7 Sharpe ratio and index tracking

7.1 Sharpe Ratio

The Sharpe ratio is a performance metric that allows us to compare the return of an asset or portfolio through a ratio of reward-to-risk. The ratio measures the excess rate of return per unit of risk. In other words, we are computing the risk-adjusted rate of return of an asset or portfolio.

Conceptually, this measures how much additional return you are receiving for the additional volatility of holding an asset i or a portfolio p over the risk-free rate, r_f. We can use this to compare two or more financial products or portfolios.

$$\mathrm{SR}_i = \frac{\mathbb{E}[r_i] - r_f}{\sqrt{\mathrm{var}(r_i - r_f)}} = \frac{\mu_i - r_f}{\sigma_i}$$
$$\mathrm{SR}_p = \frac{\mathbb{E}[r_p] - r_f}{\sqrt{\mathrm{var}(r_p - r_f)}} = \frac{\mu^T x - r_f}{\sqrt{x^T Q x}}$$

where σ_i and Q are based on the asset excess returns.

Example: Assume we have a risk-free rate of 5%. Imagine the following scenario

	Portfolio A	Portfolio B
Return	15%	12%
Std. Dev.	8%	5%

By looking at the realized return, we could argue that portfolio A had a better performance. However, if we compare their Sharpe ratios we have that

$$\mathrm{SR}_A = \frac{15-5}{8} = 1.25, \qquad \mathrm{SR}_B = \frac{12-5}{5} = 1.4$$

which means portfolio B had a better risk-adjusted performance.

We can calculate the Sharpe ratio in two ways:

1. Ex Ante: We use our estimated parameters to compute the Sharpe ratio. This gives us an idea of what our future expectation is:
$$\mathrm{SR}_p = \frac{\mu^T x - r_f}{\sqrt{x^T Q x}}$$
This is useful to make decisions about how to construct a portfolio (i.e., we can use this in an optimization model to maximize the Sharpe ratio).

2. Ex Post: We use realized values to compute this Sharpe ratio. Here, we take a look at our historical portfolio values and we compute the ratio of realized return to the volatility incurred

during a time series

$$\mathrm{SR}_p = \frac{\mu_p - r_f}{\sigma_p}$$

This is useful when we want to compare the risk-adjusted performance of two or more portfolios.

7.2 Optimizing the Sharpe Ratio

We can construct a portfolio that seeks to maximize the ex ante Sharpe ratio

$$\begin{aligned}
\max_{x} \quad & \left.\frac{\mu^T x - r_f}{\sqrt{x^T Q\ x}}\quad\right\}\ \text{Non-linear, difficult to solve} \\
\text{s.t.} \quad & 1^T x = 1 \\
& \left. Ax \le b \quad \right\}\ \text{Any additional constraints we might have} \\
& (x_i \ge 0, \quad i = 1,\ ...,\ n)
\end{aligned}$$

Note: We do not need to subtract the risk-free rate from if we are using excess returns to estimate μ. For example, in Project 1 we subtracted the historical risk-free rate from our returns before we estimated our expected returns.

We can simplify the objective through a simple transformation. First, assume there exists a feasible portfolio $\widehat{x}$ with $(\mu - \bar{r}_f)^T \widehat{x} > 0$, where $r_f \in \mathbb{R}$ is the risk-free rate, and we use $\bar{r}_f \in \mathbb{R}^n$ to denote a vector with all its elements equal to the risk-free rate. The transformation is the following

$$\kappa = \frac{1}{(\mu - \bar{r}_f)^T x}, \qquad y = \kappa x$$

where $y \in \mathbb{R}^n$ and $\kappa \in \mathbb{R}$. The objective function can now be rewritten as

$$\frac{\mu^T x - r_f}{\sqrt{x^T Q\ x}} = \frac{(\mu - \bar{r}_f)^T x}{\frac{1}{\kappa}\sqrt{y^T Q\ y}} = \frac{\frac{1}{\kappa}}{\frac{1}{\kappa}\sqrt{y^T Q\ y}} = \frac{1}{\sqrt{y^T Q\ y}}$$

Maximizing $\frac{1}{\sqrt{y^T Q\ y}}$ is equivalent to minimizing $y^T Q\ y$. Since we are replacing our decision variable x with y and κ, we also have that

$$\kappa = \frac{1}{(\mu - \bar{r}_f)^T x} \quad \Longleftrightarrow \quad (\mu - \bar{r}_f)^T y = 1$$

and

$$1^T x = 1 \quad \Longleftrightarrow \quad \frac{1^T y}{\kappa} = 1 \quad \Longleftrightarrow \quad 1^T y = \kappa$$

