CS 171: Intro to ML and DM

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UC Riverside

Slide Set 10: Neural Network I

Slides from CS 171

- From UC Riverside
  - CS 171: Introduction to Machine Learning and Data Mining
  - Professor Christian Shelton
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    - Elements of Statistical Learning (Hastie, et al.)
    - Pattern Recognition and Machine Learning (Bishop)
    - An Introduction to Machine Learning (Kubat)
    - Machine Learning: A Probabilistic Perspective (Murphy)
  - For use only by enrolled students in the course

Non-linear methods

In

$$f(x) = x^\top w \qquad\qquad f(x) = \sigma(x^\top w)$$

replace x with $\phi(x)$:

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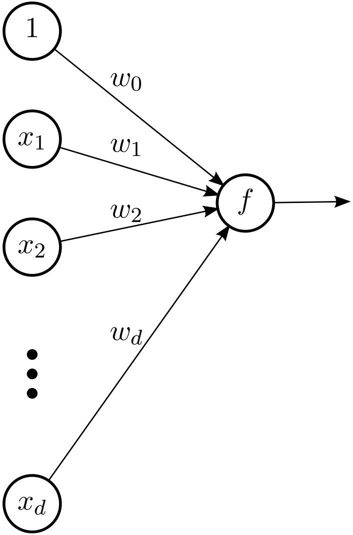
$$f(x) = \phi(x)^\top w \qquad\qquad f(x) = \sigma(\phi(x)^\top w)$$

If $\phi(x)$ selected by hand, we are done!

Learning Non-linearity



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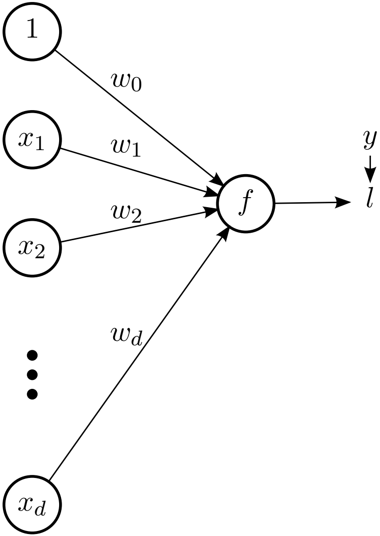

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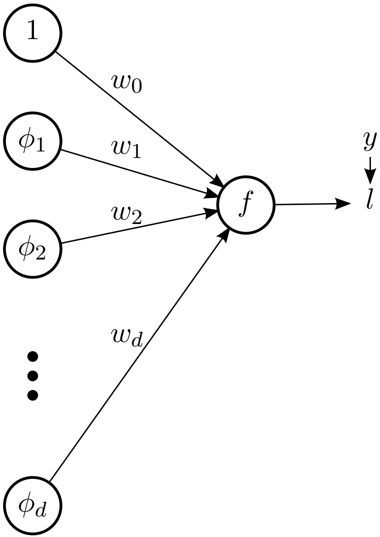

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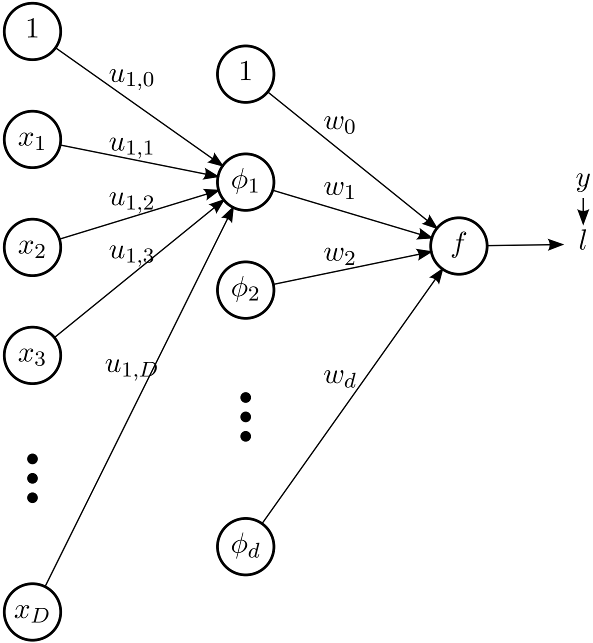

