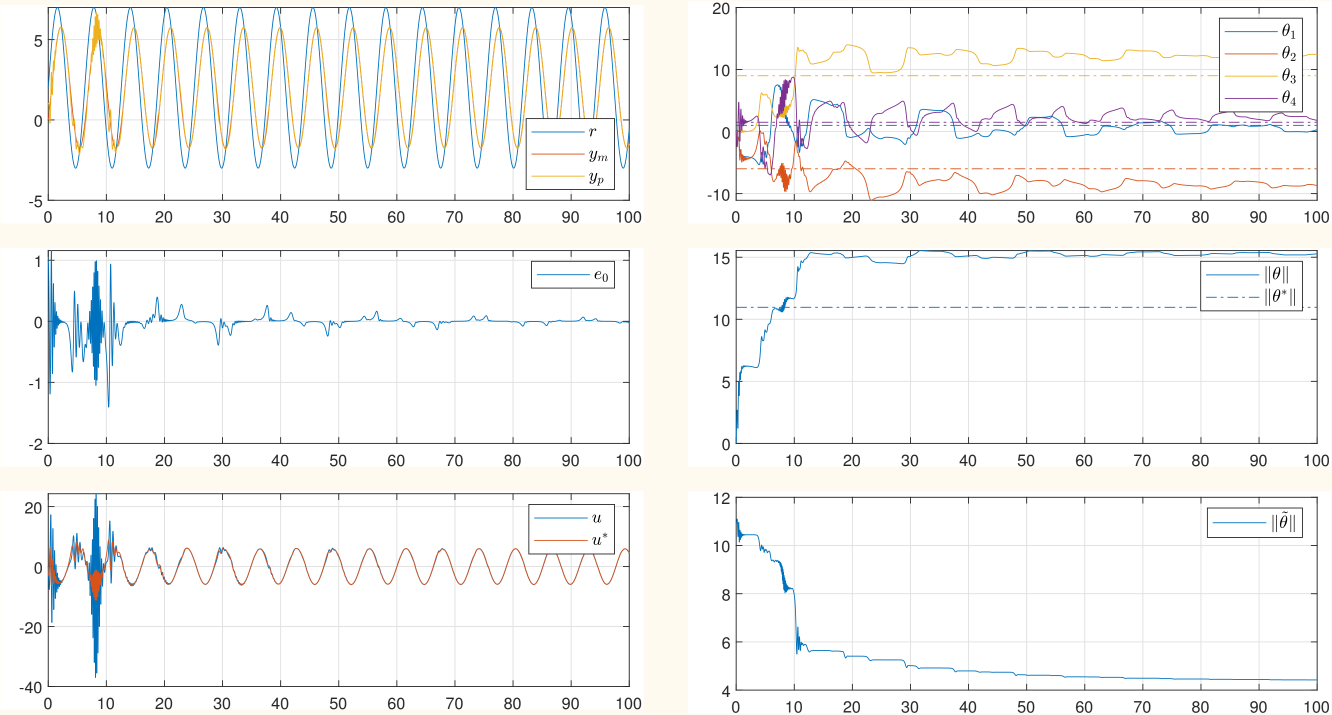


Figura 62: Resultado da simulação com algoritmo MRAC direto. (Script: `fig02.m`)

Simulation #3 Condição inicial grande.

Condições iniciais.....: $y(0) = 5$

$$y_m(0) = 0$$

$$\theta^T(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

Parâmetros..............: $\Gamma = 10\, I$

Sinal de referência....: $r = 2 + 5\,\sin(t)$

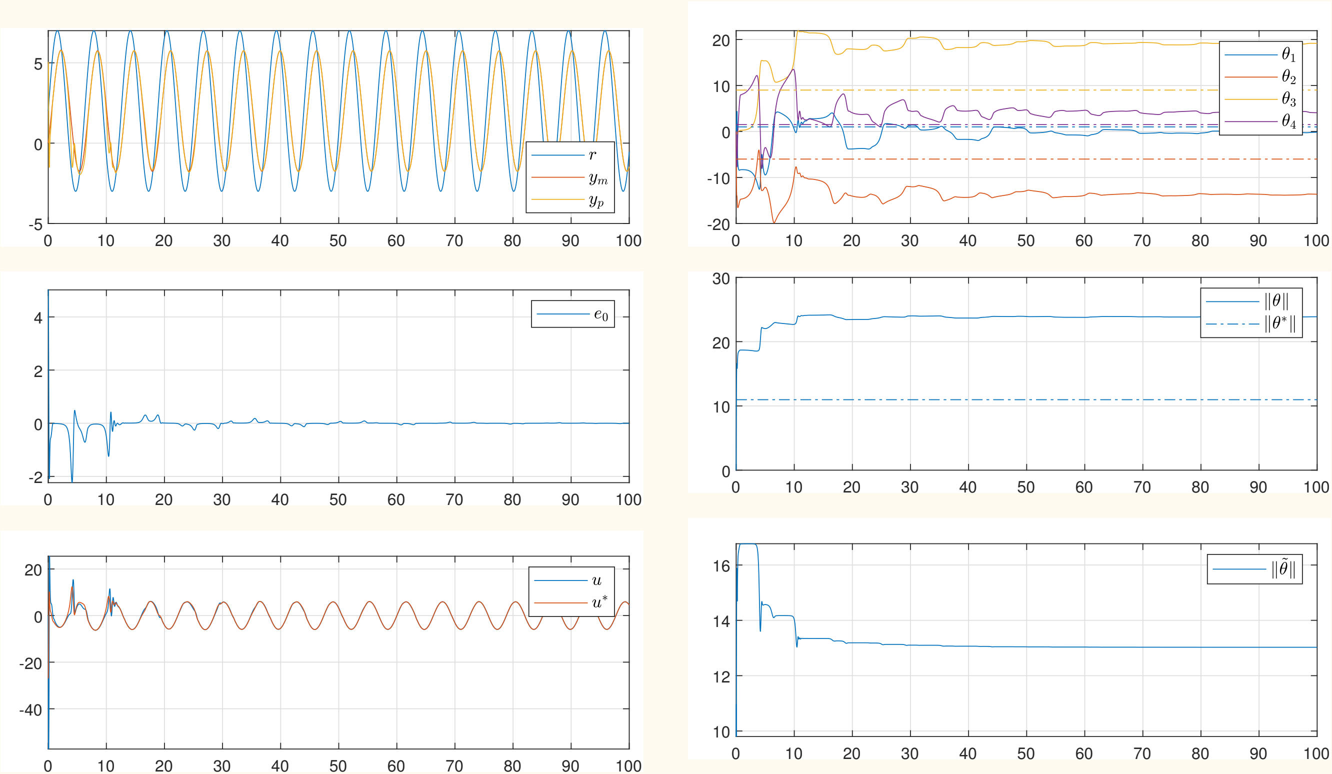


Figura 63: Resultado da simulação com algoritmo MRAC direto. (Script: `fig03.m`)

Simulation #4 Idem com excitação por onda quadrada.

Condições iniciais.....: $y(0) = 5$

$$y_m(0) = 0$$

$$\theta^T(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

Parâmetros..............: $\Gamma = 10\, I$

Sinal de referência....: $r = 2 + 5\mathrm{sqw}(\pi\, t/10)$

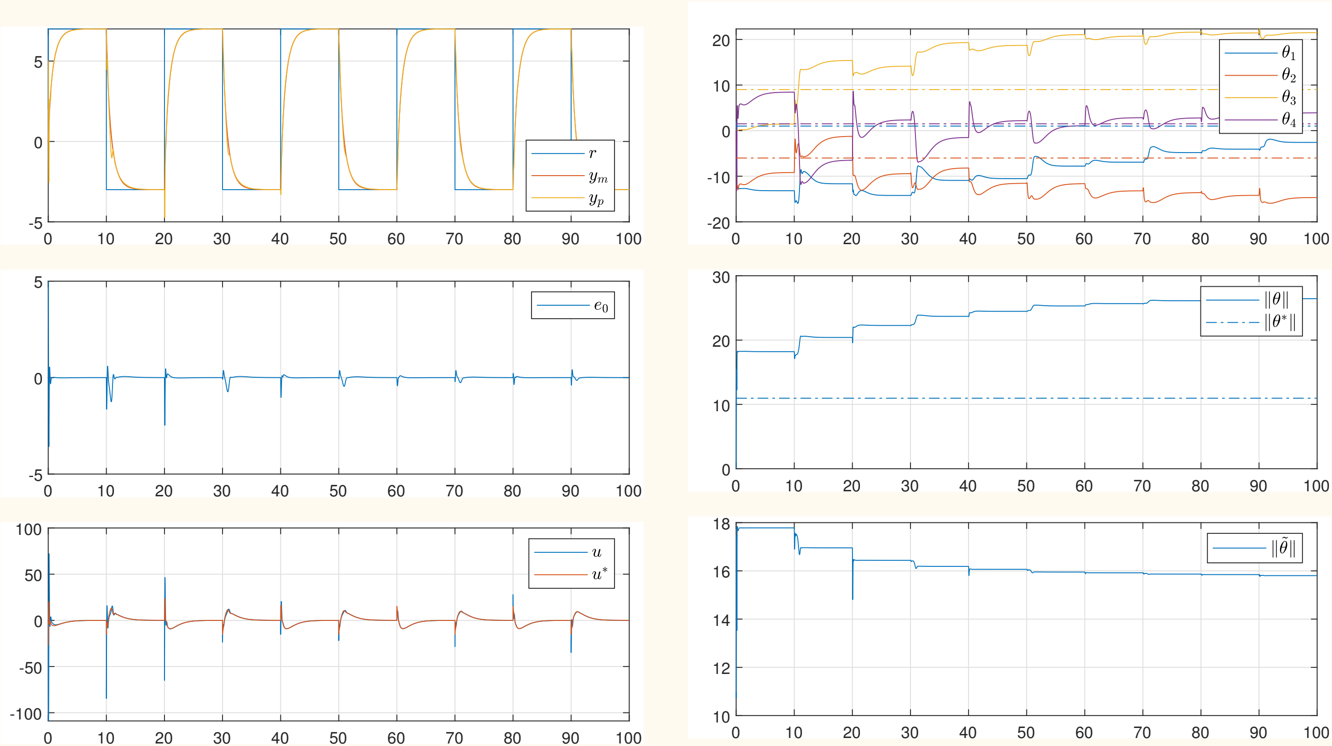


Figura 64: Resultado da simulação com algoritmo MRAC direto. (Script: `fig04.m`)

Example 17 4th order plant.

Classificação do sistema: $n = 4$ (ordem)
$n^* = 1$ (grau relativo)
$n_p = 8$ (# de parâmetros)

Planta.....................: $P(s) = \dfrac{0.1(s+2)^3}{s^4}$

Modelo...................: $M(s) = \dfrac{1}{s+1}$

Filtro......................: $\dfrac{1}{\Lambda(s)} = \dfrac{1}{(s+0.5)(s+1.5)(s+2)}$

Matching................: $\theta^{*T} = \begin{bmatrix} -6.5 & -7.25 & -2 & -50 & 60 & 175 & 112.5 & 10 \end{bmatrix}$

$$\|\theta^*\| = 222.6658$$

$$\|\theta^*_{ff}\| = 14.1\,, \qquad \|\theta^*_{fb}\| = 222.22$$

Simulation #1 Condição inicial nula & $\theta(0) \approx \theta^*$.

Condições iniciais.....: $$\begin{aligned} y(0) &= 0 \\ y_m(0) &= 0 \\ \theta(0) &= 0.99\,\theta^* \end{aligned}$$

Parâmetros...............: $\Gamma = 1\,I$

Sinal de referência....: $r = 3 + 10\mathrm{sqw}(\pi\, t/10)$

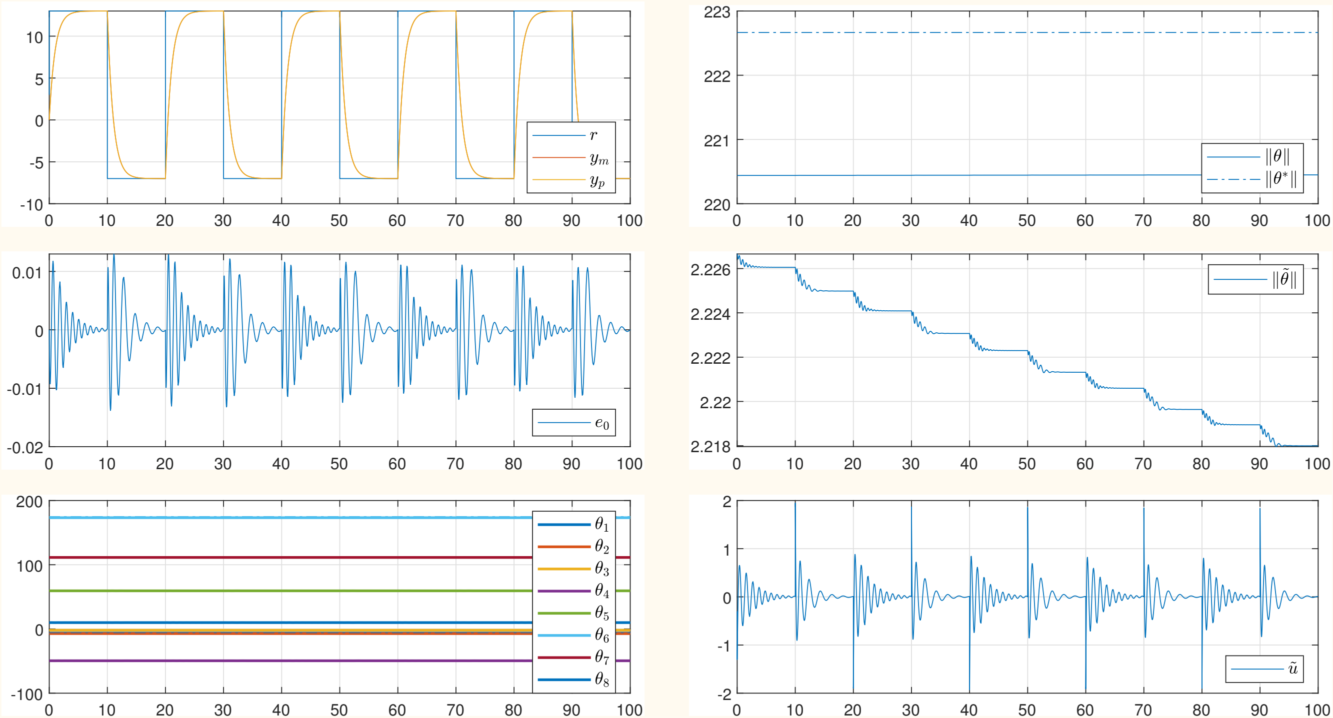


Figura 65: Resultado da simulação com algoritmo MRAC direto. (Script: `fig01.m`)

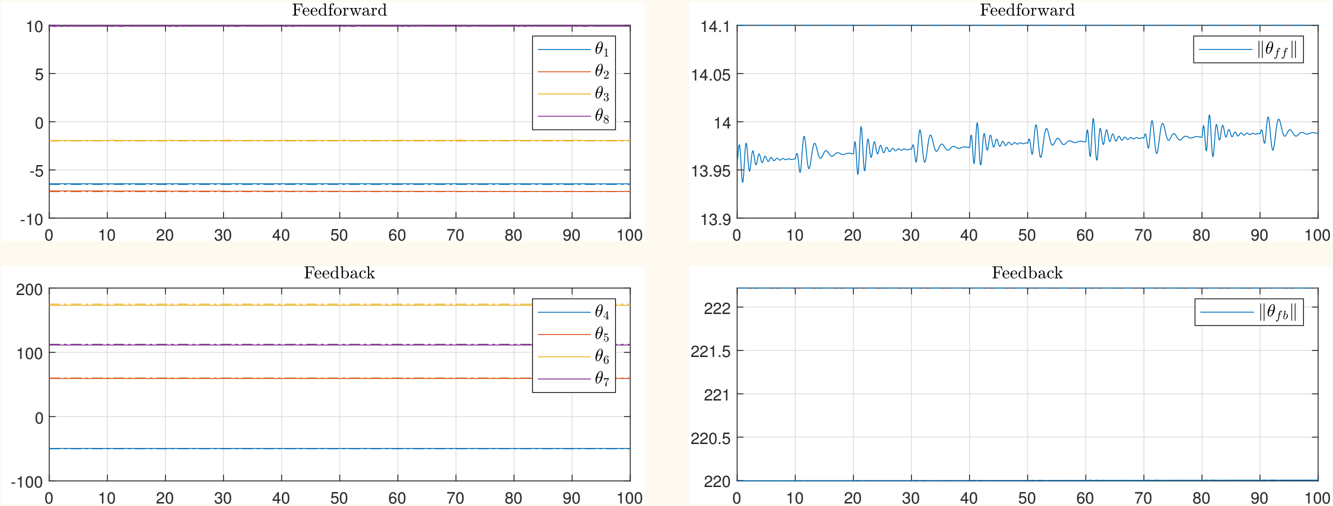


Figura 66: Resultado da simulação com algoritmo MRAC direto. (Script: `fig01.m`)

Simulation #2 Condições iniciais nulas. Ganho pequeno.

Condições iniciais.....: $y(0) = 0$

$$y_m(0) = 0$$

$$\theta^T(0) = \boxed{\begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}}$$

Parâmetros..............: $\Gamma = 1\, I$

Sinal de referência....: $r = 3 + 10\mathrm{sqw}(\pi\, t/10)$

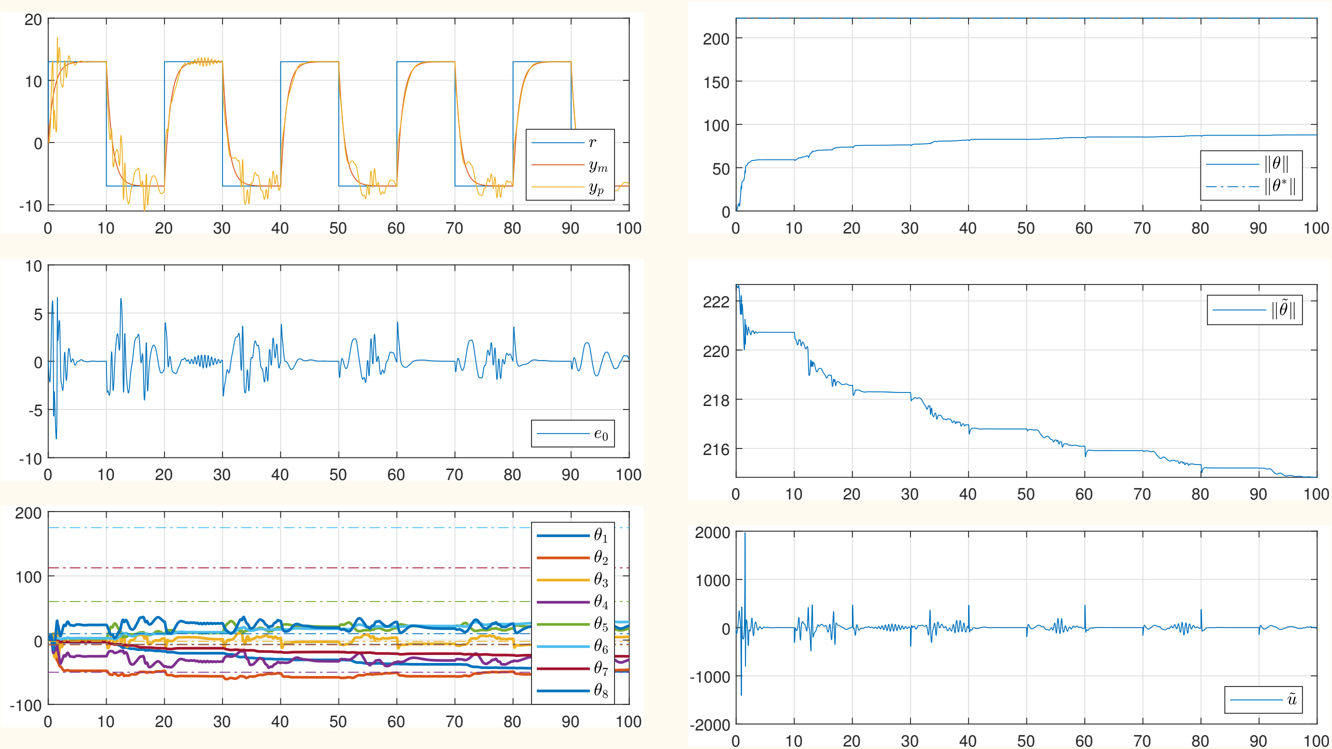


Figura 67: Resultado da simulação com algoritmo MRAC direto. (Script: `fig02.m`)

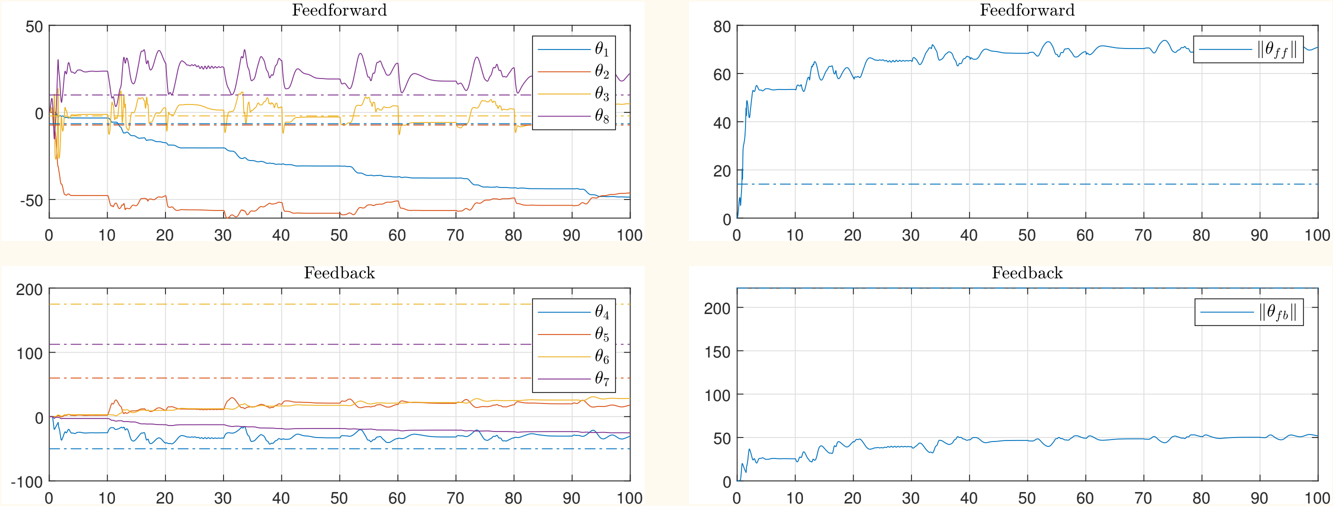


Figura 68: Resultado da simulação com algoritmo MRAC direto. (Script: `fig02.m`)

Simulation #3 Condições iniciais nulas. Ganho grande.

Condições iniciais.....: $y(0) = 0$

$$y_m(0) = 0$$

$$\theta^T(0) = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}$$

Parâmetros..............: $\Gamma = 10\, I$

Sinal de referência....: $r = 3 + 10\mathrm{sqw}(\pi\, t/10)$

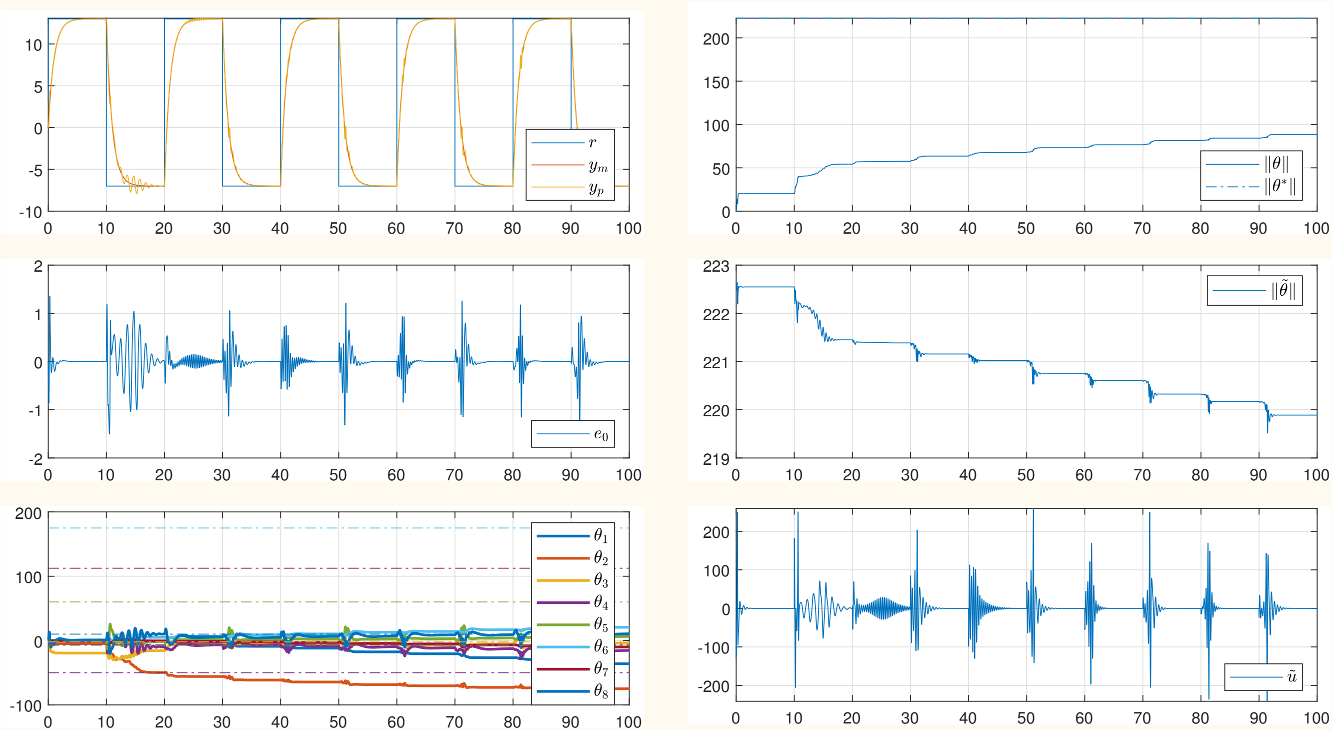


Figura 69: Resultado da simulação com algoritmo MRAC direto. (Script: `fig03.m`)

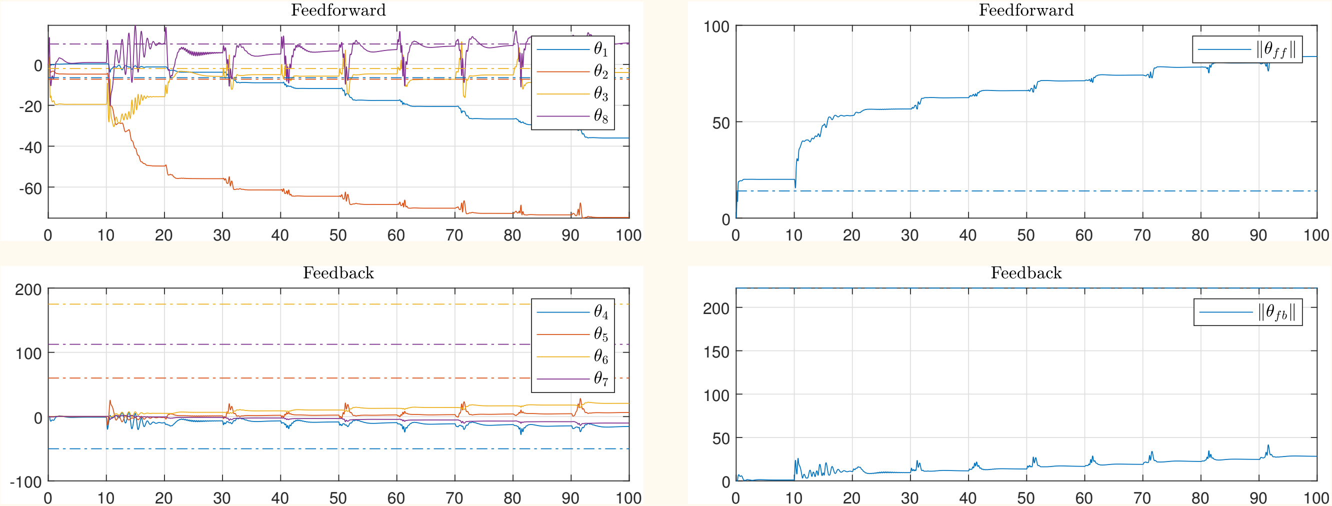


Figura 70: Resultado da simulação com algoritmo MRAC direto. (Script: `fig03.m`)

Simulation #4 Efeito da excitação contendo senóides.

Condições iniciais.....: $y(0) = 0$

$$y_m(0) = 0$$

$$\theta^T(0) = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}$$

Parâmetros..............: $\Gamma = 10\, I$

Sinal de referência....: $r = 3 + 10\mathrm{sqw}(\pi\, t/10) + S$

$$S = 2\sin(0.5t) + 2\sin(t) + 2\sin(2t) + 2\sin(3t)$$

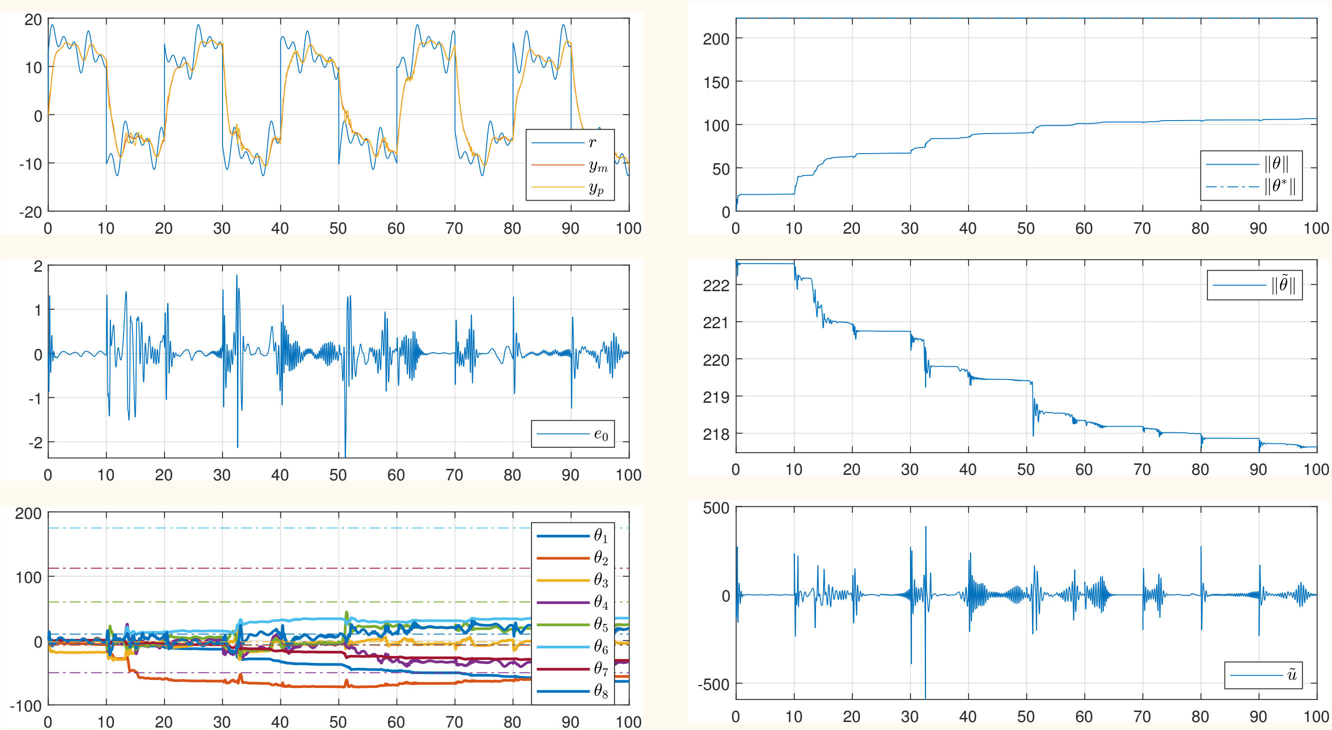


Figura 71: Resultado da simulação com algoritmo MRAC direto. (Script: `fig04.m`)

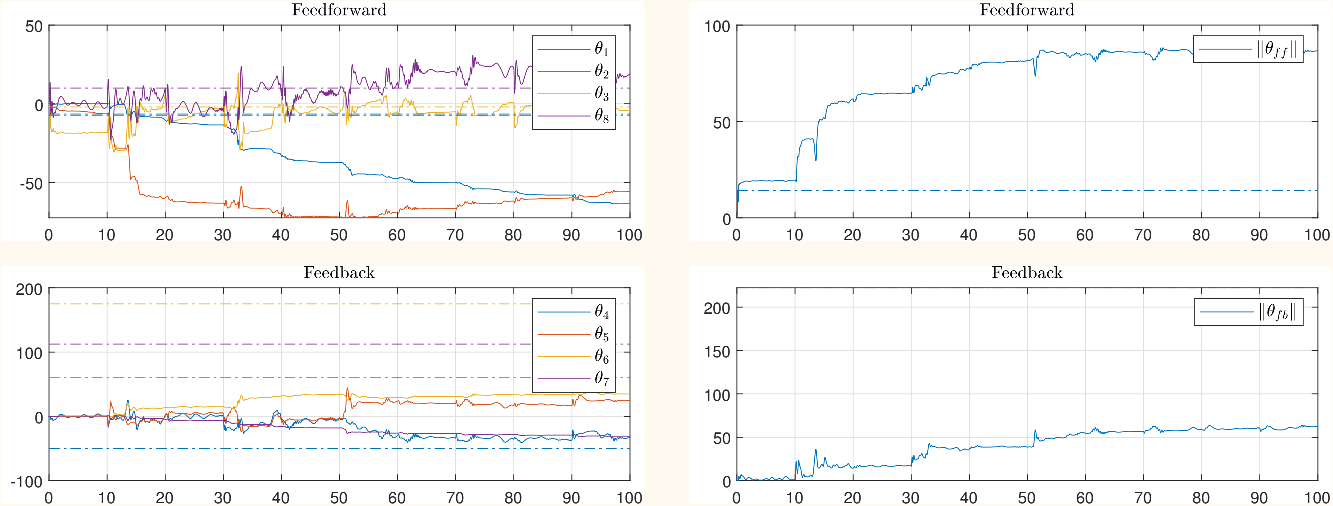


Figura 72: Resultado da simulação com algoritmo MRAC direto. (Script: `fig04.m`)

Simulation #5 Idem com condição inicial grande.

Condições iniciais.....: $y(0) = 10$

$y_m(0) = 0$

$\theta^T(0) = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}$

Parâmetros..............: $\Gamma = 10\,I$

Sinal de referência....: $r = 3 + 10\mathrm{sqw}(\pi\,t/10) + S$

$S = \sum \sin(.)$

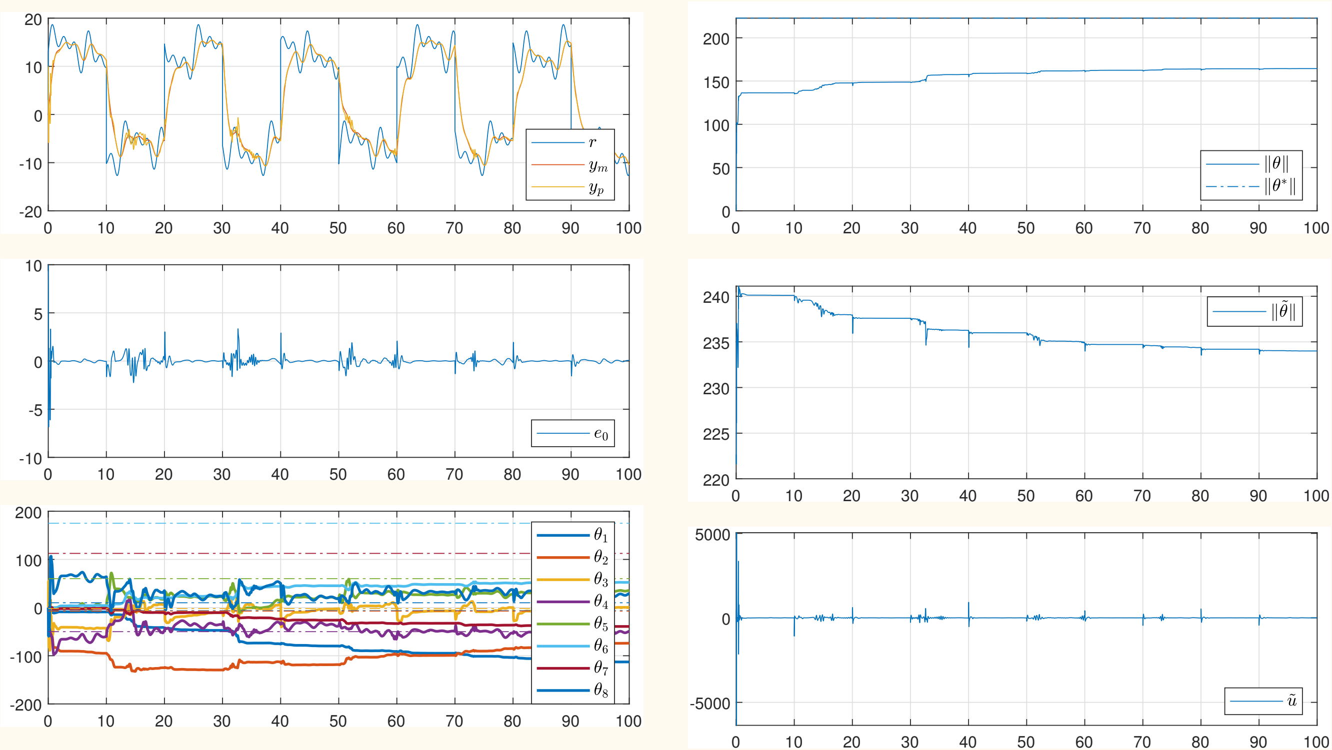


Figura 73: Resultado da simulação com algoritmo MRAC direto. (Script: `fig05.m`)

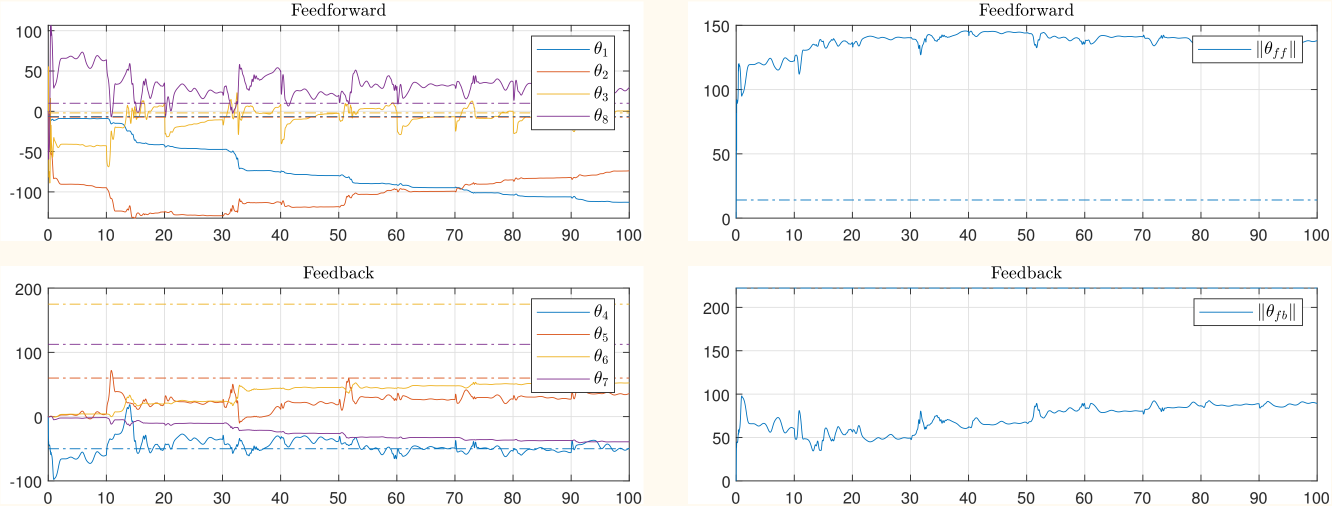


Figura 74: Resultado da simulação com algoritmo MRAC direto. (Script: `fig05.m`)

★ Note que $\|\theta\|$ é praticamente constante para $t > 50$.
No entanto, $\|\theta_{ff}\|$ e $\|\theta_{fb}\|$ variam.

Simulation #6 Idem com excitação por uma única senóide.

Condições iniciais.....: $y(0) = 10$

$$y_m(0) = 0$$

$$\theta^T(0) = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}$$

Parâmetros..............: $\Gamma = 10\, I$

Sinal de referência....: $r = 3 + 5\sin(t)$

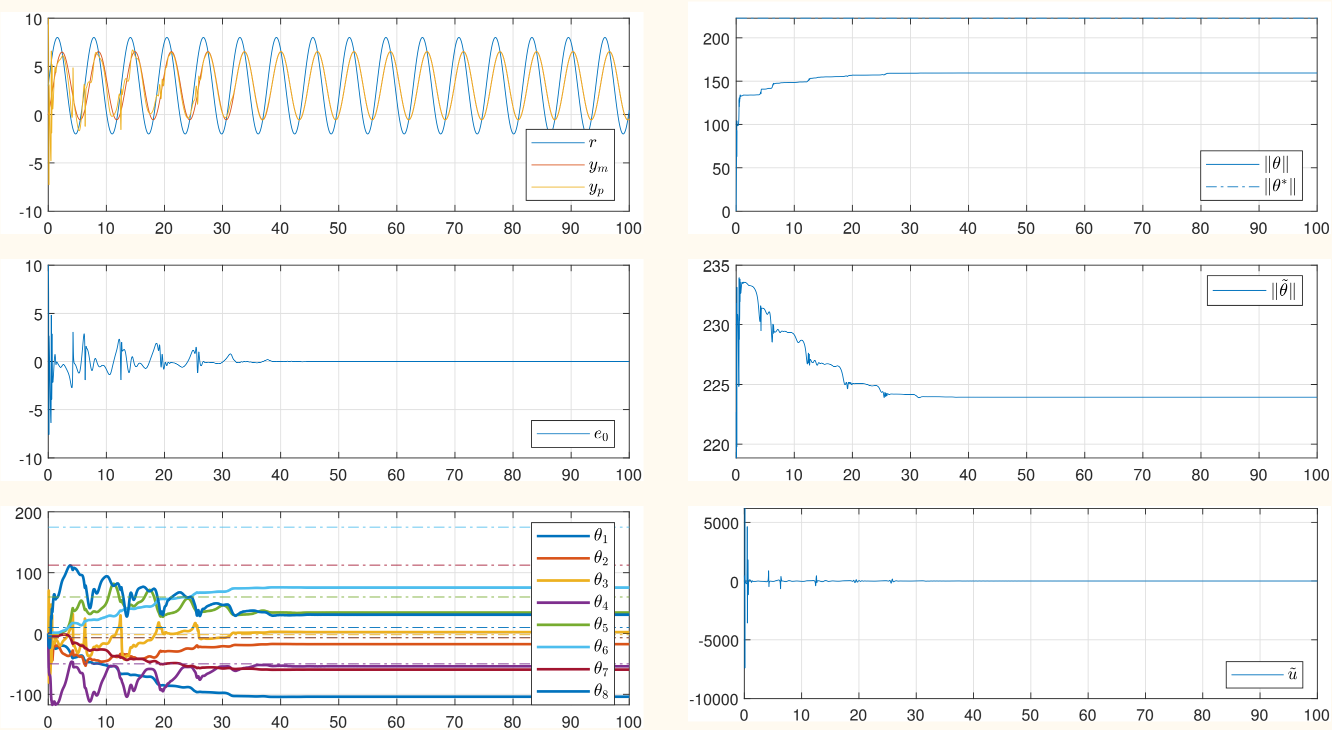


Figura 75: Resultado da simulação com algoritmo MRAC direto. (Script: `fig06.m`)

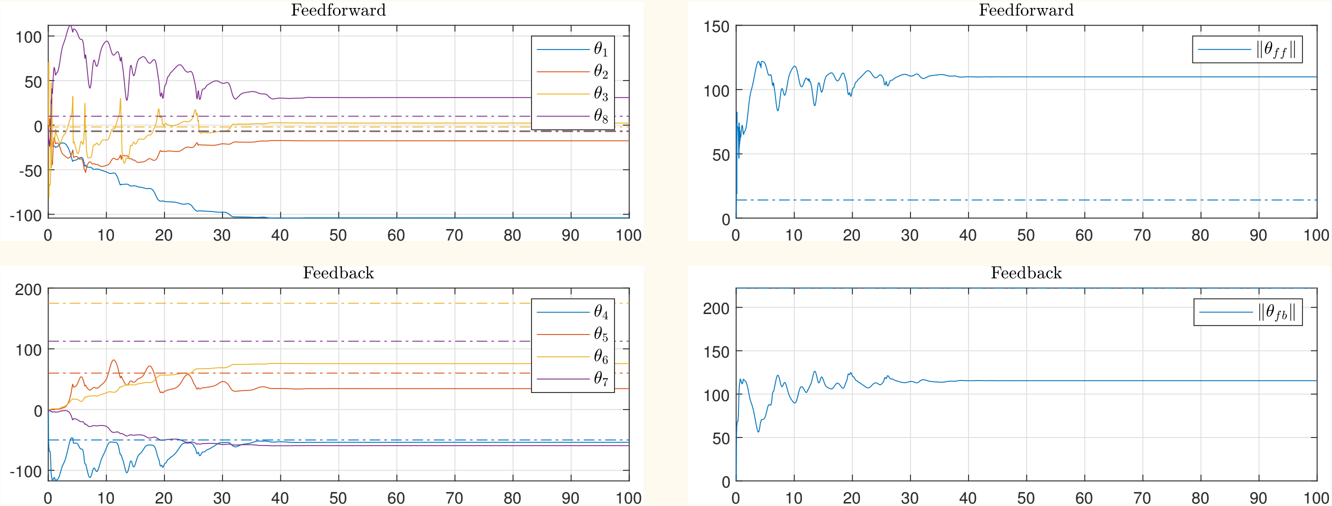


Figura 76: Resultado da simulação com algoritmo MRAC direto. (Script: `fig06.m`)

4.3 Caso $n^* = 2$

Problem: $M(s)$ cannot be chosen SPR.

Solution: (Monopoli multiplier)

Select a polynomial $L(s)$ such that $M(s)L(s)$ is SPR.

$$\Rightarrow \quad \boxed{L(s) = (s + \ell_0)} \qquad \left[\text{degree}(L) = 1\right]$$

Example 18 2nd order model.

$$M(s) = \frac{k_m}{(s+a)(s+b)}$$

$$L(s) = (s + \ell_0)$$

$$M(s)L(s) = \frac{k_m(s+\ell_0)}{(s+a)(s+b)}$$

Sufficient condition for SPR: $\boxed{a \leq \ell_0 \leq b}$

★ Poles and zeros are alternated.

The idea.

The equation error can be written as

$$\begin{aligned} e_a = \operatorname{sign}(k_p)e_0 &= |k_p|\, M\, \underbrace{L L^{-1}}_{1} \left[u - \theta^{*T}\omega\right] \\ &= \underbrace{|k_p|\, M L}_{\text{SPR}} \left[L^{-1}u - \theta^{*T}\, \underbrace{L^{-1}\omega}_{\xi}\right] \\ &= \underbrace{|k_p|\, M L}_{\text{SPR}} \left[\underbrace{L^{-1}u}_{\zeta} - \theta^{*T}\xi\right] \end{aligned}$$

Then, we select the virtual control $\boxed{\zeta = \theta^T\xi}$ $\Rightarrow$ $\boxed{u = L(s)\left[\theta^T\xi\right]}$

Filtered signal: $\boxed{\xi = L(s)^{-1}[\omega]}$ (Vector)

The equation error renders

$$e_a = \underbrace{|k_p|\, M L}_{\text{SPR}} \left[\theta^T \xi - \theta^{*T} \xi\right]$$

$$\Rightarrow \quad \boxed{e_a = |k_p|\, M L\left[\tilde{\theta}^T \xi\right]}$$

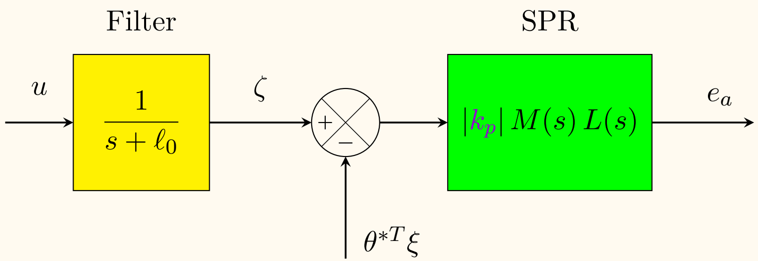


Figura 77: Interpretation.

Result: The update law for θ can be designed as in the case $n^* = 1$!

Important: The control u is easily implemented! (By *backstepping*!)

Recall that

$$\xi = \frac{1}{s+\ell_0}\omega \quad \Rightarrow \quad \dot{\xi} + \ell_0 \xi = \omega$$

The control u is

$$\begin{aligned} u = (s+\ell_0)\theta^T\xi &= \big(\dot{\theta}^T\xi + \theta^T\dot{\xi}\big) + \ell_0\big(\theta^T\xi\big) \\ &= \dot{\theta}^T\xi + \theta^T\big(\underbrace{\dot{\xi} + \ell_0\xi}_{\omega}\big) \quad \Rightarrow \quad \boxed{u = \theta^T\omega + \dot{\theta}^T\xi} \end{aligned}$$

where $\dot{\theta}$ is available!

★ Note that no differentiator is used to obtain the control u !

4.3.1 Lyapunov design

The derivation of the update law is similar to the previous case.

Error equation: $\Rightarrow$

$$\boxed{e_a = \underbrace{|k_p|\, ML}_{\text{SPR}} \left[\tilde{\theta}^T \xi\right]}$$

State space representation:

$$\begin{cases} \dot{e} = A_m e + B_m[\tilde{\theta}^T \xi] \\ e_a = C_m e \end{cases}$$

- ★ $\{A_m, B_m, C_m\}$ is a non–minimal realization of $|k_p| M(s) L(s)$.
- ★ The error vector $e \in \mathbb{R}^{5n-2}$.

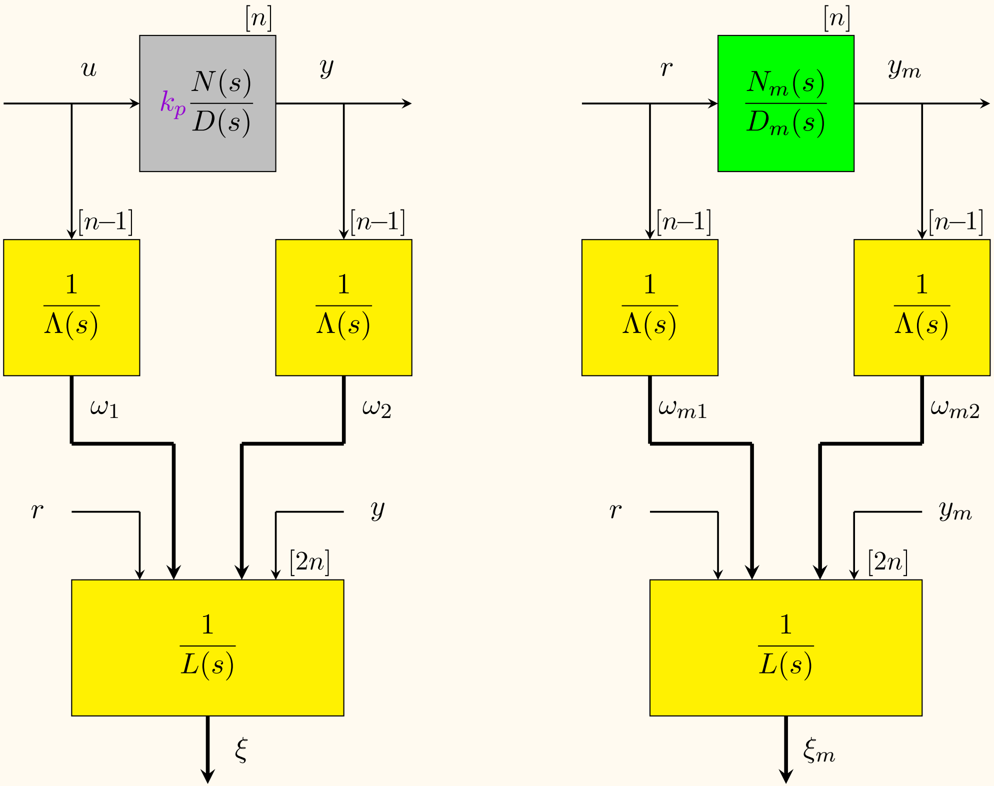


Figura 78: Non–minimal realization with $e \in \mathbb{R}^{5n-2}$.

Since the system $\{A_m, B_m, C_m\}$ is SPR then

$$\exists \begin{cases} P = P^T > 0 \\ Q = Q^T > 0 \end{cases} \quad \text{s.t.} \quad \begin{cases} A_m^T P + P A_m = -2Q \\ P B_m = C_m^T \end{cases}$$

Set of equations: $$\begin{cases} \dot{e} = A_m e + B_m[\tilde{\theta}^T \xi]\,, & e_a = C_m e \\ \dot{\theta} = ? \end{cases}$$

Now, consider the Lyapunov function

$$\boxed{2V(e, \tilde{\theta}) = e^T P e + \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}} \qquad \Gamma = \Gamma^T > 0$$

Time derivative:

$$
\begin{aligned}
2\dot{V} &= \dot{e}^T P e + e^T P \dot{e} + \left(\dot{\theta}^T \Gamma^{-1} \tilde{\theta} + \tilde{\theta}^T \Gamma^{-1} \dot{\theta}\right) \\
&= \left(A_m e + B_m \tilde{\theta}^T \xi\right)^T P e + e^T P \left(A_m e + B_m \tilde{\theta}^T \xi\right) + 2\tilde{\theta}^T \Gamma^{-1} \dot{\theta} \\
&= e^T A_m^T P e + \tilde{\theta}^T \xi B_m^T P e + e^T P A_m e + e^T P B_m \tilde{\theta}^T \xi + 2\tilde{\theta}^T \Gamma^{-1} \dot{\theta} \\
&= e^T \big(\underbrace{A_m^T P + P A_m}_{-2Q}\big) e + 2\, \tilde{\theta}^T \xi B_m^T P e + 2\tilde{\theta}^T \Gamma^{-1} \dot{\theta} \\
&= -2 e^T Q e + 2\tilde{\theta}^T \xi B_m^T P e + 2\tilde{\theta}^T \Gamma^{-1} \dot{\theta}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\dot{V} &= -e^T Q e + \tilde{\theta}^T \xi B_m^T P e + \tilde{\theta}^T \Gamma^{-1} \dot{\theta} \\
&= -e^T Q e + \tilde{\theta}^T \Gamma^{-1} \Big[\Gamma \xi \underbrace{B_m^T P e}_{C_m e = e_a} + \dot{\theta}\Big]
\end{aligned}
$$

We choose the update law: $$\boxed{\dot{\theta} = -\Gamma\, \xi e_a}$$

Result: $$\boxed{\dot{V} = -e^T Q e \leq 0}$$

- ★ The previous regressor ω has been replaced by ξ.
- ★ The dynamics of the filter $L^{-1}(s)$ has been included.

From the above analysis, we have that $\boxed{\omega \in \mathcal{L}_\infty}$.

Since $L^{-1}(s)$ is BIBO, then

$$\xi = \underbrace{L^{-1}(s)}_{\text{BIBO}} \underbrace{\omega}_{\mathcal{L}_\infty} \quad \Rightarrow \quad \boxed{\xi \in \mathcal{L}_\infty}$$

Comments

- The control signal is modified! It is not only $\theta^T\omega$!
- Note that the extra term $\dot{\theta}^T\xi \to 0$ as $\theta \to \theta^*$.
- The extra term acts only during the identification transient.

Summary

Subsystem	Equation	Dimension
Plant	$y = P(s)\, u$	n
Model	$y_m = M(s)\, r$	n
Tracking error	$e_a = \operatorname{sign}(k_p)\,(y - y_m)$	
Control law	$u = \theta^T \omega + \dot{\theta}^T \xi$	
SV filters	$\dot{\omega}_1 = A_f\, \omega_1 + b_f\, u$ $\dot{\omega}_2 = A_f\, \omega_2 + b_f\, y$	$n-1$ $n-1$
Regressor	$\omega^T = \begin{bmatrix} \omega_1^T & y & \omega_2^T & r \end{bmatrix}$	
ξ-Filter	$\dot{\xi} = -\ell_0\, \xi + \omega\,, \quad \ell_0 > 0$	$2n$
Update law	$\dot{\theta} = -\Gamma \xi\, e_a$	$2n$

System total dimension: $\boxed{N = 8n - 2}$

4.3.2 Simulações

Example 19 2nd order plant.

Classificação do sistema: $n = 2$ (ordem)
$n^* = 2$ (grau relativo)
$n_p = 4$ (# de parâmetros)

Planta......................: $P(s) = \dfrac{k_p}{s^2 - 2s}$

Modelo....................: $M(s) = \dfrac{3}{(s+1)(s+3)}$

Filtros......................: $\dfrac{1}{\Lambda(s)} = \dfrac{1}{s+1}$

Polinômio.................: $L(s) = s + \ell_0$

Matching . . . . :

$$\theta^{*T} = \begin{bmatrix} -6 & -21 & 18 & 3 \end{bmatrix}$$

$$\|\theta^*\| = 28.4605$$

Simulation #1 Condição inicial nula & $\theta(0) \approx \theta^*$.

Condições iniciais.....: $y(0) = 0$

$y_m(0) = 0$

$\theta(0) = 0.9\,\theta^*$

Parâmetros..............: $\Gamma = 1\,I$

$\ell_0 = 1$

$k_p = 1$

Sinal de referência....: $r = 2 + 5\,\sin(t)$

★ Está simulação mostra que a parte linear está correta.

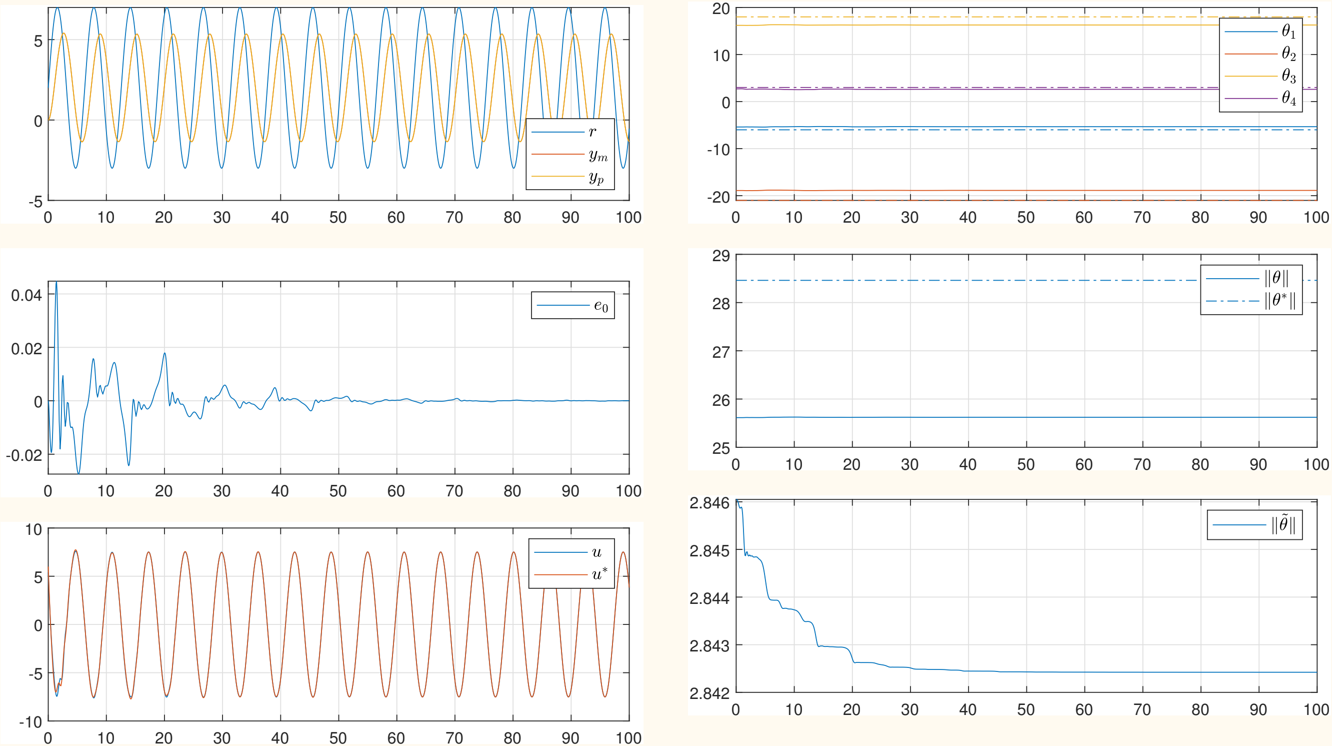


Figura 79: Resultado da simulação com algoritmo MRAC direto. (Script: `fig01.m`)

Simulation #2 Condições iniciais nulas & ganho pequeno.