Finally, our reformulated optimization model to maximize the Sharpe ratio is

$$\begin{aligned} \min_{y,\ \kappa} \quad & y^T Q\ y \\ \text{s.t.} \quad & (\mu - \bar{r}_f)^T y = 1 \\ & 1^T y = \kappa \\ & Ay \leq b \cdot \kappa \\ & \kappa \geq 0 \\ & (y_i \geq 0, \quad i = 1,\ ...,\ n) \end{aligned}$$

If we do not have any other linear constraints, then the matrix A is null and we can remove that constraint. This formulation does not impose a budget constraint. To recover our optimal portfolio weights, all we need to do is

$$x^* = \frac{y^*}{\kappa^*} = \frac{y^*}{\sum_{i=1}^n y_i^*}.$$

7.3 Index tracking

Index trackers have become increasingly popular in industry as passive management funds have gained popularity due to their low cost. Many exchange traded funds (ETFs), typically known as index funds, will attempt to mimic the price path of a specific index.

The motivation for us is to create a portfolio that behaves like a benchmark index, but we want to use fewer assets (e.g., we do not want to hold the 500 assets in the S&P 500).

We can use a cluster-based approach:

Suppose our index has $n = 500$ assets, but we only wish to hold $k < n$ assets

Diagram: Bucketing

Each cluster (or bucket) should contain assets that are most similar to each other.

$$\text{Similarity measure:} \quad \rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} = \text{correlation between } i \text{ and } j$$

We can use the following integer variables $y \in \mathbb{Z}^n$ and $z \in \mathbb{Z}^{n \times n}$ as our decision variables.

Let y_j represent if asset j is selected to be in the portfolio

$$y_j = \begin{cases} 1 & \text{if true} \\ 0 & \text{otherwise} \end{cases}$$

and let z_{ij} represent whether asset j is a representative of asset i

$$z_{ij} = \begin{cases} 1 & \text{if } j \text{ is the most similar asset in the portfolio to } i \\ 0 & \text{otherwise} \end{cases}$$

Then we can write our index tracking problem

$$\begin{array}{lll} \max\limits_{z,y} & \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} \rho_{ij} z_{ij} & \\ \text{s.t.} & \sum_{j=1}^{n} y_j = k & \Big\}\text{Portfolio size constraint} \\ & \sum_{j=1}^{n} z_{ij} = 1 \quad \text{for } i = 1, ..., n & \Big\}\begin{array}{l}\text{Each asset has exactly} \\ \text{one representative}\end{array} \\ & z_{ij} \leq y_j \quad \text{for } i = 1, ..., n;\ j = 1, ..., n & \Big\}\begin{array}{l}\text{asset must be in the portfolio} \\ \text{to be a representative}\end{array} \\ & z_{ij} \in \{0, 1\}, \quad y_j \in \{0, 1\} & \end{array}$$

After we solve this model, we can find the optimal weight of the selected assets as follows. Suppose we chose asset j, then we have that the optimal weight is the sum of the market capitalization of all assets from the bucket it is representing, and we divide this by the market capitalization of the index,

$$x_j = \frac{\sum_{i=1}^{n} V_i z_{ij}}{\sum_{i=1}^{n} V_i}$$

where V_i is the market capitalization of asset i and x_j is the optimal weight of our selected asset j.

Example: Suppose an index has 8 assets and we wish to hold only 3. This means we wish to create 3 buckets, and we will only hold the 'leader' of each bucket.

Assume Asset 1 is the most representative asset between Assets 1, 2, and 5. Assume Asset 6 is the most representative asset between Assets 3 and 6. Finally, assume Asset 7 is the most representative between Assets 4, 7 and 8.

If we optimize this problem, we would get the following optimal solution

$$y^* = \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 1 & 1 & 0\end{bmatrix}^T$$

$$z^* = \begin{array}{c c} & \begin{array}{cccccccc} j=1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{c} i=1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

where y^* indicates the assets we have selected to be part of our index tracking portfolio, and z^* shows the members of each bucket. The 1st column of z^* shows the members of our 1st bucket, where Asset 1 is our leader as indicated by y^*. The 6th column of z^* shows the members of our 2nd bucket, where Asset 6 is our leader as indicated by y^*. The 7th column of z^* shows the members of our 3rd bucket, where Asset 7 is our leader as indicated by y^*.

If we sum each row of z^* individually we can see that each asset is a member of a single bucket, i.e., each asset has only one representative.