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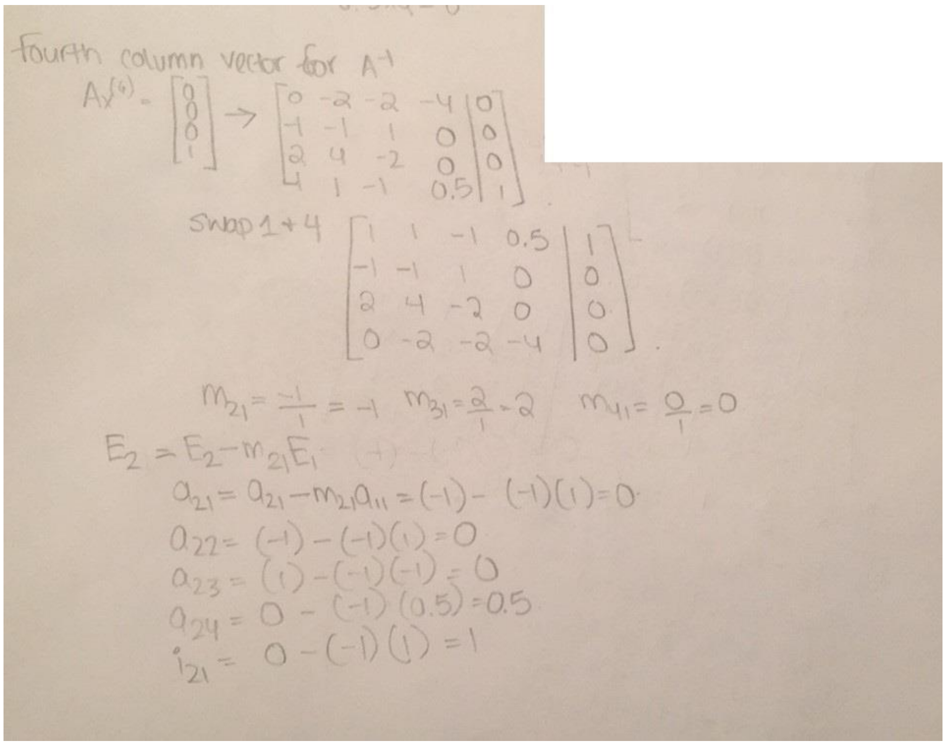
CSC 349A: Numerical Analysis
Assignment 4

2018-11-02

1. Let

$$A = \begin{pmatrix} 0 & -2 & -2 & -4 \\ -1 & -1 & 1 & 0 \\ 2 & 4 & -2 & 0 \\ 1 & 1 & -1 & 0.5 \end{pmatrix}$$

   (a) **(4 points)** Use Gaussian elimination with partial pivoting to compute the fourth column vector of A^{-1} . Do this by hand computation, no programming. (Do not compute all of A^{-1}.) Explicitly interchange rows as required, and show all of the derived linear systems and the back substitution. Do not do any scaling.



Fourth column vector for A^{-1}

$$Ax^{(4)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow \left[\begin{array}{cccc|c} 0 & -2 & -2 & -4 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 2 & 4 & -2 & 0 & 0 \\ 1 & 1 & -1 & 0.5 & 1 \end{array}\right]$$

Swap 1 + 4

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 0.5 & 1 \\ -1 & -1 & 1 & 0 & 0 \\ 2 & 4 & -2 & 0 & 0 \\ 0 & -2 & -2 & -4 & 0 \end{array}\right].$$

$m_{21} = \frac{-1}{1} = -1 \quad m_{31} = \frac{2}{1} = 2 \quad m_{41} = \frac{0}{1} = 0$

