8 Risk parity

A risk parity portfolio is analogous to the equally-weighted portfolio, except we seek to equally distribute risk instead of weight. This means we wish to diversify our risk as much as possible by having each asset contribute the same level of risk to the portfolio. In other words, risk parity solves the portfolio optimization problem by assigning wealth in such a way that the individual asset risk contributions are equalized. Risk Parity is sometimes referred to as "Equal Risk Contribution" (ERC).

Advantages:

- Fully diversified from a risk perspective.
- We do not need to use estimated expected returns (which are prone to large estimation errors).

8.1 Decomposing the Risk Measure

Consider a continuously differentiable function $f(x)$. Function $f(x)$ is homogeneous of degree one if, for any constant c,

$$f(c \cdot x_1, c \cdot x_2, ..., c \cdot x_n) = c \cdot f(x_1, x_2, ..., x_n)$$

For any continuous differentiable homogeneous function of degree one, its Euler decomposition is

$$f(x) = x_1 \cdot \frac{\partial f}{\partial x_1} + x_2 \cdot \frac{\partial f}{\partial x_2} + ... + x_n \cdot \frac{\partial f}{\partial x_n} = x^T \nabla f(x)$$

We wish to partition our risk measure in such a way that we can measure the individual risk contribution per asset. Luckily for us, the portfolio standard deviation is a homogeneous function of degree one.

Let $\sigma_p = \sqrt{x^T Q \, x}$ be the standard deviation (i.e., risk) of the portfolio. We can find the marginal risk contribution of asset i as follows

$$\frac{\partial \sigma_p}{\partial x_i} = \frac{(Q \, x)_i}{\sqrt{x^T Q x}}.$$

where $(Qx)_i$ is the i^{th} element of the vector Qx. Thus, by Euler's decomposition of σ_p, we have

$$\sigma_p = \sqrt{x^T Q x} = \sum_{i=1}^{n} x_i \frac{\partial \sigma_p}{\partial x_i} = \sum_{i=1}^{n} x_i \frac{(Qx)_i}{\sqrt{x^T Q x}}.$$

Now, consider the portfolio variance, $\sigma_p^2 = x^T Q x$, as our risk measure. From the previous derivation we can find the following useful property

$$\sigma_p^2 = x^T Q x = \sum_{i=1}^{n} x_i (Qx)_i = \sum_{i=1}^{n} R_i,$$

where $R_i = x_i(Qx)_i$ is the individual risk contribution of asset i. We can think of R_i as the variance contribution per asset. Due to the relationship outlined above, $\frac{R_i}{\sqrt{x^T Q x}}$ is the contribution of asset i to the portfolio standard deviation.

8.2 Risk Parity Portfolio Optimization

Risk parity seeks a portfolio where $R_i = R_j \ \forall \ i, j$.

Idea: Use a least-squares approach, i.e., minimize the sum of squared differences,

$$\min_x \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \big(x_i(Qx)_i - x_j(Qx)_j \big)^2$$

Problem: This is a non-convex function. This non-convexity becomes apparent if we inspect the individual risk contributions, R_i. First, recast R_i in standard quadratic notation

$$R_i = x_i(Qx)_i = x^T A_i x,$$

where $A_i \in \mathbb{R}^{n \times n}$ is a matrix that captures the individual risk contribution of asset i. The sparse symmetric matrices A_i are composed of the superposition of row i and column i from the original covariance matrix Q multiplied by one half, with all other elements in the matrix equal to zero, e.g.,

$$A_1 = \begin{bmatrix} \sigma_1^2 & \frac{1}{2}\sigma_{12} & \cdots & \frac{1}{2}\sigma_{1n} \\ \frac{1}{2}\sigma_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}\sigma_{n1} & 0 & \cdots & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & \frac{1}{2}\sigma_{12} & 0 & \cdots & 0 \\ \frac{1}{2}\sigma_{21} & \sigma_2^2 & \frac{1}{2}\sigma_{23} & \cdots & \frac{1}{2}\sigma_{2n} \\ 0 & \frac{1}{2}\sigma_{32} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{1}{2}\sigma_{n2} & 0 & \cdots & 0 \end{bmatrix},$$

$$A_n = \begin{bmatrix} 0 & \cdots & 0 & \frac{1}{2}\sigma_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \frac{1}{2}\sigma_{n-1,n} \\ \sigma_{n1} & \cdots & \frac{1}{2}\sigma_{n,n-1} & \sigma_n^2 \end{bmatrix}$$

By inspection, we can tell the sparse matrices A_i are indefinite, each having a single positive eigenvalue, a single negative eigenvalue, and all other eigenvalues being equal to zero.

This non-convex formulation has many local minima. We can find different ‘long-short’ combinations of assets that solve the risk parity problem. Moreover, any local minimum that meets the risk parity condition $R_i = R_j$ is also a global solution. Thus, we cannot guarantee the uniqueness of our solution, and our portfolio will be sensitive to our initial conditions.

Solution: Consider only the subset of ‘long-only’ portfolios, i.e., disallow short selling. Within this smaller feasible region, the problem is well-behaved. Moreover, this guarantees that our optimal portfolio is not some arbitrary ‘long-short’ combination.

