Block Properties for Biwm

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Todas as contas a seguir tem o intuito de melhorar problemas de instabilidade numérica, reduzir o custo computacional de inversas e simplesmente fazer este belíssimo código funcionar.

Contas na Log-Verossimilhança

Log do Determinante

Se

$$\mathbf{\Sigma}_{\boldsymbol{\theta}}(\mathbf{h}) = \begin{pmatrix} C_{11}(\mathbf{h}) & C_{12}(\mathbf{h}) \\ C_{21}(\mathbf{h}) & C_{22}(\mathbf{h}) \end{pmatrix} \tag{1}$$

Então

$$det(\Sigma) = |\Sigma| = \left| C_{11}(\mathbf{h}) - C_{12}(\mathbf{h}) C_{22}^{-1}(\mathbf{h}) C_{21}(\mathbf{h}) \right| * \left| C_{22}(\mathbf{h}) \right| \tag{2}$$

Portanto, com o logarítmo fica

$$log(det(\Sigma)) = log \left| C_{11}(\mathbf{h}) - C_{12}(\mathbf{h}) C_{22}^{-1}(\mathbf{h}) C_{21}(\mathbf{h}) \right| + log \left| C_{22}(\mathbf{h}) \right| \tag{3}$$

Extra

Para o nosso caso também é válido que

$$det(\Sigma) = \left| C_{11}(\mathbf{h}) C_{22}(\mathbf{h}) - C_{12}(\mathbf{h}) C_{21}(\mathbf{h}) \right|$$

pois as matrizes $C_{12}(\mathbf{h})$ e $C_{22}(\mathbf{h})$ comutam. Além disso como $C_{12}(\mathbf{h}) = C_{21}(\mathbf{h})$

$$det(\Sigma) = \left| C_{11}(\mathbf{h}) C_{22}(\mathbf{h}) - C_{12}^2(\mathbf{h}) \right|$$

Inversa

Aqui existem duas formas. Eu preferi usar a que irei apresentar a seguir pois ela encaixa bem com o resultado (3) do log do determinante[1].

Se

[1] Apesar de eu achar que não tem como o alimentar o optim com as inversas obtidas, para ganhar tempo

$$\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{-1}(\mathbf{h}) = \begin{pmatrix} C_{11}^*(\mathbf{h}) & C_{12}^*(\mathbf{h}) \\ C_{21}^*(\mathbf{h}) & C_{22}^*(\mathbf{h}) \end{pmatrix} \tag{4}$$

Então

$$\begin{aligned}
C_{11}^*(\mathbf{h}) &= \Bigg[C_{11}(\mathbf{h}) - C_{12}(\mathbf{h})C_{22}^{-1}(\mathbf{h})C_{21}(\mathbf{h})\Bigg]^{-1} \\
C_{12}^*(\mathbf{h}) &= -\Bigg[C_{11}(\mathbf{h}) - C_{12}(\mathbf{h})C_{22}^{-1}(\mathbf{h})C_{21}(\mathbf{h})\Bigg]^{-1} C_{12}(\mathbf{h})C_{22}^{-1}(\mathbf{h}) \\
C_{21}^*(\mathbf{h}) &= -C_{22}^{-1}(\mathbf{h})C_{21}(\mathbf{h})\Bigg[C_{11}(\mathbf{h}) - C_{12}(\mathbf{h})C_{22}^{-1}(\mathbf{h})C_{21}(\mathbf{h})\Bigg]^{-1} \\
C_{22}^*(\mathbf{h}) &= C_{22}^{-1}(\mathbf{h}) + C_{22}^{-1}(\mathbf{h})C_{21}(\mathbf{h})\Bigg[C_{11}(\mathbf{h}) - C_{12}(\mathbf{h})C_{22}^{-1}(\mathbf{h})C_{21}(\mathbf{h})\Bigg]^{-1} C_{12}(\mathbf{h})C_{22}^{-1}(\mathbf{h})
\end{aligned} \tag{5}$$