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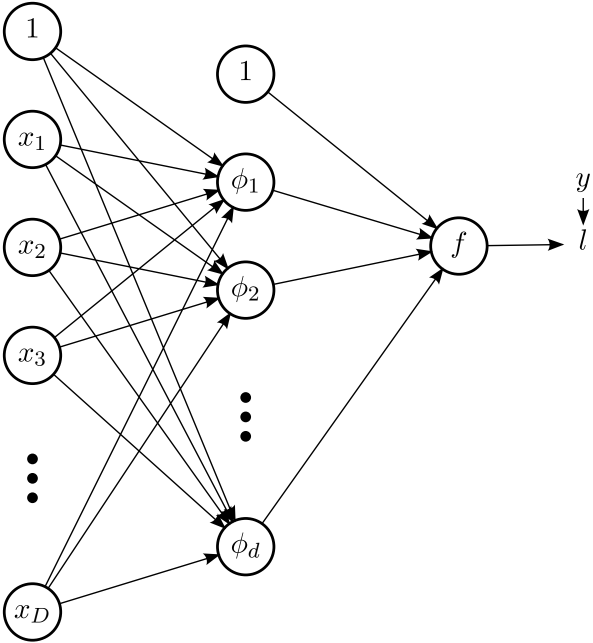

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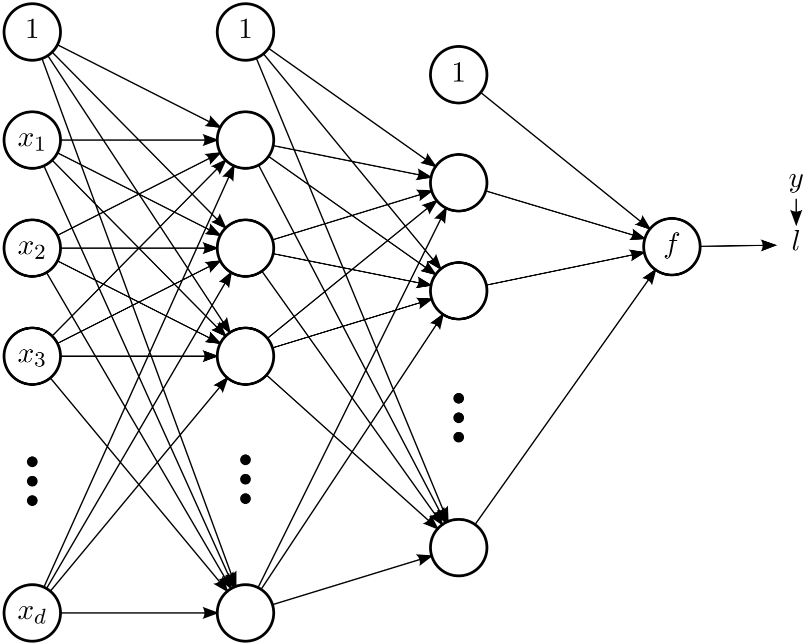

Neural Network Learning Algorithm

Stochastic Gradient Descent:

- Repeat until converged
  - For i from 1 to m (for each example)
    - Run forward propagation with x_i
    - Run backward propagation with y_i and results from forward-prop
    - Compute ∇_W for all weight layers, W, from forward-prop and backprop
    - Change weight layer W by $-\eta\nabla_W$

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- Repeat until converged
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    - Run forward propagation with x_i
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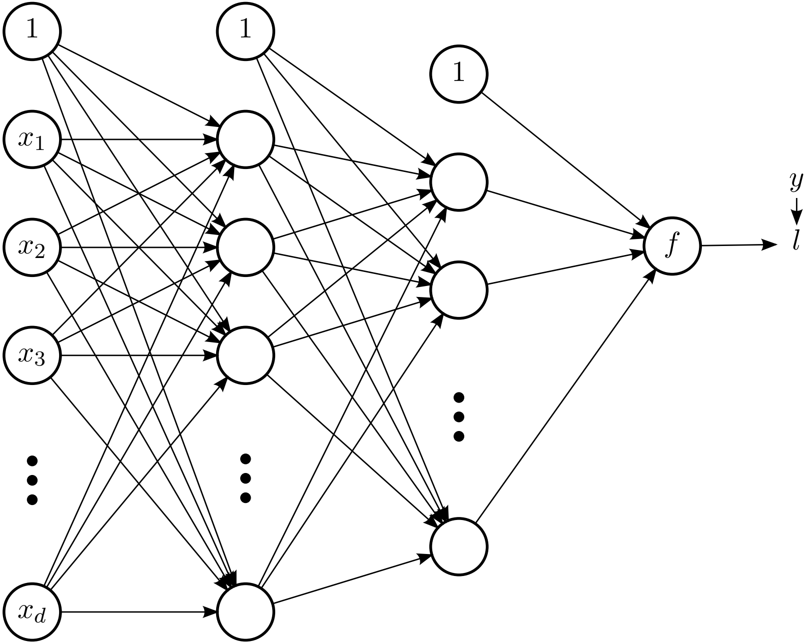
    - Compute ∇_W for all weight layers, W, from forward-prop and backprop
    - Add ∇_W to running sum, S_W for all weight layers
  - Change weight layer W by $-\eta S_W$

