Bond Price Volatility

Financial Engineering and Computations

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此章內容

- Financial Engineering & Computation教課書
- 第四章　Bond Price Volatility

- C++財務程式設計的第三章（3-4,3-5）

Outline

- Price Volatility
- Duration
- Convexity
- Immunization

Price Volatility

- Price volatility measures the sensitivity of the percentage price change to changes in interest rates (interest rate risk).
- It is key to the risk management of interest-rate-sensitive securities.
- Define price volatility by

$$-\frac{\partial P / P}{\partial y} \longrightarrow$$ It is also so-call modified duration!

$$\frac{\partial P}{P}(\text{percent price change}) \approx -D \times \partial y$$

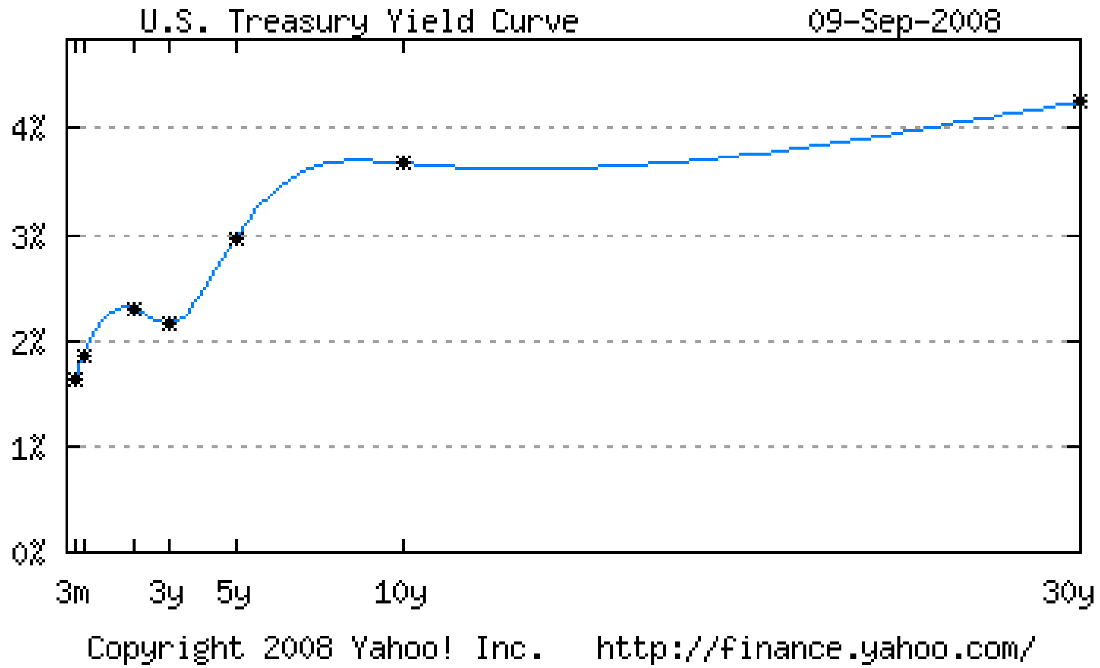
Numerical Example: Percentage Change of Bond Price

- Consider a bond whose modified duration is 11.54 with a yield of 10%.

If the yield increases instantaneously from 10% to 10.1%, the approximate percentage price change will be

$$-11.54 \times 0.001 = -0.01154 = -1.154\%.$$

General speaking, the duration we talk about is modified duration!



Maturity	Yield	Yesterday	Last Week	Last Month
3 Month	1.64	1.67	1.64	1.63
6 Month	1.86	1.87	1.88	1.87
2 Year	2.30	2.30	2.36	2.49
3 Year	2.15	2.14	2.17	2.35
5 Year	2.97	2.91	3.09	3.19
10 Year	3.67	3.70	3.81	3.93
30 Year	4.26	4.30	4.42	4.54

Behavior of Price Volatility

- Price volatility increases as the coupon rate decreases.
  - Bonds selling at a deep discount are more volatile than those selling near or above par.
  - Zero-coupon bonds are the most volatile.

- Price volatility increases as the required yield decreases.
  - So bonds traded with higher yields are less volatile.

Behavior of Price Volatility

- For bonds selling above par or at par, price volatility in creases as the term to maturity lengthens (see figure on next page).
  - Bonds with a longer maturity are more volatile.

    (But the *yields* of long-term bonds are less volatile than those of short-term bonds.)
- For bonds selling below par, price volatility first increa ses then decreases.
  - Longer maturity here cannot be equated with higher price volatility.

Figure 4.1 **(Premium bonds and par bonds)**: Volatility with respect to terms to maturity