$E_2 = E_2 - m_{21}E_1$

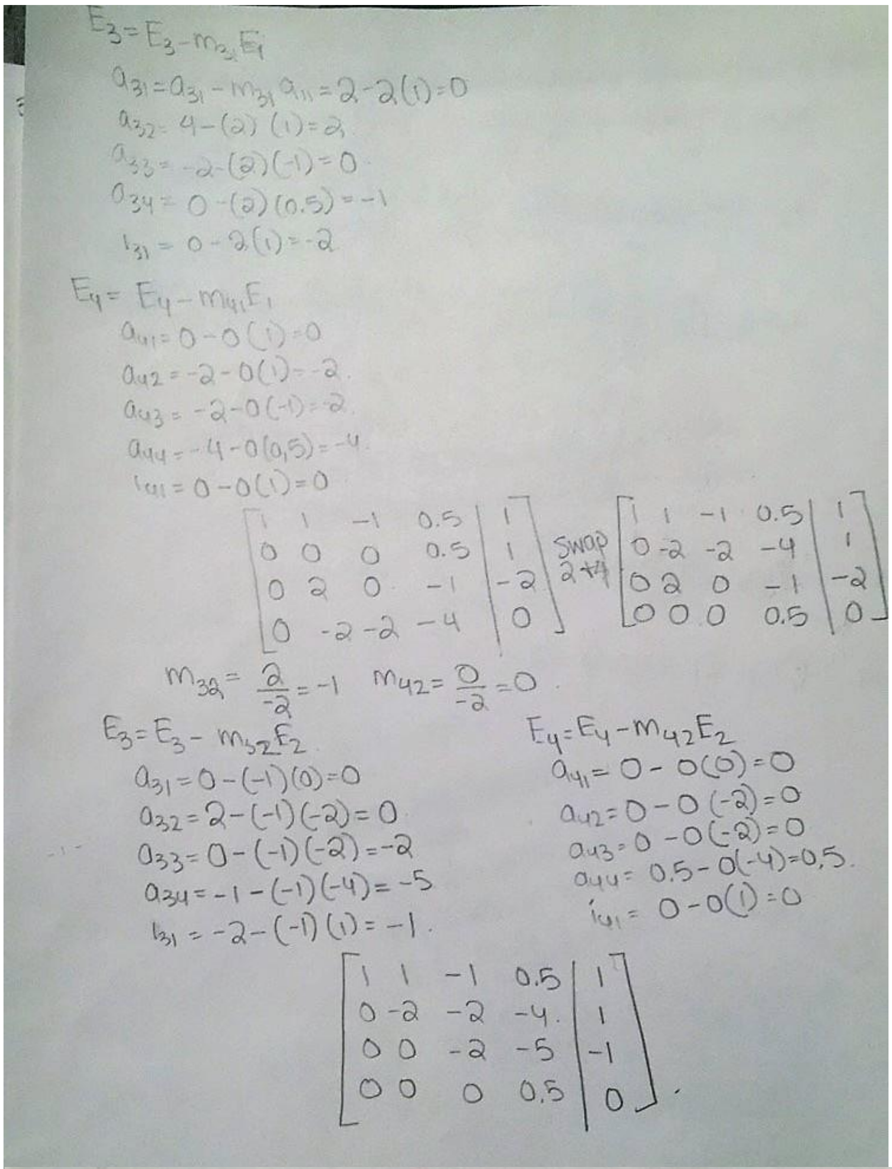
$a_{21} = a_{21} - m_{21}a_{11} = (-1) - (-1)(1) = 0$

$a_{22} = (-1) - (-1)(1) = 0$

$a_{23} = (1) - (-1)(-1) = 0$

$a_{24} = 0 - (-1)(0.5) = 0.5$

$i_{21} = 0 - (-1)(1) = 1$



$E_3 = E_3 - m_{31}E_1$

$a_{31} = a_{31} - m_{31}a_{11} = 2 - 2(1) = 0$

$a_{32} = 4 - (2)(1) = 2$

$a_{33} = -2 - (2)(-1) = 0$

$a_{34} = 0 - (2)(0.5) = -1$

$b_{31} = 0 - 2(1) = -2$

$E_4 = E_4 - m_{41}E_1$

$a_{41} = 0 - 0(1) = 0$

$a_{42} = -2 - 0(1) = -2$

$a_{43} = -2 - 0(-1) = -2$

$a_{44} = -4 - 0(0.5) = -4$

$b_{41} = 0 - 0(1) = 0$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 0.5 & 1 \\ 0 & 0 & 0 & 0.5 & 1 \\ 0 & 2 & 0 & -1 & -2 \\ 0 & -2 & -2 & -4 & 0 \end{array}\right] \overset{\text{Swap}}{\underset{2+4}{}} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 0.5 & 1 \\ 0 & -2 & -2 & -4 & 1 \\ 0 & 2 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0.5 & 0 \end{array}\right]$$

