CS 171: Intro to ML and DM

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UC Riverside

Slide Set 4: Linear Regression, II

Slides from CS 171

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Non-linear Regression

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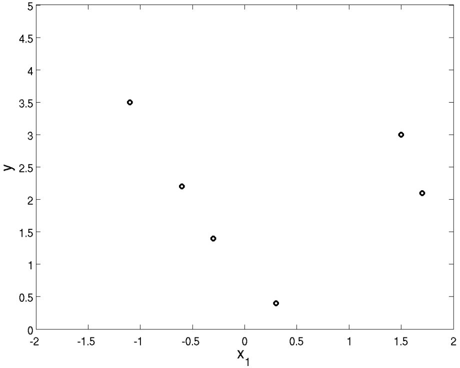
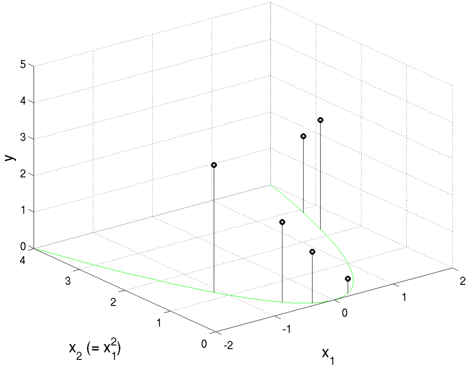
So instead of just adding the "0th" attribute (always 1)
also add other attributes that can be calculated from the given attributes.

Non-linear regression





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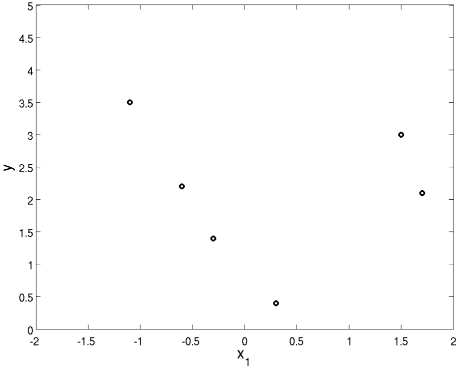

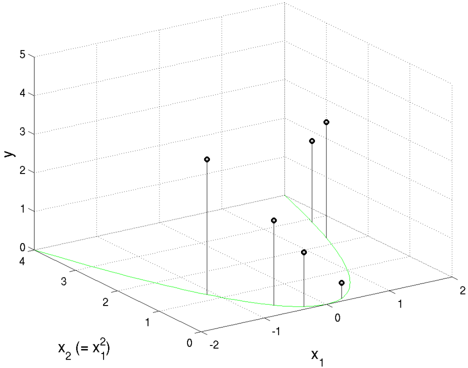

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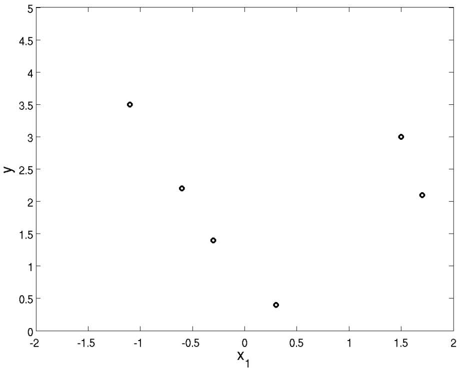

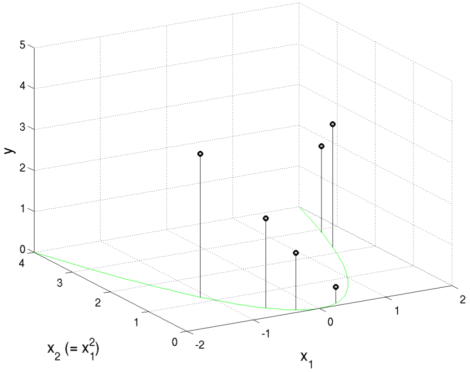

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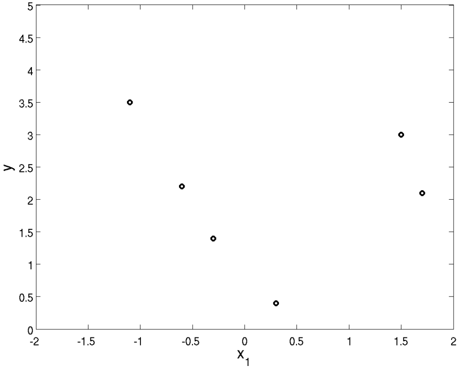

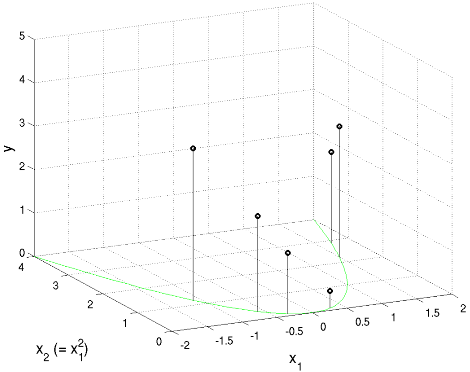

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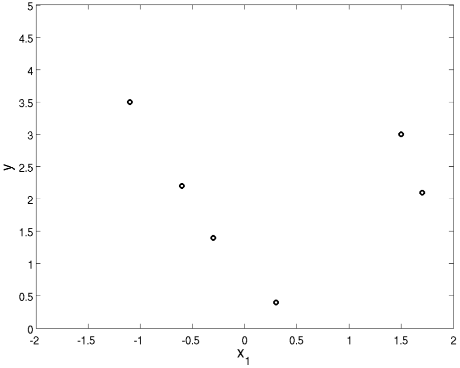

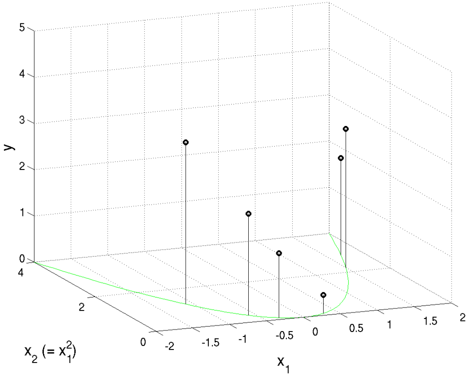

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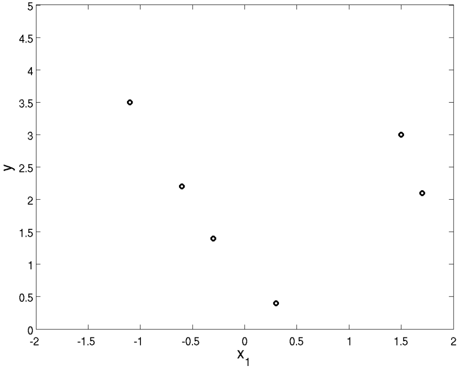

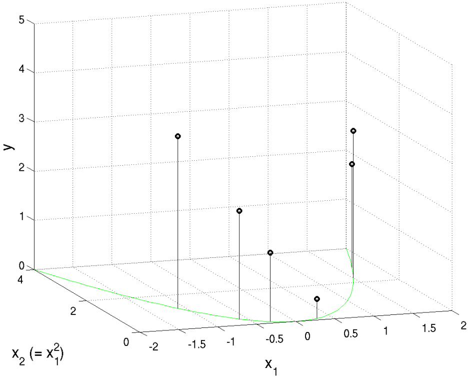

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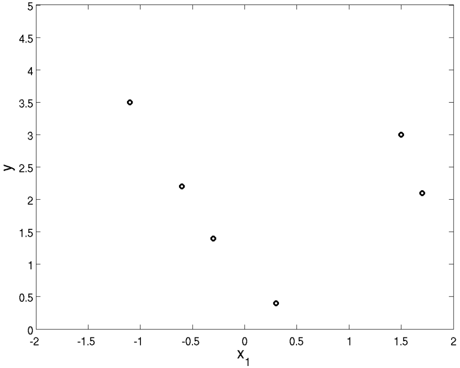

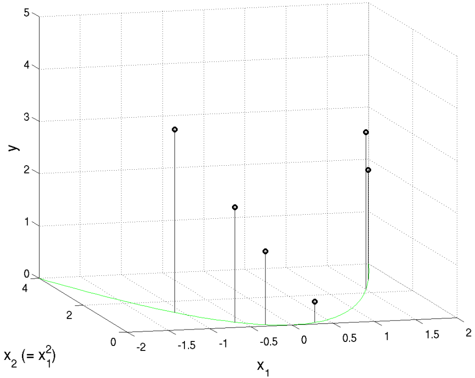

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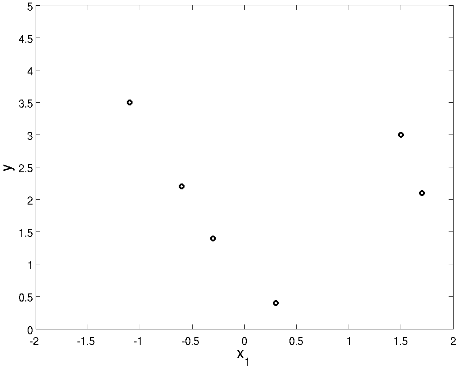

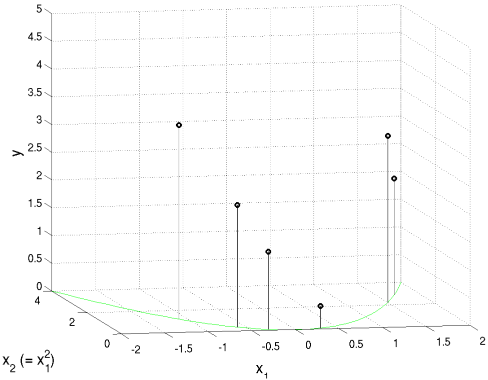

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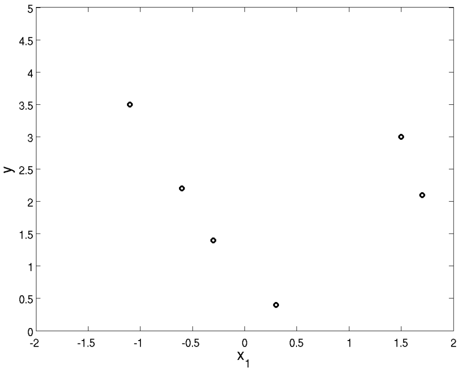

