Leoza Kabir
V00840048
Instructor: Rich Little

CSC 349A: Numerical Analysis
Assignment 3

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Question #1 - 10 marks.
Consider the function

$$f(x) = \frac{1 + \cos x}{(x - \pi)^2}$$

where x = π is in radians.

(a) Using b = 10, k = 4, idealized, floating-point arithmetic with rounding, compute fl(f(x)) at x = 3.154. Note that fl(π) = 3.142.
Furthermore, The correct value of f(3.154) to 8 correct significant digits is 0.49999359. Your floating-point approximation should have a relative error greater than 30%.

CSC 349A

1) a) $fl(\cos x) = fl(\cos 3.154) = fl(-0.999923029) = -0.9999$

$fl(1+\cos x) = fl(1+(-0.9999)) = fl(0.0001) = 0.0001$

$fl(x-\pi) = fl(3.154 - 3.142) = 0.012$

$fl((x-\pi)^2) = fl(0.012^2) = 0.000144$

$fl\left(\frac{1+\cos x}{(x-\pi)^2}\right) = fl\left(\frac{0.0001}{0.000144}\right) = 0.6944$

$|\varepsilon_t| = \left|\frac{0.49999359 - 0.6944}{0.49999359}\right| = 0.3888 \approx 39\%$

(b) Determine the fourth order Taylor polynomial approximation for cos (x) expanded about x = π (Do not include the remainder term.) Leave this polynomial in terms of expressions involving $(x - \pi)^k$, for integer values of k.

b) $f(x) = \cos x$ — $f(\pi) = \cos \pi = -1$

$f'(x) = -\sin x$ — $f'(\pi) = -\sin \pi = 0$

$f''(x) = -\cos x$ — $f''(\pi) = -\cos \pi = 1$

$f'''(x) = \sin x$ — $f'''(\pi) = \sin \pi = 0$

$f^4(x) = \cos x$ — $f^4(\pi) = \cos \pi = -1$

$$f(x) = f(\pi) + f'(\pi)(x-\pi) + \frac{f''(\pi)}{2!}(x-\pi)^2 + \frac{f'''(\pi)}{3!}(x-\pi)^3 + \frac{f^4(\pi)}{4!}(x-\pi)^4$$

$$\approx -1 + 0 + \frac{1}{2!}(x-\pi)^2 + 0 - \frac{1}{4!}(x-\pi)^4$$

$$= -1 + \frac{(x-\pi)^2}{2} - \frac{(x-\pi)^4}{24}$$

(c) Substitute the polynomial approximation from (b) into the formula for f(x), and sim- plify in order to obtain a polynomial approximation for f(x). Note: do not multiply out the remaining factor $(x-\pi)^2$; leave it in this form.

c) $f(x) = \dfrac{1 + \left(-1 + \frac{(x-\pi)^2}{2} - \frac{(x-\pi)^4}{24}\right)}{(x-\pi)^2} = \dfrac{\cancel{1} \cancel{-1} + \frac{(x-\pi)^2}{2} - \frac{(x-\pi)^4}{24}}{(x-\pi)^2}$

$= \left[\dfrac{(x-\pi)^2}{2} - \dfrac{(x-\pi)^4}{24}\right] \times \dfrac{1}{(x-\pi)^2}$

$= \dfrac{\cancel{(x-\pi)^2}}{2\cancel{(x-\pi)^2}} - \dfrac{(x-\pi)^{\cancel{4}\,2}}{24\cancel{(x-\pi)^2}}$

$\dfrac{1}{2} - \dfrac{(x-\pi)^2}{24}$

(d) Show that the problem of computing f(3.145) is well-conditioned. Use the definition of condition and the notation in Handout 6 to show this.

