6 CVaR optimization

6.1 What is VaR?

Value-at-Risk (VaR) is a popular risk metric, particularly favored by practitioners. It is referred to as a tail-based (or downside) risk measure because it measures the loss at the tail of a Profit and Loss (PnL) probability distribution.

Let x be a portfolio with random returns r

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad r = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

For any realization of the returns

$$\hat{r} = \begin{bmatrix} \hat{r}_1 \\ \vdots \\ \hat{r}_n \end{bmatrix}$$

you can compute the portfolio loss

$$f(x, \hat{r}) = -(\hat{r}_1 x_1 + \hat{r}_2 x_2 + \cdots + \hat{r}_n x_n)$$

Let's say our distribution of profits and losses is the following

Diagram: Introductory VaR diagram

VaR is related to the percentiles of the loss distribution. We must

- Specify a time horizon
- Specify a confidence level, α

The VaR for some time horizon and confidence level α is the loss (in \$ or %) that your portfolio x is likely to exceed with probability $1 - \alpha$ within a given time horizon.

Example: a 10-day $\text{VaR}_{95\%}$ equal to \$1,000,000 means our portfolio is likely to exceed this loss with a probability of 5% during the next 10 days.

Useful Notation:

- x – Portfolio
- r – Random vector of returns
- $p(r)$ – Density of our random asset returns $r \in \mathbb{R}^n$
- $f(x,r)$ – loss of portfolio for a realization of our random asset returns r
- γ – loss

The cumulative density function (CDF) of loss associated with x is

$$\Psi(x,\gamma) = \int_{f(x,r)<\gamma} p(r)\ dr$$

which is the probability that the loss of x is less than γ. Then, for a given α,

$$\text{VaR}_\alpha(x) = \min\{\gamma \in \mathbb{R} \mid \Psi(x,\gamma) \geq \alpha\}$$

6.2 VaR Drawbacks

VaR has some potential drawbacks that prohibit us from easily finding an optimal solution,

1. VaR is not sub-additive, i.e., VaR does not always respect diversification.

$$\text{VaR}(x+y) \leq \text{VaR}(x) + \text{VaR}(y)$$

   does not always hold.

2. Optimizing VaR is a non-convex activity, i.e., it is a difficult optimization problem.

6.3 CVaR

Conditional Value-at-Risk (CVaR), also known as Expected Shortfall (ES), measures the expected loss exceeding VaR.

Diagram: CVaR diagram

$$\text{CVaR}_\alpha(x) = \frac{1}{1-\alpha}\int_{f(x,r)\geq \text{VaR}_\alpha(x)} f(x,r)p(r)\ dr$$

By definition, $\text{CVaR}_\alpha(x) \geq \text{VaR}_\alpha(x)$

6.4 CVaR Optimization

Minimizing portfolio variance has the drawback of simulating an investor averse to both upside and downside movements. Instead, we could try to minimize our downside risk only, which is more realistic.

$$\begin{aligned} \min_{x} \quad & \text{CVaR}_\alpha(x) \\ \text{s.t.} \quad & x \in \mathcal{X} \end{aligned}$$

where $\mathcal{X}$ is the set of admissible portfolios (i.e., it includes our budget, short selling restriction, target return, and other linear constraints pertaining to our decision variable x).

$$\mathcal{X} = \left\{x \in \mathbb{R}^n \mid 1^T x = 1;\ ...;\ \mu^T x \geq R\right\}$$

Optimizing $\text{CVaR}_\alpha(x)$ in this form is hard because it is still defined in terms of $\text{VaR}_\alpha(x)$.

<u>Idea</u>: Introduce an auxiliary variable as a placeholder for VaR during the optimization process. Consider the following variable

$$\gamma = \text{VaR}_\alpha(x) = \min\{\gamma \in \mathbb{R} : \Psi(x,\gamma) \geq \alpha\}$$

Now we can define CVaR with respect to the variable γ. Consider the following function

$$F_\alpha(x,\gamma) = \gamma + \frac{1}{1-\alpha}\int \Big(f(x,r) - \gamma\Big)^+ p(r)\ dr$$

where the function $a^+ = \max\{a, 0\}$.

<u>Important Properties of</u> $F_\alpha(x,\gamma)$:

1. $F_\alpha(x,\gamma)$ is a convex function of γ.

2. $\text{VaR}_\alpha(x)$ is the γ that minimizes $F_\alpha(x,\gamma)$.

3. The minimum value over γ of the function $F_\alpha(x,\gamma)$ is $\text{CVaR}_\alpha(x)$,

   $\Rightarrow\ F_\alpha\big(x, \text{VaR}_\alpha(x)\big) = \text{CVaR}_\alpha(x)$

Therefore, we can simplify our optimization problem from non-convex to convex:

Non-convex:

$$\begin{aligned} \min_{x} \quad & \mathrm{CVaR}_\alpha(x) \\ \text{s.t.} \quad & x \in \mathcal{X} \end{aligned} \quad \Longleftrightarrow \quad$$

Convex:

$$\begin{aligned} \min_{x,\gamma} \quad & F_\alpha(x,\gamma) \\ \text{s.t.} \quad & x \in \mathcal{X} \end{aligned}$$

Optimizing $F_\alpha(x,\gamma)$ is still a challenge since we need to use the density function $p(r)$ (and $p(r)$ is usually difficult to solve analytically). Instead, we can use a scenario representation,

$$\hat{r}_s \quad \text{for } s = 1, ..., S$$

where $\hat{r}_s$ is the realization of scenario s. We consider each scenario to be equally likely. We can then approximate $F_\alpha(x,\gamma)$ as

$$\tilde{F}_\alpha(x,\gamma) = \gamma + \frac{1}{(1-\alpha)S} \sum_{s=1}^{S} \underbrace{\Big(f(x,\hat{r}_s) - \gamma\Big)^+}_{\text{non-linear}}$$

The loss function $f(x,\hat{r}_s)$ based on scenarios is typically a linear function of the form $-\hat{r}_s^T x$, where $\hat{r}_s \in \mathbb{R}^n$ represents the loss (or profit) attained by each asset in a given scenario. Therefore,

$$\left.\begin{aligned} \min_{x,\gamma} \quad & \tilde{F}_\alpha(x,\gamma) \\ \text{s.t.} \quad & x \in \mathcal{X} \end{aligned}\right\} \quad \text{is a Linear Program (easy to solve!)}$$

We can deal with the non-linearity as we have done before. For each $s = 1, ..., S$,

$$\left.\begin{aligned} & z_s \geq 0, \\ & z_s \geq f(x,\hat{r}_s) - \gamma \end{aligned}\right\} \quad = \Big(f(x,\hat{r}_s) - \gamma\Big)^+$$

Thus, our $\mathrm{CVaR}_\alpha(x)$ optimization problem becomes

$$\begin{aligned} \min_{x,z,\gamma} \quad & \gamma + \frac{1}{(1-\alpha)S} \sum_{s=1}^{S} z_s \\ \text{s.t.} \quad & z_s \geq 0, && s = 1, ..., S \\ & z_s \geq f(x,\hat{r}_s) - \gamma, && s = 1, ..., S \\ & x \in \mathcal{X} \end{aligned}$$

where $x \in \mathcal{X}$ includes our budget, target return, and any other constraints pertaining to our portfolio weights x. The CVaR optimization problem can be prepared by generating a sufficiently large number of scenarios using Monte Carlo simulations that simulate the changes in price over a time horizon. We can use these scenarios to construct our corresponding Linear Program.