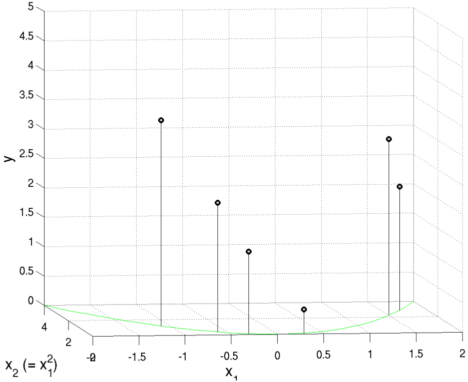

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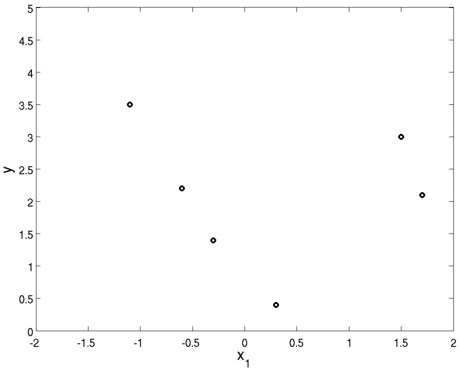

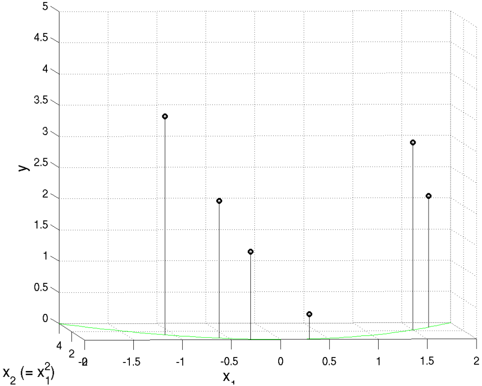

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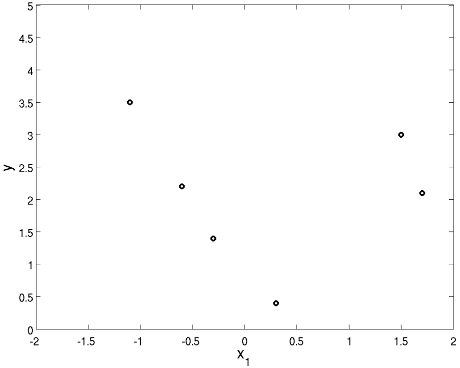

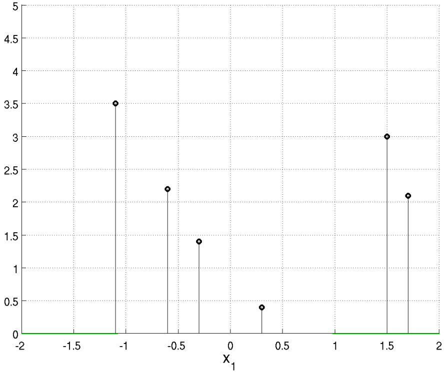

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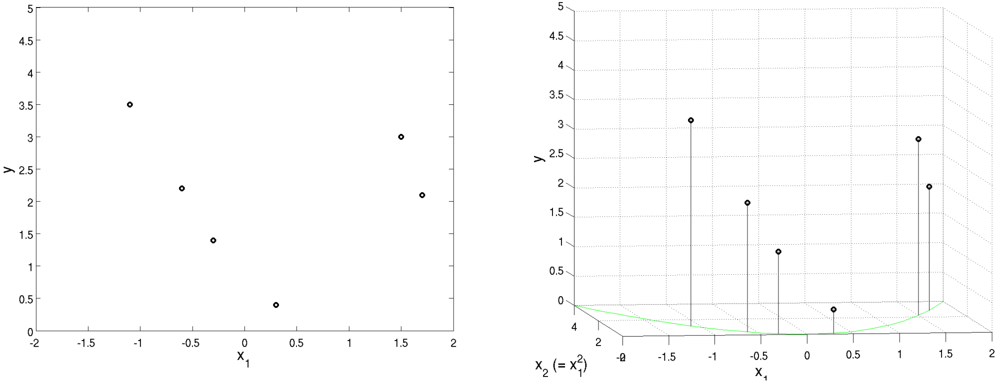

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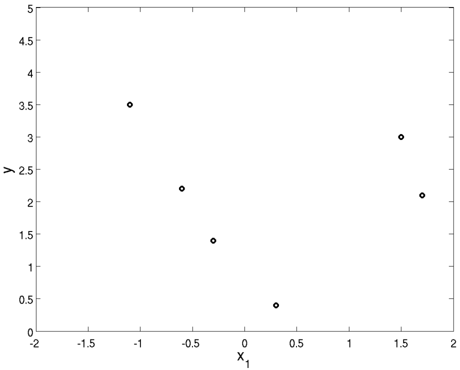

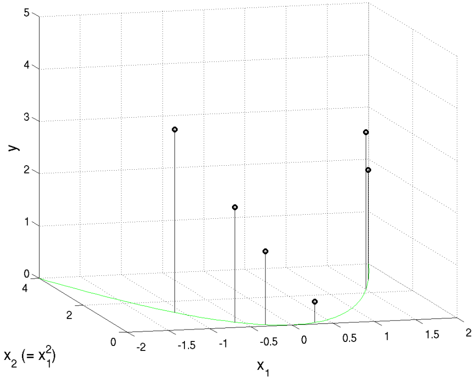

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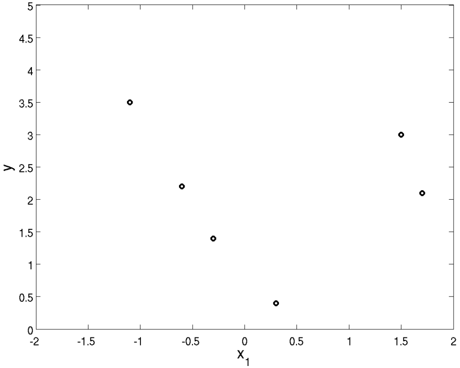

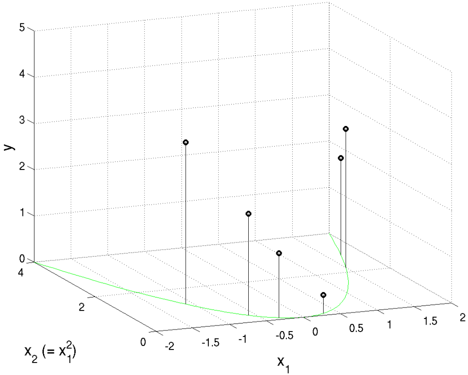

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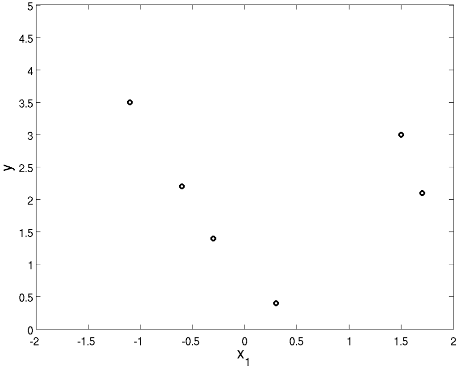

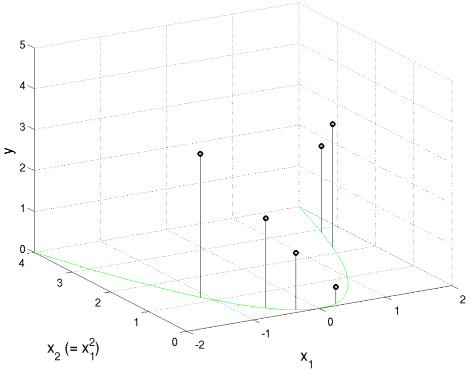

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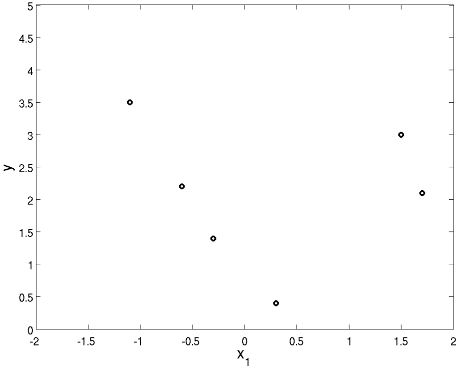

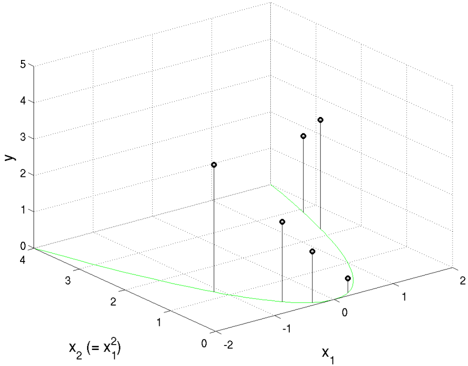

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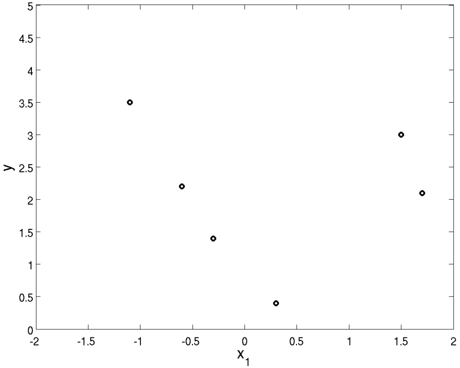

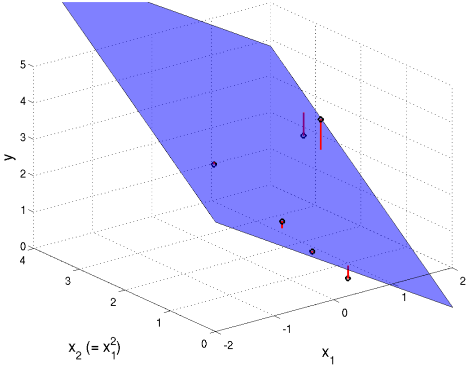

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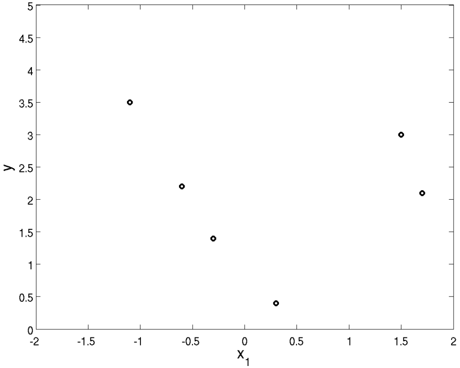