Condições iniciais.....: $y(0) = 0$

$y_m(0) = 0$

$\theta^T(0) = [0 \quad 0 \quad 0 \quad 0]$

Parâmetros..............: $\Gamma = 1\,I$

$\ell_0 = 1$

$k_p = 1$

Sinal de referência....: $r = 2 + 5\,\sin(t)$

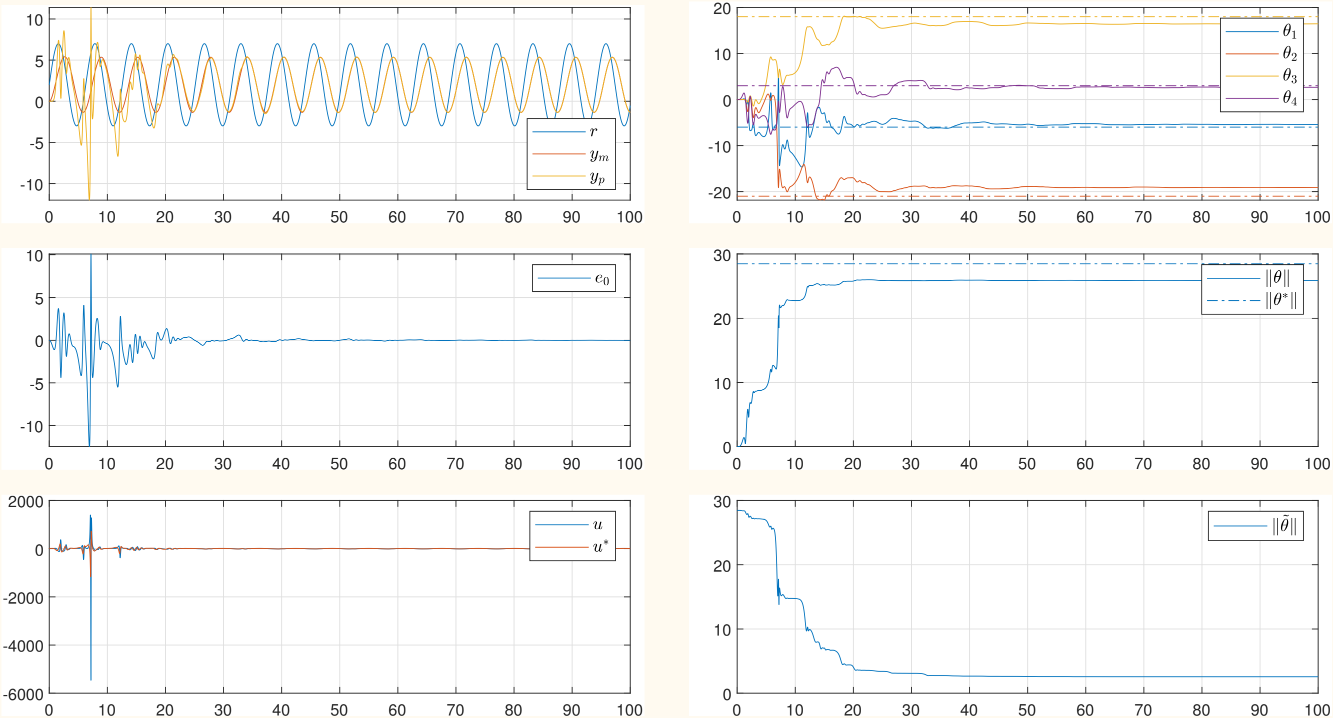


Figura 80: Resultado da simulação com algoritmo MRAC direto. (Script: `fig02.m`)

Simulation #3 Condições iniciais pequenas & ganho pequeno.

Condições iniciais.....: $y(0) = 2$

$$y_m(0) = 0$$

$$\theta^T(0) = [0 \quad 0 \quad 0 \quad 0]$$

Parâmetros..............: $\Gamma = 1\,I$

$$\ell_0 = 1$$

$$k_p = 1$$

Sinal de referência....: $r = 2 + 5\sin(t)$

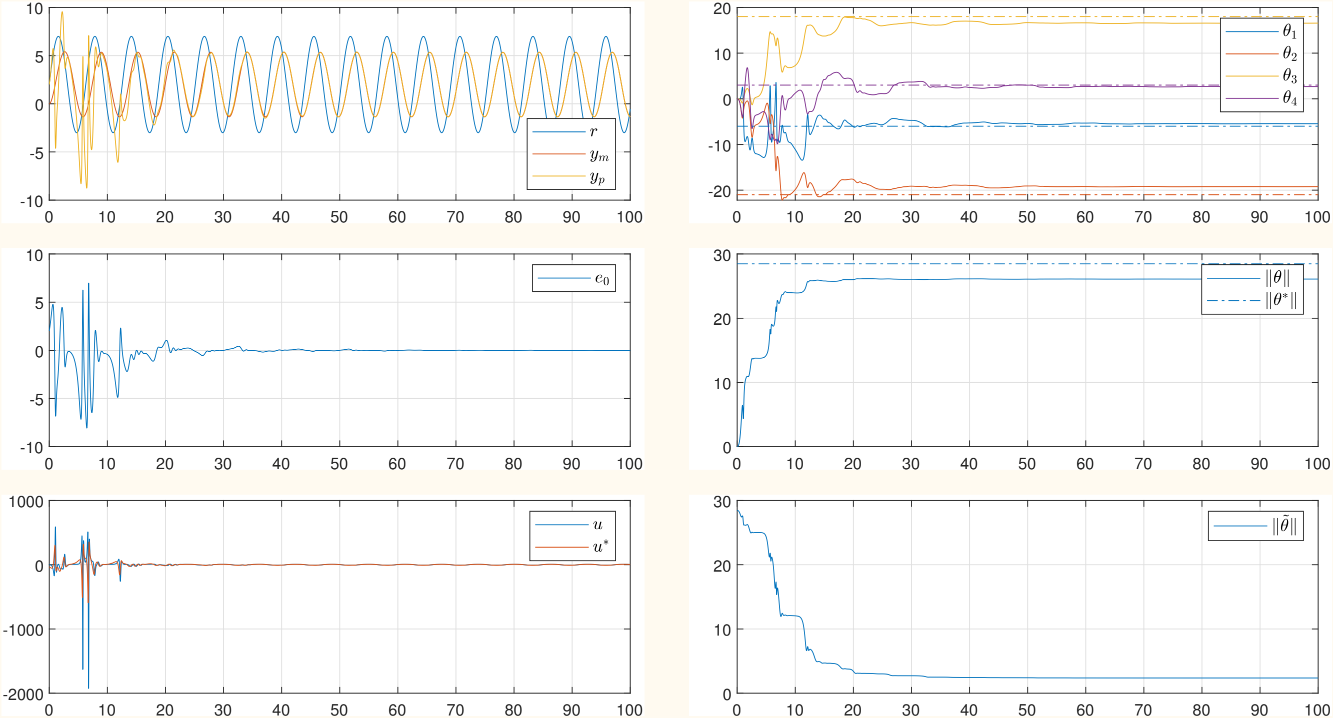


Figura 81: Resultado da simulação com algoritmo MRAC direto. (Script: `fig03.m`)

Simulation #4 Condições iniciais pequenas & ganho grande.

Condições iniciais.....: $y(0) = 2$

$$y_m(0) = 0$$

$$\theta^T(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

Parâmetros..............: $\Gamma = 10\,I$

$$\ell_0 = 1$$

$$k_p = 1$$

Sinal de referência....: $r = 2 + 5\,\sin(t)$

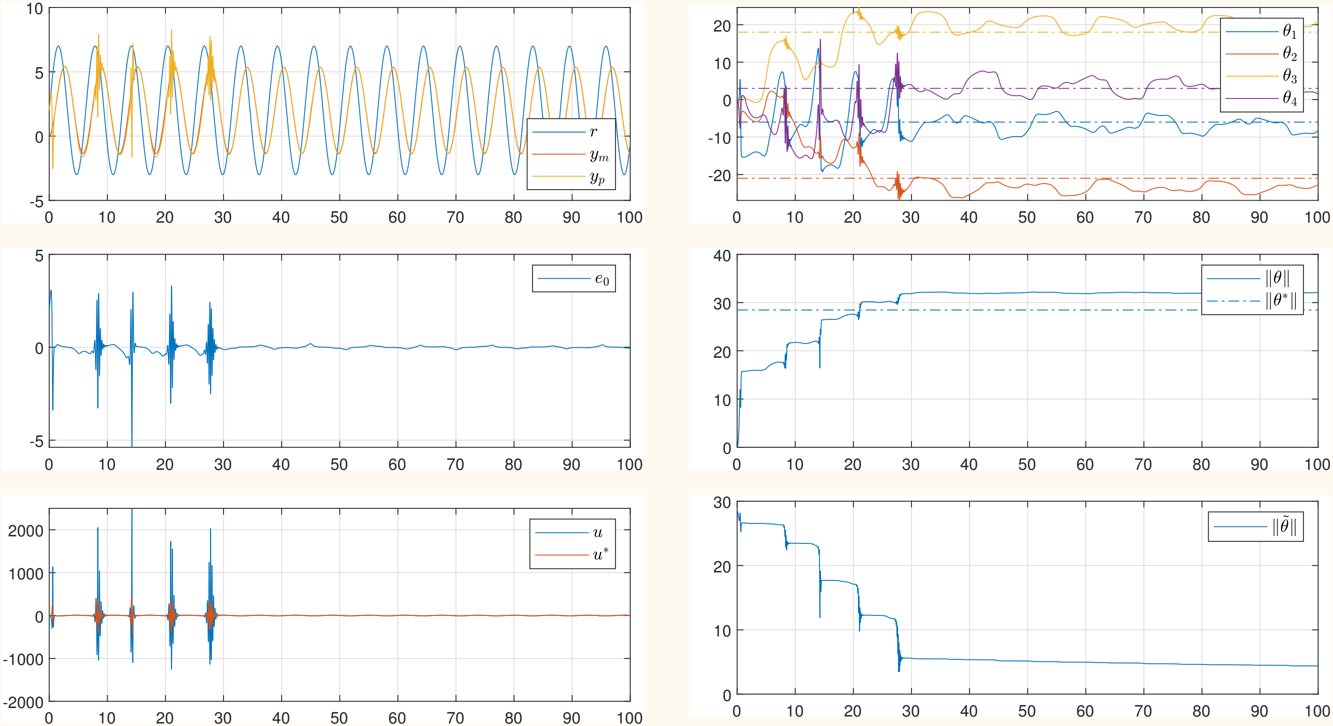


Figura 82: Resultado da simulação com algoritmo MRAC direto. (Script: `fig04.m`)

Simulation #5 Condição inicial grande & ganho grande.

Condições iniciais.....: $y(0) = 10$

$y_m(0) = 0$

$\theta^T(0) = [0 \quad 0 \quad 0 \quad 0]$

Parâmetros..............: $\Gamma = 10\, I$

$\ell_0 = 1$

$k_p = 1$

Sinal de referência....: $r = 2 + 5\, \sin(t)$

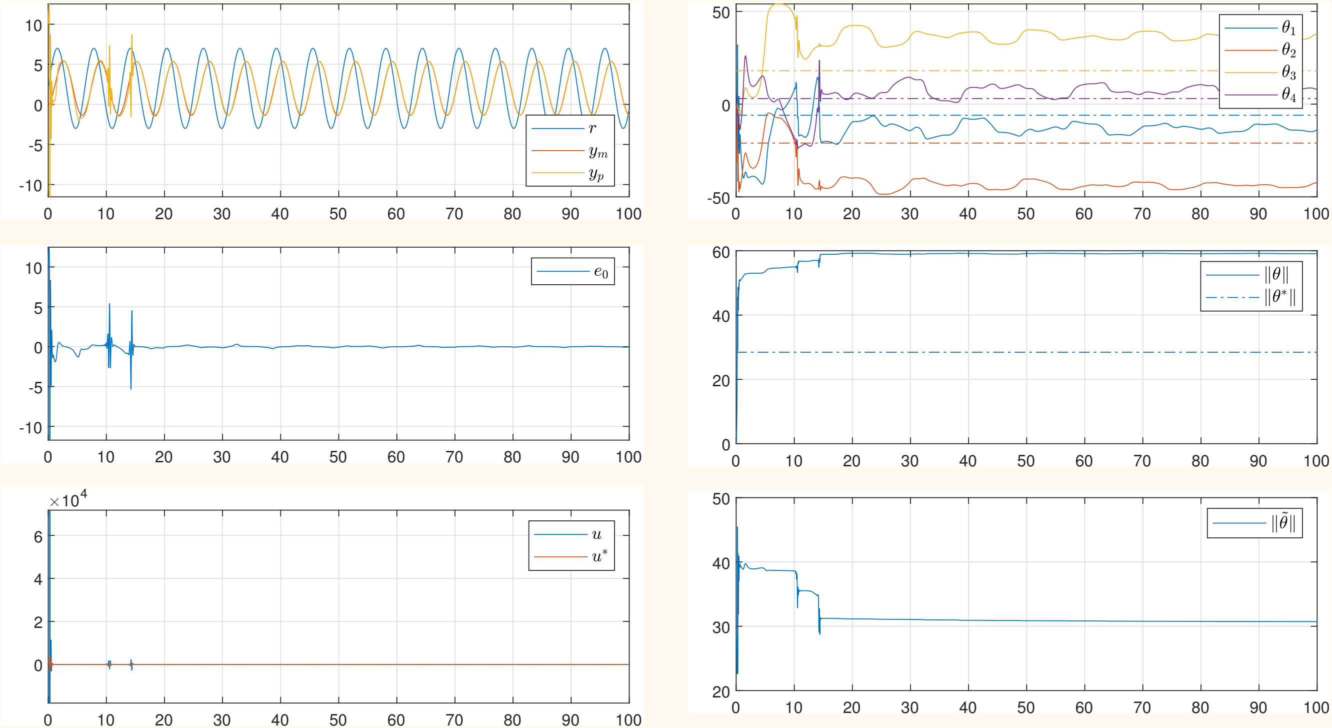


Figura 83: Resultado da simulação com algoritmo MRAC direto. (Script: `fig05.m`)

Simulation #6 Idem com excitação por onda quadrada.

Condições iniciais.....: $y(0) = 10$

$y_m(0) = 0$

$\theta^T(0) = [0 \quad 0 \quad 0 \quad 0]$

Parâmetros..............: $\Gamma = 10\, I$

$\ell_0 = 1$

$k_p = 1$

Sinal de referência....: $r = 2 + 5\mathrm{sqw}(\pi\, t/10)$

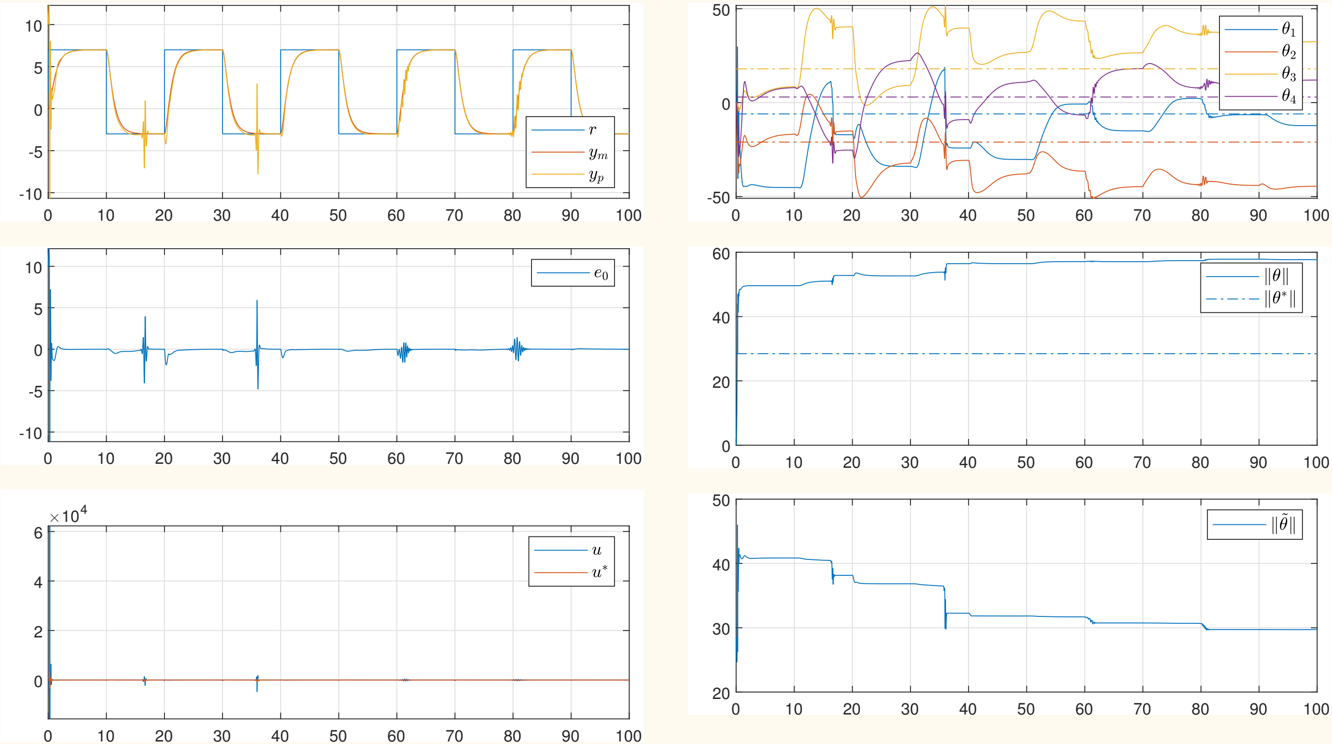


Figura 84: Resultado da simulação com algoritmo MRAC direto. (Script: `fig06.m`)

Simulation #7 Efeito da escolha inadequada do ganho de adaptação.

Condições iniciais.....: $y(0) = 2$

$$y_m(0) = 0$$

$$\theta^T(0) = [0 \quad 0 \quad 0 \quad 0]$$

Parâmetros..............: $\Gamma = \mathrm{diag}\{10,\ 0.01,\ 10,\ 10\}$

$$\ell_0 = 3$$

$$k_p = 1$$

Sinal de referência....: $r = 2 + 5\mathrm{sqw}(\pi\, t/10) + 5\sin(t)$

★ A teoria garante estabilidade do sistema $\forall\, \Gamma > 0$!

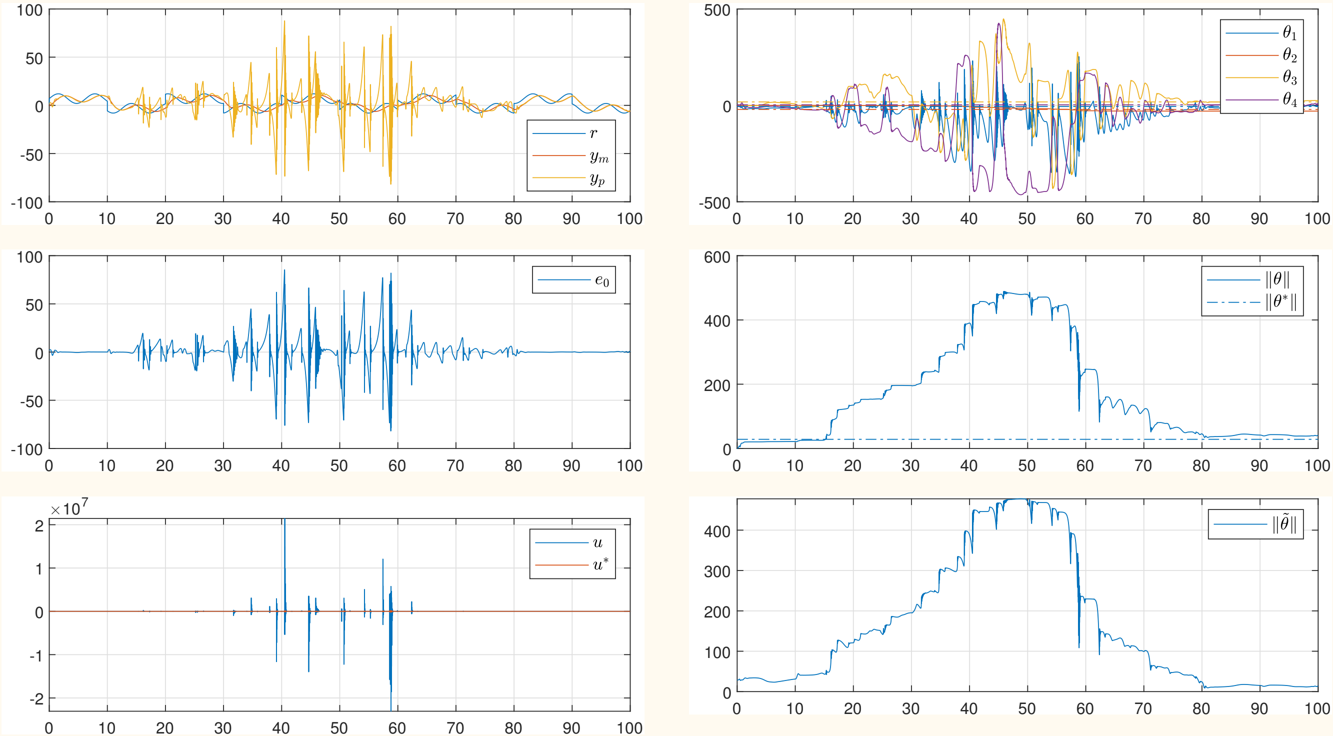


Figura 85: Resultado da simulação com algoritmo MRAC direto. (Script: `fig07.m`)

4.4 Caso geral $n^* > 1$

Problem:

★ The previous algorithm cannot be generalized to the case $n^* > 2$.

★ Monopoli multiplier trick fails.

★ The control u could not be implemented since $\ddot{\theta}$, $\dddot{\theta}$, ... are not available.

★ The update law is driven by a prediction error.

- The solution discussed in this section was developed in the 80's.

4.4.1 Método por Lyapunov ($n^* > 1$)

Ref.: [Tao:2003], (pag. 204)

Plant: $$\boxed{y = P(s)\,u}$$

Model: $$\boxed{y_m = M(s)\,r}$$

Solution: Select a polynomial $L(s)$ (Monopoli multiplier) such that

$$\boxed{M(s)L(s) \text{ is SPR}} \qquad \text{degree}(L) = n^* - 1$$

Error equation: $$e_0 = k_p\, M\big[u - \theta^{*T}\omega\big]$$

★ The unknown k_p will be estimated. $\Rightarrow$ It cannot be absorbed by the model.

Then, we can rewrite the equation as:

$$e_0 = M\,L\Big[k_p\big(\underbrace{L^{-1}u}_{\zeta} - \theta^{*T}\,\underbrace{L^{-1}\omega}_{\xi}\big)\Big]$$

Filters: $$\zeta = L^{-1}u$$ (Scalar)

$$\xi = L^{-1}\omega$$ (Vector)

Error equation: $$e_0 = ML\Big[k_p\big(\zeta - \theta^{*T}\xi\big)\Big]$$

★ A prediction can be obtained from the estimates ρ and θ.

★ An estimate ρ of k_p is essential for the prediction.

Prediction: $$\hat{e}_0 = ML\big[\rho\big(\zeta - \theta^T\xi\big) + \kappa\epsilon m_0^2\big] \qquad \boxed{\kappa > 0}$$

★ $\kappa\epsilon m_0^2$ is a stabilizing term required to prove convergence of e_0.

★ Normalizing signal: $\boxed{m_0^2 = \xi^T\xi + \chi^2}$

Definition: $$\boxed{\chi = \zeta - \theta^T\xi} \quad = L^{-1}u - \theta^T L^{-1}\omega$$

Tracking error: $$e_0 = ML\Big[k_p\big(\zeta - \theta^{*T}\xi\big)\Big]$$

Prediction: $$\hat{e}_0 = ML\big[\rho\chi + \kappa\epsilon m_0^2\big]$$

Prediction error: $$\boxed{\epsilon = e_0 - \hat{e}_0}$$

Dynamics of the prediction error:

$$\begin{aligned}
\epsilon &= e_0 - \hat{e}_0 \\
&= ML\Big[k_p\big(\zeta - \underbrace{\theta^{*T}\xi}_{(\theta-\tilde{\theta})^T\xi}\big) - \rho\chi - \kappa\epsilon m_0^2\Big] \\
&= ML\Big[k_p\,\tilde{\theta}^T\xi + k_p\big(\underbrace{\zeta - \theta^T\xi}_{\chi}\big) - \rho\chi - \kappa\epsilon m_0^2\Big] \\
&= ML\Big[k_p\,\tilde{\theta}^T\xi - \tilde{\rho}\chi - \kappa\epsilon m_0^2\Big]
\end{aligned}$$

★ The block diagram shows the stabilizing term.

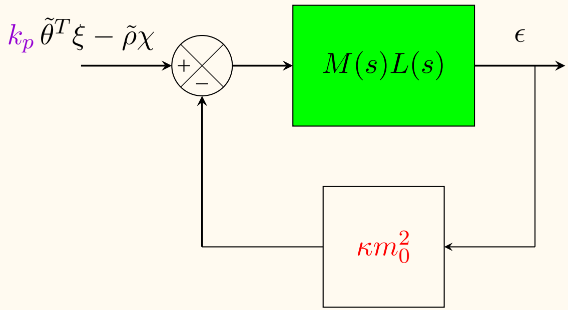


Figura 86: Stabilizing term.

State space representation:
$$\begin{cases} \dot{e} = Ae + B\Big[k_p\,\tilde{\theta}^T\xi - \tilde{\rho}\chi - \kappa\epsilon m_0^2\Big] \\ \epsilon = Ce \end{cases}$$

★ $\{A, B, C\}$ is a non–minimal realization of $M(s)L(s)$.

Since the system $\{A, B, C\}$ is SPR then

$$\exists \begin{cases} P = P^T > 0 \\ Q = Q^T > 0 \end{cases} \quad \text{s.t.} \quad \begin{cases} A^T P + PA = -2Q \\ PB = C^T \end{cases}$$

Lyapunov function:

$$\boxed{2V(e, \tilde{\theta}, \tilde{\rho}) = e^T P e + |k_p| \, \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} + \gamma^{-1} \tilde{\rho}^2}$$

Time derivative:

$$2\dot{V} = \dot{e}^T P e + e^T P \dot{e} + |k_p| \left(\dot{\theta}^T \Gamma^{-1} \tilde{\theta} + \tilde{\theta}^T \Gamma^{-1} \dot{\theta} \right) + 2\gamma^{-1} \tilde{\rho} \dot{\rho}$$

$$= \left(Ae + B\left[k_p \, \tilde{\theta}^T \xi - \tilde{\rho} \chi - \kappa \epsilon m_0^2 \right] \right)^T P e + e^T P \left(Ae + B\left[k_p \, \tilde{\theta}^T \xi - \tilde{\rho} \chi - \kappa \epsilon m_0^2 \right] \right) +$$
$$+ 2|k_p| \tilde{\theta}^T \Gamma^{-1} \dot{\theta} + 2\gamma^{-1} \tilde{\rho} \dot{\rho}$$

After rearranging the terms,

$$2\dot{V} = e^T(\underbrace{A^T P + PA}_{-2Q})e + 2e^T PB\Big[k_p\,\tilde{\theta}^T\xi - \tilde{\rho}\chi - \kappa\epsilon m_0^2\Big] + 2|k_p|\tilde{\theta}^T\Gamma^{-1}\dot{\theta} + 2\gamma^{-1}\tilde{\rho}\dot{\rho}$$

Using the relation $B^T P e = Ce = \epsilon$,

$$\dot{V} = -e^T Qe - \kappa\epsilon^2 m_0^2 + |k_p|\tilde{\theta}^T\big[\underbrace{\operatorname{sign}(k_p)\xi\epsilon + \Gamma^{-1}\dot{\theta}}\big] + \tilde{\rho}\big[\underbrace{-\chi\epsilon + \gamma^{-1}\dot{\rho}}\big]$$

We choose the update laws
$$\begin{cases} \boxed{\dot{\theta} = -\operatorname{sign}(k_p)\Gamma\xi\epsilon} \\ \boxed{\dot{\rho} = \gamma\chi\epsilon} \end{cases}$$

which gives

$$\boxed{\dot{V} = -e^T Qe - \kappa\epsilon^2 m_0^2 \leq 0}$$

Result:

- $V(t)$ is monotone non–increasing, bounded above by $V(0)$ and below by 0.
- $e,\ \theta,\ \rho \in \mathcal{L}_\infty$
- $e \in \mathcal{L}_\infty \quad \Rightarrow \quad \epsilon \in \mathcal{L}_\infty$ (since $\epsilon = Ce$)
- Integranting $\dot{V} \quad \Rightarrow \quad e\,, \epsilon m_0 \in \mathcal{L}_2$
- $e \in \mathcal{L}_2 \quad \Rightarrow \quad \epsilon \in \mathcal{L}_2$
- $\dot{\theta} = -\,\mathrm{sign}(k_p)\Gamma \frac{\xi}{\underbrace{m_0}_{\mathcal{L}_\infty}} \underbrace{\epsilon\, m_0}_{\mathcal{L}_2} \quad \Rightarrow \quad \dot{\theta} \in \mathcal{L}_2$
- $\dot{\rho} = \gamma \frac{\chi}{\underbrace{m_0}_{\mathcal{L}_\infty}} \underbrace{\epsilon\, m_0}_{\mathcal{L}_2} \quad \Rightarrow \quad \dot{\rho} \in \mathcal{L}_2$

★ We cannot conclude that $\dot{\theta},\ \dot{\rho} \in \mathcal{L}_\infty$.

Technical difficulty : How to show convergence of e_0.

Problem: $\boxed{\epsilon \in \mathcal{L}_\infty} \quad \not\Rightarrow \quad \boxed{e_0, \hat{e}_0 \in \mathcal{L}_\infty}$!!

Required steps for a demonstration:

- $y \in \mathcal{L}_\infty$ (hard part)
- $u \in \mathcal{L}_\infty$
- $\epsilon \to 0$
- $\chi \to 0$
- $e_0 \to 0$

Problems of this algorithm:

(1) It is not possible to assure that $\tilde{\rho} \to 0$ even under PE!

Note that if $\theta \equiv \theta^*$ then,

$$\chi = \zeta - \theta^T \xi = L^{-1} \theta^T \omega - \theta^T L^{-1} \omega$$
$$= L^{-1} \theta^{*T} \omega - \theta^{*T} L^{-1} \omega = 0$$

Therefore, $\theta \equiv \theta^* \quad \Rightarrow \quad \chi \equiv 0 \quad \Rightarrow \quad \boxed{\dot{\rho} = 0}$

(2) The control law $u(t)$ seems to play no role in the analysis !

It is selected as $u = \theta^T \omega$.

However, for any other selection, the analysis holds the same.

Ideally, the control should be $u = L\,\theta^T \xi$, but this cannot be implemented.

Summary

Subsystem	Equation	Dimension
Plant	$y = P(s)\,u$	n
Model	$y_m = M(s)\,r$	n
Tracking error	$e_0 = y - y_m$	
Control law	$u = \theta^T \omega$	
SV filters	$\dot{\omega}_1 = A_f \omega_1 + b_f u$	$n-1$
	$\dot{\omega}_2 = A_f \omega_2 + b_f y$	$n-1$
Regressor	$\omega^T = \begin{bmatrix} \omega_1^T & y & \omega_2^T & r \end{bmatrix}$	
ξ-Filter	$\xi = L^{-1}(s)[\omega]$	$2n(n^*-1)$
ζ-Filter	$\zeta = L^{-1}(s)[u]$	n^*-1

Subsystem	Equation	Dimension
Prediction	$\hat{e}_0 = M(s)L(s)\big[\rho\chi + \kappa\epsilon m_0^2\big]$	n
	$\chi = \zeta - \theta^T\xi$	
	$m_0^2 = \xi^T\xi + \chi^2$	
Prediction error	$\epsilon = e_0 - \hat{e}_0$	
Update law	$\dot{\theta} = -\operatorname{sign}(k_p)\Gamma\epsilon\xi$	$2n$
	$\dot{\rho} = \gamma\epsilon\chi$	1

System total dimension: $\boxed{N = (2n+1)(n^*-1) + 7n - 1}$

Warning: The signal of χ is swapped in [Tao:2003], (pag. 204) .

As a result, the sign of $\dot{\rho}$ is also swapped.

★ ξ may be obtained from much lower order filters.

4.4.2 Reduced order filters

The order of the filters in the above table are:

SV filters	$\dot{\omega}_1 = A_f \omega_1 + b_f u$	$n-1$
	$\dot{\omega}_2 = A_f \omega_2 + b_f y$	$n-1$
ξ-filter	$\xi = L^{-1}(s)[\omega]$	$2n(n^*-1)$
ζ-filter	$\zeta = L^{-1}(s)[u]$	n^*-1

Total order:

$$\boxed{N = 2(n-1) + (2n+1)(n^*-1)}$$

However, they can be implemented more efficiently.

Example 20 Plant with $n = n^* = 3$.

In this case

$$\begin{cases} \Lambda(s) = s^2 + \lambda_1 s + \lambda_0 \\ L(s) = s^2 + \ell_1 s + \ell_0 \end{cases}$$

Recall that

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} u_f \\ \dot{u}_f \\ y_f \\ \dot{y}_f \end{bmatrix} = \frac{1}{\Lambda(s)} \begin{bmatrix} u \\ s\,u \\ y \\ s\,y \end{bmatrix}, \qquad \omega_1 = \frac{1}{\Lambda(s)} \begin{bmatrix} u \\ s\,u \end{bmatrix}, \quad \omega_2 = \frac{1}{\Lambda(s)} \begin{bmatrix} y \\ s\,y \end{bmatrix}$$

$$\xi = \frac{1}{L(s)}\omega = \frac{1}{L(s)\Lambda(s)} \begin{bmatrix} u \\ s\,u \\ y \\ s\,y \end{bmatrix}, \qquad \xi_1 = \frac{1}{L(s)\Lambda(s)} \begin{bmatrix} u \\ s\,u \end{bmatrix}, \quad \xi_2 = \frac{1}{L(s)\Lambda(s)} \begin{bmatrix} y \\ s\,y \end{bmatrix}$$

Define a 4th order filter

$$\frac{1}{H(s)} = \frac{1}{L(s)\Lambda(s)} = \frac{1}{s^4 + h_3 s^3 + h_2 s^2 + h_1 s + h_0}$$

State-space realization: $$\begin{cases} \dot{\Omega}_1 = A\Omega_1 + Bu \\ \dot{\Omega}_2 = A\Omega_2 + By \end{cases}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -h_0 & -h_1 & -h_2 & -h_3 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Omega_1 = \frac{1}{L(s)\Lambda(s)} \begin{bmatrix} u \\ s\,u \\ s^2\,u \\ s^3\,u \end{bmatrix}, \qquad \Omega_2 = \frac{1}{L(s)\Lambda(s)} \begin{bmatrix} y \\ s\,y \\ s^2\,y \\ s^3\,y \end{bmatrix}$$

Then,

$$\omega_1 = \frac{L(s)}{H(s)} \begin{bmatrix} u \\ s\,u \end{bmatrix} = \begin{bmatrix} \ell_0 & \ell_1 & 1 & 0 \\ 0 & \ell_0 & \ell_1 & 1 \end{bmatrix} \Omega_1 \,, \qquad \omega_2 = \frac{L(s)}{H(s)} \begin{bmatrix} y \\ s\,y \end{bmatrix} = \begin{bmatrix} \ell_0 & \ell_1 & 1 & 0 \\ 0 & \ell_0 & \ell_1 & 1 \end{bmatrix} \Omega_2$$

$$\xi_1 = \frac{1}{H(s)} \begin{bmatrix} u \\ s\,u \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \Omega_1 \,, \qquad \xi_2 = \frac{1}{H(s)} \begin{bmatrix} y \\ s\,y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \Omega_2$$

$$\dot{\xi}_1 = \frac{1}{H(s)} \begin{bmatrix} s\,u \\ s^2\,u \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Omega_1 \,, \qquad \dot{\xi}_2 = \frac{1}{H(s)} \begin{bmatrix} s\,y \\ s^2\,y \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Omega_2$$

$$\ddot{\xi}_1 = \frac{1}{H(s)} \begin{bmatrix} s^2\,u \\ s^3\,u \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Omega_1 \,, \qquad \ddot{\xi}_2 = \frac{1}{H(s)} \begin{bmatrix} s^2\,y \\ s^3\,y \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Omega_2$$

The total order of the filters according to the table is

$$N = 2(n-1) + (2n+1)(n^*-1) = 4 + 14 = \boxed{18}$$

The efficient implementation requires only

Ω-filters	$\dot{\Omega}_1 = A\Omega_1 + Bu$	$(n-1) + (n^*-1)$
	$\dot{\Omega}_2 = A\Omega_2 + By$	$(n-1) + (n^*-1)$

Total order: $N = 2(n-1) + 2(n^*-1) = 4 + 4 = \boxed{8}$

4.4.3 Simulações

Example 21 3rd order plant.

Classificação do sistema: $n = 3$ (ordem)
$n^* = 3$ (grau relativo)
$n_p = 6$ (# de parâmetros)

Planta.....................: $P(s) = \dfrac{k_p}{s^2(s+a)}$

Modelo...................: $M(s) = \dfrac{k_m}{(s+1)(s+2)(s+3)}$

Filtro de estado.......: $\dfrac{1}{\Lambda(s)} = \dfrac{1}{(s+1)^2}$

Polinômio...............: $L(s) = (s+1.5)(s+2.5)$ $\left(\Rightarrow\ M(s)L(s) = \dfrac{k_m\,(s+1.5)(s+2.5)}{(s+1)(s+2)(s+3)}\right)$

Simulation #1 Condições iniciais nulas & $\theta(0) \approx \theta^*$.

Condições iniciais.....: $y_p(0) = 0$ $\quad$ $\theta(0) = 0.95\,\theta^*$

$y_m(0) = 0$ $\quad$ $\rho(0) = 0.95\,k_p$

$\hat{e}_0(0) = 0$

Parâmetros...............: $k_p = 1$ $\quad$ $\Gamma = 10\,I$

$a = 0.5$ $\quad$ $\gamma = 10$

$k_m = 6$ $\quad$ $\kappa = 1$

Sinal de referência....: $r = dc + 5\text{sqw}(\pi\, t/11) + 2\text{sqw}(\pi\, t/3) + 2\sin(t)$

Matching ....: $\theta^{*T} = \begin{bmatrix} -19.25 & -5.5 & -23.875 & 17.875 & 24.75 & 6 \end{bmatrix}$

$\|\theta^*\| = 44.033$

★ O transitório piora com $a \to 0$.

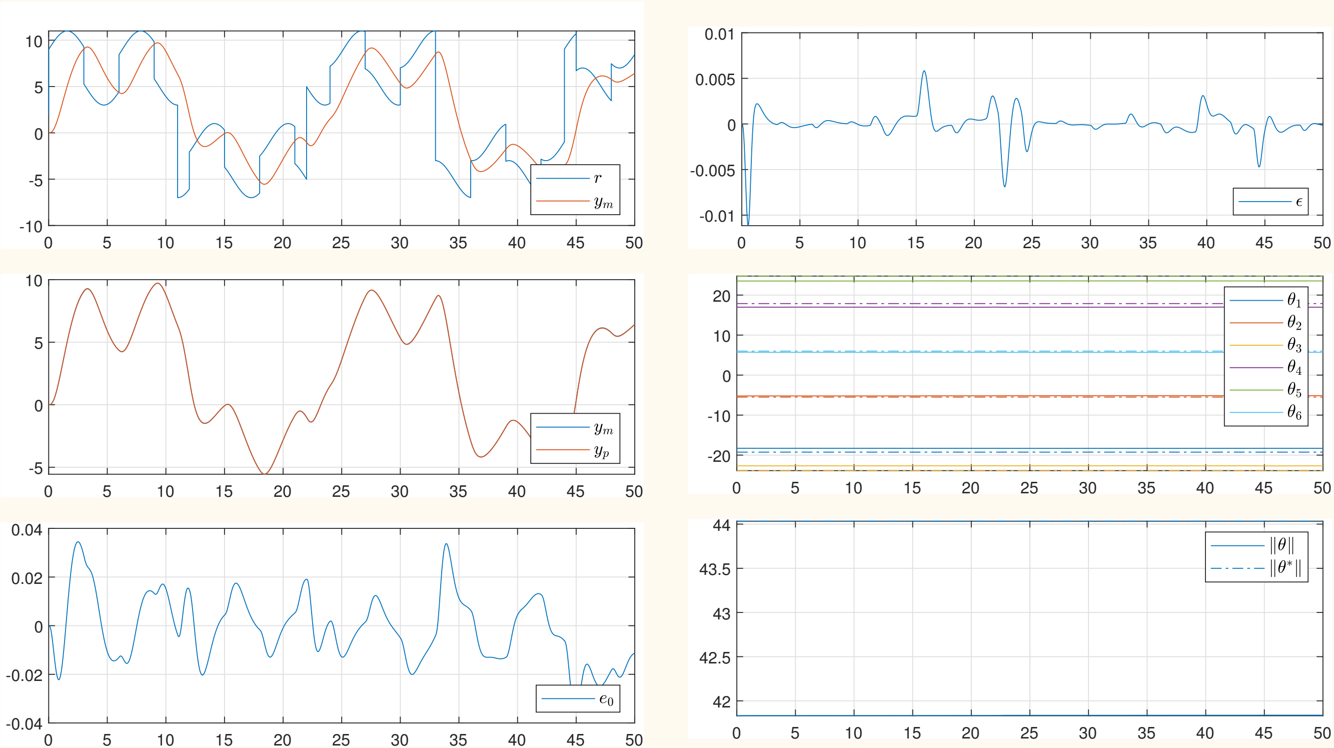


Figura 87: Resultado da simulação com algoritmo MRAC direto (caso $n^* = 3$). (`fig01.m`)

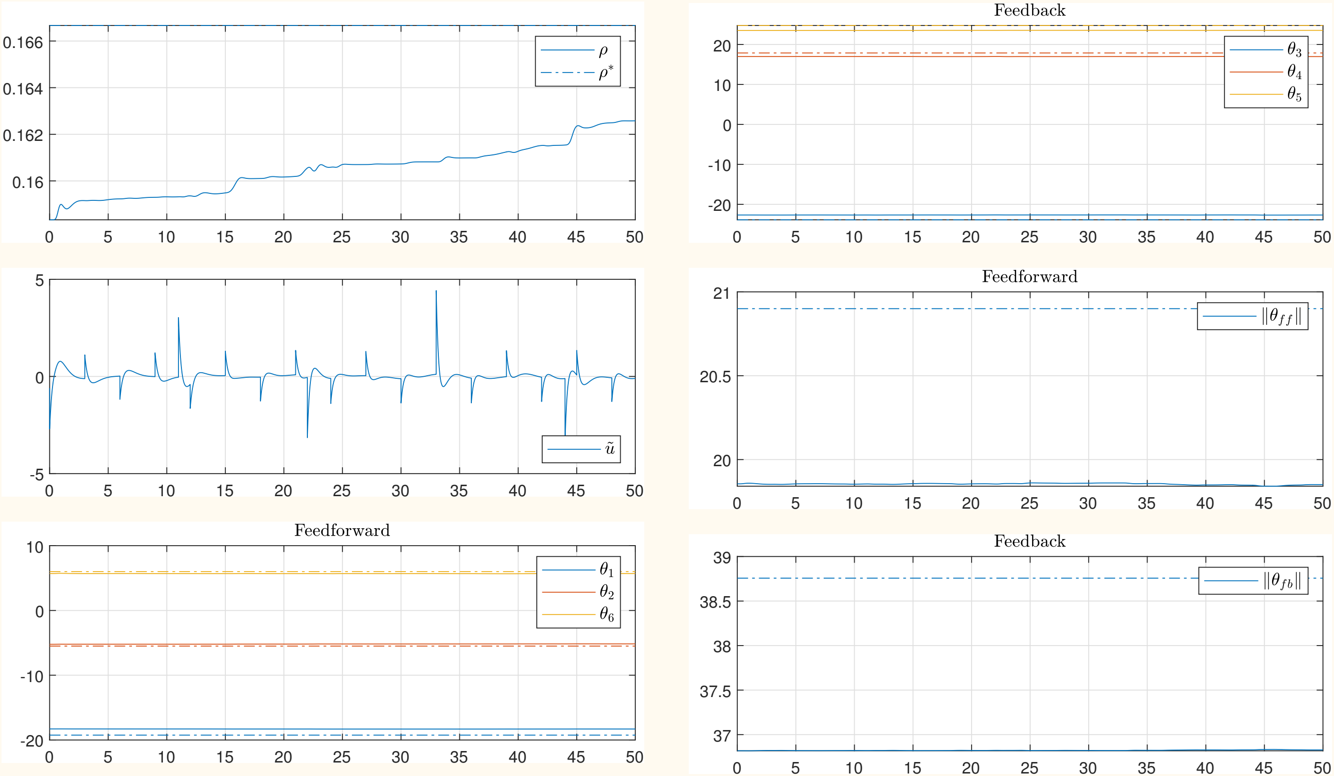


Figura 88: Resultado da simulação com algoritmo MRAC direto (caso $n^* = 3$). (`fig01.m`)

Simulação #2 Condições iniciais pequenas.

Condições iniciais.....: $y_p(0) = \boxed{2}$ $\theta(0) = \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}^T$

$y_m(0) = 0$ $\rho(0) = 0$

$\hat{e}_0(0) = 0$

Parâmetros..............: $k_p = 1$ $\Gamma = 10\, I$

$a = \boxed{0.5}$ $\gamma = 10$

$k_m = 6$ $\kappa = 1$

Sinal de referência....: $r = dc + 5\mathrm{sqw}(\pi\, t/11) + 2\mathrm{sqw}(\pi\, t/3) + 2\sin(t)$

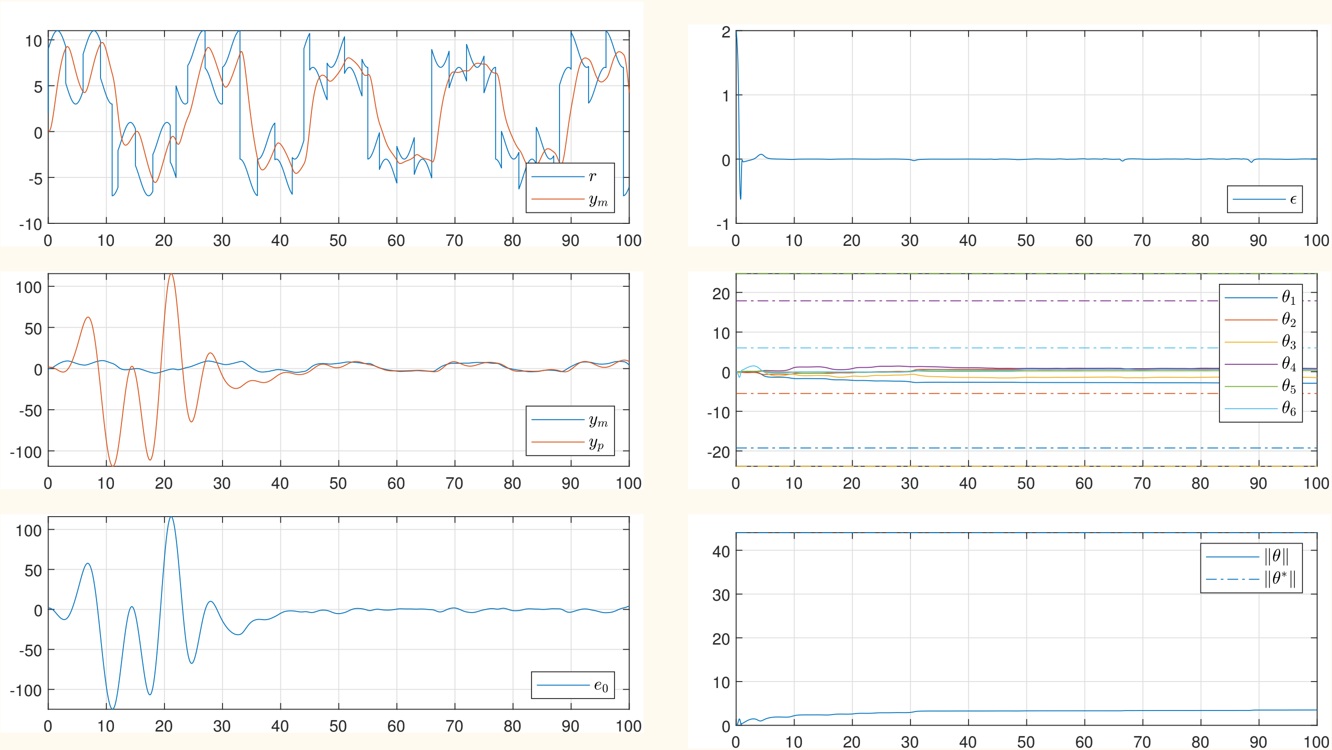


Figura 89: Resultado da simulação com algoritmo MRAC direto (caso $n^* = 3$). (`fig02.m`)

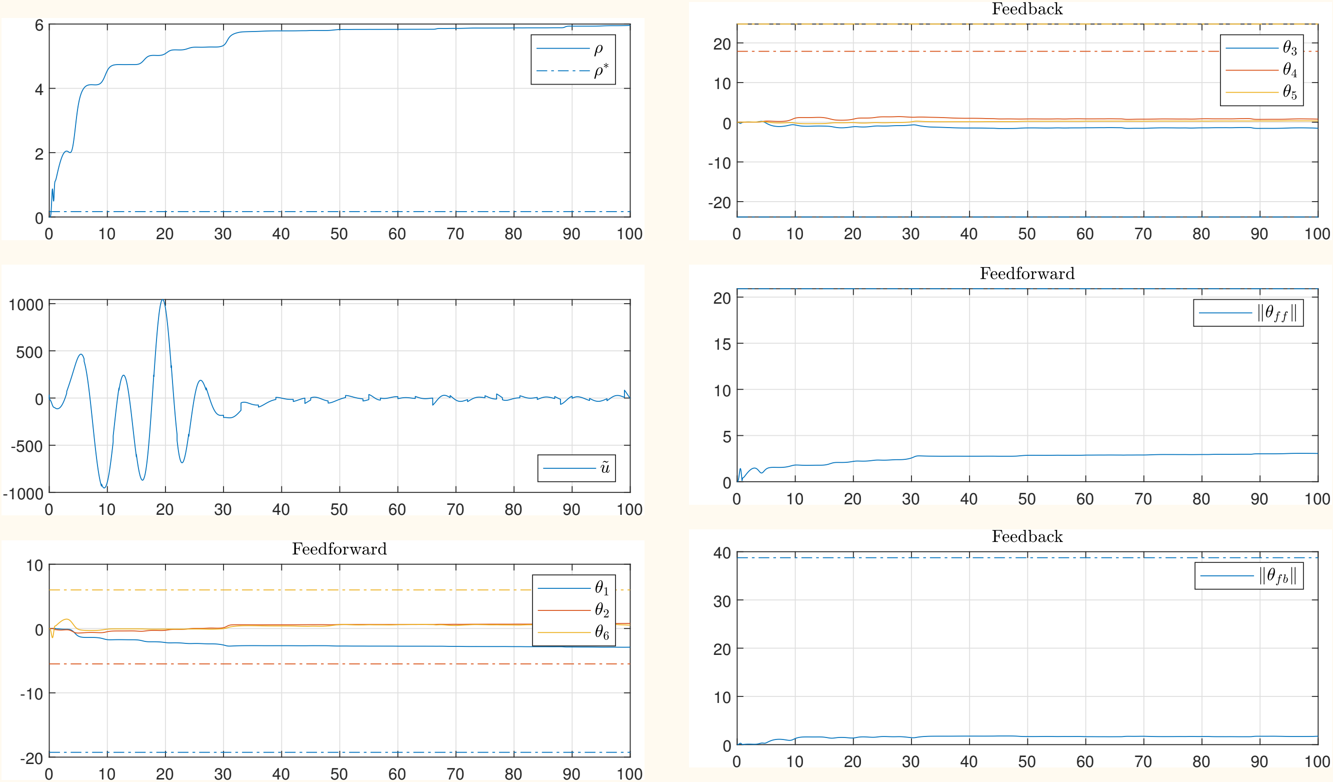


Figura 90: Resultado da simulação com algoritmo MRAC direto (caso $n^* = 3$). (fig02.m)

Simulação #3 Condições iniciais grandes.

Condições iniciais.....: $y_p(0) = \boxed{10}$ $\theta(0) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$

$y_m(0) = 0$ $\rho(0) = 0$

$\hat{e}_0(0) = 0$

Parâmetros..............: $k_p = 1$ $\Gamma = 10\, I$

$a = \boxed{0.5}$ $\gamma = 10$

$k_m = 6$ $\kappa = 1$

Sinal de referência....: $r = dc + 5\mathrm{sqw}(\pi\, t/11) + 2\mathrm{sqw}(\pi\, t/3) + 2\sin(t)$

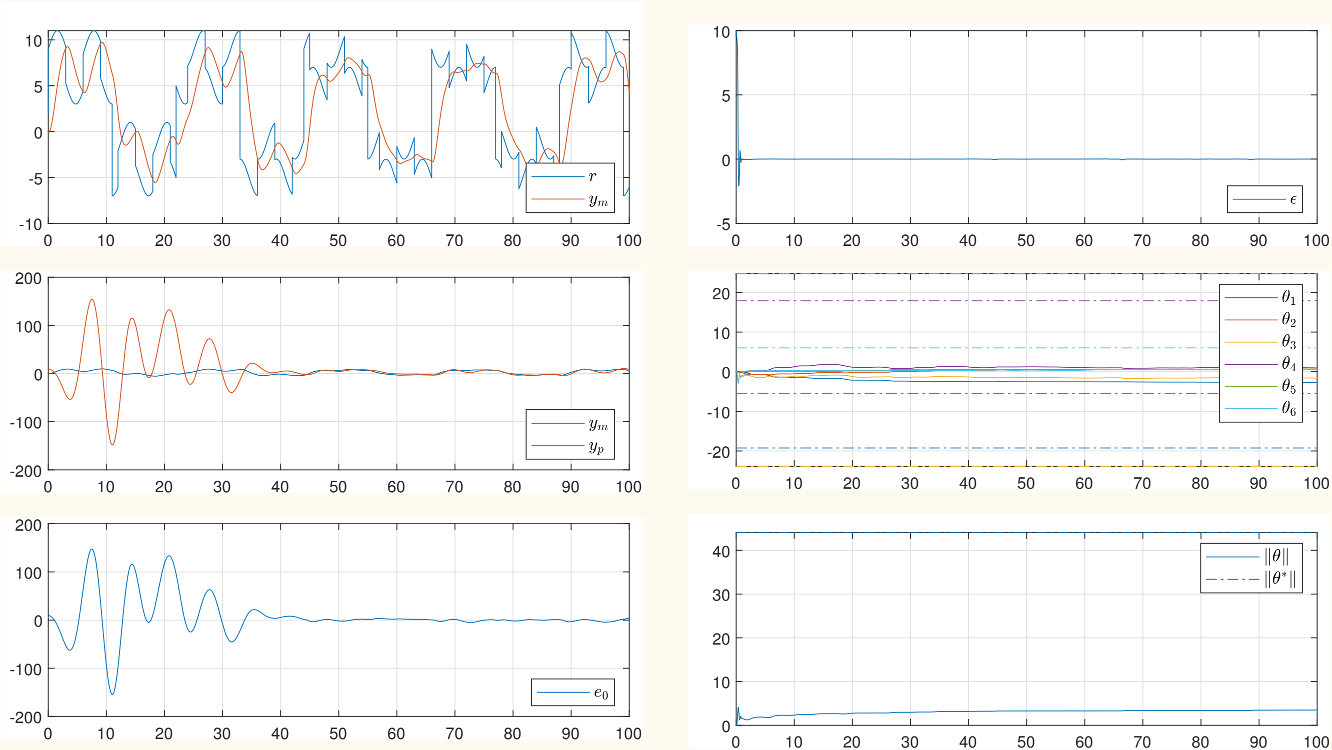


Figura 91: Resultado da simulação com algoritmo MRAC direto (caso $n^* = 3$). (`fig03.m`)

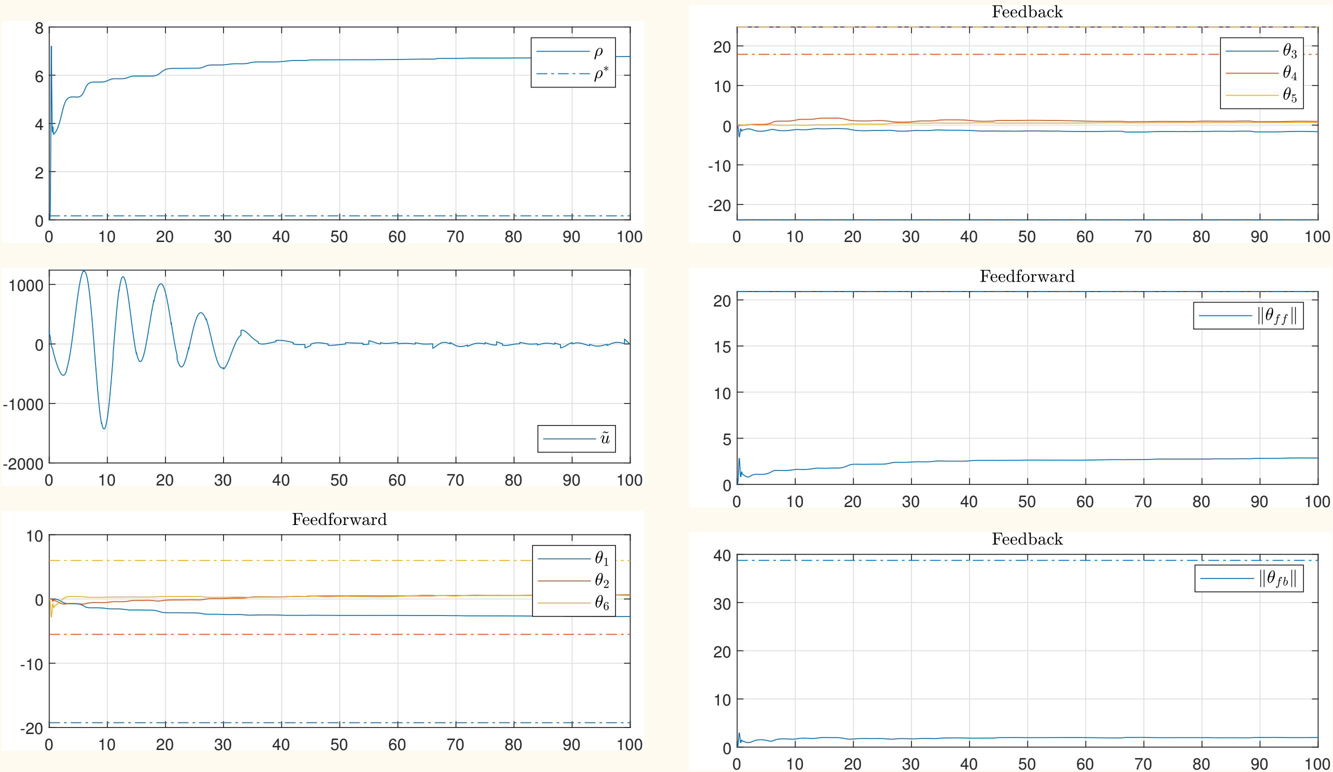


Figura 92: Resultado da simulação com algoritmo MRAC direto (caso $n^* = 3$). (`fig03.m`)

4.4.4 Método do Gradiente $(n^* > 1)$

Ref.: [Tao:2003], (pag. 211)

Planta: $$\boxed{y = P(s)\, u}$$

Modelo: $$\boxed{y_m = M(s)\, r}$$

Controle: $$\boxed{u = \theta^T \omega}$$

A equação do erro pode ser escrita como:

$$\begin{aligned} e_0 &= k_p\, M(s)\big[u - \theta^{*T}\omega\big] \\ &= k_p\Big[\underbrace{M u}_{\zeta} - \theta^{*T}\underbrace{M\omega}_{\xi}\Big] \end{aligned}$$

Filtros: $\boxed{\zeta = Mu}$ $\boxed{\xi = M\omega}$

Portanto: $\boxed{e_0 = k_p\left[\zeta - \theta^{*T}\xi\right]}$

★ Neste esquema k_p também será adaptado.