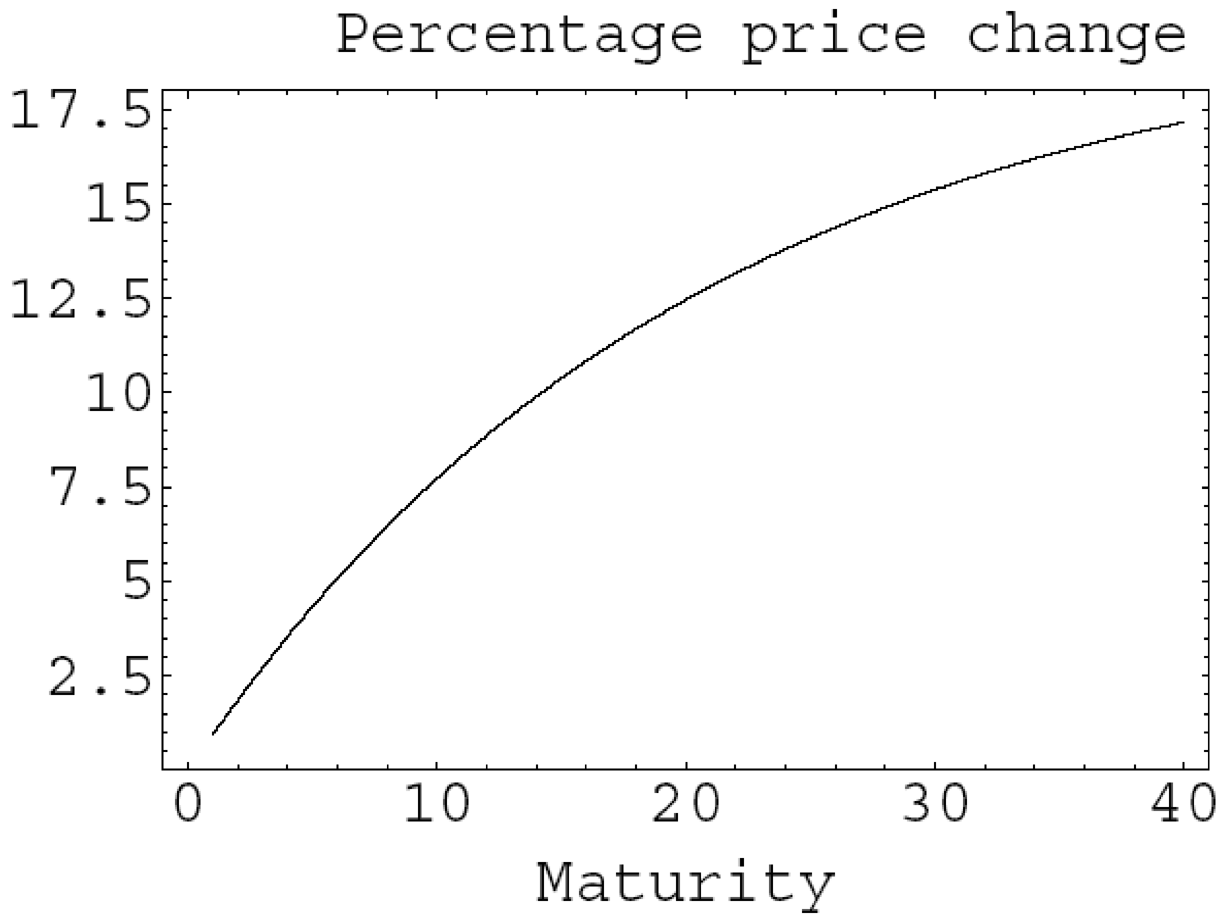
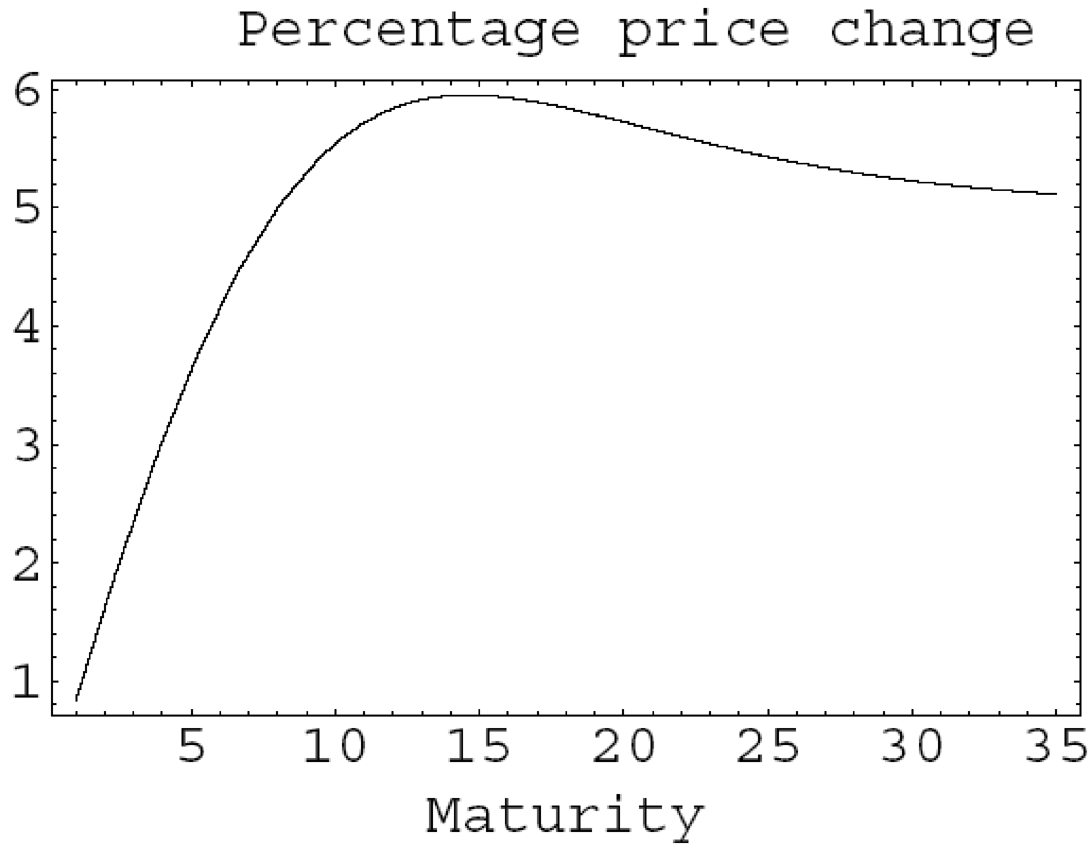


Figure 4.1 **(discount bonds)**:
Volatility with respect to terms to maturity.



Macaulay Duration

- The Macaulay duration (MD) is a weighted average of the times to an asset's cash flows.
- The weights are the cash flows' PVs divided by the asset's price. Formally,

$$MD \equiv \frac{1}{p}\sum_{i=1}^{n}\frac{iC_i}{(1+y)^i}$$

- The Macaulay duration, in periods, is equal to

$$MD = -(1+y)\frac{\partial P / P}{\partial y} \qquad (4.2)$$

The Proof

$$P = \frac{C}{1+y} + \frac{C}{(1+y)^2} + \ldots + \frac{C+F}{(1+y)^n}$$

$$\therefore \frac{\partial P}{\partial y} = \frac{-C}{(1+y)^2} + \frac{-2C}{(1+y)^3} + \ldots + \frac{-n(C+F)}{(1+y)^{n+1}}$$

$$\therefore \frac{\partial P}{\partial y} = -\frac{1}{1+y}[\frac{C}{(1+y)} + \frac{2C}{(1+y)^2} + \ldots + \frac{n(C+F)}{(1+y)^n}]$$

$$\therefore \frac{\partial P}{\partial y}\frac{1}{P} = -\frac{1}{1+y}[\frac{C}{(1+y)} + \frac{2C}{(1+y)^2} + \ldots + \frac{n(C+F)}{(1+y)^n}]\frac{1}{P}$$

$$Define: MD = \frac{\frac{C}{(1+y)} + \frac{2C}{(1+y)^2} + \ldots + \frac{n(C+F)}{(1+y)^n}}{P} = \frac{\sum_{i=1}^{n}\frac{iC_i}{(1+y)^i}}{P}$$

$$\therefore \frac{\partial P}{\partial y}\frac{1}{P} = -\frac{1}{1+y}MD \Rightarrow \frac{\partial P / P}{\partial y / 1+y} = -MD$$