Já que[2] $C_{12}(\mathbf{h}) = C_{21}^T(\mathbf{h})$, se definirmos

$$\begin{aligned}
V_1 &:= C_{22}^{-1}(\mathbf{h})C_{21}(\mathbf{h}) \\
V_2 &:= C_{11}(\mathbf{h}) - C_{12}(\mathbf{h})C_{22}^{-1}(\mathbf{h})C_{21}(\mathbf{h})
\end{aligned} \tag{6}$$

podemos obter uma forma simplificada para o sistema (5), já que como $C_{11}, C_{12}, C_{21}, C_{22}$ são simétricas

$$\begin{aligned}
C_{11}^*(\mathbf{h}) &= V_2^{-1} \\
C_{12}^*(\mathbf{h}) &= -V_2^{-1}V_1^T \\
C_{21}^*(\mathbf{h}) &= -V_1V_2^{-1} \\
C_{22}^*(\mathbf{h}) &= C_{22}^{-1}(\mathbf{h}) - V_1V_2^{-1}V_1^T
\end{aligned} \tag{7}$$

pois $V_1 = C_{22}^{-1}(\mathbf{h})C_{21}(\mathbf{h})$ fornece $V_1^T = C_{12}(\mathbf{h})C_{22}^{-1}(\mathbf{h})$.

Note ainda que $C_{12}^*(\mathbf{h}) = C_{21}^{*T}(\mathbf{h})$.

Prova:

Primeiramente

$$\begin{aligned}
V_2^T &= C_{11}(\mathbf{h}) - (C_{12}(\mathbf{h})C_{22}^{-1}(\mathbf{h})C_{12}(\mathbf{h}))^T \\
&= C_{11}(\mathbf{h}) - C_{12}(\mathbf{h})(C_{12}(\mathbf{h})C_{22}^{-1}(\mathbf{h}))^T \\
&= C_{11}(\mathbf{h}) - C_{12}(\mathbf{h})C_{22}^{-1}(\mathbf{h})C_{12}(\mathbf{h}) \\
&= V_2
\end{aligned} \tag{8}$$

Agora

$$\begin{aligned}
C_{12}^{*T}(\mathbf{h}) &= -(V_2^{-1}V_1^T)^T \\
&= V_1^T(V_2^{-1})^T \\
&= V_1^T(V_2^{-1}) \\
&= C_{21}^*(\mathbf{h})
\end{aligned} \tag{9}$$

[2] No artigo Σ é definida assim

Avaliação de $\boldsymbol{\Sigma}^{-1}$ na log-verossimilhança

Neste caso precisamos avaliar $y^T\boldsymbol{\Sigma}^{-1}y$. A estrutura de blocos também pode nos ajudar a deixar o algorítmo um pouco mais estável e rápido.

Novamente a inversa de $\boldsymbol{\Sigma}$ é denotada por

$$\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{-1}(\mathbf{h}) = \begin{pmatrix} C_{11}^*(\mathbf{h}) & C_{12}^*(\mathbf{h}) \\ C_{21}^*(\mathbf{h}) & C_{22}^*(\mathbf{h}) \end{pmatrix} \tag{10}$$

Agora se $y^T = (y_1, y_2)^T$, em que y_1 e y_2 são, respectivamente, os vetores resposta associados às primeira e segunda variáveis multivariadas, temos que

$$\begin{aligned} y^T\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{-1}(\mathbf{h})y &= y^T \begin{pmatrix} C_{11}^*(\mathbf{h}) & C_{12}^*(\mathbf{h}) \\ C_{21}^*(\mathbf{h}) & C_{22}^*(\mathbf{h}) \end{pmatrix} y \\ &= (y_1, y_2)^T \begin{pmatrix} C_{11}^*(\mathbf{h}) & C_{12}^*(\mathbf{h}) \\ C_{21}^*(\mathbf{h}) & C_{22}^*(\mathbf{h}) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &= (y_1, y_2)^T \begin{pmatrix} C_{11}^*(\mathbf{h})y_1 + C_{12}^*(\mathbf{h})y_2 \\ C_{21}^*(\mathbf{h})y_1 + C_{22}^*(\mathbf{h})y_2 \end{pmatrix} \\ &= y_1^T C_{11}^*(\mathbf{h})y_1 + y_1^T C_{12}^*(\mathbf{h})y_2 + y_2^T C_{21}^*(\mathbf{h})y_1 + y_2^T C_{22}^*(\mathbf{h})y_2 \\ &= y_1^T C_{11}^*(\mathbf{h})y_1 + 2y_1^T C_{12}^*(\mathbf{h})y_2 + y_2^T C_{22}^*(\mathbf{h})y_2 \end{aligned} \tag{11}$$

pois como $C_{12}^{*T}(\mathbf{h}) = C_{21}^*(\mathbf{h})$ obtemos que

$$y_1^T C_{12}^*(\mathbf{h})y_2 = tr(y_1^T C_{12}^*(\mathbf{h})y_2) = tr(y_1^T C_{12}^*(\mathbf{h})y_2)^T = tr((C_{12}^*(\mathbf{h})y_2)^T y_1) = tr(y_2^T C_{12}^{*T}(\mathbf{h})y_1) = y_2^T C_{21}^*(\mathbf{h})y_1$$