Learning Non-linearity



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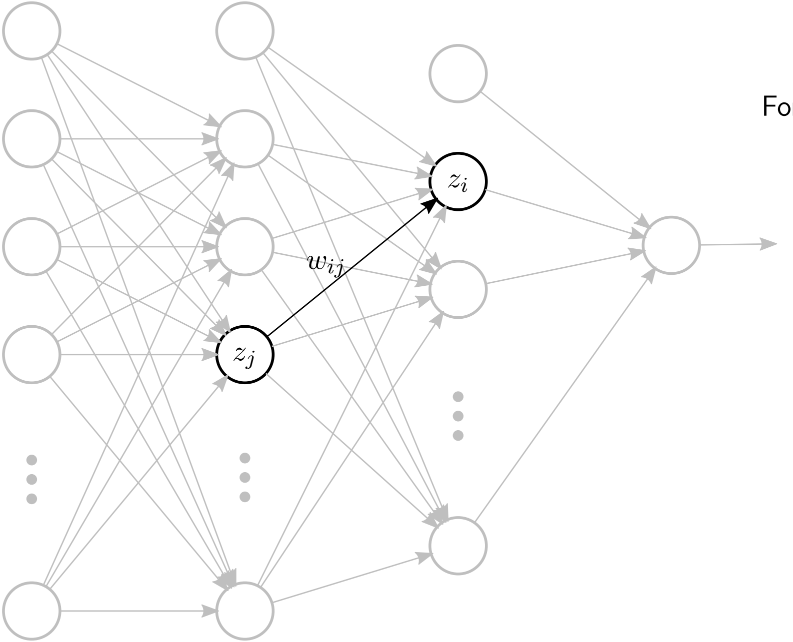
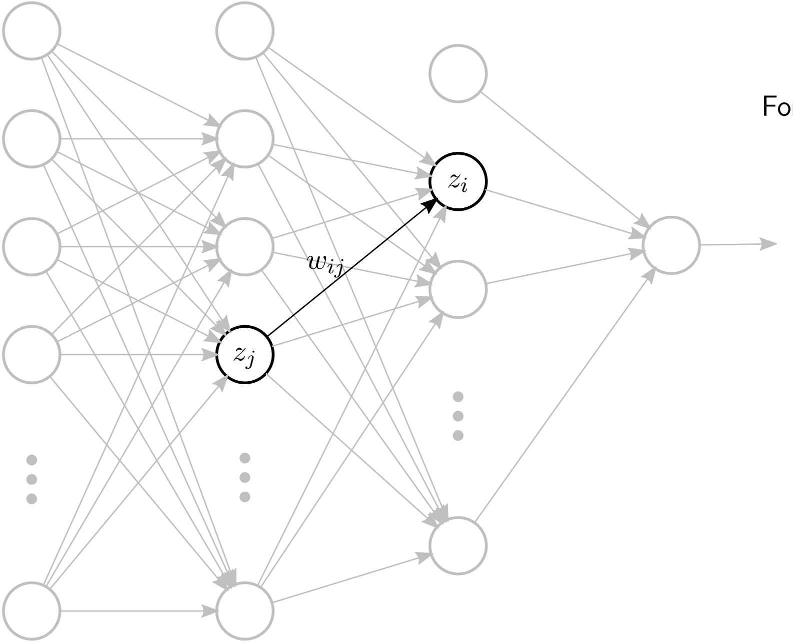


For a given x, y

$$z_i = g_i(a_i)$$

$$a_i = \sum_j w_{ij} z_j$$

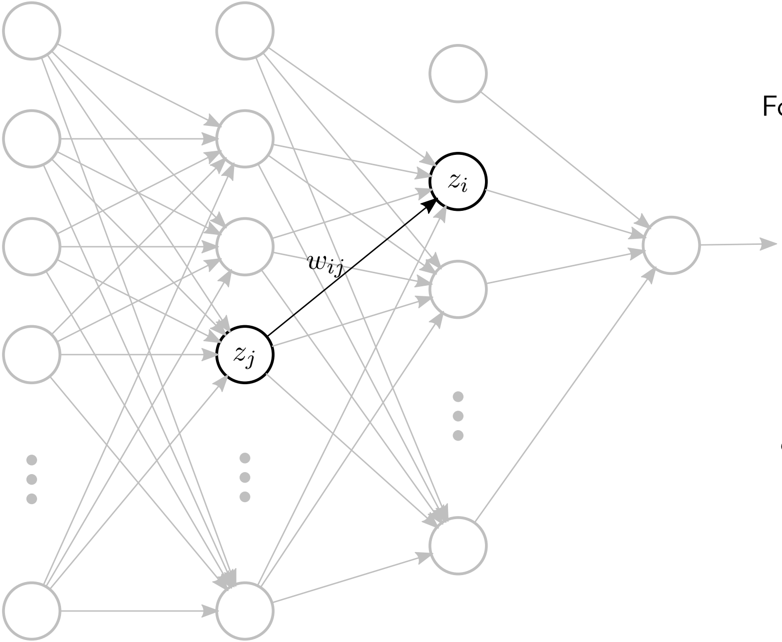
Learning Non-linearity



For a given x, y

$$
\begin{aligned}
z_i &= g_i(a_i) \\
a_i &= \sum_j w_{ij} z_j \\
\frac{\partial l}{\partial w_{ij}} &= \frac{\partial l}{\partial a_i} \frac{\partial a_i}{\partial w_{ij}} \\
&= \delta_i z_j
\end{aligned}
$$