Example:

Duration of 6-Year Eurobond, 1,000 Face Value, 8 Percent Coupon and Market Yields 8%

t	C_t	DF_t	$C_t \times DF_t$	$C_t \times DF_t \times t$
1	80	0.9259	74.07	74.07
2	80	0.8573	68.59	137.18
3	80	0.7938	63.51	190.53
4	80	0.7350	58.80	235.20
5	80	0.6806	54.45	272.25
6	1080	0.6302	680.58	4083.48
			1000	4992.71
MD=4992.71/1000=4.993 years				

C is cash flow, DF is discount factor

Bonds and Interest rates

存續期間 Duration

代號	美國公債	市價(台幣)	內扣費用(%)	存續期間(年)
00679B	元大美債20年	30.9	0.16	17.1
00687B	國泰20年美債	32.4	0.17	16.8
00764B	群益25年美債	32.5	0.17	16.8
00768B	復華20年美債	57.9	0.18	17.0
00857B	永豐20年美公債	26.9	0.22	16.9
00696B	富邦美債20年	33.1	0.20	16.8
00795B	中信美國公債20年	31.2	0.15	17.1
00931B	統一美債20年	15.4	N/A	16.8
00779B	凱基美債25+	32.6	0.14	17.6
00697B	元大美債7-10	37.0	0.55	7.2
00695B	富邦美債7-10	36.4	0.47	7.1
00719B	元大美債1-3	31.7	0.19	1.7
00694B	富邦美債1-3	41.5	0.19	1.7
00856B	永豐1-3年美公債	38.7	0.25	1.7
00864B	中信美國公債0-1	45.6	0.19	0.4
00859B	群益0-1年美債	42.1	0.21	0.5
00865B	國泰US短期公債	45.3	0.26	0.4

長天期債券：存續期間較長，價格對利率較敏感

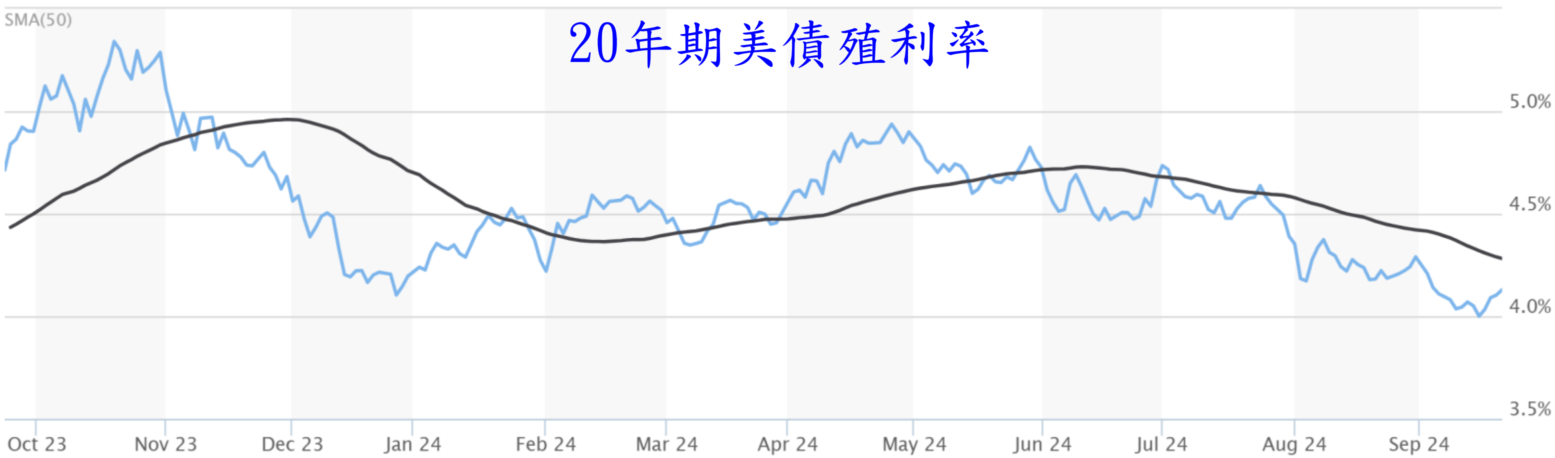
Ex. 若利率下降1%，債券價格上漲約10.多%

短天期債券：存續期間較短，價格對利率較不敏感

Ex. 若利率下降1%，債券價格上漲約1.多%

Bonds and Exchange rates

美債殖利率與美金匯率高度正相關
（在不考慮其他重大金融事件下）



Interest rates and Exchange rates

- 利率和債券價格反向連動，與匯率呈高度正相關
- 長天期的債券價格對利率波動較敏感
- 同時考慮利率與匯率對債券的影響：

	匯率和利率連動上升時	匯率和利率連動下降時
美金定存	美金價值不變，賺到台幣計價	美金價值不變，賠到台幣計價
短天期美債	美債價值小幅下降 < 匯兌利得 →賺到台幣計價	美債價值小幅上升 < 匯兌利損 →賠到台幣計價
長天期美債	美債價值大幅下降 > 匯兌利得 →賠到台幣計價	美債價值大幅上升 > 匯兌利損 →賺到台幣計價

C++: Macaulay Duration的計算

- Macaulay Duration的計算

$$MD = \frac{1}{P}\left(\sum_{i=1}^{n}\frac{ic}{(1+r)^i} + \frac{nF}{(1+r)^n}\right)$$

- 利用for loop同時求算 $\frac{1}{P}$ 和 $\sum_{i=1}^{n}\frac{ic}{(1+r)^i} + \frac{nF}{(1+r)^n}$
- 相乘即為答案