Também é possível escrever o resultado anterior em termos de (11), o que fornece

$$y_1^T V_2^{-1} y_1 - 2y_1^T V_2^{-1} V_1^T y_2 + y_2^T C_{22}^{-1} y_2 - y_2^T V_1 V_2^{-1} V_1^T y_2$$

Contas no Gradiente

Aqui a idéia é mitigar o custo computacional de $tr\left(\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{-1}(\mathbf{h})\frac{\partial \Sigma}{\partial \theta_k}\right) + y^T\left[\frac{\partial \Sigma(\mathbf{h})}{\partial \theta_k}\right]y$ já que vão aparecer muitas matrizes nulas bem grandes em $\partial\Sigma/\partial\boldsymbol{\theta}_k$

Relembrar é viver: Produto de Matrizes

Se

$$\boldsymbol{\Sigma}_{\boldsymbol{\theta}}(\mathbf{h}) = \begin{pmatrix} C_{11}(\mathbf{h}) & C_{12}(\mathbf{h}) \\ C_{21}(\mathbf{h}) & C_{22}(\mathbf{h}) \end{pmatrix} \tag{12}$$

e

$$\boldsymbol{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{13}$$

Então desde as matrizes $\boldsymbol{M}$ e $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}(\mathbf{h})$ sejam conformes

$$\boldsymbol{\Sigma}_{\boldsymbol{\theta}}(\mathbf{h})\boldsymbol{M} = \begin{pmatrix} C_{11}(\mathbf{h})A + C_{12}(\mathbf{h})C & C_{11}(\mathbf{h})B + C_{12}(\mathbf{h})D \\ C_{21}(\mathbf{h})A + C_{22}(\mathbf{h})C & C_{21}(\mathbf{h})B + C_{22}(\mathbf{h})D \end{pmatrix} \tag{14}$$

Caso em que $\theta_k = \sigma_1^2$

Traço

$$
\begin{aligned}
\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{-1}(\mathbf{h})\frac{\partial\Sigma}{\partial\sigma_1^2} &= \begin{pmatrix} C_{11}^*(\mathbf{h}) & C_{12}^*(\mathbf{h}) \\ C_{21}^*(\mathbf{h}) & C_{22}^*(\mathbf{h}) \end{pmatrix} \begin{pmatrix} M(\mathbf{h}|\nu_1,a) & \\ \frac{\rho\sigma_2}{2\sigma_1}M(\mathbf{h}|\nu_3,a) & \mathbf{O} \end{pmatrix} \\
&= \begin{pmatrix} C_{11}^*(\mathbf{h})M(\mathbf{h}|\nu_1,a) + C_{12}^*(\mathbf{h})\frac{\rho\sigma_2}{2\sigma_1}M(\mathbf{h}|\nu_3,a) & C_{11}^*(\mathbf{h})\frac{\rho\sigma_2}{2\sigma_1}M(\mathbf{h}|\nu_3,a) \\ C_{21}^*(\mathbf{h})M(\mathbf{h}|\nu_1,a) + C_{22}^*(\mathbf{h})\frac{\rho\sigma_2}{2\sigma_1}M(\mathbf{h}|\nu_3,a) & C_{21}^*(\mathbf{h})\frac{\rho\sigma_2}{2\sigma_1}M(\mathbf{h}|\nu_3,a) \end{pmatrix}
\end{aligned} \tag{15}
$$