Utilizando estimas ρ e θ, calculamos a predição: $\hat{e}_0 = \rho\Big[\underbrace{\zeta - \theta^T\xi}_{-\chi}\Big]$

Definindo o sinal: $\boxed{\chi = -\zeta + \theta^T\xi}$

A predição é escrita como: $\boxed{\hat{e}_0 = -\rho\chi}$

Erro de predição:

$$
\begin{aligned}
\epsilon &= e_0 - \hat{e}_0 \\
&= k_p\Big[\zeta - \theta^{*T}\xi\Big] + \rho\chi \\
&= k_p\Big[\zeta - \big(\theta - \tilde{\theta}\big)^T\xi\Big] + \rho\chi \\
&= k_p\,\tilde{\theta}^T\xi + k_p\Big[\underbrace{\zeta - \theta^T\xi}_{-\chi}\Big] + \rho\chi \\
&= k_p\,\tilde{\theta}^T\xi + \tilde{\rho}\chi
\end{aligned}
$$

★ Note que não é uma equação dinâmica.

As leis de adaptação são dadas por:

$$\boxed{\dot{\theta} = -\frac{\mathrm{sign}(k_p)\Gamma\xi\epsilon}{m^2}}$$

$$\boxed{\dot{\rho} = -\frac{\gamma\chi\epsilon}{m^2}}$$

Onde: $\boxed{m^2 = 1 + \xi^T\xi + \chi^2}$ (Sinal normalizante.)

Análise de estabilidade

Considere a função de Lyapunov (parcial):

$$2V(\tilde{\theta}, \tilde{\rho}) = |k_p|\tilde{\theta}^T \Gamma^{-1}\tilde{\theta} + \gamma^{-1}\tilde{\rho}^2$$

Derivando,

$$\dot{V} = -\frac{\epsilon^2}{m^2} \leq 0$$

Teorema. O algoritmo do Gradiente assegura que:

- $\theta, \rho \in \mathcal{L}_\infty$
- $\dfrac{\epsilon}{m} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$
- $\dot{\theta} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$
- $\dot{\rho} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$

★ Lembrar que $\mathcal{L}_2$ significa “quase convergência”.

Dificuldade técnica: Mostrar convergência de e.

Problema: $\boxed{\epsilon \in \mathcal{L}_\infty} \not\Rightarrow \boxed{e, \hat{e} \in \mathcal{L}_\infty}$!!

Passos necessários para a demonstração:

- $e \in \mathcal{L}_2 \cap \mathcal{L}_\infty$
- $\xi \in \mathcal{L}_\infty$ (parte difícil)
- $\dot{e} \in \mathcal{L}_\infty$
- Usando Barbalat, $\boxed{e \to 0}$

Problems of this algorithm:

(1) It is not possible to assure that $\tilde{\rho} \to 0$ even under PE!

(2) The control law $u(t)$ seems to play no role in the analysis !

Resumo do algoritmo

Subsistema	Equação	Ordem
Planta	$y = P(s)\,u$	n
Modelo	$y_m = M(s)\,r$	n
Erro	$e = y - y_m$	
Controle	$u = \theta^T\omega$	
Λ-Filters	$\dot{\omega}_1 = A_f\omega_1 + b_f u$	$n-1$
	$\dot{\omega}_2 = A_f\omega_2 + b_f y$	$n-1$
Regressor	$\omega^T = \begin{bmatrix}\omega_1^T & y & \omega_2^T & r\end{bmatrix}$	
ξ-Filter	$\xi = M(s)[\omega]$	$2n^2$
ζ-Filter	$\zeta = M(s)[u]$	n

Subsistema	Equação	Ordem
Predição	$\hat{e} = \rho\chi$ $\chi = \theta^T\xi - \zeta$	
Erro de predição	$\epsilon = e_0 - \hat{e}_0$	
Adaptação	$\dot{\theta} = -\dfrac{\text{sign}(k_p)\Gamma\epsilon\xi}{m^2}$ $\dot{\rho} = -\dfrac{\gamma\epsilon\chi}{m^2}$ $m^2 = 1 + \xi^T\xi + \chi^2$	$2n$ 1

Ordem total do sistema: $\boxed{N = 2n^2 + 7n - 1}$

4.4.5 Simulações

(...)

5 Least-squares MRAC

Contents

5.1 Introduction

Reference:

1. Ramon R. Costa
Lyapunov design of least-squares model-reference adaptive control.
IFAC 2020, Berlin, Germany, July 11-17, **2020**.

5.1.1 Motivation

This chapter is about the design of a model-reference adaptive control algorithm (MRAC) where a least-squares update law replaces the usual gradient law.

- The idea is not new.
- It was introduced by [Goodwin & Mayne:1987]

.

★ The least-squares algorithm has a vastly superior convergence rate.

- Designs of MRAC with least-squares.

For example:

★ [Sastry & Bodson:1989]

★ [Ioannou & Sun:1996]

The algorithms in these references do not require the state of the plant.

- There are many other results that do require the state of the plant.

What is new here?

- Here the update law is driven by the tracking error.

  Usually it is driven by some prediction error.

- The stability analysis is simple and elegant.

  It is very similar to the analysis of the conventional MRAC.

- There is no need to introduce normalization or projection.

- Remarkable improvement of the tracking error and parameter convergence.

The proposed solution is simple:

(1) Apply the Monopoli's multiplier
This reduces the relative degree of $M(s)L(s)$ to zero.

(2) Decompose $M(s)L(s)$ as a sum of an SPR transfer function and a direct term.
This direct term is the *trick* to assure an $\mathcal{L}_2$ property for the parameter error.

(3) Introduce a least-squares update law.
Possible thanks to the $\mathcal{L}_2$ property.

Assumptions

The prior available information regarding the plant $P(s)$ is:

(1) the order n is known

(2) relative degree $n^* = 1$

(3) $P(s)$ is minimum phase

(4) the sign of the high frequency gain k_p is known

For the proposed least-squares MRAC,

(5) a lower bound for k_p is known

5.2 Review of conventional MRAC

Control law : $\boxed{u = \theta^T \omega}$

Update law : $\boxed{\dot{\theta} = -\Gamma \omega e_a}$

Parameter convergence

- If $\tilde{u} = \tilde{\theta}^T \omega \approx 0$ then $e_0 \approx 0$.
- This is achieved when
  - $\tilde{\theta} \approx 0$ (good identification) OR
  - $\tilde{\theta} \approx \perp \omega$ (close to orthogonal).
- $\tilde{\theta}(t) \to 0$ requires *persistent excitation* condition.

5.3 Modified MRAC design (M-MRAC)

1st step : Apply the Monopoli's multiplier.

The control law is now : $\boxed{u = L(s)\theta^T \xi} \rightarrow \boxed{u = \theta^T \omega + \dot{\theta}^T \xi}$

Polynomial : $\boxed{L(s) = s + \ell_0}$, $\ell_0 > 0$

Filtered signal: $\boxed{\xi = L^{-1}(s)\omega}$

The error equation becomes

$$e_a = |k_p| M(s) \big[L(s) \theta^T \xi - \theta^{*T} \omega \big] \; = \; |k_p| \underbrace{M(s) L(s)}_{n^*=0} \big[\tilde{\theta}^T \xi \big]$$

★ $M(s)L(s)$ can be decomposed as

$$M(s)L(s) = \frac{s+\ell_0}{s+a_m} = \underbrace{\frac{\ell_0 - a_m}{s+a_m}}_{SPR \text{ if } \ell_0 > a_m} + 1 = \alpha M(s) + 1$$

Result,

$$e_a = \underbrace{\alpha |k_p| M(s)}_{\text{SPR}} \big[\tilde{\theta}^T \xi \big] + |k_p| \big[\tilde{\theta}^T \xi \big] \, ,$$

State space realization : $\begin{cases} \dot{e} = A_m e + B_m [\tilde{\theta}^T \xi] \\ e_a = C_m e + |k_p| [\tilde{\theta}^T \xi] \end{cases}$ $\quad (e \in \mathbb{R}^{5n-2})$

Lyapunov function: $2V = e^T e + \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$

Deriving,

$$\dot{V} = -e^T Q e + \underbrace{e^T P B_m}_{e_a - |k_p|[\tilde{\theta}^T \xi]} \left[\tilde{\theta}^T \xi\right] + \tilde{\theta}^T \Gamma^{-1} \dot{\theta}$$

After rearranging,

$$\dot{V} = -e^T Q e - |k_p| \left(\tilde{\theta}^T \xi\right)^2 + \tilde{\theta}^T \left[\xi e_a + \Gamma^{-1} \dot{\theta}\right]$$

Update law : $\boxed{\dot{\theta} = -\Gamma \xi e_a}$

Result : $\boxed{\dot{V} = -e^T Q e - |k_p| \left(\tilde{\theta}^T \xi\right)^2 \leq 0}$

Alternative analisys: Including $|k_p|$ and recalling that $B_m = B_m' k_p$.

Lyapunov function: $2V = e^T e + |k_p|\tilde{\theta}^T \Gamma^{-1}\tilde{\theta}$

Deriving,

$$\dot{V} = -e^T Q' e + \underbrace{e^T P' B_m'}_{e_a - |k_p|[\tilde{\theta}^T \xi]} |k_p|\left[\tilde{\theta}^T \xi\right] + |k_p|\tilde{\theta}^T \Gamma^{-1}\dot{\theta}$$

After rearranging,

$$\dot{V} = -e^T Q' e - |k_p|^2 \left(\tilde{\theta}^T \xi\right)^2 + |k_p|\tilde{\theta}^T \left[\xi e_a + \Gamma^{-1}\dot{\theta}\right]$$

Update law : $\boxed{\dot{\theta} = -\Gamma\, \xi e_a}$

Result : $\boxed{\dot{V} = -e^T Q' e - |k_p|^2 \left(\tilde{\theta}^T \xi\right)^2 \leq 0}$

5.3.1 Stability analysis

We can conclude that

- $e \in \mathcal{L}_\infty \quad , \quad \tilde{\theta} \in \mathcal{L}_\infty$
- $e \in \mathcal{L}_2 \quad , \quad \tilde{\theta}^T \xi \in \mathcal{L}_2$
- $\omega,\, \xi,\, \dot{\xi} \in \mathcal{L}_\infty \quad \Rightarrow \quad \dot{e},\, \dot{\theta} \in \mathcal{L}_\infty$
- $u \in \mathcal{L}_\infty \quad \Rightarrow$ Global uniform stability

Moreover,

- $e \in \mathcal{L}_2$ and $\dot{e} \in \mathcal{L}_\infty \quad \Rightarrow \quad \boxed{\lim_{t\to\infty} e(t) = 0}$
- $\tilde{\theta}^T \xi \in \mathcal{L}_2$ and $\dfrac{d}{dt}\tilde{\theta}^T \xi \in \mathcal{L}_\infty \quad \Rightarrow \quad \boxed{\lim_{t\to\infty} \tilde{\theta}^T \xi = 0}$

From the control law,

$$\boxed{u = \theta^T \omega + \dot{\theta}^T \xi}$$

- There is an extra degree of freedom for the control action.
- Now $\tilde{u} \approx 0$ and $e_0 \approx 0$ even with large $\tilde{\theta}^T \omega$.
- That is, $\tilde{u} = \tilde{\theta}^T \omega + \dot{\theta}^T \xi \approx 0$ when $\boxed{\tilde{\theta}^T \omega \approx -\dot{\theta}^T \xi}$
- $\tilde{\theta} \to 0$ requires persistent excitation.

Rate of convergence

The error equation

$$e_a = M(s)L(s)|k_p|\big[\tilde{\theta}^T\xi\big] = \frac{s+\ell_0}{s+a_m}|k_p|\big[\tilde{\theta}^T\xi\big]$$

can be rewritten as

$$(s+a_m)e_a = |k_p|(s+\ell_0)\big[\tilde{\theta}^T\xi\big]$$

$$\begin{aligned}
\Rightarrow \qquad \dot{e}_0 &= -a_m e_0 + \frac{d}{dt}\big(k_p\tilde{\theta}^T\xi\big) + \ell_0 k_p\tilde{\theta}^T\xi \\
&= -a_m e_0 + k_p\dot{\theta}^T\xi + k_p\tilde{\theta}^T\dot{\xi} + \ell_0 k_p\tilde{\theta}^T\xi \\
&= -a_m e_0 - k_p\big[\operatorname{sign}(k_p)\Gamma\xi e_0\big]^T\xi + k_p\tilde{\theta}^T\big(\dot{\xi} + \ell_0\xi\big) \\
&= -\big(a_m + |k_p|\xi^T\Gamma\xi\big)e_0 + k_p\tilde{\theta}^T\omega
\end{aligned}$$

★ The rate of convergence of $e_0 \to 0$ now depends explicitly on Γ.

5.3.2 Summary of the M-MRAC

Subsystem	Equation	Order
Plant	$y = P(s)\, u$	n
Model	$y_m = M(s)\, r$	n
Track. error	$e_a = \operatorname{sign}(k_p)(y - y_m)$	
Λ-Filters	$\dot{\omega}_1 = A_f \omega_1 + b_f u$ $\dot{\omega}_2 = A_f \omega_2 + b_f y$	$n-1$ $n-1$
Regressor	$\omega^T = \begin{bmatrix} \omega_1^T & y & \omega_2^T & r \end{bmatrix}$	
ξ-Filter	$\xi = L^{-1}(s)[\omega]\,, \quad L(s) = s + \ell_0$	$2n$
Control law	$u = \theta^T \omega + \dot{\theta}^T \xi$	
Update law	$\dot{\theta} = -\Gamma\, \xi e_a$	$2n$

System total dimension: $\boxed{N = 8n - 2}$

5.4 Least-squares MRAC design (LS-MRAC)

3rd step : Introduce least-squares update law.

Lyapunov function :

$$\boxed{2V(e,\tilde{\theta}) = \gamma e^T P e + \tilde{\theta}^T R^{-1}(t)\tilde{\theta}} \qquad R(0) = R^T(0) > 0$$

The derivative is

$$\dot{V} = -\gamma e^T Q e - \gamma |k_p| \left(\tilde{\theta}^T \xi\right)^2 + \gamma \tilde{\theta}^T \xi e_a + \tilde{\theta}^T R^{-1}\dot{\theta} + \frac{1}{2}\tilde{\theta}^T \dot{R}^{-1}\tilde{\theta}$$

Recall that

$$\frac{d}{dt}\left[RR^{-1}\right] = \dot{R}R^{-1} + R\dot{R}^{-1} = 0 \qquad \Rightarrow \qquad \boxed{\dot{R}^{-1} = -R^{-1}\dot{R}R^{-1}}$$

Thus,

$$\dot V = -\gamma e^T Q e - \gamma |k_p| \left(\tilde\theta^T \xi\right)^2 + \tilde\theta^T \left[\gamma\, \xi e_a + R^{-1}\dot\theta\right] - \frac{1}{2}\tilde\theta^T \underbrace{R^{-1}\dot R R^{-1}}_{-\xi\xi^T}\tilde\theta$$

Update laws

$$\boxed{\dot\theta = -\gamma\, R\xi e_a}$$

$$\boxed{R^{-1}\dot R R^{-1} = -\xi\xi^T} \quad \Rightarrow \quad \boxed{\dot R = -R\xi\xi^T R}$$

Result :

$$\begin{aligned}\dot V &= -\gamma e^T Q e - \gamma |k_p| \left(\tilde\theta^T \xi\right)^2 + \frac{1}{2}\left(\tilde\theta^T \xi\right)^2 \\ &= -\gamma e^T Q e - \Big(\underbrace{\gamma |k_p| - \frac{1}{2}}_{>0 \text{ if } \gamma > \frac{1}{2|k_p|}}\Big)\left(\tilde\theta^T \xi\right)^2 \ \leq 0\end{aligned}$$

Alternative analisys: Including $|k_p|$ and recalling that $B_m = B'_m k_p$.

Lyapunov function: $2V(e,\tilde{\theta}) = \gamma e^T P' e + |k_p| \tilde{\theta}^T R^{-1}(t) \tilde{\theta}$

The derivative is

$$\begin{aligned}\dot{V} &= -\gamma e^T Q' e - \gamma |k_p|^2 \left(\tilde{\theta}^T \xi\right)^2 + \gamma |k_p| \tilde{\theta}^T \xi e_a + |k_p| \tilde{\theta}^T R^{-1} \dot{\theta} + \frac{|k_p|}{2} \tilde{\theta}^T \dot{R}^{-1} \tilde{\theta} \\ &= -\gamma e^T Q' e - \gamma |k_p|^2 \left(\tilde{\theta}^T \xi\right)^2 + |k_p| \tilde{\theta}^T \left[\gamma \xi e_a + R^{-1} \dot{\theta}\right] - \frac{|k_p|}{2} \tilde{\theta}^T \underbrace{R^{-1} \dot{R} R^{-1}}_{-\xi\xi^T} \tilde{\theta}\end{aligned}$$

Update laws: $\boxed{\dot{\theta} = -\gamma R \xi e_a}$ $\boxed{\dot{R} = -R\xi\xi^T R}$

Result :

$$\begin{aligned}\dot{V} &= -\gamma e^T Q e - \gamma |k_p|^2 \left(\tilde{\theta}^T \xi\right)^2 + \frac{|k_p|}{2} \left(\tilde{\theta}^T \xi\right)^2 \\ &= -\gamma e^T Q e - |k_p|^2 \Big(\underbrace{\gamma - \frac{1}{2|k_p|}}_{>0 \text{ if } \gamma > \frac{1}{2|k_p|}} \Big) \left(\tilde{\theta}^T \xi\right)^2 \ \leq 0\end{aligned}$$

5.4.1 Stability analysis

★ Stability condition: $\boxed{\gamma > \frac{1}{2|k_p|}} \quad \Rightarrow \quad \dot{V}(e, \tilde{\theta}) \leq 0$

Therefore,

- $e \in \mathcal{L}_\infty \quad , \quad \tilde{\theta}^T R^{-1} \tilde{\theta} \in \mathcal{L}_\infty$
- $e \in \mathcal{L}_2 \quad , \quad \tilde{\theta}^T \xi \in \mathcal{L}_2$
- $\omega,\ \xi,\ \dot{\xi} \in \mathcal{L}_\infty$

Boundedness of R and θ are obtained as follows:

Since $R(0) = R^T(0) > 0$, then

$$\dot{R}^{-1}(t) = \xi\xi^T$$

By integrating,

$$R^{-1}(t) = R^{-1}(0) + J(t) > 0\,, \quad t \geq 0$$

where

$$J(t) = \int_0^t \xi(\tau)\xi^T(\tau)d\tau$$

Therefore, $R^{-1}(t) > R^{-1}(0)$, and so $R(t) > 0$, $\forall t \geq 0$, and $R \in \mathcal{L}_\infty$.

This means that $\dot{\theta}, \dot{R} \in \mathcal{L}_\infty$.

From the Lyapunov function,

$$2V = \underbrace{\gamma e^T P e}_{\in \mathcal{L}_\infty} + \underbrace{\tilde{\theta}^T R^{-1}(0)\tilde{\theta}}_{\in \mathcal{L}_\infty} + \underbrace{\tilde{\theta}^T J(t)\tilde{\theta}}_{\in \mathcal{L}_\infty}$$

- $V \in \mathcal{L}_\infty \quad \Rightarrow \quad \tilde{\theta}^T R^{-1}(0)\tilde{\theta} \in \mathcal{L}_\infty \quad \Rightarrow \quad \tilde{\theta} \in \mathcal{L}_\infty$

Convergence

- $\dot{e} \in \mathcal{L}_\infty \quad \Rightarrow \quad \lim_{t\to\infty} e(t) = 0$
- $\dfrac{d}{dt}\tilde{\theta}^T \xi \in \mathcal{L}_\infty \quad \Rightarrow \quad \lim_{t\to\infty} \tilde{\theta}^T \xi = 0$
- Therefore, all signals are bounded $\quad \Rightarrow \quad$ Global uniform stability

5.4.2 Summary of the LS-MRAC

Subsystem	Equation	Order
Plant	$y = P(s)\,u$	n
Model	$y_m = M(s)\,r$	n
Track. error	$e_a = \mathrm{sign}(k_p)(y - y_m)$	
SV Filters	$\dot{\omega}_1 = A_f\omega_1 + b_f u$ $\dot{\omega}_2 = A_f\omega_2 + b_f y$	$n-1$ $n-1$
Regressor	$\omega^T = \begin{bmatrix}\omega_1^T & y & \omega_2^T & r\end{bmatrix}$	
ξ-Filter	$\xi = L^{-1}(s)[\omega]$	$2n$
Control law	$u = \theta^T\omega + \dot{\theta}^T\xi$	
Update law	$\dot{\theta} = -\gamma\, R\xi e_a\,, \quad \gamma > \dfrac{1}{2\lvert k_p\rvert}$ $\dot{R} = -R\xi\xi^T R\,, \quad R(0) = R^T(0) > 0$	$2n$ $4n^2$

System total dimension: $\boxed{N = 8n + 4n^2 - 2}$

5.5 Simulation results

Example 22 4th order plant.

System class: $n = 4$ (order)
$n^* = 1$ (relative degree)
$n_p = 8$ (# of parameters)

Plant ....: $$P(s) = \frac{(s+2)^3}{3s^4}$$

Model ....: $$M(s) = \frac{1}{s+1}$$

SV filter ....: $$\frac{1}{\Lambda(s)} = \frac{1}{(s+0.5)(s+1)(s+1.5)}$$

Matching ....: $$\theta^{*T} = [-7.25 \quad -9.25 \quad -3 \quad -12 \quad 6.75 \quad 22.5 \quad 18.75 \quad 3]$$
$$\|\theta^*\| = 34.6915$$

Simulation #1 Result using the conventional MRAC.

Initial condition........: $y(0) = 10$

$y_m(0) = 0$

$\theta(0) = 0$

Parameters..............: $\Gamma = 20\,I$

Reference signal.......: $r_{sin}(t) = 3 + \sin(t) + \sin(3t) + \sin(5t) + \sin(7t)$

★ ...

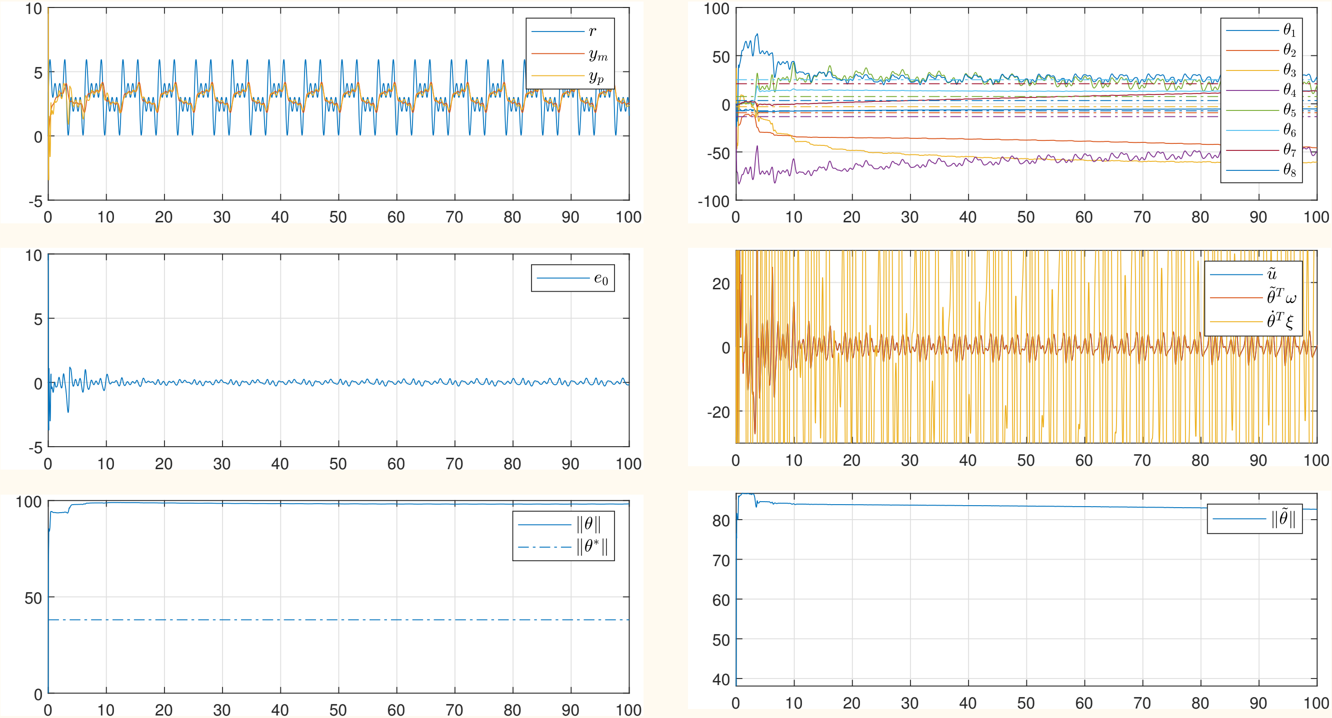


Figura 93: Conversional MRAC. $r = r_{sin}(t)$.

Simulation #2 Result using the M-MRAC.

Initial condition........: $y(0) = \boxed{10}$

$y_m(0) = 0$

$\theta(0) = \boxed{0}$

Filter ....: $L(s) = s + 2$

Parameters..............: $\Gamma = 20\, I$

Reference signal.......: $r_{sin}(t) = 3 + \sin(t) + \sin(3t) + \sin(5t) + \sin(7t)$

★ Note the fast convergence of the $\tilde{u}$ signal.

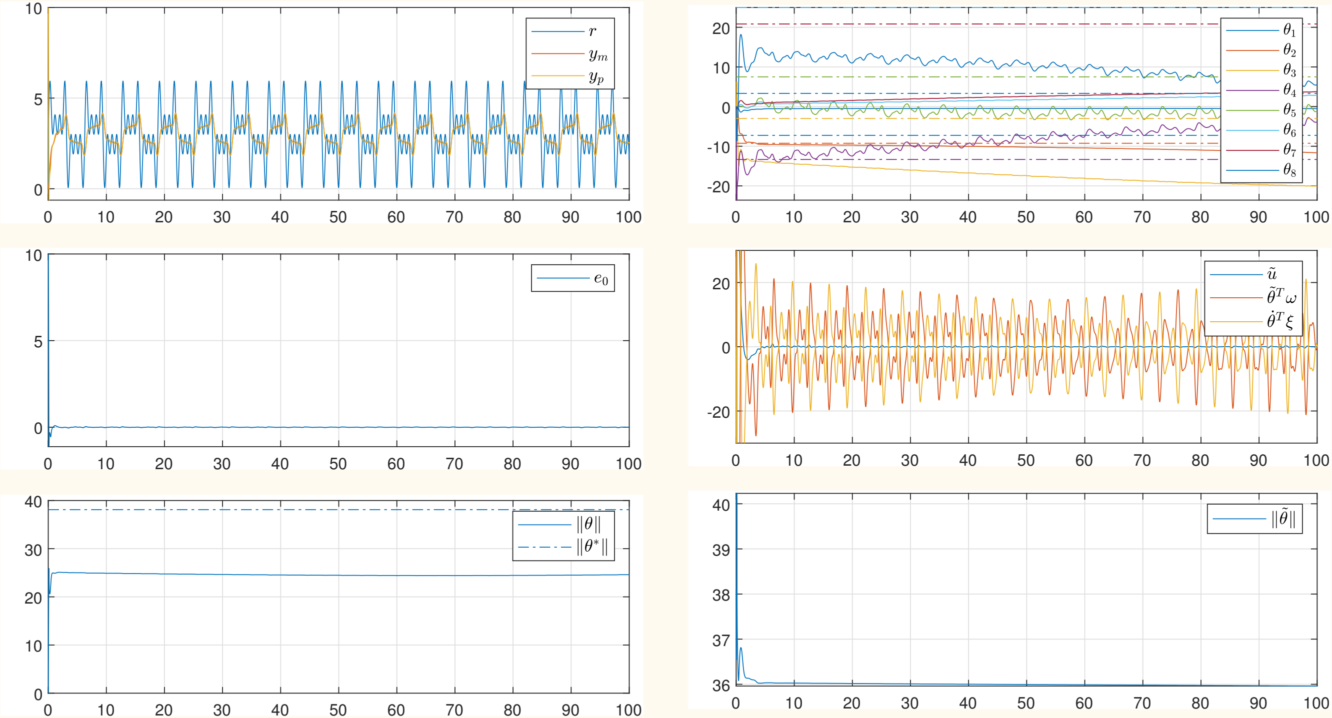


Figura 94: M-MRAC. $r = r_{sin}(t)$.

Simulation #3 Result using the LS-MRAC and $r = r_{\sin}(t)$.

Initial condition........: $y(0) = \boxed{10}$

$y_m(0) = 0$

$\theta(0) = \boxed{0}$

Filter ....: $L(s) = s + 2$

Parameters..............: $\gamma = 20$

$R(0) = 20\,I$

Reference signal.......: $r_{sin}(t) = 3 + \sin(t) + \sin(3t) + \sin(5t) + \sin(7t)$

★ Note the fast convergence of the $\tilde{\theta}^T \omega$.

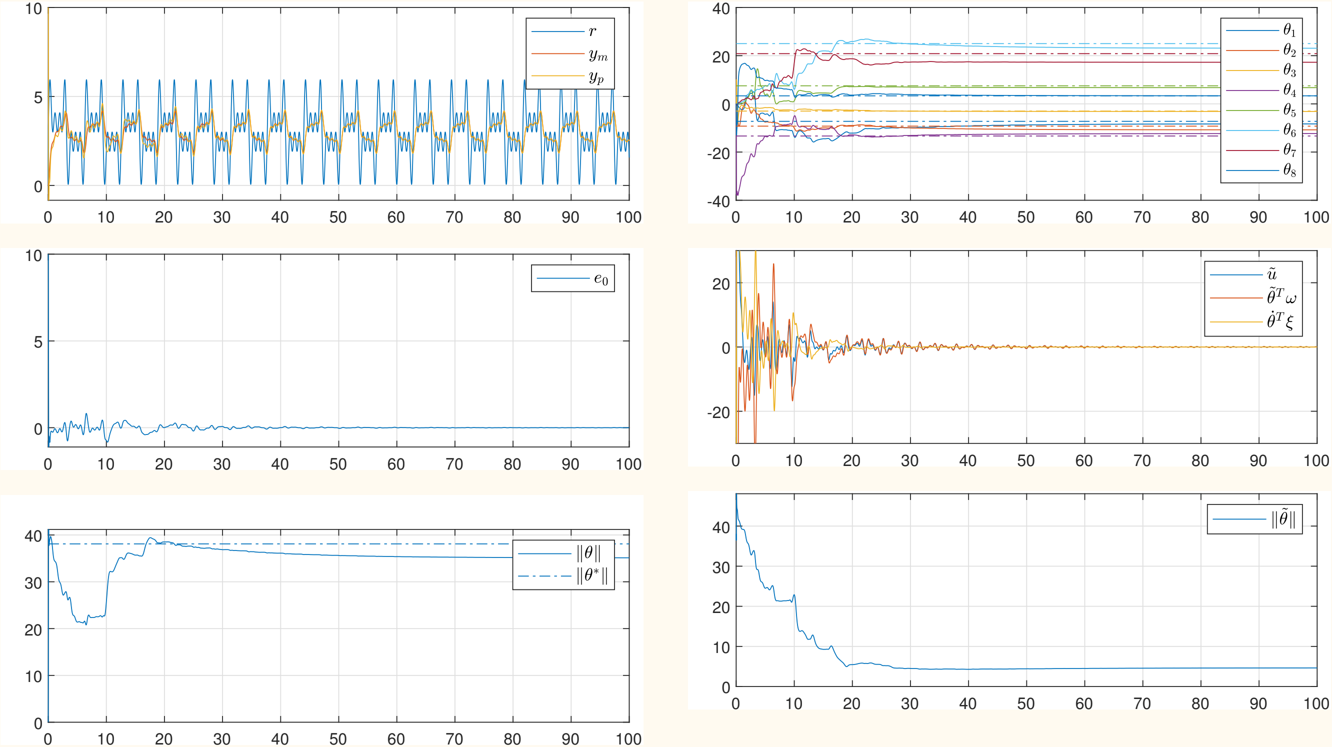


Figura 95: LS-MRAC. $r = r_{sin}(t)$.

Simulation #4 Result using the LS-MRAC and $r = r_{sqw}(t)$.

Initial condition........: $y(0) = 10$
$y_m(0) = 0$
$\theta(0) = 0$

Filter ....: $L(s) = s + 2$

Parameters..............: $\gamma = 20$
$R(0) = 50\, I$

Reference signal.......: $r_{sqw}(t) = 3 + 10\,\mathrm{sign}(\sin(0.1\pi t))$

- ★ In this case, $\tilde{\theta} \to 0$.
- ★ Note that the high adaptation gain assures a high sensitivity.

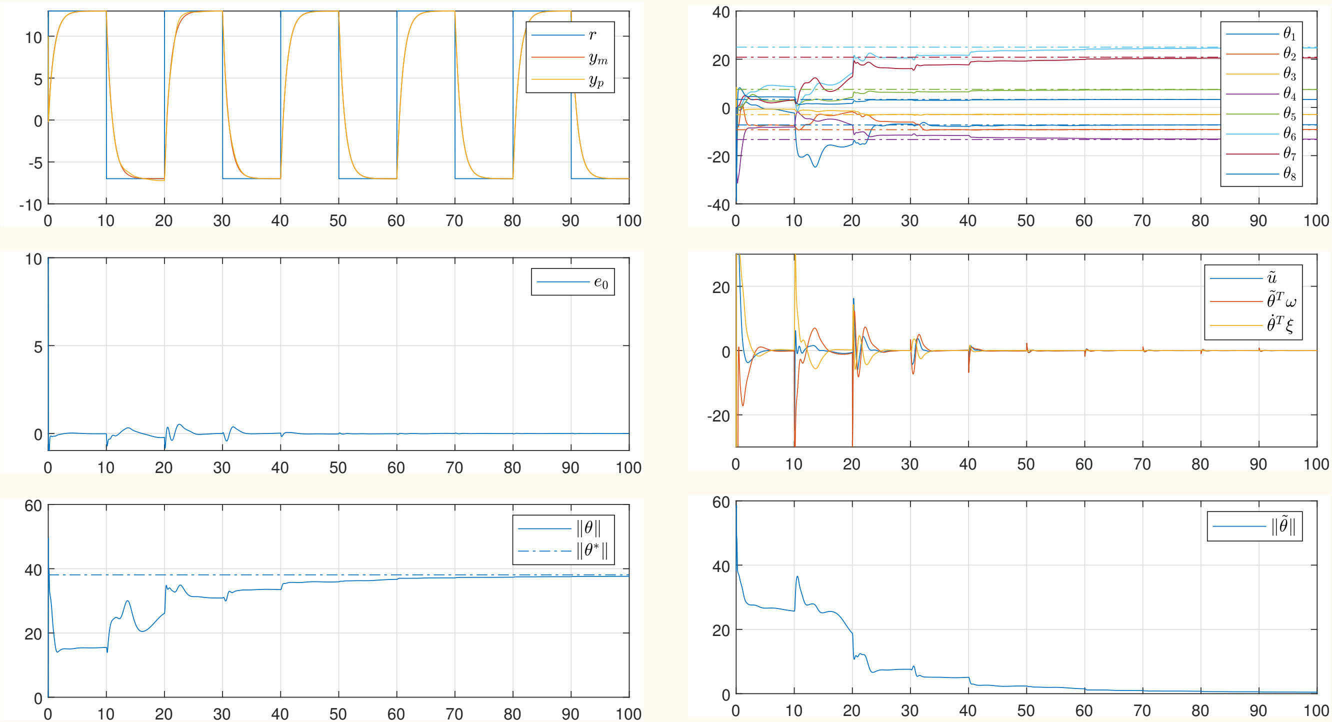


Figura 96: LS-MRAC. $r = r_{sqw}(t)$.

5.6 Conclusions

- Two algorithms are proposed and analysed:
  - Monopoli Modified MRAC (M-MRAC)
  - Least-squares MRAC (LS-MRAC)
- Both show remarkable improvement on the tracking error behavior.
- As expected, the LS-MRAC shows a superior convergence rate of the parameters.

6 Combined/composite M-MRAC+LS estimator

Contents

6.1 Introduction

References:

1. [Duarte and Narendra (1989)]
2. [Slotine and Li (1989)]
3. [Costa (2023)]

★ Combined OR composite?

- One combines 2 parameter estimates.
- The other uses 2 errors for the estimate.

★ The idea is not new.

Objective:

- Preserve the nice tracking property of the M-MRAC.
- Improve parameter convergence with an LS estimator.
- Remove the need to know an upper bound of $|k_p|$!

Combined MRAC

- Proposed by [Duarte and Narendra (1989)]
- The algorithm employs 2 parameter estimators:
  - one for the control parameter θ AND
  - one for the plant parameter ψ
- The estimates θ and ψ are combined to form a mismatch signal Δ.
  - Δ is a nonlinear function of θ and ψ.
- 2 gradient update laws are introduced:
  - one for θ driven by the tracking error e_0 and Δ.
  - one for ψ driven by the prediction error ϵ and Δ.

Composite MRAC

- Proposed by [Slotine and Li (1989)]
- The algorithm employs only 1 parameter estimator:
  - for the control parameter θ OR
  - for the plant parameter ψ
- The least-squares update law is driven by both errors:
  - tracking error e_0 AND prediction error ϵ.

What is new here?

- We employ the M-MRAC algorithm which has better tracking properties.
- The proposed algorithm is a mixing of the previous approaches.
- Employs 2 estimators for the same control parameter: θ AND ψ
- The mismatch signal is linear in the parameters: $\Delta = \theta - \psi$.
- 2 update laws are introduced:
  - one gradient law for θ driven by the tracking error e_0 and Δ ($\rightarrow$ M-MRAC)
  - one least-squares law for ψ driven by the prediction error ϵ and Δ
- The parameter θ is forced to track the much better LS estimates ψ.

6.2 Review of the combined MRAC

6.2.1 First order case

Plant : $\dot{y} = -a_p y + k_p u$

Model : $\dot{y}_m = -a_m y_m + k_m r, \quad a_m > 0$

Error : $e = y - y_m$

Regressor : $\omega = \begin{bmatrix} y & r \end{bmatrix}^T$

Control parameters estimate : $\theta = \begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix}^T$

Control law : $u = \theta^T \omega$

Matching parameters : $\theta_1^* = \dfrac{a_p - a_m}{k_p}, \qquad \theta_2^* = \dfrac{k_m}{k_p}$

Plant parameters estimate : $\psi = \begin{bmatrix}\psi_1 & \psi_2\end{bmatrix}^T$

Matching parameters : $\psi^* = \begin{bmatrix}a_p & k_p\end{bmatrix}^T$

Identification model : $\dot{\hat{y}} = -a_i\epsilon + \psi_1 y + \psi_2 u, \quad a_i > 0$

Identification error : $\epsilon = y - \hat{y}$

Algebraic relations : $$\psi_2^*\theta_1^* - \psi_1^* + a_m = 0$$ $$\psi_2^*\theta_2^* - k_m = 0$$

Estimate errors (nonlinear) : $$\Delta_1 = \psi_2\theta_1 - \psi_1 + a_m$$ $$\Delta_2 = \psi_2\theta_2 - k_m$$

Error equation : $\dot{e} = -a_m e + k_p\tilde{\theta}^T\omega$

Identification equation : $\dot{\epsilon} = -a_i\epsilon + \tilde{\psi}_1 y + \tilde{\psi}_2 u$

Control update laws : $\dot{\theta} = -\operatorname{sign}(k_p)\big[\gamma_1 e\omega + \gamma_2 \Delta\big]$

Identification update laws :
$$\begin{aligned}\dot{\psi}_1 &= \kappa_1 \epsilon y + \kappa_2 \Delta_1 \\ \dot{\psi}_2 &= \kappa_1 \epsilon u - \kappa_2\, \theta^T \Delta\end{aligned}$$

Stability

The Δ functions can be rewritten as

$$\begin{aligned}\Delta_1 &= \psi_2\theta_1 - \psi_1 + a_m - \big(\psi_2^*\theta_1^* - \psi_1^* + a_m\big) \\ &= \psi_2\theta_1 - (\psi_1 - \psi_1^*) - k_p\big(\theta_1 - \tilde{\theta}_1\big) \\ &= \theta_1\tilde{\psi}_2 - \tilde{\psi}_1 + k_p\tilde{\theta}_1\end{aligned}$$

$$\begin{aligned}\Delta_2 &= \psi_2\theta_2 - k_m - \big(\psi_2^*\theta_2^* - k_m\big) \\ &= \psi_2\theta_2 - k_p\big(\theta_2 - \tilde{\theta}_2\big) \\ &= \theta_2\tilde{\psi}_2 + k_p\tilde{\theta}_2\end{aligned}$$

Using the Lyapunov function

$$V = \frac{\gamma_1}{2\gamma_2}e^2 + \frac{|k_p|}{\gamma_2}\tilde{\theta}^T\tilde{\theta} + \frac{\kappa_1}{2\kappa_2}\epsilon^2 + \frac{1}{\kappa_2}\tilde{\psi}^T\tilde{\psi}$$

we obtain

$$\dot{V} = -\frac{\gamma_1}{2\gamma_2}a_m e^2 - \frac{\kappa_1}{2\kappa_2}a_i\epsilon^2 - \Delta^T\Delta \leq 0$$

Note: Next assignment: Verify this proof!

Note: The notation adopted in [Duarte and Narendra (1989)] is:

$$\eta_a = \tilde{\psi}_1\,, \qquad \eta_b = \tilde{\psi}_2\,, \qquad \phi_\theta = \tilde{\theta}_1\,, \qquad \phi_k = \tilde{\theta}_2\,.$$

6.2.2 Simulation

(...)

6.3 Review of the composite MRAC

(...)

Control update laws : $\dot{\theta} = -P(t)\big[\operatorname{sign}(k_p)e\,\omega + \phi\epsilon\big]$

(...)

6.3.1 First order case

Plant : $\dot{y} = -a_p y + k_p u$

Model : $\dot{y}_m = -a_m y_m + k_m r, \quad a_m > 0$

Error : $e = y - y_m$

Control parameter : $\theta = \begin{bmatrix}\theta_1 & \theta_2\end{bmatrix}^T$

Control law : $u = \theta^T \omega$

Error equation: $\boxed{e = k_p M(s)\left[\tilde{\theta}^T \omega\right]}$

State-space realization: $\dot{e} = -a_m e + k_p \tilde{\theta}^T \omega$

Linear parameterization:

$$\begin{aligned} e &= k_p M(s)\big[u - \theta^{*T}\omega\big] \\ &= k_p \underbrace{M(s)[u]}_{\zeta} - k_p \theta^{*T} \underbrace{M(s)[\omega]}_{\xi} \quad \Rightarrow \quad \underbrace{\frac{1}{k_p}}_{\theta_2^*} e = \zeta - \theta^{*T}\xi \end{aligned}$$

Therefore,

$$\zeta = \theta^{*T}\xi + \theta_2^* e = \theta^{*T} \underbrace{\begin{bmatrix} \xi_1 \\ \xi_2 + e \end{bmatrix}}_{\phi} \quad \Rightarrow \quad \boxed{\zeta = \theta^{*T}\phi}$$

Prediction: $\boxed{\hat{\zeta} = \theta^T \phi}$

Prediction error: $\epsilon = \hat{\zeta} - \zeta = (\theta - \theta^*)^T \phi \quad \Rightarrow \quad \boxed{\epsilon = \tilde{\theta}^T \phi}$

Lyapunov function: $2V = e^2 + |k_p|\tilde{\theta}^T R^{-1}\tilde{\theta}$

Therefore,

$$\begin{aligned}
\dot{V} &= e\dot{e} + |k_p|\tilde{\theta}^T R^{-1}\dot{\theta} + \frac{1}{2}|k_p|\tilde{\theta}^T \dot{R}^{-1}\tilde{\theta} \\
&= -a_m e^2 + k_p\tilde{\theta}^T \omega e + |k_p|\tilde{\theta}^T R^{-1}\dot{\theta} - \frac{1}{2}|k_p|\tilde{\theta}^T R^{-1}\dot{R}R^{-1}\tilde{\theta} \\
&= -a_m e^2 + |k_p|\tilde{\theta}^T\big[\operatorname{sign}(k_p)\omega e + R^{-1}\dot{\theta}\big] - \frac{1}{2}|k_p|\tilde{\theta}^T R^{-1}\dot{R}R^{-1}\tilde{\theta}
\end{aligned}$$

We select $\boxed{\dot{\theta} = -R\big[\operatorname{sign}(k_p)\omega e + \gamma\phi\epsilon\big]}$ $\boxed{\dot{R} = -R\phi\phi^T R}$

Result,

$$\begin{aligned}
\dot{V} &= -a_m e^2 - |k_p|\tilde{\theta}^T\big[\gamma\phi\epsilon\big] + \frac{1}{2}|k_p|\tilde{\theta}^T\phi\phi^T\tilde{\theta} \\
&= -a_m e^2 - |k_p|\Big(\gamma\tilde{\theta}^T\phi\epsilon - \frac{1}{2}\tilde{\theta}^T\phi\phi^T\tilde{\theta}\Big) \\
&= -a_m e^2 - |k_p|\Big(\gamma - \frac{1}{2}\Big)\epsilon^2
\end{aligned}$$

Therefore, if $\gamma > \frac{1}{2}$,

$$\dot{V} = -a_m e^2 - |k_p|\big(\gamma - \frac{1}{2}\big)\epsilon^2 \leq 0$$

★ Note that the algorithm does not require any a priori knowledge regarding $|k_p|$.

Summary of the algorithm

Subsystem	Equation	Order
Plant	$y = P(s)\, u$	n
Model	$y_m = M(s)\, r$	n
Error	$e = y - y_m$	
Control	$u = \theta^T \omega$	
Λ-Filters	$\dot{\omega}_1 = A_f \omega_1 + b_f u$	$n-1$
	$\dot{\omega}_2 = A_f \omega_2 + b_f y$	$n-1$
Regressor	$\omega^T = \begin{bmatrix} \omega_1^T & y & \omega_2^T & r \end{bmatrix}$	
ξ-Filter	$\xi = M(s)[\omega]$	$2n^2$
ζ-Filter	$\zeta = M(s)[u]$	n

Subsystem	Equation	Order
Prediction	$\hat{\zeta} = \theta^T \phi$ $\phi = \xi + [\mathbf{0} \quad 1]^T e$	
Prediction error	$\epsilon = \hat{\zeta} - \zeta$	
Update laws	$\dot{\theta} = -R\big[\mathrm{sign}(k_p)\omega e + \gamma \phi \epsilon\big]$ $\dot{R} = -R\phi\phi^T R$	$2n$ n^2

Order of the system: $\boxed{N = 6n - 2}$

6.3.2 Simulation

(...)

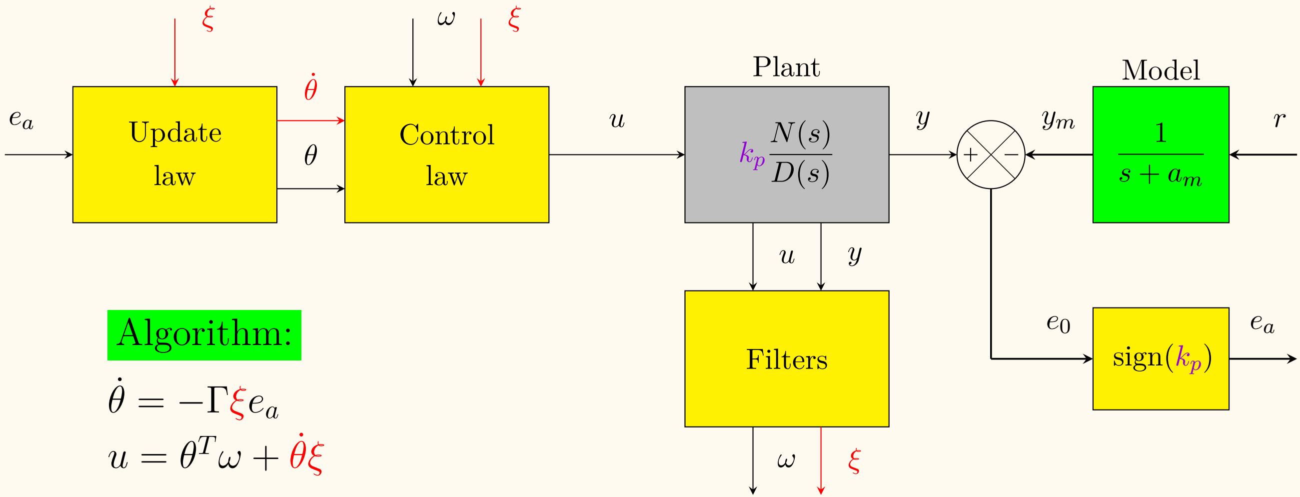
6.4 Review of the M-MRAC

The M-MRAC algorithm was recently introduced in [Costa (2020)]

- ★ This is the same algorithm used for the plant with $n^* = 2$.
- ★ Main feature: Remarkable tracking properties.

Ref.: *Lyapunov design of least-squares model-reference adaptive control*
Ramon R. Costa, IFAC 2020, Berlin, Germany

Block diagram: The M-MRAC algorithm



Algorithm:

$$\dot{\theta} = -\Gamma \xi e_a$$

$$u = \theta^T \omega + \dot{\theta} \xi$$

Required knowledge regarding the plant:

(1) the order n is known,

(2) the relative degree $n^* = 1$,

(3) the zeros are stable, and

(4) $\operatorname{sign}(k_p)$ is known.

★ There is no need to know an upper bound of $|k_p|$!

The control law is defined as

$$u = \theta^T\omega + \dot{\theta}^T\xi = L(s)\big[\theta^T\xi\big]$$

where $\boxed{L(s) = s + \ell_0}$ (Monopoli multiplier)

$\boxed{\xi = L^{-1}(s)[\omega]}$ (filtered signal)

The error equation becomes

$$e_0 = k_p M(s)[u - \theta^{*T}\omega] \quad \Rightarrow \quad e_a = |k_p| \overbrace{\underbrace{M(s)L(s)}_{(\ell_0-a_m)M(s)+1}}^{n^*=0\,!} \big[\underbrace{(\theta-\theta^*)}_{\tilde{\theta}}{}^T \underbrace{L(s)\xi}_{\xi}\big]$$

$$= \underbrace{(\overbrace{\ell_0 - a_m}^{\alpha})\,|k_p|\,M(s)}_{\text{SPR if } \alpha>0}\big[\tilde{\theta}^T\xi\big] + |k_p|\,\big[\tilde{\theta}^T\xi\big]$$

Features

(1) The control law has a high-gain feedback

$$u = \theta^T \omega + \dot{\theta}^T \xi = \theta^T \omega - \underbrace{\xi^T \Gamma \xi}_{\text{high-gain}} \, e_a$$

(2) As a result, the error dynamics become

$$\begin{aligned}
\dot{e}_a &= -a_m e_a + \frac{d}{dt}\Big(|k_p| \tilde{\theta}^T \xi\Big) + \ell_0 |k_p| \tilde{\theta}^T \xi \\
&= -a_m e_a + |k_p| \dot{\theta}^T \xi + |k_p| \tilde{\theta}^T \dot{\xi} + \ell_0 |k_p| \tilde{\theta}^T \xi \\
&= -a_m e_a - |k_p| \big(\Gamma \, \xi e_a\big)^T \xi + |k_p| \tilde{\theta}^T \big(\dot{\xi} + \ell_0 \xi\big) \\
&= -\underbrace{\big(a_m + |k_p| \xi^T \Gamma \xi\big)}_{\text{fast dynamics}} e_a + |k_p| \, \tilde{\theta}^T \omega
\end{aligned}$$

(3) For the conventional MRAC

$$\tilde{u} = \underbrace{u - \theta^{*T}\omega}_{\tilde{\theta}^T\omega} \approx 0 \quad \Rightarrow \quad \begin{cases} \tilde{\theta} \approx 0 \\ \tilde{\theta} \ \text{ and } \ \omega \ \text{ are } \ \approx \perp \end{cases}$$

However, for the M-MRAC

$$\tilde{u} = \tilde{\theta}^T\omega + \dot{\theta}^T\xi \approx 0 \quad \Rightarrow \quad \boxed{\tilde{\theta}^T\omega \approx -\dot{\theta}^T\xi}$$

★ $\tilde{u} \approx 0$ and $e_a \approx 0$ even with large $\tilde{\theta}$ or large $\tilde{\theta}^T\omega$.

6.4.1 Simulation

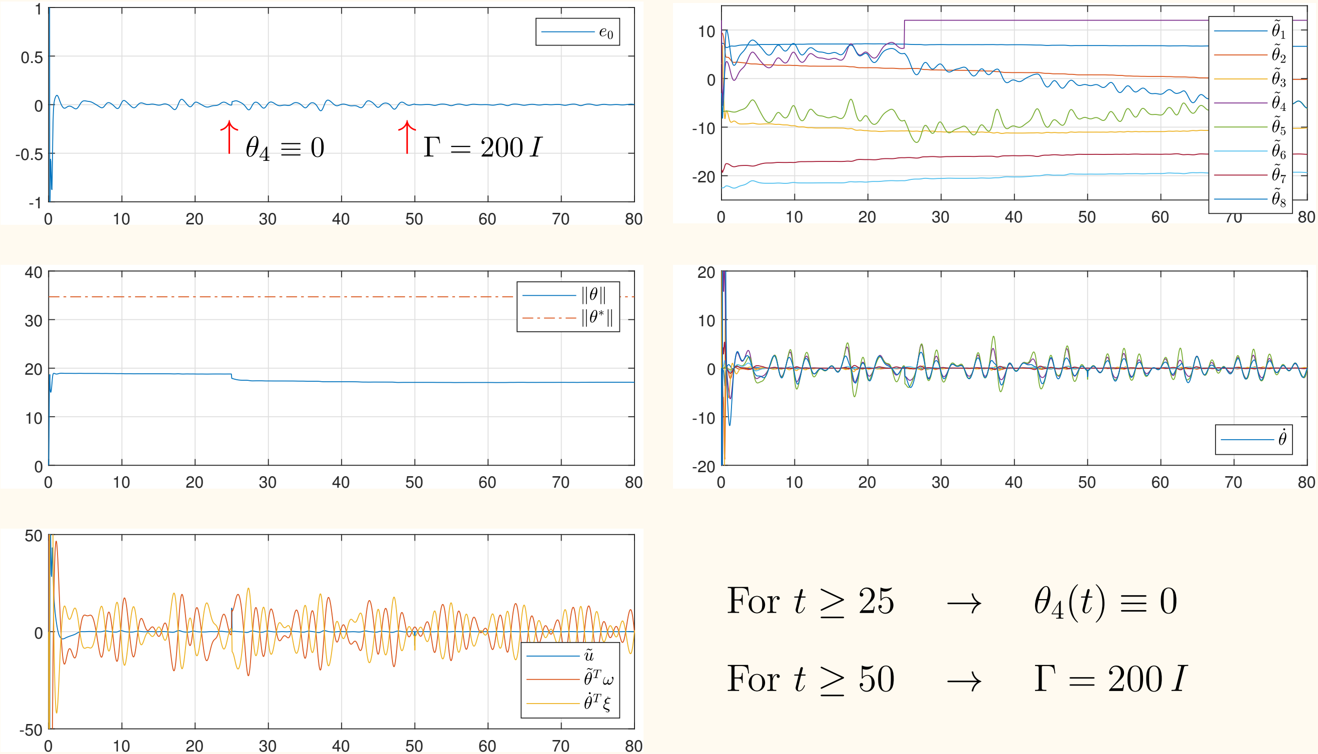
Data used in this simulation:

$$\boxed{P(s) = \frac{(s+2)^3}{3s^4}}, \quad y(0) = 2$$

$$\Gamma = 50\, I$$
$$r(t) = 3 + \sin(1t) + \sin(1.3t) + \sin(2.5t) + \sin(3.2t)$$

Matching parameter:

$$\theta^{*T} = \begin{bmatrix} -7.25 & -9.25 & -3 & -12 & 6.75 & 22.5 & 18.75 & 3 \end{bmatrix}$$
$$\|\theta^*\| = 34.7$$



$$\text{For } t \geq 25 \quad \rightarrow \quad \theta_4(t) \equiv 0$$

$$\text{For } t \geq 50 \quad \rightarrow \quad \Gamma = 200\, I$$

★ This simulation illustrates the properties of the M-MRAC algorithm.

Reference signals

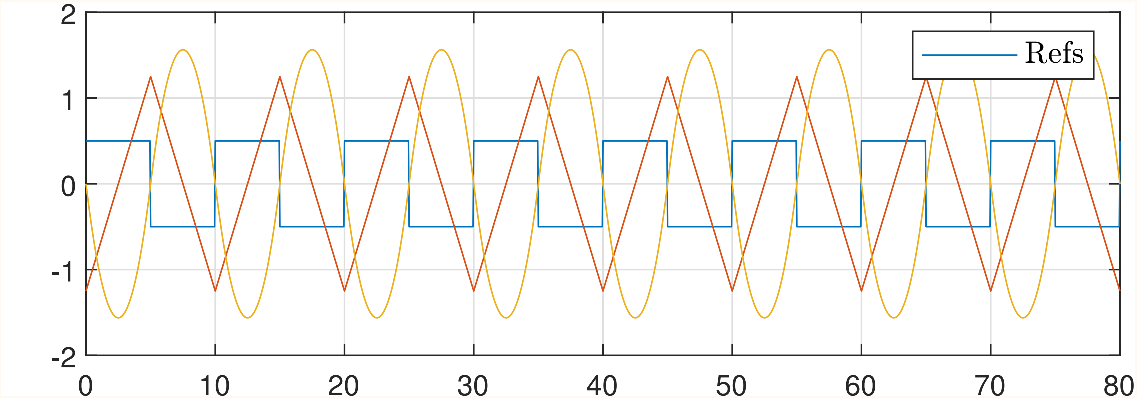


Figura 97: Reference signals: Square, triangular, and parabolic waves.

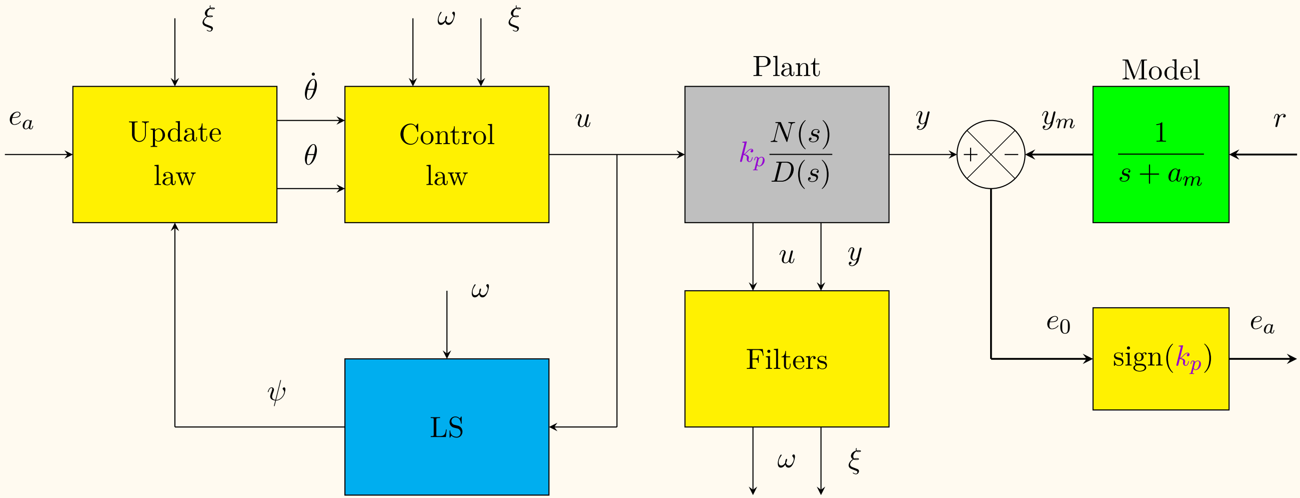
6.5 M-MRAC + LS estimator

Idea: Since $\tilde{u} \approx 0$, then

$$u = \theta^{*T}\omega + \tilde{u} \quad \Rightarrow \quad \boxed{u \approx \theta^{*T}\omega}$$

Then, we can apply a LS algorithm to estimate θ^* from u.

Block diagram of the M-MRAC with LS



★ ψ is a 2nd estimate of θ^*.

Problem: Design a stable algorithm.

Recall the error equation

$$e_a = \alpha\,|k_p|\,M(s)\big[\tilde{\theta}^T\xi\big] + |k_p|\,\big[\tilde{\theta}^T\xi\big]$$

The total order of the system composed by the plant and filters is $5n-2$.

Non–minimal state space realization:

$$\begin{cases} \dot{e} = A_m e + B_m\big[\tilde{\theta}^T\xi\big] \\ e_a = C_m e + |k_p|\big[\tilde{\theta}^T\xi\big], \qquad e \in \mathbb{R}^{5n-2} \end{cases}$$

MKY lemma: The system $\{A_m, B_m, C_m\}$ is SPR then

$$\exists \begin{cases} P = P^T > 0 \\ Q = Q^T > 0 \end{cases} \text{ such that } \boxed{\begin{aligned} A_m^T P + P A_m &= -2Q \\ P B_m &= C_m^T \end{aligned}}$$

6.5.1 Modified update law

The LS estimate ψ is included in the update law of θ as

$$\boxed{\dot{\theta} = -\Gamma\,\xi e_a - \sigma\Delta} \qquad \Gamma = \Gamma^T > 0, \qquad \sigma > 0$$

where $$\boxed{\Delta = \theta - \psi}$$ (Parameters mismatch)

For $e_a \equiv 0$,

$$\dot{\theta} = -\sigma(\theta - \psi) \quad \Rightarrow \quad \boxed{\theta = \frac{1}{(1/\sigma)\,s + 1}\psi}$$

★ That is, θ tracks ψ with a time constant $1/\sigma$.