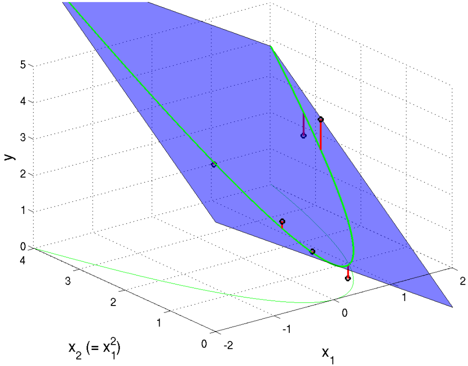

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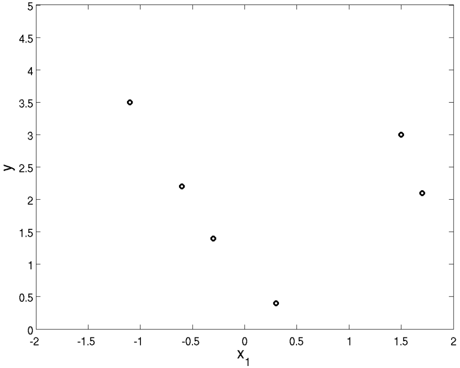

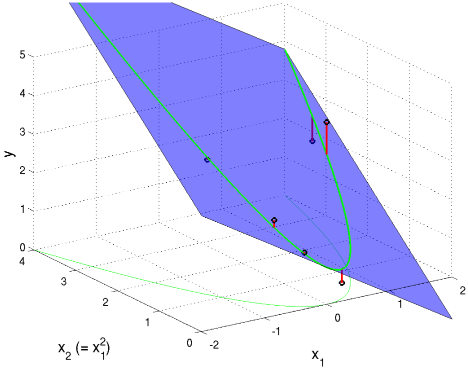

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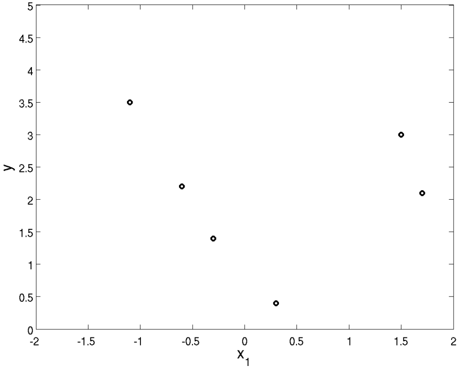

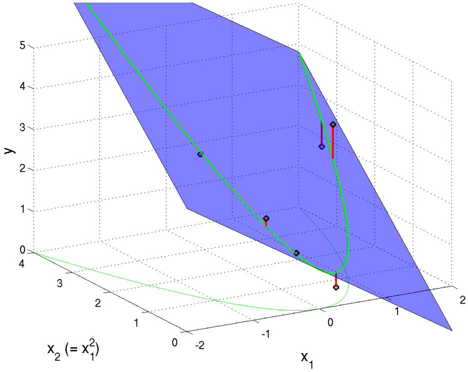

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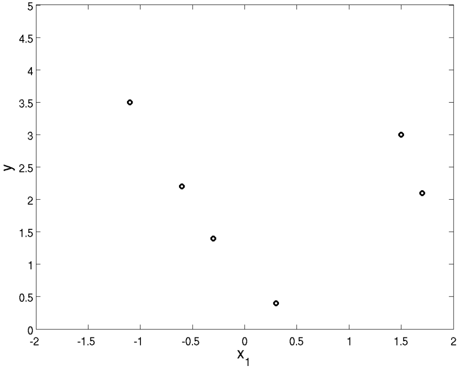

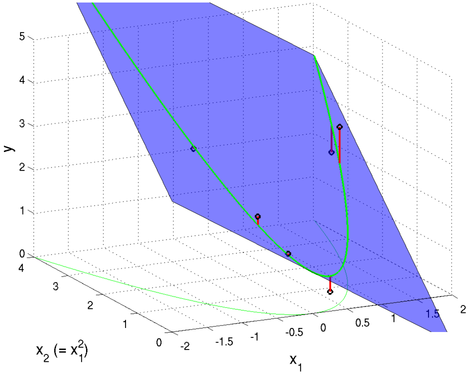

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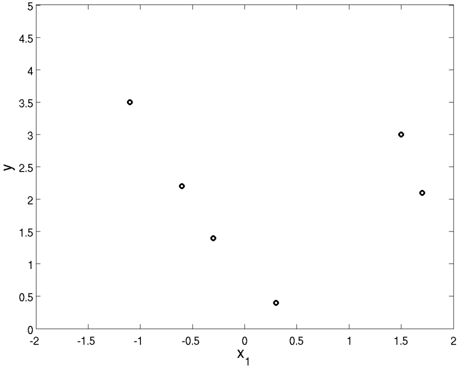

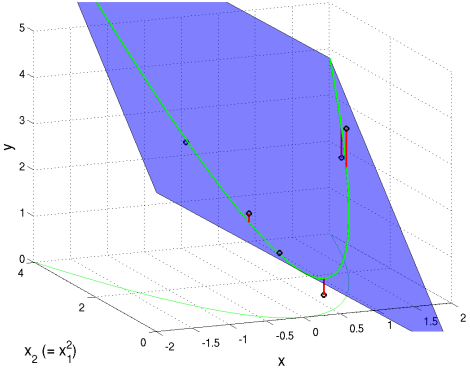

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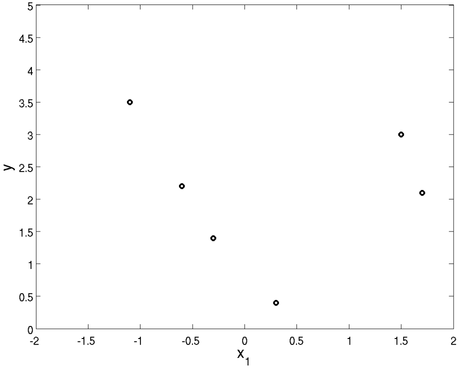

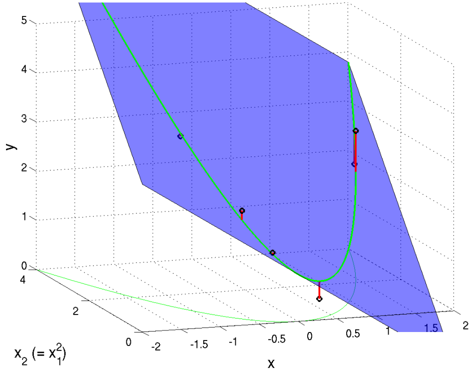

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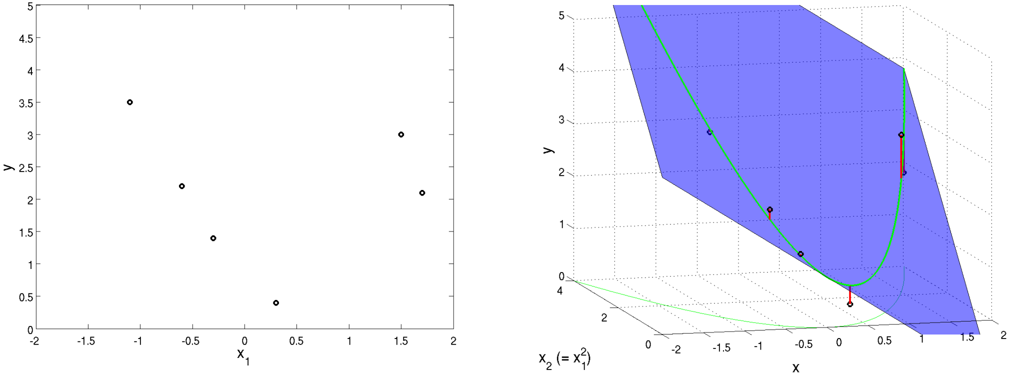

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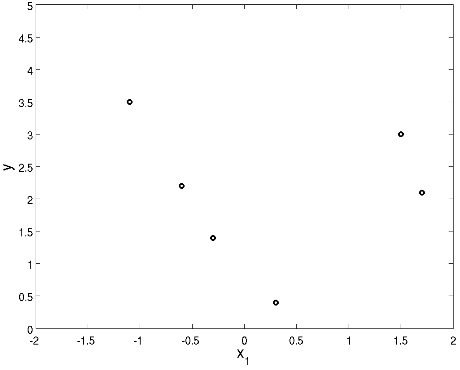

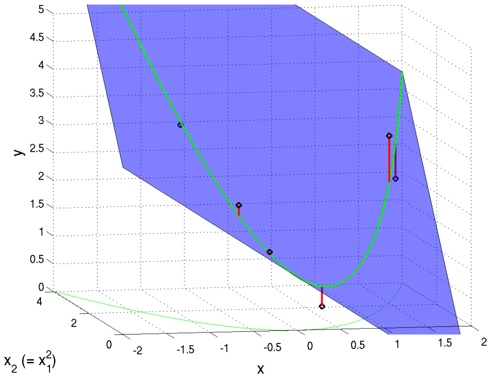

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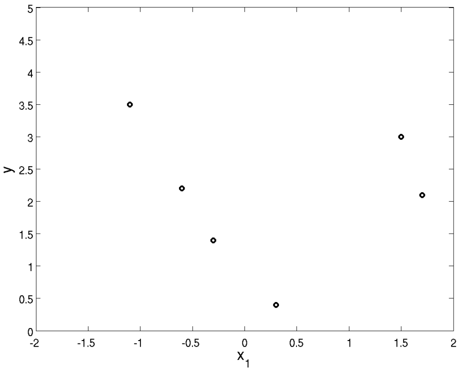

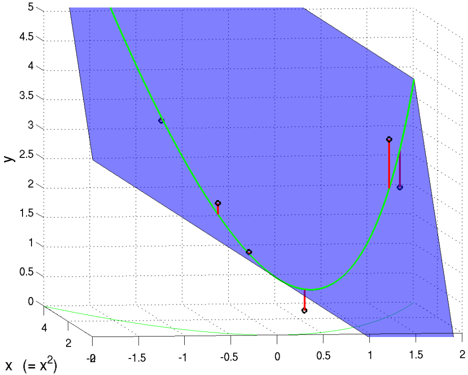

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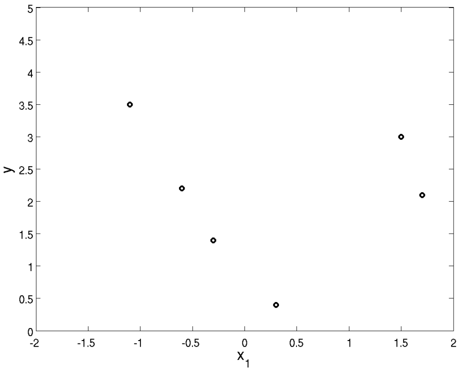

