3 MVO variants and Black–Litterman

3.1 Additional MVO constraints

We can add more constraints to our standard MVO problem if we want to enforce or restrict our resulting portfolio under specific criteria.

- **Enforce diversification**

  → We can add constraints to limit our exposure on a certain asset (or industry), or to force us to have at least some exposure to it.

  Example: Imagine we solved our MVO problem and we now have the following portfolio

$$x = \begin{bmatrix} \text{AAPL} \\ \text{MSFT} \\ \text{TD} \\ \text{RY} \\ \text{IBM} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.2 \\ 0.08 \\ 0.12 \\ 0.35 \end{bmatrix}$$

  and we want to limit our exposure to the technology sector to invest at least 5% of our wealth, but not more than 50%. We also want to maintain a minimum investment of 10% on TD. Thus, we can add the following constraints to our portfolio

$$\begin{array}{lll} 1) \quad L_1 \leq x_1 + x_2 + x_5 \leq U_1 & \rightarrow & 0.05 \leq x_1 + x_2 + x_5 \leq 0.5\% \\ 2) \quad L_2 \leq x_3 & \rightarrow & 0.1 \leq x_3 \end{array}$$

  where L_i and U_i are our upper and lower bounds on our desired exposure. If we were to re-optimize, we would have the following portfolio

$$x = \begin{bmatrix} 0.15 \\ 0.12 \\ 0.15 \\ 0.35 \\ 0.23 \end{bmatrix}$$

  When several of these constraints are present. If we impose several of these constraints, some of them may not be active (e.g., limiting our exposure to the technology sector had the unforeseen effect of increasing our investment on TD to 0.15, meaning our constraint $x_3 \geq 0.1$ is inactive).

- **Transaction costs**

  Every time we create or rebalance a portfolio we have to engage a broker to place our buy and sell orders. This process is not free, and we must pay a fee to the broker for its service. Moreover, we must pay the "bid-ask" spread in order to buy or sell our assets. The bid-ask spread is the difference between the how much is a buyer willing to pay for a stock vs. how much is a seller willing to accept for it. This market friction comes at a cost to us, the investor.

Example: We have a portfolio x^0 and we wish to rebalance it. We can model the transaction costs linearly as follows

$$\sum_{i=1}^{n} c_i \cdot |x_i^* - x_i^0| \le T$$

where c is the fee we must pay during the transaction, T is a limit we want to impose on our total transaction costs, and x^* is our portfolio after re-optimizing. However, the absolute value function $|\cdot|$ is computationally problematic because it is not differentiable everywhere.

We can introduce n auxiliary variables to deal with this

$$\begin{aligned} & x_i^* - x_i^0 \le y_i \\ & x_i^0 - x_i^* \le y_i \\ \rightarrow \quad & \sum_{i=1}^{n} c_i \cdot y_i \le T \end{aligned}$$

which allows us to limit our transaction costs while only adding n auxiliary variables and $2n+1$ linear constraints.

3.2 Mixed-Integer Programs and discrete choice constraints

MIPs (or MILPs for linear programs, MIQPs for quadratic programs) are optimization problems that include a combination of both continuous and integer variables. The inclusion of integer variables allows us to add discrete choice variables to our models.

- Integer programs can sometimes increase the complexity of an optimization problem significantly from a computational standpoint.
- The solution algorithms require additional steps and a different formulation to tackle this type of problem.
- The general topic of Integer Programming is beyond the scope of this course. However, for the problems we will study, they remain conceptually easy to formulate and computationally easy to solve.
- We will study how adding integer variables can be used to make discrete choices when computing optimal portfolios.

- **Cardinality constraints and Buy-in thresholds**

  $\rightarrow$ Limit the number of assets in the portfolio.

Example: We have the following portfolio

$$x = \begin{bmatrix} 0.25 \\ 0.41 \\ 0.01 \\ 0.02 \\ 0.31 \end{bmatrix}$$

However, we want to make our portfolio easy to manage, avoiding the need to rebalance and trade a large number of assets. We decide to limit our exposure to only 2 assets:

$$x = \begin{bmatrix} 0 \\ 0.59 \\ 0 \\ 0 \\ 0.41 \end{bmatrix}$$

This is useful when we might want to track an index, such as the S&P 500, but we do not want to hold 500 different assets. Instead, we might want to hold a basket of only 50 assets to reduce our portfolio management cost.