6.5.2 Linear parametrization

★ Introduced in [Sastry & Bodson 1989].

The error equation is rewritten as

$$\frac{1}{k_p} e_0 = M(s)[u] - \theta^{*T} M(s)[\omega]$$

Applying $M^{-1}(s)L^{-1}(s)$ to both sides, yields

$$\underbrace{\frac{1}{k_p}}_{\theta^*_{2n}} \underbrace{M^{-1}L^{-1}}[e_0] = \underbrace{L^{-1}[u]}_{\zeta} - \theta^{*T} \underbrace{L^{-1}[\omega]}_{\xi}$$

that is

$$\zeta = \theta^{*T}\xi + \theta^*_{2n} \underbrace{M^{-1}L^{-1}}[e_0]$$

Using the decomposition

$$M^{-1}(s)L^{-1}(s) = -\underbrace{(\ell_0 - a_m)}_{\alpha} L^{-1}(s) + 1$$

leads to

$$\begin{aligned}
\zeta &= \theta^{*T}\xi + \theta^*_{2n}\underbrace{M^{-1}L^{-1}}[e_0] \\
&= \theta^{*T}\xi + \theta^*_{2n}\left(1 - \alpha L^{-1}\right)[e_0] \\
&= \theta^{*T}\xi + \theta^*_{2n}\left(e_0 - \alpha\varphi\right) \\
&= \theta^{*T}\left(\xi + \begin{bmatrix} \mathbf{0} \\ e_0 - \alpha\varphi \end{bmatrix}\right) \quad \Rightarrow \quad \boxed{\zeta = \theta^{*T}\phi}
\end{aligned}$$

where

$$\boxed{\varphi = L^{-1}(s)[e_0]} \qquad \text{(Filtered signal)}$$

$$\boxed{\phi = \xi + \begin{bmatrix} \mathbf{0} \\ e_0 - \alpha\varphi \end{bmatrix}} \qquad (\mathbf{0} \in \mathbb{R}^{2n-1})$$

6.5.3 Proposed LS algorithm

New estimate : ψ

Prediction : $\hat{\zeta} = \psi^T \phi$

Prediction error : $\varepsilon = \zeta - \hat{\zeta} = (\psi - \theta^*)^T \phi = \underbrace{\tilde{\psi}^T \phi}$

$$\dot{\psi} = -R\Big(\frac{\tau \phi \varepsilon}{m^2} - \frac{\sigma}{\beta}\Gamma^{-1}\Delta\Big) \qquad \tau \geq \frac{1}{2}, \quad \beta > 0$$

$$\dot{R} = -\frac{R\phi\phi^T R}{m^2} \qquad R(0) = R^T(0) > 0$$

$$m^2 = 1 + \kappa \phi^T R \phi \qquad \kappa > 0$$

6.5.4 Design procedure

Lyapunov function

$$2V(e,\tilde{\theta},\tilde{\psi}) = e^T P e + \tilde{\theta}^T \Gamma^{-1}\tilde{\theta} + \beta\tilde{\psi}^T R^{-1}(t)\tilde{\psi}$$

The derivative of V is given by

$$\dot{V} = -e^T Q e + \underbrace{e^T P B_m}\tilde{\theta}^T\xi + \tilde{\theta}^T\Gamma^{-1}\dot{\theta} + \beta\tilde{\psi}^T R^{-1}\dot{\psi} - \frac{1}{2}\beta\tilde{\psi}^T \underbrace{R^{-1}\dot{R}R^{-1}}_{\dot{R}^{-1}}\tilde{\psi}$$

However,

$$e^T P B_m = e^T C_m^T = e_a - |k_p|\left[\tilde{\theta}^T\xi\right]$$

Therefore,

$$\dot{V} = -e^T Q e - |k_p|\left(\tilde{\theta}^T\xi\right)^2 + \tilde{\theta}^T\left[\xi e_a + \Gamma^{-1}\dot{\theta}\right] + \beta\tilde{\psi}^T R^{-1}\dot{\psi} - \frac{1}{2}\beta\tilde{\psi}^T R^{-1}\dot{R}R^{-1}\tilde{\psi}$$

Using the update laws $\dot{\theta}$, $\dot{\psi}$, and $\dot{R}$,

$$\begin{aligned}
\dot{V} &= -e^T Q e - |k_p|\left(\tilde{\theta}^T \xi\right)^2 - \sigma\tilde{\theta}^T\Gamma^{-1}\Delta - \beta\tilde{\psi}^T\left(\frac{\tau\phi\varepsilon}{m^2} - \frac{\sigma}{\beta}\Gamma^{-1}\Delta\right) + \frac{1}{2}\beta\tilde{\psi}^T\frac{\phi\phi^T}{m^2}\tilde{\psi} \\
&= -e^T Q e - |k_p|\left(\tilde{\theta}^T \xi\right)^2 - \underbrace{\sigma\tilde{\theta}^T\Gamma^{-1}\Delta + \sigma\tilde{\psi}^T\Gamma^{-1}\Delta}\, - \frac{\beta\tau}{m^2}\tilde{\psi}^T\phi\underbrace{\phi^T\tilde{\psi}}_{\varepsilon} + \frac{\beta}{2m^2}\tilde{\psi}^T\phi\phi^T\tilde{\psi} \\
&= -e^T Q e - |k_p|\left(\tilde{\theta}^T \xi\right)^2 - \sigma\left(\tilde{\theta} - \tilde{\psi}\right)^T\Gamma^{-1}\Delta - \frac{\beta}{2m^2}(2\tau - 1)\left(\tilde{\psi}^T\phi\right)^2
\end{aligned}$$

However,

$$\Delta = \theta - \psi = \left(\tilde{\theta} + \theta^*\right) - \left(\tilde{\psi} + \theta^*\right) \quad \Rightarrow \quad \boxed{\Delta = \tilde{\theta} - \tilde{\psi}}$$

Thus,

$$\dot{V} = -e^T Q e - |k_p|\left(\tilde{\theta}^T \xi\right)^2 - \sigma\Delta^T\Gamma^{-1}\Delta - \frac{\beta}{2m^2}(2\tau - 1)\varepsilon^2$$

Stability analysis

★ Stability condition: $\boxed{\tau > \frac{1}{2}} \quad \Rightarrow \quad \dot{V}(e, \tilde{\theta}, \tilde{\psi}) \leq 0$

Therefore,

- $\dot{V}(e, \tilde{\theta}, \tilde{\psi}) \leq 0 \quad \Rightarrow \quad e\,,\, \theta \in \mathcal{L}_\infty\,, \quad \tilde{\psi}^T R^{-1} \tilde{\psi} \in \mathcal{L}_\infty$
- $e\,,\, \tilde{\theta}^T \xi\,,\, \Delta\,,\, \varepsilon \in \mathcal{L}_2$
- $e \in \mathcal{L}_\infty \quad \Rightarrow \quad \omega\,,\, \xi\,,\, \dot{\xi}\,,\, \varphi\,,\, \dot{\varphi} \in \mathcal{L}_\infty$
- $\theta\,,\, \xi \in \mathcal{L}_\infty \quad \Rightarrow \quad \dot{e} \in \mathcal{L}_\infty$
- $e\,,\, \dot{e}\,,\, \xi\,,\, \dot{\xi} \in \mathcal{L}_\infty \quad \Rightarrow \quad \phi\,,\, \dot{\phi} \in \mathcal{L}_\infty$

Since $R(0) = R^T(0) > 0$, then

$$\dot{R}^{-1}(t) = \phi\phi^T \geq 0$$

By integrating,

$$R^{-1}(t) = R^{-1}(0) + \underbrace{\int_0^t \phi(\tau)\phi^T(\tau)d\tau}_{J(t)} > 0\,, \quad t \geq 0$$

Therefore,

- $R^{-1}(t) > R^{-1}(0)$ (Monotonically increasing)
- $R(t) > 0$, $\forall t \geq 0$ (Monotonically decreasing)
- $R \in \mathcal{L}_\infty$
- $R\,,\ \phi \in \mathcal{L}_\infty \quad \Rightarrow \quad \dot{R} \in \mathcal{L}_\infty$

From the Lyapunov function,

$$2V = \underbrace{e^T P e}_{\in \mathcal{L}_\infty} + \underbrace{\tilde{\theta}^T \Gamma^{-1} \tilde{\theta}}_{\in \mathcal{L}_\infty} + \underbrace{\beta \tilde{\psi}^T R^{-1}(0) \tilde{\psi}}_{\in \mathcal{L}_\infty} + \underbrace{\beta \tilde{\psi}^T J(t) \tilde{\psi}}_{\in \mathcal{L}_\infty}$$

Conclusion:

- $V \in \mathcal{L}_\infty \quad \Rightarrow \quad \tilde{\psi}^T R^{-1}(0) \tilde{\psi} \in \mathcal{L}_\infty \quad \Rightarrow \quad \boxed{\tilde{\psi} \in \mathcal{L}_\infty}$
- $\tilde{\psi} \in \mathcal{L}_\infty \quad \Rightarrow \quad \hat{\zeta}\,,\ \Delta\,,\ \dot{\theta} \in \mathcal{L}_\infty$
- $\dot{\theta} \in \mathcal{L}_\infty \quad \Rightarrow \quad \boxed{u \in \mathcal{L}_\infty}$
- $u \in \mathcal{L}_\infty \quad \Rightarrow \quad \zeta\,,\ \dot{\zeta} \in \mathcal{L}_\infty$
- $\zeta\,,\ \hat{\zeta} \in \mathcal{L}_\infty \quad \Rightarrow \quad \varepsilon \in \mathcal{L}_\infty$
- $\varepsilon \in \mathcal{L}_\infty \quad \Rightarrow \quad \dot{\psi}\,,\ \dot{\Delta} \in \mathcal{L}_\infty$

Convergence

- $\dot{e} \in \mathcal{L}_\infty \quad \Rightarrow \quad \lim_{t\to\infty} e(t) = 0$
- $\frac{d}{dt}\tilde{\theta}^T\xi \in \mathcal{L}_\infty \quad \Rightarrow \quad \lim_{t\to\infty} \tilde{\theta}^T\xi = 0$
- $\dot{\Delta} \in \mathcal{L}_\infty \quad \Rightarrow \quad \lim_{t\to\infty} \Delta(t) = 0$
- $\dot{\varepsilon} = \frac{d}{dt}\tilde{\psi}^T\phi \in \mathcal{L}_\infty \quad \Rightarrow \quad \lim_{t\to\infty} \varepsilon = 0$
- Therefore, all signals are bounded $\quad \Rightarrow \quad$ Global uniform stability

Summarizing,

$$\boxed{e, \theta, \psi, \varepsilon \in \mathcal{L}_\infty} \quad \text{and} \quad \boxed{e, \tilde{\theta}^T\xi, \Delta, \varepsilon \to 0}$$

- ★ Note the effect of the parameter β.
- ★ For large β the term $\dfrac{\sigma}{\beta}\Gamma^{-1}\Delta$ becomes negligible!
- ★ This means that the update law $\dot{\psi}$ becomes decoupled from θ.

Summary of the M-MRAC + LS estimator

Subsystem	Equation	Order
Plant	$y = P(s)\, u$	n
Model	$y_m = M(s)\, r$	n
Track. error	$e_a = \operatorname{sign}(k_p)(y - y_m)$	
SV-filters	$\dot{\omega}_1 = A_f \omega_1 + b_f u$ $\dot{\omega}_2 = A_f \omega_2 + b_f y$	$n-1$ $n-1$
Regressor	$\omega^T = \begin{bmatrix} \omega_1^T & y & \omega_2^T & r \end{bmatrix}$	
ξ-filter	$\dot{\xi} = -\ell_0 \xi + \omega\,, \qquad \ell_0 > a_m$	$2n$
Control	$u = \theta^T \omega + \dot{\theta}^T \xi$	
Update law	$\dot{\theta} = -\Gamma\, \xi e_a - \sigma \Delta, \qquad \Gamma = \Gamma^T > 0, \quad \sigma > 0$ $\Delta = \theta - \psi$	$2n$

Subsystem	Equation	Order
Filters	$\dot{\zeta} = -\ell_0\zeta + u$	1
	$\dot{\varphi} = -\ell_0\varphi + e_0$	1
Prediction	$\hat{\zeta} = \psi^T\phi$	
	$\phi = \xi + \left[\mathbf{0}^T \quad (e_0 - \alpha\varphi)\right]^T, \qquad \mathbf{0} \in \mathbb{R}^{2n-1}$	
Prediction error	$\varepsilon = \hat{\zeta} - \zeta$	
LS estimator	$\dot{\psi} = -R\Big(\frac{\tau\phi\varepsilon}{m^2} - \frac{\sigma}{\beta}\Gamma^{-1}\Delta\Big), \quad \tau > \frac{1}{2}, \quad \beta > 0$	$2n$
	$\dot{R} = -\frac{R\phi\phi^T R}{m^2}, \qquad R(0) = R^T(0) > 0$	$4n^2$
	$m^2 = 1 + \kappa\phi^T R\phi, \qquad \kappa \geq 0$	

System total dimension: $\boxed{N = 10n + 4n^2}$

6.5.5 Simulation

Example 23 4th order plant.

System class: $n = 4$ (order)
$n^* = 1$ (relative degree)
$n_p = 8$ (# of parameters)

Plant .... : $P(s) = \dfrac{(s+2)^3}{3s^4}$

Model .... : $M(s) = \dfrac{1}{s+1}$

SV filter .... : $\dfrac{1}{\Lambda(s)} = \dfrac{1}{(s+0.5)(s+1)(s+1.5)}$

Matching .... : $\theta^{*T} = [-7.25 \quad -9.25 \quad -3 \quad -12 \quad 6.75 \quad 22.5 \quad 18.75 \quad 3]$
$\|\theta^*\| = 34.7$

Initial condition........:

$$y(0) = \boxed{5}$$
$$y_m(0) = 0$$
$$\theta(0) = \boxed{0}$$
$$R(0) = \boxed{200\, I}$$

Parameters..............:

$$\Gamma = 5\, I$$
$$\tau = 5\,, \qquad \beta = \boxed{1000}\,, \qquad \kappa = 0.1\,, \qquad \sigma = 0.5$$

Reference signal.......: $r_{sqw}(t) = 3 + 10\,\mathrm{sqw}(0.1\pi t)$

★ ...

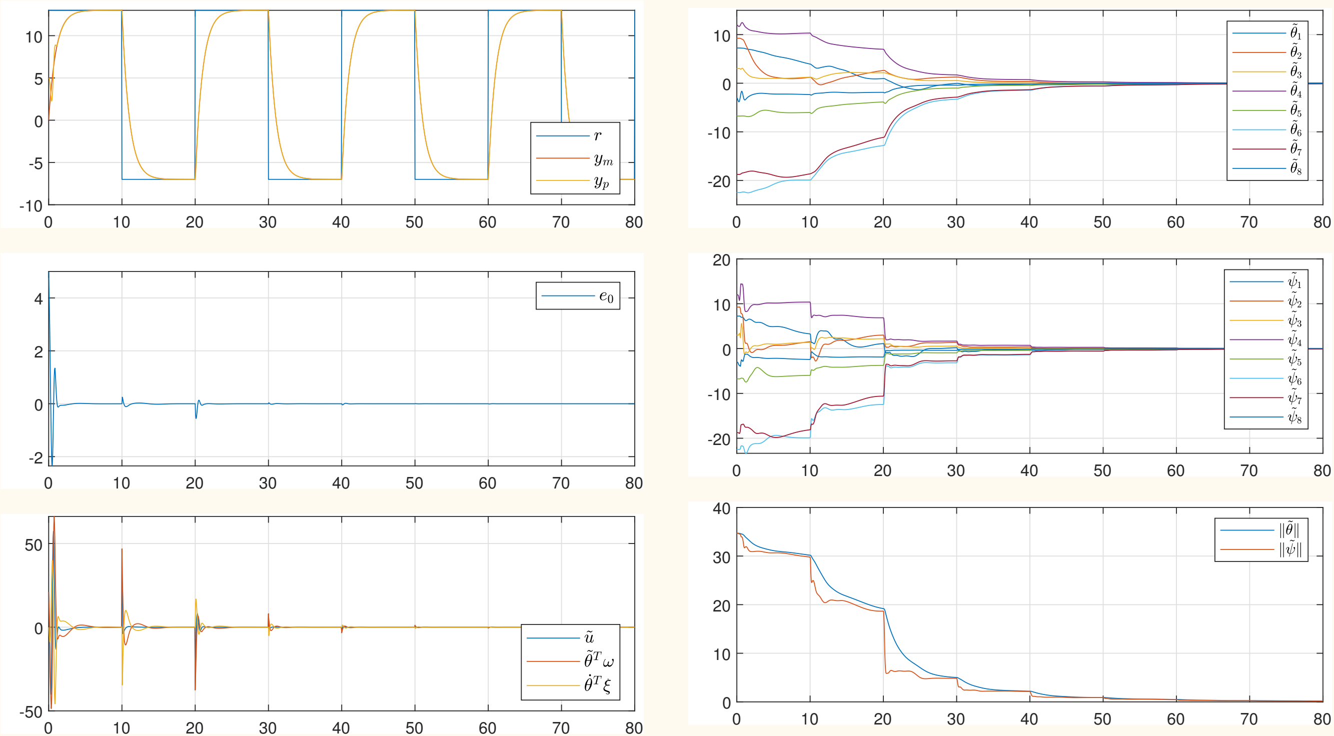


Figura 98: Composite M-MRAC with a LS estimator. $r = r_{sqw}(t)$.

6.6 Conclusion

★ The nice tracking performance of the M-MRAC algorithm is preserved.

★ The convergence of the parameters is remarkably improved.

★ A lower bound of $|k_p|$ is not required !

The price:

- The implementation of the LS algorithm requires a large number of state variables.

8 Backstepping

Contents

8.1 Introduction

Reference: [Krstić et al. (1995), pag.417]

- Nova abordagem para o problema.
- Originalmente desenvolvido para o controle adaptativo de sistemas não lineares.
- Abandona a condição SPR.
- Utiliza observador!
- Important concepts:
  - *Virtual control*
  - *Stabilizing function*
  - *Tuning function*

8.2 Exemplos preliminares

Nesta seção são apresentados 3 exemplos de projeto.

Em todos os casos, a planta é de 1a. ordem com um único parâmetro θ.

Casos considerados:

(1) Regulação com θ conhecido.

(2) Rastreamento com θ conhecido.

(3) Rastreamento com θ desconhecido.

8.2.1 Regulação com parâmetro conhecido

Planta: $\boxed{\dot{y} = u + \theta f(y)}$ (θ é conhecido)

★ $f(y)$ é uma função não linear de y.

Para estabilizar este sistema, escolhemos $\boxed{u = -\theta f - y}$

Resultado: $\boxed{\dot{y} = -y} \Rightarrow \boxed{y \to 0}$

8.2.2 Rastreamento com parâmetro conhecido

Planta: $\boxed{\dot{y} = u + \theta f(y)}$ (θ é conhecido)

Objetivo: $\boxed{y \to y_r}$ y_r é a trajetória desejada.

Transformação: $\boxed{z = y - y_r}$ $\Rightarrow$ $\dot{z} = \dot{y} - \dot{y}_r = u + \theta f - \dot{y}_r$

Escolhemos $\boxed{u = -z - \theta f + \dot{y}_r}$ (z = erro de rastreamento)

Resultado: $\boxed{\dot{z} = -z}$ $\Rightarrow$ $\boxed{z \to 0}$ e $\boxed{y \to y_r}$

Análise via Lyapunov:

$$2V = z^2$$

$$\begin{aligned}\dot{V} &= z\dot{z} = z\big(\underbrace{u + \theta f - \dot{y}_r}_{-z}\big) \\ &= z(-z) \\ &= -z^2 \ \boxed{< 0}\end{aligned}$$

Resultado: O sistema realimentado é exponencialmente estável.

8.2.3 Rastreamento com parâmetro desconhecido

Planta: $\boxed{\dot{y} = u + \theta f(y)}$ (θ é desconhecido)

Objetivo: $\boxed{y \to y_r}$

Transformação: $\boxed{z = y - y_r}$ $\Rightarrow$ $\dot{z} = \dot{y} - \dot{y}_r = u + \theta f - \dot{y}_r$

Estimativa de θ: $\boxed{\hat{\theta}}$

Erro paramétrico: $\boxed{\tilde{\theta} = \theta - \hat{\theta}}$

★ Para projetar o controle agora precisamos usar Lyapunov !!

Função de Lyapunov: $$\boxed{2V = z^2 + \gamma^{-1}\tilde{\theta}^2}$$

Derivando, $$\begin{aligned}\dot{V} &= z\dot{z} + \gamma^{-1}\tilde{\theta}\dot{\theta} \\ &= z\big(u + \theta f - \dot{y}_r\big) + \gamma^{-1}\tilde{\theta}\big(-\dot{\hat{\theta}}\big)\end{aligned}$$

Escolhemos $$\boxed{u = -z - \hat{\theta}f + \dot{y}_r}$$

Substituindo, $$\begin{aligned}\dot{V} &= z\big(-z + \theta f - \hat{\theta}f\big) - \gamma^{-1}\tilde{\theta}\dot{\hat{\theta}} \\ &= -z^2 + z\tilde{\theta}f - \gamma^{-1}\tilde{\theta}\dot{\hat{\theta}} \\ &= -z^2 + \gamma^{-1}\tilde{\theta}\left[\gamma z f - \dot{\hat{\theta}}\right]\end{aligned}$$

Escolhemos $\boxed{\dot{\hat{\theta}} = \gamma z f}$ $\Rightarrow$ $\boxed{\dot{V} = -z^2 \leq 0}$

Resultado: O sistema realimentado é estável.

8.3 Caso $n^* = 2$

- Nesta seção serão apresentados 2 exemplos de projeto.
- Planta de 2a. ordem com um único parâmetro θ.
- Casos considerados:
  1. Rastreamento com θ conhecido.
  2. Rastreamento com θ desconhecido.

8.3.1 Rastreamento com parâmetro conhecido

Planta: $$\begin{cases} \dot{x}_1 = x_2 + \theta f(x_1) \\ \dot{x}_2 = u \\ y = x_1 \end{cases}$$ (θ conhecido)

Objetivo: $\boxed{y \to y_r}$

Ideia: Usar x_2 como controle virtual para a 1a. equação !!

- ★ Note que x_2 não pode ser forçado a ser identicamente igual ao controle ideal desejado.
- ★ Função estabilizante α. É o sinal de controle que resolveria o problema !!

Transformação:
$$\begin{cases} z_1 = y - y_r & \Rightarrow \quad \text{Rastrear } y_r \\ z_2 = x_2 - \alpha & \Rightarrow \quad \text{Forçar } \; x_2 \to \alpha \end{cases}$$

Novo objetivo: $\boxed{z \to 0}$

Reescrevemos o sistema de equações como:

$$\begin{cases} \dot{z}_1 = \dot{x}_1 - \dot{y}_r = x_2 + \theta f - \dot{y}_r \\ \dot{z}_2 = \dot{x}_2 - \dot{\alpha} = u - \dot{\alpha} \end{cases}$$

Função estabilizante para a 1a. equação: $\boxed{\alpha = -z_1 - \theta f + \dot{y}_r}$

Como $\alpha = g(x_1, y_r, \dot{y}_r)$ $\Rightarrow$ $$\boxed{\dot{\alpha} = \frac{\partial \alpha}{\partial x_1}\dot{x}_1 + \frac{\partial \alpha}{\partial y_r}\dot{y}_r + \frac{\partial \alpha}{\partial \dot{y}_r}\ddot{y}_r}$$

1a. função de Lyapunov: $$\boxed{2V_1 = z_1^2}$$

Derivando,
$$\begin{aligned}
\dot{V}_1 = z_1\dot{z}_1 &= z_1\big(x_2 + \theta f - \dot{y}_r\big) \\
&= z_1\big(z_2 + \alpha + \theta f - \dot{y}_r\big) \\
&= z_1\big(z_2 - z_1\big) \\
&= -z_1^2 + z_1 z_2
\end{aligned}$$

2a. função de Lyapunov: $$\boxed{2V = 2V_1 + z_2^2}$$

Derivando, $$\dot V = \dot V_1 + z_2 \dot z_2 = -z_1^2 + z_1 z_2 + z_2\big(u - \dot\alpha\big)$$

Escolhemos $$\boxed{u = -z_1 - z_2 + \dot\alpha}$$

Resultado: $$\boxed{\dot V = -z_1^2 - z_2^2 < 0}$$

★ O sistema é exponencialmente estável.

Resumo do algoritmo

Erros	$z_1 = y - y_r$ $z_2 = x_2 - \alpha$
Função estabilizante	$\alpha = -z_1 - \theta f + \dot{y}_r$
Derivada	$\dot{\alpha} = \dfrac{\partial \alpha}{\partial x_1}\dot{x}_1 + \dfrac{\partial \alpha}{\partial y_r}\dot{y}_r + \dfrac{\partial \alpha}{\partial \dot{y}_r}\ddot{y}_r$
Controle	$u = -z_1 - z_2 + \dot{\alpha}$

★ O algoritmo necessita do estado para ser implementado !

★ Mais adiante será introduzido um observador para contornar esse problema.

8.3.2 Rastreamento com parâmetro desconhecido

Planta: $\begin{cases} \dot{x}_1 = x_2 + \theta f(x_1) \\ \dot{x}_2 = u \\ y = x_1 \end{cases}$ (θ desconhecido)

Transformação: $\begin{cases} z_1 = y - y_r \\ z_2 = x_2 - \alpha \end{cases}$

Sistema transformado: $\begin{cases} \dot{z}_1 = x_2 + \theta f - \dot{y}_r \\ \dot{z}_2 = u - \dot{\alpha} \end{cases}$

Função estabilizante para a 1a. equação: $\boxed{\alpha = -z_1 - \hat{\theta} f + \dot{y}_r}$

Substituindo,
$$
\begin{aligned}
\dot{z}_1 &= x_2 + \theta f - \dot{y}_r \\
&= \big(z_2 + \alpha\big) + \theta f - \dot{y}_r \\
&= z_2 + \big(-z_1 - \hat{\theta} f + \dot{y}_r\big) + \theta f - \dot{y}_r \\
&= z_2 - z_1 + \tilde{\theta} f
\end{aligned}
$$

1a. função de Lyapunov: $$\boxed{2V_1 = z_1^2 + \gamma^{-1}\tilde{\theta}^2}$$

Derivando,
$$
\begin{aligned}
\dot{V}_1 &= z_1\dot{z}_1 + \gamma^{-1}\tilde{\theta}\dot{\hat{\theta}} \\
&= z_1\big(z_2 - z_1 + \tilde{\theta} f\big) - \gamma^{-1}\tilde{\theta}\dot{\hat{\theta}} \\
&= -z_1^2 + z_1 z_2 + \gamma^{-1}\tilde{\theta}\big[\gamma z_1 f - \dot{\hat{\theta}}\big]
\end{aligned}
$$

Escolhemos $$\boxed{\dot{\hat{\theta}} = \gamma z_1 f}$$

- ★ Porém, não podemos implementar esta lei de adaptação !!
- ★ Ainda falta considerar a dinâmica de z_2.
- ★ θ irá reaparecer na análise.

Ideia: Deixar a escolha da lei de adaptação em aberto.

A dinâmica de z_2 é dada por

$$\dot{z}_2 = u - \dot{\alpha}$$

★ Como obter $\dot{\alpha}$?

Derivando diretamente a expressão $\boxed{\alpha = -z_1 - \hat{\theta} f + \dot{y}_r}$

Resultado:

$$\begin{aligned}\dot{\alpha} &= -\dot{z}_1 - \dot{\hat{\theta}} f + \hat{\theta}\dot{f} + \ddot{y}_r \\ &= -\dot{y} + \dot{y}_r - \dot{\hat{\theta}} f + \hat{\theta}\dot{f} + \ddot{y}_r \\ &= -\left(x_2 + \theta f\right) - \dot{\hat{\theta}} f + \hat{\theta}\dot{f} + \dot{y}_r + \ddot{y}_r\end{aligned}$$

Notation: Uma forma mais compacta & sistemática utiliza derivadas parciais.

Como $\alpha = g(x_1, y_r, \dot{y}_r, \hat{\theta})$, então

$$\dot{\alpha} = \frac{\partial \alpha}{\partial x_1}\big(x_2 + \theta f\big) + \frac{\partial \alpha}{\partial y_r}\dot{y}_r + \frac{\partial \alpha}{\partial \dot{y}_r}\ddot{y}_r + \frac{\partial \alpha}{\partial \hat{\theta}}\dot{\hat{\theta}}$$

★ Note here that

- θ is unknown
- $\dot{\hat{\theta}}$ is still not defined

$$\dot{\alpha} = \frac{\partial \alpha}{\partial x_1}\big(x_2 + \theta f\big) + \frac{\partial \alpha}{\partial y_r}\dot{y}_r + \frac{\partial \alpha}{\partial \dot{y}_r}\ddot{y}_r + \frac{\partial \alpha}{\partial \hat{\theta}}\dot{\hat{\theta}}$$

Para simplificar o desenvolvimento, definimos

$$\boxed{\beta = \frac{\partial \alpha}{\partial x_1}\big(x_2 + \hat{\theta} f\big) + \frac{\partial \alpha}{\partial y_r}\dot{y}_r + \frac{\partial \alpha}{\partial \dot{y}_r}\ddot{y}_r + \frac{\partial \alpha}{\partial \hat{\theta}}\gamma z_1 f}$$

- ★ β agrupa somente sinais conhecidos.
- ★ Note que usamos $\gamma z_1 f$ no lugar de $\dot{\hat{\theta}}$ na definição de β !!
- ★ Usamos a informação disponível.

Substituindo,

$$\dot{\alpha} = \beta + \frac{\partial \alpha}{\partial x_1}\tilde{\theta} f - \frac{\partial \alpha}{\partial \hat{\theta}}\big(\gamma z_1 f - \dot{\hat{\theta}}\big)$$

Portanto,

$$\dot{z}_2 = u - \beta - \frac{\partial \alpha}{\partial x_1}\tilde{\theta} f + \frac{\partial \alpha}{\partial \hat{\theta}}\left(\gamma z_1 f - \dot{\hat{\theta}}\right)$$

2a. função de Lyapunov: $\boxed{2V = 2V_1 + z_2^2}$

Derivando,

$$\dot{V} = -z_1^2 + z_1 z_2 + \gamma^{-1}\tilde{\theta}\left(\gamma z_1 f - \dot{\hat{\theta}}\right) + z_2\left[u - \beta - \frac{\partial \alpha}{\partial x_1}\tilde{\theta} f + \frac{\partial \alpha}{\partial \hat{\theta}}\left(\gamma z_1 f - \dot{\hat{\theta}}\right)\right]$$

Após rearranjar os termos,

$$\dot{V} = -z_1^2 + z_1 z_2 + \gamma^{-1}\tilde{\theta}\left(\gamma z_1 f - \gamma \frac{\partial \alpha}{\partial x_1} z_2 f - \dot{\hat{\theta}}\right) + z_2\left[u - \beta + \frac{\partial \alpha}{\partial \hat{\theta}}\left(\gamma z_1 f - \dot{\hat{\theta}}\right)\right]$$

1a. escolha:
$$\boxed{\dot{\hat{\theta}} = \gamma\Big(z_1 - \frac{\partial\alpha}{\partial x_1} z_2\Big) f}$$

Portanto,

$$\begin{aligned}
\dot{V} &= -z_1^2 + z_1 z_2 + z_2\Big[u - \beta + \frac{\partial\alpha}{\partial\hat{\theta}}\Big(\gamma z_1 f - \gamma z_1 f + \gamma \frac{\partial\alpha}{\partial x_1} z_2 f\Big)\Big] \\
&= -z_1^2 + z_1 z_2 + z_2\Big[u - \beta + \gamma \frac{\partial\alpha}{\partial\hat{\theta}} \frac{\partial\alpha}{\partial x_1} z_2 f\Big]
\end{aligned}$$

2a. escolha:
$$\boxed{u = -z_1 - z_2 + \beta - \gamma \frac{\partial\alpha}{\partial\hat{\theta}} \frac{\partial\alpha}{\partial x_1} z_2 f}$$

Resultado: $$\dot{V} = -z_1^2 - z_2^2 \leq 0$$

- ★ O sistema é estável.
- ★ O algoritmo necessita o estado para ser implementado!

Resumo do algoritmo

Erros	$z_1 = y - y_r$ $z_2 = x_2 - \alpha$
Função estabilizante	$\alpha = -z_1 - \hat{\theta} f + \dot{y}_r$
Sinal auxiliar	$\beta = \dfrac{\partial \alpha}{\partial x_1}\left(x_2 + \hat{\theta} f\right) + \dfrac{\partial \alpha}{\partial y_r}\dot{y}_r + \dfrac{\partial \alpha}{\partial \dot{y}_r}\ddot{y}_r + \dfrac{\partial \alpha}{\partial \hat{\theta}}\gamma z_1 f$
Adaptação	$\dot{\hat{\theta}} = \gamma\left(z_1 - \dfrac{\partial \alpha}{\partial x_1} z_2\right) f$
Controle	$u = -z_1 - z_2 + \beta - \gamma \dfrac{\partial \alpha}{\partial \hat{\theta}} \dfrac{\partial \alpha}{\partial x_1} z_2 f$

Exercício. O termo β foi definido como:

$$\boxed{\beta = \frac{\partial \alpha}{\partial x_1}\left(x_2 + \hat{\theta} f\right) + \frac{\partial \alpha}{\partial y_r}\dot{y}_r + \frac{\partial \alpha}{\partial \dot{y}_r}\ddot{y}_r + \frac{\partial \alpha}{\partial \hat{\theta}}\gamma z_1 f}$$

O que aconteceria se

$$\boxed{\beta = \frac{\partial \alpha}{\partial x_1}\left(x_2 + \hat{\theta} f\right) + \frac{\partial \alpha}{\partial y_r}\dot{y}_r + \frac{\partial \alpha}{\partial \dot{y}_r}\ddot{y}_r}$$

isto é, sem o termo $\dfrac{\partial \alpha}{\partial \hat{\theta}}\gamma z_1 f$?

8.4 State observer

8.4.1 Observable canonical form

Plant :

$$\boxed{y = \frac{B(s)}{A(s)}u}$$

$$B(s) = b_m s^m + \cdots + b_1 s + b_0$$

$$A(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

- n is the order of the plant
- $\rho = n - m$ is the relative degree
- b_m is the high frequency gain

The observable canonical form is given by

$$
\begin{cases}
\dot{x}_1 = x_2 - a_{n-1}y \\
\quad\vdots \\
\dot{x}_{\rho-1} = x_\rho - a_{m-1}y \\
\dot{x}_\rho = x_{\rho+1} - a_m y + b_m u & \leftarrow \text{ u appears !!} \\
\quad\vdots \\
\dot{x}_{n-1} = x_n - a_1 y + b_1 u \\
\dot{x}_n = \quad -a_0 y + b_0 u \\
y = x_1
\end{cases}
$$

Using matricial notation,

$$
\begin{cases}
\dot{x} = Ax - ay + \left[\begin{array}{c} 0 \\ \hline b \end{array}\right] u \\
y = e_1^T x
\end{cases}
$$

Using matricial notation,

$$\begin{cases} \dot{x} = Ax - ay + \left[\begin{array}{c} 0 \\ \hline b \end{array}\right] u \\ y = e_1^T x \end{cases}$$

where

$$A = \left[\begin{array}{c|c} 0 & I \\ \hline & 0 \end{array}\right], \qquad a = \begin{bmatrix} a_{n-1} \\ \vdots \\ a_0 \end{bmatrix}, \qquad b = \begin{bmatrix} b_m \\ \vdots \\ b_0 \end{bmatrix}$$

$$e_1^T = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

8.4.2 Full order observer

Fundamental hypotheses:
- The model of the plant is known.
- The plant is completely observable.

★ Only the initial condition $x(0)$ is unknown !!

★ Observer $=$ copy of the plant $+$ feedback of the output error.

Plant .... : $\dot{x} = Ax + Bu$

Observer .... : $\dot{\hat{x}} = A\hat{x} + Bu + \underbrace{K\big(y - \hat{y}\big)}_{\text{Feedback}}$

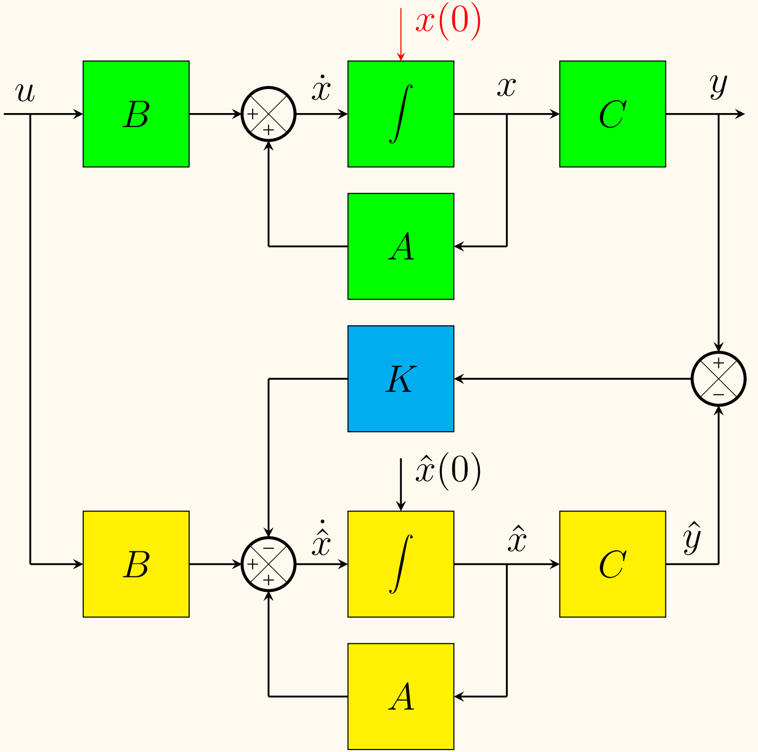


Figura 106: Full order observer.

Example 24 Full order observer for a 2nd order plant.

Plant:
$$\boxed{\; y = \frac{k_p}{s^2 + a_1 s + a_0} u \;}$$

Observable canonical form:
$$\begin{cases} \dot{x}_1 = -a_1 x_1 + x_2 \\ \dot{x}_2 = -a_0 x_1 \qquad + k_p u \\ \ \ y = x_1 \end{cases}$$

Using matrix notation:
$$\begin{cases} \dot{x} = \begin{bmatrix} -a_1 & 1 \\ -a_0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ k_p \end{bmatrix} u \\ y = e_1^T x = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$$

The model coefficients can the isolated into a vector by rewriting the equations as:

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x - \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} y + \begin{bmatrix} 0 \\ k_p \end{bmatrix} u \\ \\ y = e_1^T x = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$$

Defining $\quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F(y,u)^T = \left[\begin{array}{c|cc} 0 & -y & 0 \\ u & 0 & -y \end{array}\right], \quad \theta = \left[\begin{array}{c} k_p \\ \hline a_1 \\ a_0 \end{array}\right]$

one gets $\quad \boxed{\dot{x} = Ax + F^T\theta}$

★ This notation is convenient for an adaptive observer.

Upon adding and subtracting $ke_1^T x$ (i.e., introducing feedback), gives

$$\dot{x} = Ax + ke_1^T x - ke_1^T x + F^T\theta \quad \Rightarrow \quad \boxed{\dot{x} = A_0 x + ky + F^T\theta}$$

where

$$k = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}, \qquad A_0 = A - k\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix} \quad \text{(Hurwitz)}$$

Now, introducing the K filters

$$\begin{cases} \dot{\xi} = A_0\xi + ky \\ \dot{\Omega}^T = A_0\Omega^T + F^T \end{cases} \tag{116}$$

the estimate of x is given by:

$$\boxed{\hat{x} = \xi + \Omega^T\theta} \quad \text{(Full order observer)} \tag{117}$$

★ See (Kreisselmeier, 1977)

Checking out:

$$
\begin{aligned}
\dot{\hat{x}} &= \dot{\xi} + \dot{\Omega}^T\theta \\
&= \big(A_0\xi + ky\big) + \big(A_0\Omega^T + F^T\big)\theta \\
&= A_0\big(\underbrace{\xi + \Omega^T\theta}_{\hat{x}}\big) + ky + F^T\theta
\end{aligned}
$$

Therefore:

$$\boxed{\dot{\hat{x}} = A_0\hat{x} + ky + F^T\theta}$$

★ Observer is a copy of the plant + error feedback.

Checking out:

$$
\begin{aligned}
\dot{\hat{x}} &= A_0\hat{x} + ky + F^T\theta \\
&= \big(A - ke_1^T\big)\hat{x} + ky + F^T\theta \\
&= \underbrace{A\hat{x} + F^T\theta}_{\text{Copy}} + \underbrace{k\big(y - \hat{y}\big)}_{\text{Feedback !}}
\end{aligned}
$$

Example 25 **Full order observer** for a 3rd order plant.

Plant:
$$\boxed{y = \frac{b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} u}$$
(Here $b_1 = k_p$)

Observable canonical form:
$$\begin{cases} \dot{x}_1 = -a_2 x_1 + x_2 \\ \dot{x}_2 = -a_1 x_1 \qquad + x_3 + b_1 u \\ \dot{x}_3 = -a_0 x_1 \qquad\qquad + b_0 u \\ \ \ y = x_1 \end{cases}$$

Using matrix notation:
$$\begin{cases} \dot{x} = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ b_1 \\ b_0 \end{bmatrix} u \\ y = e_1^T x = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x \end{cases}$$

The model coefficients can the isolated into a vector by rewriting the equations as:

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x - \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} y + \begin{bmatrix} 0 \\ b_1 \\ b_0 \end{bmatrix} u \\ \\ y = e_1^T x = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x \end{cases}$$

Defining $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad F(y,u)^T = \left[\begin{array}{cc|ccc} 0 & 0 & -y & 0 & 0 \\ u & 0 & 0 & -y & 0 \\ 0 & u & 0 & 0 & -y \end{array}\right], \quad \theta = \left[\begin{array}{c} b_1 \\ b_0 \\ \hline a_2 \\ a_1 \\ a_0 \end{array}\right]$

Then, one gets $\boxed{\dot{x} = Ax + F^T\theta}$

Upon adding and subtracting $ke_1^T x$, gives

$$\dot{x} = Ax + ke_1^T x - ke_1^T x + F^T\theta \quad \Rightarrow \quad \boxed{\dot{x} = A_0 x + ky + F^T\theta}$$

where

$$k = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}, \qquad A_0 = A - k\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -k_1 & 1 & 0 \\ -k_2 & 0 & 1 \\ -k_3 & 0 & 0 \end{bmatrix}$$

Now, introducing the K filters
$$\begin{cases} \dot{\xi} = A_0\xi + ky \\ \dot{\Omega}^T = A_0\Omega^T + F^T \end{cases}$$

the estimate of x is given by:

$$\boxed{\hat{x} = \xi + \Omega^T\theta}$$

(Full order observer)

8.4.3 Reducing the filters order

K filters:
$$\begin{cases} \dot{\xi} = A_0\xi + ky \\ \dot{\Omega}^T = A_0\Omega^T + F^T \end{cases}$$

★ The filters can be implemented in a more efficient way.

Example 26 3rd order plant with one zero ($n = 3$ and $m = 1$).

$$F(y,u)^T = \left[\begin{array}{cc|ccc} 0 & 0 & -y & 0 & 0 \\ u & 0 & 0 & -y & 0 \\ 0 & u & 0 & 0 & -y \end{array}\right]$$

In this case,

$$\dot{\Omega}^T = A_0\Omega^T + F^T = A_0\Omega^T + \left[\begin{array}{cc|ccc} 0 & 0 & -y & 0 & 0 \\ u & 0 & 0 & -y & 0 \\ 0 & u & 0 & 0 & -y \end{array}\right]$$

Denote

$$\boxed{\Omega^T \equiv \left[\begin{array}{cc|c} v_1 & v_0 & \Xi \end{array}\right]} \tag{118}$$

Then, one has that

$$\dot{\Omega}^T = A_0 \left[\begin{array}{cc|c} v_1 & v_0 & \Xi \end{array}\right] + \left[\begin{array}{cc|ccc} 0 & 0 & -y & 0 & 0 \\ u & 0 & 0 & -y & 0 \\ 0 & u & 0 & 0 & -y \end{array}\right]$$

$$\dot{v}_0 = A_0 v_0 + \begin{bmatrix} 0 \\ 0 \\ u \end{bmatrix} = A_0 v_0 + e_3 u \quad \Rightarrow \quad \boxed{\dot{v}_0 = A_0 v_0 + e_3 u} \tag{119}$$

where $e_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$.

Properties:

$$A_0 e_3 = \begin{bmatrix} -k_1 & 1 & 0 \\ -k_2 & 0 & 1 \\ -k_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{e_2} \quad \Rightarrow \quad \boxed{A_0 e_3 = e_2} \quad (***)$$

Upon introducing the variables

$$\begin{cases} v_0 = \lambda & (= A_0^0 \lambda) \\ v_1 = A_0 \lambda & (= A_0^1 \lambda) \end{cases} \tag{120}$$

gives,

$$\boxed{\dot{v}_0 = A_0 v_0 + e_3 u} \quad \Rightarrow \quad \boxed{\dot{\lambda} = A_0 \lambda + e_3 u}$$

(***) Attention to the notation. Here $e_2 = [0\ 1\ 0]^T$ and not $e_2 = [0\ 1]^T$!!

Therefore,

$$\begin{aligned}\dot{v}_1 &= A_0\dot{\lambda} \\ &= A_0\big(A_0\lambda + e_3 u\big) \\ &= A_0\big(\underbrace{A_0\lambda}_{v_1}\big) + \underbrace{A_0 e_3}_{e_2} u = \boxed{A_0 v_1 + e_2 u}\end{aligned}$$

As a result, all vectors v_i can be obtained from only one filter:

$$\boxed{\dot{\lambda} = A_0\lambda + e_n u} \qquad \boxed{v_i = A_0^i\lambda} \qquad (i = 0, \cdots, m) \tag{121}$$

A similar procedure can be applied for Ξ,

$$\dot{\Xi} = A_0 \Xi - I y$$

All column vectors of Ξ can be obtained from only one filter:

$$\boxed{\dot{\eta} = A_0 \eta + e_n y} \qquad \boxed{\Xi = -\begin{bmatrix} A_0^{n-1}\eta & \cdots & A_0 \eta & \eta \end{bmatrix}} \tag{122}$$

For this 3rd order example,

$$\Xi = -\begin{bmatrix} A_0^2 \eta & A_0 \eta & \eta \end{bmatrix}$$

Finally, after observing that

$$\begin{aligned}
A_0 e_3 &= e_2 \\
A_0 e_2 &= A_0\big(A_0 e_3\big) = A_0^2 e_3 = e_1 \\
A_0 e_1 &= A_0\big(A_0^2 e_3\big) = A_0^3 e_3 = -k
\end{aligned}$$

then,

$$\boxed{\xi = -A_0^3 \eta} \tag{123}$$

In fact,

$$\begin{aligned}
\dot{\xi} &= -A_0^3 \dot{\eta} \\
&= -A_0^3\big(A_0 \eta + e_3 y\big) \\
&= A_0\big(\underbrace{-A_0^3 \eta}_{\xi}\big) - \underbrace{A_0^3 e_3}_{-k}\, y = \boxed{A_0 \xi + k y}
\end{aligned}$$

8.4.4 Reduced order observer

Ref. : [Friedland:2012], (pag. 261)

★ This is also known as Luenberger Observer.

Consider a plant in the form

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

where $\boxed{x \in \mathbb{R}^n}$ and $\boxed{y \in \mathbb{R}^m}$

Suppose that the first m state variables can be obtained directly from the measurements y.

In other words, suppose that the state can be partitioned as

$$\begin{cases} \dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1u \\ \dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2u \\ y = C_1x_1 \end{cases}$$

and

$$\boxed{x_1 = C_1^{-1}y} \in \mathbb{R}^m$$

A reduced order observer can be used to estimate the remaining $x_2 \in \mathbb{R}^{n-m}$.

A reduced order system may be obtained by defining

$$\boxed{\chi = x_2 + Ny} \quad \Rightarrow \quad \boxed{x_2 = \chi - Ny} \quad \chi \in \mathbb{R}^{n-m}$$

It can be easily shown that the dynamics of χ is given by

$$\boxed{\dot{\chi} = Q\chi + Ry + Su} \tag{124}$$

where,

$$\boxed{\begin{aligned} Q &= A_{22} + NC_1A_{12} \\ R &= -QN + \big(A_{21} + NC_1A_{11}\big)C_1^{-1} \\ S &= B_2 + NC_1B_1 \end{aligned}}$$

Checking out

$$
\begin{aligned}
\dot{\chi} &= \dot{x}_2 + NC_1\dot{x}_1 \\
&= \big(A_{21}x_1 + A_{22}x_2 + B_2u\big) + NC_1\big(A_{11}x_1 + A_{12}x_2 + B_1u\big) \\
&= \big(\underbrace{A_{22} + NC_1A_{12}}_{Q}\big)x_2 + \big(A_{21} + NC_1A_{11}\big)x_1 + \big(\underbrace{B_2 + NC_1B_1}_{S}\big)u
\end{aligned}
$$

Adding and subtracting the term QNy,

$$
\begin{aligned}
\dot{\chi} &= Qx_2 + QNy - QNy + \big(A_{21} + NC_1A_{11}\big)x_1 + Su \\
&= Q(\underbrace{x_2 + Ny}_{\chi}) - QNy + \big(A_{21} + NC_1A_{11}\big)C_1^{-1}y + Su \\
&= Q\chi + \big[\underbrace{-QN + \big(A_{21} + NC_1A_{11}\big)C_1^{-1}}_{R}\big]y + Su \\
&= Q\chi + Ry + Su
\end{aligned}
$$

The reduced order system is

$$\boxed{\dot{\chi} = Q\chi + Ry + Su}$$

Therefore, the reduced order observer is given by

$$\boxed{\dot{\hat{\chi}} = Q\hat{\chi} + Ry + Su} \tag{125}$$

$$\boxed{\hat{x}_1 = x_1 = C_1^{-1}y} \tag{126}$$

$$\boxed{\hat{x}_2 = \hat{\chi} - Ny} \tag{127}$$

To assure stability of the observer, N must be designed such that

$$\boxed{Q = A_{22} + NC_1A_{12}}$$ is Hurwitz !

Example 27 Reduced order observer for a 2nd order plant.

Plant: $$\boxed{\; y = \frac{k_p}{s^2 + a_1 s + a_0} u \;}$$

Observable canonical form : $$\begin{cases} \dot{x}_1 = -a_1 x_1 + x_2 \\ \dot{x}_2 = -a_0 x_1 \qquad + k_p u \\ \;\; y = x_1 \end{cases}$$

Using matrix notation: $$\begin{cases} \dot{x} = \begin{bmatrix} -a_1 & 1 \\ -a_0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ k_p \end{bmatrix} u \\ \\ y = e_1^T x = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$$

Define $\boxed{\chi = x_2 + Ny}$ $\Rightarrow$ $\boxed{x_2 = \chi - Ny}$

The dynamics of χ is given by

$$
\begin{aligned}
\dot{\chi} &= (-a_0 y + k_p u) + N(x_2 - a_1 y) \\
&= N x_2 - (a_0 + N a_1) y + k_p u \\
&= N(\chi - Ny) - (a_0 + N a_1) y + k_p u \\
&= N\chi - N^2 y \underbrace{-(a_0 + N a_1) y + k_p u}_{F^T \theta}
\end{aligned}
$$

Result: $\boxed{\dot{\chi} = N\chi - N^2 y + F^T \theta}$

where $F^T = \begin{bmatrix} u & -Ny & -y \end{bmatrix}, \qquad \theta = \begin{bmatrix} k_p \\ a_1 \\ a_0 \end{bmatrix}$

Reduced order system: $$\boxed{\dot{\chi} = N\chi - N^2 y + F^T\theta}$$

The K filters are: $$\begin{cases} \dot{\xi} = N\xi - N^2 y \\ \dot{\Omega}^T = N\Omega + F^T \end{cases}$$

The estimate of χ is obtained as: $$\boxed{\hat{\chi} = \xi + \Omega^T\theta}$$

and the estimate of x_2 is obtained as: $$\boxed{\hat{x}_2 = \hat{\chi} - Ny}$$

★ Of course, one should select $$\boxed{N < 0}$$

Checking out:

$$
\begin{aligned}
\dot{\hat{\chi}} &= \dot{\xi} + \dot{\Omega}^T\theta \\
&= \big(N\xi - N^2 y\big) + \big(N\Omega^T + F^T\big)\theta \\
&= N\big(\underbrace{\xi + \Omega^T\theta}_{\hat{\chi}}\big) - N^2 y + F^T\theta
\end{aligned}
$$

$$
\Rightarrow \quad \boxed{\dot{\hat{\chi}} = N\hat{\chi} - N^2 y + F^T\theta}
$$

which is similar to the reduced order system.

8.5 Caso $n^* = 2$ com observador

Esta seção introduz o método BKST completo, com observador.

Serão apresentados 2 exemplos de projeto:

(1) Exemplo mais simples possível.

- Problema de rastreamento
- Planta de 2a. ordem não linear (mesma planta da seção anterior)

$$\begin{cases} \dot{x}_1 = x_2 + \theta f(x_1) \\ \dot{x}_2 = u \\ y = x_1 \end{cases}$$

- Um único parâmetro θ conhecido

(2) Exemplo mais completo.

- Problema de rastreamento
- Planta de 2a. ordem linear

$$y = \frac{k_p}{s^2 + a_1 s + a_0}\, u$$

- Vetor de parâmetros $\theta^T = [k_p \ a_1 \ a_0]$ desconhecido

★ A partir desse ponto, somente serão consideradas plantas lineares.

8.5.1 Tracking with known parameter

Plant: $$\begin{cases} \dot{x}_1 = x_2 + \theta f(x_1) \\ \dot{x}_2 = u \\ \quad y = x_1 \end{cases}$$ (θ known !)

Objective: $\boxed{y \to y_r}$

- ★ The variable x_2 is not available, so we will employ an state observer.
- ★ Notice that the plant is already in the observer canonical form.

The plant can be rewritten as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x - \begin{bmatrix} \theta \\ 0 \end{bmatrix} f(y) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = e_1^T x = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

To introduce an observer, we define

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad F^T = \begin{bmatrix} 0 & f(y) \\ u & 0 \end{bmatrix}, \qquad \Theta = \begin{bmatrix} 1 \\ \theta \end{bmatrix}$$

which gives

$$\dot{x} = Ax + \begin{bmatrix} 0 & f(y) \\ u & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \theta \end{bmatrix} = Ax + F^T \Theta$$

Adding and subtracting the term $ke_1^T x$,

$$\dot{x} = Ax - ke_1^T x + ke_1^T x + F^T\Theta \quad \Rightarrow \quad \boxed{\dot{x} = A_0 x + ky + F^T\Theta}$$

where

$$k = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \qquad A_0 = A - k\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix} \quad \text{(Hurwitz)}$$

Introduce the K filters:
$$\begin{cases} \dot{\xi} = A_0\xi + ky \\ \dot{\Omega}^T = A_0\Omega + F^T \end{cases}$$

An estimate of x is obtained by:
$$\boxed{\hat{x} = \xi + \Omega^T\Theta} \tag{128}$$

Very important: The same parameter vector Θ is used by both, the observer and the controller !

Define the estimate error

$$\boxed{\varepsilon = x - \hat{x}} \tag{129}$$

Its dynamics verifies

$$\begin{aligned}\dot{\varepsilon} &= \left(A_0 x + ky + F^T \Theta\right) - \left(A_0 \hat{x} + ky + F^T \Theta\right) \\ &= A_0(x - \hat{x}) \quad \Rightarrow \quad \boxed{\dot{\varepsilon} = A_0 \varepsilon}\end{aligned}$$

Property: If A_0 is Hurwitz

then $\varepsilon(t) \to 0$ exponentially

$\Rightarrow \quad \exists P = P^T > 0$ tal que $\boxed{PA_0 + A_0^T P = -I}$

Let's partition Ω^T as

$$\Omega^T \equiv \left[\ v \ \middle|\ \eta\ \right]$$

Recalling that

$$F(y,u)^T = \begin{bmatrix} 0 & f(y) \\ u & 0 \end{bmatrix}$$

then, we can writte

$$\boxed{\dot{v} = A_0 v + e_2 u}$$

$$\boxed{\dot{\eta} = A_0 \eta + e_1 f(y)}$$

De (129) e (128) obtemos a seguinte relação:

$$x = \hat{x} + \varepsilon \quad \Rightarrow \quad \boxed{x = \xi + \Omega^T \Theta + \varepsilon} \tag{130}$$

Expandindo,

$$x = \xi + \begin{bmatrix} v & \eta \end{bmatrix} \begin{bmatrix} 1 \\ \theta \end{bmatrix} + \varepsilon \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} v_1 & \eta_1 \\ v_2 & \eta_2 \end{bmatrix} \begin{bmatrix} 1 \\ \theta \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$$

ou seja,

$$\begin{cases} x_1 = \xi_1 + v_1 + \theta\eta_1 + \varepsilon_1 \\ x_2 = \xi_2 + v_2 + \theta\eta_2 + \varepsilon_2 \end{cases}$$

★ Resultado: escrevemos o estado da planta em função do estado do observador.

Relembrando as equações originais

Planta: $$\begin{cases} \dot{x}_1 = x_2 + \theta f(x_1) \\ \dot{x}_2 = u \end{cases}$$

Como x_2 não está disponível, vamos utilizar o estado do observador.

Substituindo x_2,

$$\begin{aligned} \dot{y} &= \big(\xi_2 + v_2 + \theta\eta_2 + \varepsilon_2\big) + \theta f(y) \\ &= \xi_2 + v_2 + \theta\big[\underbrace{\eta_2 + f(y)}_{\omega}\big] + \varepsilon_2 \\ &= \xi_2 + v_2 + \theta\omega + \varepsilon_2 \end{aligned}$$

onde $$\boxed{\omega = \eta_2 + f(y)}$$ (Regressor)

Resultado:

$$\boxed{\dot{y} = \xi_2 + v_2 + \theta\omega + \varepsilon_2} \tag{131}$$

Pergunta: Qual variável vamos utilizar como controle virtual?

★ Note que todas as variáveis são disponíveis, exceto ε_2.

Ideia: Utilizar a cascata $y \leftarrow v_2 \leftarrow u.$

Solução: Utilizamos a variável v_2 do observador

$$\begin{cases} \dot{y} = v_2 + \xi_2 + \theta\omega + \varepsilon_2 \\ \dot{v}_2 = -k_2 v_1 + u \end{cases}$$

★ Lembrar que : $\boxed{\dot{v} = A_0 v + e_2 u}$ $\boxed{\dot{\xi} = A_0 \xi + k y}$

★ Neste sistema o estado está disponível !!

★ Observe que o erro ε_2 será considerado na análise!

Objetivo do controle: Rastrear o sinal de referência y_r.

Transformações:

$$\boxed{z_1 = y - y_r} \to 0$$

$$\boxed{z_2 = v_2 - \alpha} \to 0$$

★ α é uma função estabilizante que resolve o problema de rastreamento.

Novo objetivo: $\boxed{z \to 0}$

Começamos a análise pela dinâmica de z_1:

$$\begin{aligned}\dot{z}_1 &= \dot{y} - \dot{y}_r \\ &= \underbrace{v_2} + \xi_2 + \theta\omega + \varepsilon_2 - \dot{y}_r \\ &= \big(z_2 + \alpha\big) + \xi_2 + \theta\omega + \varepsilon_2 - \dot{y}_r\end{aligned}$$

Em vista disto, escolhemos a função estabilizante

$$\boxed{\alpha = -c_1 z_1 - d_1 z_1 - \xi_2 - \theta\omega + \dot{y}_r} \qquad (132)$$

★ A utilização de c_1 e d_1 é apenas por conveniência !!

Substituindo...

$$\boxed{\dot{z}_1 = -c_1 z_1 - d_1 z_1 + z_2 + \varepsilon_2}$$

Escolhemos a 1a. função de Lyapunov,

$$\boxed{2V_1 = z_1^2 + \frac{1}{2d_1}\varepsilon^T P\varepsilon}$$

Derivando, temos:

$$\begin{aligned}
\dot{V}_1 &= z_1\dot{z}_1 + \frac{1}{4d_1}\left(\dot{\varepsilon}^T P\varepsilon + \varepsilon^T P\dot{\varepsilon}\right)\\
&= z_1\left(-c_1z_1 - d_1z_1 + z_2 + \varepsilon_2\right) + \frac{1}{4d_1}\left(\dot{\varepsilon}^T P\varepsilon + \varepsilon^T P\dot{\varepsilon}\right)\\
&= -c_1z_1^2 + z_1z_2 - d_1z_1^2 + z_1\varepsilon_2 + \frac{1}{4d_1}\varepsilon^T\Big(\underbrace{A_0^T P + PA_0}_{-I}\Big)\varepsilon\\
&= -c_1z_1^2 + z_1z_2 \underbrace{- d_1z_1^2 + z_1\varepsilon_2 - \frac{1}{4d_1}\varepsilon^T\varepsilon}_{X}
\end{aligned}$$

O termo X pode ser escrito como:

$$
\begin{aligned}
X &= -d_1 z_1^2 + z_1 \varepsilon_2 - \frac{1}{4d_1}\varepsilon^T \varepsilon \\
&= -d_1 z_1^2 + z_1 \varepsilon_2 - \frac{1}{4d_1}\left(\varepsilon_1^2 + \varepsilon_2^2\right) \\
&= -d_1 \left[z_1^2 - \frac{z_1 \varepsilon_2}{d_1} + \left(\frac{\varepsilon_2}{2d_1}\right)^2 \right] - \frac{\varepsilon_1^2}{4d_1} \\
&= -d_1 \left[z_1 - \frac{\varepsilon_2}{2d_1} \right]^2 - \frac{\varepsilon_1^2}{4d_1} \leq 0 \qquad \text{(!!)}
\end{aligned}
$$

★ O termo $d_1 z_1$ introduzido em (132) completa o quadrado!