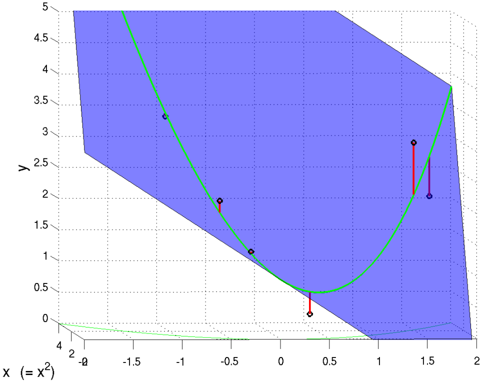

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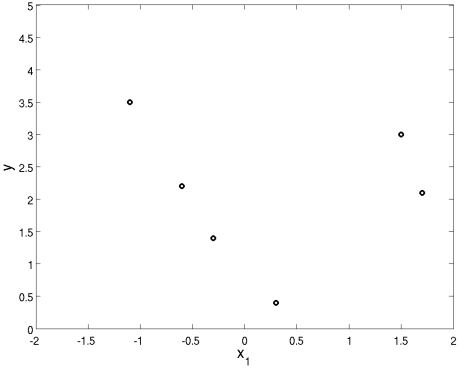

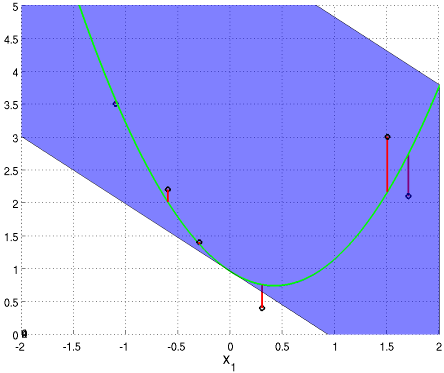

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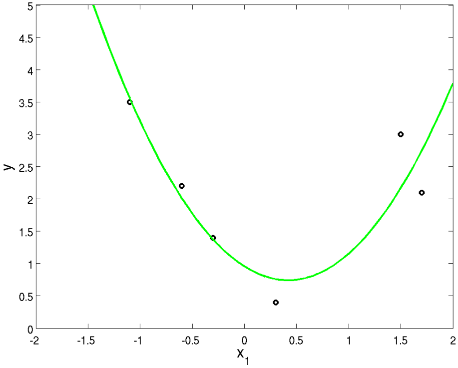

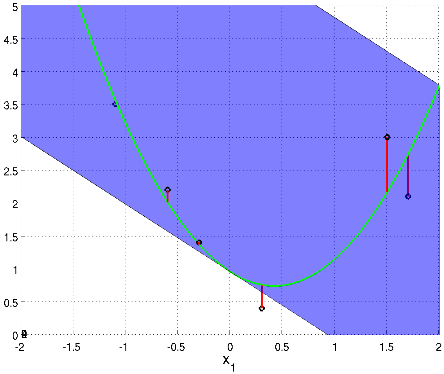

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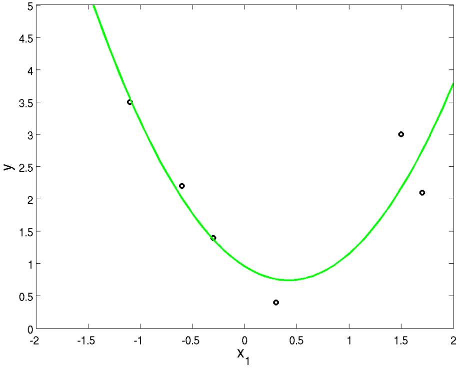

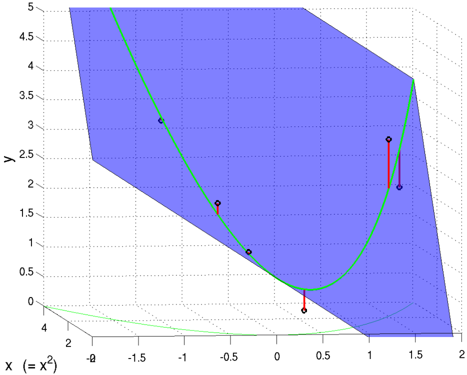

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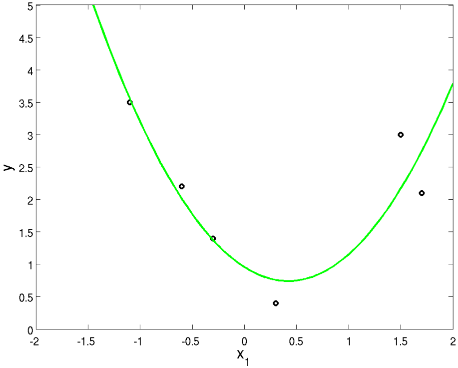

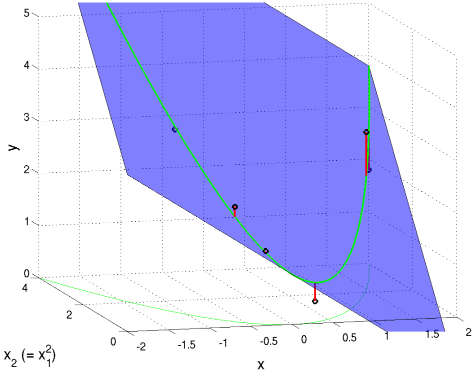

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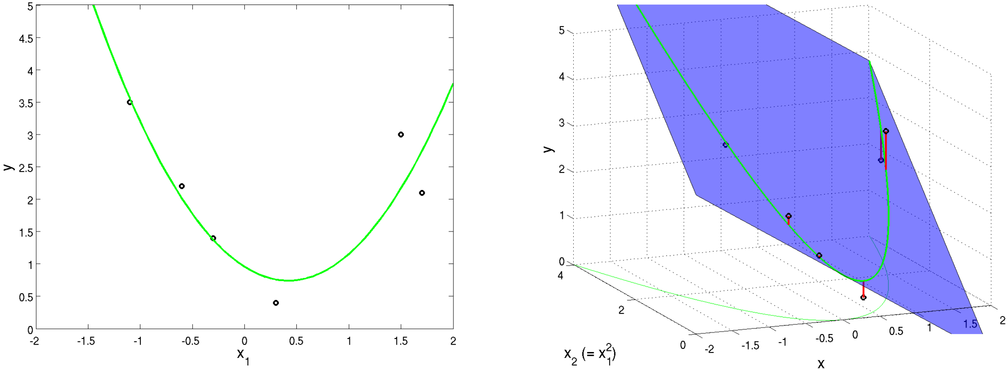

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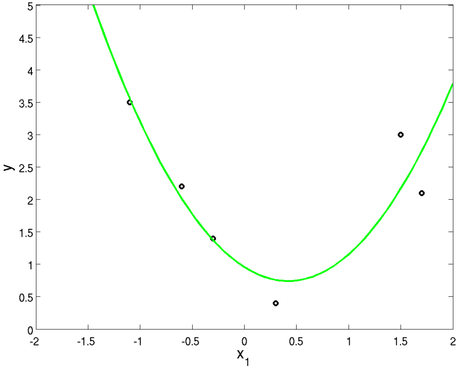

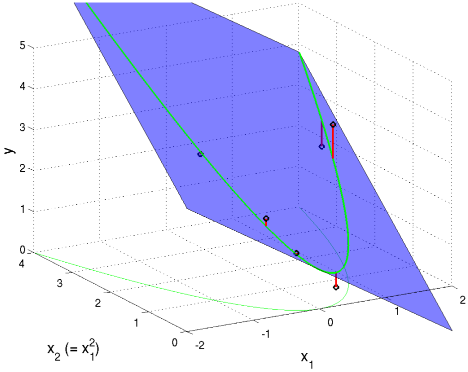

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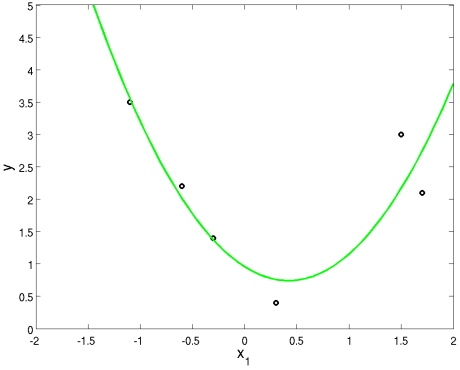

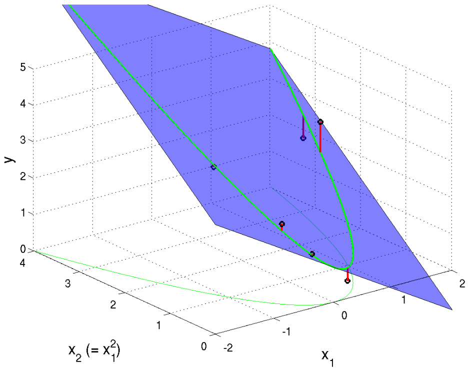

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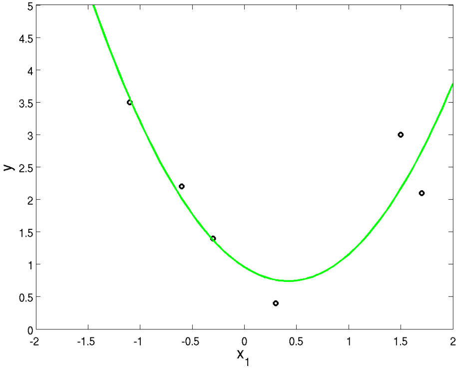

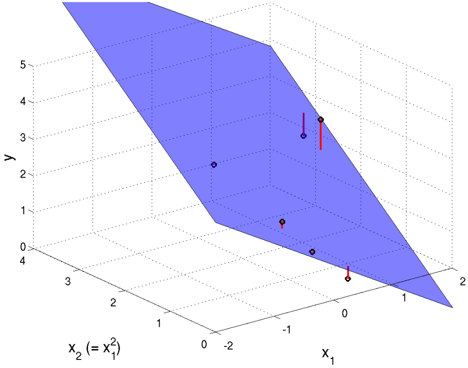

Feature Mapping

The function that takes the raw attributes and creates features from them is often written $\phi(x)$. For instance

$$\phi(x) = \begin{bmatrix}1 & x_1 & x_2\end{bmatrix} \qquad \phi(x) = \begin{bmatrix}1 & x_1 & x_2 & x_1^2 & x_2^2 & x_1x_2\end{bmatrix}$$

$$\phi(x) = \begin{bmatrix}1 & x_1 & x_1^2\end{bmatrix} \qquad \phi(x) = \begin{bmatrix}1 & x_1 & x_2 & \sin(x_1) & \sin(x_2)\end{bmatrix}$$

