Writing

$+y_1y_2 - y_1$

$-y_1y_2 - y_1(1-y_2)$

$V(-\frac{V}{2}-1)$ ~~$V(\ldots)$~~

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12. $\because X_1, X_2$ 獨立

$f(x_1,x_2)=f(x_1)f(x_2)=e^{-x_1}e^{-x_2}=e^{-(x_1+x_2)}$

$\begin{cases} y_1 = x_1+x_2 \\ y_2 = \frac{x_1}{x_1+x_2} \end{cases} \Rightarrow \begin{cases} x_1 = y_1y_2 \\ x_2 = y_1(1-y_2) \end{cases} \Rightarrow \because X_1, X_2 > 0, \therefore y_1 > 0, 0<y_2<1$

$f(y_1y_2, y_1(1-y_2)) = e^{-(y_1y_2+y_1(1-y_2))} = e^{-y_1}$

$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 1-y_2 & -y_1 \end{vmatrix} = -y_1$

$g(y_1,y_2) = f(y_1y_2, y_1(1-y_2))|J| = y_1e^{-y_1}$

又 $g(y_1) = \int_0^1 y_1e^{-y_1}dy_2 = y_1e^{-y_1}.\ y_1>0$

$g(y_2) = \int_0^\infty y_1e^{-y_1}dy_1 = \Gamma(2) = 1,\ 0<y_2<1$

故 $g(y_1,y_2) = g(y_1)g(y_2) \Rightarrow Y_1, Y_2$ 獨立

22、

$M_x(t) = (1-2t)^{-\frac{V}{2}}$

$E(x) = \mu_1' = \frac{dM_x(t)}{dt}\Big|_{t=0}$

$= -\frac{V}{2}(1-2t)^{-\frac{V}{2}-1}(-2)\Big|_{t=0}$

$= V$

$E(x^2) = \mu_2' = \frac{d^2M_x(t)}{dt}\Big|_{t=0}$

$= V(-\frac{V}{2}-1)(1-2t)^{-\frac{V}{2}-2}(-2)\Big|_{t=0}.$

$= V(V+2)$

$\sigma^2 = E(x^2)-(E(x))^2 = V(V+2)-V^2 = 2V$ #.

$\Rightarrow x_1 = +\sqrt{y},\ x_2 = -\sqrt{y}$

14. $y = x^2 \Rightarrow x = \pm\sqrt{y}.$

$J_1 = \frac{dx_1}{dy} = \frac{1}{2\sqrt{y}}\quad J_2 = \frac{dx_2}{dy} = -\frac{1}{2\sqrt{y}}.$

$g(y) = f(x_1)J_1 + f(x_2)J_2$

$= f(\sqrt{y})\cdot\frac{1}{2\sqrt{y}} + f(-\sqrt{y})\cdot\frac{1}{2\sqrt{y}}$

$= \frac{1}{2\sqrt{y}}\left(\frac{1+\sqrt{y}}{2} + \frac{1-\sqrt{y}}{2}\right) = \frac{1}{2\sqrt{y}},\ 0<y<1$

$(-1<x<1 \Rightarrow 0\le y<1$

又 $y\ne 0 \Rightarrow 0<y<1)$

18 $M_x(t) = E(e^{tx}) = \sum_{x=1}^{\infty} e^{tx}f(x)$ $\qquad S=\frac{a_1}{1-r}$