Portanto, X pode ser desconsiderado o que resulta

$$
\boxed{\dot{V}_1 \leq -c_1 z_1^2 + z_1 z_2}
$$

Próximo passo: Análise da dinâmica de z_2.

Temos que

$$\begin{aligned}\dot{z}_2 &= \dot{v}_2 - \dot{\alpha} \\ &= -k_2 v_1 + u - \dot{\alpha}\end{aligned}$$

Pode-se verificar facilmente que α é função de y, ξ, η, y_r e $\dot{y}_r$:

$$\boxed{\alpha(y, \xi, \eta, y_r, \dot{y}_r)}$$

Verificação:

$$\alpha = -c_1 z_1 - d_1 z_1 - \xi_2 - \theta\omega - \dot{y}_r$$

$$z_1 = y - y_r$$

$$\omega = \eta_2 + f(y)$$

$$\dot{\xi} = A_0 \xi + k y$$

$$\dot{\eta} = A_0 \eta + e_1 f(y)$$

Portanto,

$$\boxed{\begin{aligned} \dot{\alpha} = {} & \frac{\partial \alpha}{\partial y}\big(\xi_2 + v_2 + \theta\omega + \varepsilon_2\big) + \frac{\partial \alpha}{\partial \xi}\big(A_0\xi + ky\big) + \\ & + \frac{\partial \alpha}{\partial \eta}\big(A_0\eta + e_1 f(y)\big) + \frac{\partial \alpha}{\partial y_r}\dot{y}_r + \frac{\partial \alpha}{\partial \dot{y}_r}\ddot{y}_r \end{aligned}}$$

★ A única variável não disponível é ε_2.

Podemos escrever a equação para $\dot{z}_2$ como:

$$\dot{z}_2 = u - \beta - \frac{\partial \alpha}{\partial y}\varepsilon_2$$

onde β é uma função que reune somente sinais disponíveis :

$$\boxed{\begin{aligned} \beta = k_2 v_1 + \frac{\partial \alpha}{\partial y}\big(\xi_2 + v_2 + \theta\omega\big) + \frac{\partial \alpha}{\partial \xi}\big(A_0\xi + ky\big) + \\ + \frac{\partial \alpha}{\partial \eta}\big(A_0\eta + e_1 f(y)\big) + \frac{\partial \alpha}{\partial y_r}\dot{y}_r + \frac{\partial \alpha}{\partial \dot{y}_r}\ddot{y}_r \end{aligned}}$$

- ★ Note que o termo que depende de ε_2 permanece na equação de $\dot{z}_2$.
- ★ Neste ponto estamos prontos para continuar a análise via Lyapunov.

Aumentamos a função de Lyapunov da seguinte forma:

$$\boxed{V_2 = V_1 + \frac{1}{2}z_2^2 + \frac{1}{4d_2}\varepsilon^T P \varepsilon}$$

Derivando,

$$\begin{aligned}
\dot{V}_2 &\leq -c_1 z_1^2 + z_1 z_2 + z_2\Big(u - \beta - \frac{\partial \alpha}{\partial y}\varepsilon_2\Big) + \frac{1}{4d_2}\Big(\dot{\varepsilon}^T P \varepsilon + \varepsilon^T P \dot{\varepsilon}\Big) \\
&\leq -c_1 z_1^2 + z_2\Big(z_1 + u - \beta\Big) - z_2 \frac{\partial \alpha}{\partial y}\varepsilon_2 - \frac{1}{4d_2}\varepsilon^T \varepsilon
\end{aligned}$$

Escolhemos a lei de controle

$$\boxed{u = -c_2 z_2 - z_1 + \beta - d_2\Big(\frac{\partial \alpha}{\partial y}\Big)^2 z_2}$$

★ O termo $d_2\Big(\frac{\partial \alpha}{\partial y}\Big)^2 z_2$ irá completar o quadrado !

Substituindo,

$$\dot{V}_2 \leq -c_1 z_1^2 + z_2\Big[-c_2 z_2 - d_2\Big(\frac{\partial\alpha}{\partial y}\Big)^2 z_2\Big] - z_2\frac{\partial\alpha}{\partial y}\varepsilon_2 - \frac{1}{4d_2}\varepsilon^T\varepsilon$$

$$\leq -c_1 z_1^2 - c_2 z_2^2 \underbrace{-d_2\Big(\frac{\partial\alpha}{\partial y}\Big)^2 z_2^2 - z_2\frac{\partial\alpha}{\partial y}\varepsilon_2 - \frac{1}{4d_2}\varepsilon^T\varepsilon}_{X}$$

O termo X pode ser escrito como:

$$X = -d_2\Big(\frac{\partial\alpha}{\partial y}\Big)^2 z_2^2 - z_2\frac{\partial\alpha}{\partial y}\varepsilon_2 - \frac{1}{4d_2}\big(\varepsilon_1^2 + \varepsilon_2^2\big)$$

$$= -d_2\Big[\Big(\frac{\partial\alpha}{\partial y}\Big)^2 z_2^2 + \frac{z_2}{d_2}\frac{\partial\alpha}{\partial y}\varepsilon_2 + \frac{\varepsilon_2^2}{4d_2^2}\Big] - \frac{\varepsilon_1^2}{4d_2}$$

$$= -d_2\Big[z_2\frac{\partial\alpha}{\partial y} + \frac{\varepsilon_2}{2d_2}\Big]^2 - \frac{\varepsilon_1^2}{4d_2} \leq 0 \qquad \text{(!!)}$$

★ O termo $X \leq 0$ pode ser descartado.

Resultado:

$$\boxed{\dot{V}_2 \leq -c_1 z_1^2 - c_2 z_2^2}$$

Resumo do algoritmo

Filtros	$\dot{v} = A_0 v + e_2 u$
Filtros	$\dot{\eta} = A_0 \eta + e_1 f(y)$ $\dot{\xi} = A_0 \xi + ky$
Regressor	$\omega = \eta_2 + f(y)$
Erro z_1	$z_1 = y - y_r$
Função estabilizante	$\alpha = -c_1 z_1 - d_1 z_1 - \xi_2 - \theta\omega + \dot{y}_r$
Erro z_2	$z_2 = v_2 - \alpha$
Sinal auxiliar	$\beta = k_2 v_1 + \frac{\partial \alpha}{\partial y}\left(\xi_2 + v_2 + \theta\omega\right) + \frac{\partial \alpha}{\partial \xi}\left(A_0\xi + ky\right) + \frac{\partial \alpha}{\partial \eta}\left(A_0\eta + e_1 f(y)\right) + \frac{\partial \alpha}{\partial y_r}\dot{y}_r + \frac{\partial \alpha}{\partial \dot{y}_r}\ddot{y}_r$
Controle	$u = -c_2 z_2 - z_1 + \beta - d_2 \left(\frac{\partial \alpha}{\partial y}\right)^2 z_2$

8.5.2 Simulações

(...)

8.5.3 Rastreamento com parâmetro desconhecido

Example 28 Sistema de 2a. ordem.

Plant:
$$\boxed{y = \frac{k_p}{s^2 + a_1 s + a_0} u}$$

Neste exemplo:
$$m = 0$$
$$n = 2$$
$$\rho = n - m = 2$$
$$b_0 = k_p$$

Forma canônica observável:

$$\begin{cases} \dot{x}_1 = x_2 - a_1 y \\ \dot{x}_2 = - a_0 y + k_p u \\ y = x_1 \end{cases}$$

Utilizando notação matricial,

$$\begin{cases} \dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} x - \underbrace{\begin{bmatrix} a_1 \\ a_0 \end{bmatrix} y + \begin{bmatrix} 0 \\ k_p \end{bmatrix} u}_{F(y,u)^T \theta} \\ y = e_1^T x \end{cases}$$

Para definir o observador, escrevemos o sistema na forma

$$\begin{cases} \dot{x} = Ax + F(y,u)^T\theta \\ y = e_1^T x \end{cases}$$

onde,

$$F(y,u)^T = \left[\begin{array}{c|cc} 0 & -y & 0 \\ u & 0 & -y \end{array}\right], \qquad \theta = \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} k_p \\ a_1 \\ a_0 \end{bmatrix}$$

- ★ Nesta forma, os parâmetros da planta são arranjados no vetor θ.
- ★ Em consequência, o regressor é escrito na forma de uma matrix $F(y,u)^T$.

Adding and subtracting the term $ke_1^T x$,

$$\dot{x} = Ax - ke_1^T x + ke_1^T x + F^T\theta \quad \Rightarrow \quad \boxed{\dot{x} = A_0 x + ky + F^T\theta}$$

Para estimar o estado, utilizamos os filtros abaixo:

$$\text{Filtros } K \quad \begin{cases} \dot{\xi} = A_0\xi + ky \\ \dot{\Omega}^T = A_0\Omega^T + F^T \end{cases} \qquad (\Omega \text{ é matrix ! })$$

onde,

$$k = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}, \qquad A_0 = A - ke_1^T = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix} \quad \text{(Hurwitz)}$$

O estado estimado é dado por:

$$\boxed{\hat{x} = \xi + \Omega^T\theta} \tag{133}$$

★ Este estimador não pode ser implementado. θ é desconhecido !!

Definindo o erro de estima

$$\boxed{\varepsilon = x - \hat{x}} \tag{134}$$

verifica-se que sua dinâmica satisfaz

$$\boxed{\dot{\varepsilon} = A_0 \varepsilon}$$

Property: If A_0 is Hurwitz

then $\varepsilon(t) \to 0$ exponencialmente

$\Rightarrow \quad \exists P = P^T > 0$ tal que $\boxed{PA_0 + A_0^T P = -I}$

De (134) e (133) obtemos a seguinte relação:

$$x = \hat{x} + \varepsilon \quad \Rightarrow \quad \boxed{x = \xi + \Omega^T \theta + \varepsilon}$$

Redução da ordem dos filtros

Renomeamos as colunas de Ω^T: $\Omega^T \equiv \left[\begin{array}{c|c} v_0 & \Xi \end{array}\right]$

Relembrando que, $F(y,u)^T = \left[\begin{array}{c|cc} 0 & -y & 0 \\ u & 0 & -y \end{array}\right]$

Introduzimos os filtros $\begin{cases} \dot{\lambda} = A_0\lambda + e_2 u \\ \dot{\eta} = A_0\eta + e_2 y \end{cases}$

e obtemos os sinais

$$v_0 = \lambda$$

$$\Xi = -\begin{bmatrix} A_0\eta & \eta \end{bmatrix}$$

$$\xi = -A_0^2\eta$$

O projeto BKST começa pela equação da saída, i.e., pela equação para $\dot{x}_1$,

$$\begin{aligned}\dot{x}_1 &= x_2 - a_1 y \\ &= x_2 - e_1^T a y\end{aligned} \tag{135}$$

Porém,

$$x = \xi + \Omega^T\theta + \varepsilon = \begin{bmatrix}\xi_1 \\ \xi_2\end{bmatrix} + \begin{bmatrix}v_{0,1} & \Xi_{(1)} \\ v_{0,2} & \Xi_{(2)}\end{bmatrix}\begin{bmatrix}k_p \\ a\end{bmatrix} + \begin{bmatrix}\varepsilon_1 \\ \varepsilon_2\end{bmatrix}$$

Para a 2a. linha dessa equação, temos

$$\begin{aligned}x_2 &= \xi_2 + \begin{bmatrix}v_{0,2} & \Xi_{(2)}\end{bmatrix}\begin{bmatrix}k_p \\ a\end{bmatrix} + \varepsilon_2 \\ &= k_p v_{0,2} + \xi_2 + \begin{bmatrix}0 & \Xi_{(2)}\end{bmatrix}\begin{bmatrix}k_p \\ a\end{bmatrix} + \varepsilon_2\end{aligned} \tag{136}$$

Substituindo (136) em (135), (lembrar que $x_1 = y$)

$$\begin{aligned}
\dot{y} &= x_2 - e_1^T a y \\
&= k_p v_{0,2} + \xi_2 + \begin{bmatrix} 0 & \Xi_{(2)} \end{bmatrix} \begin{bmatrix} k_p \\ a \end{bmatrix} + \varepsilon_2 - e_1^T a y \\
&= k_p v_{0,2} + \xi_2 + \underbrace{\begin{bmatrix} 0 & \left(\Xi_{(2)} - y e_1^T\right) \end{bmatrix}}_{\bar{\omega}^T} \begin{bmatrix} k_p \\ a \end{bmatrix} + \varepsilon_2
\end{aligned}$$

Definindo

$$\boxed{\omega^T = \begin{bmatrix} v_{0,2} & \left(\Xi_{(2)} - y e_1^T\right) \end{bmatrix}} \quad \text{Vetor regressor} \tag{137}$$

$$\boxed{\bar{\omega}^T = \begin{bmatrix} 0 & \left(\Xi_{(2)} - y e_1^T\right) \end{bmatrix}} \quad \text{Vetor regressor truncado} \tag{138}$$

temos que

$$\begin{aligned}
\dot{y} &= \xi_2 + \omega^T \theta + \varepsilon_2 \\
&= k_p v_{0,2} + \xi_2 + \bar{\omega}^T \theta + \varepsilon_2
\end{aligned} \tag{139}$$

Algoritmo de projeto

Sistema original:
$$\begin{cases} \dot{x}_1 = x_2 - a_1 y \\ \dot{x}_2 = \qquad - a_0 y + k_p u \end{cases}$$

Ideia do BKST:

- Utilizar x_2 como controle *virtual* para a 1a. equação.
- Utilizar u como controle para a 2a. equação.

Problema: O estado não é medido !!

Solução: Utilizamos o sistema com o estado do observador:

$$\begin{cases} \dot{y} = k_p\, v_{0,2} + \xi_2 + \bar{\omega}^T\theta + \varepsilon_2 \\ \dot{v}_{0,2} = -k_2 v_{0,1} + u \end{cases} \tag{140}$$

- ★ Lembrar que : $\dot{v}_0 = A_0 v_0 + e_2 u$ $\dot{\xi} = A_0 \xi + ky$
- ★ Neste sistema o estado está disponível !!

Objetivo do controle: Rastrear o sinal de referência y_r.

Pelo método BSKT, a variável de controle para a equação de $\dot{y}$ é $v_{0,2}$.

- Vamos projetar $v_{0,2}$ tal que $y \to y_r$, isto é, vamos fazer

$$\boxed{z_1 = y - y_r} \to 0$$

- Do mesmo modo, o projeto de $v_{0,2}$ é tal que

$$\boxed{z_2 = v_{0,2} - \alpha} \to 0$$

★ α é uma função estabilizante.

Por conveniência, vamos utilizar a seguinte transformação de coordenadas:

$$\begin{cases} z_1 = y - y_r \\ z_2 = v_{0,2} - \hat{\rho}\dot{y}_r - \alpha \end{cases} \tag{141}$$

onde $\hat{\rho}$ é uma estima de $\boxed{\rho = \frac{1}{k_p}}$.

- ★ O termo $\hat{\rho}\dot{y}_r$ é introduzido somente por conveniência.
- ★ Este termo em vermelho poderia estar contido em α.

Objetivo do projeto: Fazer $z \to 0$.

1o. Passo Começamos pela dinâmica de z_1:

$$\begin{aligned}\dot{z}_1 &= \dot{y} - \dot{y}_r \\ &= k_p v_{0,2} + \xi_2 + \bar{\omega}^T\theta + \varepsilon_2 - \dot{y}_r\end{aligned}$$

Substituindo $v_{0,2}$, tem-se

$$\begin{aligned}\dot{z}_1 &= k_p \underbrace{v_{0,2}} + \xi_2 + \bar{\omega}^T\theta + \varepsilon_2 - \dot{y}_r \\ &= k_p\big(z_2 + \hat{\rho}\dot{y}_r + \alpha\big) + \xi_2 + \bar{\omega}^T\theta + \varepsilon_2 - \dot{y}_r \\ &= k_p z_2 + k_p\,\alpha + \xi_2 + \bar{\omega}^T\theta + \varepsilon_2 + \big(\underbrace{k_p\hat{\rho} - 1}_{X}\big)\dot{y}_r\end{aligned}$$

Porém,

$$X = k_p\hat{\rho} - 1 = k_p\big(\rho - \tilde{\rho}\big) - 1 = \underbrace{k_p\rho}_{1} - k_p\tilde{\rho} - 1 = -k_p\,\tilde{\rho}$$

Portanto,

$$\dot{z}_1 = \underbrace{k_p\,\alpha}_{} + \xi_2 + \bar{\omega}^T\theta + \varepsilon_2 - k_p\,\tilde{\rho}\,\dot{y}_r + k_p z_2$$

A função α deveria ser da forma $\alpha = \frac{1}{k_p}\big(\underbrace{-c_1 z_1 - \xi_2 + \cdots}_{\bar{\alpha}}\big)$

Porém, como $\frac{1}{k_p}$ é desconhecido, vamos empregar a sua estimativa $\hat{\rho}$,

$$\boxed{\alpha = \hat{\rho}\bar{\alpha}}$$

Então,

$$\dot{z}_1 = k_p\,\hat{\rho}\bar{\alpha} + \xi_2 + \bar{\omega}^T\theta + \varepsilon_2 - k_p\,\tilde{\rho}\,\dot{y}_r + k_p z_2$$

Porém,

$$\begin{aligned} k_p\,\hat{\rho}\,\bar{\alpha} &= k_p\big(\rho - \tilde{\rho}\big)\bar{\alpha} \\ &= \underbrace{k_p\,\rho}_{1}\,\bar{\alpha} - k_p\tilde{\rho}\bar{\alpha} \\ &= \bar{\alpha} - k_p\tilde{\rho}\bar{\alpha} \end{aligned}$$

Resultado:

$$\dot{z}_1 = \bar{\alpha} + \xi_2 + \bar{\omega}^T\theta + \varepsilon_2 - k_p\left(\dot{y}_r + \bar{\alpha}\right)\tilde{\rho} + k_p z_2$$

Resultado:

$$\dot{z}_1 = \bar{\alpha} + \xi_2 + \bar{\omega}^T\theta + \varepsilon_2 - k_p\left(\dot{y}_r + \bar{\alpha}\right)\tilde{\rho} + k_p z_2$$

Em vista disto, escolhemos a função estabilizante:

$$\boxed{\bar{\alpha} = -c_1 z_1 - d_1 z_1 - \xi_2 - \bar{\omega}^T\hat{\theta}} \tag{142}$$

★ $\hat{\theta}$ é uma estimativa de θ.

★ O termo $d_1 z_1$ será necessário para compensar o erro ε_2 !!

Substituindo...

$$\dot{z}_1 = -c_1 z_1 - d_1 z_1 + \varepsilon_2 + \bar{\omega}^T\tilde{\theta} - k_p\left(\dot{y}_r + \bar{\alpha}\right)\tilde{\rho} + \underbrace{k_p z_2}_{\text{problema!}}$$

★ O termo $k_p z_2$ é um problema porque k_p é desconhecido...

Obs.: O termo $\hat{k}_p z_2$ poderia ser incluído em $\bar{\alpha}$?

O termo $\bar{\omega}^T\tilde{\theta} + k_p z_2$ pode ser escrito como:

$$
\begin{aligned}
\bar{\omega}^T\tilde{\theta} + k_p z_2 &= \bar{\omega}^T\tilde{\theta} + \big(\tilde{k}_p + \hat{k}_p\big) z_2 \\
&= \bar{\omega}^T\tilde{\theta} + \big(\underbrace{e_1^T\tilde{\theta}}_{\tilde{k}_p}\big)\big(\underbrace{v_{0,2} - \hat{\rho}\dot{y}_r - \alpha}_{z_2}\big) + \hat{k}_p z_2 \\
&= \underbrace{\bar{\omega}^T\tilde{\theta} + e_1^T\tilde{\theta}v_{0,2}}_{\omega^T\tilde{\theta}} - e_1^T\tilde{\theta}\hat{\rho}\big(\dot{y}_r + \bar{\alpha}\big) + \hat{k}_p z_2 \\
&= \big[\omega - \hat{\rho}\big(\dot{y}_r + \bar{\alpha}\big)e_1\big]^T\tilde{\theta} + \hat{k}_p z_2
\end{aligned}
$$

Substituindo...

$$
\boxed{\dot{z}_1 = -c_1 z_1 - d_1 z_1 + \varepsilon_2 + \big[\omega - \hat{\rho}\big(\dot{y}_r + \bar{\alpha}\big)e_1\big]^T\tilde{\theta} - k_p\,\big(\dot{y}_r + \bar{\alpha}\big)\tilde{\rho} + \hat{k}_p z_2}
$$

★ Note que esta equação contém somente “erros” !!

Estamos prontos para iniciar o processo de síntese das leis de adaptação para $\hat{\theta}$ e $\hat{\rho}$.

Dinâmica de z_1:

$$\dot{z}_1 = -c_1 z_1 - d_1 z_1 + \varepsilon_2 + \big[\underbrace{\omega - \hat{\rho}(\dot{y}_r + \bar{\alpha})e_1}_{\text{regressor}}\big]^T \tilde{\theta} - k_p\,(\underbrace{\dot{y}_r + \bar{\alpha}}_{\text{regressor}})\tilde{\rho} + \hat{k}_p z_2$$

★ Note que a equação acima está no formato apropriado para a análise.

Escolhemos a função de Lyapunov,

$$\boxed{2V_1 = z_1^2 + \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} + |k_p| \gamma^{-1} \tilde{\rho}^2 + \frac{1}{2d_1} \varepsilon^T P \varepsilon}$$

Lembrar que: $\boxed{\tilde{\theta} = \theta - \hat{\theta}}$ e $\boxed{\tilde{\rho} = \rho - \hat{\rho}}$

Derivando, temos:

$$
\begin{aligned}
\dot{V}_1 = {} & z_1\Big(-c_1 z_1 - d_1 z_1 + \varepsilon_2 + \big[\omega - \hat{\rho}\big(\dot{y}_r + \bar{\alpha}\big)e_1\big]^T\tilde{\theta} - k_p\,\big(\dot{y}_r + \bar{\alpha}\big)\tilde{\rho} + \hat{k}_p z_2\Big) - \\
& - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} - |k_p|\gamma^{-1}\tilde{\rho}\dot{\hat{\rho}} + \frac{1}{2d_1}\big(\dot{\varepsilon}^T P\varepsilon + \varepsilon^T P\dot{\varepsilon}\big) \\
= {} & -c_1 z_1^2 + \hat{k}_p z_1 z_2 - |k_p|\gamma^{-1}\tilde{\rho}\Big[\underbrace{\gamma\,\mathrm{sign}(k_p)\big(\dot{y}_r + \bar{\alpha}\big)z_1 + \dot{\hat{\rho}}}\Big] + \\
& + \tilde{\theta}^T\Gamma^{-1}\Big[\underbrace{\Gamma\Big(\omega - \hat{\rho}\big(\dot{y}_r + \bar{\alpha}\big)e_1\Big)z_1 - \dot{\hat{\theta}}}\Big] - d_1 z_1^2 + z_1\varepsilon_2 + \frac{1}{4d_1}\varepsilon^T\big(\underbrace{A_0^T P + P A_0}_{-I}\big)\varepsilon
\end{aligned}
$$

Para eliminar os termos em $\tilde{\theta}$ e $\tilde{\rho}$ escolhemos:

$$\boxed{\dot{\hat{\rho}} = -\gamma\,\mathrm{sign}(k_p)\big[\dot{y}_r + \bar{\alpha}\big]z_1}$$

$$\boxed{\dot{\hat{\theta}} = \Gamma\tau_1} \qquad \boxed{\tau_1 = \big[\omega - \hat{\rho}\big(\dot{y}_r + \bar{\alpha}\big)e_1\big]z_1} \qquad \text{(\textit{Tuning function \#1})}$$

Aqui aparece um ponto chave do método:

★ Não vamos implementar $\boxed{\dot{\hat{\theta}} = \Gamma \tau_1}$.

★ A variável θ ainda vai reaparecer nos próximos passos !!

Estratégia:

- Deixamos a escolha de $\dot{\hat{\theta}}$ em aberto.
- Utilizamos τ_1 como nossa 1a. função de sintonia (*Tuning function #1*).

★ $\Gamma \tau_1$ é uma primeira estimativa para $\dot{\hat{\theta}}$!

Como resultado, temos que

$$\dot{V}_1 = -c_1 z_1^2 + \hat{k}_p z_1 z_2 + \tilde{\theta}^T\left(\tau_1 - \Gamma^{-1}\dot{\hat{\theta}}\right) \underbrace{-d_1 z_1^2 + z_1\varepsilon_2 - \frac{1}{4d_1}\varepsilon^T\varepsilon}_{X}$$

O termo X pode ser escrito como:

$$\begin{aligned} X &= -d_1 z_1^2 + z_1\varepsilon_2 - \frac{1}{4d_1}\varepsilon^T\varepsilon \\ &= -d_1 z_1^2 + z_1\varepsilon_2 - \frac{1}{4d_1}\left(\varepsilon_1^2 + \varepsilon_2^2\right) \\ &= -d_1\left[z_1^2 - \frac{z_1\varepsilon_2}{d_1} + \left(\frac{1}{2d_1}\varepsilon_2\right)^2\right] - \frac{\varepsilon_1^2}{4d_1} \\ &= -d_1\left[z_1 - \frac{1}{2d_1}\varepsilon_2\right]^2 - \frac{\varepsilon_1^2}{4d_1} \leq 0 \qquad \text{(!!)} \end{aligned}$$

Portanto,

$$\boxed{\dot{V}_1 \leq -c_1 z_1^2 + \hat{k}_p z_1 z_2 + \tilde{\theta}^T\left(\tau_1 - \Gamma^{-1}\dot{\hat{\theta}}\right)}$$

2o. Passo Análise da dinâmica de z_2.

Já definimos

$$z_2 = v_{0,2} - \hat{\rho}\dot{y}_r - \alpha$$

Portanto,

$$\dot{z}_2 = \dot{v}_{0,2} - \hat{\rho}\ddot{y}_r - \dot{\hat{\rho}}\dot{y}_r - \dot{\alpha}$$

Pode-se verificar facilmente que α é função de $y, \eta, \hat{\theta}, \hat{\rho}$ e y_r :

$$\boxed{\alpha(y, \eta, \hat{\theta}, \hat{\rho}, y_r)}$$

Verificação:

$$\alpha = \hat{\rho}\bar{\alpha} = \hat{\rho}\Big[-c_1 z_1 - d_1 z_1 - \xi_2 - \bar{\omega}^T \hat{\theta}\Big]$$

$$z_1 = y - y_r$$

$$\xi = -A_0 \eta$$

$$\bar{\omega}^T = \Big[0 \quad \big(\Xi_{(2)} - y e_1^T\big)\Big]$$

$$\Xi = -\big[A_0\eta \quad \eta\big]$$

Portanto,

$$\boxed{\dot{\alpha} = \frac{\partial \alpha}{\partial y}\big(\xi_2 + \omega^T \theta + \varepsilon_2\big) + \frac{\partial \alpha}{\partial \eta}\big(A_0\eta + e_2 y\big) + \frac{\partial \alpha}{\partial y_r}\dot{y}_r + \frac{\partial \alpha}{\partial \hat{\theta}}\dot{\hat{\theta}} + \frac{\partial \alpha}{\partial \hat{\rho}}\dot{\hat{\rho}}}$$

★ Reaparece θ !! Vai ser substituído por $\tilde{\theta} + \hat{\theta}$.

Podemos escrever a equação para $\dot{z}_2$ como:

$$\begin{aligned}
\dot{z}_2 &= \underbrace{\dot{v}_{0,2}} - \hat{\rho}\ddot{y}_r - \dot{\hat{\rho}}\dot{y}_r - \dot{\alpha} \\
&= \big(u - k_2 v_{0,1}\big) - \hat{\rho}\ddot{y}_r - \dot{\hat{\rho}}\dot{y}_r - \dot{\alpha} \\
&= u - \beta - \frac{\partial \alpha}{\partial y}\big(\omega^T \tilde{\theta} + \varepsilon_2\big) - \frac{\partial \alpha}{\partial \hat{\theta}}\dot{\hat{\theta}}
\end{aligned}$$

onde β é uma função que reune somente sinais disponíveis :

$$\boxed{\beta = k_2 v_{0,1} + \frac{\partial \alpha}{\partial y}\big(\xi_2 + \omega^T \hat{\theta}\big) + \frac{\partial \alpha}{\partial \eta}\big(A_0 \eta + e_2 y\big) + \frac{\partial \alpha}{\partial y_r}\dot{y}_r + \left(\dot{y}_r + \frac{\partial \alpha}{\partial \hat{\rho}}\right)\dot{\hat{\rho}} + \hat{\rho}\ddot{y}_r}$$

★ Note que os termos que dependem de sinais não definidos $(\dot{\hat{\theta}})$ ou desconhecidos $(\tilde{\theta}$ e $\varepsilon_2)$ permanecem explícitos na equação de $\dot{z}_2$.

Neste ponto estamos prontos para continuar a análise.

Aumentamos a função de Lyapunov da seguinte forma:

$$\boxed{V_2 = V_1 + \frac{1}{2}z_2^2 + \frac{1}{4d_2}\varepsilon^T P\varepsilon}$$

Derivando,

$$\dot{V}_2 \le -c_1 z_1^2 + \hat{k}_p z_1 z_2 + \tilde{\theta}^T\left(\tau_1 - \Gamma^{-1}\dot{\hat{\theta}}\right) + \\ + z_2\left(u - \beta - \frac{\partial\alpha}{\partial y}\left(\omega^T\tilde{\theta} + \varepsilon_2\right) - \frac{\partial\alpha}{\partial\hat{\theta}}\dot{\hat{\theta}}\right) + \frac{1}{4d_2}\varepsilon^T\left(\underbrace{A_0^T P + PA_0}_{-I}\right)\varepsilon$$

Rearranjando os termos, temos

$$\dot{V}_2 \le -c_1 z_1^2 + \tilde{\theta}^T\left(\tau_1 - \frac{\partial\alpha}{\partial y}\omega z_2 - \Gamma^{-1}\dot{\hat{\theta}}\right) + \\ + z_2\left(u + \hat{k}_p z_1 - \beta - \frac{\partial\alpha}{\partial\hat{\theta}}\dot{\hat{\theta}}\right) - z_2\frac{\partial\alpha}{\partial y}\varepsilon_2 - \frac{1}{4d_2}\varepsilon^T\varepsilon$$

Repetindo...

$$\dot{V}_2 \leq -c_1 z_1^2 + \tilde{\theta}^T \Big(\tau_1 - \frac{\partial \alpha}{\partial y}\omega z_2 - \Gamma^{-1}\dot{\hat{\theta}}\Big) + \\ + z_2\Big(u + \hat{k}_p z_1 - \beta - \frac{\partial \alpha}{\partial \hat{\theta}}\dot{\hat{\theta}}\Big) - z_2 \frac{\partial \alpha}{\partial y}\varepsilon_2 - \frac{1}{4d_2}\varepsilon^T \varepsilon$$

Em vista da expressão acima, escolhemos a 2a. função de sintonia

$$\boxed{\dot{\hat{\theta}} = \Gamma \tau_2} \qquad \boxed{\tau_2 = \tau_1 - \frac{\partial \alpha}{\partial y}\omega z_2}$$

e o controle

$$\boxed{u = -c_2 z_2 - \hat{k}_p z_1 + \beta + \frac{\partial \alpha}{\partial \hat{\theta}}\Gamma \tau_2 - d_2\Big(\frac{\partial \alpha}{\partial y}\Big)^2 z_2}$$

Resultado,

$$\dot{V}_2 \leq -c_1 z_1^2 - c_2 z_2^2 \underbrace{-d_2\left(\frac{\partial \alpha}{\partial y}\right)^2 z_2^2 - z_2 \frac{\partial \alpha}{\partial y}\varepsilon_2 - \frac{1}{4d_2}\varepsilon^T\varepsilon}_{X}$$

O termo X pode ser escrito como:

$$\begin{aligned} X &= -d_2\left(\frac{\partial \alpha}{\partial y}\right)^2 z_2^2 - z_2 \frac{\partial \alpha}{\partial y}\varepsilon_2 - \frac{1}{4d_2}\varepsilon^T\varepsilon \\ &= -d_2\left(z_2 \frac{\partial \alpha}{\partial y} + \frac{1}{2d_2}\varepsilon_2\right)^2 - \frac{1}{4d_2}\varepsilon_1^2 \leq 0 \qquad \text{(!!)} \end{aligned}$$

★ O termo X pode ser descartado !

Como resultado, obtém-se:

$$\boxed{\dot{V}_2 \leq -c_1 z_1^2 - c_2 z_2^2}$$

Resumo do algoritmo

Filtros	$\dot{\lambda} = A_0\lambda + e_2 u\,, \qquad v_0 = \lambda\,,$ $\dot{\eta} = A_0\eta + e_2 y\,, \qquad \Xi = -\begin{bmatrix} A_0\eta & \eta \end{bmatrix}, \quad \xi = -A_0^2\eta$
Regressores	$\omega^T = \begin{bmatrix} v_{0,2} & (\Xi_{(2)} - y e_1^T) \end{bmatrix}$ $\bar{\omega}^T = \begin{bmatrix} 0 & (\Xi_{(2)} - y e_1^T) \end{bmatrix}$
Erros	$z_1 = y - y_r$ $z_2 = v_{0,2} - \hat{\rho}\dot{y}_r - \alpha$
Função estabilizante	$\bar{\alpha} = -c_1 z_1 - d_1 z_1 - \xi_2 - \bar{\omega}^T\hat{\theta}$ $\alpha = \hat{\rho}\bar{\alpha}$
Sinal auxiliar	$\beta = k_2 v_{0,1} + \dfrac{\partial\alpha}{\partial y}(\xi_2 + \omega^T\hat{\theta}) + \dfrac{\partial\alpha}{\partial\eta}(A_0\eta + e_2 y) + \dfrac{\partial\alpha}{\partial y_r}\dot{y}_r + \left(\dot{y}_r + \dfrac{\partial\alpha}{\partial\hat{\rho}}\right)\dot{\hat{\rho}} + \hat{\rho}\ddot{y}_r$

Tuning functions	$\tau_1 = [\omega - \hat{\rho}(\dot{y}_r + \bar{\alpha})e_1]z_1$ $\tau_2 = \tau_1 - \dfrac{\partial \alpha}{\partial y}\omega z_2$
Adaptação	$\dot{\hat{\rho}} = -\gamma\,\mathrm{sign}(k_p)[\dot{y}_r + \bar{\alpha}]z_1$ $\dot{\hat{\theta}} = \Gamma\tau_2$
Controle	$u = -c_2 z_2 - \hat{k}_p z_1 + \beta + \dfrac{\partial \alpha}{\partial \hat{\theta}}\Gamma\tau_2 - d_2\left(\dfrac{\partial \alpha}{\partial y}\right)^2 z_2$

8.5.4 Simulações

(...)

8.6 Backstepping with reduced order observer

So far, only full order observers are being used.

8.6.1 Indirect adaptation

- ★ This is the usual approach for the Backstepping method.
- ★ The plant parameters are identified by the algorithm and then used to generate the control action.

Example 29 2nd order plant.

Plant:
$$\boxed{y = \frac{k_p}{s^2 + a_1 s + a_0} u}$$

In this case:
$$\begin{cases} m = 0 \\ n = 2 \\ \rho = n - m = 2 \\ b_0 = k_p \end{cases}$$

Observable canonical form:
$$\begin{cases} \dot{x}_1 = -a_1 x_1 + x_2 \\ \dot{x}_2 = -a_0 x_1 \qquad + k_p u \\ y = x_1 \end{cases}$$

Using matrix notation:
$$\begin{cases} \dot{x} = \begin{bmatrix} -a_1 & 1 \\ -a_0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ k_p \end{bmatrix} u \\ y = e_1^T x = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$$

Define

$$\boxed{\chi = x_2 + Ny} \quad \Rightarrow \quad \boxed{x_2 = \chi - Ny} \tag{143}$$

The dynamics of χ is given by

$$\begin{aligned}
\dot{\chi} &= (-a_0 y + k_p u) + N(x_2 - a_1 y) \\
&= N x_2 - (a_0 + N a_1) y + k_p u \\
&= N(\chi - Ny) - (a_0 + N a_1) y + k_p u \\
&= N\chi - N^2 y \underbrace{-(a_0 + N a_1) y + k_p u}_{F^T\theta} \quad \Rightarrow \quad \boxed{\dot{\chi} = N\chi - N^2 y + F^T\theta}
\end{aligned}$$

where $F^T = \begin{bmatrix} u & \vert & -Ny & -y \end{bmatrix}, \qquad \theta = \begin{bmatrix} k_p \\ \hline a \end{bmatrix} = \begin{bmatrix} k_p \\ \hline a_1 \\ a_0 \end{bmatrix}$

Reduced order system: $$\boxed{\dot{\chi} = N\chi - N^2 y + F^T\theta}$$

The K filters are:
$$\begin{cases} \dot{\xi} = N\xi - N^2 y \\ \dot{\Omega}^T = N\Omega + F^T \end{cases}$$

The reduced order observer is given by:
$$\begin{cases} \hat{\chi} = \xi + \Omega^T\theta \\ \hat{x}_1 = x_1 \\ \hat{x}_2 = \hat{\chi} - Ny \end{cases} \tag{144}$$

The estimate error is defined as $$\boxed{\varepsilon = \chi - \hat{\chi}}$$

$$\Rightarrow \quad \boxed{\dot{\varepsilon} = N\varepsilon} \quad \text{with} \quad \boxed{N < 0}$$

★ Recall that θ in unknown. The observer cannot be implemented.

Reduced order filters

Recall that $F^T = \left[u \;\middle|\; -Ny \quad -y\right]$

Let's partition the filtered F^T as: $\Omega^T \equiv \left[\; v_0 \;\middle|\; \Xi \;\right]$

Introduce the filters
$$\begin{cases} \dot{\lambda} = N\lambda + u & (145) \\ \dot{\eta} = N\eta + y & (146) \end{cases}$$

Then,
$$\begin{cases} v_0 = \lambda \\ \Xi = \left[-N\eta \quad -\eta\right] \\ \xi = -N^2\eta \end{cases}$$

As a result,

$$\Omega^T \equiv \left[\; v_0 \;\middle|\; -N\eta \quad -\eta \;\right] \tag{147}$$

From (144) one gets

$$\begin{aligned} x_2 &= \chi - Ny \\ &= \hat{\chi} + \varepsilon - Ny \\ &= \xi + \Omega^T \theta - Ny + \varepsilon \end{aligned}$$

Since $\Omega^T = \begin{bmatrix} v_0 & \Xi \end{bmatrix}$, then

$$\boxed{x_2 = \xi + \begin{bmatrix} v_0 & \Xi \end{bmatrix} \begin{bmatrix} k_p \\ a \end{bmatrix} - Ny + \varepsilon} \tag{148}$$

The BKST design starts from $\dot{x}_1$ equation:

$$\begin{aligned}
\dot{y} = \dot{x}_1 &= x_2 - a_1 y \\
&= \xi + \begin{bmatrix} v_0 & \Xi \end{bmatrix} \begin{bmatrix} k_p \\ a \end{bmatrix} - Ny + \varepsilon - a_1 y \\
&= k_p v_0 + \xi + \begin{bmatrix} 0 & -(N\eta + y) & -\eta \end{bmatrix} \begin{bmatrix} k_p \\ a_1 \\ a_0 \end{bmatrix} - Ny + \varepsilon
\end{aligned} \tag{149}$$

Define

$$\bar{\omega}^T = \begin{bmatrix} 0 & -(N\eta + y) & -\eta \end{bmatrix} \tag{150}$$

then

$$\dot{y} = -Ny + k_p v_0 + \xi + \bar{\omega}^T \theta + \varepsilon \tag{151}$$

Tuning functions design

Design system:

$$\dot{y} = -Ny + k_p v_0 + \xi + \bar{\omega}^T \theta + \varepsilon \tag{152}$$

$$\dot{v}_0 = N v_0 + u \tag{153}$$

Let's introduce the transformation

$$z_1 = y - y_r \tag{154}$$

$$z_2 = v_0 - \hat{\rho}\dot{y}_r - \alpha_1 \tag{155}$$

where $\hat{\rho}$ is an estimate of $\boxed{\rho = \dfrac{1}{k_p}}$.

★ $\hat{\rho}\dot{y}_r$ is included in z_2 only for convenience.

Step 1 From (154):

$$\begin{aligned}\dot{z}_1 &= \dot{y} - \dot{y}_r \\ &= -Ny + k_p v_0 + \xi + \bar{\omega}^T\theta - \dot{y}_r + \varepsilon\end{aligned}$$

Replacing v_0,

$$\begin{aligned}\dot{z}_1 &= -Ny + k_p\big(z_2 + \hat{\rho}\dot{y}_r + \alpha_1\big) + \xi + \bar{\omega}^T\theta - \dot{y}_r + \varepsilon \\ &= -Ny + k_p z_2 + k_p\,\alpha_1 + \xi + \bar{\omega}^T\theta + \big(\underbrace{k_p\hat{\rho} - 1}_{X}\big)\dot{y}_r + \varepsilon\end{aligned}$$

However,

$$X = k_p\hat{\rho} - 1 = k_p\Big(\hat{\rho} - \underbrace{\frac{1}{k_p}}_{\rho}\Big) = -k_p\,\tilde{\rho} \qquad \text{(where } \tilde{\rho} = \rho - \hat{\rho}\text{)}$$

Therefore,

$$\dot{z}_1 = -Ny + k_p z_2 + \underbrace{k_p\,\alpha_1} + \xi + \bar{\omega}^T\theta - k_p\,\tilde{\rho}\,\dot{y}_r + \varepsilon$$

Define

$$\boxed{\alpha_1 = \hat{\rho}\bar{\alpha}_1} \tag{156}$$

Then,

$$\dot{z}_1 = -Ny + k_p z_2 + \underbrace{k_p\,\hat{\rho}\bar{\alpha}_1}_{X} + \xi + \bar{\omega}^T\theta - k_p\,\tilde{\rho}\,\dot{y}_r + \varepsilon$$

However,

$$X = k_p\,\hat{\rho}\,\alpha_1 = k_p\big(\rho - \tilde{\rho}\big)\bar{\alpha}_1 = \underbrace{k_p\,\rho}_{1}\,\bar{\alpha}_1 - k_p\tilde{\rho}\bar{\alpha}_1 = \bar{\alpha}_1 - k_p\tilde{\rho}\bar{\alpha}_1$$

Replacing,

$$\dot{z}_1 = -Ny + k_p z_2 + \bar{\alpha}_1 - k_p\tilde{\rho}\bar{\alpha}_1 + \xi + \bar{\omega}^T\theta - k_p\,\tilde{\rho}\,\dot{y}_r + \varepsilon$$

After rearranging terms,

$$\dot{z}_1 = -Ny + k_p z_2 + \bar{\alpha}_1 - k_p\big(\bar{\alpha}_1 + \dot{y}_r\big)\tilde{\rho} + \xi + \bar{\omega}^T\theta + \varepsilon$$

In view of the above equation, we chose

$$\boxed{\bar{\alpha}_1 = -c_1 z_1 - d_1 z_1 - \xi - \bar{\omega}^T\hat{\theta} + Ny} \tag{157}$$

where $\hat{\theta}$ is an estimate of θ.

Replacing,

$$\dot{z}_1 = -c_1 z_1 - d_1 z_1 + \underbrace{k_p z_2}_{\text{problem!}} + \bar{\omega}^T\tilde{\theta} - k_p\left(\dot{y}_r + \bar{\alpha}_1\right)\tilde{\rho} + \varepsilon$$

★ $k_p z_2$ is a problem since k_p is not available.

However, the terms $\bar{\omega}^T\tilde{\theta} + k_p z_2$ can be rearranged as:

$$
\begin{aligned}
\bar{\omega}^T\tilde{\theta} + k_p z_2 &= \underbrace{\begin{bmatrix}0 & -(N\eta+y) & -\eta\end{bmatrix}}_{\bar{\omega}^T}\begin{bmatrix}\tilde{k}_p \\ \tilde{a}\end{bmatrix} + \big(\tilde{k}_p + \hat{k}_p\big)\Big(\underbrace{v_0 - \hat{\rho}\dot{y}_r - \alpha_1}_{z_2}\Big) \\
&= \underbrace{\begin{bmatrix}v_0 & -(N\eta+y) & -\eta\end{bmatrix}}_{\omega^T}\tilde{\theta} - \underbrace{\tilde{k}_p}_{e_1^T\tilde{\theta}}\big(\hat{\rho}\dot{y}_r + \underbrace{\alpha_1}_{\hat{\rho}\bar{\alpha}_1}\big) + \hat{k}_p z_2 \\
&= \omega^T\tilde{\theta} - e_1^T\tilde{\theta}\hat{\rho}\big(\dot{y}_r + \bar{\alpha}_1\big) + \hat{k}_p z_2 \\
&= \Big(\underbrace{\omega - \hat{\rho}\big(\dot{y}_r + \bar{\alpha}_1\big)e_1}_{\omega_0}\Big)^T\tilde{\theta} + \hat{k}_p z_2
\end{aligned}
$$

where

$$\boxed{\omega = \begin{bmatrix}v_0 & -(N\eta+y) & -\eta\end{bmatrix}} \tag{158}$$

$$\boxed{\omega_0 = \omega - \hat{\rho}\big(\dot{y}_r + \bar{\alpha}_1\big)e_1} \tag{159}$$

Replacing,

$$\dot{z}_1 = -c_1 z_1 - d_1 z_1 + \omega_0^T \tilde{\theta} - k_p \left(\dot{y}_r + \bar{\alpha}_1\right) \tilde{\rho} + \hat{k}_p z_2 + \varepsilon$$

★ This equation has only errors !

Now, consider the Lyapunov function

$$\boxed{2V_1 = z_1^2 + \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} + |k_p| \gamma^{-1} \tilde{\rho}^2 + \frac{1}{2d_1}\Big(\frac{1}{-N}\Big)\varepsilon^2}$$

Recall that $\boxed{\tilde{\theta} = \theta - \hat{\theta}}$ and $\boxed{\tilde{\rho} = \rho - \hat{\rho}}$

The derivative is given by

$$\dot{V}_1 = z_1\Big(- c_1 z_1 - d_1 z_1 + \omega_0^T \tilde{\theta} - k_p \left(\dot{y}_r + \bar{\alpha}_1\right)\tilde{\rho} + \hat{k}_p z_2 + \varepsilon\Big) - \\ - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} - |k_p| \gamma^{-1} \tilde{\rho}\dot{\hat{\rho}} + \frac{1}{4d_1}\Big(\frac{1}{-N}\Big)\varepsilon\dot{\varepsilon}$$

$$= -c_1 z_1^2 + \hat{k}_p z_1 z_2 - |k_p|\gamma^{-1}\tilde{\rho}\Big[\gamma \operatorname{sign}(k_p)\big(\dot{y}_r + \bar{\alpha}_1\big) z_1 + \dot{\hat{\rho}}\Big] + \\ + \tilde{\theta}^T\Big[\omega_0 z_1 - \Gamma^{-1}\dot{\hat{\theta}}\Big] - d_1 z_1^2 + z_1\varepsilon - \frac{1}{4d_1}\varepsilon^2$$

We choose

$$\boxed{\dot{\hat{\rho}} = -\gamma \operatorname{sign}(k_p)\big(\dot{y}_r + \bar{\alpha}\big) z_1}$$

$$\boxed{\tau_1 = \omega_0 z_1}$$

(*Tuning function #1*)

As a result, one has

$$\dot{V}_1 = -c_1 z_1^2 + \hat{k}_p z_1 z_2 + \tilde{\theta}^T \left(\tau_1 - \Gamma^{-1}\dot{\hat{\theta}}\right) \underbrace{-d_1 z_1^2 + z_1 \varepsilon - \frac{1}{4d_1}\varepsilon^2}_{X}$$

The term X can be rewritten as

$$\begin{aligned} X &= -d_1 z_1^2 + z_1 \varepsilon - \frac{1}{4d_1}\varepsilon^2 \\ &= -\begin{bmatrix} z_1 & \varepsilon \end{bmatrix} \begin{bmatrix} d_1 & 1/2 \\ 1/2 & 1/(4d_1) \end{bmatrix} \begin{bmatrix} z_1 \\ \varepsilon \end{bmatrix} \leq 0 \qquad \text{(!!)} \end{aligned}$$

Therefore, the term X can be neglected which gives

$$\boxed{\dot{V}_1 \leq -c_1 z_1^2 + \hat{k}_p z_1 z_2 + \tilde{\theta}^T \left(\tau_1 - \Gamma^{-1}\dot{\hat{\theta}}\right)}$$

Step 2 From (155)

$$z_2 = v_0 - \hat{\rho}\dot{y}_r - \alpha_1$$

Deriving,

$$\dot{z}_2 = \dot{v}_0 - \hat{\rho}\ddot{y}_r - \dot{\hat{\rho}}\dot{y}_r - \dot{\alpha}_1$$

The stabilizing function (157) can be expanded as

$$\begin{aligned}
\bar{\alpha}_1 &= -c_1 z_1 - d_1 z_1 - \xi - \bar{\omega}^T \hat{\theta} + Ny \\
&= -c_1(y - y_r) - d_1(y - y_r) - N^2\eta - \begin{bmatrix} 0 & -(N\eta + y) & -\eta \end{bmatrix} \hat{\theta} + Ny \\
&= (N - c_1 - d_1)y + (c_1 + d_1)y_r - N^2\eta + (N\eta + y)\hat{a}_1 + \eta\hat{a}_0 \qquad (160)
\end{aligned}$$

This means that

$$\boxed{\alpha_1 = \hat{\rho}\bar{\alpha}_1 = \alpha_1\big(y, \eta, \hat{\theta}, \hat{\rho}, y_r\big)}$$

The signal $\dot{\alpha}_1$ can be expressed as

$$\dot{\alpha}_1 = \frac{\partial \alpha_1}{\partial y}\dot{y} + \frac{\partial \alpha_1}{\partial \eta}\dot{\eta} + \frac{\partial \alpha_1}{\partial y_r}\dot{y}_r + \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}} + \frac{\partial \alpha_1}{\partial \hat{\rho}}\dot{\hat{\rho}}$$

Now, using (152), (153) and (146), the expression for $\dot{z}_2$ becomes

$$\begin{aligned}
\dot{z}_2 &= \dot{v}_0 - \hat{\rho}\ddot{y}_r - \dot{\hat{\rho}}\dot{y}_r - \dot{\alpha}_1 \\
&= \big(Nv_0 + u\big) - \hat{\rho}\ddot{y}_r - \dot{\hat{\rho}}\dot{y}_r - \frac{\partial \alpha_1}{\partial y}\big(- Ny + \xi + \underbrace{k_p v_0 + \bar{\omega}^T\theta}_{\omega^T\theta} + \varepsilon\big) - \frac{\partial \alpha_1}{\partial y_r}\dot{y}_r - \\
&\qquad - \frac{\partial \alpha_1}{\partial \eta}\big(N\eta + y\big) - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}} - \frac{\partial \alpha_1}{\partial \hat{\rho}}\dot{\hat{\rho}}
\end{aligned}$$

We can collect all available signals as

$$\boxed{\beta = -Nv_0 + \frac{\partial \alpha_1}{\partial y}\big(- Ny + \xi + \omega^T\hat{\theta}\big) + \frac{\partial \alpha_1}{\partial \eta}\big(N\eta + y\big) + \frac{\partial \alpha_1}{\partial y_r}\dot{y}_r + \Big(\dot{y}_r + \frac{\partial \alpha_1}{\partial \hat{\rho}}\Big)\dot{\hat{\rho}} + \hat{\rho}\ddot{y}_r}$$

Therefore,

$$\dot{z}_2 = u - \beta - \frac{\partial \alpha_1}{\partial y}\big(\omega^T \tilde{\theta} + \varepsilon\big) - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}}$$

★ All the terms that depend on signals still not defined $(\dot{\hat{\theta}})$ or unknown $(\tilde{\theta}$ and $\varepsilon_2)$ remain in the $\dot{z}_2$ equation.

Now, consider a second Lyapunov function

$$\boxed{V_2 = V_1 + \frac{1}{2}z_2^2 + \frac{1}{4d_2}\Big(\frac{1}{-N}\Big)\varepsilon^2}$$

Deriving,

$$\dot{V}_2 \leq -c_1 z_1^2 + \hat{k}_p z_1 z_2 + \tilde{\theta}^T\big(\tau_1 - \Gamma^{-1}\dot{\hat{\theta}}\big) + \\ + z_2\Big(u - \beta - \frac{\partial \alpha_1}{\partial y}\big(\omega^T \tilde{\theta} + \varepsilon\big) - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}}\Big) - \frac{1}{4d_2}\varepsilon^2$$

Rearranging,

$$\dot{V}_2 \leq -c_1 z_1^2 + \tilde{\theta}^T \Big(\tau_1 - \frac{\partial \alpha_1}{\partial y} \omega z_2 - \Gamma^{-1} \dot{\hat{\theta}} \Big) +$$
$$+ z_2 \Big(u + \hat{k}_p z_1 - \beta - \frac{\partial \alpha_1}{\partial \hat{\theta}} \dot{\hat{\theta}} \Big) - z_2 \frac{\partial \alpha_1}{\partial y} \varepsilon - \frac{1}{4 d_2} \varepsilon^2$$

In view of the above expression, we select

$$\boxed{\dot{\hat{\theta}} = \Gamma \tau_2}$$

$$\boxed{\tau_2 = \tau_1 - \frac{\partial \alpha_1}{\partial y} \omega z_2}$$

2nd tuning function

$$\boxed{u = -c_2 z_2 - \hat{k}_p z_1 + \beta + \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma \tau_2 - d_2 \Big(\frac{\partial \alpha_1}{\partial y} \Big)^2 z_2}$$

As a result one has

$$\dot{V}_2 \leq -c_1 z_1^2 - c_2 z_2^2 \underbrace{-d_2\left(\frac{\partial \alpha_1}{\partial y}\right)^2 z_2^2 - z_2 \frac{\partial \alpha_1}{\partial y}\varepsilon - \frac{1}{4d_2}\varepsilon^2}_{X}$$

The term X can be expressed as

$$\begin{aligned} X &= -d_2\left(\frac{\partial \alpha_1}{\partial y}\right)^2 z_2^2 - z_2 \frac{\partial \alpha}{\partial y}\varepsilon_2 - \frac{1}{4d_2}\varepsilon^T \varepsilon \\ &= -d_2\left(z_2 \frac{\partial \alpha_1}{\partial y} + \frac{1}{2d_2}\varepsilon_2\right)^2 - \frac{1}{4d_2}\varepsilon_1^2 \leq 0 \qquad \text{(!!)} \end{aligned}$$

★ The term X can be omitted in the $\dot{V}_2$.

The result is

$$\boxed{\dot{V}_2 \leq -c_1 z_1^2 - c_2 z_2^2}$$

Summary of the algorithm

Filters	$\dot{\lambda} = N\lambda + u\,, \qquad v_0 = \lambda\,,$ $\dot{\eta} = N\eta + y\,, \qquad \Xi = \begin{bmatrix} -N\eta & -\eta \end{bmatrix}, \quad \xi = -N^2\eta$
Regressors	$\Omega^T = \begin{bmatrix} v_0 & \Xi \end{bmatrix}$ $\bar{\omega}^T = \begin{bmatrix} 0 & -(N\eta + y) & -\eta \end{bmatrix}$ $\omega^T = \begin{bmatrix} v_0 & -(N\eta + y) & -\eta \end{bmatrix}$ $\omega_0 = \omega - \hat{\rho}\Big(\dot{y}_r + \bar{\alpha}_1\Big)e_1$
Errors	$z_1 = y - y_r$ $z_2 = v_0 - \hat{\rho}\dot{y}_r - \alpha_1$
Stabilizing function	$\bar{\alpha}_1 = -c_1 z_1 - d_1 z_1 - \xi - \bar{\omega}^T\hat{\theta} + Ny$ $\alpha_1 = \hat{\rho}\bar{\alpha}_1$

Auxiliary signal	$\beta = -Nv_0 + \frac{\partial \alpha_1}{\partial y}\big(-Ny + \xi + \omega^T\hat{\theta}\big) + \frac{\partial \alpha_1}{\partial \eta}\big(N\eta + y\big) + \frac{\partial \alpha_1}{\partial y_r}\dot{y}_r + \left(\dot{y}_r + \frac{\partial \alpha_1}{\partial \hat{\rho}}\right)\dot{\hat{\rho}} + \hat{\rho}\ddot{y}_r$
Tuning functions	$\tau_1 = \omega_0 z_1$ $\tau_2 = \tau_1 - \frac{\partial \alpha_1}{\partial y}\omega z_2$
Update laws	$\dot{\hat{\rho}} = -\gamma \operatorname{sign}(k_p)\big[\dot{y}_r + \bar{\alpha_1}\big]z_1$ $\dot{\hat{\theta}} = \Gamma\tau_2$
Control	$u = -c_2 z_2 - \hat{k}_p z_1 + \beta + \frac{\partial \alpha_1}{\partial \hat{\theta}}\Gamma\tau_2 - d_2\left(\frac{\partial \alpha_1}{\partial y}\right)^2 z_2$

Implementation

From (156) and (160),

$$
\begin{aligned}
\alpha_1 = \hat{\rho}\bar{\alpha}_1 &= \hat{\rho}\big(- c_1 z_1 - d_1 z_1 - \xi - \bar{\omega}^T \hat{\theta} + N y \big) \\
&= -c_1 \hat{\rho}(y - y_r) - d_1 \hat{\rho}(y - y_r) - N^2 \hat{\rho}\eta - \hat{\rho}\begin{bmatrix} 0 & -(N\eta + y) & -\eta \end{bmatrix} \hat{\theta} + N\hat{\rho} y \\
&= (N - c_1 - d_1)\hat{\rho} y + (c_1 + d_1)\hat{\rho} y_r - N^2 \hat{\rho}\eta + (N\eta + y)\ \hat{\rho}\hat{a}_1 + \eta\hat{\rho}\hat{a}_0
\end{aligned}
$$

Then,

$$
\frac{\partial \alpha_1}{\partial y} = \big(N - c_1 - d_1 + \hat{a}_1\big)\hat{\rho} \qquad \frac{\partial \alpha_1}{\partial y_r} = \big(c_1 + d_1\big)\hat{\rho}
$$

$$
\frac{\partial \alpha_1}{\partial \eta} = \big(N^2 + N\hat{a}_1 + \hat{a}_0\big)\hat{\rho} \qquad \frac{\partial \alpha_1}{\partial \hat{\theta}} = -\hat{\rho}\bar{\omega}^T
$$

$$
\frac{\partial \alpha_1}{\partial \hat{\rho}} = \bar{\alpha}_1
$$

8.6.2 Simulations

Plant :
$$\boxed{P(s) = \frac{1}{s^2}}$$

Model :
$$\boxed{M(s) = \frac{2}{(s+1)(s+2)}}$$

Simulation #1

Parameters..............: $\gamma = 1$

$\Gamma = 1\, I$

$N = -1$

$c_1 = 1\,, \qquad d_1 = 1\,, \qquad c_2 = 1\,, \qquad d_2 = 1$

Initial conditions......: $y(0) = 3$

$\theta(0) = [0 \;\; 0 \;\; 0]^T$

$\rho(0) = 0$

Reference signal.......: $r = 2 + 5\sin(t) + 5\sin(3.7t)$

Corrigir. y_r é a saída do modelo de referência. Juntamente com $\dot{y}_r$ e $\ddot{y}_r$.