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Instead of X, we sometimes call the new matrix Φ:

$$X = \begin{bmatrix} \text{---}x_1\text{---} \\ \text{---}x_2\text{---} \\ \vdots \\ \text{---}x_m\text{---} \end{bmatrix} \qquad \Phi = \begin{bmatrix} \text{---}\phi(x_1)\text{---} \\ \text{---}\phi(x_2)\text{---} \\ \vdots \\ \text{---}\phi(x_m)\text{---} \end{bmatrix}$$

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The function that takes the raw attributes and creates features from them is often written $\phi(x)$. For instance

$$\phi(x) = \begin{bmatrix} 1 & x_1 & x_2 \end{bmatrix} \qquad \phi(x) = \begin{bmatrix} 1 & x_1 & x_2 & x_1^2 & x_2^2 & x_1 x_2 \end{bmatrix}$$

$$\phi(x) = \begin{bmatrix} 1 & x_1 & x_1^2 \end{bmatrix} \qquad \phi(x) = \begin{bmatrix} 1 & x_1 & x_2 & \sin(x_1) & \sin(x_2) \end{bmatrix}$$

Instead of X, we sometimes call the new matrix Φ:

$$X = \begin{bmatrix} \text{---} x_1 \text{---} \\ \text{---} x_2 \text{---} \\ \vdots \\ \text{---} x_m \text{---} \end{bmatrix} \qquad \Phi = \begin{bmatrix} \text{---} \phi(x_1) \text{---} \\ \text{---} \phi(x_2) \text{---} \\ \vdots \\ \text{---} \phi(x_m) \text{---} \end{bmatrix}$$

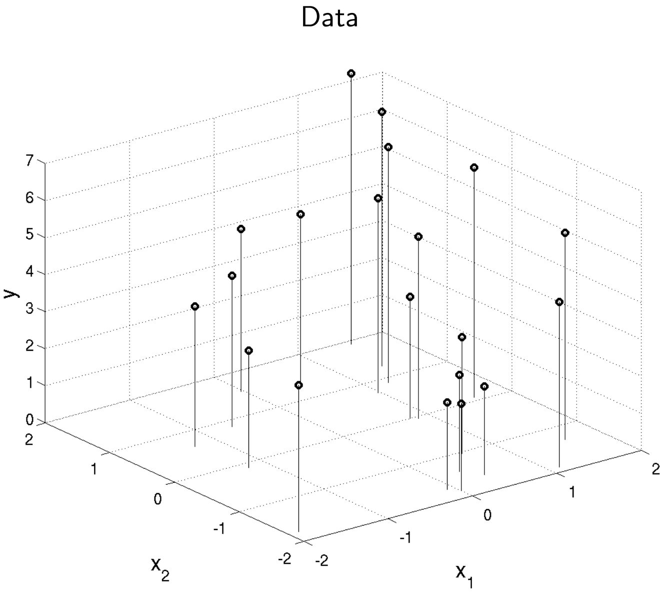
The learning equation correspondingly changes notation:

$$\hat{w} = (X^\top X)^{-1}(X^\top Y) \qquad \hat{w} = (\Phi^\top \Phi)^{-1}(\Phi^\top Y)$$

And of course the resulting function changes equation too:

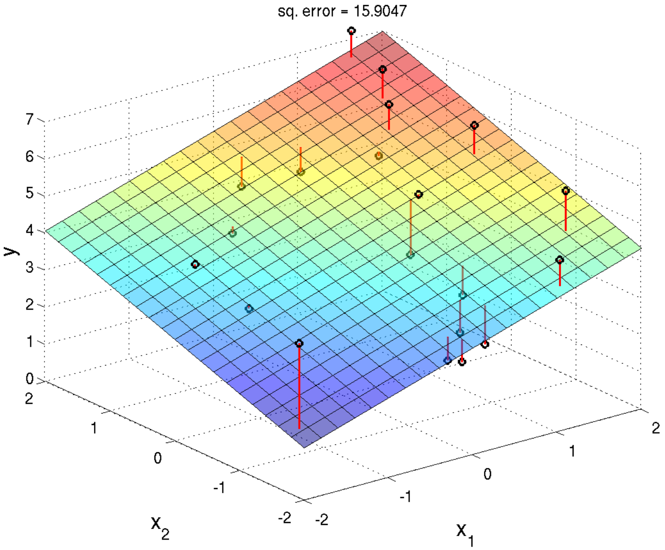
$$f(x) = x^\top \hat{w} \qquad f(x) = \phi(x)^\top \hat{w}$$

Overfitting



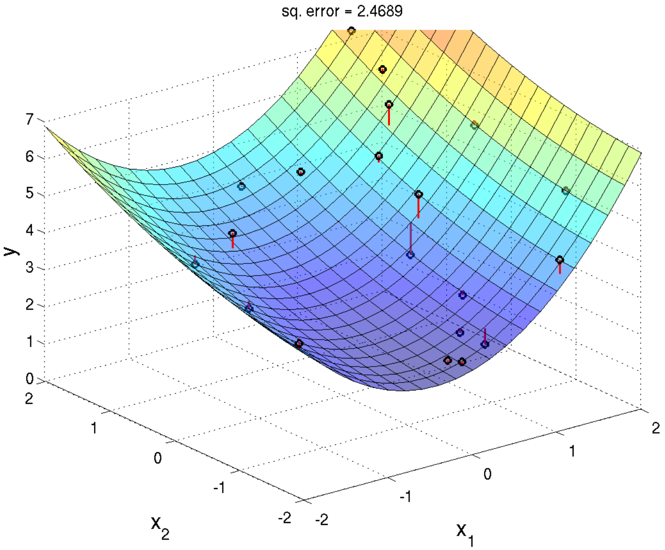
Overfitting

Linear: $\phi(x) = \begin{bmatrix} 1 & x_1 & x_2 \end{bmatrix}$



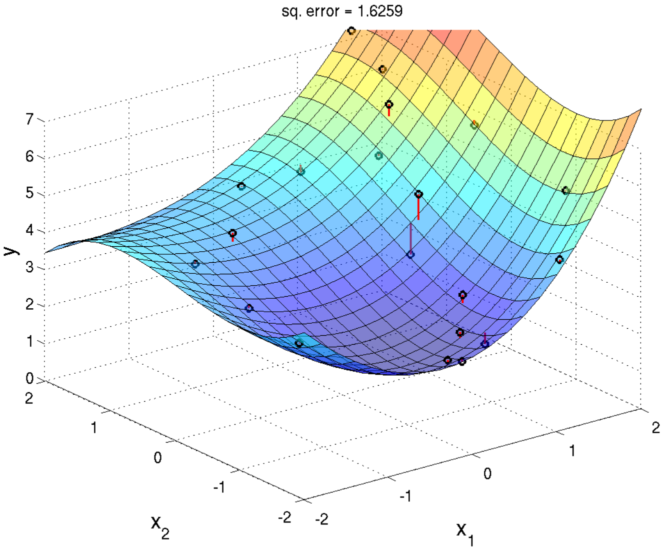
Overfitting

2nd order: $\phi(x) = \begin{bmatrix}1 & x_1 & x_2 & x_1^2 & x_1 x_2 & x_2^2\end{bmatrix}$



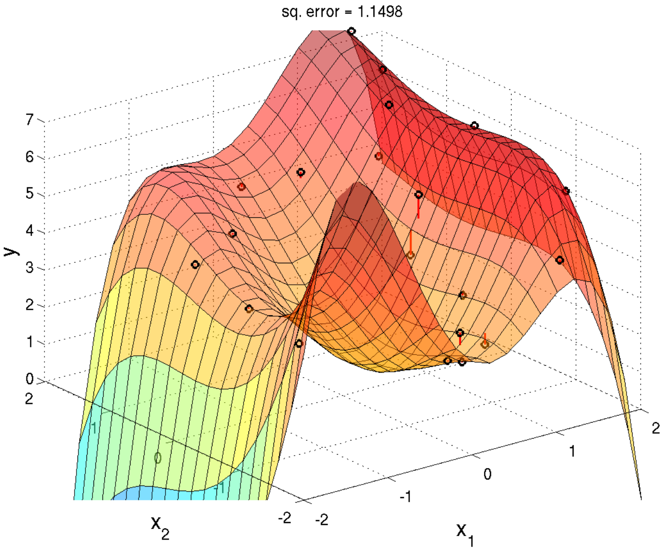
Overfitting

3rd order: $\phi(x) = [1 \quad x_1 \quad x_2 \quad x_1^2 \quad x_1x_2 \quad x_2^2 \quad x_1^3 \quad x_1^2x_2 \quad x_1x_2^2 \quad x_2^3]$



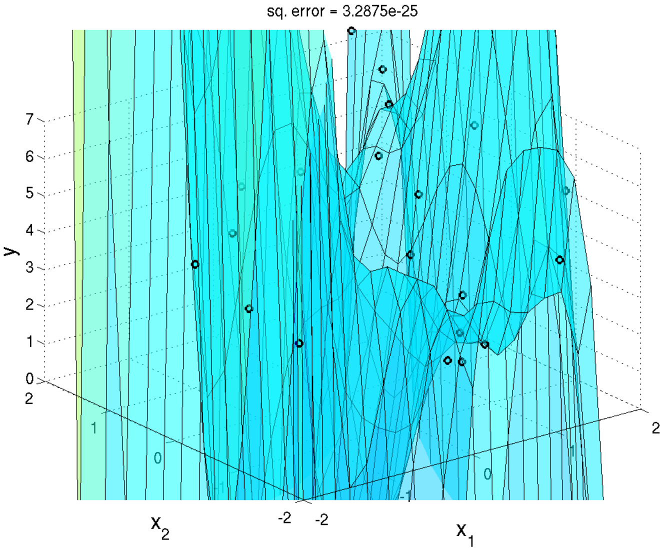
Overfitting

4th order: $\phi(x) = \begin{bmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 & x_2^4 \end{bmatrix}$