Hint: Use the polynomial approximation for f(x) from (c) to show that the exact value of f(3.154 + ε) is approximately equal to 0.49999 (that is, approximately equal to the

3.154

exact value of f(3.154)) whenever ε —— is small.

d) $x = 3.154$

perturb

$x + \varepsilon = 3.154 + \varepsilon \xrightarrow{\text{exact}} \dfrac{1}{2} - \dfrac{1}{24}(3.154 + \varepsilon - \pi)^2$

$= \dfrac{1}{2} - \dfrac{1}{24}(\varepsilon + 0.012407346)(\varepsilon + 0.012407346)$

$= \dfrac{1}{2} - \dfrac{1}{24}(\varepsilon^2 + 0.024814692\varepsilon + 0.000153942)$

$= \dfrac{1}{2} - \dfrac{\varepsilon^2}{24} - \dfrac{0.024814692\varepsilon}{24} - \dfrac{0.000153942}{24}$

$= \underbrace{-\dfrac{\varepsilon^2}{24} - \dfrac{0.024814692\varepsilon}{24}}_{\text{become very small, when } \varepsilon \text{ small}} + 0.499993585$

$\approx 0.4999 \rightarrow$ well conditioned

(e) Show that the computation of fl(f(3.154)) in (a) is unstable. Use the definition of stability and the notation in Handout 7 to show this.

Hint: Use the result in (c) to show that when $\frac{\varepsilon}{\ }$ is small, the exact value of f(3.154 + ε), done in (d), cannot be close to the floating-point calculation fl(f(3.154)) in (a), for any ε.

e) Stability Analysis.

given problem data $x = 3.154$ $\xrightarrow[\text{computation}]{\text{fl. pt}}$ 0.6944

perturbed problem $\hat{x} = 3.154 + \varepsilon$ with $\left|\frac{\varepsilon}{3.154}\right|$ small $\xrightarrow[\text{computation}]{\text{exact}}$ 0.4999.

$3.154 + \varepsilon \approx 0.4999$ is very close to 0.4999935 for all small values of ε.

Hence, there are no small values of ε that is close to 0.6944, thus the computation is unstable.

Question #2 - 10 Marks

(a) Write a MATLAB function M-file with header

function root = Bisect (xl , xu , eps , imax, f, enablePlot)

corresponding to the pseudocode given in Handout #8 for the Bisection method (in xl it is an "ell" not a "one"). Note, in the pseudocode "exit" is used to break out of the function. In Matlab, this will cause the program to close. You want to use "return" in implementation.

The only differences from that given algorithm are the following:

- **print a caption for your computed approximations by inserting the following statement just before the <u>while</u> statement:**

fprintf (' iteration approximation \n')

- **print each successive computed approximation by inserting the following statement after the computation of x_r at the beginning of the while loop:**

  fprintf (' %6.0f %18.8f \n', i, xr)

- **print a message to indicate that the algorithm has failed to converge in imax steps by replacing the last statement in the pseudocode by the following:**

  fprintf (' failed to converge in %g iterations\n', imax)

- **The extra argument enablePlot is used to select an optional plot of showing each iteration of the bisection method when enablePlot is set to 1. When enablePlot is set to 0 no figure will be generated. The command hold on can be used to in the same figure using the plot command. For example try the following to understand how this works:**

  hold on;
  x = [0:0.1:1]; plot(x,
  exp(x)); plot(x, log(x));

  For each iteration you should plot the function between xl and xu as well as stars indicating on the graph the values of f(xl), f(xu) and f(xr). The following example MATLAB code (continuation of the previous example) can help you understand the syntax to accomplish this.

  z = [0.2, 0.4, 0.8];
  fz = exp(z);
  plot(x, exp(x), z, fz, '*g');
  hold off;

  DO NOT INCLUDE ALL THE ENDPOINTS IN YOUR ANSWER. DO INCLUDE THE ENDPOINTS FOR ITERATIONS 1, 2, 4, 6.

DELIVARABLES: A copy of your MATLAB M-file.

```
function root = Bisect(xl,xu,eps, imax, f, enablePlot)
i = 1;
fl = f(xl);
fprintf('iteration approximation \n')
while (i<=imax)
    xr = (xl + xu)/2
    fprintf ('%6.0f %18.8f \n', i, xr )
    fr = f(xr)
    if fr == 0 || ((xu-xl)/abs(xu+xl)) < eps
        root = xr
        return
    end
    %graph
    if enablePlot == 1 && (i==1 || i == 2 || i == 4 || i==6)
        hold on;
        x = [xl:0.1:xu]
        z = [xl, xu, xr]
        fu = f(xu)
        fz = [fl,fu,fr]
        plot(x,z,fu,fz,'*g');
        hold off;
    end
    i = i + 1
    if fl*fr < 0
```

```
        xu = xr
    else
        xl = xr
        fl = fr
    end
end
fprintf ('failed to converge in %g iterations\n', imax )
root = 'failed to converge'
end
```