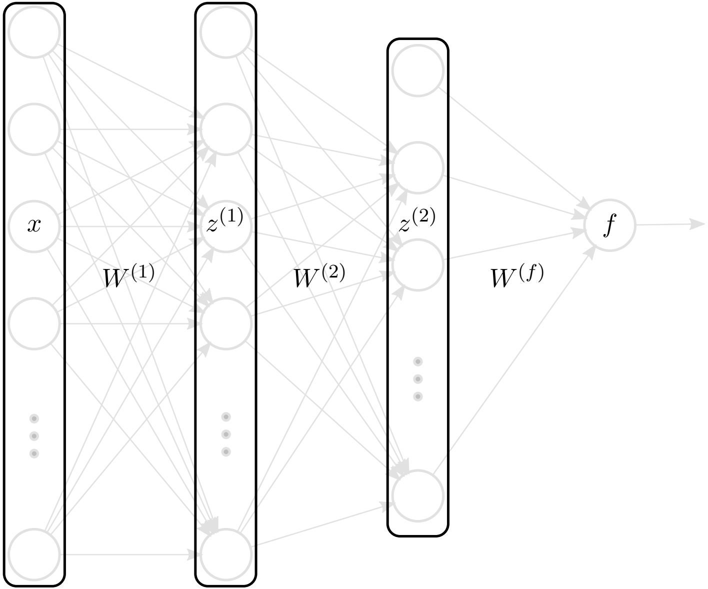
Learning Non-linearity



For a given x, y

$$\begin{aligned}
z_i &= g_i(a_i) \\
a_i &= \sum_j w_{ij} z_j \\
\frac{\partial l}{\partial w_{ij}} &= \frac{\partial l}{\partial a_i}\frac{\partial a_i}{\partial w_{ij}} \\
&= \delta_i z_j \\
\delta_j = \frac{\partial l}{\partial a_j} &= \frac{\partial l}{\partial z_j}\frac{\partial z_j}{\partial a_j} \\
&= \sum_i \frac{\partial l}{\partial a_i}\frac{\partial a_i}{\partial z_j}\frac{\partial z_j}{\partial a_j} \\
&= \sum_i \delta_i w_{ij} g'(a_j)
\end{aligned}$$

Layered Neural Network



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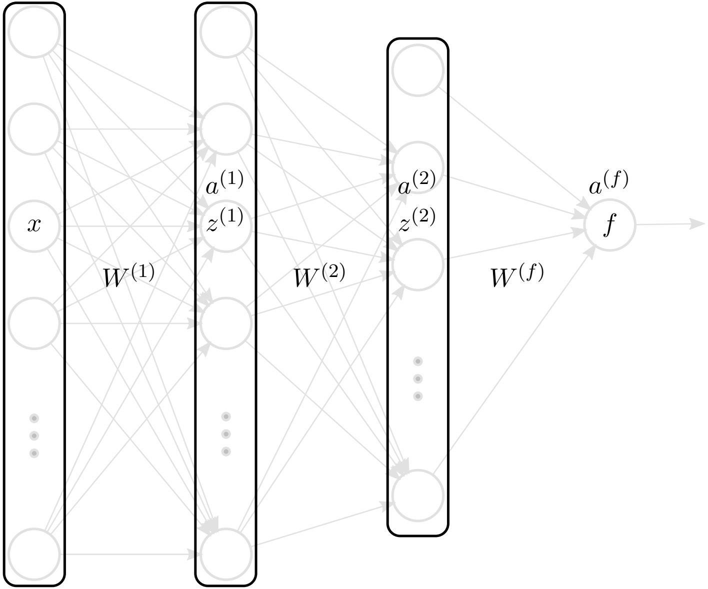

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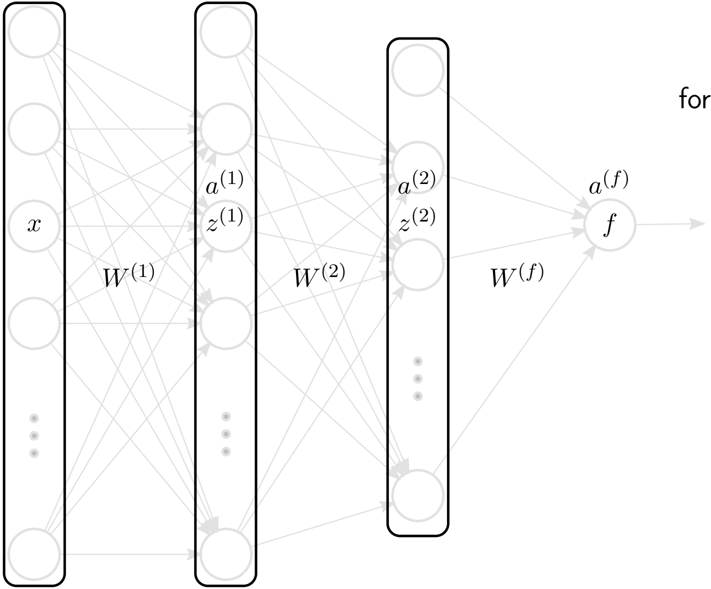