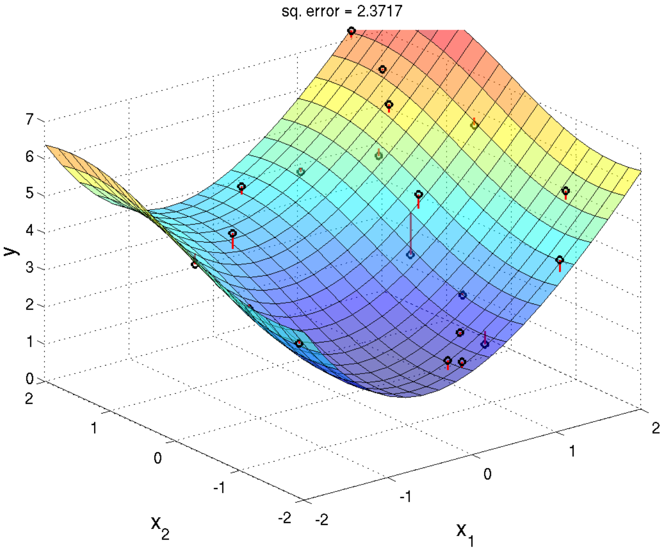
Overfitting

5th order: $\phi(x) = \begin{bmatrix} 1 & x_1 & x_2 & x_1^2 & x_1 x_2 & x_2^2 & x_1^3 & x_1^2 x_2 & \ldots & x_2^5 \end{bmatrix}$



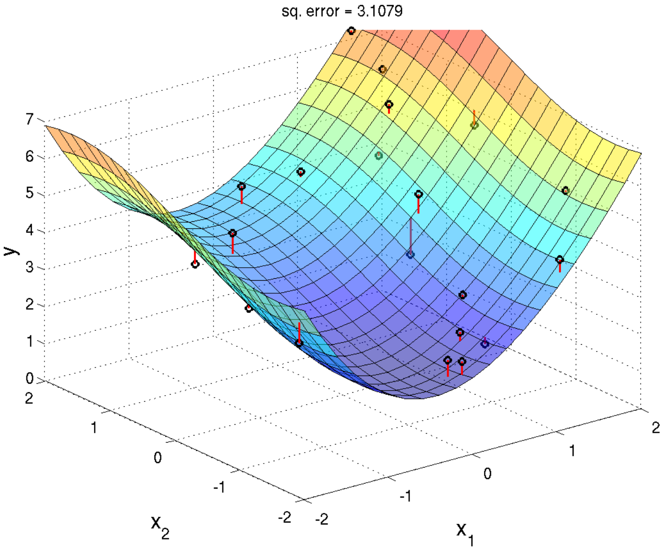
Overfitting

"correct": $\phi(x) = \begin{bmatrix} 1 & \sin(x_1) & \sin(x_2) & x_1^2 \end{bmatrix}$

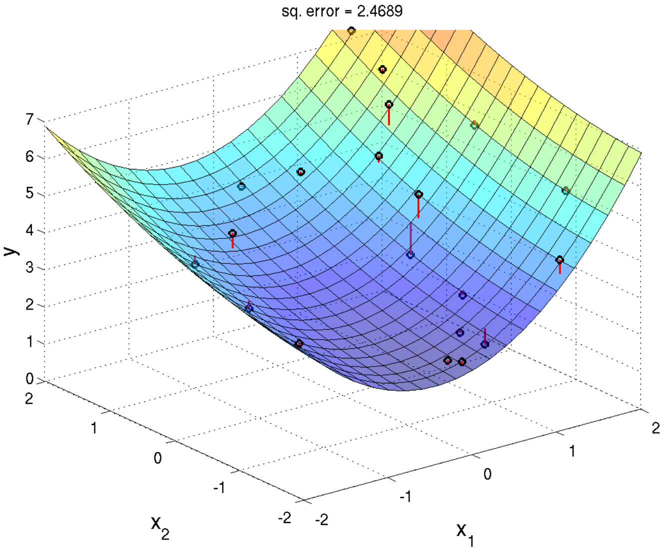


Overfitting

really "correct": $\phi(x) = \left[3 + \sin(x_1) + \sin(x_2) + x_1^2\right]$

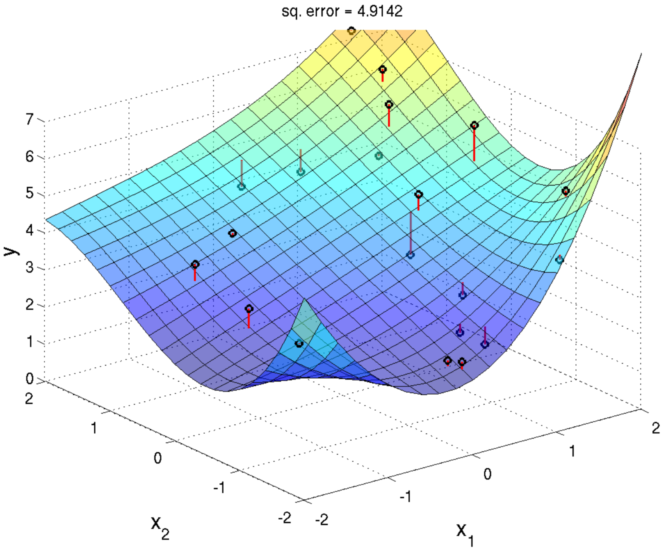


2nd order: $\phi(x) = \begin{bmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{bmatrix}$



Overfitting

some 5th order: $\phi(x) = \begin{bmatrix} 1 & x_1 & x_2 & x_1^2 x_2 & x_1^3 x_2^2 & x_1^2 x_2^3 \end{bmatrix}$



Regularization

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This is called "ridge regression" or "LLS with L_2 regularization."

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This is called "ridge regression" or "LLS with L_2 regularization."
If we have a constant feature, we generally do not include it in the regularization.

Ridge Regression

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Ridge Regression

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$$w = \left(X^\top X + \lambda I \right)^{-1} X^\top Y$$

Ridge Regression

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$$w = \left(X^\top X + \lambda \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & & 1 \end{bmatrix} \right)^{-1} X^\top Y$$

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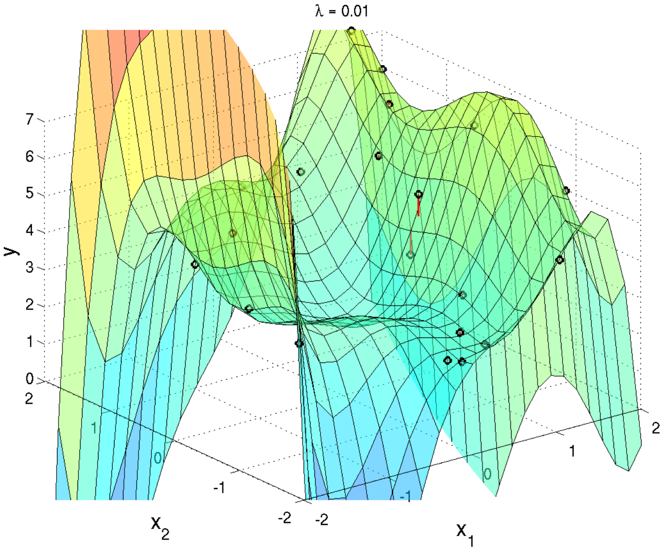
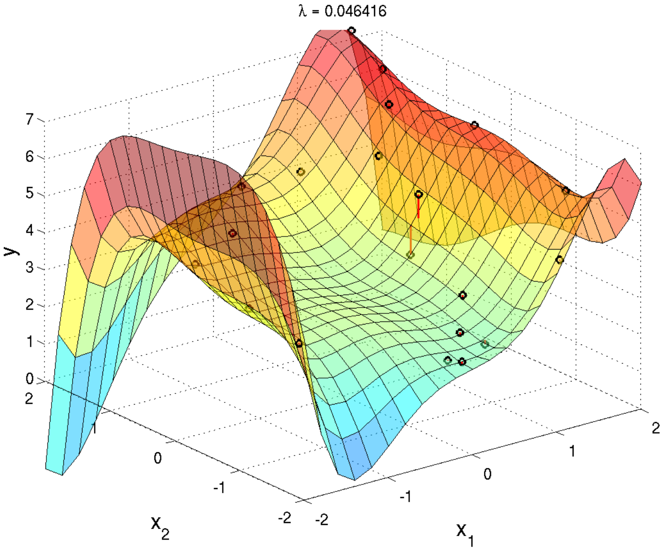
Regularization

5th order polynomial



Regularization

5th order polynomial



Regularization

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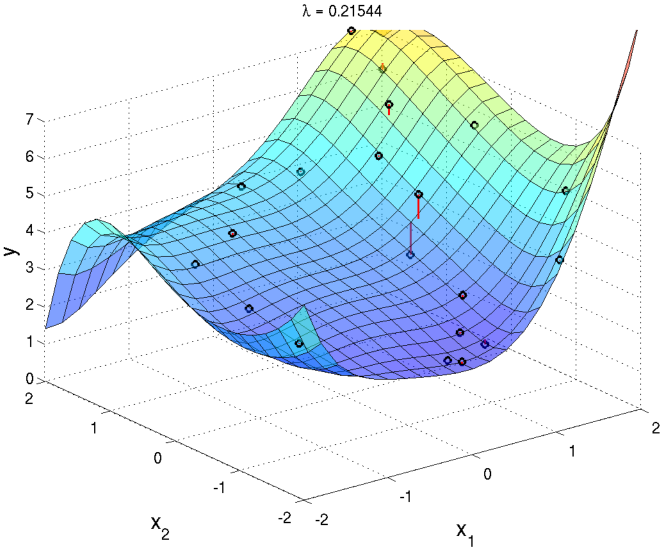
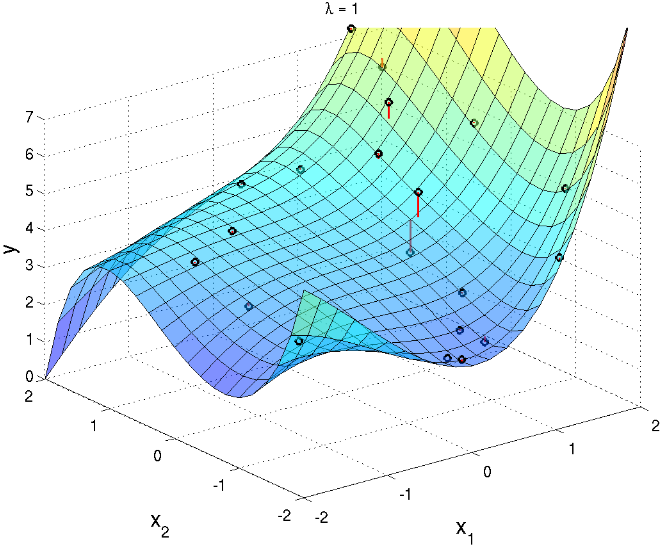
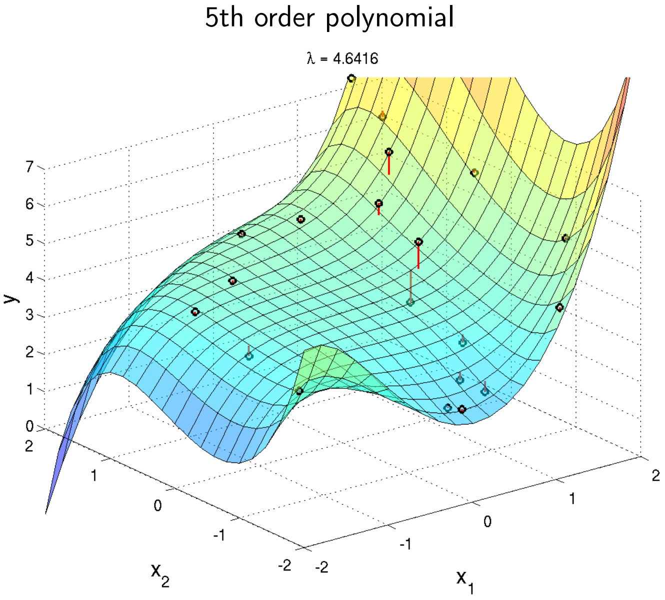