完整程式碼

```
#include <stdio.h>
void main()
{
     int n;
     float c, r, Value=0,Discount,Duration=0;
     printf("請輸入期數:");
     scanf("%d",&n);
     printf("請輸入債息:");
     scanf("%f",&c);
     printf("請輸入利率:");
     scanf("%f",&r);
     for(int i=1;i<=n;i=i+1)
     {
      Discount=1;
      for(int j=1;j<=i;j++)
      {
       Discount=Discount/(1+r);
      }
      Duration=Duration+i*Discount*c;
      Value=Value+Discount*c;
      if(i==n)
      {
       Value=Value+Discount*100;
       Duration=Duration+n*Discount*100;
      }
     }
     Duration=Duration/Value;
     printf("Duration=%f",Duration);
}
```

For迴圈: 計算 Duration,Value

For迴圈: 計算Discount factor

If 條件式: i等於n時,考慮face value

Homework 3

- Program Exercise

a. 課本(C++財務程式設計)第三章習題8, 9。

b. Use the data in FISD to calculate Duration.

issuer_id = 34696

- *OFFERING_DATE,*
- *MATURITY,*
- *COUPON, (coupon rate)*
- *OFFERING_PRICE,(Present value)*
- *OFFERING_YIELD.*

Homework 3

- Program Exercise

a. 課本(C++財務程式設計)第三章習題8, 9。

b. Use the data in FISD to calculate Duration.

8.請嘗試使用一些簡單的財務知識，來驗證本章的計算存續期間的範例程式產生的答案是否合理。假定債券支付的債息為0，則其存續期間應為多少？請問當債息提高（或下降），存續期間應提高還是下降？並將推論的結果輸入範例程式中，驗證推論的結果是否和程式的輸出相符合。

Homework 3

9.在本章實例演練中討論的存續期間（duration）稱為 Macaulay duration (MD)，其定義經化簡可得 $\dfrac{-\partial P / \partial r \times (1+r)}{P}$

為了討論債券價格的變化和殖利率變動的關係，

可定義Modified duration為 $\dfrac{-\partial P / P}{\partial r} = \dfrac{MD}{(1+r)}$

請修改本章計算存續期間的範例程式，計算 Modified duration。請利用程式中已計算出的 Modified duration，計算當殖利率變動一個 basis point時，該債券價格變動的百分比

Macaulay Duration

- The MD of a coupon bond is less than its maturity.
- The MD of a zero-coupon bond
- The MD of a consol bond

MD of a Coupon Bond

- The MD of a coupon bond is

$$MD = \frac{1}{P}\left(\sum_{i=1}^{n}\frac{iC}{(1+y)^i} + \frac{nF}{(1+y)^n}\right) \qquad (4.3)$$

Where C is the period fixed interest flow.

The MD of a zero-coupon bond

- MD of a zero-coupon bond is it's final maturity (n).
- Proof: because no cash flows before maturity, the MD is

$$MD = \frac{\sum_{i=1}^{n} iC_i (1+y)^{-i}}{\sum_{t=1}^{n} C_t (1+y)^{-i}} = \frac{nC_n (1+y)^{-n}}{C_n (1+y)^{-n}} = n$$

The MD of a Consol Bond

- A consol bond pay a fixed coupon each period but it never matures. $(Maturity \;\; date = \infty)$
- The duration of a consol bond is: $MD_c = 1 + \dfrac{1}{y}$

$$\because P = \frac{C}{y} \Rightarrow C = Py$$

$$\because MD = \frac{\dfrac{C}{(1+y)} + \dfrac{2C}{(1+y)^2} + \dfrac{3C}{(1+y)^3} + \ldots.\infty}{P} = \frac{\dfrac{Py}{(1+y)} + \dfrac{2Py}{(1+y)^2} + \dfrac{3Py}{(1+y)^3} + \ldots.\infty}{P}$$

$$= \frac{y}{(1+y)} + \frac{2y}{(1+y)^2} + \frac{3y}{(1+y)^3} + \ldots\ldots\infty$$

$$\because \frac{1}{(1+y)} MD = \frac{y}{(1+y)^2} + \frac{2y}{(1+y)^3} + \frac{3y}{(1+y)^4} + \ldots\ldots\infty$$

$$\Rightarrow MD - \frac{1}{(1+y)} MD = \frac{y}{(1+y)} + \frac{y}{(1+y)^2} + \frac{y}{(1+y)^3} + \ldots.\infty \Rightarrow \frac{y}{(1+y)} MD = \frac{y/(1+y)}{1 - \dfrac{1}{(1+y)}} = 1 \Rightarrow MD = \frac{1+y}{y} = 1 + \frac{1}{y}$$

Where y is yield to maturity (YTM)

The MD of Floating-rate instruments

- A floating-rate instrument makes interest rate payments based on some publicized index such as the London Interbank Offered Rate (LIBOR), the U.S. T-bill rate.
- Instead of being locked into a number, the coupon rate is reset periodically to reflect the prevailing interest rate.
- Floating-rate instrument are typically less sensitive to interest rate changes than are fixed-rate instrument.

The MD of Floating-rate instruments

- Assume that the fixed coupon rate ***c*** in first *j* period, **y** in *n-j* period, also market yield is **y** now. The first reset date is ***j*** period from now, and reset will be performed thereafter.
- Let the principal be \$1 for simplicity. The cash flow of the floating-rate instrument is

$$\overbrace{c, c, \ldots\ldots, c}^{j},\ \overbrace{y, \ldots\ldots, y, y+1}^{n-j}$$

- The MD of a floating-rate instrument is $MD_{Fix} - \sum_{i=j+1}^{n} \frac{1}{(1+y)^{i-1}}$

Denote the MD of an otherwise identical fixed-rate bond.

#Homework 4

- Prove that

$$MD_{floating} = MD_{Fix} - \sum_{i=j+1}^{n} \frac{1}{(1+y)^{i-1}}$$

Where the bond is priced at par ,and the principal be $1 for simplicity.

Conversion

- To convert the MD to be year based, modify(4.3) as follow:

$$\frac{1}{p}\left(\sum_{i=1}^{n}\frac{i}{k}\frac{C}{(1+\frac{y}{k})^{i}}+\frac{n}{k}\frac{F}{(1+\frac{y}{k})^{n}}\right)$$

Where ***y*** is the *annual yield* and ***k*** is the compounding frequency per annum.