$m_{32} = \frac{2}{-2} = -1 \quad m_{42} = \frac{0}{-2} = 0$

$E_3 = E_3 - m_{32}E_2$

$a_{31} = 0 - (-1)(0) = 0$

$a_{32} = 2 - (-1)(-2) = 0$

$a_{33} = 0 - (-1)(-2) = -2$

$a_{34} = -1 - (-1)(-4) = -5$

$b_{31} = -2 - (-1)(1) = -1$

$E_4 = E_4 - m_{42}E_2$

$a_{41} = 0 - 0(0) = 0$

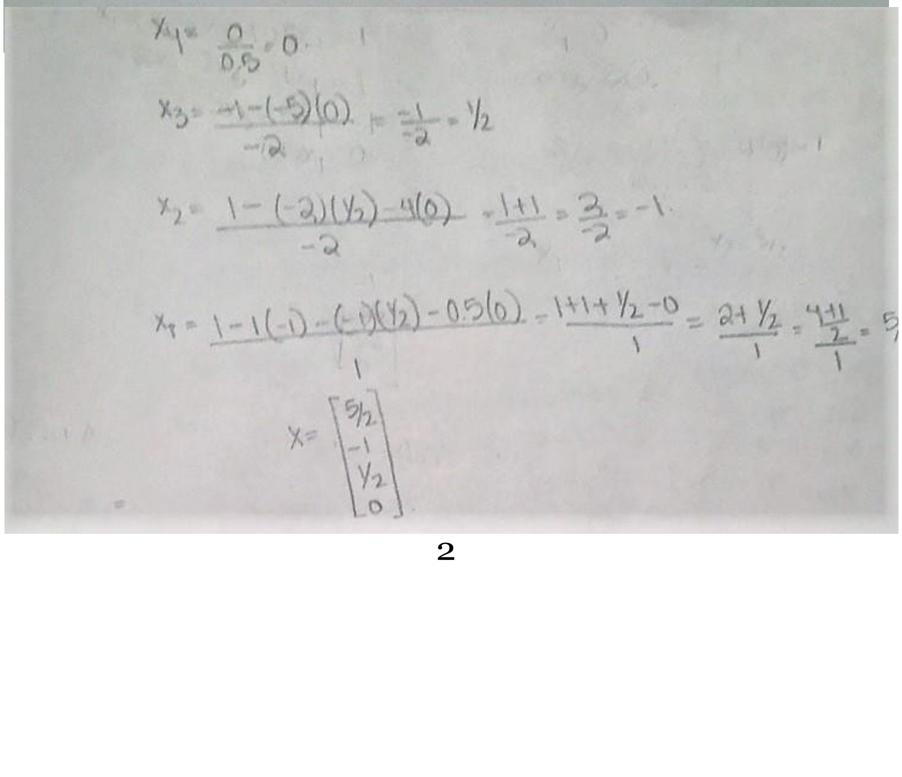
$a_{42} = 0 - 0(-2) = 0$

$a_{43} = 0 - 0(-2) = 0$

$a_{44} = 0.5 - 0(-4) = 0.5$

$b_{41} = 0 - 0(1) = 0$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 0.5 & 1 \\ 0 & -2 & -2 & -4 & 1 \\ 0 & 0 & -2 & -5 & -1 \\ 0 & 0 & 0 & 0.5 & 0 \end{array}\right].$$



$x_4 = \frac{0}{0.5} = 0$

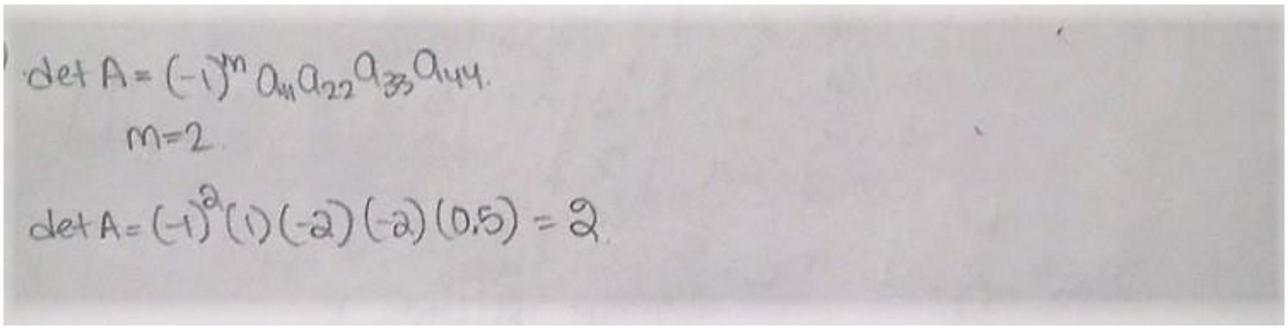
$x_3 = \frac{-1-(-5)(0)}{-2} = \frac{-1}{-2} = 1/2$