Regularization

5th order polynomial



Regularization



Regularization

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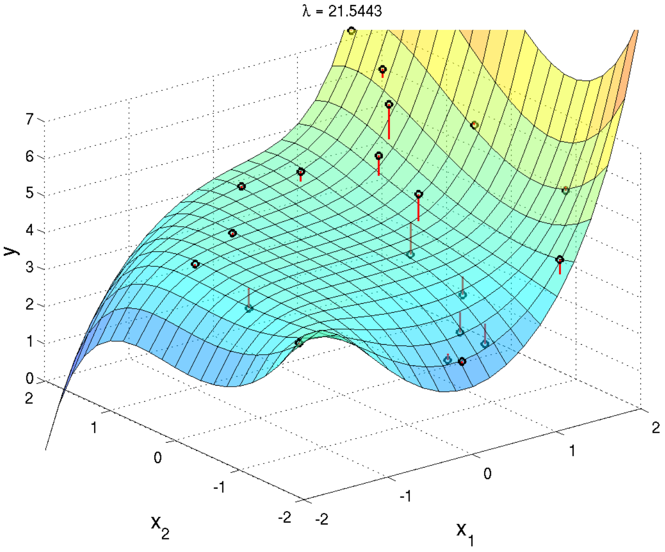

Regularization

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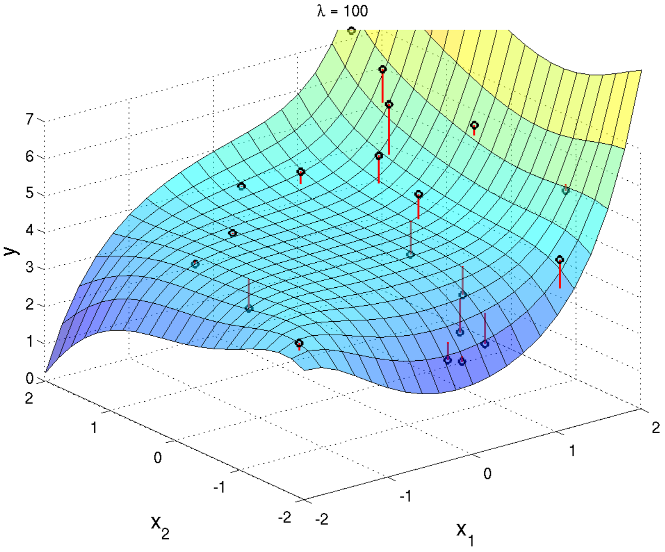

Regularization

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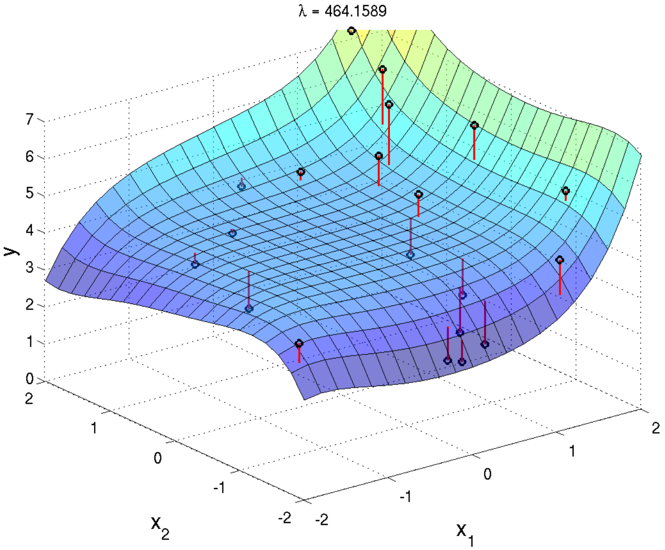
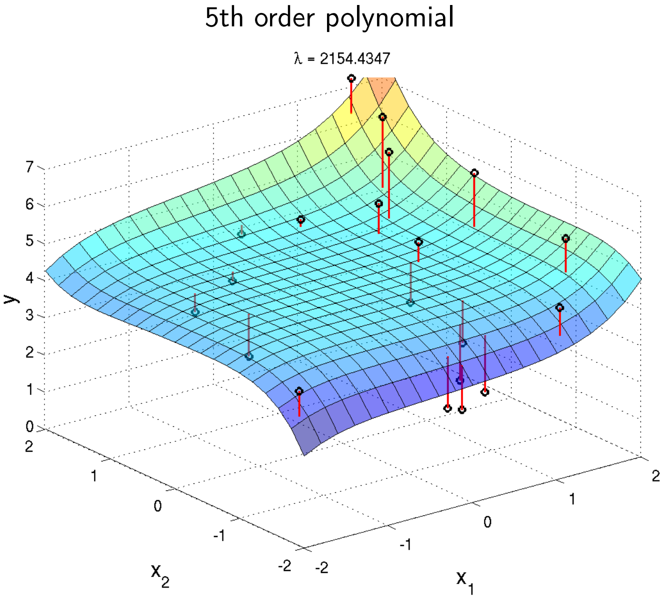
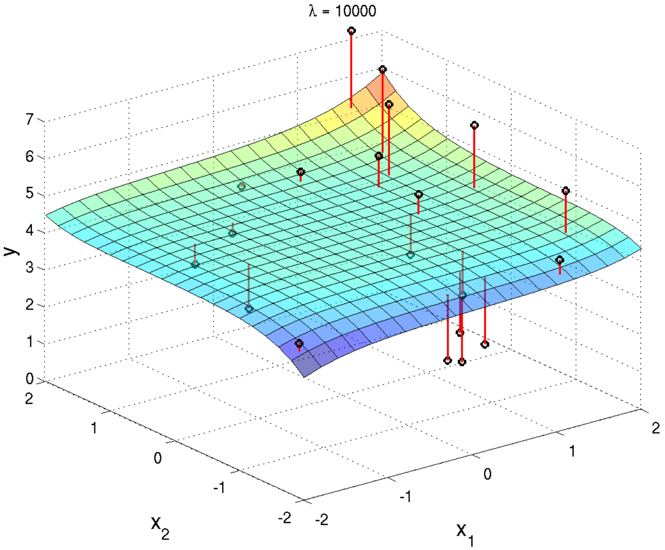


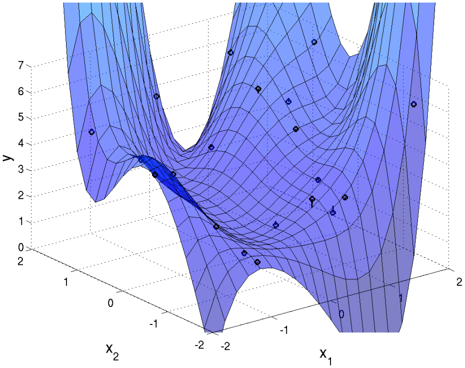
Regularization



5th order polynomial

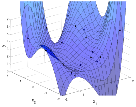


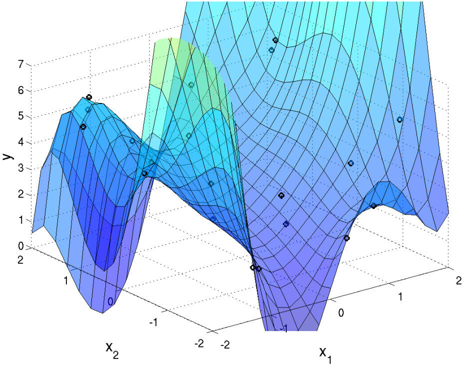
$$\lambda = 0.02$$



Bias-Variance Trade-off

$$\lambda = 0.02$$





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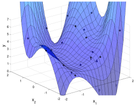

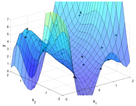

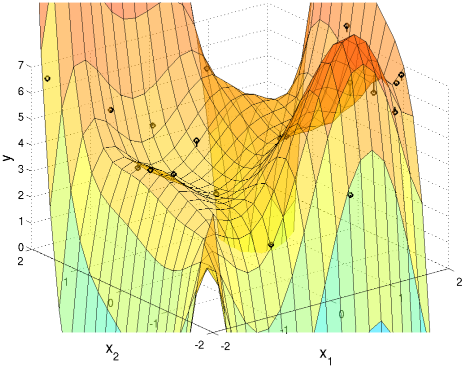

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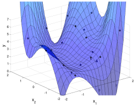

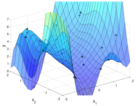

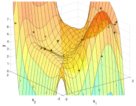

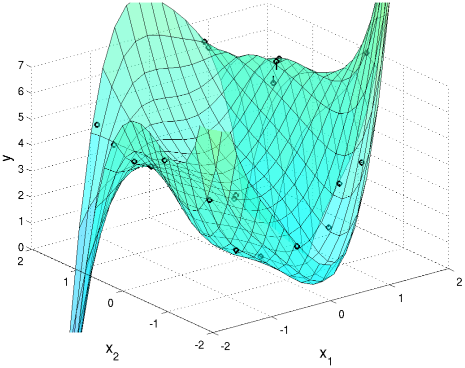

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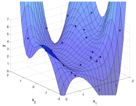