$\sum_{x=1}^{\infty} e^{tx}pq^{x-1} = \frac{p}{q}\sum_{x=1}^{\infty}(e^tq)^x$

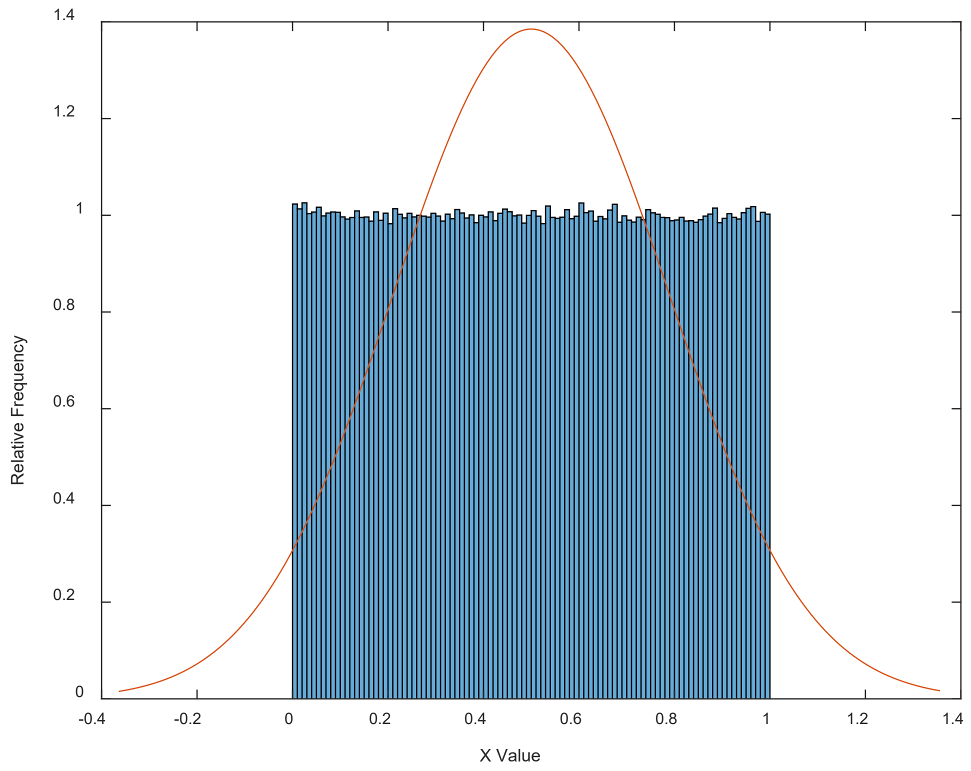
$= \frac{p}{q}\left(\frac{e^tq}{1-e^tq}\right) = \frac{pe^t}{1-e^tq}$

$\mu_1' = \frac{dM_x(t)}{dt}\Big|_{t=0} = \frac{(1-e^tq)pe^t-(-e^tq)pe^t}{(1-e^tq)^2}\Big|_{t=0} = \frac{pe^t}{(1-e^tq)^2}\Big|_{t=0} = \frac{p}{(1-q)^2} = \frac{1}{p} = E(X)$

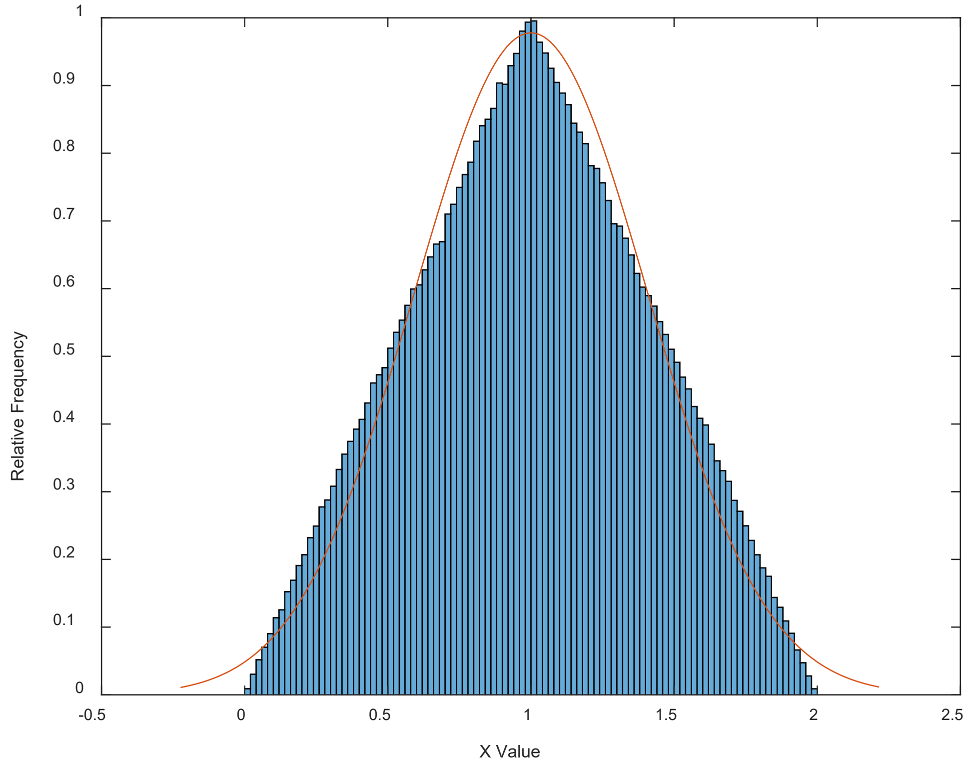
$\mu_2' = \frac{d^2M_x(t)}{dt^2}\Big|_{t=0} = \frac{(1-e^tq)^2pe^t-2(1-e^tq)(-qe^t)pe^t}{(1-e^tq)^4}\Big|_{t=0} = \frac{2-p}{p^2} = E(X^2)$