Portanto como $C_{12}^*(\mathbf{h}) = C_{21}^{*T}(\mathbf{h})$

$$
\begin{aligned}
tr\left(\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{-1}(\mathbf{h})\frac{\partial\Sigma}{\partial\sigma_1^2}\right) &= tr\left(C_{11}^*(\mathbf{h})M(\mathbf{h}|\nu_1,a)\right) + 2tr\left(C_{12}^*(\mathbf{h})\frac{\rho\sigma_2}{2\sigma_1}M(\mathbf{h}|\nu_3,a)\right) \\
&= tr\left(C_{11}^*(\mathbf{h})M(\mathbf{h}|\nu_1,a)\right) + \frac{\rho\sigma_2}{\sigma_1}tr\left(C_{12}^*(\mathbf{h})M(\mathbf{h}|\nu_3,a)\right) \\
&= \frac{1}{\sigma_1^2}tr\left(C_{11}^*(\mathbf{h})C_{11}(\mathbf{h})\right) + \frac{1}{\sigma_1^2}tr\left(C_{12}^*(\mathbf{h})C_{12}(\mathbf{h})\right) \\
&= \frac{1}{\sigma_1^2}\left[tr\left(C_{11}^*(\mathbf{h})C_{11}(\mathbf{h})\right) + tr\left(C_{12}^*(\mathbf{h})C_{12}(\mathbf{h})\right)\right]
\end{aligned} \tag{16}
$$

Forma Quadrática

$$
\begin{aligned}
y^T\left[\frac{\partial\Sigma(\mathbf{h})}{\partial\theta_k}\right]y &= \begin{pmatrix} y_1^T & y_2^T \end{pmatrix} \begin{pmatrix} M(\mathbf{h}|\nu_1,a) & \\ \rho\frac{\sigma_2}{2\sigma_1}M(\mathbf{h}|\nu_3,a) & \mathbf{O} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\
&= \begin{pmatrix} y_1^T & y_2^T \end{pmatrix} \begin{pmatrix} M(\mathbf{h}|\nu_1,a)y_1 + \rho\frac{\sigma_2}{2\sigma_1}M(\mathbf{h}|\nu_3,a)y_2 \\ \rho\frac{\sigma_2}{2\sigma_1}M(\mathbf{h}|\nu_3,a)y_1 \end{pmatrix} \\
&= y_1^TM(\mathbf{h}|\nu_1,a)y_1 + y_1^T\rho\frac{\sigma_2}{2\sigma_1}M(\mathbf{h}|\nu_3,a)y_2 + y_2^T\rho\frac{\sigma_2}{2\sigma_1}M(\mathbf{h}|\nu_3,a)y_1 \\
&= y_1^TM(\mathbf{h}|\nu_1,a)y_1 + \rho\frac{\sigma_2}{\sigma_1}y_1^TM(\mathbf{h}|\nu_3,a)y_2 \\
&= \frac{1}{\sigma_1^2}y_1^T\left[C_{11}(\mathbf{h}|\nu_1,a)y_1 + C_{12}(\mathbf{h}|\nu_3,a)y_2\right]
\end{aligned} \tag{17}
$$

Caso em que $\theta_k = \sigma_2^2$

Traço

$$
\begin{aligned}
\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{-1}(\mathbf{h})\frac{\partial\Sigma}{\partial\sigma_2^2} &= \begin{pmatrix} C_{11}^*(\mathbf{h}) & C_{12}^*(\mathbf{h}) \\ C_{21}^*(\mathbf{h}) & C_{22}^*(\mathbf{h}) \end{pmatrix} \begin{pmatrix} \mathbf{O} & \\ \frac{\rho\sigma_1}{2\sigma_2}M(\mathbf{h}|\nu_3,a) & M(\mathbf{h}|\nu_2,a) \end{pmatrix} \\
&= \begin{pmatrix} C_{12}^*(\mathbf{h})\frac{\rho\sigma_1}{2\sigma_2}M(\mathbf{h}|\nu_3,a) & C_{11}^*(\mathbf{h})\frac{\rho\sigma_1}{2\sigma_2}M(\mathbf{h}|\nu_3,a) + C_{12}^*(\mathbf{h})M(\mathbf{h}|\nu_2,a) \\ C_{22}^*(\mathbf{h})\frac{\rho\sigma_1}{2\sigma_2}M(\mathbf{h}|\nu_3,a) & C_{21}^*(\mathbf{h})\frac{\rho\sigma_1}{2\sigma_2}M(\mathbf{h}|\nu_3,a) + C_{22}^*(\mathbf{h})M(\mathbf{h}|\nu_2,a) \end{pmatrix}
\end{aligned} \tag{18}
$$