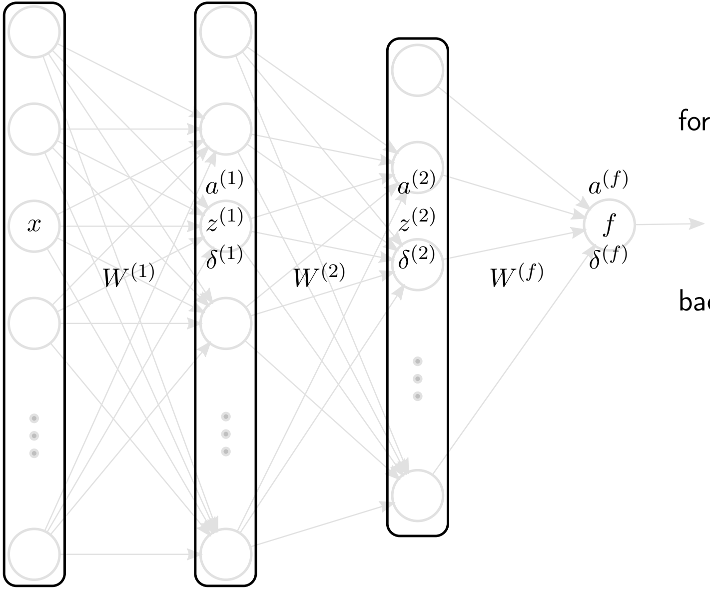
forward:

$$a^{(i)} = W^{(i)} z^{(i-1)}$$

$$z^{(i)} = g(a^{(i)})$$

(element-wise application of g)

Layered Neural Network



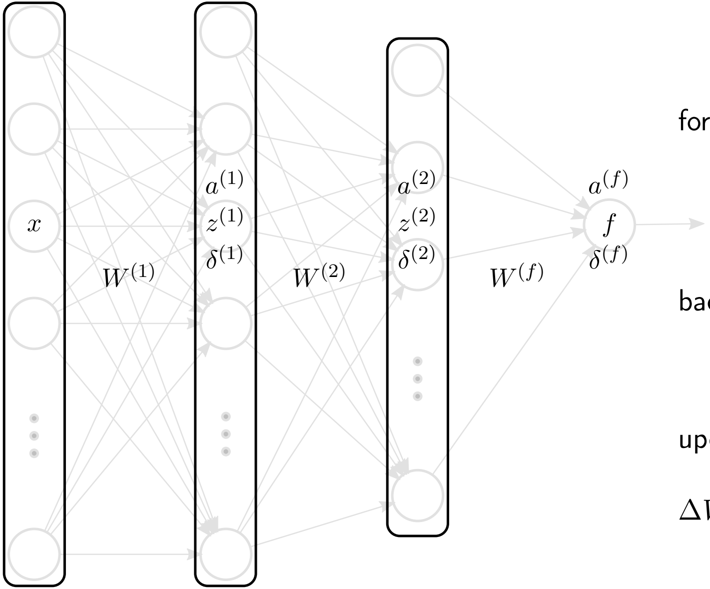
forward:

$$a^{(i)} = W^{(i)} z^{(i-1)}$$
$$z^{(i)} = g(a^{(i)})$$

backward:

$$\delta^{(i)} = g'(a^{(i)}) \odot \left({W^{(i+1)}}^\top \delta^{(i+1)} \right)$$

Layered Neural Network



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$$z^{(i)} = g(a^{(i)})$$

backward:

$$\delta^{(i)} = g'(a^{(i)}) \odot \left({W^{(i+1)}}^\top \delta^{(i+1)} \right)$$

update:

$$\Delta W^{(i)} = -\eta \delta^{(i)} {z^{(i-1)}}^\top$$

Output Layer

For regression:

$$g_f(a) = a$$

$$\delta^{(f)} = (f - y)$$

$$l(f, y) = \frac{1}{2}(f - y)^2$$

Output Layer

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$$g_f(a) = a \qquad\qquad l(f, y) = \frac{1}{2}(f - y)^2$$

$$\delta^{(f)} = (f - y)$$

For binary classification, $y \in \{0, 1\}$:

$$g_f(a) = \sigma(a) \qquad\qquad l(f, y) = -(y \log(f) + (1 - y) \log(1 - f))$$

$$\delta^{(f)} = (f - y)$$

Non-linearities

Historically most common:

$$g(a) = \sigma(a) = 1/(1 + e^{-a})$$

$$g'(a) = g(a)\,(1 - g(a))$$

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(we will always use this in this course)