$\sigma^2 = E(X^2)-(E(X))^2 = \mu_2'-(\mu_1')^2 =$ (2-p)/p^2 – 1/(p^2) = q/(p^2)

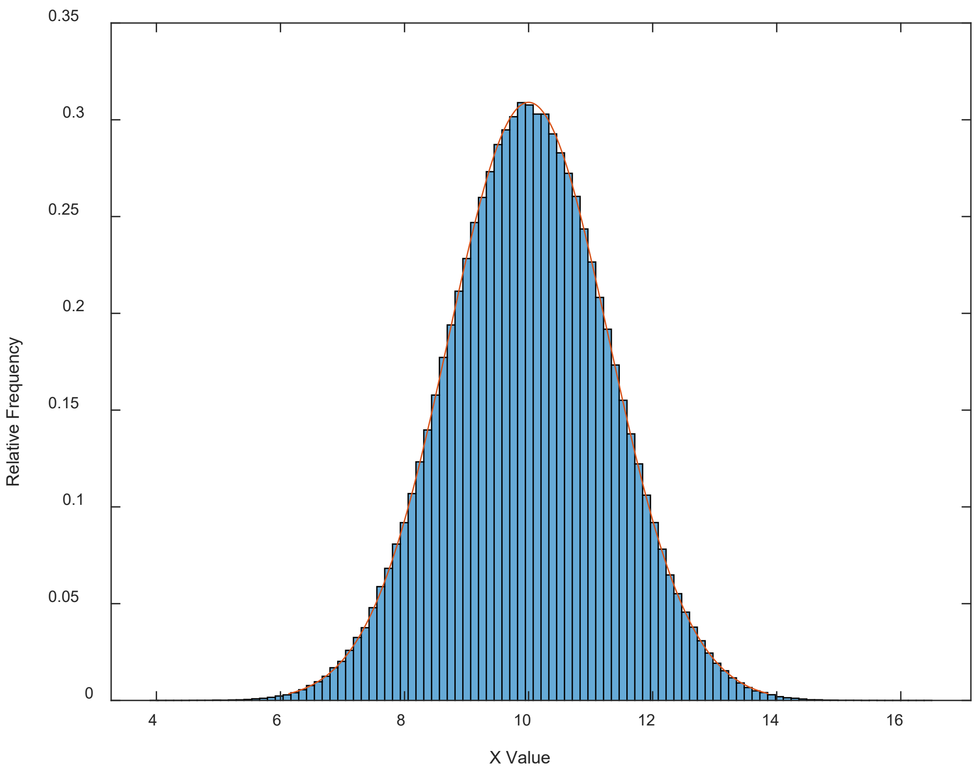
Matlab



↑ n = 1，Irwin-Hall 分布就是 uniform 分布。與常態分佈相差甚遠。



↑ n = 2，Irwin-Hall 分布是一個三角形。與常態分佈已經比較接近了。



↑ n = 20，Irwin-Hall 分布已經趨近於常態分佈。這也可以驗證中央極限定理。