$x_2 = \frac{1-(-2)(1/2)-4(0)}{-2} = \frac{1+1}{-2} = \frac{2}{-2} = -1$

$x_1 = \frac{1-1(-1)-(-1)(1/2)-0.5(0)}{1} = \frac{1+1+1/2-0}{1} = \frac{2+1/2}{1} = \frac{\frac{4+1}{2}}{1} = 5$

$$x = \begin{bmatrix} 5/2 \\ -1 \\ 1/2 \\ 0 \end{bmatrix}$$

(b) **(2 points)** From the computations in (a), what is the determinant of A? (Show explicitly how you obtain this, including how the sign of the determinant is obtained.)



$\det A = (-1)^m a_{11}a_{22}a_{33}a_{44}.$

$m=2.$

$\det A = (-1)^2(1)(-2)(-2)(0.5) = 2$

2. (a) **(2 points)** Let A be an $n \times n$ nonsingular lower triangular matrix with all of its nonzero entries on three diagonals of the matrix: the main diagonal, the first sub-diagonal and the second sub-diagonal; that is,

$$\begin{bmatrix} a_{1,1} & & & & & & \\ a_{2,1} & a_{2,2} & & & & & \\ a_{3,1} & a_{3,2} & a_{3,3} & & & & \\ & a_{4,2} & a_{4,3} & a_{4,4} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & a_{n-1,n-3} & a_{n-1,n-2} & a_{n-1,n-1} & \\ & & & & a_{n,n-2} & a_{n,n-1} & a_{n,n} \end{bmatrix}$$

Note that A is nonsingular if and only if all of the entries on its main diagonal are nonzero, that each row of A contains at most 3 nonzero entries, and that entries on the first sub-diagonal and the second sub-diagonal may be nonzero or zero. For example, if $n = 6$, then

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1.1 & & 0 & 0 & 0 \\ 2.3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2.2 & 3 & -2.11 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 2.4 & 1.1 & -2.5 \end{bmatrix}$$

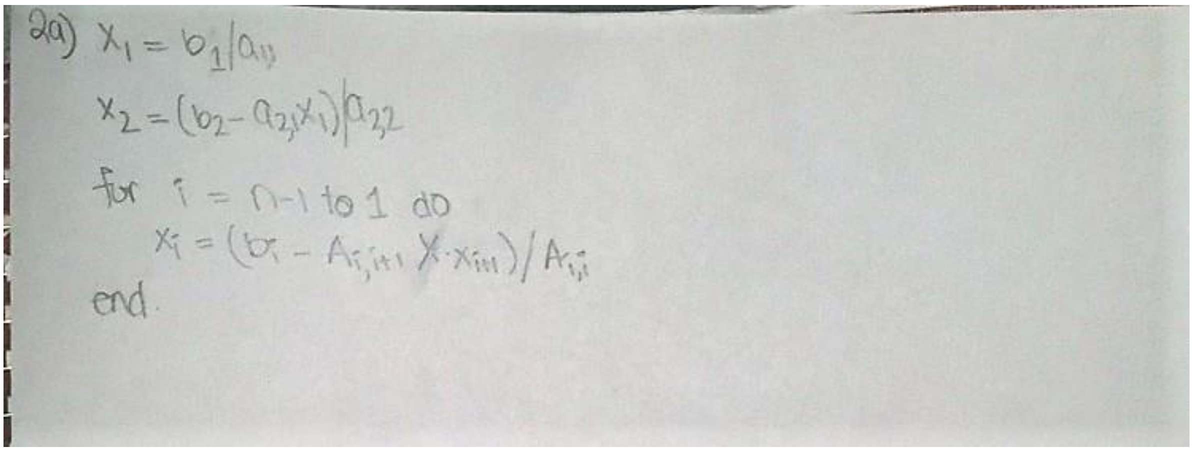
is such a lower triangular matrix.