- Equation (4.2) also becomes $MD=-(1+\frac{y}{k})\frac{\partial P/p}{\partial y}$

- Note from the definition that $MD(in\ years)=\frac{MD(in\ periods)}{k}$

Difference of formulas

- Macaulay Duration: $MD = -\dfrac{\partial P / P}{\partial y / (1+y)}$
- Modified Duration: $D = \dfrac{MD}{1+y} = -\dfrac{\partial P / P}{\partial y}$
- Dollar Duration: $DD = D \times P = -\dfrac{\partial P}{\partial y}$

Effective Duration

- A general numerical formula for volatility is the effective duration, defined as

$$\frac{P_- - P_+}{P_0(y_+ - y_-)} \qquad (4.5)$$

where P_- is the price if the yield is decreased by Δy, P^+ is the price if the yield is increased by Δy, P_0 is the initial price, y is the initial yield, and Δy is small.

- We can compute the effective duration of just about any financial instrument.

Effective Duration

- Most useful where yield changes alter the cash flow or securities whose cash flow is so complex that simple formulas are unavailable
- Duration of a security can be longer than its maturity or negative.
  - Consider a cash flow: -1 @ time 1 and 2 @time 2
  - Consider a cap
- Neither makes sense under the maturity interpretation.
- An alternative is to use

$$\frac{P_0 - P_+}{P_0 \Delta y}.$$

Immunization and MD

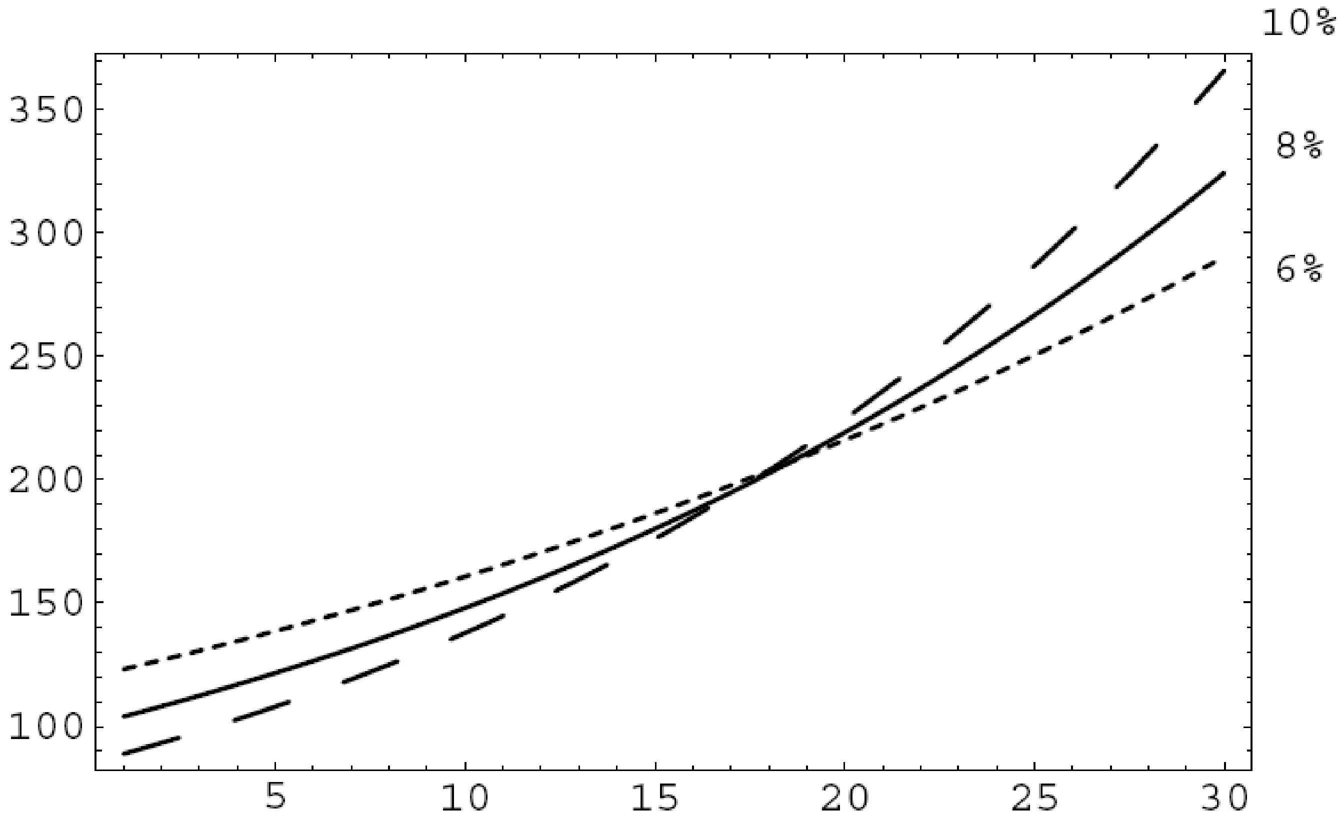
- A portfolio immunizes a liability if its value at horizon covers the liability for small rate changes now.
- How do we find such a bond portfolio?

→A bond portfolio whose MD equals the horizon and whose PV equals the PV of the single future liability.

– At horizon, losses from the interest on interest will be compensated by gains in the sale price when interest rates fall.

– Losses from the sale price will be compensated by the gains in the interest on interest when interest rates rise

Bond value under three rate scenarios



Immunization

- Assume the liability is ***L*** at time ***m*** and the current interest rate is ***y***. We are looking for a portfolio such that

  (1) FV is L at the horizon m;

  (2) $\partial \mathrm{FV}/\partial y = 0$;

  (3) FV is convex around y.

- Condition (1) says the obligation is met.
- Conditions (2) and (3) mean ***L*** is the portfolio's minimum **FV** at horizon for small rate changes.

The Proof (1)

- Let $\mathrm{FV} \equiv P(1+\mathrm{y})^{\mathrm{m}}$, where P is the PV of the portfolio. Now,

$$\frac{\partial FV}{\partial y} = m(1+y)^{m-1}P + (1+y)^{m}\frac{\partial P}{\partial y} \qquad (4.8)$$

- Imposing Condition (2) leads to

$$m = -(1+y)\frac{\partial P/P}{\partial y} \qquad (4.9)$$

- The MD is equal to the horizon *m*.

The Proof (2)

- Employ coupon bond for immunization, because

$$FV = \sum_{i=1}^{n} \frac{C}{(1+y)^{i-m}} + \frac{F}{(1+y)^{n-m}}$$

- It follows that

$$\frac{\partial^2 FV}{\partial^2 y} > 0, \ for \ y > -1 \qquad (4.10)$$

- Because the FV is convex for y>1, the minimum value of FV is indeed **L**.