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Historically most common:

$$g(a) = \sigma(a) = 1/(1 + e^{-a})$$

$$g'(a) = g(a)\,(1 - g(a))$$

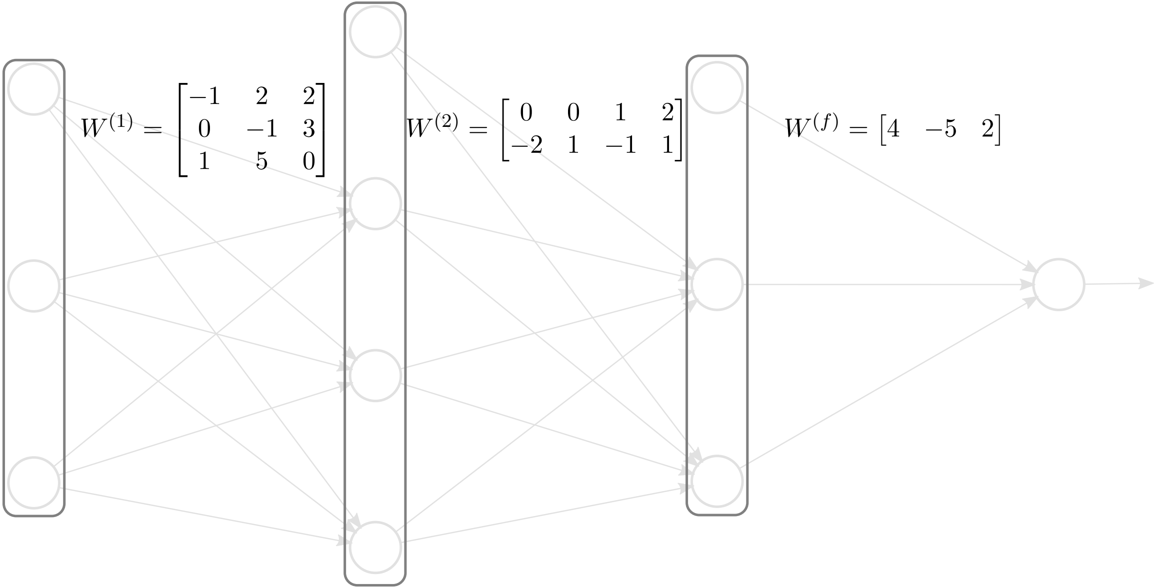
(we will always use this in this course)

Now common (for hidden units):

$$g(a) = \max(0, a)$$

$$g'(a) = \begin{cases} a < 0 & 0 \\ a \geq 0 & 1 \end{cases}$$

Layered Neural Network Example

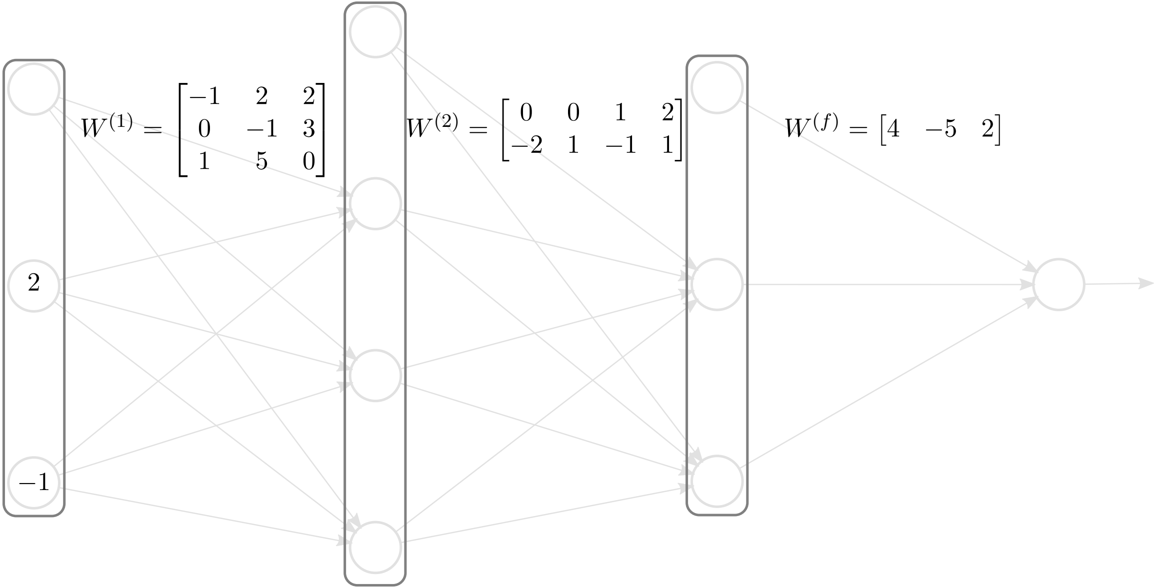


$$W^{(1)} = \begin{bmatrix} -1 & 2 & 2 \\ 0 & -1 & 3 \\ 1 & 5 & 0 \end{bmatrix}$$