Let $b = [b_1, b_2, \ldots, b_n]^T$ denote a (column) vector with n entries. An $n \times n$ system of linear equations $Ax = b$, where A is as described above, can be efficiently solved by forward substitution, which is similar to back-substitution but starts with the first equation. That is, the first equation can be used to solve for x_1, the second equation can be used to solve for x_2, the third equation for x_3, and so on.

Fill in the blanks in the following algorithm (use pseudocode, not MATLAB) so that it will solve such a system of linear equations.

```
$x_1 \leftarrow$ ______________________

$x_2 \leftarrow$ ______________________

for $i$ = ______________________

    ___________$\leftarrow$ ______________________

end
```

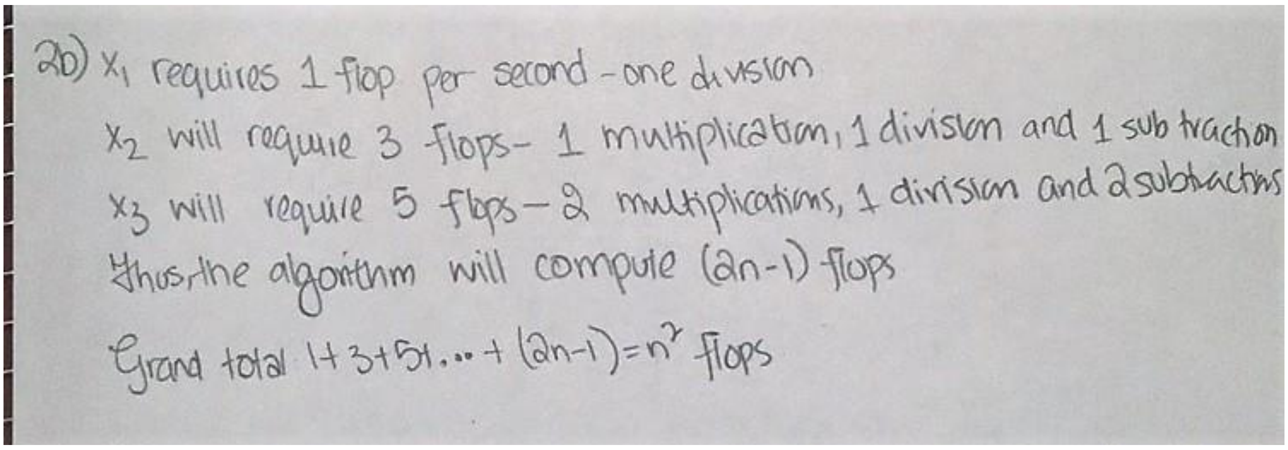
DELIVERABLES: Your complete pseudocode, it can be hand written or typed.



```
2a) $x_1 = b_1/a_{1,1}$
    $x_2 = (b_2 - a_{2,1}x_1)/a_{2,2}$
    for $i$ = $n-1$ to 1 do
        $x_i = (b_i - A_{i,i+1} \cancel{X} \cdot x_{i+1})/A_{i,i}$
    end.
```

(b) **(2 points)** Give a floating-point operation (flop) count for this forward substitution algorithm. That is, determine the total number of floating-point additions, subtractions, multiplications and divisions (as a function of n) that this algorithm will execute.

DELIVERABLES: All the steps in your count, including your reasoning.