Example: Immunization by using duration technique

- Suppose that we are in 2007, and the insurer has to make a guarantee payment **$1,469** to a policyholder in 5 years, 2012. The amount is equivalent to investing $1,000 at an annually compound rate of 8% over 5 years.

- Strategy1: Buy five-year maturity discount bonds.
- Strategy2: Buy five-year duration coupon bonds.

Strategy1: Buy five-year maturity discount bonds

- If the insurer buy 1.469 units of these bonds at a total cost of $1000 in 2007, these investment would produce exactly $1469 on maturity in five years.

- The reason is that the duration of this bond portfolio exactly matches the target horizon for the insurer's future liability.

$$P = \frac{1000}{1.08^5} = 680.58 \Rightarrow total\ \cos t = 1.469 \times 680.58 = 1000$$

$$cash\ \ flow\ in\ \ five\ years = \$1000 \times 1.469 = \$1469$$

Strategy2: Buy five-year duration coupon bonds.

The gain or losses on reinvestment income that result from an interest rate change are exactly offset by losses or gains from the bond proceeds on sale.

	YTM fall to 7%	YTM is 8%	YTM rise to 9%
Coupons (5×$80)	400	400	400
Reinvestment income	60	69	78
Sale of bond at end of the 5th year	1009	1000	991
	$1469	$1469	$1469

Cash matching

Immunization

- If there is no single bond whose MD match the horizon, a portfolio of two bonds A and B, can be assembled by the solution of

$$1 = \omega_A + \omega_B$$
$$D = \omega_A D_A + \omega_B D_B \quad (\textit{See} \text{ next page})$$

Here, $\mathbf{D_i}$ is the MD of bond i and $\boldsymbol{\omega_i}$ is the weight of bond i in the portfolio.

- Make sure that D falls between D_A and D_B to guarantee $\omega_A>0, \omega_B>0$, and positive portfolio convexity.

$$Set\ \ D = \frac{1}{P}\sum_{i=1}^{n}\frac{iC_i}{(1+y)^i}$$

$$D_A = \frac{1}{P_A}\sum_{i=1}^{n_A}\frac{iA_i}{(1+y)^i}\ ,\ D_B = \frac{1}{P_B}\sum_{i=1}^{n_B}\frac{iB_i}{(1+y)^i}$$

(A_i, B_i : cashflow of A and B at i-th period)

$$\therefore W_A D_A + W_B D_B = \frac{W_A}{P_A}\sum_{i=1}^{n_A}\frac{iA_i}{(1+y)^i} + \frac{W_B}{P_B}\sum_{i=1}^{n_B}\frac{iB_i}{(1+y)^i}$$

$$Set\ \ P = W_A P + W_B P$$

$$we\ can\ buy\ \ \frac{W_A P}{P_A}\ units\ of\ \ A,\ and\ \ \frac{W_B P}{P_B}\ units\ of\ \ B.$$

$$then\ D = \frac{1}{P}\sum_{i=1}^{n_A}\frac{i\frac{W_A P}{P_A}A_i}{(1+y)^i} + \frac{1}{P}\sum_{i=1}^{n_B}\frac{i\frac{W_B P}{P_B}B_i}{(1+y)^i} = \frac{W_A}{P_A}\sum_{i=1}^{n_A}\frac{iA_i}{(1+y)^i} + \frac{W_B}{P_B}\sum_{i=1}^{n_B}\frac{iB_i}{(1+y)^i}$$

$$\therefore\ D\ =\ W_A D_A + W_B D_B$$

In Class Exercise

- The liability has an MD of 3 years, but the money manager has access to only two kinds of bonds with MDs of 1 year and 4 years. What is the right proportion of each bond in the portfolio in order to match the liability’s MD?

Limitations of Duration

- Duration matching can be costly.
- Immunization is a dynamic problem.
  - Because continuous rebalancing may not be easy to do and involves costly transaction fees.
  - There is a trade-off between being perfectly immunized and the transaction costs of maintaining.
- Large interest rate and convexity (see next figure).
  - Duration accurately measures the price sensitivity of fixed-income securities for small change in interest rates.