How can we formulate our MVO to limit our number of assets? We can introduce binary variables to our problem

$$y_i = \begin{cases} 1 & \text{if asset } i \text{ is in the portfolio,} \\ 0 & \text{otherwise.} \end{cases}$$

Now we can setup our MVO problem as a MIQP

$$\begin{aligned} \min_{x,y} \quad & x^T Q \, x \\ \text{s.t.} \quad & \mu^T x \geq R \\ & 1^T x = 1 \\ & 1^T y = K \qquad \Longleftrightarrow \ \Big(\sum_i^n y_i = K \Big) \\ & L_i \cdot y_i \leq x_i \leq U_i \cdot y_i \quad i = 1, \ ..., \ n \\ & \underbrace{y_i = \{0, 1\}}_{\text{Binary variable}}, \quad i = 1, \ ..., \ n \\ & (x_i \geq 0, \quad i = 1, \ ..., \ n) \end{aligned}$$

where L_i and U_i are lower and upper bounds on our variables. These can be thought of as buy-in thresholds, where we may want to impose a threshold before we invest in a stock (e.g., we may not want to invest in a stock unless we have to put at least 5% of our wealth in it). Alternatively, if we do not wish to impose a threshold, we can set L_i and U_i to arbitrarily large negative and positive numbers, respectively, i.e., $L_i << 0$ and $U_i >> 0$.

- **Round lots** (trades done in units of round lots)

  $\rightarrow$ Trade in multiples of a unit size.

  Round lot constraints are usually imposed for practical reasons.

  - As our previous example showed, sometimes our optimal weights are 1% or less in a single asset. Depending on our budget, this might mean we have to buy a fraction of a stock (e.g., 8.72 Apple shares), which is not possible.
  - Moreover, brokers and other agents in the market will usually only allow us to buy in lots, such as 100 shares of Apple at a time (otherwise we might face steep transaction costs).

  We can incorporate this into our portfolio by forcing our decision variable x to allocate wealth by round lot.

  How do we discretize our decision variable?

  Parameter: f_i = fraction of wealth of 1 lot of asset i

$$\rightarrow \quad f_i = \frac{\text{Price per share} \times \text{No. of Shares per lot}}{\text{Total Budget}}$$

  Decision: y_i = number of roundlots of asset i $\quad \rightarrow \quad x_i = y_i f_i$

  Example: We have a budget of \$20,000. There are three assets and we are allowed to buy and sell in round lots of 10 shares. The prices today are the following:

$$P_1 = \$45$$
$$P_2 = \$70$$
$$P_3 = \$32$$

  Therefore, our fractions of wealth are

$$f_1 = \frac{45 \times 10}{20,000} = 0.0225$$
$$f_2 = \frac{70 \times 10}{20,000} = 0.035$$
$$f_3 = \frac{32 \times 10}{20,000} = 0.016$$

Setting up MVO:

$$
\begin{aligned}
\min_{y} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}\,\underbrace{y_i f_i}_{x_i}\underbrace{y_j f_j}_{x_j} \\
\text{s.t.} \quad & \sum_{i=1}^{n}\mu_i y_i f_i \geq R \\
& \sum_{i=1}^{n} y_i f_i = 1 \\
& \underbrace{y \in \mathbb{Z}^n}_{\text{Integer variable}}
\end{aligned}
$$

However, this might not be feasible!

Example:

$$0.27y_1 + 0.55y_2 = 1 \quad \text{for } y_1, y_2 \in \mathbb{Z} \quad \text{has no solution}$$

We can fix this by adding slack and surplus variables $\varepsilon^- \geq 0,\ \varepsilon^+ \geq 0$

$$
\begin{aligned}
\min_{y,\varepsilon^+,\varepsilon^-} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij} y_i f_i y_j f_j + p^+\varepsilon^+ + p^-\varepsilon^- \\
\text{s.t.} \quad & \sum_{i=1}^{n}\mu_i y_i f_i \geq R \\
& \sum_{i=1}^{n} y_i f_i + \varepsilon^- - \varepsilon^+ = 1 \\
& y \in \mathbb{Z}^n \\
& \varepsilon^- \geq 0,\ \varepsilon^+ \geq 0
\end{aligned}
$$

where p^+ and p^- are positive penalty values. The larger we set them, the more we emphasize we want to be close to the original budget.