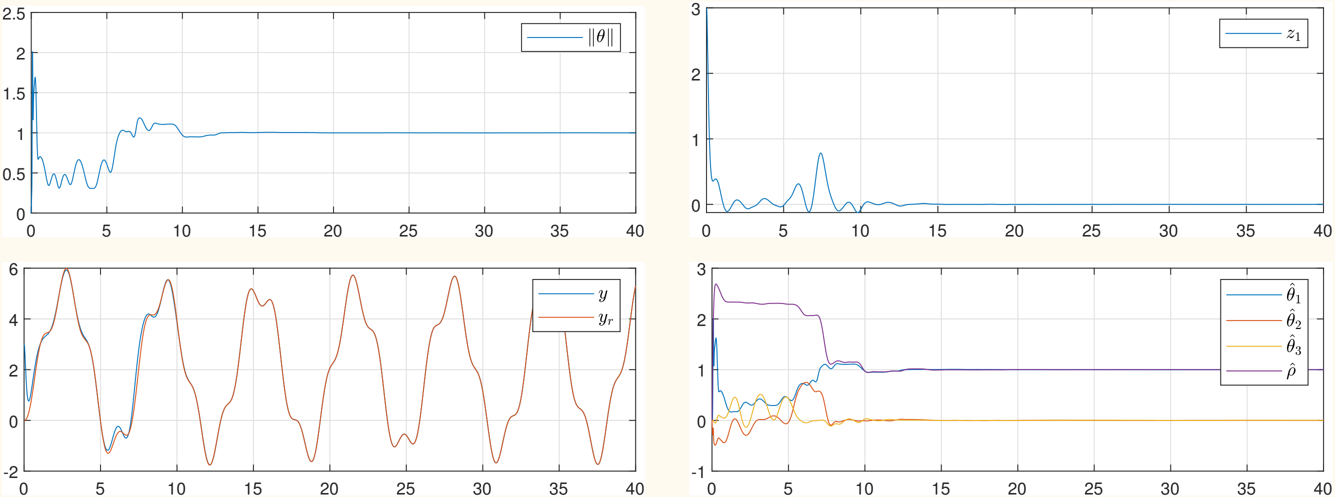


Figura 107: Simulation results with BKST using reduced order observer. (Script: `fig01.m`)

Simulation #2

Parameters...............:

$$\gamma = 10$$

$$\Gamma = 10\, I$$

$$N = -1$$

$$c_1 = 1\,, \qquad d_1 = 1\,, \qquad c_2 = 1\,, \qquad d_2 = 1$$

Initial conditions.......:

$$y(0) = 3$$

$$\theta(0) = [0 \quad 0 \quad 0]^T$$

$$\rho(0) = 0$$

Reference signal........:

$$y_r = 2 + 5\sin(t) + 5\sin(3.7t)$$

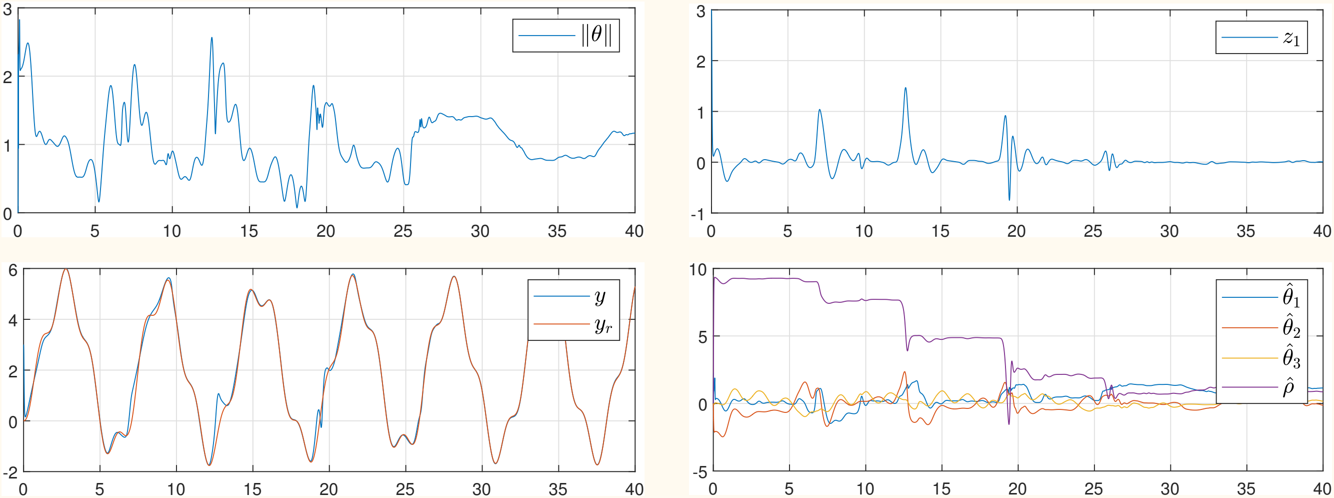


Figura 108: Simulation results with BKST using reduced order observer. (Script: `fig02.m`)

Simulation #3

Parameters..............:

$$\gamma = \boxed{10}$$

$$\Gamma = \boxed{1\, I}$$

$$N = -1$$

$$c_1 = 1\,, \qquad d_1 \;= 1\,, \qquad c_2 = 1\,, \qquad d_2 \;= 1$$

Initial conditions......:

$$y(0) = 3$$

$$\theta(0) = [0 \quad 0 \quad 0]^T$$

$$\rho(0) = 0$$

Reference signal.......: $y_r = 2 + 5\sin(t) + 5\sin(3.7t)$

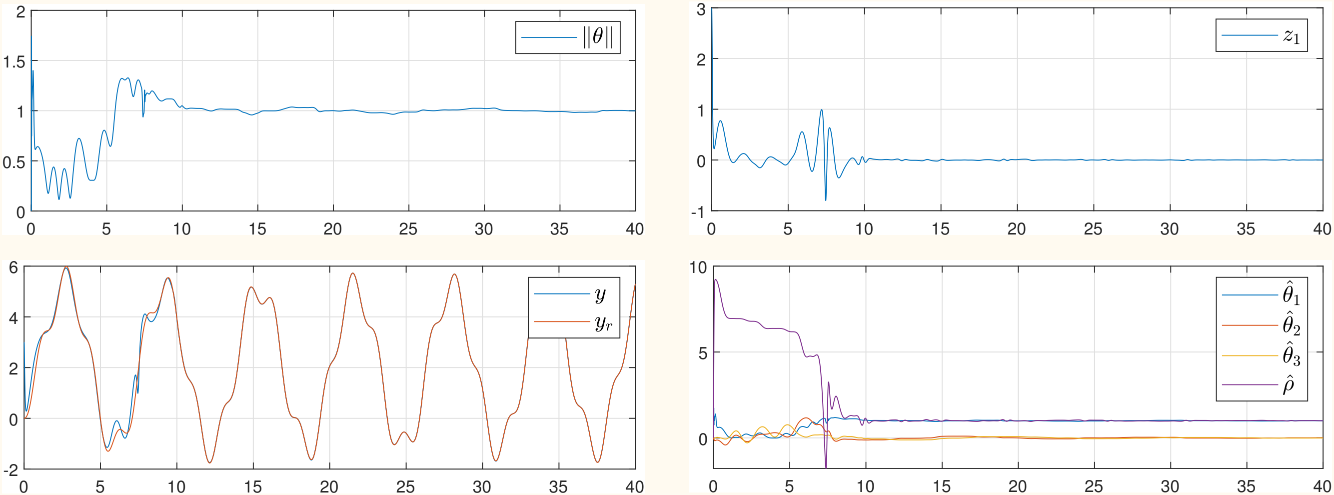


Figura 109: Simulation results with BKST using reduced order observer. (Script: `fig03.m`)

8.6.3 Direct adaptation

- ★ In this case, the controller's parameters are identified.
- ★ The idea is to apply the BKST methodology to the error equation.

Example 30 2nd order plant.

Plant:
$$y = \frac{k_p}{s^2 + a_1 s + a_0} u$$

Case: $m = 0$

$n = 2$ (plant order)

$\rho = 2$ (relative degree)

Formulação com observador de ordem completa

Modelo: $M(s) = \dfrac{1}{s^2 + a_{m_1} s + a_{m_0}}$

Eq. do erro: $e_0 = M(s) k_p \big[u - \theta^{*T}\omega\big]$

Realização na forma canônica observável:

$$\begin{cases} \dot{\epsilon} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \epsilon - \begin{bmatrix} a_{m_1} \\ a_{m_0} \end{bmatrix} e_0 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} k_p \big(u - \theta^{*T}\omega\big) \, , \\ e_0 = \begin{bmatrix} 1 & 0 \end{bmatrix} \epsilon = e_1^T \epsilon = \epsilon_1 \, , \end{cases}$$

onde

$$\theta^* = \begin{bmatrix} \theta_1^* \\ \theta_2^* \\ \theta_3^* \\ 1/k_p \end{bmatrix}, \quad \omega = \begin{bmatrix} \omega_1 \\ y \\ \omega_2 \\ r \end{bmatrix}.$$

Observar que

$$k_p\theta^* = \begin{bmatrix} k_p\theta_1^* \\ k_p\theta_2^* \\ k_p\theta_3^* \\ 1 \end{bmatrix} := \begin{bmatrix} \psi_1^* \\ \psi_2^* \\ \psi_3^* \\ 1 \end{bmatrix}.$$

Seguindo a formulação do algoritmo, os parâmetros a serem identificados são:

$$\Psi^* = \begin{bmatrix} k_p & \psi_1^* & \psi_2^* & \psi_3^* \end{bmatrix}^T.$$

Definindo

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad a_m = \begin{bmatrix} a_{m_0} \\ a_{m_1} \end{bmatrix},$$

a equação do erro pode ser escrita como

$$\begin{aligned} \dot{e} &= Ae - a_m e_0 + \begin{bmatrix} 0 \\ k_p \end{bmatrix} u - \begin{bmatrix} 0 \\ k_p \theta^{*T} \end{bmatrix} \omega \\ &= A_0 e + k e_0 - a_m e_0 + \begin{bmatrix} 0 \\ k_p \end{bmatrix} u - \begin{bmatrix} 0 \\ \psi^{*T} \end{bmatrix} \omega_r - \begin{bmatrix} 0 \\ 1 \end{bmatrix} r, \end{aligned}$$

onde

$$k = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}, \qquad A_0 = A - k e_1^T = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix},$$

$$\omega_r = \begin{bmatrix} \omega_1 \\ y \\ \omega_2 \end{bmatrix} \quad \Rightarrow \quad \omega = \begin{bmatrix} \omega_r^T & r \end{bmatrix}.$$

Introduzindo os sinais

$$F^T = \begin{bmatrix} 0 & 0 \\ u & -\omega_r^T \end{bmatrix}, \qquad \Psi = \begin{bmatrix} k_p \\ \psi^* \end{bmatrix},$$

a equação do erro resulta

$$\dot{e} = A_0 e + (k - a_m)e_0 - e_2^T r + F^T \Psi .$$

(...)

Observador de ordem reduzida

Modelo: $M(s) = \dfrac{1}{s^2 + a_{m_1} s + a_{m_0}}$

Eq. do erro: $e_0 = M(s) k_p \big[u - \theta^{*T} \omega \big]$

Realização na forma canônica observável:

$$\begin{cases} \dot{\epsilon} = \begin{bmatrix} -a_{m_1} & 1 \\ -a_{m_0} & 0 \end{bmatrix} \epsilon + \begin{bmatrix} 0 \\ 1 \end{bmatrix} k_p \big(u - \theta^{*T} \omega \big) , \\ e_0 = \begin{bmatrix} 1 & 0 \end{bmatrix} \epsilon = e_1^T \epsilon = \epsilon_1 , \end{cases}$$

onde

$$\theta^* = \begin{bmatrix} \theta_1^* \\ \theta_2^* \\ \theta_3^* \\ 1/k_p \end{bmatrix}, \quad \omega = \begin{bmatrix} \omega_1 \\ y \\ \omega_2 \\ r \end{bmatrix}.$$

Definindo

$$\chi = \epsilon_2 + N e_0 \qquad \epsilon_2 = \chi - N e_0 \,,$$

tem-se que

$$\begin{aligned}
\dot{\chi} &= \dot{\epsilon_2} + N\dot{\epsilon_1} \\
&= \big[- a_{m_0} e_0 + k_p (u - \theta^{*T} \omega) \big] + N \big(- a_{m_1} e_0 + \epsilon_2 \big) \\
&= -\big(a_{m_0} + N a_{m_1} \big) e_0 + N(\chi - N e_0) + k_p (u - \theta^{*T} \omega) \\
&= -\big(a_{m_0} + N a_{m_1} + N^2 \big) e_0 + N\chi + k_p (u - \theta^{*T} \omega) \,.
\end{aligned}$$

Observando que

$$k_p\theta^* = \begin{bmatrix} k_p\theta_1^* \\ k_p\theta_2^* \\ k_p\theta_3^* \\ 1 \end{bmatrix} := \begin{bmatrix} \psi_1^* \\ \psi_2^* \\ \psi_3^* \\ 1 \end{bmatrix} = \begin{bmatrix} \psi^* \\ 1 \end{bmatrix},$$

então

$$\begin{aligned} k_p(u - \theta^{*T}\omega) &= k_p u - k_p\theta^{*T}\omega \\ &= k_p u - \psi^{*T}\omega_r - r \\ &= F^T\Psi^* - r\,, \end{aligned}$$

onde

$$F^T = \begin{bmatrix} u & -\omega_r^T \end{bmatrix}, \qquad \Psi^* = \begin{bmatrix} k_p \\ \psi^* \end{bmatrix},$$

$$\omega_r = \begin{bmatrix} \omega_1 \\ y \\ \omega_2 \end{bmatrix} \quad \Rightarrow \quad \omega = \begin{bmatrix} \omega_r^T & r \end{bmatrix}^T.$$

Portanto,

$$
\begin{aligned}
\dot{\chi} &= N\chi - \big(\underbrace{a_{m_0} + N a_{m_1} + N^2}_{c}\big)e_0 + F^T\Psi^* - r \\
&= N\chi - c e_0 - r + F^T\Psi^* \,.
\end{aligned}
$$

(...)

8.7 Caso $n^* = 3$

Ref.: [KKK:1995], (pag. 417)

★ *Backstepping* adaptativo para sistemas lineares.

8.7.1 Rastreamento com parâmetro desconhecido

Example 31 Sistema de 3a. ordem.

Planta:
$$\boxed{\; y = \frac{k_p}{s^3 + a_2 s^2 + a_1 s + a_0} u \;}$$

Neste exemplo:
$$m = 0$$
$$n = 3$$
$$\rho = n - m = 3$$
$$b_0 = k_p$$

Forma canônica observável:

$$
\begin{cases}
\dot{x}_1 = x_2 - a_2 y \\
\dot{x}_2 = x_3 - a_1 y \\
\dot{x}_3 = \quad - a_0 y + k_p u \\
\ y = x_1
\end{cases}
$$

Ou melhor,

$$
\begin{cases}
\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{A} x - \underbrace{\begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} y + \begin{bmatrix} 0 \\ 0 \\ k_p \end{bmatrix} u}_{F(y,u)^T\theta} \\
y = e_1^T x
\end{cases}
$$

Para definir o observador, escrevemos o sistema na forma

$$\begin{cases} \dot{x} = Ax + F(y,u)^T\theta \\ y = e_1^T x \end{cases} \tag{161}$$

onde,

$$F(y,u)^T = \left[\begin{array}{c|ccc} 0 & -y & 0 & 0 \\ 0 & 0 & -y & 0 \\ u & 0 & 0 & -y \end{array}\right], \qquad \theta = \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} k_p \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}$$

★ Nesta forma, os parâmetros da planta são arranjados no vetor θ.

★ Em consequência, o regressor é escrito na forma de uma matrix $F(y,u)^T$.

Importante: O mesmo parâmetro θ é usado para o observador e para o controle !

Para estimar o estado, utilizamos os filtros abaixo:

$$\text{Filtros} \quad \begin{cases} \dot{\xi} = A_0\xi + ky \\ \dot{\Omega}^T = A_0\Omega^T + F^T \qquad (\Omega \text{ é matrix ! }) \end{cases} \tag{162}$$

onde,

$$k = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \qquad \text{deve ser escolhido tal que}$$

$$A_0 = A - ke_1^T = \begin{bmatrix} -k_1 & 1 & 0 \\ -k_2 & 0 & 1 \\ -k_3 & 0 & 0 \end{bmatrix} \qquad \text{seja Hurwitz !}$$

O estado estimado é dado por:

$$\boxed{\hat{x} = \xi + \Omega^T \theta} \tag{163}$$

★ Este estimador **não pode ser implementado**. θ é desconhecido !!

Verificação: Escrevemos a equação (161) na forma

$$\begin{aligned} \dot{x} &= Ax + F^T \theta \\ &= Ax - \underbrace{k e_1^T x + k e_1^T x}_{0} + F^T \theta \\ &= A_0 x + ky + F^T \theta \end{aligned} \tag{164}$$

A dinâmica do observador é dada por:

$$
\begin{aligned}
\dot{\hat{x}} = \dot{\xi} + \dot{\Omega}^T\theta &= \big(A_0\xi + ky\big) + \big(A_0\Omega^T + F^T\big)\theta \\
&= A_0\big(\underbrace{\xi + \Omega^T\theta}_{\hat{x}}\big) + ky + F^T\theta \\
&= A_0\hat{x} + ky + F^T\theta \qquad (165)
\end{aligned}
$$

★ Compare (164) e (165) !!

A equação (165) também pode ser escrita como:

$$
\begin{aligned}
\dot{\hat{x}} &= A_0\hat{x} + ky + F^T\theta \\
&= \big(A - ke_1^T\big)\hat{x} + ky + F^T\theta \\
&= \underbrace{A\hat{x} + F^T\theta}_{\text{Cópia}} + \underbrace{k\big(y - \hat{y}\big)}_{\text{Realimentação !}}
\end{aligned}
$$

Definindo o erro de estima

$$\boxed{\varepsilon = x - \hat{x}} \tag{166}$$

verifica-se que sua dinâmica satisfaz

$$\boxed{\dot{\varepsilon} = A_0 \varepsilon} \tag{167}$$

Propriedade importante: $\varepsilon(t) \to 0$ exponencialmente

$\exists P = P^T > 0$ tal que $\boxed{PA_0 + A_0^T P = -I}$

Da equação (166) obtemos a seguinte relação:

$$x = \hat{x} + \varepsilon \quad \Rightarrow \quad \boxed{x = \xi + \Omega^T \theta + \varepsilon} \tag{168}$$

Redução da ordem dos filtros:

$$\boxed{\dot{\Omega}^T = A_0 \Omega^T + F^T}$$

Renomeamos as colunas de Ω^T como: $\Omega^T \equiv \left[\, v_0 \;\middle|\; \Xi \,\right]$

e lembrando que $$F(y,u)^T = \left[\begin{array}{c|ccc} 0 & -y & 0 & 0 \\ 0 & 0 & -y & 0 \\ u & 0 & 0 & -y \end{array}\right]$$

Definam-se os filtros

$$\boxed{\dot{\lambda} = A_0\lambda + e_3 u} \quad \text{Filtro-}\lambda \tag{169}$$

$$\boxed{\dot{\eta} = A_0\eta + e_3 y} \quad \text{Filtro-}\eta \tag{170}$$

De onde se tem:

$$\boxed{v_0 = \lambda} \tag{171}$$

$$\boxed{\xi = -A_0^3\eta} \tag{172}$$

$$\boxed{\Xi = -\begin{bmatrix} A_0^2\eta & A_0\eta & \eta \end{bmatrix}} \tag{173}$$

O projeto BKST começa pela equação da saída, i.e., pela equação para $\dot{x}_1$,

$$\begin{aligned} \dot{x}_1 &= x_2 - a_2 y \\ &= x_2 - e_1^T a y \end{aligned} \tag{174}$$

Porém, (vide equação (168))

$$x = \xi + \Omega^T \theta + \varepsilon = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} + \begin{bmatrix} v_{0,1} & \Xi_{(1)} \\ v_{0,2} & \Xi_{(2)} \\ v_{0,3} & \Xi_{(3)} \end{bmatrix} \begin{bmatrix} k_p \\ a \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix}$$

Para a 2a. linha dessa equação, temos

$$\begin{aligned} x_2 &= \xi_2 + \begin{bmatrix} v_{0,2} & \Xi_{(2)} \end{bmatrix} \begin{bmatrix} k_p \\ a \end{bmatrix} + \varepsilon_2 \\ &= k_p v_{0,2} + \xi_2 + \begin{bmatrix} 0 & \Xi_{(2)} \end{bmatrix} \begin{bmatrix} k_p \\ a \end{bmatrix} + \varepsilon_2 \end{aligned} \tag{175}$$

Substituindo (175) em (174), (lembrar que $x_1 = y$)

$$
\begin{aligned}
\dot{y} &= x_2 - e_1^T a y \\
&= k_p v_{0,2} + \xi_2 + \begin{bmatrix} 0 & \Xi_{(2)} \end{bmatrix} \begin{bmatrix} k_p \\ a \end{bmatrix} + \varepsilon_2 - e_1^T a y \\
&= k_p v_{0,2} + \xi_2 + \underbrace{\begin{bmatrix} 0 & \left(\Xi_{(2)} - y e_1^T\right) \end{bmatrix}}_{\bar{\omega}^T} \begin{bmatrix} k_p \\ a \end{bmatrix} + \varepsilon_2
\end{aligned}
$$

Definindo

$$\boxed{\omega^T = \begin{bmatrix} v_{0,2} & \left(\Xi_{(2)} - y e_1^T\right) \end{bmatrix}} \quad \text{Vetor regressor} \tag{176}$$

$$\boxed{\bar{\omega}^T = \begin{bmatrix} 0 & \left(\Xi_{(2)} - y e_1^T\right) \end{bmatrix}} \quad \text{Vetor regressor truncado} \tag{177}$$

temos que

$$
\begin{aligned}
\dot{y} &= \xi_2 + \omega^T \theta + \varepsilon_2 \\
&= k_p v_{0,2} + \xi_2 + \bar{\omega}^T \theta + \varepsilon_2
\end{aligned} \tag{178}
$$

Algoritmo de projeto

Sistema original:

$$\text{Planta} \begin{cases} \dot{x}_1 = x_2 - a_2 y \\ \dot{x}_2 = x_3 - a_1 y \\ \dot{x}_3 = \quad - a_0 y + k_p u \end{cases}$$

Ideia do BKST:

- Utilizar x_2 como controle *virtual* para a 1a. equação.
- Utilizar x_3 como controle *virtual* para a 2a. equação.
- Utilizar u como controle para a 3a. equação.

Problema: O estado não é medido !!

Solução: Utilizamos o sistema com o estado do observador:

$$\begin{cases} \dot{y} = k_p\, v_{0,2} + \xi_2 + \bar{\omega}^T\theta + \varepsilon_2 \\ \dot{v}_{0,2} = v_{0,3} - k_2 v_{0,1} \\ \dot{v}_{0,3} = \qquad -k_3 v_{0,1} + u \end{cases} \tag{179}$$

★ Lembrar que : $\boxed{\dot{v}_0 = A_0 v_0 + e_3 u}$ $\boxed{\dot{\xi} = A_0 \xi + k y}$

★ Neste sistema o estado está disponível !!

Objetivo do controle: Rastrear o sinal de referência y_r.

Pelo método BSKT, a variável de controle para a equação de $\dot{y}$ é $v_{0,2}$.

- Vamos projetar $v_{0,2}$ tal que $y \to y_r$, isto é, vamos fazer

$$\boxed{z_1 = y - y_r} \to 0$$

- Do mesmo modo, o projeto de $v_{0,2}$ é tal que

$$\boxed{z_2 = v_{0,2} - \alpha_1} \to 0$$

- E o projeto de $v_{0,3}$ é tal que

$$\boxed{z_3 = v_{0,3} - \alpha_2} \to 0$$

★ α_1 e α_2 são funções estabilizantes.

Por conveniência, vamos utilizar a seguinte transformação de coordenadas:

$$\boxed{\begin{aligned} z_1 &= y - y_r \\ z_2 &= v_{0,2} - \hat{\rho}\dot{y}_r - \alpha_1 \\ z_3 &= v_{0,3} - \hat{\rho}\ddot{y}_r - \alpha_2 \end{aligned}} \tag{180}$$

onde $\hat{\rho}$ é uma estima de $\boxed{\rho = \frac{1}{k_p}}$.

- ★ Os termos $\hat{\rho}\dot{y}_r$ e $\hat{\rho}\ddot{y}_r$ são introduzidos somente por conveniência.
- ★ Estes termos em vermelho poderiam estar contidos em α_1 e α_2.

Objetivo do projeto: Fazer $z \to 0$.

1o. Passo Começamos pela dinâmica de z_1:

$$\begin{aligned}\dot{z}_1 &= \dot{y} - \dot{y}_r \\ &= k_p v_{0,2} + \xi_2 + \bar{\omega}^T\theta + \varepsilon_2 - \dot{y}_r\end{aligned}$$

Substituindo $v_{0,2}$, tem-se

$$\begin{aligned}\dot{z}_1 &= k_p \underbrace{v_{0,2}} + \xi_2 + \bar{\omega}^T\theta + \varepsilon_2 - \dot{y}_r \\ &= k_p\big(z_2 + \hat{\rho}\dot{y}_r + \alpha_1\big) + \xi_2 + \bar{\omega}^T\theta + \varepsilon_2 - \dot{y}_r \\ &= k_p z_2 + k_p\,\alpha_1 + \xi_2 + \bar{\omega}^T\theta + \varepsilon_2 + \big(\underbrace{k_p\hat{\rho} - 1}\big)\dot{y}_r\end{aligned}$$

Porém,

$$k_p\hat{\rho} - 1 = k_p\big(\rho - \tilde{\rho}\big) - 1 = \underbrace{k_p\rho}_{1} - k_p\tilde{\rho} - 1 = -k_p\,\tilde{\rho}$$

Portanto,

$$\dot{z}_1 = \underbrace{k_p\,\alpha_1} + \xi_2 + \bar{\omega}^T\theta + \varepsilon_2 - k_p\,\tilde{\rho}\,\dot{y}_r + k_p z_2$$

A função α_1 deveria ser da forma $\alpha_1 = \dfrac{1}{k_p}\big(\underbrace{-c_1 z_1 - \xi_2 + \cdots}_{\bar{\alpha}_1}\big)$

Porém, como $\dfrac{1}{k_p}$ é desconhecido, vamos empregar a sua estimativa $\hat{\rho}$,

$$\boxed{\alpha_1 = \hat{\rho}\bar{\alpha}_1}$$

Então,

$$\dot{z}_1 = k_p\,\hat{\rho}\bar{\alpha}_1 + \xi_2 + \bar{\omega}^T\theta + \varepsilon_2 - k_p\,\tilde{\rho}\,\dot{y}_r + k_p z_2$$

Porém,

$$\begin{aligned} k_p\,\hat{\rho}\,\alpha_1 &= k_p\big(\rho - \tilde{\rho}\big)\bar{\alpha}_1 \\ &= \underbrace{k_p\,\rho}_{1}\,\bar{\alpha}_1 - k_p\tilde{\rho}\bar{\alpha}_1 \\ &= \bar{\alpha}_1 - k_p\tilde{\rho}\bar{\alpha}_1 \end{aligned}$$

Resultado:

$$\dot{z}_1 = \bar{\alpha}_1 + \xi_2 + \bar{\omega}^T\theta + \varepsilon_2 - k_p\left(\dot{y}_r + \bar{\alpha}_1\right)\tilde{\rho} + k_p z_2$$

Resultado:

$$\dot{z}_1 = \bar{\alpha}_1 + \xi_2 + \underbrace{\bar{\omega}^T\theta + \varepsilon_2} - k_p\left(\dot{y}_r + \bar{\alpha}_1\right)\tilde{\rho} + k_p z_2$$

Em vista disto, escolhemos a 1a. função estabilizante:

$$\boxed{\bar{\alpha}_1 = -c_1 z_1 - d_1 z_1 - \xi_2 - \bar{\omega}^T\hat{\theta}} \tag{181}$$

- ★ $\hat{\theta}$ é uma estimativa de θ.
- ★ O termo $d_1 z_1$ é introduzido para completar o quadrado. !!

Substituindo...

$$\dot{z}_1 = -c_1 z_1 - d_1 z_1 + \underbrace{\bar{\omega}^T\tilde{\theta} + \varepsilon_2} - k_p\left(\dot{y}_r + \bar{\alpha}_1\right)\tilde{\rho} + \underbrace{k_p z_2}_{\text{problema!}}$$

- ★ O termo $k_p z_2$ é um problema porque k_p é desconhecido...

O termo $\bar{\omega}^T\tilde{\theta} + k_p z_2$ pode ser escrito como:

$$
\begin{aligned}
\bar{\omega}^T\tilde{\theta} + k_p z_2 &= \bar{\omega}^T\tilde{\theta} + \big(\tilde{k}_p + \hat{k}_p\big) z_2 \\
&= \bar{\omega}^T\tilde{\theta} + \big(\underbrace{e_1^T\tilde{\theta}}_{\tilde{k}_p}\big)\big(\underbrace{v_{0,2} - \hat{\rho}\dot{y}_r - \alpha_1}_{z_2}\big) + \hat{k}_p z_2 \\
&= \underbrace{\bar{\omega}^T\tilde{\theta} + e_1^T\tilde{\theta}v_{0,2}}_{\omega^T\tilde{\theta}} - e_1^T\tilde{\theta}\hat{\rho}\big(\dot{y}_r + \bar{\alpha}_1\big) + \hat{k}_p z_2 \\
&= \big[\omega - \hat{\rho}\big(\dot{y}_r + \bar{\alpha}_1\big)e_1\big]^T\tilde{\theta} + \hat{k}_p z_2
\end{aligned}
$$

Substituindo...

$$
\boxed{\dot{z}_1 = -c_1 z_1 - d_1 z_1 + \varepsilon_2 + \big[\omega - \hat{\rho}\big(\dot{y}_r + \bar{\alpha}_1\big)e_1\big]^T\tilde{\theta} - k_p\,\big(\dot{y}_r + \bar{\alpha}_1\big)\tilde{\rho} + \hat{k}_p z_2}
$$

★ Note que esta equação contém somente “erros” !!

Estamos prontos para iniciar o processo de síntese das leis de adaptação para $\hat{\theta}$ e $\hat{\rho}$.

Dinâmica de z_1:

$$\dot{z}_1 = -c_1 z_1 - d_1 z_1 + \varepsilon_2 + \Big[\underbrace{\omega - \hat{\rho}(\dot{y}_r + \bar{\alpha}_1)e_1}_{\text{regressor}}\Big]^T \tilde{\theta} - k_p\,\big(\underbrace{\dot{y}_r + \bar{\alpha}_1}_{\text{regressor}}\big)\tilde{\rho} + \hat{k}_p z_2$$

★ Note que a equação acima está no formato apropriado para a análise.

Escolhemos a função de Lyapunov,

$$\boxed{2V_1 = z_1^2 + \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} + |k_p|\gamma^{-1}\tilde{\rho}^2 + \frac{1}{2d_1}\varepsilon^T P \varepsilon}$$

Lembrar que: $\boxed{\tilde{\theta} = \theta - \hat{\theta}}$ e $\boxed{\tilde{\rho} = \rho - \hat{\rho}}$

Derivando, temos:

$$\dot V_1 = z_1\Big(-c_1 z_1 - d_1 z_1 + \varepsilon_2 + \big[\omega - \hat\rho(\dot y_r + \bar\alpha_1)e_1\big]^T\tilde\theta - k_p\,(\dot y_r + \bar\alpha_1)\tilde\rho + \hat k_p z_2\Big) -$$

$$- \tilde\theta^T\Gamma^{-1}\dot{\hat\theta} - |k_p|\gamma^{-1}\tilde\rho\dot{\hat\rho} + \frac{1}{2d_1}\big(\dot\varepsilon^T P\varepsilon + \varepsilon^T P\dot\varepsilon\big)$$

$$= -c_1 z_1^2 + \hat k_p z_1 z_2 - |k_p|\tilde\rho\Big[\underbrace{\operatorname{sign}(k_p)\big(\dot y_r + \bar\alpha_1\big)z_1 + \gamma^{-1}\dot{\hat\rho}}\Big] +$$

$$+ \tilde\theta^T\Big[\underbrace{\Big(\omega - \hat\rho\big(\dot y_r + \bar\alpha_1\big)e_1\Big)z_1 - \Gamma^{-1}\dot{\hat\theta}}\Big] - d_1 z_1^2 + z_1\varepsilon_2 + \frac{1}{4d_1}\varepsilon^T\big(\underbrace{A_0^T P + P A_0}_{-I}\big)\varepsilon$$

Para eliminar os termos em $\tilde\theta$ e $\tilde\rho$ escolhemos:

$$\boxed{\dot{\hat\rho} = -\gamma\,\operatorname{sign}(k_p)\big(\dot y_r + \bar\alpha_1\big)z_1}$$

$$\boxed{\dot{\hat\theta} = \Gamma\tau_1} \qquad \boxed{\tau_1 = \Big(\omega - \hat\rho\big(\dot y_r + \bar\alpha_1\big)e_1\Big)z_1} \qquad \textit{(Tuning function \#1)}$$

Aqui aparece um ponto chave do método:

★ Não vamos implementar $\boxed{\dot{\hat{\theta}} = \Gamma \tau_1}$.

★ A variável θ ainda vai reaparecer nos próximos passos !!

Estratégia:

(1) Deixamos a escolha de $\dot{\hat{\theta}}$ em aberto.
Isto é, $\dot{\hat{\theta}}$ permanece com o *status* de indeterminado.

(2) Utilizamos τ_1 como nossa 1a. função de sintonia (*Tuning function #1*).

Como resultado, temos que

$$\dot{V}_1 = -c_1 z_1^2 + \hat{k}_p z_1 z_2 + \tilde{\theta}^T \left(\tau_1 - \Gamma^{-1} \dot{\hat{\theta}}\right) \underbrace{-d_1 z_1^2 + z_1 \varepsilon_2 - \frac{1}{4d_1} \varepsilon^T \varepsilon}_{X}$$

O termo X pode ser escrito como:

$$\begin{aligned} X &= -d_1 z_1^2 + z_1 \varepsilon_2 - \frac{1}{4d_1} \varepsilon^T \varepsilon \\ &= -d_1 z_1^2 + z_1 \varepsilon_2 - \frac{1}{4d_1} \left(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2\right) \\ &= -d_1 \left[z_1^2 - \frac{z_1 \varepsilon_2}{d_1} + \left(\frac{1}{2d_1} \varepsilon_2\right)^2\right] - \frac{1}{4d_1} \left(\varepsilon_1^2 + \varepsilon_3^2\right) \\ &= -d_1 \left[z_1 - \frac{1}{2d_1} \varepsilon_2\right]^2 - \frac{1}{4d_1} \left(\varepsilon_1^2 + \varepsilon_3^2\right) \leq 0 \qquad \text{(!!)} \end{aligned}$$

Portanto,

$$\boxed{\dot{V}_1 \leq -c_1 z_1^2 + \hat{k}_p z_1 z_2 + \tilde{\theta}^T \left(\tau_1 - \Gamma^{-1} \dot{\hat{\theta}}\right)}$$

2o. Passo Análise da dinâmica de z_2.

Já definimos

$$z_2 = v_{0,2} - \hat{\rho}\dot{y}_r - \alpha_1$$

Portanto,

$$\dot{z}_2 = \dot{v}_{0,2} - \hat{\rho}\ddot{y}_r - \dot{\hat{\rho}}\dot{y}_r - \dot{\alpha}_1$$

Pode-se verificar facilmente que α_1 é função de $y, \eta, \hat{\theta}, \hat{\rho}$ e y_r :

$$\boxed{\alpha_1(y, \eta, \hat{\theta}, \hat{\rho}, y_r)}$$

Verificação:

$$\alpha_1 = \hat{\rho}\bar{\alpha}_1 = \hat{\rho}\Big[-c_1 z_1 - d_1 z_1 - \xi_2 - \bar{\omega}^T\hat{\theta}\Big]$$

$$z_1 = y - y_r$$

$$\xi = -A_0\eta$$

$$\bar{\omega}^T = \begin{bmatrix}0 & \left(\Xi_{(2)} - y e_1^T\right)\end{bmatrix}$$

$$\Xi = -\begin{bmatrix}A_0^2\eta & A_0\eta & \eta\end{bmatrix}$$

Portanto,

$$\boxed{\dot{\alpha}_1 = \frac{\partial \alpha_1}{\partial y}\left(\xi_2 + \omega^T\theta + \varepsilon_2\right) + \frac{\partial \alpha_1}{\partial \eta}\left(A_0\eta + e_3 y\right) + \frac{\partial \alpha_1}{\partial y_r}\dot{y}_r + \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}} + \frac{\partial \alpha_1}{\partial \hat{\rho}}\dot{\hat{\rho}}}$$

★ Reaparece θ !! Vai ser substituído por $\tilde{\theta} + \hat{\theta}$.

Podemos escrever a equação para $\dot{z}_2$ como:

$$
\begin{aligned}
\dot{z}_2 &= \dot{v}_{0,2} - \hat{\rho}\ddot{y}_r - \dot{\hat{\rho}}\dot{y}_r - \dot{\alpha}_1 \\
&= \big(v_{0,3} - k_2 v_{0,1}\big) - \hat{\rho}\ddot{y}_r - \dot{\hat{\rho}}\dot{y}_r - \dot{\alpha}_1 \\
&= v_{0,3} - \hat{\rho}\ddot{y}_r - \beta_2 - \frac{\partial \alpha_1}{\partial y}\big(\omega^T \tilde{\theta} + \varepsilon_2\big) - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}}
\end{aligned}
$$

onde β_2 é uma função que reune somente sinais disponíveis :

$$
\boxed{\beta_2 = k_2 v_{0,1} + \frac{\partial \alpha_1}{\partial y}\big(\xi_2 + \omega^T \hat{\theta}\big) + \frac{\partial \alpha_1}{\partial \eta}\big(A_0 \eta + e_3 y\big) + \frac{\partial \alpha_1}{\partial y_r}\dot{y}_r + \Big(\dot{y}_r + \frac{\partial \alpha_1}{\partial \hat{\rho}}\Big)\dot{\hat{\rho}}}
$$

★ Note que os termos que dependem de sinais ainda não definidos $(\dot{\hat{\theta}})$ ou desconhecidos ($\tilde{\theta}$ e ε_2) permanecem explícitos na equação de $\dot{z}_2$.

★ Os sinais incluídos em β_2 serão derivados no próximo passo!

Utilizando a relação

$$z_3 = v_{0,3} - \hat{\rho}\ddot{y}_r - \alpha_2 \quad \Rightarrow \quad \boxed{v_{0,3} - \hat{\rho}\ddot{y}_r = z_3 + \alpha_2}$$

finalmente, temos que

$$\boxed{\dot{z}_2 = \alpha_2 - \beta_2 - \frac{\partial \alpha_1}{\partial y}\left(\omega^T \tilde{\theta} + \varepsilon_2\right) - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}} + z_3}$$

★ Neste ponto estamos prontos para continuar a análise via Lyapunov.

Aumentamos a função de Lyapunov da seguinte forma:

$$\boxed{V_2 = V_1 + \frac{1}{2} z_2^2 + \frac{1}{4 d_2} \varepsilon^T P \varepsilon}$$

Derivando,

$$\dot{V}_2 \leq -c_1 z_1^2 + \hat{k}_p z_1 z_2 + \tilde{\theta}^T \left(\tau_1 - \Gamma^{-1} \dot{\hat{\theta}}\right) +$$
$$+ z_2 \left(\alpha_2 - \beta_2 - \frac{\partial \alpha_1}{\partial y} (\omega^T \tilde{\theta} + \varepsilon_2) - \frac{\partial \alpha_1}{\partial \hat{\theta}} \dot{\hat{\theta}} + z_3\right) - \frac{1}{4 d_2} \varepsilon^T \varepsilon$$

Rearranjando os termos, temos

$$\dot{V}_2 \leq -c_1 z_1^2 + z_2 z_3 + \tilde{\theta}^T \left(\tau_1 - \frac{\partial \alpha_1}{\partial y} \omega z_2 - \Gamma^{-1} \dot{\hat{\theta}}\right) +$$
$$+ z_2 \left(\alpha_2 + \hat{k}_p z_1 - \beta_2 - \frac{\partial \alpha_1}{\partial \hat{\theta}} \dot{\hat{\theta}}\right) - z_2 \frac{\partial \alpha_1}{\partial y} \varepsilon_2 - \frac{1}{4 d_2} \varepsilon^T \varepsilon$$

Copiando...

$$\dot{V}_2 \leq -c_1 z_1^2 + z_2 z_3 + \tilde{\theta}^T \Big(\tau_1 - \frac{\partial \alpha_1}{\partial y}\omega z_2 - \Gamma^{-1}\dot{\hat{\theta}}\Big) +$$

$$+ z_2\Big(\alpha_2 + \hat{k}_p z_1 - \beta_2 - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}}\Big) - z_2 \frac{\partial \alpha_1}{\partial y}\varepsilon_2 - \frac{1}{4 d_2}\varepsilon^T \varepsilon$$

Nesse ponto escolhemos a 2a. função de sintonia

$$\boxed{\dot{\hat{\theta}} = \Gamma \tau_2} \qquad \boxed{\tau_2 = \tau_1 - \frac{\partial \alpha_1}{\partial y}\omega z_2}$$

e a 2a. função estabilizante

$$\boxed{\alpha_2 = -c_2 z_2 - \hat{k}_p z_1 + \beta_2 + \frac{\partial \alpha_1}{\partial \hat{\theta}}\Gamma \tau_2 - d_2 \Big(\frac{\partial \alpha_1}{\partial y}\Big)^2 z_2}$$

★ Trick! Usamos a expressão $\dot{\hat{\theta}} = \Gamma \tau_2$ na escolha de α_2!

Resultado,

$$\dot{V}_2 \leq -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 + \tilde{\theta}^T \left(\tau_2 - \Gamma^{-1}\dot{\hat{\theta}}\right) + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}}\left(\Gamma \tau_2 - \dot{\hat{\theta}}\right) - \underbrace{-d_2\left(\frac{\partial \alpha_1}{\partial y}\right)^2 z_2^2 - z_2 \frac{\partial \alpha_1}{\partial y}\varepsilon_2 - \frac{1}{4d_2}\varepsilon^T \varepsilon}_{X}$$

O termo X pode ser escrito como:

$$X = -d_2\left(\frac{\partial \alpha_1}{\partial y}\right)^2 z_2^2 - z_2 \frac{\partial \alpha_1}{\partial y}\varepsilon_2 - \frac{1}{4d_2}\varepsilon^T \varepsilon$$

$$X = -d_2\left(z_2 \frac{\partial \alpha_1}{\partial y} + \frac{1}{2d_2}\varepsilon_2\right)^2 - \frac{1}{4d_2}\left(\varepsilon_1^2 + \varepsilon_3^2\right) \leq 0 \qquad \text{(!!)}$$

Resultado:

$$\boxed{\dot{V}_2 \leq -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 + \tilde{\theta}^T \left(\tau_2 - \Gamma^{-1}\dot{\hat{\theta}}\right) + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}}\left(\Gamma \tau_2 - \dot{\hat{\theta}}\right)}$$

Importante: Ainda temos a equação de z_3 para analisar !!

- ★ Novamente, não vamos implementar $\boxed{\dot{\hat{\theta}} = \Gamma\tau_2}$.

- ★ A variável θ ainda vai reaparecer mais uma vez !!

Estratégia:

1. Deixamos a escolha de $\dot{\hat{\theta}}$ em aberto.
2. Utilizamos τ_2 como nossa 2a. função de sintonia (*Tuning function #2*).
3. Utilizamos $\Gamma\tau_2$ na definição de α_2.

3o. Passo Análise da dinâmica de z_3.

★ Este é um passo crucial !!

Já definimos

$$z_3 = v_{0,3} - \hat{\rho}\ddot{y}_r - \alpha_2$$

Portanto,

$$\dot{z}_3 = \dot{v}_{0,3} - \hat{\rho}\,\dddot{y}_r - \dot{\hat{\rho}}\ddot{y}_r - \dot{\alpha}_2$$

Pode-se verificar que α_2 é função de $y, \eta, \hat{\theta}, \hat{\rho}, \lambda_1, \lambda_2, y_r$ e $\dot{y}_r$:

$$\boxed{\alpha_2(y, \eta, \hat{\theta}, \hat{\rho}, \lambda_1, \lambda_2, y_r, \dot{y}_r)}$$

Portanto,

$$\boxed{\begin{aligned}\dot{\alpha}_2 = {} & \frac{\partial \alpha_2}{\partial y}\big(\xi_2 + \omega^T\theta + \varepsilon_2\big) + \frac{\partial \alpha_2}{\partial \eta}\big(A_0\eta + e_3 y\big) + \frac{\partial \alpha_2}{\partial \hat{\theta}}\dot{\hat{\theta}} + \frac{\partial \alpha_2}{\partial \hat{\rho}}\dot{\hat{\rho}} + \\ & + \frac{\partial \alpha_2}{\partial \lambda_1}\big(-k_1\lambda_1 + \lambda_2\big) + \frac{\partial \alpha_2}{\partial \lambda_2}\big(-k_2\lambda_2 + \lambda_3\big) + \frac{\partial \alpha_2}{\partial y_r}\dot{y}_r + \frac{\partial \alpha_2}{\partial \dot{y}_r}\ddot{y}_r\end{aligned}}$$

★ Novamente, reaparece θ !! Vai ser substituído por $\tilde{\theta} + \hat{\theta}$.

Lembrando que

$$\dot{v}_{0,3} = -k_3 v_{0,1} + u$$

podemos escrever:

$$\boxed{\dot{z}_3 = u - \beta_3 - \hat{\rho}\,\dddot{y}_r - \frac{\partial \alpha_2}{\partial y}\big(\omega^T\tilde{\theta} + \varepsilon_2\big) - \frac{\partial \alpha_2}{\partial \hat{\theta}}\dot{\hat{\theta}}}$$

onde β_3 é uma função somente de sinais disponíveis:

$$\boxed{\begin{aligned}\beta_3 = k_3 v_{0,1} + \frac{\partial \alpha_2}{\partial y}\big(\xi_2 + \omega^T\hat{\theta}\big) + \frac{\partial \alpha_2}{\partial \eta}\big(A_0\eta + e_3 y\big) + \frac{\partial \alpha_2}{\partial \lambda_1}\big(-k_1\lambda_1 + \lambda_2\big) + \\ + \frac{\partial \alpha_2}{\partial \lambda_2}\big(-k_2\lambda_1 + \lambda_3\big) + \frac{\partial \alpha_2}{\partial y_r}\dot{y}_r + \frac{\partial \alpha_2}{\partial \dot{y}_r}\ddot{y}_r + \Big(\ddot{y}_r + \frac{\partial \alpha_2}{\partial \hat{\rho}}\Big)\dot{\hat{\rho}}\end{aligned}}$$

3a. função de Lyapunov:

$$\boxed{V_3 = V_2 + \frac{1}{2} z_3^2 + \frac{1}{4 d_3} \varepsilon^T P \varepsilon}$$

Derivando,

$$\dot{V}_3 \leq -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 + \tilde{\theta}^T \left(\tau_2 - \Gamma^{-1} \dot{\hat{\theta}} \right) + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} \left(\Gamma \tau_2 - \dot{\hat{\theta}} \right) + \\ + z_3 \left[u - \beta_3 - \hat{\rho}\, \dddot{y}_r - \frac{\partial \alpha_2}{\partial y} \left(\omega^T \tilde{\theta} + \varepsilon_2 \right) - \frac{\partial \alpha_2}{\partial \hat{\theta}} \dot{\hat{\theta}} \right] - \frac{1}{4 d_3} \varepsilon^T \varepsilon$$

Rearranjando,

$$\dot{V}_3 \leq -c_1 z_1^2 - c_2 z_2^2 + \tilde{\theta}^T \left(\tau_2 - \frac{\partial \alpha_2}{\partial y} \omega z_3 - \Gamma^{-1} \dot{\hat{\theta}} \right) + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} \left(\Gamma \tau_2 - \dot{\hat{\theta}} \right) + \\ + z_3 \left[u - \beta_3 - \hat{\rho}\, \dddot{y}_r - \frac{\partial \alpha_2}{\partial \hat{\theta}} \dot{\hat{\theta}} \right] - z_3 \frac{\partial \alpha_2}{\partial y} \varepsilon_2 - \frac{1}{4 d_3} \varepsilon^T \varepsilon$$

Copiando...

$$\dot{V}_3 \leq -c_1 z_1^2 - c_2 z_2^2 + \tilde{\theta}^T \Big(\tau_2 - \frac{\partial \alpha_2}{\partial y} \omega z_3 - \Gamma^{-1} \dot{\hat{\theta}} \Big) + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} \Big(\Gamma \tau_2 - \dot{\hat{\theta}} \Big) + \\ + z_3 \Big[u - \beta_3 - \hat{\rho}\, \dddot{y}_r - \frac{\partial \alpha_2}{\partial \hat{\theta}} \dot{\hat{\theta}} \Big] - z_3 \frac{\partial \alpha_2}{\partial y} \varepsilon_2 - \frac{1}{4 d_3} \varepsilon^T \varepsilon$$

Para eliminar o termo em $\tilde{\theta}$, escolhemos

$$\boxed{\dot{\hat{\theta}} = \Gamma \tau_3} \qquad \boxed{\tau_3 = \tau_2 - \frac{\partial \alpha_2}{\partial y} \omega z_3}$$

Portanto,

$$\dot{V}_3 \leq -c_1 z_1^2 - c_2 z_2^2 + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} \Big(\underbrace{\Gamma \tau_2 - \Gamma \tau_3}_{X} \Big) + z_3 \Big[u - \beta_3 - \hat{\rho}\, \dddot{y}_r - \frac{\partial \alpha_2}{\partial \hat{\theta}} \Gamma \tau_3 \Big] - \\ - z_3 \frac{\partial \alpha_2}{\partial y} \varepsilon_2 - \frac{1}{4 d_3} \varepsilon^T \varepsilon$$

Importante: O termo X pode ser escrito como:

$$X = \Gamma\tau_2 - \Gamma\tau_3$$
$$= \Gamma\tau_2 - \Gamma\Big(\tau_2 - \frac{\partial\alpha_2}{\partial y}\omega z_3\Big) = \Gamma\frac{\partial\alpha_2}{\partial y}\omega z_3$$

★ O termo em X possui z_3 como fator !! Vai ser absorvido por u !!

Substituindo e rearranjando,

$$\dot{V}_3 \leq -c_1 z_1^2 - c_2 z_2^2 + z_3\Big[u - \beta_3 - \hat{\rho}\,\ddot{y}_r - \frac{\partial\alpha_2}{\partial\hat{\theta}}\Gamma\tau_3 + z_2\frac{\partial\alpha_1}{\partial\hat{\theta}}\Gamma\frac{\partial\alpha_2}{\partial y}\omega\Big] - $$
$$- z_3\frac{\partial\alpha_2}{\partial y}\varepsilon_2 - \frac{1}{4d_3}\varepsilon^T\varepsilon$$

Em vista da expressão acima, escolhemos a lei de controle

$$\boxed{u = -c_3 z_3 + \beta_3 + \hat{\rho}\,\ddot{y}_r + \frac{\partial\alpha_2}{\partial\hat{\theta}}\Gamma\tau_3 - d_3\Big(\frac{\partial\alpha_2}{\partial y}\Big)^2 z_3 - z_2\frac{\partial\alpha_1}{\partial\hat{\theta}}\Gamma\frac{\partial\alpha_2}{\partial y}\omega}$$

Resultado:

$$\dot{V}_3 \leq -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 \underbrace{- d_3 \left(\frac{\partial \alpha_2}{\partial y}\right)^2 z_3^2 - z_3 \frac{\partial \alpha_2}{\partial y} \varepsilon_2 - \frac{1}{4 d_3} \varepsilon^T \varepsilon}_{X}$$

O termo X pode ser escrito como:

$$X = -d_3 \left(z_3 \frac{\partial \alpha_2}{\partial y} + \frac{1}{2 d_3} \varepsilon_2 \right)^2 - \frac{1}{4 d_3} \left(\varepsilon_1^2 + \varepsilon_3^2 \right) \leq 0 \quad !!$$

e, portanto, pode ser descartado.

Como resultado, obtém-se:

$$\boxed{\dot{V}_3 \leq -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2}$$

Resumo do algoritmo

Filtros	$\dot{\lambda} = A_0\lambda + e_3 u\,, \qquad v_0 = \lambda$ $\dot{\eta} = A_0\eta + e_3 y\,, \qquad \Xi = -\begin{bmatrix} A_0^2\eta & A_0\eta & \eta \end{bmatrix}, \qquad \xi = -A_0^3\eta$
Regressores	$\omega^T = \begin{bmatrix} v_{0,2} & (\Xi_{(2)} - y e_1^T) \end{bmatrix}$ $\bar{\omega}^T = \begin{bmatrix} 0 & (\Xi_{(2)} - y e_1^T) \end{bmatrix}$
Erros	$z_1 = y - y_r$ $z_2 = v_{0,2} - \hat{\rho}\dot{y}_r - \alpha_1$ $z_3 = v_{0,3} - \hat{\rho}\ddot{y}_r - \alpha_2$
Funções estabilizante	$\bar{\alpha}_1 = -c_1 z_1 - d_1 z_1 - \xi_2 - \bar{\omega}^T\hat{\theta}$ $\alpha_1 = \hat{\rho}\bar{\alpha}_1$ $\alpha_2 = -c_2 z_2 - \hat{k}_p z_1 + \beta_2 + \dfrac{\partial \alpha_1}{\partial \hat{\theta}}\Gamma\tau_2 - d_2\left(\dfrac{\partial \alpha_1}{\partial y}\right)^2 z_2$
Sinais auxiliares	$\beta_2 = k_2 v_{0,1} + \dfrac{\partial \alpha_1}{\partial y}(\xi_2 + \omega^T\hat{\theta}) + \dfrac{\partial \alpha_1}{\partial \eta}(A_0\eta + e_3 y) + \dfrac{\partial \alpha_1}{\partial y_r}\dot{y}_r + \left(\dot{y}_r + \dfrac{\partial \alpha_1}{\partial \hat{\rho}}\right)\dot{\hat{\rho}}$ $\beta_3 = k_3 v_{0,1} + \dfrac{\partial \alpha_2}{\partial y}(\xi_2 + \omega^T\hat{\theta}) + \dfrac{\partial \alpha_2}{\partial \eta}(A_0\eta + e_3 y) + \dfrac{\partial \alpha_2}{\partial \lambda_1}(-k_1\lambda_1 + \lambda_2) +$ $+ \dfrac{\partial \alpha_2}{\partial \lambda_2}(-k_2\lambda_1 + \lambda_3) + \dfrac{\partial \alpha_2}{\partial y_r}\dot{y}_r + \dfrac{\partial \alpha_2}{\partial \dot{y}_r}\ddot{y}_r + \left(\ddot{y}_r + \dfrac{\partial \alpha_2}{\partial \hat{\rho}}\right)\dot{\hat{\rho}}$

Tuning function	$\tau_1 = [\omega - \hat{\rho}(\dot{y}_r + \bar{\alpha}_1)e_1]z_1$ $\tau_2 = \tau_1 - \frac{\partial \alpha_1}{\partial y}\omega z_2$ $\tau_3 = \tau_2 - \frac{\partial \alpha_2}{\partial y}\omega z_3$
Adaptação	$\dot{\hat{\rho}} = -\gamma \operatorname{sign}(k_p)[\dot{y}_r + \bar{\alpha}_1]z_1$ $\dot{\hat{\theta}} = \Gamma\tau_3$
Controle	$u = -c_3 z_3 + \beta_3 + \hat{\rho}\,\dddot{y}_r + \frac{\partial \alpha_2}{\partial \hat{\theta}}\Gamma\tau_3 - d_3\left(\frac{\partial \alpha_2}{\partial y}\right)^2 z_3 - z_2\frac{\partial \alpha_1}{\partial \hat{\theta}}\Gamma\frac{\partial \alpha_2}{\partial y}\omega$

8.7.2 Simulações

(...)

10 MIMO MRAC

Contents

10.1 Introduction

Reference : [Tao:2003], (pag. 371)

- ★ Adaptive control of multivariable linear systems is a relatively old problem.
- ★ The first attempts were made around 1975.

  The solution, however, soon proved to be surprisingly difficult.

★ One of the main obstacles is the high-frequency gain matrix (K_p).

★ The generalization of the SISO case hypothesis, that $\operatorname{sign}(k_p)$ is known, is not obvious.

★ Several restrictive, if not impractical, conditions have been proposed to derive a solution.

★ Solution in SISO case:

$$e_0 = M(s)\, k_p\big[\tilde{\theta}^{*T}\omega\big] \quad \Rightarrow \quad \underbrace{\mathrm{sign}(k_p)e_0}_{e_a} = \underbrace{M(s)\,|k_p|}_{SPR}\,\big[\tilde{\theta}^{T}\omega\big]$$

★ $|k_p|$ is absorbed by the model: $\underbrace{M(s)}_{SPR} \quad \Rightarrow \quad \underbrace{M(s)\,|k_p|}_{SPR}$

★ State realization:

$$\underbrace{M(s)\,|k_p|}_{SPR} = C_m\big(sI - A_m\big)^{-1}B_m \quad \Rightarrow \quad \begin{cases} e = A_m e + B_m\big[\tilde{\theta}^{T}\omega\big] \\ e_a = C_m e \end{cases}$$

★ Problem in MIMO case: SPR only if $K_p = K_p^T > 0$.

★ This section presents a solution based on the factorization of K_p.

★ In the case $n^* = 1$, the factorization used is SDU,

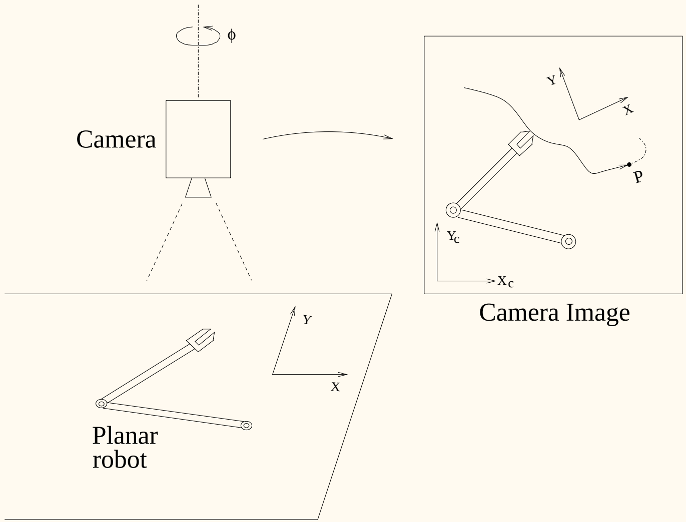
$$\boxed{K_p = SDU}$$

- S = Symmetric
- D = Diagonal
- U = Unitary upper triangular

★ In the case $n^* > 1$ other factorizations are possible (LDU and LDS).

★ To solve the problem, it is enough to know the signs of the leading principal minors (leading principal minors) of the gain matrix.

10.1.1 Motivation



ϕ = camera misalignment angle (unknown).

Figura 114: Visual servoing of a planar robot using a fixed camera.

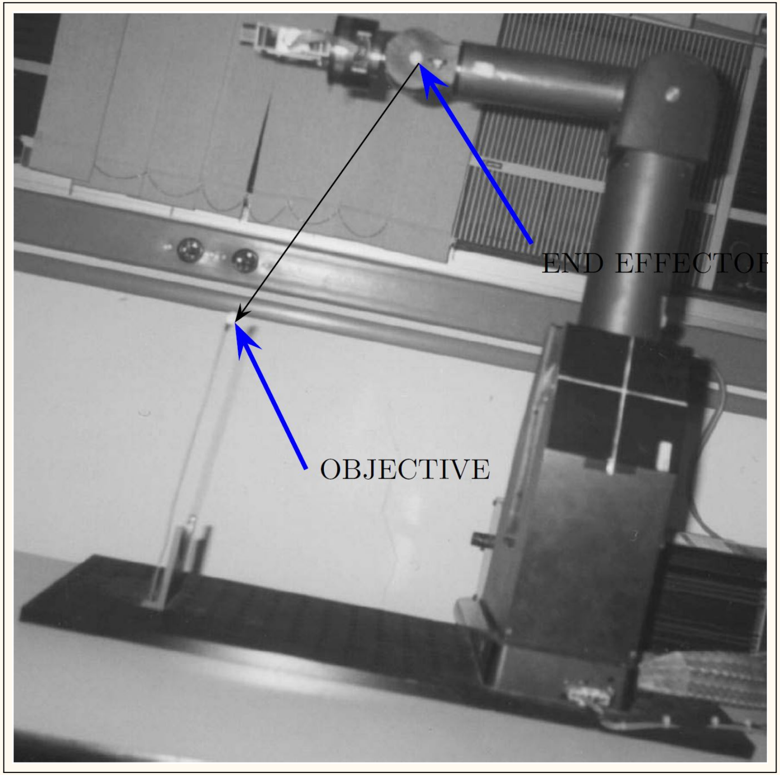


Figura 115: Experiment setup.