$$W^{(2)} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ -2 & 1 & -1 & 1 \end{bmatrix}$$

$$W^{(f)} = \begin{bmatrix} 4 & -5 & 2 \end{bmatrix}$$

Layered Neural Network Example, forward propagation



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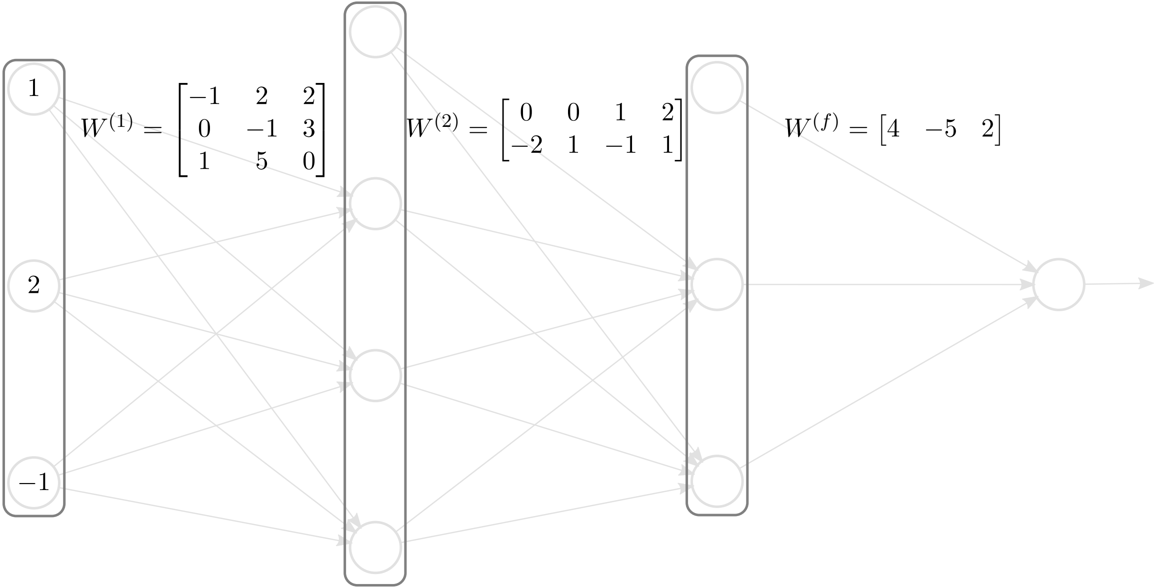

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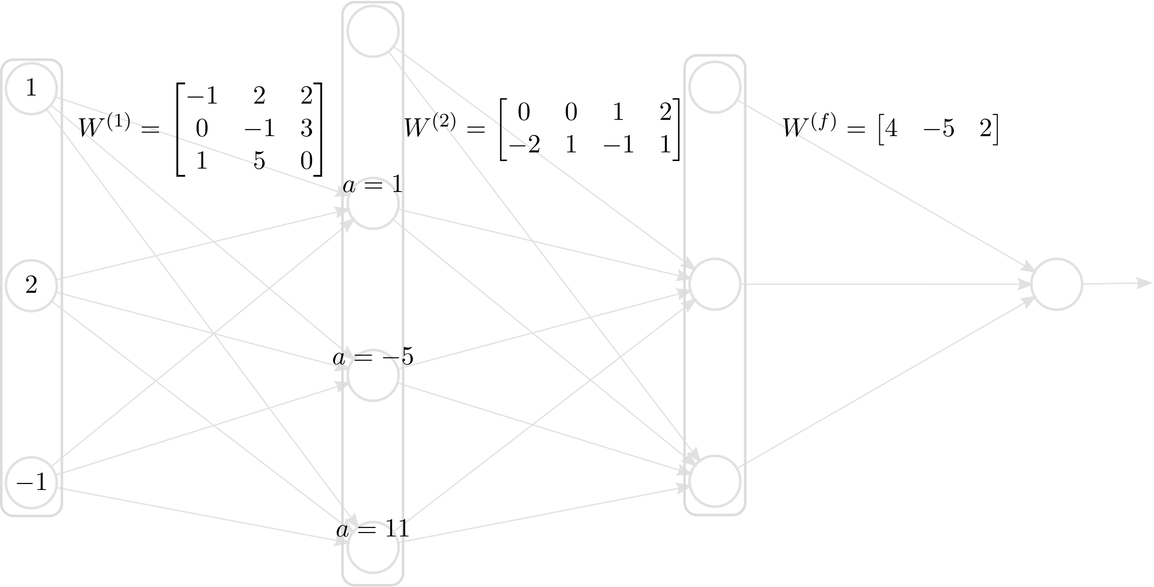