2b) x_1 requires 1 flop per second - one division
x_2 will require 3 flops - 1 multiplication, 1 division and 1 subtraction
x_3 will require 5 flops - 2 multiplications, 1 division and 2 subtractions
Thus, the algorithm will compute $(2n-1)$ flops
Grand total $1+3+5+\dots+(2n-1)=n^2$ flops

(c) **(4 points)** Convert your pseudocode to MATLAB code and write a function that takes as input the nonsingular lower triangular matrix *A*, and column vector *b* and returns the results of "forward" substitution. The input matrix should be the full matrix including the zeros. Show how your function can be called and what the result is for the following matrices *A* and *b*:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 0 & 7 & 8 & 9 \\ 2 & 3 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 15 \\ 24 \\ 5 \end{bmatrix}$$

DELIVERABLES: A copy of the m-file function, a diary of the steps taken to solve for the given *A* and *b* and the solution *x*.

```
function x = BackwardSub_TriangularMatrix(a,b)
n=length(b);
x(n,1)  = b(1)/a(1,1);
x(n,2)=(b(2)-(a(2,1)*x(n,1)))/a(2,2);
for i=n-1:-1:1
    x(i)=(b(i)-a(i,i+1:n)*x(i+1:n,1))/a(i,i);
end
```

```
>> x=BackwardSub_TriangularMatrix(A,b)

x =

   1.0000        0
   1.6667        0
   2.5000        0
   1.0000   1.0000
```

3. The following system of linear equations $Ax = b$

$$\begin{bmatrix} -0.2345 & 2.107 \\ 0.1234 & -1.115 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2.345 \\ 1.001 \end{bmatrix}$$

has exact solution $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 345.404\ldots \\ 37.329\ldots \end{bmatrix}$. This problem is **ill-conditioned**; for example, if the (2,2)-entry of A is perturbed to be -1.111, then the exact solution of this perturbed linear system is

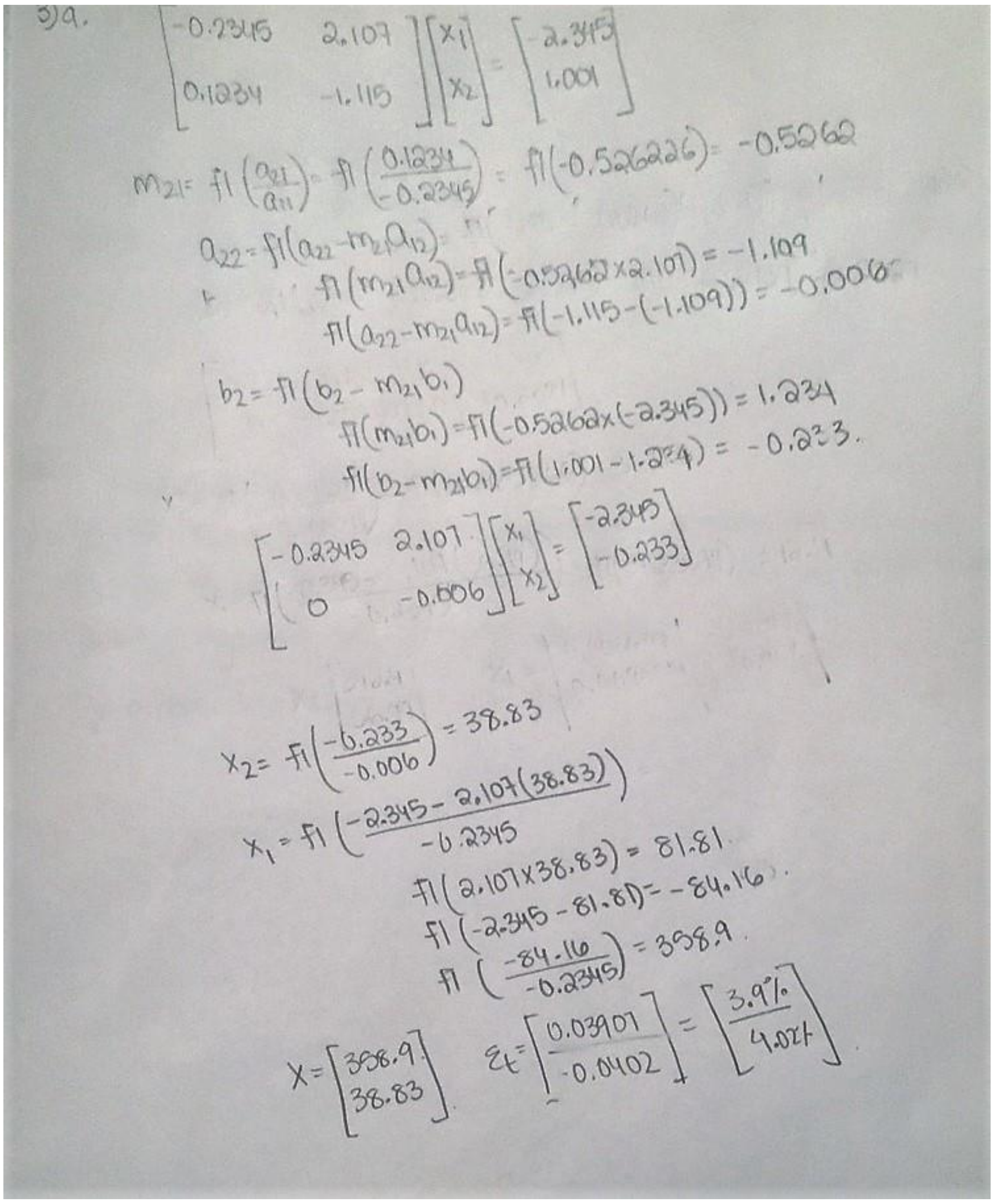
$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 943.861... \\ 103.934... \end{bmatrix}$$