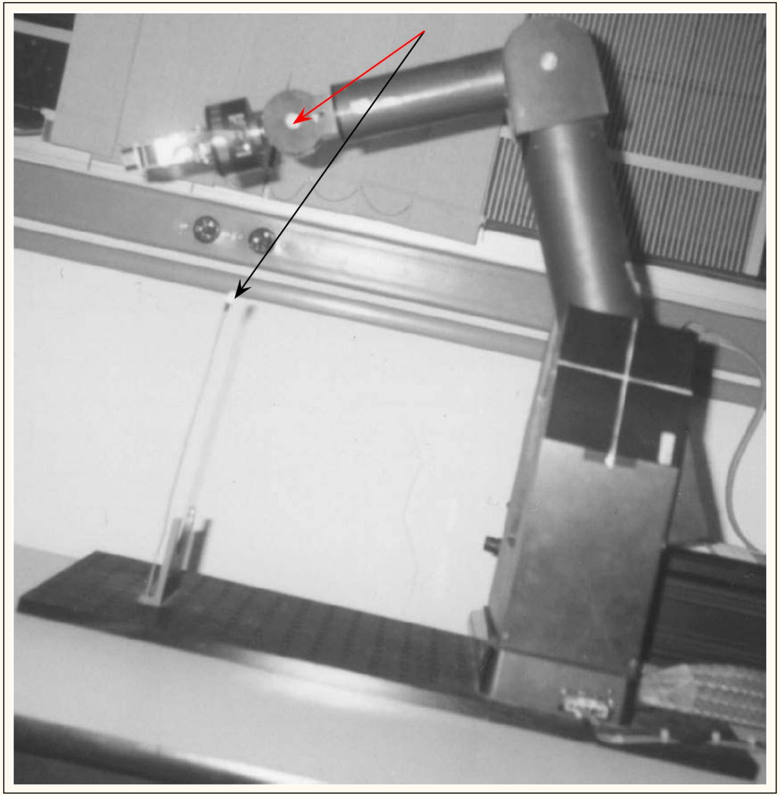


Figura 116: Step 1. Move 1/2 of the distance.

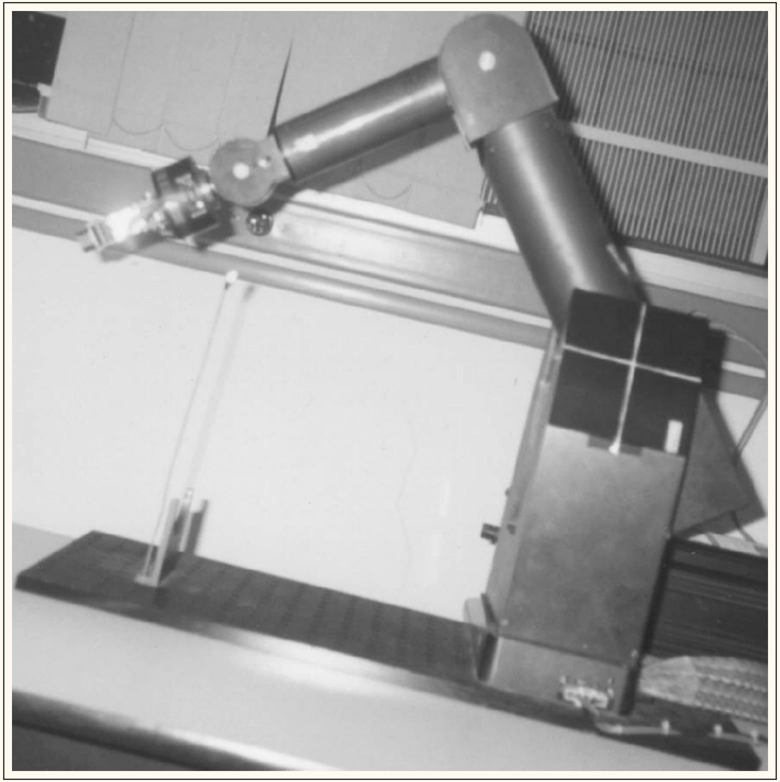


Figura 117: Step 2. Move 1/2 of the distance.

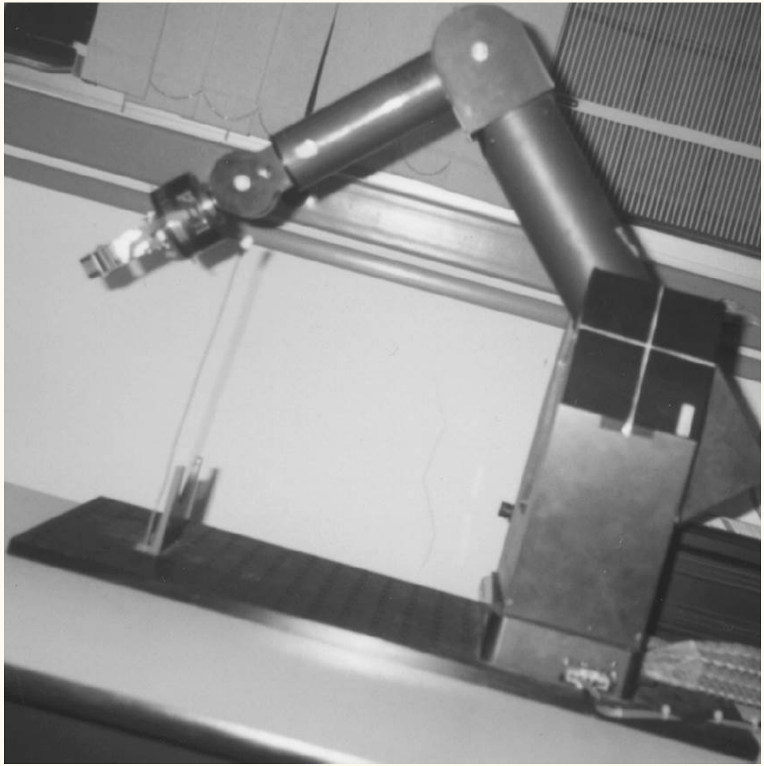


Figura 118: Step 3. Move 1/2 of the distance.

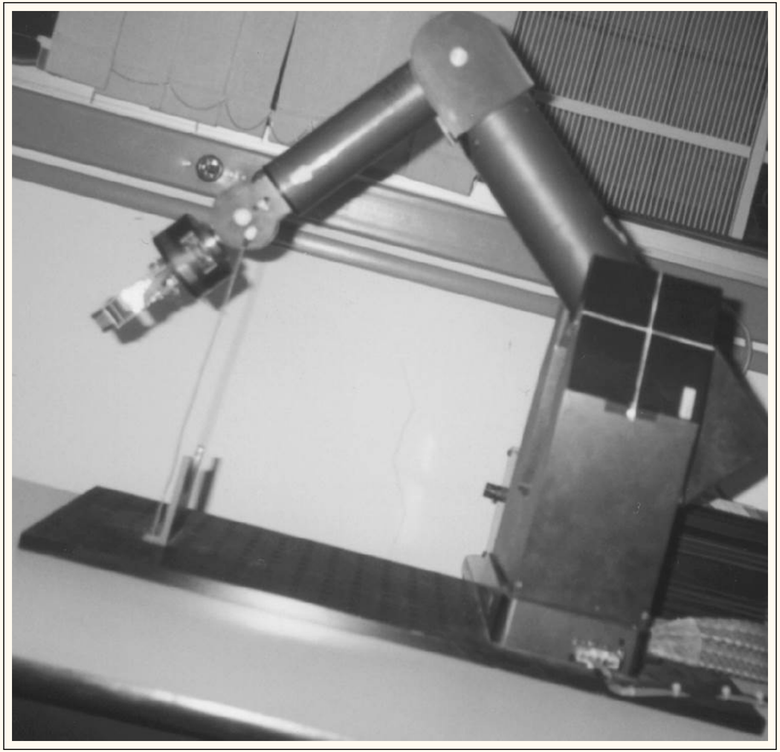


Figura 119: Step 4. Move 1/2 of the distance.

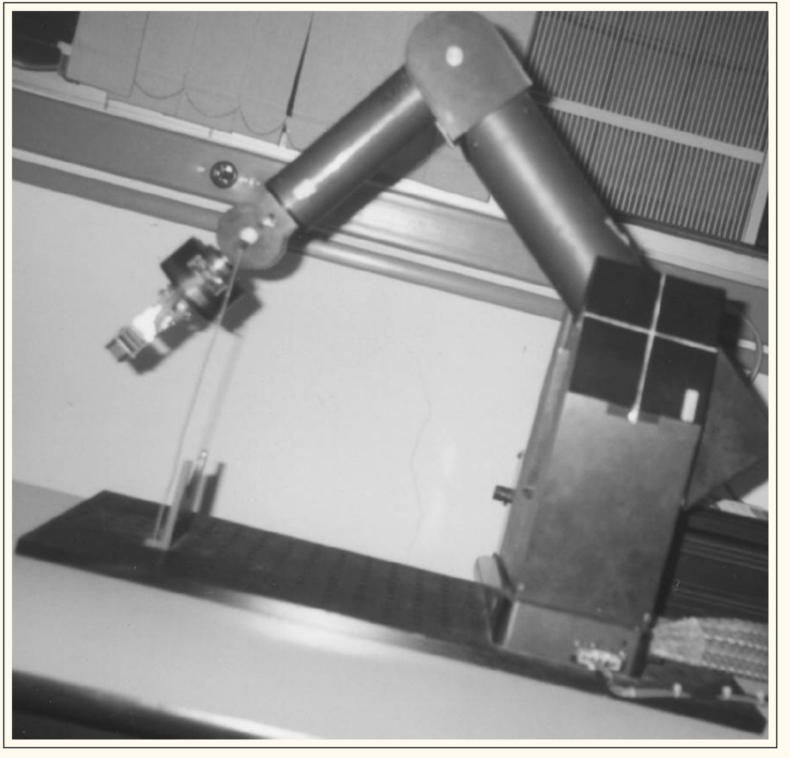


Figura 120: Easy !!

Equivalent problem : Model reference adaptive control (MRAC) of a 2×2 plant.

Plant :

$$\boxed{y = M(s) K_p u}\ , \quad K_p = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

K_p = high frequency gain matrix = rotation matrix.

$M(s)$ is known !

Model :

$$\boxed{y_M = M(s) r}\ , \quad M(s) = \begin{bmatrix} \frac{1}{s+\lambda} & 0 \\ 0 & \frac{1}{s+\lambda} \end{bmatrix}$$

$$\boxed{K_m = I}$$

(For simplicity.)

★ The reference model is the plant with $\phi = 0$.

Solutions found in textbooks :

- [Sastry & Bodson:1989], (pag. 277)

★ Not applicable because K_p is assumed to be known!

Textbook Solutions :

- [Narendra & Annaswamy:1989]

  Condition : Knowledge of a matrix Γ such that

$$\boxed{\Gamma K_p + (\Gamma K_p)^T > 0}$$

★ Incorrect stability proof!

★ Instability confirmed by simulation.

Textbook Solutions :

- [Ioannou & Sun:1996], (pag. 740)

  Condition: Knowledge of a matrix S_p such that

$$\boxed{K_p S_p = (K_p S_p)^T > 0}$$

★ Too restrictive ! Impractical for the servo-vision problem.

★ Non-generic condition !

Example 34 Servo-vision problem: $K_p = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

So that $K_p S_p = (K_p S_p)^T$, independent of a and b, the matrix S_p must have the form:

$$S_p = \begin{bmatrix} s_1 & s_2 \\ s_2 & -s_1 \end{bmatrix}$$

Symmetry : $K_p S_p = (K_p S_p)^T = \begin{bmatrix} as_1 - bs_2 & as_2 + bs_1 \\ as_2 + bs_1 & bs_2 - as_1 \end{bmatrix}$

However, $\boxed{\det(K_p S_p) = -(s_1^2 + s_2^2)(a^2 + b^2) < 0}$ $\forall s_1,\ s_2$

★ $\det(K_p S_p) = \lambda_1 \lambda_2 < 0 \quad \Rightarrow \quad \lambda_1 > 0$ and $\lambda_2 < 0$.

- ★ Bibliographic survey revealed that the problem is old.
- ★ First attempt at solution was made by [Monopoli & Hsing:1975]
- ★ ... lack of theory !!

Recent solutions to the servo-vision problem :

- Use of a SU factorization of K_p (non-explicit)

  [Zergeroglu, Dawson, Queiroz & Behal:1999]

- Use of a LDU factorization of K_p (non-explicit)

  [Hsu & Aquino:1999]

Common idea : Use of some kind of factorization of K_p.

Use of factorizations

LU factorization : Idea introduced by [Weller & Goodwin:1994] for the case of indirect adaptation.

Problem : Obtain a non-singular estimate of $\hat{K}_p$.

$$\boxed{K_p = LU} \Rightarrow \boxed{\hat{K}_p = \hat{L}\hat{U}} \Rightarrow \hat{K}_p \text{ is non-singular}$$

L is a unitary lower $\triangle$
U is an upper $\triangle$

★ Just make sure that the diagonal elements of $\hat{U}$ are not close to 0.

SDU Factorization : Introduced by [Morse:1993]

$S =$ symmetric positive definite

$D =$ diagonal

$U = \triangle$ upper unitary

★ Just recorded the idea. Didn't apply to MIMO MRAC.

10.2 Review and Nomenclature

Fato. Every transfer function matrix $G(s)$ can be factored as

$$\boxed{G(s) = N_R(s)D_R^{-1}(s)} \quad : \text{ right fraction}$$

$$\boxed{G(s) = D_L^{-1}(s)N_L(s)} \quad : \text{ left fraction}$$

- N_R, D_R, D_L, and N_L are polynomial matrices
- D_R and D_L are nonsingular

Example 35

$$G(s) = \begin{bmatrix} s+a & 0 \\ 0 & s+b \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \text{(Left fraction)}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} s-2a+3b & 4(-a+b) \\ 3(a-b)/2 & s+3a-2b \end{bmatrix}^{-1} \qquad \text{(Right fraction)}$$

Definição. The poles of $G(s)$ are the roots of

$$\boxed{\det(D_R) = 0} \quad \text{or} \quad \boxed{\det(D_L) = 0}$$

Definição. The zeros of $G(s)$ are the roots of

$$\boxed{\det(N_R) = 0} \quad \text{or} \quad \boxed{\det(N_L) = 0}$$

Definição. Row degrees of $D_L(s)$

$$\boxed{\partial_{ri}(D_L) = \max_j \ \partial(D_L)_{ij}}$$

Definição. For a proper left factorization of $G(s)$

$\nu_i = \partial_{ri}(D_L) =$ observability index

$n =$ system order $= \sum \nu_i$

$\nu = \max_i(\nu_i) =$ *observability index*

★ ν_i corresponds to the minimum order of an observer from output i.

Example 36

$$G(s) = \begin{bmatrix} \frac{1}{(s+a)^3} & 0 \\ 0 & \frac{1}{(s+b)^2} \end{bmatrix} = \begin{bmatrix} (s+a)^3 & 0 \\ 0 & (s+b)^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In this case : $\boxed{\nu_1 = 3}$, $\boxed{\nu_2 = 2}$ $\Rightarrow$ $\boxed{\nu = 3}$

Interpretation :

- ★ Reduced-order observers : order $\nu_1 - 1$ and $\nu_2 - 1$
- ★ Input/output filters : order $\nu - 1$

Definição. The polynomial matrix $\xi(s)$ is a *interactor* matrix of $G(s)$ if

$$\boxed{\lim_{s\to\infty} \xi(s)G(s) = K_p}$$ is non-singular.

★ The *interactor* matrix defines the multivariable relative degree.

Example 37

$$G(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{2}{s+2} \\ \dfrac{1}{s^2+5s+4} & \dfrac{1}{s+3} \end{bmatrix}$$

$$\xi(s) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$$

$$\lim_{s\to\infty} \xi(s)G(s) = \lim_{s\to\infty} \begin{bmatrix} \dfrac{s}{s+1} & \dfrac{2s}{s+2} \\ \dfrac{s}{s^2+5s+4} & \dfrac{s}{s+3} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

10.3 Problem formulation

Plant: $\boxed{y = P(s)u}$

Hypotheses : **(H1)** $P(s)$ has full rank and relative degree 1

(H2) Zeros have negative real part

(H3) *Observability index* ν of $P(s)$ is known

Fato. Relative degree 1 $\Leftrightarrow$ $\boxed{\xi(s) = sI}$.

Fato. $\xi(s)$ diagonal $\Leftrightarrow$ $P(s)$ can be decoupled by dynamic feedback.

Reference model : $\boxed{y_M = M(s)r}$ $\boxed{M(s) = \operatorname{diag}\left\{\frac{1}{s+a_i}\right\}}$

Control objective : $\boxed{e(t) = y(t) - y_M(t) \to 0}$

If $P(s)$ were perfectly known,

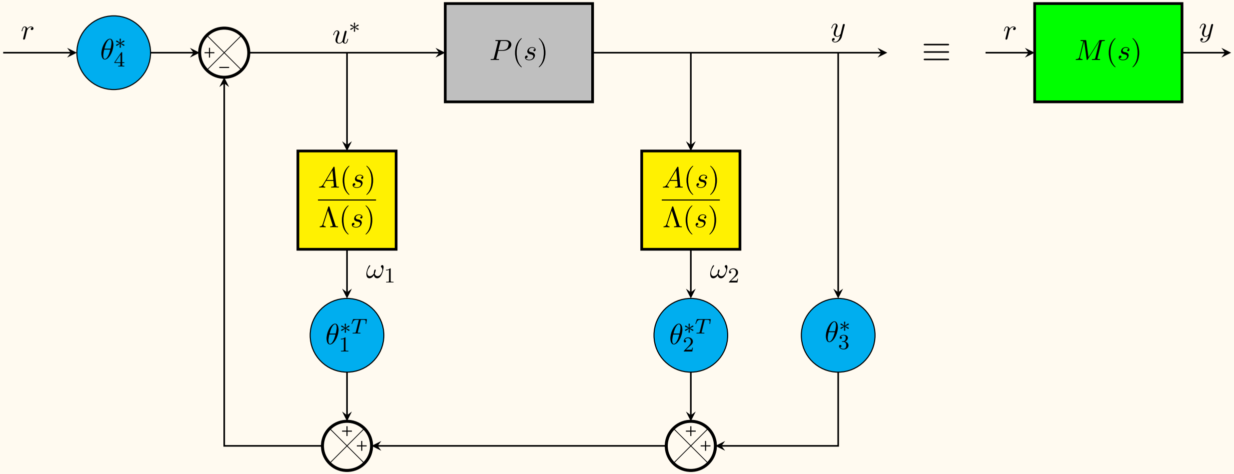


Figure. Controller structure.

Filters : $\boxed{\omega_1 = \dfrac{A(s)}{\Lambda(s)} u^*}$, $\boxed{\omega_2 = \dfrac{A(s)}{\Lambda(s)} y}$, $\partial[\Lambda(s)] = \nu - 1$

Regressor vector : $\omega = \begin{bmatrix} \omega_1^T & \omega_2^T & y^T & r^T \end{bmatrix}^T$ (Vector!!)

Ideal parameter matrix : $\theta^* = \begin{bmatrix} \theta_1^{*T} & \theta_2^{*T} & \theta_3^* & \theta_4^* \end{bmatrix}^T$ (Matrix!!)

Optimal control law : $\boxed{u^* = \theta_1^{*T}\omega_1 + \theta_2^{*T}\omega_2 + \theta_3^* y + \theta_4^* r = \theta^{*T}\omega}$

★ u^* guarantees *matching* : $y = P(s)u^* = M(s)r = y_M$

10.3.1 Equation of *matching*

$$u^* = \theta_1^{*T}\omega_1 + \theta_2^{*T}\omega_2 + \theta_3^* y + \theta_4^* r$$

Putting ω_1, ω_2, y, and r in terms of u^*,

$$u^* = \theta_1^{*T}\frac{A}{\Lambda}u^* + \theta_2^{*T}\frac{A}{\Lambda}Pu^* + \theta_3^* Pu^* + \theta_4^* M^{-1}Pu^*$$

we obtain the following *matching* equation:

$$\boxed{I - \theta_1^{*T}\frac{A(s)}{\Lambda(s)} - \theta_2^{*T}\frac{A(s)}{\Lambda(s)}P(s) - \theta_3^{*T}P(s) = \theta_4^* M^{-1}(s)P(s)}$$

$$\boxed{\theta_4^* = {K_p}^{-1}}$$

10.4 Error equation

After replacing u^* with u, we have

$$u = \theta_1^{*T}\frac{A}{\Lambda}u + \theta_2^{*T}\frac{A}{\Lambda}Pu + \theta_3^{*}Pu + K_p{}^{-1}M^{-1}Pu$$

$$MK_pu = MK_p\left[\theta_1^{*T}\omega_1 + \theta_2^{*T}\omega_2 + \theta_3^{*}y\right] + Pu$$

Using $\qquad e = y - y_M = Pu - Mr,$

Error equation : $\boxed{e = M(s)K_p\left[u - \theta^{*T}\omega\right]}$

10.4.1 MRAC SISO Project Recap

Error equation : $\boxed{e = M(s)K_p\left[u - \theta^{*T}\omega\right]}$

1. $M(s)$ is SPR

2. Linear parameterization : $\boxed{u = \theta^T\omega}$ $\Rightarrow$ $\boxed{e = M(s)K_p\left[\tilde{\theta}^T\omega\right]}$

   Parametric error : $\tilde{\theta} = \theta - \theta^*$

3. $\mathrm{sign}(K_p)$ is known

4. Adaptation law : $\boxed{\dot{\theta} = -\gamma\ \mathrm{sign}(K_p)\omega e}$ stability and convergence $e(t) \to 0$

Problem: Generalize $\operatorname{sign}(K_p)$ for the MIMO case.

If K_p is diagonal,

$$\boxed{\operatorname{sign}(K_p) = \operatorname{diag}\{\operatorname{sign}(k_{ii})\}}$$

Solution: Use SDU factorization of K_p.

★ Why use $\boxed{K_p = SDU}$?

$\boxed{S = \text{ symmetric positive definite}}$ is absorbed by the analysis

$\boxed{D = \text{ diagonal}}$ $\Rightarrow$ $\mathrm{sign}(D) = \mathrm{diag}\{\,\mathrm{sign}(d_{ii})\}$

$\boxed{U = \triangle \text{ upper unitary}}$ is absorbed by the parameterization of $u(t)$

10.4.2 SDU factorization of K_p

Ref.: [Morse:1993]

Lemma. If the *leading principal minors* $\Delta_i(K_p) \neq 0$, then

$$\boxed{K_p = SDU}$$

S = symmetric positive definite

D = diagonal

U = $\triangle$ upper unitary

- ★ This factorization is not unique!
- ★ In this paper, D was defined as $\mathrm{diag}\{+1's \text{ or } -1's\}$ ($\Rightarrow$ Uniqueness).

Prova. $\boxed{K_p = L_p D_p U_p}$: LDU factorization

$L_p = \triangle$ lower unit

$U_p = \triangle$ upper unit

$$D_p = \operatorname{diag}\left\{\Delta_1, \frac{\Delta_2}{\Delta_1}, \cdots, \frac{\Delta_m}{\Delta_{m-1}}\right\}$$

$$\Rightarrow \quad K_p = L_p \underbrace{D_+ D}_{D_p} U_p = \underbrace{L_p D_+ L_p^T}_{S} \; D \; \underbrace{D^{-1} L_p^{-T} D U_p}_{U}$$

Then $\boxed{S = L_p D_+ L_p^T}$ $\boxed{U = D^{-1} L_p^{-T} D U_p}$

★ D_+ diagonal > 0 is arbitrary !! (scaling factor)

★ This factorization is not unique.

Example 38 To illustrate the format of the factors S, D, U, consider the matrix

$$K_p = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

LDU factorization: $$L_p = \begin{bmatrix} 1 & 0 \\ l_1 & 1 \end{bmatrix}, \quad D_p = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2/\Delta_1 \end{bmatrix}, \quad U_p = \begin{bmatrix} 1 & u_1 \\ 0 & 1 \end{bmatrix}$$

where: $$l_1 = k_{21}/\Delta_1\,, \qquad u_1 = k_{12}/\Delta_1\,, \qquad \Delta_1 = k_{11}\,, \qquad \Delta_2 = k_{11}k_{22} - k_{12}k_{21}$$

SDU Factorization:

$$S = \begin{bmatrix} d_1^+ & d_1^+ l_1 \\ d_1^+ l_1 & d_2^+ + d_1^+ l_1^2 \end{bmatrix}, \quad D = D_+^{-1} D_p = \begin{bmatrix} \dfrac{\Delta_1}{d_1^+} & 0 \\ 0 & \dfrac{\Delta_2}{d_2^+ \Delta_1} \end{bmatrix}, \quad U = \begin{bmatrix} 1 & u_1 - \dfrac{d_1^+ l_1 \Delta_2}{d_2^+ \Delta_1^2} \\ 0 & 1 \end{bmatrix}$$

where: $$D_+ = \begin{bmatrix} d_1^+ & 0 \\ 0 & d_2^+ \end{bmatrix}$$

10.4.3 New parameterization

★ The parameterization will absorb the factor U.

Factorization : $\boxed{K_p = SDU}$

$$and = M(s) \underbrace{K_p}_{SDU} \left[u - \theta^{*T}\omega\right]$$

$$= M(s)SD\left[Uu - \underbrace{U\theta_1^{*T}}_{K_1}\omega_1 - \underbrace{U\theta_2^{*T}}_{K_2}\omega_2 - \underbrace{U\theta_3^{*}}_{K_3}y - \underbrace{U\theta_4^{*}}_{K_4}r\right]$$

Key point: $\boxed{Uu = u - K_5 u}$ K_5 is strictly $\triangle$ superior

Therefore, $$e = M(s)SD\big[u - \underbrace{K_1\omega_1 - K_2\omega_2 - K_3 y - K_4 r - K_5 u}_{u^*}\big]$$

★ K_5 has lots of zeros.

Defining yourself: $$u^* = K_1\omega_1 + K_2\omega_2 + K_3 y + K_4 r + K_5 u \equiv \begin{bmatrix} \Theta_1^{*T}\Omega_1 \\ \Theta_2^{*T}\Omega_2 \\ \vdots \\ \Theta_m^{*T}\Omega_m \end{bmatrix}$$

New regressors: $$\begin{cases} \Omega_1^T = \begin{bmatrix}\omega^T & u_2 & u_3 & \cdots & u_m\end{bmatrix} \\ \Omega_2^T = \begin{bmatrix}\omega^T & u_3 & \cdots & u_m\end{bmatrix} \\ \quad\vdots \\ \Omega_m^T = \begin{bmatrix}\omega^T\end{bmatrix} \end{cases}$$

To make the notation more compact, we define

Regression Matrix : $$\Omega^T = \begin{bmatrix} \Omega_1^T & 0 & \cdots & 0 \\ 0 & \Omega_2^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Omega_m^T \end{bmatrix}$$

Parameter Vector : $$\Theta^* = \begin{bmatrix} \Theta_1^* \\ \Theta_2^* \\ \vdots \\ \Theta_m^* \end{bmatrix}$$

Optimal control: $$\boxed{u^* = \Omega^T \Theta^*}$$

New error equation: $$\boxed{e = \Big(M(s)S\Big) D \left(u - \Omega^T \Theta^*\right)}$$

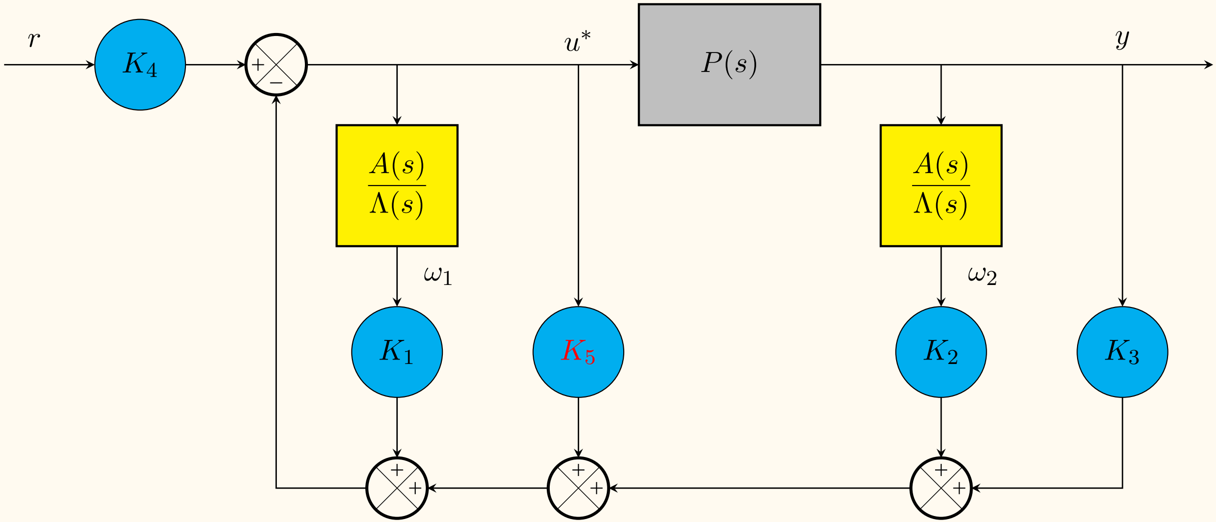


Figure. Controller structure with new parameterization.

★ K_5 is strictly △ upper $\Rightarrow$ no algebraic loop!

Error equation :

$$e = \Big(M(s) S \Big) D \left(u - \Omega^T \Theta^* \right)$$

Returning to the analogy with the SISO case :

1. Linear parameterization

   New control law :

   $$u = \Omega^T \Theta$$

2. $\text{sign}(D)$ replaces $\text{sign}(K_p)$

3. ... a problem appears with the SPR condition of the term $M(s)S$!!

Error equation : $$e = \Big(\underbrace{M(s)S}_{\text{SPR?}}\Big) D \left(u - \Omega^T \Theta^*\right)$$

SPR Condition

Fato. If $\boxed{M(s)S \text{ is SPR}}$ $\Rightarrow$ $\boxed{S = S^T > 0}$

Although : $\boxed{M(s) \;\; \text{SPR} \quad \text{e} \quad S = S^T > 0}$ $\not\Rightarrow$ $\boxed{M(s)S \;\; \text{is SPR}}$

Fato. $\{A, B, C\}$ com $\left(n^* = 1\right)$ $\Rightarrow$ $\boxed{K_p = CB}$

Verification:

$$K_p = \lim_{s\to\infty} sP(s) = \lim_{s\to\infty} s\,C\big(sI - A\big)^{-1}B$$
$$= \lim_{s\to\infty} C\big(I - s^{-1}A\big)^{-1}B = CB$$

Fato. $\{A, B, C\}$ SPR $\Rightarrow$ $\boxed{K_p = {K_p}^T > 0}$

Verification:

$\{A, B, C\}$ SPR $\Rightarrow$ $\exists P = P^T > 0$ such that $PB = C^T$

$$K_p = CB = (PB)^T B = B^T PB = {K_p}^T > 0$$

Example 39 $$M(s) = \begin{bmatrix} \dfrac{1}{s+a_1} & 0 \\ 0 & \dfrac{1}{s+a_2} \end{bmatrix}$$

Let $\{A, B, C\}$ be a realization of $M(s)S$:

$$A = \begin{bmatrix} -a_1 & 0 \\ 0 & -a_2 \end{bmatrix}, \qquad B = S = S^T > 0, \qquad C = I$$

Lemma SPR : $\boxed{M(s)S \text{ é SPR}} \Leftrightarrow \begin{cases} AP + AP = -2Q < 0 \\ PS = I \;\Rightarrow\; \boxed{P = S^{-1}} \end{cases}$

- For $a_1 = a_2 = a$, $M(s)S = \dfrac{1}{s+a} S$ is SPR for all $S = S^T > 0$.

- However, for $a_1 \neq a_2$, the SPR condition constrains S.

$$P = S^{-1} = \left(L_p D_+ L_p^T\right)^{-1} = \begin{bmatrix} d_1^+ & d_1^+ l_1 \\ d_1^+ l_1 & d_2^+ + d_1^+ l_1^2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{d_1^+} + \frac{l_1^2}{d_2^+} & -\frac{l_1}{d_2^+} \\ -\frac{l_1}{d_2^+} & \frac{1}{d_2^+} \end{bmatrix}$$

$$Q = -\frac{PA + AP}{2} = \begin{bmatrix} \frac{a_1}{d_1^+} + \frac{a_1 l_1^2}{d_2^+} & -\frac{l_1(a_1+a_2)}{2d_2^+} \\ -\frac{l_1(a_1+a_2)}{2d_2^+} & \frac{a_2}{d_2^+} \end{bmatrix}$$

Then $Q > 0$ if $\boxed{d_2^+ > 0}$ and $\boxed{0 < d_1^+ < \dfrac{4a_1 a_2 d_2^+}{l_1^2 (a_1 - a_2)^2}}$

★ The matrix $D_+ = \mathrm{diag}\{d_1, d_2\}$ is a *scale factor*.

Conclusion : There exists D_+ such that $S = L_p D_+ L_p^T$ and $M(s)S$ is SPR.

Lemma. Given A and L_p, there exists D_+ such that

$$\boxed{M(s)S = (sI - A)^{-1} L_p D_+ L_p^T} \quad \text{is SPR}$$

★ Reminder: $\boxed{S = L_p D_+ L_p^T}$

10.5 Design of the update law

As in the SISO case: $\Big\{\ X := \big[X_M =$ state of a non-minimal realization of $M(s)S$

State error : $\boxed{z = X - X_M}$

Error equation : $\boxed{e = M(s)SD\left(u - \Omega^T\Theta^*\right)}$

Non-minimal realization : $\begin{cases} \dot{z} = A_M z + B_M D\left(u - \Omega^T\Theta^*\right) \\ e = C_M z \end{cases}$

SPR Lemma : $\begin{cases} A_M^T P + P A_M = -2Q \\ P B_M = C_M^T \end{cases}$

Lyapunov function : $\boxed{2V(z, \tilde{\Theta}) = z^T P z + \tilde{\Theta}^T |\mathbb{D}| \Gamma^{-1} \tilde{\Theta}}$

where:

$$\mathbb{D} = \operatorname{diag}\{d_1 I_1, \quad d_2 I_2, \quad \cdots, \quad d_m I_m\}$$

$$\Gamma = \operatorname{diag}\{\Gamma_1 I_1, \quad \Gamma_2 I_2, \quad \cdots, \quad \Gamma_m I_m\}$$

★ $|\mathbb{D}|\Gamma^{-1} = \left(|\mathbb{D}|\Gamma^{-1}\right)^T > 0.$

Differentiating,

$$
\begin{aligned}
2\dot{V} &= \dot{z}^T P z + z^T P \dot{z} + \dot{\tilde{\Theta}}^T |\mathbb{D}| \Gamma^{-1} \tilde{\Theta} + \tilde{\Theta}^T |\mathbb{D}| \Gamma^{-1} \dot{\tilde{\Theta}} \\
&= \left[A_M z + B_M D \left(\Omega^T \tilde{\Theta} \right) \right]^T P z + z^T P \left[A_M z + B_M D \left(\Omega^T \tilde{\Theta} \right) \right] + 2\tilde{\Theta}^T |\mathbb{D}| \Gamma^{-1} \dot{\tilde{\Theta}} \\
&= z^T \big(\underbrace{A_M^T P + P A_M}_{-2Q} \big) z + 2\tilde{\Theta}^T \Omega D \underbrace{B_M^T P z}_{e} + 2\tilde{\Theta}^T |\mathbb{D}| \Gamma^{-1} \dot{\tilde{\Theta}}
\end{aligned}
$$

Therefore, $\dot{V} = -z^T Q z + \tilde{\Theta}^T \Omega D e + \tilde{\Theta}^T |\mathbb{D}| \Gamma^{-1} \dot{\tilde{\Theta}}$

Using the relation $\boxed{\Omega D = \mathbb{D}\,\Omega}$

we have that $\dot{V} = -z^T Q z + \tilde{\Theta}^T \, |\mathbb{D}| \, \Gamma^{-1} \Big[\operatorname{sign}\big(\mathbb{D}\big) \, \Gamma \, \Omega \, e + \dot{\tilde{\Theta}} \Big]$

Adaptation : $\boxed{\dot{\Theta} = -\operatorname{sign}(\mathbb{D}) \, \Gamma \, \Omega \, e}$

Result: $\boxed{V(z, \tilde{\Theta}) = -z^T Q z \le 0}$

Boundedness : $\boxed{z \in \mathcal{L}_\infty \cap \mathcal{L}_2}$, $\boxed{\Theta_i \in \mathcal{L}_\infty}$, $\boxed{\omega \in \mathcal{L}_\infty}$

By the definition of Ω: $\boxed{\Omega_i \in \mathcal{L}_\infty}$, $\boxed{u_i \in \mathcal{L}_\infty}$, $\boxed{\dot{e} \in \mathcal{L}_\infty}$

Convergence : $\boxed{e \in \mathcal{L}_2}$ e $\boxed{\dot{e} \in \mathcal{L}_\infty}$ $\Rightarrow$ $\boxed{e \to 0}$

Reminder :
$$\begin{cases} \Omega_1^T & = \begin{bmatrix} \omega^T & u_2 & u_3 & \cdots & u_m \end{bmatrix} \\ \vdots & \\ \Omega_{m-2}^T & = \begin{bmatrix} \omega^T & u_{m-1} & u_m \end{bmatrix} \\ \Omega_{m-1}^T & = \begin{bmatrix} \omega^T & u_m \end{bmatrix} \\ \Omega_m^T & = \begin{bmatrix} \omega^T \end{bmatrix} \end{cases}$$

Result: $\boxed{\dot{V} = -e^T Q e \leq 0}$

- $V(t) > 0$ and $\dot{V} \leq 0 \quad \Rightarrow \quad z \in \mathcal{L}_\infty, \quad \tilde{\Theta} \in \mathcal{L}_\infty$
- Integrating $\dot{V} \quad \Rightarrow \quad z \in \mathcal{L}_2$
- $\begin{cases} z = X - X_M \in \mathcal{L}_\infty \\ \qquad\quad X_M \in \mathcal{L}_\infty \end{cases} \Rightarrow \quad \begin{cases} X = \begin{bmatrix} x^T & \omega_1^T & \omega_1^T \end{bmatrix} \in \mathcal{L}_\infty \\ \qquad y,\ \omega_1,\ \omega_2 \in \mathcal{L}_\infty \end{cases}$
- $r(t) \in \mathcal{L}_\infty \quad \Rightarrow \quad \omega^T = \begin{bmatrix} \omega_1^T & \omega_2^T & y^T & r^T \end{bmatrix} \in \mathcal{L}_\infty$
- Sequentially, we show that $u_m,\ u_{m-1}, \cdots, u_2, u_1 \in \mathcal{L}_\infty$
- $\Omega_m = \omega \in \mathcal{L}_\infty \quad \Rightarrow \quad u_m = \Theta_m^T \Omega_m \in \mathcal{L}_\infty$
- $\Omega_{m-1}^T = [\omega^T \;\; u_m] \in \mathcal{L}_\infty \quad \Rightarrow \quad u_{m-1} = \Theta_{m-1}^T \Omega_{m-1} \in \mathcal{L}_\infty$
- $\Omega_{m-2}^T = [\omega^T \;\; u_m \;\; u_{m-1}] \in \mathcal{L}_\infty \quad \Rightarrow \quad u_{m-2} = \Theta_{m-2}^T \Omega_{m-2} \in \mathcal{L}_\infty$

- $\cdots$

- Result: all signals in the closed-loop system are bounded.

- $\begin{cases} z \in \mathcal{L}_2 \\ \dot{z} \in \mathcal{L}_\infty \end{cases} \Rightarrow \boxed{z, \; e \to 0}$

Summary

Subsystem	Equation	Order
Plant	$y = P(s)\,u$	n
Model	$y_M = M(s)\,r$	n
Error	$e = y - y_M$	
Control	$u = \Omega^T \Theta$	
Filtros	$\dot{\omega}_1 = A_f\,\omega_1 + B_f\,u$ $\dot{\omega}_2 = A_f\,\omega_2 + B_f\,y$	$m(\nu - 1)$ $m(\nu - 1)$
Regressor	$\omega^T = \begin{bmatrix}\omega_1^T & \omega_2^T & y^T & r^T\end{bmatrix}$ $\Omega_1^T = \begin{bmatrix}\omega^T & u_2 & u_3 & \cdots & u_m\end{bmatrix}$ $\Omega_2^T = \begin{bmatrix}\omega^T & u_3 & \cdots & u_m\end{bmatrix}$ $\vdots$ $\Omega_m^T = \begin{bmatrix}\omega^T\end{bmatrix}$	
Update law	$\dot{\Theta}_i = -\gamma_i\ \mathrm{sign}(d_i) e_i\,\Omega_i$	$m(4\nu + m - 1)/2$

Total order of the system: $$\boxed{N = 2n + m(8\nu + m - 5)/2}$$

Reminder: $m =$ number of inputs/outputs

$\nu =$ *observability index*

Teorema. If :

- hypotheses (H1), (H2), (H3) are verified,
- $\operatorname{sign}(d_i)$ are known and
- $r(t)$ is bounded,

then the adaptive control algorithm shown in the table ensures that all signals of the closed-loop system are uniformly bounded and the tracking error $e(t)$ converges to 0.

10.6 Simulations

Case 1 : servo-vision problem. Only K_p is unknown.

Plant: $$P(s) = \begin{bmatrix} \frac{1}{s+2} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix} K_p$$

$$K_p = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -h\sin(\phi) & h\cos(\phi) \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

Model: $$M(s) = \begin{bmatrix} \frac{2}{s+2} & 0 \\ 0 & \frac{2}{s+2} \end{bmatrix}$$

★ Only K_p is unknown $\Rightarrow$ 5 parameters are adapted !

If : $\boxed{\;-90^\circ < \phi < 90^\circ\;}$ $\Rightarrow \begin{cases} d_1 > 0 \\ d_2 > 0 \end{cases}$

Adaptive Control Algorithm:

$$u_1 = \Theta_1^T \Omega_1\,, \qquad \Omega_1^T = [2r_1 \quad 2r_2 \quad u_2]\,, \qquad \dot{\Theta}_1 = -\gamma_1 \Omega_1 e_1$$

$$u_2 = \Theta_2^T \Omega_2\,, \qquad \Omega_2^T = [2r_1 \quad 2r_2]\,, \qquad \dot{\Theta}_2 = -\gamma_2 \Omega_2 e_2$$

Simulation #1 Zero initial conditions.

Initial conditions . . . . :

$$y(0) = [0 \quad 0]^T$$
$$y_m(0) = [0 \quad 0]^T$$
$$\Theta_1(0) = [0 \quad 0 \quad 0]^T$$
$$\Theta_2(0) = [0 \quad 0]^T$$

Parameters . . . . :

$$\gamma_1 = 1\,I$$
$$\gamma_2 = 1\,I$$

Reference signal . . . . :

$$r1 = 1 + 10\,\sin(5t)$$
$$r2 = -1 + 5\,\sin(3t)$$

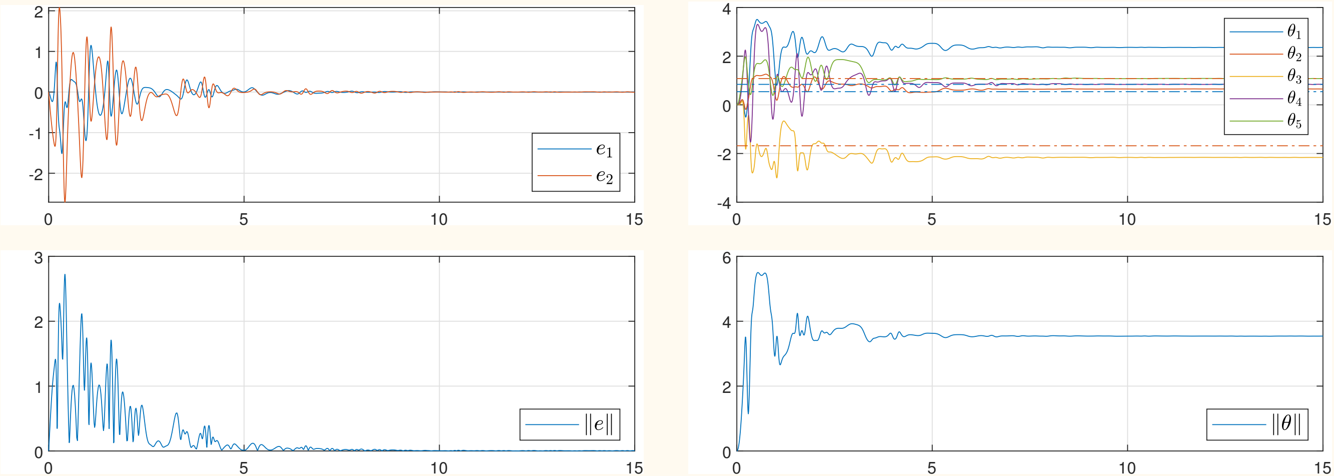


Figura 121: MIMO MRAC simulation. Case $n^* = 1$ and 5 parameters. (Script: `fig1.m`)

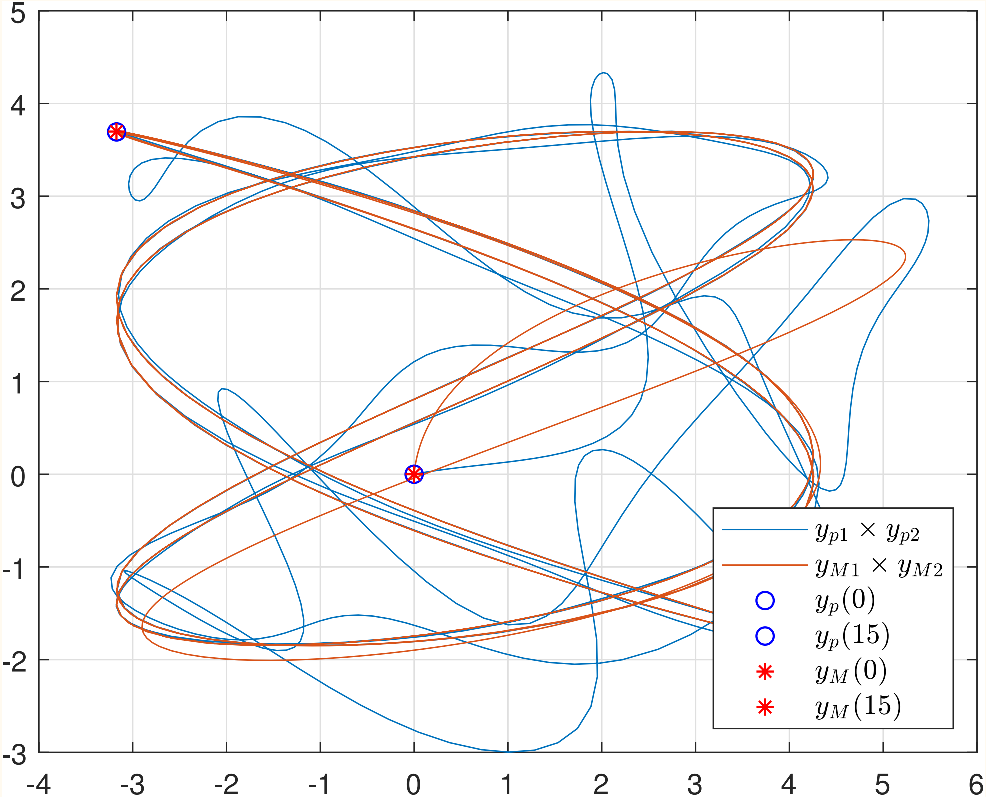


Figura 122: MIMO MRAC simulation. Case $n^* = 1$ and 5 parameters. (Script: `fig1.m`)

Simulation #2. Small initial conditions.

Initial conditions . . . . :

$$y(0) = \begin{bmatrix} 2 & 2 \end{bmatrix}^T$$
$$y_m(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$$
$$\Theta_1(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$$
$$\Theta_2(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$$

Parameters . . . . :

$$\gamma_1 = 10\, I$$
$$\gamma_2 = 10\, I$$

Reference signal . . . . :

$$r1 = 1 + 10 \sin(5t)$$
$$r2 = -1 + 5 \sin(3t)$$

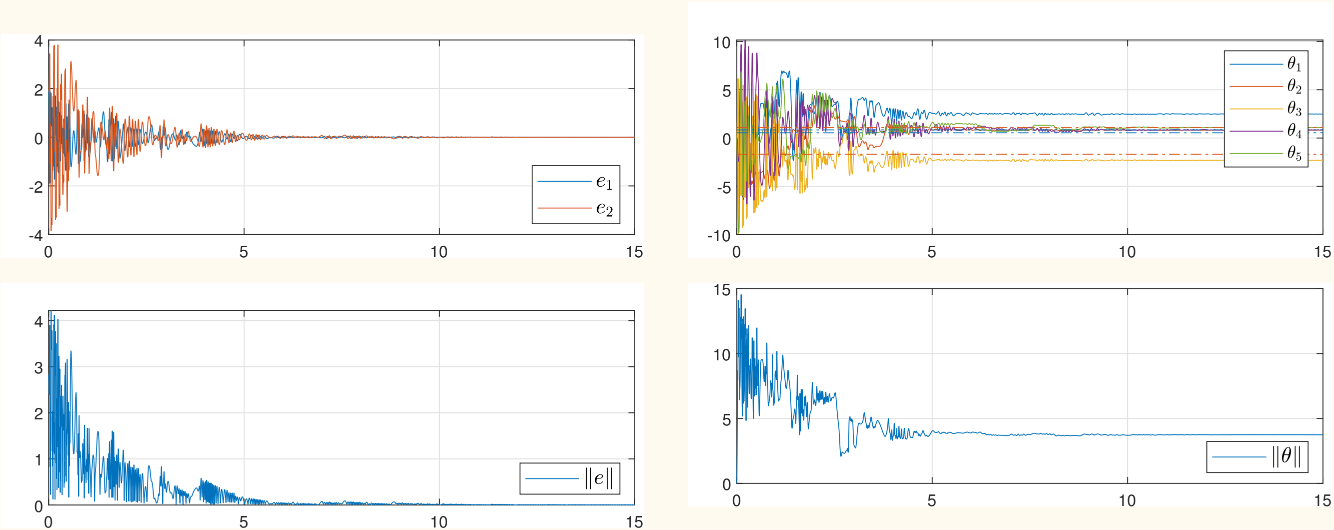


Figura 123: MIMO MRAC simulation. Case $n^* = 1$ and 5 parameters. (Script: `fig2.m`)

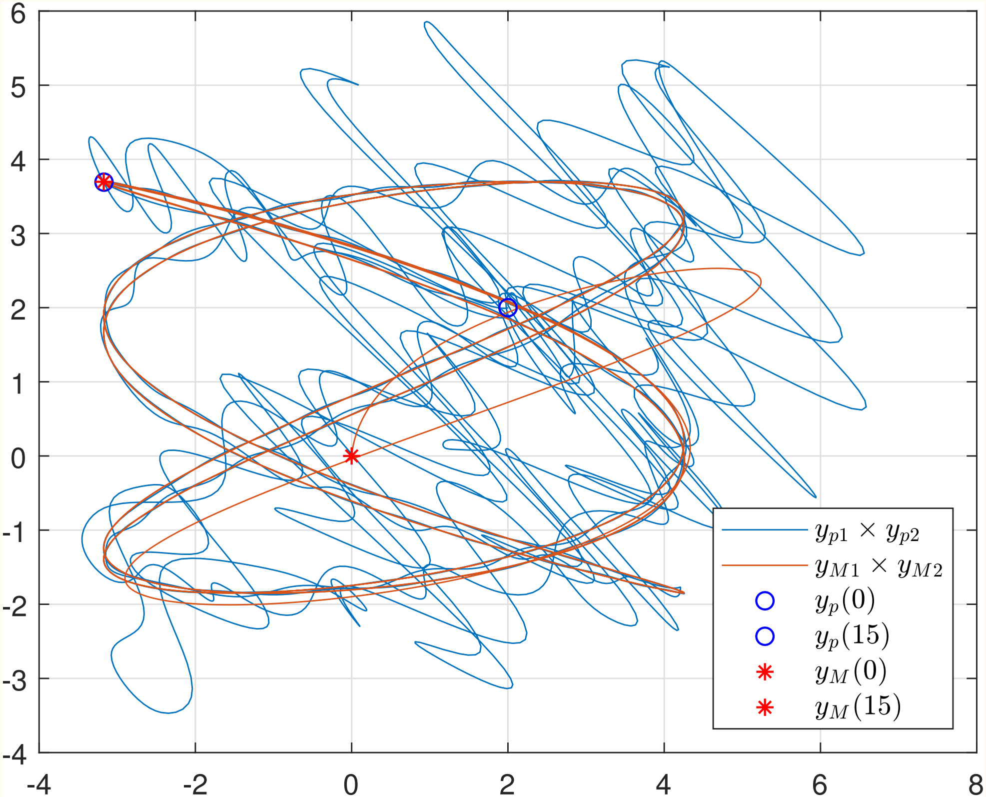


Figura 124: MIMO MRAC simulation. Case $n^* = 1$ and 5 parameters. (Script: `fig2.m`)

Case 2 : All plant parameters are unknown.

Plant : $$P(s) = \frac{1}{s^2-1}\begin{bmatrix} s+3 & 2s \\ -2s-4 & s+3 \end{bmatrix}$$

$$K_p = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \Rightarrow \begin{cases} d_1 > 0 \\ d_2 > 0 \end{cases}$$

Model: $$M(s) = \begin{bmatrix} \dfrac{2}{s+2} & 0 \\ 0 & \dfrac{1}{s+1} \end{bmatrix}$$

★ In this case, 17 parameters are adapted $\begin{cases} \omega_1, \omega_2, y, r, u \in \mathbb{R}^2 \\ \Omega_1 \in \mathbb{R}^9 \\ \Omega_2 \in \mathbb{R}^8 \end{cases}$

Simulation #1. Initial conditions null.

Initial conditions . . . . :

$$y(0) = [0 \;\; 0]^T$$
$$y_m(0) = [0 \;\; 0]^T$$
$$\Theta_1(0) = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]^T \;\; \in \mathbb{R}^9$$
$$\Theta_2(0) = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]^T \;\; \in \mathbb{R}^8$$

Parameters . . . . :

$$\gamma_1 = 1\, I$$
$$\gamma_2 = 1\, I$$

Reference signal . . . . :

$$r1 = 1 + 10\,\sin(5t)$$
$$r2 = -1 + 5\,\sin(3t)$$

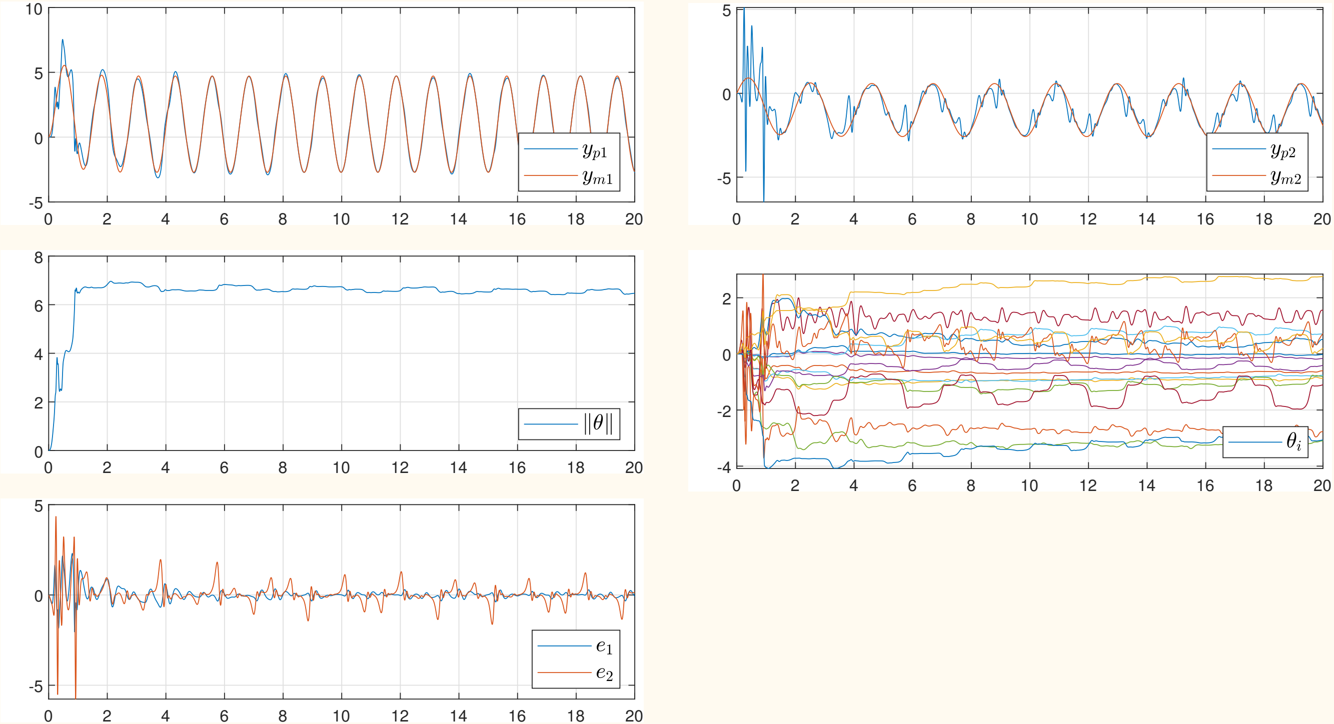


Figura 125: MIMO MRAC simulation. Case $n^* = 1$ and 17 parameters. (Script: `fig1.m`)

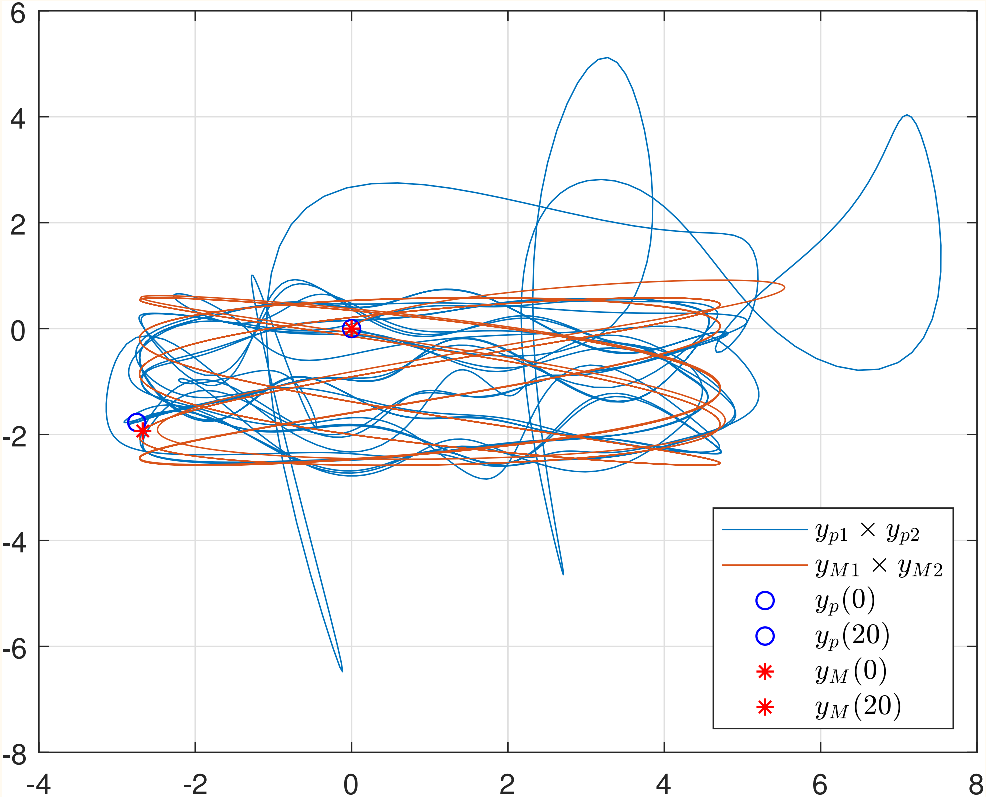


Figura 126: MIMO MRAC simulation. Case $n^* = 1$ and 17 parameters. (Script: `fig1.m`)

10.7 Conclusions

- The MIMO MRAC algorithm is analogous to the SISO MRAC.
- In the SISO case the sign of the high frequency gain is known.

  The MIMO analog is the knowledge of the signs of the leading principal minors (*leading principal minors*) of K_p.
- Key points :
  - SDU factorization
  - Decomposition $Uu = u - (I - U)u$
  - Non-uniqueness of SDU factorization
- Price of the solution found : over-parameterization

10.8 Extensions

- Case with relative degree > 1 : Hsu, Costa, Imai, Tao & Kokotović [CDC'01]

  3 possible factorizations of K_p : LDU, SDU, or LDS

- Multivariate Nussbaum Gain : Imai, Costa & Hsu [Symposium'01]

- Multivariate Backstepping : Costa, Hsu, Imai & Tao [ACC'02]

Other solutions

- (Ortega, Hsu & Astolfi [CDC'01])

  Using the approach called *Immersion & Invariance* (I&I)

12 MIMO LS-MRAC

Contents

12.1 Introduction

The problem of extending the Lyapunov-based design of model-reference adaptive control (MRAC) to multivariable (MIMO) plants was resolved in (Costa et al., 2003). The central difficulty of this problem was how to deal with the plant's high frequency gain matrix K_p. For a diagonal matrix the solution is the same as for the SISO version. That is, it is enough to know the signs of its diagonal elements. For a generic matrix, however, we need to know the signs of all leading principal minors of K_p. The technique used to achieve this result is based on the existence of a SDU factorization of K_p, where S is symmetric positive definite, D is diagonal, and U is a unitary upper triangular matrix. The matrix S is absorbed by the error model, matrix U is absorbed by a reparametrization, and D is the remaining matrix which the signs are required.