3.3 Black–Litterman Model

MVO has the drawback of yielding concentrated portfolios, which sometimes gives us too much exposure to a single asset (i.e., we have too much idiosyncratic risk). To overcome this we can use the Black–Litterman model, which estimates the expected returns μ_i in such a way that if used in MVO we would get the observed market cap weights.

Consider the following version of MVO

$$
\begin{aligned}
\max_{x} \quad & \mu^T x - \frac{\lambda}{2} x^T Q x \\
\text{s.t.} \quad & 1^T x = 1
\end{aligned}
$$

Given the estimated parameters μ and Q, we can calculate an optimal portfolio. What if we already have a portfolio $\bar{x}$ and the asset covariance matrix Q, but we do not know the estimated returns μ that led us to this portfolio?

Reverse optimization

We can think of μ as our decision variable, and solve the problem in reverse

$$\max_{\mu} \quad \bar{x}^T\mu - \frac{\lambda}{2}\bar{x}^T Q\bar{x}$$

We can use the FONC to find a closed-form solution that maximizes the objective of our original problem

$$\mu = \lambda Q\bar{x}$$

This is the motivation behind the B–L method. The goal is to derive a set of expected returns $\bar{\mu}$ that implicitly incorporates certain desirable properties. This $\bar{\mu}$ is the expected value of the combination of two distinct distributions. These distributions represent the market equilibrium and the investor's own views about the market.

1st Distribution: Market equilibrium distribution

$$\pi \sim \mathcal{N}(\bar{\pi}, \tau Q)$$

- Q is our covariance matrix.
- $\bar{\pi}$ are the true (but unknown) expected market equilibrium returns. If we are given the market portfolio, x_{mkt}, what are the associated expected returns that, if used in MVO, would yield this same portfolio?

  → a set of returns $\bar{\pi}$ such that when used with MVO, will yield the asset weights of the market portfolio, i.e., x_{mkt}.

  If we assume the market is in equilibrium (i.e., conditions of CAPM / one-fund theorem hold), we can observe the 'optimal' market portfolio x_{mkt}, which are just the cap weights of assets,

$$x_{\text{mkt}} \;\Rightarrow\; x_i = \frac{\text{market cap } i}{\text{market cap of the market}}$$

- π are our implied expected market equilibrium returns, that differ from the true expected market equilibrium returns $\bar{\pi}$.
- τ is a small positive constant, and dampens the variability of π (i.e., τ reduces the variance of each π_i relative to the random asset returns r_i. In other words, the implied expected returns are random variables themselves, but they are less noisy than the random asset returns r_i).

How to find π?

The 'reverse optimization' is the same as before, except now we wish to solve for π,

$$\max_{\pi} \quad x_{\text{mkt}}^T \pi - \frac{1}{2}\lambda x_{\text{mkt}}^T Q x_{\text{mkt}}$$

where

$$\lambda = \frac{\mathbb{E}[r_{\text{mkt}}] - R_f}{\sigma_{\text{mkt}}^2} = \text{Risk aversion coefficient}$$

Thus, the closed-form solution is

$$\pi = \lambda Q x_{\text{mkt}}$$

How to calculate the rest of the parameters?

The expected market excess return $\mathbb{E}[r_{\text{mkt}}]$ (not to be confused with the B–L market equilibrium expected returns π) and market variance (based on excess returns) can be found in the same way we find a portfolio's return and variance,

$$\begin{aligned} \mathbb{E}[r_{\text{mkt}}] - r_f &= \mu_{\text{raw}}^T x_{\text{mkt}} - r_f, \\ Q &= \text{cov}(r_i - r_f, r_j - r_f) \quad \text{for all } i,\ j \\ \sigma_{\text{mkt}}^2 &= x_{\text{mkt}}^T Q x_{\text{mkt}} \end{aligned}$$

where $\mu_{\text{raw}} \in \mathbb{R}^n$ is the expected returns of each asset estimated as the geometric mean from raw data (as we have done for the typical MVO problem, and should not be confused with μ from the B–L model), and $Q \in \mathbb{R}^{n\times n}$ is our covariance matrix (estimated from excess returns).

2nd Distribution: Investor's subjective views on the assets.

Examples of views:

1. We have a strong view that the money market rate will be 2%.
2. We have a weaker view that the S&P 500 will outperform US bonds by 5%.