Portanto como $C_{12}^*(\mathbf{h}) = C_{21}^{*T}(\mathbf{h})$

$$
\begin{aligned}
tr\left(\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{-1}(\mathbf{h})\frac{\partial\Sigma}{\partial\sigma_2^2}\right) &= tr\left(C_{22}^{*}(\mathbf{h})M(\mathbf{h}|\nu_2,a)\right) + 2tr\left(C_{12}^{*}(\mathbf{h})\frac{\rho\sigma_1}{2\sigma_2}M(\mathbf{h}|\nu_3,a)\right) \\
&= tr\left(C_{22}^{*}(\mathbf{h})M(\mathbf{h}|\nu_2,a)\right) + \frac{\rho\sigma_1}{\sigma_2}tr\left(C_{12}^{*}(\mathbf{h})M(\mathbf{h}|\nu_3,a)\right) \\
&= \frac{1}{\sigma_2^2}tr\left(C_{22}^{*}(\mathbf{h})C_{22}(\mathbf{h})\right) + \frac{1}{\sigma_2^2}tr\left(C_{12}^{*}(\mathbf{h})C_{12}(\mathbf{h})\right) \\
&= \frac{1}{\sigma_2^2}\left[tr\left(C_{22}^{*}(\mathbf{h})C_{22}(\mathbf{h})\right) + tr\left(C_{12}^{*}(\mathbf{h})C_{12}(\mathbf{h})\right)\right]
\end{aligned} \tag{19}
$$

Forma Quadrática

$$
\begin{aligned}
y^T\left[\frac{\partial\Sigma(\mathbf{h})}{\partial\theta_k}\right]y &= \begin{pmatrix} y_1^T & y_2^T \end{pmatrix}\begin{pmatrix} \multicolumn{2}{c}{\mathbf{O}} \\ \rho\frac{\sigma_1}{2\sigma_2}M(\mathbf{h}|\nu_3,a) & M(\mathbf{h}|\nu_2,a) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\
&= \begin{pmatrix} y_1^T & y_2^T \end{pmatrix}\begin{pmatrix} \rho\frac{\sigma_1}{2\sigma_2}M(\mathbf{h}|\nu_3,a)y_2 \\ \rho\frac{\sigma_1}{2\sigma_2}M(\mathbf{h}|\nu_3,a)y_1 + M(\mathbf{h}|\nu_2,a)y_2 \end{pmatrix} \\
&= y_2^T M(\mathbf{h}|\nu_1,a)y_2 + y_1^T\rho\frac{\sigma_1}{2\sigma_2}M(\mathbf{h}|\nu_3,a)y_2 + y_2^T\rho\frac{\sigma_1}{2\sigma_2}M(\mathbf{h}|\nu_3,a)y_1 \\
&= y_2^T M(\mathbf{h}|\nu_1,a)y_2 + \rho\frac{\sigma_1}{\sigma_2}y_1^T M(\mathbf{h}|\nu_3,a)y_2 \\
&= \frac{1}{\sigma_2^2}\left[y_2^T C_{22}(\mathbf{h}|\nu_1,a) + y_1^T C_{12}(\mathbf{h}|\nu_3,a)\right]y_2
\end{aligned} \tag{20}
$$

Caso em que $\theta_k = \rho$

Traço

$$
\begin{aligned}
\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{-1}(\mathbf{h})\frac{\partial\Sigma}{\partial\rho} &= \begin{pmatrix} C_{11}^{*}(\mathbf{h}) & C_{12}^{*}(\mathbf{h}) \\ C_{21}^{*}(\mathbf{h}) & C_{22}^{*}(\mathbf{h}) \end{pmatrix}\begin{pmatrix} \multicolumn{2}{c}{\mathbf{O}} \\ \sigma_1\sigma_2 M(\mathbf{h}|\nu_3,a) & \mathbf{O} \end{pmatrix} \\
&= \begin{pmatrix} C_{12}^{*}(\mathbf{h})\sigma_1\sigma_2 M(\mathbf{h}|\nu_3,a) & C_{11}^{*}(\mathbf{h})\sigma_1\sigma_2 M(\mathbf{h}|\nu_3,a) \\ C_{22}^{*}(\mathbf{h})\sigma_1\sigma_2 M(\mathbf{h}|\nu_3,a) & C_{21}^{*}(\mathbf{h})\sigma_1\sigma_2 M(\mathbf{h}|\nu_3,a) \end{pmatrix}
\end{aligned} \tag{21}
$$