Other possible factorizations and extensions were introduced in (**?**) (Tao, 2003). The survey (**?**) provides a detailed and comprehensive account of MRAC theory and design techniques with focus on the MIMO case.

(...)

- (Annaswamy and Fradkov, 2021) stress the importance of parameter estimation. Learning = parameter estimation !
- (Costa, 2020) showed the importance of reducing the relative degree of the error equation to zero. This allows the introduction of a LS update law.

——×——

12.2 The SISO case

This section reviews the LS-MRAC algorithm developed in (Costa, 2020) for the case of linear SISO plants.

12.3 Problem formulation

Consider a linear plant modeled as

$$y = P(s)u\,, \quad P(s) = k_p \frac{N_p(s)}{D_p(s)}\,, \tag{329}$$

where u is the control signal and y is the output. The high frequency gain k_p and all the coefficients of the polynomials $N_p(s)$ and $D_p(s)$ are unknown. The only available information regarding $P(s)$ are the following:

1) the order of the plant n is known,

2) the plant has relative degree 1,

3) $N(s)$ is Hurwitz, i.e., $P(s)$ is minimum phase,

4) the $\mathrm{sign}(k_p)$ is known, and

5) a lower bound for $|k_p|$ is known.

Then, the adaptive control problem can be stated as follows. Given a *reference model* with the same relative degree $n^* = 1$ and, without loss of generality, described by

$$y_m = M(s)r\,, \quad M(s) = \frac{1}{s + a_m}\,, \quad a_m > 0\,, \tag{330}$$

where r is a reference input signal, find a control signal $u(t)$ that stabilizes the closed loop system and such that the *output error* (or *tracking error*)

$$e_0 = y - y_m\,, \tag{331}$$

tends to zero asymptotically for arbitrary initial conditions and arbitrary piece–wise continuous uniformly bounded reference signals r. Good convergence properties for the controller parameters are aimed as well, which is the motivation for pursuing a least-squares approach to the problem.

As usual, *state variable filters* (SVF's)

$$\dot{v}_1 = \Lambda v_1 + g u\,, \tag{332}$$

$$\dot{v}_2 = \Lambda v_2 + g y\,, \tag{333}$$

where Λ is Hurwtiz matrix, and v_1, $v_2 \in \mathbb{R}^{n-1}$, are used to assemble the *regressor vector*

$$\omega = [v_1^T \quad y \quad v_2^T \quad r]^T \in \mathbb{R}^{2n}\,. \tag{334}$$

This control structure assures the existence and uniqueness of a constant parameter vector $\theta^{*T} = [\theta_1^{*T} \quad \theta_n^* \quad \theta_2^{*T} \quad \theta_{2n}^*]$, with $\theta_{2n}^* = 1/k_p$, such that the transfer function of the closed–loop system with $u = \theta^{*T}\omega$ matches $M(s)$ exactly, i.e., $y = P(s)u = P(s)\theta^{*T}\omega = M(s)r$ (Tao, 2003, p. 197).

12.3.1 The error equation

The dynamics of the tracking error e_0, or *error equation*, is given by

$$e_0 = k_p\, M(s)\big[u - \theta^{*T}\omega\big]\,, \tag{335}$$

irrespective of how u is defined. This equation is the basis for developing the adaptive laws. A simple derivation is given in (Slotine and Li, 1991, p. 345).

Defining the signal

$$e_a = \operatorname{sign}(k_p) e_0 \,, \tag{336}$$

the error equation can be conveniently rewritten as

$$e_a = |k_p| \, M(s) \big[u - \theta^{*T} \omega \big] \,. \tag{337}$$

Note that for a SPR model $M(s)$, $|k_p| \, M(s)$ is also SPR. Therefore, the unknown parameter k_p can be absorbed by the error model which makes it linear in the parameter θ^*. This is the fundamental idea behind the solution to the MIMO case briefly reviewed in the next section.

12.3.2 Design procedure

In the conventional MRAC algorithm, the control law is chosen as

$$u = \theta^T \omega\,, \tag{338}$$

where the controller parameter $\theta \in \mathbb{R}^{2n}$ should be adaptively adjusted to ensure desired tracking performance. For the LS-MRAC algorithm, the control law is modified as (Costa, 2020)

$$u = L(s)\big[\theta^T \xi\big] = \theta^T \omega + \dot{\theta}^T \xi\,, \tag{339}$$

where $L(s) = s + \ell_0$, $\ell_0 > 0$, and

$$\xi = L^{-1}(s)[\omega]\,. \tag{340}$$

The objective of introducing a Monopoli's multiplier $L(s)$ (Narendra and Valavani, 1978; Ioannou and Sun, 1996) in (339) is to achieve an error equation with relative degree zero. With this control law, the error equation (337) becomes

$$\begin{aligned} e_a &= |k_p|\, M(s)\big[L(s)\theta^T \xi - \theta^{*T} \omega\big] \\ &= |k_p|\, M(s)L(s)\big[\tilde{\theta}^T \xi\big]\,, \end{aligned} \tag{341}$$

where $\tilde{\theta} = \theta - \theta^*$. The transfer function $M(s)L(s)$ can be decomposed as

$$M(s)L(s) = \frac{s+\ell_0}{s+a_m} = \alpha\, M(s) + 1\,, \tag{342}$$

where $\alpha = \ell_0 - a_m$. Thus, we get

$$e_a = \alpha\, |k_p|\, M(s)\big[\tilde{\theta}^T \xi\big] + |k_p|\, \big[\tilde{\theta}^T \xi\big]\,. \tag{343}$$

The total order of the system comprising the plant (329) and filters (332)-(333) and (340) is $5n-2$. Then, defining the error vector $e \in \mathbb{R}^{5n-2}$, we can write the following non–minimal state space realization of (343)

$$\dot{e} = A_m e + B_m\big[\tilde{\theta}^T \xi\big]\,, \tag{344}$$

$$e_a = C_m e + |k_p|\big[\tilde{\theta}^T \xi\big]\,, \tag{345}$$

where $\alpha\,|k_p|\,M(s) = C_m(sI - A_m)^{-1}B_m$. Selecting $\ell_0 > a_m$ ensures that $\alpha > 0$, and the transfer function $\alpha\,|k_p|\,M(s)$ is SPR. Then, from the Meyer–Kalman–Yakubovich (MKY) Lemma (Narendra and Annaswamy, 1989, p. 66), there exist matrices $P =$

$P^T > 0$ and $Q = Q^T > 0$ such that the realization $\{A_m, B_m, C_m\}$ satisfies

$$A_m^T P + P A_m = -2Q\,, \tag{346}$$

$$P B_m = C_m^T\,. \tag{347}$$

Consider the first Lyapunov function

$$2V_1(e) = e^T P e\,. \tag{348}$$

Using (346), the derivative of V along (344) yields

$$\dot{V}_1 = -e^T Q e + e^T P B_m \tilde{\theta}^T \xi\,.$$

From (347) and (345) we have

$$e^T P B_m = e^T C_m^T = e_a - |k_p| \left[\tilde{\theta}^T \xi\right]. \tag{349}$$

Therefore,

$$\begin{aligned}\dot{V}_1 &= -e^T Q e + \big(e_a - |k_p|\tilde{\theta}^T\xi\big)\tilde{\theta}^T\xi \\ &= -e^T Q e - |k_p|\left(\tilde{\theta}^T\xi\right)^2 + \tilde{\theta}^T\xi e_a\,. \end{aligned} \tag{350}$$

The term $-|k_p|\left(\tilde{\theta}^T\xi\right)^2$ obtained in (350) is the key to introduce a least-squares update law in the algorithm. Now consider the second Lyapunov function

$$V_2(e,\tilde{\theta}) = \gamma V_1(e) + \frac{1}{2}\tilde{\theta}^T R^{-1}(t)\tilde{\theta}\,, \tag{351}$$

where $\gamma > 0$ and $R(t)$ is a *covariance matrix* with $R(0) = R^T(0) > 0$. Using the fact that

$$\dot{R}^{-1} = -R^{-1}\dot{R}R^{-1}\,, \tag{352}$$

we obtain

$$\begin{aligned}\dot{V}_2 = -\gamma e^T Q e - \gamma|k_p|\left(\tilde{\theta}^T\xi\right)^2 + \\ + \tilde{\theta}^T\big(\gamma\,\xi e_a + R^{-1}\dot{\theta}\big) + \frac{1}{2}\tilde{\theta}^T R^{-1}\dot{R}R^{-1}\tilde{\theta}\,.\end{aligned}$$

Choosing the update laws

$$\dot{\theta} = -\gamma\, R\xi e_a\,, \tag{353}$$

$$\dot{R} = -R\xi\xi^T R\,, \tag{354}$$

we get

$$\begin{aligned}\dot{V}_2 &= -\gamma e^T Q e - \gamma|k_p|\left(\tilde{\theta}^T\xi\right)^2 + \frac{1}{2}\tilde{\theta}^T\xi\xi^T\tilde{\theta}\\ &= -\gamma e^T Q e - \left(\gamma|k_p| - \frac{1}{2}\right)\left(\tilde{\theta}^T\xi\right)^2 .\end{aligned} \tag{355}$$

12.3.3 Summary of the LS-MRAC

Table 1 summarizes the LS-MRAC algorithm.

Tracking error	$e_a = \operatorname{sign}(k_p)\,(y - y_m)$	(336)
SV filters	$\dot{v}_1 = \Lambda\, v_1 + g\, u$	(332)
	$\dot{v}_2 = \Lambda\, v_2 + g\, y$	(333)
	$\omega^T = [v_1^T \quad y \quad v_2^T \quad r]$	
ξ-filter	$\dot{\xi} = -\ell_0\, \xi + \omega\,, \qquad \ell_0 > a_m$	(340)
Control	$u = \theta^T \omega + \dot{\theta}^T \xi$	(339)
Update laws	$\dot{\theta} = -\gamma\, R\xi e_a$	(353)
	$\dot{R} = -R\xi\xi^T R$	(354)
	$R(0) = R^T(0) > 0$	

Tabela 1: LS-MRAC algorithm.

The M-MRAC algorithm described in (Costa, 2020) is a particular case of the LS-

MRAC. It is easily obtained by making the following simplification

$$\dot{R} = 0\,,$$
$$R(0) = \gamma^{-1}\Gamma\,, \quad \Gamma = \Gamma^T > 0\,.$$

12.3.4 Stability

Theorem: For the closed-loop system consisting of the plant (329), reference model (330), and the LS-MRAC algorithm summarized in Table 1, if

$$\gamma > \frac{1}{2|k_p|}\,, \tag{356}$$

then all its signals are globally uniformly bounded and $e, \tilde{\theta}^T\xi \to 0$ as $t \to \infty$.

Proof If conditions (356) is satisfied, then from (355)

$$\dot{V}_2(e,\tilde{\theta}) \le 0\,.$$

This proves that $e, \tilde{\theta}^T R^{-1}\tilde{\theta} \in \mathcal{L}_\infty$ and $e, \tilde{\theta}^T\xi \in \mathcal{L}_2$. Since ω and ξ are part of the error signal e, then $\omega, \xi \in \mathcal{L}_\infty$. Also, from (340), $\dot{\xi} \in \mathcal{L}_\infty$.

Boundedness of R and θ are obtained as in (Tao, 2003, p. 104). Since $R(0) = R^T(0) > 0$, then

$$\dot{R}^{-1}(t) = \xi\xi^T.$$

By integrating, one has that

$$R^{-1}(t) = R^{-1}(0) + J(t) \geq 0\,, \quad t \geq 0\,, \tag{357}$$

where

$$J(t) = \int_0^t \xi(\tau)\xi^T(\tau)d\tau\,.$$

Therefore, $R^{-1}(t) > R^{-1}(0)$, and so $R(t) > 0$, $\forall t \geq 0$, and $R \in \mathcal{L}_\infty$. This means that $\dot{\theta}, \dot{R} \in \mathcal{L}_\infty$. From (351) and (357),

$$2V_2(e, \tilde{\theta}) = 2V_1(e) + \tilde{\theta}^T R^{-1}(0)\tilde{\theta} + \tilde{\theta}^T J(t)\tilde{\theta}\,.$$

Since $V_2(e, \tilde{\theta}) \in \mathcal{L}_\infty$, then the term $\tilde{\theta}^T R^{-1}(0)\tilde{\theta} \in \mathcal{L}_\infty$ and, hence, $\theta \in \mathcal{L}_\infty$. From (353), $\dot{\theta} \in \mathcal{L}_\infty$. Therefore, it can be concluded that $u \in \mathcal{L}_\infty$ and, consequently, all system signals are bounded.

Moreover, since $e, \tilde{\theta}^T \xi \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\dot{e}, \frac{d}{dt}\tilde{\theta}^T \xi \in \mathcal{L}_\infty$ then, from Barbălat Lemma (Tao, 2003, p. 81), it follows that $\lim_{t\to\infty} e(t) = 0$ and $\lim_{t\to\infty} \tilde{\theta}^T \xi(t) = 0$. ■

The control law (339) can be written as

$$\begin{aligned} u &= \theta^T \omega + \dot{\theta}^T \xi \\ &= \theta^T \omega - \gamma \xi^T R \xi e_a \,, \end{aligned}$$

which clearly shows a feedback of the tracking error e_a. This is the feature that accounts for the algorithm remarkable tracking error dynamics. From (337), we obtain

$$\dot{e}_a = -\big(a_m + \gamma\,|k_p|\xi^T R\,\xi\big)e_a + |k_p|\tilde{\theta}^T \omega\,.$$

Here we see that the feedback term $\gamma \xi^T R \xi e_a$ acts in the way to increase the convergence rate of e_a and to attenuate the effect of the term $\tilde{\theta}^T \omega$.

The control mismatch is given by

$$\begin{aligned} \tilde{u} &= u - \theta^{*T} \omega \\ &= \tilde{\theta}^T \omega + \dot{\theta}^T \xi \,. \end{aligned}$$

This means that it is possible to attain $\tilde{u} \approx 0$ and, consequently, $e_a \approx 0$ even with large $\tilde{\theta}^T \omega$. In other words, $e_a \approx 0$ when $\tilde{\theta}^T \omega \approx -\dot{\theta}^T \xi$. This feature is illustrated by simulation results in (Costa, 2020).

12.4 The MIMO case

The LS-MRAC algorithm presented in the previous section is now extended to the MIMO case. This extension is achieved quite straightforwardly by applying the design procedure of the SISO case to the MIMO MRAC algorithm introduced in (Costa et al., 2003).

12.4.1 Preliminaries

In the MIMO case the plant and the reference model are modeled as

$$y = P(s)\,u\,, \quad P(s) = K_p N(s) D^{-1}(s)\,, \tag{358}$$

$$y_m = M(s)\,r\,, \tag{359}$$

where $P(s)$ and $M(s)$ are $m \times m$ transfer matrices, and $y, u, y_m, r \in \mathbb{R}^m$. The high frequency gain matrix K_p and the coefficients of the $m \times m$ polynomial matrices $D(s)$ and $N(s)$ are unknown.

The following assumptions regarding $P(s)$ are required:

1) the observability index ν of $P(s)$ is known,

2) $P(s)$ has relative degree 1,

3) the transmission zeros of $P(s)$ have negative real parts,

4) the signs of the leading principal minors $\{\Delta_i\}_{i=1}^m$ of the matrix K_p are known, and

5) upper and lower bounds for $\{|\Delta_i|\}_{i=1}^m$ are known.

Assumption 5) is a counterpart of a similar assumption made for the SISO LS-MRAC. In view of assumption 2), $M(s)$ is chosen as

$$M(s) = \operatorname{diag}\left\{\frac{1}{s+a_i}\right\}_{i=1}^m, \quad a_i > 0\,, \tag{360}$$

which can be realized by $\{A, I, I\}$ where

$$A = \operatorname{diag}\{-a_i\}_{i=1}^m\,. \tag{361}$$

The *state variable filters* are given by

$$\dot{v}_{1,i} = \Lambda v_{1,i} + g u_i\,, \quad v_{1,i} \in \mathbb{R}^{\nu-1}\,, \tag{362}$$

$$\dot{v}_{2,i} = \Lambda v_{2,i} + g y_i\,, \quad v_{2,i} \in \mathbb{R}^{\nu-1}\,, \tag{363}$$

$$\begin{aligned} v_1^T &= \begin{bmatrix} v_{1,1}^T & v_{1,2}^T & \cdots & v_{1,m}^T \end{bmatrix} \in \mathbb{R}^{m(\nu-1)}\,, \\ v_2^T &= \begin{bmatrix} v_{2,1}^T & v_{2,2}^T & \cdots & v_{2,m}^T \end{bmatrix} \in \mathbb{R}^{m(\nu-1)}\,, \end{aligned}$$

where Λ is a Hurwitz matrix. The regressor vector is set as

$$\omega^T = \begin{bmatrix} v_1^T & v_2^T & y^T & r^T \end{bmatrix} \in \mathbb{R}^{2m\nu} .$$

If $P(s)$ is known, then a control law which achieves matching between the closed-loop transfer matrix and $M(s)$, i.e. $y = P(s)u^* = M(s)r = y_m$, is given by (Tao, 2003)

$$u^* = \theta_1^{*T} v_1 + \theta_2^{*T} v_2 + \theta_3^* y + \theta_4^* r = \theta^{*T} \omega \,, \tag{364}$$

where $\theta^{*T} = \begin{bmatrix} \theta_1^{*T} & \theta_2^{*T} & \theta_3^* & \theta_4^* \end{bmatrix}$, $\theta_1^*, \theta_2^* \in \mathbb{R}^{m(\nu-1)\times m}$, $\theta_3^* \in \mathbb{R}^{m\times m}$, and $\theta_4^* = K_p^{-1}$.

The dynamics of the tracking error is given by (Tao, 2003, p. 201)

$$e_0 = M(s) K_p \left[u - \theta^{*T} \omega \right] . \tag{365}$$

Except for the fact that $M(s)$ and K_p are matrices and e_0 and u are vectors, this MIMO error equation has the same form as the SISO error equation (335). Another important difference with respect to the SISO case is that, in general, the MIMO case lacks the uniqueness of the matching parameter θ^*.

12.4.2 SDU factorization

The MIMO MRAC algorithm proposed in (Costa et al., 2003) relies on a control parametrization derived from an SDU factorization of the matrix K_p. The following Lemma is central for this algorithm. It assures the existence of a SDU factorization.

Lemma 1 *Every $m \times m$ real matrix K_p with nonzero leading principal minors $\{\Delta_i\}_{i=1}^{m}$ can be factored as*

$$K_p = SDU, \tag{366}$$

where S is symmetric positive definite, D is diagonal, and U is unity upper triangular.

Proof 1 *(From (Costa et al., 2003).)*

Since all Δ_i are nonzero, there exists a unique factorization (?),

$$K_p = L_p D_p U_p\,, \tag{367}$$

where L_p is unity lower triangular, U_p is unity upper triangular, and

$$D_p = \; diag\left\{\Delta_1, \frac{\Delta_2}{\Delta_1}, \ldots, \frac{\Delta_m}{\Delta_{m-1}}\right\} \,. \tag{368}$$

Factoring D_p as

$$D_p = D_+ D \,, \tag{369}$$

where D_+ is a diagonal matrix with positive entries, (367) is rewritten as

$$K_p = \left(L_p D_+ \, L_p^T\right) D \left(D^{-1} L_p^{-T} D \, U_p\right) ,$$

so that (366) is satisfied by

$$S = L_p \, D_+ \, L_p^T \,, \tag{370}$$

$$U = D^{-1} L_p^{-T} D \, U_p \,. \tag{371}$$

In the SISO case, the unknown term $|k_p|$ is easily absorbed by the error model, that is, for a given SPR model $M(s)$, $|k_p|M(s)$ is still SPR. The idea in the MIMO case is similar. However, a symmetry condition also shows up. Thus, only the factor S can be absorbed, that is, only $M(s)S$ can remain SPR.

12.4.3 Control parametrization

Applying the SDU factorization (366), the error equation (365) is rewritten as

$$\begin{aligned} e_0 &= M(s)SDU[u - \theta^{*T}\omega] \\ &= M(s)SD\left[Uu - U\theta^{*T}\omega\right] \\ &= M(s)SD\left[u - U\theta^{*T}\omega - (I-U)u\right]. \end{aligned} \tag{372}$$

Note that the matrix $(I-U)$ is strictly upper triangular. Therefore, the control signal u can be defined as a function of $(I-U)u$ without given rise to any static loop since no individual control signal u_i would depend on itself.

The unknown U is incorporated in the parametrization by introducing the parameter vector Θ^* via the identity

$$\Theta^{*T}\Omega \equiv U\theta^{*T}\omega + (I-U)u\,, \tag{373}$$

where

$$\Theta^{*T} = \begin{bmatrix} \Theta_1^{*T} & \Theta_2^{*T} & \cdots & \Theta_m^{*T} \end{bmatrix}, \tag{374}$$

$$\Omega^T = \operatorname{diag}\{\Omega_i^T\}_{i=1}^m, \tag{375}$$

and

$$\begin{aligned} \Omega_1^T &= [\omega^T \quad u_2 \quad u_3 \ \ldots \ u_m], \\ \Omega_2^T &= [\omega^T \quad u_3 \ \ldots \ u_m], \\ &\vdots \\ \Omega_{m-1}^T &= [\omega^T \quad u_m], \\ \Omega_m^T &= [\omega^T]. \end{aligned} \tag{376}$$

Each vector Θ_i^{*T} concatenates the i-th row of the matrix $U\theta^{*T}$ and also the non zero entries of the i-th row of $(I-U)$. The error equation (372) has thus been brought to the form

$$e_0 = M(s)S\,D\left[u - \Omega^T\Theta^*\right]. \tag{377}$$

Note that this parametrization introduces $m(m-1)/2$ additional adaptive parameters.

12.4.4 SPR PROPERTY

It is fundamental that $M(s)S$ satisfy the SPR property. This means that a minimal realization $\{A, S, I\}$ of $M(s)S$ should satisfy the Popov-Kalman-Yakubovich (PKY) Lemma (?)

$$A^T P + PA = -2Q\,, \tag{378}$$

$$PS = I\,, \tag{379}$$

with $P = P^T > 0$ and $Q = Q^T > 0$. In (Costa et al., 2003) this property is ensured by the following Lemma.

Lemma 2 *For any $A = \; diag\{-a_i\}_{i=1}^m$, $a_i > 0$, and any $m \times m$ unity lower triangular matrix L_p, there is a matrix $D_+ = \; diag\{d_i^+\}_{i=1}^m$, $d_i^+ > 0$, such that $(sI - A)^{-1}S$ is SPR with $S = L_p\, D_+\, L_p^T$.*

The proof is fairly simple (see (Costa et al., 2003)). It relies on the fact that the factor S is not unique and can be scaled by a matrix D_+. From (379), it follows that

$P = S^{-1}$. Then, from (378), the condition

$$2Q = -(AS^{-1} + S^{-1}A) > 0 \tag{380}$$

is satisfied by consecutively selecting $d_m^+, d_{m-1}^+, \ldots, d_1^+$, such that the leading principal minors of Q are all positive.

An alternative parametrization of the matrix D_+ is proposed in Lemma 3 bellow. This result will be useful to conclude the stability of the MIMO extension of the LS-MRAC algorithm developed in the next.

Lemma 3 *For any* $A = diag\{-a_i\}_{i=1}^m$, $a_i > 0$, *and any* $m \times m$ *unity lower triangular matrix* L_p, *there is a positive constant* d^* *such that*

$$2Q = -(AS^{-1} + S^{-1}A) > 0$$

with $S = L_p D_+ L_p^T$ *and*

$$D_+ = \, diag\Big\{1,\ d_+^2,\ d_+^4,\ \cdots,\ d_+^{2(m-1)}\Big\}, \quad \forall d_+ > d^*\,.$$

Proof 2 *Using the factorization $D_+ = D_+^{1/2} D_+^{1/2}$, we have*

$$
\begin{aligned}
2Q &= -S^{-1}\big(SA + AS\big)S^{-1} \\
&= -S^{-1}\big(L_p D_+ L_p^T A + A L_p D_+ L_p^T\big)S^{-1} \\
&= -S^{-1} D_+^{1/2}\big(D_+^{-1/2} L_p D_+^{1/2} D_+^{1/2} L_p^T D_+^{-1/2} A + \\
&\qquad + A D_+^{-1/2} L_p D_+^{1/2} D_+^{1/2} L_p^T D_+^{-1/2}\big) D_+^{1/2} S^{-1} \\
&= -S^{-1} D_+^{1/2}\big(L_+ L_+^T A + A L_+ L_+^T\big) D_+^{1/2} S^{-1}\,,
\end{aligned}
\tag{381}
$$

where,

$$
L_+ = D_+^{-1/2} L_p D_+^{1/2}\,. \tag{382}
$$

Denoting the matrix L_p as

$$L_p = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \ell_{21} & 1 & 0 & \cdots & 0 \\ \ell_{31} & \ell_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \ell_{m1} & \ell_{m2} & \ell_{m3} & \cdots & 1 \end{bmatrix},$$

we get the following pattern from (382)

$$L_+ = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \dfrac{\ell_{21}}{d_+} & 1 & 0 & 0 & \cdots & 0 \\ \dfrac{\ell_{31}}{d_+^2} & \dfrac{\ell_{32}}{d_+} & 1 & 0 & \cdots & 0 \\ \dfrac{\ell_{41}}{d_+^3} & \dfrac{\ell_{42}}{d_+^2} & \dfrac{\ell_{43}}{d_+} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{\ell_{m1}}{d_+^{m-1}} & \dfrac{\ell_{m2}}{d_+^{m-2}} & \dfrac{\ell_{m3}}{d_+^{m-3}} & \dfrac{\ell_{m4}}{d_+^{m-4}} & \cdots & 1 \end{bmatrix},$$

which clearly shows that

$$\lim_{d_+ \to \infty} L_+ = I\,.$$

This means that for a sufficiently large d_+, $L_+ L_+^T$ *is diagonal dominant. Consequently,* *there exists* $d^* > 0$ *such that*

$$L_+ L_+^T A + A L_+ L_+^T < 0\,, \quad \forall d_+ > d^*\,,$$

and, as a result, $Q > 0$*.*

12.4.5 Design procedure

Following the steps of the SISO case design, the control law is chosen as

$$u = L(s)\big[\Xi^T \Theta\big] = \Omega^T \Theta + \Xi^T \dot{\Theta}\,, \tag{383}$$

where

$$L(s) = (s + \ell_0)\, I\,, \tag{384}$$

with $\ell_0 > 0$ and $I \in \mathbb{R}^{m\times m}$, and

$$\Xi^T = L^{-1}(s)\Omega^T\,. \tag{385}$$

Applying the control (383), the error equation (377) becomes

$$\begin{aligned} e_0 &= M(s)L(s)SD\left[\,\Xi^T\Theta - \Xi^T\Theta^*\right] \\ &= M(s)L(s)SD\left[\,\Xi^T\tilde{\Theta}\,\right], \end{aligned} \tag{386}$$

where $\tilde{\Theta} = \Theta - \Theta^*$. Now, introduce the decomposition

$$\begin{aligned} M(s)L(s) &= \operatorname{diag}\Big\{\frac{s+\ell_0}{s+a_i}\Big\}_{i=1}^{m} \\ &= \operatorname{diag}\Big\{\frac{\alpha_i}{s+a_i}+1\Big\}_{i=1}^{m} \\ &= \mathcal{A}M(s)+I\,, \end{aligned}$$

where

$$\mathcal{A} = \operatorname{diag}\{\alpha_i\}_{i=1}^{m}\,, \quad \alpha_i = \ell_0 - a_i\,.$$

Then, the error equation becomes

$$\begin{aligned} e_0 &= \big(\mathcal{A}M(s)+I\big)SD\big[\Xi^T\tilde{\Theta}\big] \\ &= M(s)\,\mathcal{A}SD\big[\Xi^T\tilde{\Theta}\big] + SD\big[\Xi^T\tilde{\Theta}\big]\,. \end{aligned} \tag{387}$$

The design parameter ℓ_0 can be selected such that

$$\mathcal{A} = \ell_0 I + A > 0\,.$$

However, the matrix $\mathcal{A}S$ is not symmetric in general. Symmetry is ensured only when $\mathcal{A}$ and, consequently, $M(s)$ are identity matrices, a restriction that we want to avoid. To preserve the symmetry property, the error equation is rewritten as

$$\mathcal{A}^{-1}e_0 = M(s)SD\big[\Xi^T\tilde{\Theta}\big] + \mathcal{A}^{-1}SD\big[\Xi^T\tilde{\Theta}\big]\,. \tag{388}$$

The total order of the system comprising the plant (358) and filters (362)-(363) and (385) is $N = n + 4m\nu - m - 1$. Note that we need to filter only Ω_1 to form Ξ. Then, defining the error vector $e \in \mathbb{R}^N$, we can write the following non–minimal state space realization of (388)

$$\dot{e} = A_m e + B_m D\big[\Xi^T\tilde{\Theta}\big]\,, \tag{389}$$

$$\mathcal{A}^{-1}e_0 = C_m e + \mathcal{A}^{-1}SD\big[\Xi^T\tilde{\Theta}\big]\,, \tag{390}$$

where $M(s)S = C_m(sI - A_m)^{-1}B_m$, $C_m B_m = S$, and $\{A_m, B_m, C_m\}$ satisfies the MKY Lemma

$$A_m^T P' + P' A_m = -2Q'\,, \tag{391}$$

$$P' B_m = C_m^T\,, \tag{392}$$

with $P' = P'^T > 0$ and $Q' = Q'^T > 0$. Consider the first Lyapunov function

$$2V_1(e) = e^T P' e \,. \tag{393}$$

Using (391), the derivative of V_1 along the trajectories of (389) is given by

$$\dot{V}_1 = -e^T Q' e + \tilde{\Theta}^T \Xi D B_m^T P' e \,.$$

From (392) and (390) one has that

$$B_m^T P' e = C_m e = \mathcal{A}^{-1} e_0 - \mathcal{A}^{-1} S D \, \Xi^T \tilde{\Theta} \,, \tag{394}$$

which, after replaced in the above $\dot{V}_1$ expression, gives

$$\dot{V}_1 = -e^T Q' e + \tilde{\Theta}^T \Xi D \mathcal{A}^{-1} e_0 - \tilde{\Theta}^T \Xi D Q_2 D \Xi^T \tilde{\Theta} \,, \tag{395}$$

where

$$Q_2 = \left(\mathcal{A}^{-1} S + S \mathcal{A}^{-1} \right) / 2 \tag{396}$$

is the symmetric part of $\mathcal{A}^{-1} S$. Now define the matrix

$$\mathbb{D} = \operatorname{diag}\left\{ d_i \alpha_i^{-1} I_i \right\}_{i=1}^{m} \,, \tag{397}$$

where d_i are the diagonal elements of matrix D, $I_i \in \mathbb{R}^{N_i \times N_i}$, $N_i = (2m\nu + m - i)$, and consider a second Lyapunov function

$$2V_2(e, \tilde{\Theta}) = 2\gamma V_1(e) + \tilde{\Theta}^T |\mathbb{D}|\, R^{-1}(t)\, \tilde{\Theta}\,, \quad \gamma > 0\,, \tag{398}$$

where

$$\begin{aligned} R(t) &= \operatorname{diag}\big\{R_i(t)\big\}_{i=1}^{m}\,, \\ R_i(0) &= R_i(0)^T > 0\,, \quad R_i(0) \in \mathbb{R}^{N_i \times N_i}\,. \end{aligned}$$

Note that $|\mathbb{D}|R^{-1} = R^{-1}|\mathbb{D}|$. Using the fact that $\dot{R}^{-1} = -R^{-1}\dot{R}R^{-1}$, the derivative of V_2 results

$$\dot{V}_2 = \gamma \dot{V}_1 + \tilde{\Theta}^T |\mathbb{D}| R^{-1} \dot{\Theta} - \frac{1}{2} \tilde{\Theta}^T |\mathbb{D}| R^{-1} \dot{R} R^{-1} \tilde{\Theta}\,. \tag{399}$$

In view of the above equation, we choose the update law

$$\dot{R} = -R\, \Xi\, \Xi^T R\,. \tag{400}$$

Since Ξ is block diagonal, then $\Xi\,\Xi^T$ is also block diagonal. Hence, for $R(0)$ block diagonal, $\dot{R}$ and R result block diagonal.

Thus, the derivative (399) becomes

$$\dot{V}_2 = \gamma \dot{V}_1 + \tilde{\Theta}^T |\mathbb{D}| R^{-1} \dot{\Theta} + \frac{1}{2} \tilde{\Theta}^T |\mathbb{D}| \Xi\,\Xi^T \tilde{\Theta}\,. \tag{401}$$

Now, using the fact that $\mathbb{D}\Xi = \Xi\,D\mathcal{A}^{-1}$, $\dot{V}_2$ becomes

$$\begin{aligned}
\dot{V}_2 &= -\gamma e^T Q' e + \gamma \tilde{\Theta}^T \mathbb{D} \Xi e_0 + \tilde{\Theta}^T |\mathbb{D}| R^{-1} \dot{\Theta} - \\
&\qquad - \gamma \tilde{\Theta}^T \Xi D Q_2 D \Xi^T \tilde{\Theta} + \frac{1}{2} \tilde{\Theta}^T \Xi\,|D|\,\mathcal{A}^{-1} \Xi^T \tilde{\Theta} \\
&= -\gamma e^T Q' e + \tilde{\Theta}^T |\mathbb{D}| \Big(\gamma \operatorname{sign}(\mathbb{D}) \Xi e_0 + R^{-1} \dot{\Theta}\Big) - \\
&\qquad - \tilde{\Theta}^T \Xi D \Big(\gamma Q_2 - \frac{1}{2} |D|^{-1}\,\mathcal{A}^{-1}\Big) D \Xi^T \tilde{\Theta}\,,
\end{aligned} \tag{402}$$

which, upon setting the update law

$$\begin{aligned}
\dot{\Theta} &= -\gamma\,R \operatorname{sign}(\mathbb{D}) \Xi e_0 \\
&= -\gamma\,R\,\Xi \operatorname{sign}(D) e_0\,,
\end{aligned} \tag{403}$$

is reduced to

$$\dot{V}_2 = -\gamma e^T Q' e - \tilde{\Theta}^T \Xi D \Big(\gamma Q_2 - \frac{1}{2} |D|^{-1} \mathcal{A}^{-1} \Big) D \Xi^T \tilde{\Theta} \,. \tag{404}$$

12.4.6 Summary of the MIMO LS-MRAC

Table 2 summarizes the MIMO LS-MRAC algorithm.

Tracking error	$e_0 = y - y_m$
SV filters	$\dot{v}_{1,i} = \Lambda v_{1,i} + g u_i$ (362) $\dot{v}_{2,i} = \Lambda v_{2,i} + g y_i$ (363) $v_1^T = [v_{1,1}^T \;\; v_{1,2}^T \;\; \cdots \;\; v_{1,m}^T]$ $v_2^T = [v_{2,1}^T \;\; v_{2,2}^T \;\; \cdots \;\; v_{2,m}^T]$ $\omega^T = [v_1^T \;\; y^T \;\; v_2^T \;\; r^T]$ $\Omega_i^T = [\omega^T \;\; u_{i+1} \;\; \cdots \;\; u_m]$ (376) $\Omega^T = \operatorname{diag}\{\Omega_i^T\}_{i=1}^m$ (375)
Ξ-filter	$\dot{\Xi} = -\ell_0\, \Xi + \Omega$ (385) $\ell_0 > a_i\,, \quad \forall i \in [1, m]$
Control	$u = \Omega^T \Theta + \Xi^T \dot{\Theta}$ (383)
Update laws	$\dot{\Theta} = -\gamma\, R\, \Xi\, \operatorname{sign}(D)\, e_0$ (403) $\dot{R} = -R\, \Xi\, \Xi^T R$ (400) $R(0) = \operatorname{diag}\{R_i(0)\}_{i=1}^m$ $R_i(0) = R_i^T(0) > 0$ $R_i(0) \in \mathbb{R}^{N_i \times N_i}$

Tabela 2: MIMO LS-MRAC algorithm.

The MIMO version of the M-MRAC algorithm described in (Costa, 2020) can be obtained from Table 2 by making

$$\begin{aligned}\dot{R} &= 0\,,\\ R(0) &= \gamma^{-1}\,\mathrm{diag}\{\Gamma_i\}_{i=1}^{m}\,,\\ \Gamma_i &= \Gamma_i^T > 0\,, \quad \Gamma_i \in \mathbb{R}^{N_i\times N_i}\,.\end{aligned}$$

12.4.7 Stability

Theorem 1 *For the closed-loop system consisting of the plant (358), reference model (359), and the MIMO LS-MRAC algorithm summarized in Table 2, if*

$$\gamma > \frac{1}{2} \max_i \left(|d_{p_i}|^{-1}\right) , \quad i \in [1, m] , \tag{405}$$

where $\{d_{p_i}\}_{i=1}^m$ are the diagonal elements of matrix D_p, then all its signals are globally uniformly bounded and $e, \tilde{\Theta}^T \Xi(t) \to 0$ as $t \to \infty$.

Proof 3 *Lemma 3 ensures the existence of a constant $d^* > 0$ such that $Q > 0$, $\forall d_+ > d^*$ and $M(s)S$ is SPR. Similarly to (381), we write matrix Q_2 as*

$$2Q_2 = D_+^{1/2}\left(\mathcal{A}^{-1} L_+ L_+^T + L_+ L_+^T \mathcal{A}^{-1}\right) D_+^{1/2} ,$$

where L_+ is given by (382). Thus, Lemma 3 also ensures the existence of a constant $d_2^* > 0$ such that $Q_2 > 0$, $\forall d_+ > d_2^*$. Therefore, if $d_+ > \max\{d^*,\ d_2^*\}$, then $M(s)S$ is SPR and, simultaneously, $Q_2 > 0$.

Denote the matrix

$$Q_3 = \gamma Q_2 - \frac{1}{2}|D|^{-1}\,\mathcal{A}^{-1}\,. \tag{406}$$

Using (369), we obtain

$$\begin{aligned} 2Q_3 = D_+^{1/2}\Big[\gamma\big(\mathcal{A}^{-1}L_+L_+^T + L_+L_+^T\mathcal{A}^{-1}\big) - \\ - |D_p|^{-1}\mathcal{A}^{-1}\Big]D_+^{1/2}\,. \end{aligned}$$

Since $\lim_{d_+\to\infty} L_+L_+^T = I$, *we have that, in the limit,*

$$2Q_3 = D_+^{1/2}\Big[2\gamma\mathcal{A}^{-1} - |D_p|^{-1}\mathcal{A}^{-1}\Big]D_+^{1/2}\,.$$

Therefore, if conditions (405) is satisfied, then $Q_3 > 0$ *and*

$$\dot{V}_2(e,\tilde{\Theta}) = -\gamma e^T Q' e - \tilde{\Theta}^T \Xi D Q_3 D \Xi^T \tilde{\Theta} \le 0\,. \tag{407}$$

This guarantees that $e, \Theta \in \mathcal{L}_\infty$*, the filtered signals* $\omega, \Xi \in \mathcal{L}_\infty$*, and* $e, \tilde{\Theta}^T \Xi \in \mathcal{L}_2$*. Since* $R(0) = R^T(0) > 0$*, then from (400)*

$$\dot{R}^{-1}(t) = \Xi \Xi^T \,,$$

which by integrating gives

$$R^{-1}(t) = R^{-1}(0) + J(t) \geq 0 \,, \quad t \geq 0 \,, \tag{408}$$

where

$$J(t) = \int_0^t \Xi(\tau) \Xi^T(\tau) d\tau \,.$$

Therefore, $R^{-1}(t) > R^{-1}(0)$*, and so* $R(t) > 0$*,* $\forall t \geq 0$*, and* $R \in \mathcal{L}_\infty$*. This implies that* $\dot{R}, \dot{\Theta} \in \mathcal{L}_\infty$*. From (398) and (408),*

$$2V_2(e, \tilde{\Theta}) = 2\gamma V_1(e) + \tilde{\Theta}^T |\mathbb{D}| R^{-1}(0) \tilde{\Theta} + \tilde{\Theta}^T |\mathbb{D}| J(t) \tilde{\Theta} \,.$$

Since $V_2 \in \mathcal{L}_\infty$*, then the term* $\tilde{\Theta}^T |\mathbb{D}| R^{-1}(0) \tilde{\Theta} \in \mathcal{L}_\infty$ *and, consequently,* $\Theta \in \mathcal{L}_\infty$*. Taking the equations in (376), from the bottom up, we easily get that* $(\Omega_m \in \mathcal{L}_\infty) \Rightarrow$

$(u_m \in \mathcal{L}_\infty) \Rightarrow (\Omega_{m-1} \in \mathcal{L}_\infty) \Rightarrow (u_{m-1} \in \mathcal{L}_\infty) \Rightarrow \ldots \Rightarrow (u_2 \in \mathcal{L}_\infty) \Rightarrow (\Omega_1 \in \mathcal{L}_\infty) \Rightarrow (u_1 \in \mathcal{L}_\infty)$. *This allows to conclude that* $\Omega, \dot{\Xi} \in \mathcal{L}_\infty$, *and* $u \in \mathcal{L}_\infty$. *Therefore, all system signals are globally uniformly bounded.*

Moreover, since $e, \tilde{\Theta}^T \Xi \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ *and* $\dot{e}, \frac{d}{dt}\tilde{\Theta}^T \Xi \in \mathcal{L}_\infty$ *then, from Barbălat Lemma (Tao, 2003, p. 81), it follows that* $\lim_{t\to\infty} e = 0$ *and* $\lim_{t\to\infty} \tilde{\Theta}^T \Xi = 0$.

12.5 Simulation results

To illustrate the improvement introduced by the presented MIMO algorithms two plants are considered (?). The first example corresponds to the simplest possible 2I2O adaptive control problem that appears in visual servoing with an uncalibrated camera. The system is described by

$$P(s) = \operatorname{diag}\left\{\frac{1}{s+2}, \frac{1}{s+2}\right\} K_p\,, \tag{409}$$

$$K_p = \begin{bmatrix} \cos\phi & \sin\phi \\ -h\sin\phi & h\cos\phi \end{bmatrix}, \tag{410}$$

$$M(s) = \operatorname{diag}\left\{\frac{2}{s+2}, \frac{2}{s+2}\right\}, \tag{411}$$

where the only unknown parameters are ϕ and h. The simulation results are obtained with $\phi = 1$ and $h = 0.5$. The corresponding matching parameter for this system is $\Theta_1^{*T} = [0.54 \;\; -1.68 \;\; 0]$, $\Theta_2^{*T} = [0.84 \;\; 1.08]$, $\|\Theta^*\| = 2.34$.

The second example is a 2I2O plant of third order. The simulations are performed with

$$P(s) = \frac{1}{s^2-1}\begin{bmatrix} s+3 & 2s \\ -2s-4 & s+3 \end{bmatrix}, \tag{412}$$

$$M(s) = \operatorname{diag}\left\{\frac{1}{s+1}, \frac{2}{s+2}\right\}. \tag{413}$$

This plant has poles at $s = \{1, 1, -1\}$, transmission zero at $s = -1.8$, and

$$K_p = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

This case requires 17 parameters to be adapted, that is, $\Omega_1 \in \mathbb{R}^9$, and $\Omega_2 \in \mathbb{R}^8$. For all simulations

$$y_p(0) = [1 \quad 1]^T, \qquad y_M(0) = [0 \quad 0]^T,$$
$$r(t) = [1 + 10\sin(5t) \quad -1 + 5\sin(3t)]^T.$$

All initial conditions not mentioned are zero.

SIMULATION 1. For comparison, Fig. 140 shows the simulation results of the plant (409) with the MIMO MRAC algorithm of (Costa et al., 2003). The following data is used:

$$\Gamma = 10\, I\,.$$

Note the large transient tracking error behavior. The plane view given by Fig. 140(e) makes clear that this error is quite unacceptable. Increasing the gain Γ in this algorithm does not help.

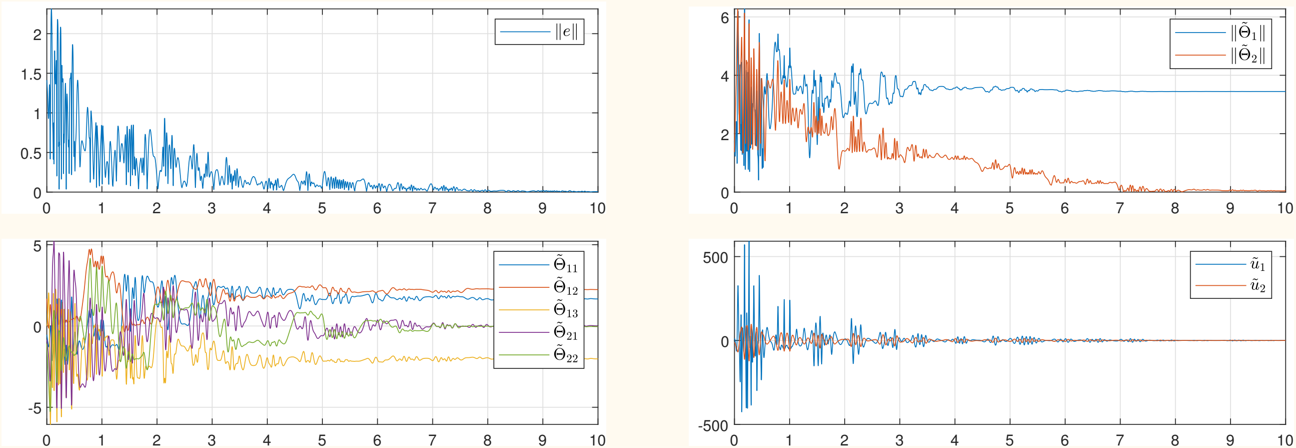


Figura 140: Simulation result of the first order plant (409) with the MIMO MRAC algorithm.

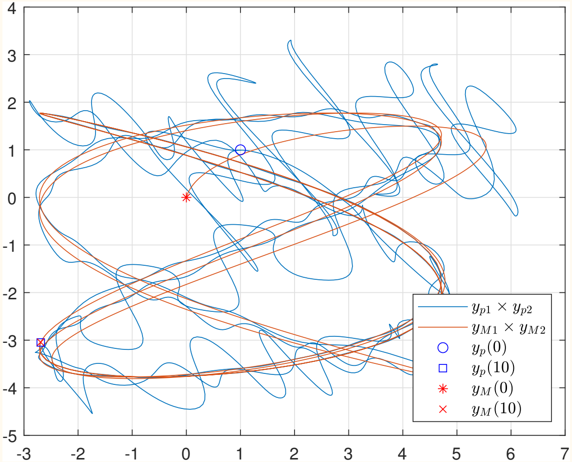


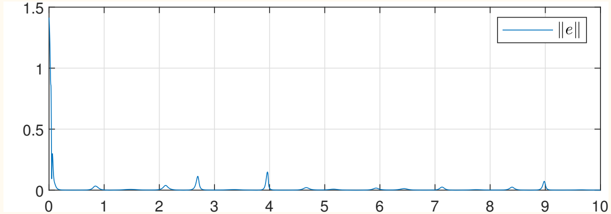
Figura 141: Simulation result of the first order plant (409) with the MIMO MRAC algorithm.

Simulation 2. Figure 142 shows the simulation results of the plant (409) with the MIMO M-MRAC algorithm summarized in Table 2 and remark ??. The following data are used:

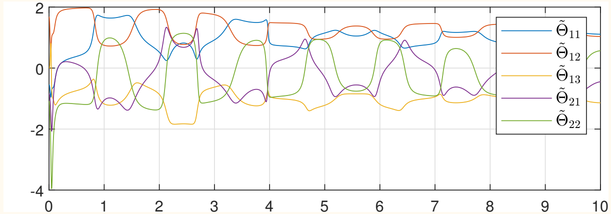
$$\ell_0 = 3\,, \quad \Gamma = 500\, I\,.$$

The tracking error transient and the control mismatch are remarkably improved. Here we can verify the effect of a large adaptation gain Γ, as pointed out in Remarks ?? and ??. Large Γ reduces the tracking error and the control mismatch, but also reduces the convergence rate of the parameter (due to small e_0).

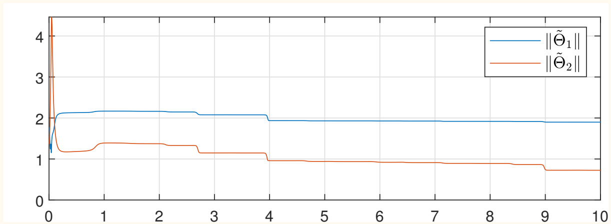
a)


$\|e\|$

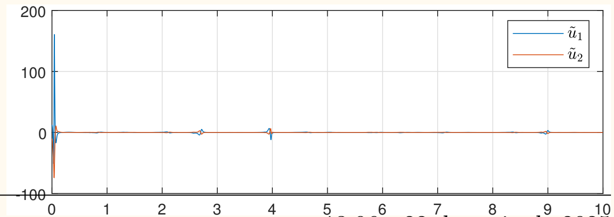
b)


$\tilde{\Theta}_{11}$
$\tilde{\Theta}_{12}$
$\tilde{\Theta}_{13}$
$\tilde{\Theta}_{21}$
$\tilde{\Theta}_{22}$

c)


$\|\tilde{\Theta}_1\|$
$\|\tilde{\Theta}_2\|$

d)

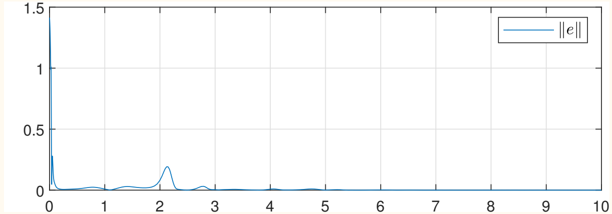

$\tilde{u}_1$
$\tilde{u}_2$

Simulation 3. Figure 143 shows the simulation results of the plant (409) with the MIMO LS-MRAC algorithm summarized in Table 2. The following data are used:

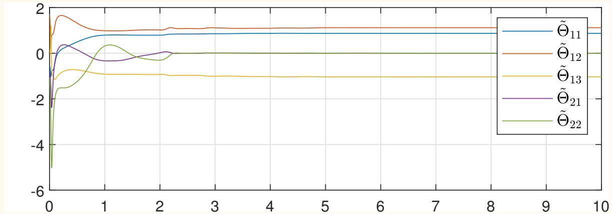
$$\ell_0 = 3\,, \quad \gamma = 50\,, \quad R(0) = 20\,I\,.$$

Note the fast and smooth convergence of the parameter even when employing large gains.

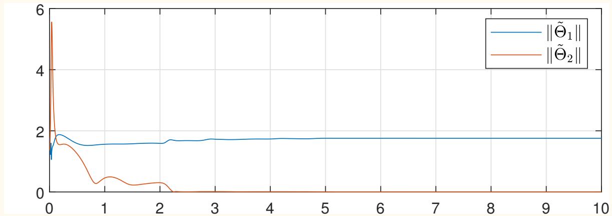
a)


$\|e\|$

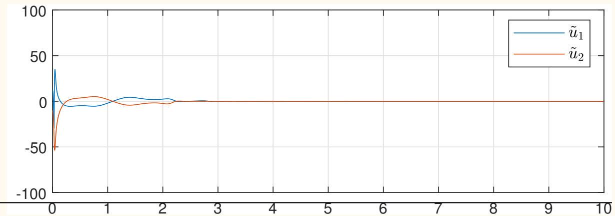
b)

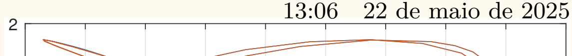

$\tilde{\Theta}_{11}$
$\tilde{\Theta}_{12}$
$\tilde{\Theta}_{13}$
$\tilde{\Theta}_{21}$
$\tilde{\Theta}_{22}$

c)


$\|\tilde{\Theta}_1\|$
$\|\tilde{\Theta}_2\|$

d)

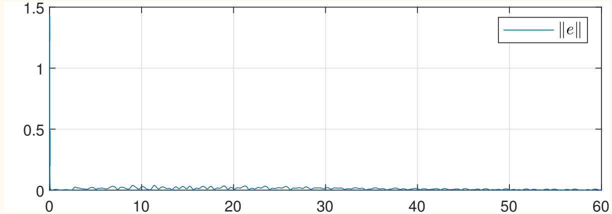

$\tilde{u}_1$
$\tilde{u}_2$



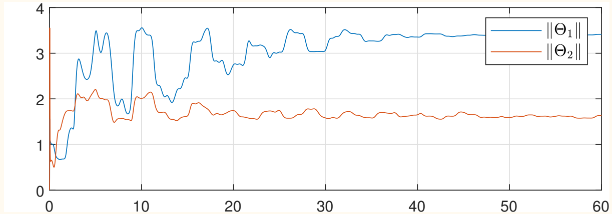
Simulation 4. Figure 144 shows the simulation results of the plant (412) with the MIMO LS-MRAC algorithm. This is a case with 17 parameters. The following data are used:

$$\nu = 2\,, \quad \ell_0 = 2\,, \quad \Lambda = -2\,I\,, \quad g = I\,,$$
$$\gamma = 100\,, \quad R(0) = 20\,I\,.$$

a)



b)



c)

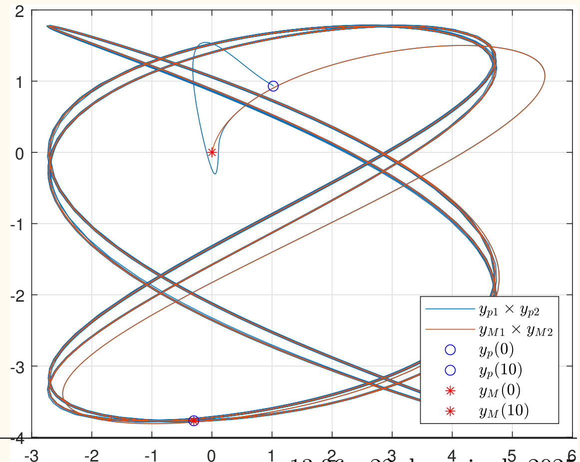


Figura 144: Simulation result of the third order plant (412) with the MIMO LS-MRAC algorithm.

12.6 Conclusion

- The MIMO LS-MRAC inherits the remarkable tracking error performance of the SISO version.
- The parameter convergence cannot be guaranteed even with persistence of excitation, since in the MIMO case the control parameter lacks uniqueness in general.
- However, the behavior of the parameter is much better than the original MIMO MRAC algorithm (Costa et al., 2003).
- The implementation is more elaborated. It requires upper and lower bounds for the leading principal minors of the HFG matrix K_p.
- One possible solution is to combine/composite the control algorithm with an identifier algorithm (Costa, 2023).

(...)

——×——

Referências

Anuradha M. Annaswamy and Alexander L. Fradkov. A historical perspective of adaptive control and learning. *Annual Reviews in Control*, 2021.

Girish Chowdhary and Eric Johnson. Least squares based modification for adaptive control. In *49th IEEE Conference on Decision and Control (CDC)*, pages 1767–1772. IEEE, 2010.

Ramon R. Costa. Lyapunov design of least-squares model reference adaptive control. In *1st Virtual IFAC World Congress (IFAC-V 2020)*, Berlin, Germany, July 11-17 2020.

Ramon R. Costa. Model-reference adaptive control with high-order parameter tuners. In *American Control Conference (ACC2022)*, Atlanta, USA, June 08-10 2022.

Ramon R. Costa. Composite model-reference adaptive control with least-squares estimator. In *IFAC World Congress*, Yokohama, Japan, July 09-14 2023.

Ramon R. Costa. Least-squares model-reference adaptive control with high-order parameter tuners. *Automatica*, 163: 111544, 2024.

Ramon R. Costa, Liu Hsu, Alvaro K. Imai, and Petar Kokotović. Lyapunov-based adaptive control of MIMO systems. *Automatica*, 39(7):1251–1257, Jul. 2003. **[doi]** .

Dimitrios Dimogianopoulos and Rogelio Lozano. Adaptive control for linear time-varying systems using direct least squares estimation. In *Proceedings of the 38th IEEE Conference on Decision and Control*, volume 4, pages 3309–3314. IEEE, 1999.

Manuel A. Duarte and Kumpati S. Narendra. Combined direct and indirect approach to adaptive control. *IEEE Transactions on Automatic Control*, 34(10):1071–1075, 1989.

S. Evesque, A. M. Annaswamy, S. Niculescu, and A. P. Dowling. Adaptive control of a class of time-delay systems. *J. Dyn. Sys., Meas., Control*, 125(2):186–193, 2003.

Thor I. Fossen and Jann Peter Strand. Passive nonlinear observer design for ships using lyapunov methods: Full-scale experiments with a supply vessel. *Automatica*, 35(1):3–16, 1999.

Joseph E. Gaudio, Anuradha M. Annaswamy, José M. Moreu, Michael A. Bolender, and Travis E. Gibson. Accelerated learning with robustness to adversarial regressors. In *Learning for Dynamics and Control*, pages 636–650. PMLR, 2021.

G. C. Goodwin and D. Q. Mayne. A parameter estimation perspective of continuous time model reference adaptive control. *Automatica*, 3(1):57–70, 1987.

Petros A. Ioannou and Jing Sun. *Robust Adaptive Control*. Prentice Hall PTR, 1996.

Iasson Karafyllis and Miroslav Krstic. Adaptive certainty-equivalence control with regulation-triggered finite-time least-squares identification. *IEEE Transactions on Automatic Control*, 63(10):3261–3275, 2018.

Gerhard Kreisselmeier. Adaptive observers with exponential rate of convergence. *IEEE transactions on automatic control*, 22(1):2–8, 1977. **[doi]** .

M. Krstić, I. Kanellakopoulos, and P. Kokotović. *Nonlinear and Adaptive Control Design*. John Wiley & Sons, 1995.

Miroslav Krstic. On using least-squares updates without regressor filtering in identification and adaptive control of nonlinear systems. *Automatica*, 45(3):731–735, 2009.

Wuquan Li and Miroslav Krstic. Stochastic adaptive nonlinear control with filterless least-squares. *IEEE Transactions on Automatic Control*, 2020.

R. V. Monopoli. Model reference adaptive control with an augmented error signal. *IEEE Trans. Aut. Contr.*, 19(5): 474–484, Oct. 1974. **[doi]** .

José M. Moreu and Anuradha M. Annaswamy. A stable high-order tuner for general convex functions. *IEEE Control Systems Letters*, 6:566–571, 2021.

A. Stephen Morse. High-order parameter tuners for the adaptive control of linear and nonlinear systems. In *Systems, Models and Feedback: Theory and Applications*, pages 339–364. Springer, 1992.

D. R. Mudgett. High-order parameter adjustment laws for adaptive stabilization. In *Proc. 1987 Conf. Info. Sci. Sys.*, 1987.

David Richard Mudgett. *Problems in parameter adaptive control*. PhD thesis, Yale University, 1988.

Kumpati S. Narendra and Anuradha M. Annaswamy. *Stable Adaptive Systems*. Prentice–Hall, 1989.

Kumpati S. Narendra and Lena S. Valavani. Stable adaptive controller design - direct control. *IEEE Trans. Aut. Contr.*, 23(8):570–583, 1978. **[doi]** .

Vladimir O. Nikiforov, Dmitry Gerasimov, and Artem Pashenko. Modular adaptive backstepping design with a high-order tuner. *IEEE Transactions on Automatic Control*, 2021.

R. Ortega. On Morse's new adaptive controller: Parameter convergence and transient performance. *IEEE Trans. Aut. Contr.*, 38(8), Aug. 1993.

Yongping Pan, Tairen Sun, and Haoyong Yu. On parameter convergence in least squares identification and adaptive control. *International Journal of Robust and Nonlinear Control*, 29(10):2898–2911, 2019.

Patrick C. Parks. A new proof of the Routh-Hurwitz stability criterion using the second method of Liapunov. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 58, pages 694–702. Cambridge University Press, 1962.

N. N. Puri and C. N. Weygandt. Second method of Liapunov and Routh's canonical form. *Journal of the Franklin Institute*, 276(5):365–384, 1963.

S. S. Sastry and M. Bodson. *Adaptive Control: Stability, Convergence and Robustness*. Prentice Hall, 1989.

J. J. E. Slotine and Weiping Li. *Applied Nonlinear Control*. Prentice–Hall International, Inc., 1991.

Jean-Jacques E. Slotine and Weiping Li. Adaptive robot control: A new perspective. In *26th IEEE conference on decision and control*, volume 26, pages 192–198. IEEE, 1987.

Jean-Jacques E. Slotine and Weiping Li. Composite adaptive control of robot manipulators. *Automatica*, 25(4):509–519, 1989.

Gilbert Strang. *Introduction to Linear Algebra*. Wellesley-Cambridge Press, 5th edition, 2016.

Gang Tao. *Adaptive Control Design and Analysis*. John Wiley & Sons, 2003.

Li Wang and David R. Mudgett. Improvement of transient response in adaptive control using modified high order tuning. In *Proc. American Contr. Conf.*, pages 282–286, San Francisco, June 1993.

Fuzhen Zhang. *The Schur Complement and Its Applications*. Springer, 2005.

Yang Zhu, Miroslav Krstic, Hongye Su, and Chao Xu. Linear backstepping output feedback control for uncertain linear systems. *International Journal of Adaptive Control and Signal Processing*, 30(8-10):1080–1098, 2016. **[doi]** .