(b) Water is flowing in a trapezoidal channel at a rate of $Q = 20m^3/s$. The critical depth y for such a channel must satisfy the equation:

$$0 = 1 - \frac{Q^2}{gA_c^3} B$$

where $g = 9.81m/s^2$, A_c = the cross-sectional area (m^2), and B = the width of the channel at the surface (m). For this case, the width and the cross-sectional area can be related to depth y by

$$B = 3 + y \quad \text{and} \quad A_c = 3y + \frac{y^2}{2}$$

Express this problem as a root finding problem for an appropriately defined function of the critical depth y. DELIVARABLES: Show all your work in deriving your formula.

$$0 = 1 - \frac{Q^2 B}{gA_c^3}$$

$$0 = 1 - \frac{(20)^2 (3+y)}{9.81\left(3y+\frac{y^2}{2}\right)^3}$$

$$0 = 1 - \frac{400(3+y)}{9.81(3y+y^2/2)^3}$$

$$= 1 - \frac{1200+400y}{9.81(3y+y^2/2)^3}$$

$$= 9.81(3y+y^2/2)^3 - 1200 + 400y$$

$$= 9.81\left(27y^3 + 9y^4 + 9y^4 + 3y^5 + \frac{3y^5}{4} + \frac{y^6}{4}\right) - 1200 + 400y$$

$$= 9.81\left(27y^3 + 18y^4 + \frac{15y^5}{4} + \frac{y^6}{4}\right) - 1200 + 400y$$

$$= 264.87y^3 + 176.58y^4 + \frac{147.15y^5}{4} + \frac{9.81y^6}{4} - 1200 + 400y$$

$$= \frac{9.81y^6}{4} + \frac{147.15y^5}{4} + 176.58y^4 + 264.87y^3 + 400y - 1200$$

$$\left(3y+\frac{y^2}{2}\right)\left(3y+\frac{y^2}{2}\right)$$

$$9y^2 + \frac{3y^3}{2} + \frac{3y^3}{2} + \frac{y^4}{4}\left(3y+y^2\right)$$

$$\left(9y^2 + \frac{6y^3}{2} + \frac{y^4}{4}\right)\left(3y+y^2\right)$$

$$27y^3 + 9y^4 + \frac{18y^4}{2} + \frac{6y^5}{2} + \frac{3y^5}{4} + \ldots$$

(c) Use the function M-file Bisect to solve the above problem (b) with initial guesses of $x_l = 0.5$ and $x_u = 2.5$, and itereate until the approximate error falls below 1% or the number of iterations exceeds 10. You will need to write an additional MATLAB function M-file. If

function y = your_function(x)

corresponds to the function of which you are computing a zero call this with

Bisect (xl , xu , eps , imax, @your_function, enable_plot)

with the appropriate parameter values.
DELIVARABLES:

- **the additional function M-file.**

```
function [fh] = depth_channel(y)
    fh = 1 - (400/9.81*(3*y+(y.^2/2)).^3)*(3+y);
end
```

- **a copy of the MATLAB statement(s) you used to call Bisect.**

```
>> Bisect (0.5, 2.5, 10^(-2),10,@depth_channel,1)
```

- **your output from Bisect including the figure with the endpoints for iterations (1,2,4,6).**

```
iteration approximation

xr =

    1.5000

     1          1.50000000

fr =

  -3.2656e+04


x =

  Columns 1 through 12

    0.5000    0.6000    0.7000    0.8000    0.9000    1.0000    1.1000    1.2000
1.3000    1.4000    1.5000    1.6000

  Columns 13 through 21

    1.7000    1.8000    1.9000    2.0000    2.1000    2.2000    2.3000    2.4000
2.5000


z =

    0.5000    2.5000    1.5000


fu =

  -2.6899e+05


fz =

   1.0e+05 *

   -0.0061   -2.6899   -0.3266
```

```
Error using plot
Vectors must be the same length.

Error in Bisect (line 20)
        plot(x,z,fu,fz,'*g');
```