Portanto como $C_{12}^{*}(\mathbf{h}) = C_{21}^{*T}(\mathbf{h})$

$$
tr\left(\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{-1}(\mathbf{h})\frac{\partial\Sigma}{\partial\rho}\right) = 2\sigma_1\sigma_2 tr\left(C_{12}^{*}(\mathbf{h})M(\mathbf{h}|\nu_3,a)\right) \tag{22}
$$

Forma Quadrática

$$
\begin{aligned}
y^T\left[\frac{\partial\Sigma(\mathbf{h})}{\partial\theta_k}\right]y &= \begin{pmatrix} y_1^T & y_2^T \end{pmatrix}\begin{pmatrix} \multicolumn{2}{c}{\mathbf{O}} \\ \sigma_1\sigma_2 M(\mathbf{h}|\nu_3,a) & \mathbf{O} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\
&= \begin{pmatrix} y_1^T & y_2^T \end{pmatrix}\begin{pmatrix} \sigma_1\sigma_2 M(\mathbf{h}|\nu_3,a)y_2 \\ \sigma_1\sigma_2 M(\mathbf{h}|\nu_3,a)y_1 \end{pmatrix} \\
&= y_1^T\sigma_1\sigma_2 M(\mathbf{h}|\nu_3,a)y_2 + y_2^T\sigma_1\sigma_2 M(\mathbf{h}|\nu_3,a)y_1 \\
&= 2\sigma_1\sigma_2 y_1^T M(\mathbf{h}|\nu_3,a)y_2
\end{aligned} \tag{23}
$$

Caso em que $\theta_k = a$

Traço

$$
\begin{aligned}
\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{-1}(\mathbf{h})\frac{\partial \Sigma}{\partial a} &= \begin{pmatrix} C_{11}^{*}(\mathbf{h}) & C_{12}^{*}(\mathbf{h}) \\ C_{21}^{*}(\mathbf{h}) & C_{22}^{*}(\mathbf{h}) \end{pmatrix} \begin{pmatrix} \sigma_1^2 M^{'}(\mathbf{h}|\nu_1, a) & \rho\sigma_1\sigma_2 M^{'}(\mathbf{h}|\nu_3, a) \\ \rho\sigma_1\sigma_2 M^{'}(\mathbf{h}|\nu_3, a) & \sigma_2^2 M^{'}(\mathbf{h}|\nu_2, a) \end{pmatrix} \\
&= \begin{pmatrix} C_{11}^{*}(\mathbf{h})\sigma_1^2 M^{'}(\mathbf{h}|\nu_1, a) + C_{12}^{*}(\mathbf{h})\rho\sigma_1\sigma_2 M^{'}(\mathbf{h}|\nu_3, a) & C_{11}^{*}(\mathbf{h})\rho\sigma_1\sigma_2 M^{'}(\mathbf{h}|\nu_3, a) + C_{12}^{*}(\mathbf{h})\sigma_2^2 M^{'}(\mathbf{h}|\nu_2, a) \\ C_{21}^{*}(\mathbf{h})\sigma_1^2 M^{'}(\mathbf{h}|\nu_1, a) + C_{22}^{*}(\mathbf{h})\rho\sigma_1\sigma_2 M^{'}(\mathbf{h}|\nu_3, a) & C_{21}^{*}(\mathbf{h})\rho\sigma_1\sigma_2 M^{'}(\mathbf{h}|\nu_3, a) + C_{22}^{*}(\mathbf{h})\sigma_2^2 M^{'}(\mathbf{h}|\nu_2, a) \end{pmatrix}
\end{aligned} \tag{24}
$$

Portanto como $C_{12}^{*}(\mathbf{h}) = C_{21}^{*T}(\mathbf{h})$

$$
\begin{aligned}
tr\left(\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{-1}(\mathbf{h})\frac{\partial \Sigma}{\partial a}\right) = & \ \sigma_1^2 tr\left(C_{11}^{*}(\mathbf{h})M^{'}(\mathbf{h}|\nu_1, a)\right) + \\
& \sigma_2^2 tr\left(C_{22}^{*}(\mathbf{h})M^{'}(\mathbf{h}|\nu_2, a)\right) + \\
& 2\rho\sigma_2\sigma_2 tr\left(C_{12}^{*}(\mathbf{h})M^{'}(\mathbf{h}|\nu_3, a)\right)
\end{aligned} \tag{25}
$$