After adding our budget constraint, the risk parity optimization problem is

$$\begin{aligned} \min_{x} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\big(x_i(Qx)_i - x_j(Qx)_j\big)^2 \\ \text{s.t.} \quad & 1^T x = 1, \\ & x \geq 0. \end{aligned}$$

<u>Issues with Risk Parity Optimization</u>:

- Uniqueness of the optimal solution is only guaranteed for 'long-only' portfolios.
- Highly non-linear objective function (we have a 4^{th} degree polynomial).
- It is difficult to find the gradient of this objective analytically.

8.3 Alternative Risk Parity Formulations

<u>Alternative 1</u>: A numerically efficient model

The objective function in our original Risk Parity model has n^2 elements. However, we can reduce this number to only n. All we need to do is introduce an auxiliary variable

$$\begin{aligned} \min_{x,\theta} \quad & \sum_{i=1}^{n}\big(x_i(Qx)_i - \theta\big)^2 \\ \text{s.t.} \quad & 1^T x = 1, \\ & x \geq 0, \end{aligned}$$

where $\theta \in \mathbb{R}$ is an unconstrained variable, giving us added flexibility during optimization. Thus, at optimality, we have that

$$x_i(Qx)_i = \theta \quad \forall\ i,$$

i.e., we must have that $R_i = R_j\ \forall i,\ j$, regardless of the value of θ.

The above formulation is numerically efficient, but it still requires us to solve a highly non-linear problem.

<u>Alternative 2</u>: An analytically intuitive model

Depending on the solver we use, we must sometimes supply both the objective function and its gradient. However, deriving the gradient of our original Risk Parity problem is difficult.

Consider the following function

$$f(y) = \frac{1}{2}y^T Q\ y - c\sum_{i=1}^{n}\ln y_i$$

where c is some positive scalar. Since Q is PSD and $c\sum_{i=1}^{n}\ln y_i$ is strictly concave, then $f(y)$ is strictly convex. We can attain the minimum of this function and get a unique solution y^* by setting the gradient to zero

$$\nabla f(y) = Q\ y - cy^{-1} = 0$$

where $y^{-1} = [1/y_1, 1/y_2, ..., 1/y_n]^T$. Hence, we must have that

$$\begin{aligned} (Qy)_i &= \frac{c}{y_i} && \forall\ i, \\ y_i(Qy)_i &= c && \forall\ i, \\ y_i(Qy)_i &= y_j(Qy)_j && \forall\ i,\ j. \end{aligned}$$

By definition of logarithms, we must also have that $y > 0$ to guarantee feasibility. Thus, to find our optimal risk parity portfolio, we have the following optimization problem

$$\begin{aligned} \min_{y} \quad & \frac{1}{2}y^T Q\ y - c\sum_{i=1}^{n}\ln y_i \\ \text{s.t.} \quad & y \geq 0, \end{aligned}$$

Finally, we can recover our optimal portfolio weights

$$x_i^* = \frac{y_i^*}{\sum_{i=1}^{n} y_i^*}$$

We should realize that x^* is unique and independent of the initial choice of c.

8.4 Properties of Risk Parity Portfolios

Let us compare the equally-weighted (EW), Minimum Variance (MV), and risk parity (RP) portfolios. By design, we know that

$$\begin{aligned} \text{EW}: \quad & x_i = x_j && \forall\ i,\ j \\ \text{MV}: \quad & \frac{\partial\sigma_p}{\partial x_i} = \frac{\partial\sigma_p}{\partial x_j} && \forall\ i,\ j \\ \text{RP}: \quad & x_i\frac{\partial\sigma_p}{\partial x_i} = x_j\frac{\partial\sigma_p}{\partial x_j} && \forall\ i,\ j \end{aligned}$$

This serves to provide some insight about the overall risk of the RP portfolio. It tells us that the RP portfolio's risk sits somewhere between the MV and EW portfolios.

Consider the following formulation of the risk parity problem:

$$
\begin{aligned}
\min_{x} \quad & x^T Q\, x \\
\text{s.t.} \quad & \left.\begin{aligned} & \textstyle\sum_{i=1}^{n} \ln x_i \geq c, \\ & 1^T x = 1, \end{aligned}\right\} \text{Only a unique value of } c \text{ guarantees we get RP} \\
& x \geq 0
\end{aligned}
$$

The equality budget constraint above is very restrictive. Notice that if $c_1 \leq c_2$, we have $\sigma_{p1} \leq \sigma_{p2}$ because this helps to relax the problem.

Let $c = -\infty$. This would, in essence, render the inequality constraint null. Thus, the solution to this problem would give us the minimum variance (MV) portfolio.

Now, let $c = -n \ln n$. The only possible solution for this value of c is $x_i = 1/n$ for all i. In particular, the quantity $\sum_{i=1}^{n} \ln x_i$, under the constraint $1^T x = 1$, is maximized for $x_i = 1/n$ (a larger value of c would be infeasible).

Therefore, we know that there exists a constant c^* for which x^* is the RP portfolio. Thus, we must have that

$$\sigma_{MV} \leq \sigma_{RP} \leq \sigma_{1/n}$$