The views are of the form

$$P\mu = q + \varepsilon$$

where $P\mu = q$ codes our views into the optimization problem and ε is the noise (or confidence level) we have on our views.

- P is a $k \times n$ matrix with k views for n assets
- q is a $k \times 1$ vector
- $\varepsilon \sim \mathcal{N}(0, \Omega)$ is the noise vector of the views

→ where Ω_{ii} is the strength of view i

$$\Omega = \begin{bmatrix} \Omega_{11} & 0 & ... & 0 \\ 0 & \Omega_{22} & ... & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & ... & \Omega_{kk} \end{bmatrix}$$

As presented by Black and Litterman, the mean of the combination of distributions 1 and 2 is

$$\bar{\mu} = \left[(\tau Q)^{-1} + P^T \Omega^{-1} P\right]^{-1} \left[(\tau Q)^{-1}\pi + P^T \Omega^{-1} q\right]$$

We can then use $\bar{\mu}$ in MVO to get our portfolio. This portfolio would combine our views of the market with the market portfolio.

Note: If we do not have any views, then $\bar{\mu} = \pi$.

How does $\bar{\mu}$ arise?

Consider a restricted case of B–L where you have views and you are 100% confident in these views.

$$\begin{aligned} \min_{\mu} \quad & \underbrace{(\mu - \pi)^T (\tau Q)^{-1} (\mu - \pi)}_{\text{Measure of distance between } \mu \text{ and } \pi} \\ s.t. \quad & P\mu = q \end{aligned}$$

where μ is the vector of decision variables, and $\varepsilon = 0$ since we are 100% confident in our views. The optimal solution from this problem is $\bar{\mu}$ and can now be used to solve our typical MVO problem.

Example: Consider an investment universe where we have stocks, bonds, and a money market. We have the following views

$$P\mu = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \mu_S \\ \mu_B \\ \mu_M \end{bmatrix} = \begin{bmatrix} 2\% \\ 5\% \end{bmatrix}$$

These views can also be described as

1. $\mu_M = 2\%$
2. $\mu_S - \mu_B = 5\%$

However, we have some doubts about our views:

1. $\Omega_{11} = 0.00001$ (i.e., we have a strong / confident view)
2. $\Omega_{22} = 0.001$ (i.e., we have a weak view)

The larger the value of Ω_{ii}, the less confidence we have on view i. Assume that the mean vector of market equilibrium returns is the following

$$\pi_S = 10.73\%$$
$$\pi_B = 7.37\%$$
$$\pi_M = 6.27\%$$

And we have the covariance matrix

$$Q = \begin{bmatrix} 0.02778 & 0.00387 & 0.00021 \\ 0.00387 & 0.01112 & -0.00020 \\ 0.00021 & -0.00020 & 0.00115 \end{bmatrix}$$

and the parameter $\tau = 0.1$. We can find $\bar{\mu}$ by using the closed-form expression from before

$$\bar{\mu} = \begin{bmatrix} \bar{\mu}_S \\ \bar{\mu}_B \\ \bar{\mu}_M \end{bmatrix} = \begin{bmatrix} 11.77\% \\ 7.51\% \\ 2.34\% \end{bmatrix}$$

Compare this result to our expected market equilibrium returns, π

$$\pi_S = 10.73\%$$
$$\pi_B = 7.37\%$$
$$\pi_M = 6.27\%$$

Note that our B–L expected returns $\bar{\mu}_S$ and $\bar{\mu}_B$ are very similar to the market equilibrium expected returns since we had a weak view on these two. On the other hand, $\bar{\mu}_M$ is very close to our strong view that $\mu_M = 2\%$.

Suggested ways to set τ and Ω_{ii}

$$0.01 \le \tau \le 0.05,$$
$$\Omega_{ii} = \tau P_i Q P_i^T \quad \text{where } P_i \text{ is the } i^{\text{th}} \text{ row of } P$$

How do we calculate our desired portfolio?

Finally, we can calculate our portfolio through the MVO problem

$$\begin{aligned} \min_x \quad & \frac{1}{2}\lambda x^T Q\, x - \bar{\mu}^T x \\ \text{s.t.} \quad & 1^T x = 1 \\ & (x \ge 0) \end{aligned}$$

where λ is the same risk aversion coefficient as before.