In general, it is very difficult to determine an accurate computed solution to an ill-conditioned linear system using floating-point arithmetic (even if a good algorithm such as Gaussian elimination with partial pivoting is used). This is illustrated by the following computation.

(a) (4 points)

Use base 10, precision $k = 4$, idealized rounding floating-point arithmetic and Gaussian elimination with partial pivoting to solve the above linear system. Specifically, using floating-point arithmetic, show the computed approximations for the multiplier, and the entries a_{22} and b_2 of the first derived system (that is, when A has been reduced to upper triangular form). (There is no need to compute a_{21} as it will become 0.) Then, using floating-point arithmetic and back-substitution to compute x_2 and x_1. Explain the results in context of the above.

DELIVERABLES: Your floating-point calculations and results plus the explanation.


3)a.

$$\begin{bmatrix}-0.2345 & 2.107\\ 0.1234 & -1.115\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}-2.345\\ 1.001\end{bmatrix}$$

$m_{21}=fl\left(\frac{a_{21}}{a_{11}}\right)=fl\left(\frac{0.1234}{-0.2345}\right)=fl(-0.526226)=-0.5262$

$a_{22}=fl(a_{22}-m_{21}a_{12})$

$fl(m_{21}a_{12})=fl(-0.5262\times 2.107)=-1.109$

$fl(a_{22}-m_{21}a_{12})=fl(-1.115-(-1.109))=-0.006$

$b_2=fl(b_2-m_{21}b_1)$

$fl(m_{21}b_1)=fl(-0.5262\times(-2.345))=1.234$

$fl(b_2-m_{21}b_1)=fl(1.001-1.234)=-0.233$.

$$\begin{bmatrix}-0.2345 & 2.107\\ 0 & -0.006\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}-2.345\\ -0.233\end{bmatrix}$$

$x_2=fl\left(\frac{-0.233}{-0.006}\right)=38.83$

$x_1=fl\left(\frac{-2.345-2.107(38.83)}{-0.2345}\right)$

$fl(2.107\times 38.83)=81.81$

$fl(-2.345-81.81)=-84.16$

$fl\left(\frac{-84.16}{-0.2345}\right)=358.9$

$X=\begin{bmatrix}358.9\\ 38.83\end{bmatrix}$ $\varepsilon_t=\begin{bmatrix}0.03901\\ -0.0402\end{bmatrix}=\begin{bmatrix}3.9\%\\ 4.02\%\end{bmatrix}$

Since the problem is ill conditioned, small changes in the data will cause the exact answer to be greatly affected, for instance the round off associated with floating point values.

(b) **(2 points)** Use the MATLAB built-in function *cond* to compute a condition number for the above coefficient matrix *A*. We wont be discussing the details of this condition number in this course, but there is some discussion of this on page 294 of the 7th edition of the textbook (page 290 of the 6th edition). Roughly speaking, if this condition number is between 1 and 10, then *A* is well conditioned, and if this condition number is greater than 1000, then *A* is ill-conditioned.

DELIVERABLES: Show your MATLAB input and output.

```
>> A=[-0.2345 2.107; 0.1234 -1.115]

A =

  -0.2345    2.1070
   0.1234   -1.1150

>> cond(A)

ans =

  3.9304e+03
```