Por outro lado, temos que

$$
M^{'}(\mathbf{h}|\nu, a) = \frac{\partial M(\mathbf{h}|\nu, a)}{\partial a} = \frac{(\frac{2}{a})^{1-\nu} d^{\nu}}{\Gamma(\nu)}\left[2\nu K_{\nu}(ad) - adK_{\nu+1}(ad)\right] \tag{26}
$$

em que $d := ||\mathbf{h}||$. Assim, se definirmos

$$
\begin{aligned}
\phi_i &:= \frac{d(\frac{2}{ad})^{1-\nu_i}}{\Gamma(\nu_i)} \\
\Phi_i(\mathbf{h}) &:= \left[2\nu_i K_{\nu_i}(ad) - adK_{\nu_i+1}(ad)\right]
\end{aligned} \tag{27}
$$

e desenvolvermos um pouco mais as contas, temos que

$$
\begin{aligned}
tr\left(\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{-1}(\mathbf{h})\frac{\partial \Sigma}{\partial a}\right) = & \ \sigma_1^2 \phi_1 tr\left(C_{11}^{*}(\mathbf{h})\Phi_1(\mathbf{h})\right) + \\
& \sigma_2^2 \phi_2 tr\left(C_{22}^{*}(\mathbf{h})\Phi_2(\mathbf{h})\right) + \\
& 2\rho\sigma_1\sigma_2\phi_3 tr\left(C_{12}^{*}(\mathbf{h})\Phi_3(\mathbf{h})\right)
\end{aligned} \tag{28}
$$

Forma Quadrática

$$
\begin{aligned}
y^T\left[\frac{\partial\Sigma(\mathbf{h})}{\partial\theta_k}\right]y &= \begin{pmatrix} y_1^T & y_2^T \end{pmatrix}\begin{pmatrix} \sigma_1^2 M^{'}(\mathbf{h}|\nu_1,a) & \\ \rho\sigma_1\sigma_2 M^{'}(\mathbf{h}|\nu_3,a) & \sigma_2^2 M^{'}(\mathbf{h}|\nu_2,a)\end{pmatrix}\begin{pmatrix} y_1 \\ y_2\end{pmatrix} \\
&= \begin{pmatrix} y_1^T & y_2^T \end{pmatrix}\begin{pmatrix} \sigma_1^2 M^{'}(\mathbf{h}|\nu_1,a)y_1 + \rho\sigma_1\sigma_2 M^{'}(\mathbf{h}|\nu_3,a)y_2 \\ \rho\sigma_1\sigma_2 M^{'}(\mathbf{h}|\nu_3,a)y_1 + \sigma_2^2 M^{'}(\mathbf{h}|\nu_2,a)y_2\end{pmatrix} \\
&= y_1^T\sigma_1^2 M^{'}(\mathbf{h}|\nu_1,a)y_1 + y_1^T\rho\sigma_1\sigma_2 M^{'}(\mathbf{h}|\nu_3,a)y_2 + y_2^T\rho\sigma_1\sigma_2 M^{'}(\mathbf{h}|\nu_3,a)y_1 + y_2^T\sigma_2^2 M^{'}(\mathbf{h}|\nu_2,a)y_2 \\
&= \sigma_1^2 y_1^T M^{'}(\mathbf{h}|\nu_1,a)y_1 + \sigma_2^2 y_2^T M^{'}(\mathbf{h}|\nu_2,a)y_2 + 2\rho\sigma_1\sigma_2 y_1^T M^{'}(\mathbf{h}|\nu_3,a)y_2
\end{aligned} \tag{29}
$$

Se utilizarmos (26) e (27) então obtemos que

$$
y^T\left[\frac{\partial\Sigma(\mathbf{h})}{\partial\theta_k}\right]y = \sigma_1^2\phi_1 y_1^T\Phi_1(\mathbf{h})y_1 + \sigma_2^2\phi_2 y_2^T\Phi_2(\mathbf{h})y_2 + 2\rho\sigma_1\sigma_2\phi_3 y_1^T\Phi_3(\mathbf{h})y_2 \tag{30}
$$