1.

1) FSPA, $r \geq 0$, $n \geq 2$. Winner pays second-highest if exists, o.w, own.

$$m(x) = \underbrace{\int_{\underline{v}}^{v_0} \beta(x)\, dG(z)}_{i \text{ wins, no other}} + \underbrace{\int_{v_0}^{x} \beta(z)\, dG(z)}_{i \text{ wins, others}} + \underbrace{\int_{x}^{\bar{v}} 0\, dG(z)}_{i \text{ loses}}$$

$$= \beta(x) G(v_0) + \int_{v_0}^{x} \beta(z)\, dG(z)$$

BC: marginal type v_0, assume $m(x)$ increasing in $\beta(x)$

$$\beta(v_0) \overset{*}{=} r \quad \text{lowest possible bid.}$$

$$m(v_0) = \beta(v_0) G(v_0) + \underbrace{\int_{v_0}^{v_0} \beta(z)\, dG(z)}_{=0}$$

$$\overset{*}{=} r\, G(v_0)$$

IC: $U(v_0; v_0) = v_0 G(v_0) - m(v_0) = 0$

$$\Rightarrow v_0 G(v_0) - r G(v_0) = 0$$

$$\Rightarrow (v_0 - r) G(v_0) = 0$$

so, $v_0 = r$, same as in FPA, $r \geq 0$

$$U(x; v) = v G(x) - m(x)$$

$$= v G(x) - \left(\beta(x) G(v_0) + \int_{v_0}^{x} \beta(z)\, dG(z)\right)$$

done above

2) 3) 4) 5)

LEIBNIZ RULE:
$\frac{d}{dx} \int_{a(x)}^{b(x)} f(x,t)\, dt$
$= f(x, b(x))\, b'(x) - f(x, a(x))\, a'(x) + \int_{a(x)}^{b(x)} \frac{d}{dx} f(x,t)\, dt$

RPC: $U'(x; v) = v G' - \beta' G(v_0) - \underbrace{\frac{d}{dx} \int_{v_0}^{x} \beta(z)\, dG(z)}_{= \text{Leibniz: } \beta(x) G'(x)}$

$$= v G' - \beta' G(v_0) - \beta(x) G'(x)$$

Letting $U'(x;x)=0$, $n=2$, $F(x)=x$, $G(x)=x$, $G'(x)=1$

gives: $\beta'(x) = -\frac{1}{v_0}\beta(x) + \frac{1}{v_0}v$ (2)

By appendix A.3, $y'(x) = a(x)y(x)+b(x)$ giving:

$$y(x) = \exp\{\textstyle\int_{x_0}^{x} a(t)\,dt\}\left(y(x_0) + \int_{x_0}^{x} b(z)\exp\{-\int_{x_0}^{z} a(t)\,dt\}\,dz\right)$$

$$= \exp\{\textstyle\int_{v_0}^{x} -\frac{1}{v_0}\,dt\}\left(\beta(v_0) + \int_{v_0}^{x}\frac{1}{v_0}y\exp\{-\int_{v_0}^{y}-\frac{1}{v_0}\,dt\}\,dy\right)$$

$$\overset{BC}{=} \exp\{-\tfrac{1}{v_0}(x-v_0)\}\left(r + \textstyle\int_{v_0}^{x}\frac{1}{v_0}y\exp\{\frac{1}{v_0}(y-v_0)\}\,dy\right)$$

$$= \exp\{1-\tfrac{x}{v_0}\}\left(r + \textstyle\int_{v_0}^{x}\frac{1}{v_0}(y\exp\{\frac{y}{v_0}-1)\,dy\right)$$

$*$ $\int_{v_0}^{x} y\exp\{\frac{y}{v_0}-1\}\,dy = \frac{1}{e}\int_{v_0}^{x} y e^{y/v_0}\,dy$

$\overset{S}{=} \frac{1}{e}\left(\left[v_0 y e^{y/v_0}\right]_{v_0}^{x} - \int_{v_0}^{x}(v_0 e^{y/v_0})\,dy\right)$

$= \frac{1}{e}v_0\left(xe^{x/v_0} - v_0 e - \left[v_0 e^{y/v_0}\right]_{v_0}^{x}\right)$

$= \frac{1}{e}v_0\left(xe^{x/v_0} - v_0 e - v_0 e^{x/v_0} + v_0 e\right)$

$= v_0 x e^{\frac{x}{v_0}-1} - v_0^2 e^{\frac{x}{v_0}-1}$

$= v_0 e^{\frac{x}{v_0}-1}(x-v_0)$

$\overset{*}{=} e^{1-\frac{x}{v_0}}\left(r + e^{\frac{x}{v_0}-1}(x-v_0)\right)$

$= re^{1-\frac{x}{v_0}} + x - v_0$

Hence $\beta(v,v_0,r) = v - v_0 + re^{1-\frac{v}{v_0}}$

$\beta(v,r) = v - r + re^{1-\frac{v}{r}}$ w/ $v \geq r$ (3)

$$m(x) = \beta(x)G(v_0) + \int_{v_0}^{x}\beta(z)\,dG(z)$$

$$= xr - r^2 + r^2e^{1-\frac{x}{r}} + \int_{r}^{x}\left(z - r + re^{1-\frac{z}{r}}\right)dz$$

$$= xr - r^2 + r^2e^{1-\frac{x}{r}} + \left[\tfrac{1}{2}z^2 - rz\right]_{r}^{x} - \left[r^2e^{1-\frac{z}{r}}\right]_{r}^{x}$$

$$= xr - r^2 + r^2e^{1-\frac{x}{r}} + \tfrac{1}{2}x^2 - rx + \tfrac{1}{2}r^2 + r^2\left(1 - e^{1-\frac{x}{r}}\right)$$

$$= \tfrac{1}{2}(x^2+r^2)$$

Prop. 3.1 says

PROPOSITION 3.1 (The Expected Payment Result). In an MSPE of any standard auctions with marginal type v_0, a bidder of type $v \in [v_0, \overline{v}]$ in expectation pays:

$$m(v) = v_0 G(v_0) + \int_{v_0}^{v} x dG(x) = vG(v) - \int_{v_0}^{v} G(x) dx.$$

$$m(v) = v_0^2 + \int_{v_0}^{v} x \, dx$$

$$= v_0^2 + \frac{1}{2}[x^2]_{v_0}^{v}$$

$$= \frac{1}{2}v^2 + \frac{1}{2}v_0^2$$

since in our question $v_0 = r$, $m(v)$ equivalent as above 5