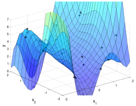

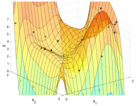

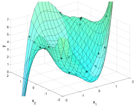

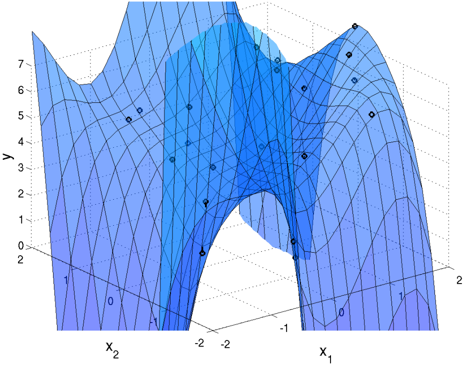

Bias-Variance Trade-off

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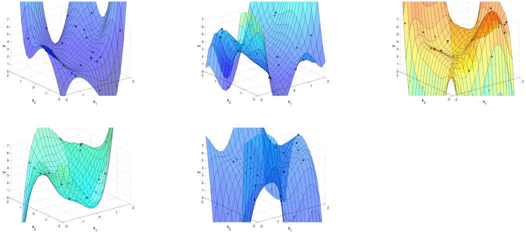

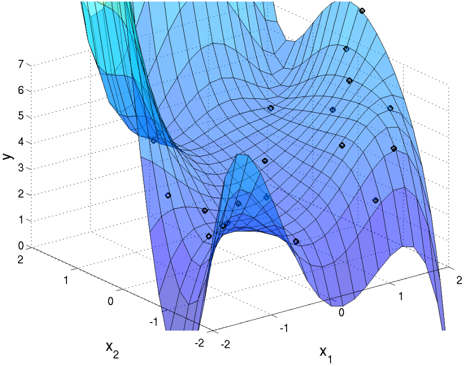

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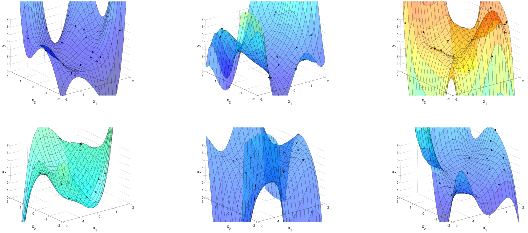

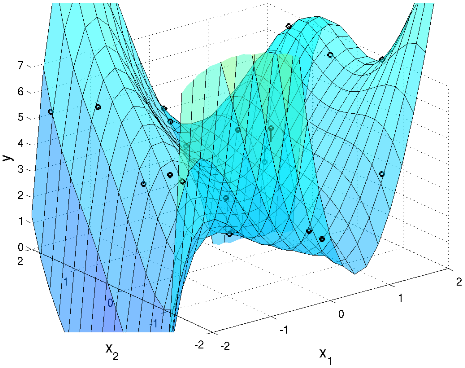

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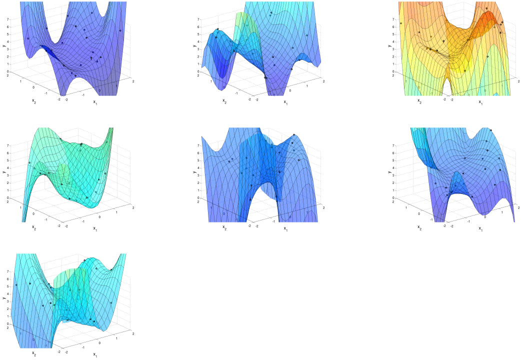

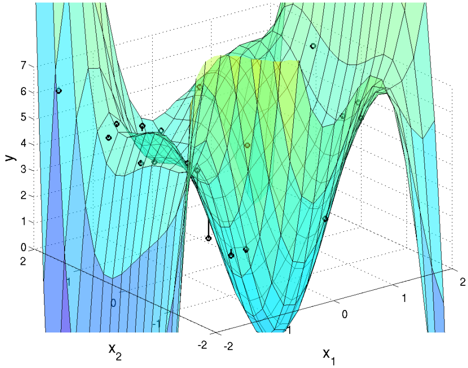

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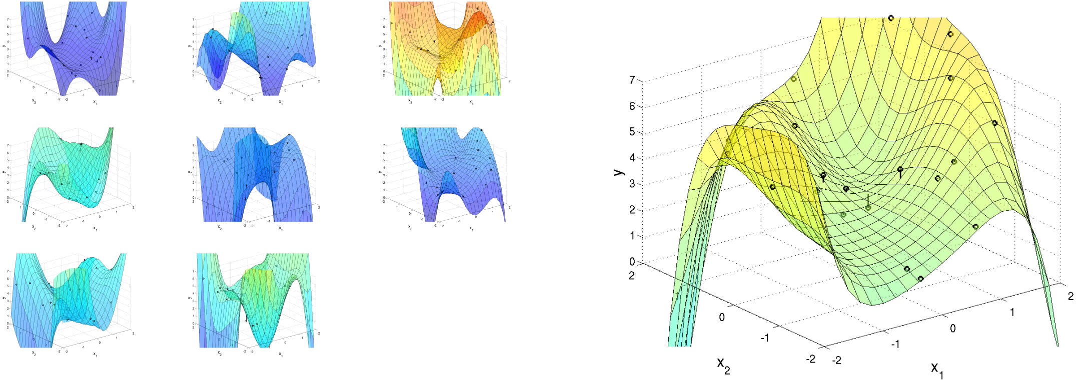
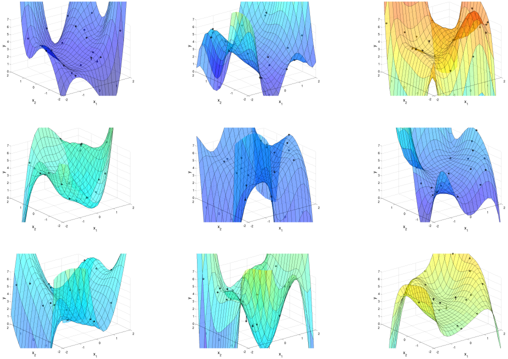


Bias-Variance Trade-off

$$\lambda = 0.02$$



Bias-Variance Trade-off

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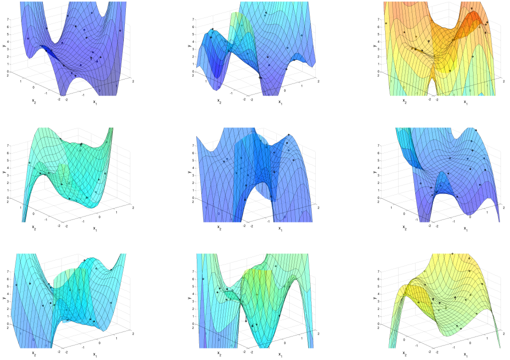
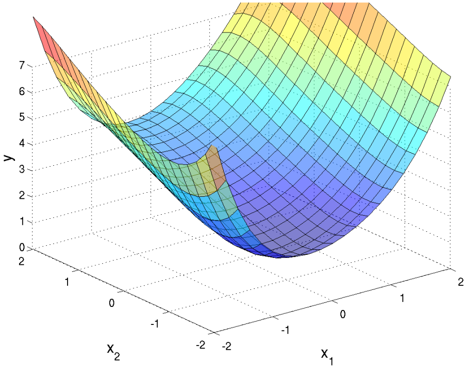
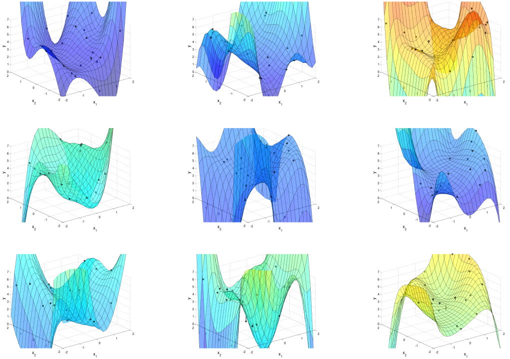
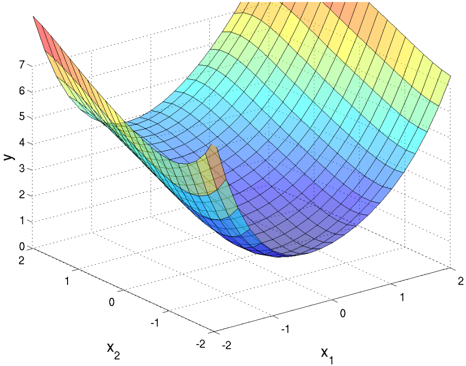




Bias-Variance Trade-off

$$\lambda = 0.02$$



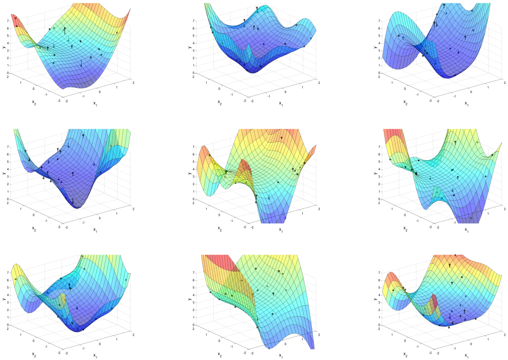


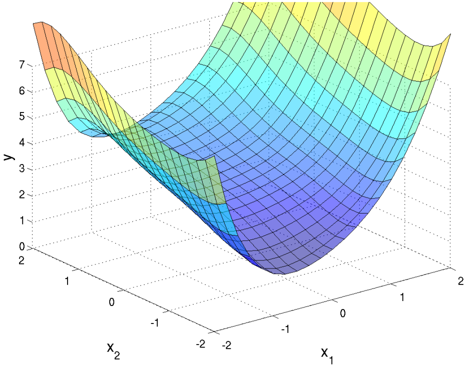
High Variance

Low Bias

Bias-Variance Trade-off

$\lambda = 1$



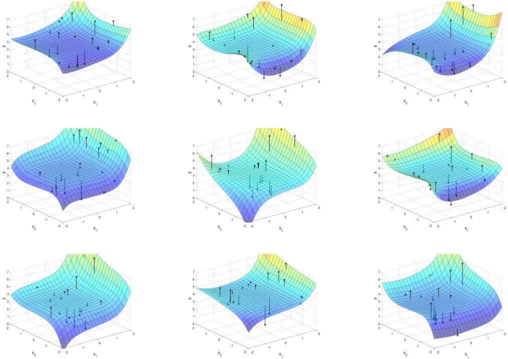


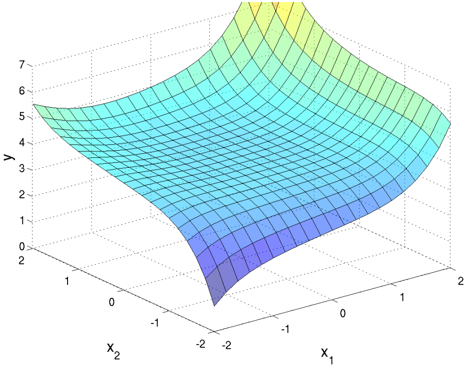
Moderate Variance

Lowish Bias

Bias-Variance Trade-off

$$\lambda = 500$$





Low Variance

High Bias