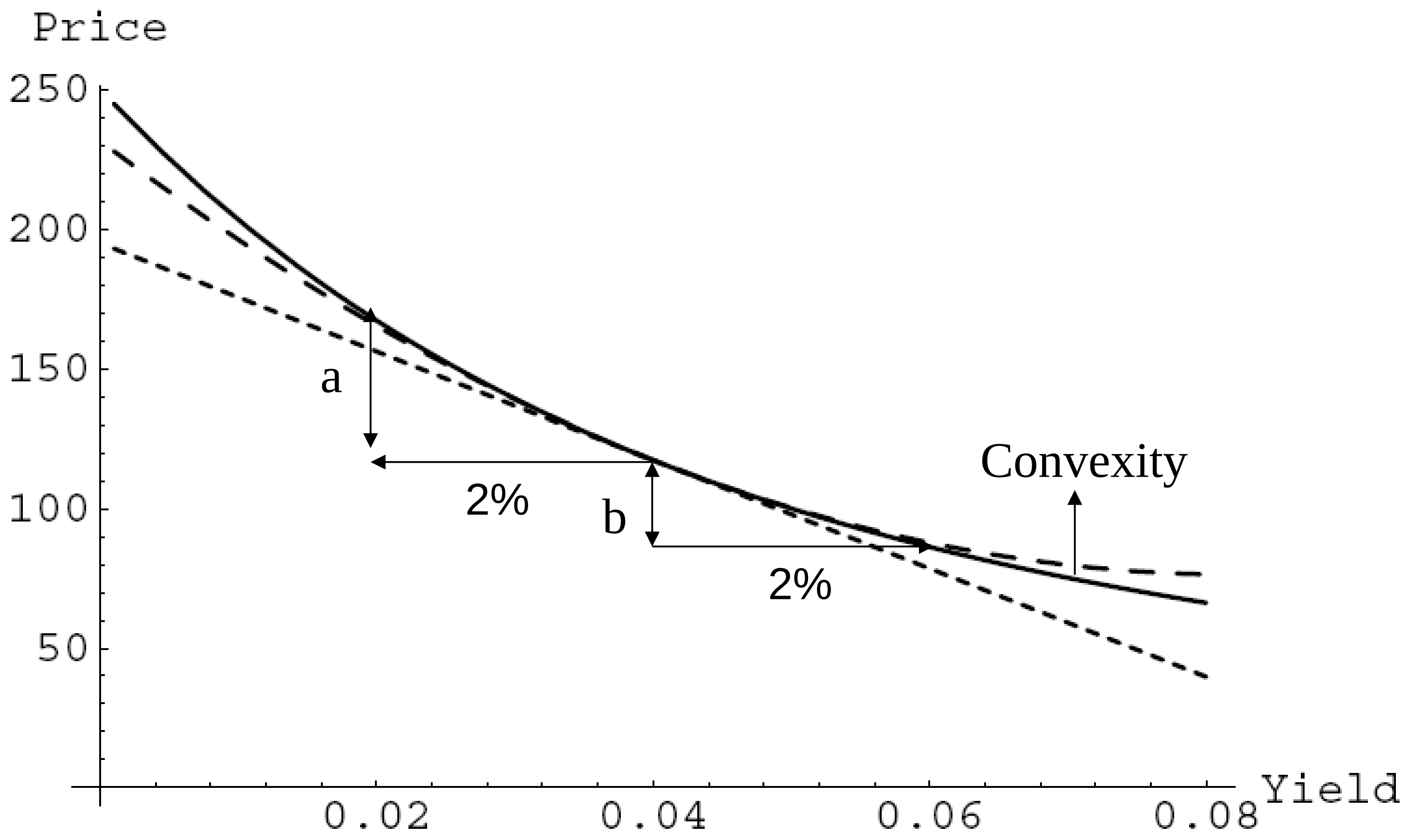
Convexity

- Convexity is defined as

$$convexity\ (in\ \ period) \equiv \frac{\partial^2 P}{\partial y^2}\frac{1}{P} \qquad (4.14)$$

- The convexity of a coupon bond is positive.
- For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude.
- Between two bonds with the same duration, the one with a higher convexity is more valuable.

Figure 4.6: Linear and quadratic approximation to bond price changes.



Convexity

- The approximation $\Delta P/P \approx -$ (modified) duration x yield change works for small yield changes.
- To improve upon it for larger yield changes, use

$$\frac{\Delta P}{P} \approx \frac{\partial P}{\partial y}\frac{1}{P}\Delta y + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}\frac{1}{P}(\Delta y)^2 = -duration \times \Delta y + \frac{1}{2} \times convexity \times (\Delta y)^2$$

Calculation of Convexity

- Formula:

$$CX = \text{Scaling factor}\left(\text{The capital loss from 1bp rise} + \text{The capital gain from 1bp fall}\right)$$

$$= 10^8\left(\frac{\Delta P^-}{P} + \frac{\Delta P^+}{P}\right)$$

- **Example:**

To calculate convexity of the 8 percent coupon, 8 percent yield, six-year maturity Eurobond that had a price of $1000:

$$CX = 10^8\left(\frac{999.53785 - 1000}{1000} + \frac{1000.46243 - 1000}{1000}\right)$$

$$= 10^8\left(0.00000028\right) = 28$$

Example

- Given convexity C , the percentage price change expressed in percentage terms is approximated by $-D \times \Delta r + C \times (\Delta r)^2 / 2$ when the yield increases instantaneously by Δr%.

- **For example：**

  if $D = 10$, $C = 150$, and $\Delta r = 2\%$, price will drop by

  17% because $\Delta P/P = -10 \times 2\% + 1/2 \times 150 \times (2\%)^2 = -17\%$

In Class Exercise

- Show that the convexity of a n-period zero-coupon bond is $n(n+1)/(1+y)^2$

Immunization (barbell Portfolio)

- Two bond portfolios with varying duration pairs D_A,D_B can be assembled to satisfy $D = \omega_A D_A + \omega_B D_B$
  However, which one is to be preferred?
- Let there be n kinds of bonds, with bond i having duration D_i and convexity C_i, where $D_1<D_2<\ldots<D_n$. We then solve the follow constrained optimization problem:

$$\begin{aligned} &\text{maximize} \quad \omega_1 C_1 + \omega_2 C_2 + \ldots\ldots + \omega_n C_n \\ &\text{subject to} \quad \omega_1 + \omega_2 + \ldots\ldots \omega_n = 1 \\ &\qquad\qquad\quad \omega_1 D_1 + \omega_2 D_2 + \ldots\ldots + \omega_n D_n = D \end{aligned}$$

The solution usually implies a barbell portfolio, which consists of very short-term bonds and very long-term bonds..

Proof of Convexity

$$C = W_A C_A + W_B C_B \qquad \frac{\partial^2 P}{\partial y^2} = \sum_{i=1}^{n} \frac{i(i+1)C_i}{(1+y)^{i+2}}$$

$$Convexity = \frac{1}{P}\frac{\partial^2 P}{\partial y^2} = \frac{1}{P}\sum_{i=1}^{n} \frac{i(i+1)C_i}{(1+y)^{i+2}} \equiv C$$

(Let A_i, B_i : cash flow of A and B at i-th period)

$$C_A = \frac{1}{P_A}\sum_{i=1}^{n_A} \frac{i(i+1)A_i}{(1+y)^{i+2}}, \quad C_B = \frac{1}{P_B}\sum_{i=1}^{n_B} \frac{i(i+1)B_i}{(1+y)^{i+2}}$$

$$\therefore W_A C_A + W_B C_B = \frac{W_A}{P_A}\sum_{i=1}^{n_A} \frac{i(i+1)A_i}{(1+y)^{i+2}} + \frac{W_B}{P_B}\sum_{i=1}^{n_B} \frac{i(i+1)B_i}{(1+y)^{i+2}}$$

Proof of Convexity

已知 $W_A C_A + W_B C_B = \frac{W_A}{P_A}\sum_{i=1}^{n_A}\frac{i(i+1)A_i}{(1+y)^{i+2}} + \frac{W_B}{P_B}\sum_{i=1}^{n_B}\frac{i(i+1)B_i}{(1+y)^{i+2}}$ 下

$$Set\ P = W_A P + W_B P$$

$$we\ can\ buy\ \frac{W_A P}{P_A}\ units\ of\ A,\ and\ \frac{W_B P}{P_B}\ units\ of\ B$$

$$then\ C = \frac{1}{P}\sum_{i=1}^{n_A}\frac{i(i+1)\frac{W_A P}{P_A}A_i}{(1+y)^{i+2}} + \frac{1}{p}\sum_{i=1}^{n_B}\frac{i(i+1)\frac{W_B P}{P_B}B_i}{(1+y)^{i+2}}$$

$$= \frac{W_A}{P_A}\sum_{i=1}^{n_A}\frac{i(i+1)A_i}{(1+y)^{i+2}} + \frac{W_B}{P_B}\sum_{i=1}^{n_B}\frac{i(i+1)B_i}{(1+y)^{i+2}} = W_A C_A + W_B C_B$$

Lagrange Multiplier Method

$$function\ f(x_1, x_2, \ldots., x_n)$$

$$subject\ to\ g(x_1, x_2, \ldots., x_n) = 0$$

$$F(x_1, x_2, \ldots., x_n, \lambda) = f(x_1, x_2, \ldots., x_n) + \lambda \bullet g(x_1, x_2, \ldots., x_n)$$

$$Fx_1(x_1, x_2, \ldots., x_n, \lambda) = 0$$

$$Fx_2(x_1, x_2, \ldots., x_n, \lambda) = 0$$

$$\vdots$$

$$Fx_n(x_1, x_2, \ldots., x_n, \lambda) = 0$$

$$g(x_1, x_2, \ldots., x_n) = 0$$

A Simple Example

$$\min \quad f(x,y) = 5x^2 + 6y^2 - xy$$
$$s.t \quad x + 2y = 24$$

$$g(x,y) = x + 2y - 24 = 0$$

$$F(x,y,\lambda) = 5x^2 + 6y^2 - xy + \lambda(x + 2y - 24)$$
$$F_x(x,y,\lambda) = 10x - y + \lambda = 0....(1)$$
$$F_y(x,y,\lambda) = 12y - x + 2\lambda = 0..(2)$$
$$g = x + 2y - 24 = 0..................(3)$$

根據(1)、(2) 得$x = \dfrac{2}{3}y$

$x = \dfrac{2}{3}y$ 代入(3),得$y = 9, x = 6$

Use Lagrange Multiplier Method to obtain the optimal bond portfolio

$$\max \ \omega_1 C_1 + \omega_2 C_2 + \ldots + \omega_n C_n$$

$$s.t \quad \omega_1 + \omega_2 + \ldots + \omega_n = 1$$

$$\omega_1 D_1 + \omega_2 D_2 + \ldots + \omega_n D_n = D$$

$$g_1(\omega_1, \omega_2, \ldots, \omega_n) = \omega_1 + \omega_2 + \ldots + \omega_n - 1 = 0$$

$$g_2(\omega_1, \omega_2, \ldots, \omega_n) = \omega_1 D_1 + \omega_2 D_2 + \ldots + \omega_n D_n - D = 0$$

$$F(\omega_1, \omega_2, \ldots, \omega_n) = \omega_1 C_1 + \omega_2 C_2 + \ldots + \omega_n C_n + \lambda_1(\omega_1 + \omega_2 + \ldots + \omega_n - 1) + \lambda_2(\omega_1 D_1 + \omega_2 D_2 + \ldots + \omega_n D_n - D)$$

Example: Immunization (Convexity is desirable)

- Consider a pension fund manger with a 15-year payout horizon. To immunize the risk of interest rate changes, the manger purchase bonds with a 15-year duration. Consider two alternative strategies to achieve this:

- Strategy1: Invest 100 percent of resources in a 15-year deep-discount bond with an 8 percent yield. (Bullet portfolio)

- Strategy2: Invest 50 percent in the very short-term money market and 50 percent in 30-year deep-discount bond with an 8 percent yield. (Barbell portfolio)

Example: Immunization (Convexity is desirable)

- **Strategy1:**

  $\triangle y = 5\%$

  Duration =15

  Convexity =206

  value of the convexity $= 1/2 \times \text{convexity} \times \triangle y^2 = 25.75\%$

- **Strategy2:**

  Duration $= 1/2 \times 0 + 1/2 \times 30 = 15$

  Convexity $= 1/2 \times 0 + 1/2 \times 797 = 398.5$

  Value of the convexity $= 1/2 \times \text{convexity} \times \triangle y^2 = 49.81\%$

High convexity is more valuable

The manger may seek to attain greater convexity in the asset portfolio than in the liability portfolio, as a result, both positive and negative shocks to interest rates would have beneficial effects on the net worth.

Categories of Immunization

- Cash matching
- Rebalancing

Cash matching

- Cash matching is the approach that a stream of liability can always be immunized with a matching stream of zero-coupon bonds.

- Two problem with this approach are that (1) zero-coupon bonds may be missing for certain matruity.(2) they typically carry lower yield.

- Recall example
(Immunization by using duration technique).

Rebalancing

- Immunization has to be rebalanced constantly to ensure that the MD remains matched to the horizon.
- The MD decreases as time passes.
- But, except for zero-coupon bonds, the decrement is not identical to that in the time to maturity.
  - Consider a coupon bond whose MD matches horizon.
  - Since the bond's maturity date lies beyond the horizon date, its MD will remain positive at horizon.
  - So immunization needs to be reestablished even if interest rates never change.

Hedging

- Hedging aims to offsets the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged.
- Define dollar duration as

$$DD \equiv modified\ duration \times price(\%\ of\ \ par) = -\frac{\partial P}{\partial y}$$

- The approximate dollar price change per $100 of par value is

$$price\ change \approx -dollar\ duration \times yield\ change$$

Hedging

- Because securities may react to interest rate changes differently, we define yield beta to measure relative yield changes.

$$Yield\ beta = \frac{Change\ in\ yield\ for\ the\ hedged\ security}{Change\ in\ yield\ for\ the\ hedging\ security}$$

- Let the hedge ratio be

$$h \equiv \frac{dollar\ duration\ of\ the\ hedged\ security}{dollar\ duration\ of\ the\ hedging\ security} \times yield\ beta \qquad (4.13)$$

- Then hedging is accomplished when the value of the hedging security is h times that of the hedged security.

Example 4.2.2

- Suppose we want to hedge bond A with a duration of seven by using bond B with a duration of eight. Under the assumption that yield beta is one and both bonds are selling at par, the hedge ratio is 7/8, This means that an investor who is long $1 million of bond A should short $7/8 million of bond B.