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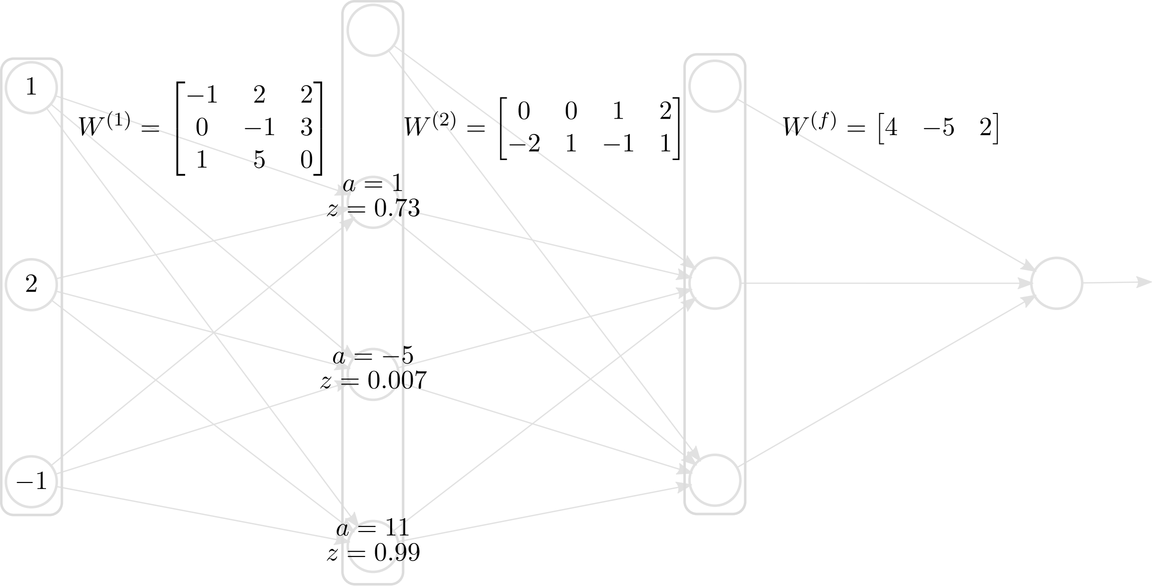

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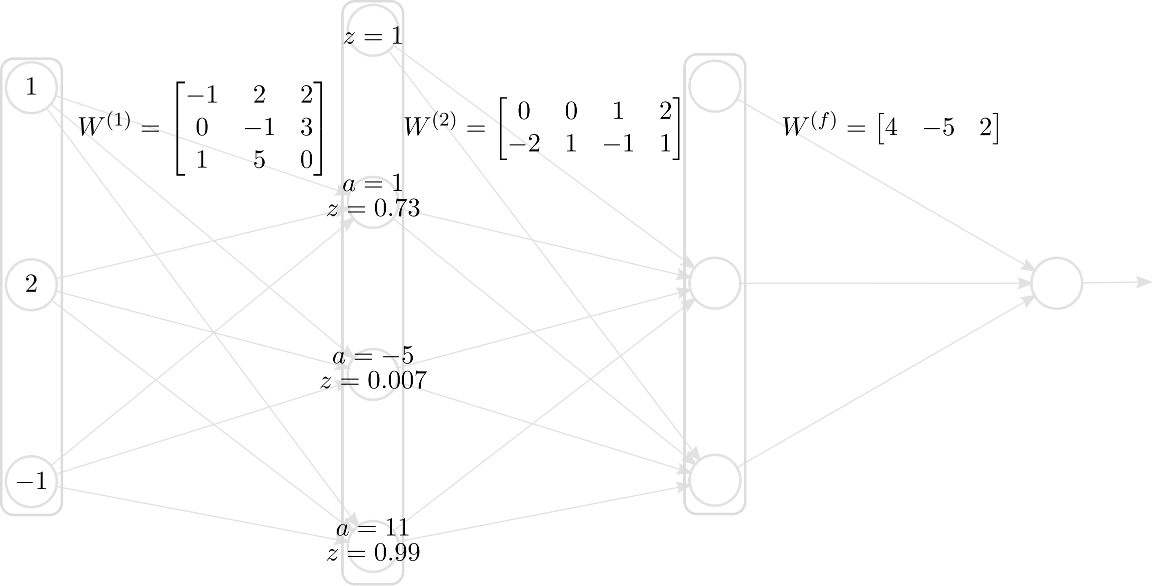

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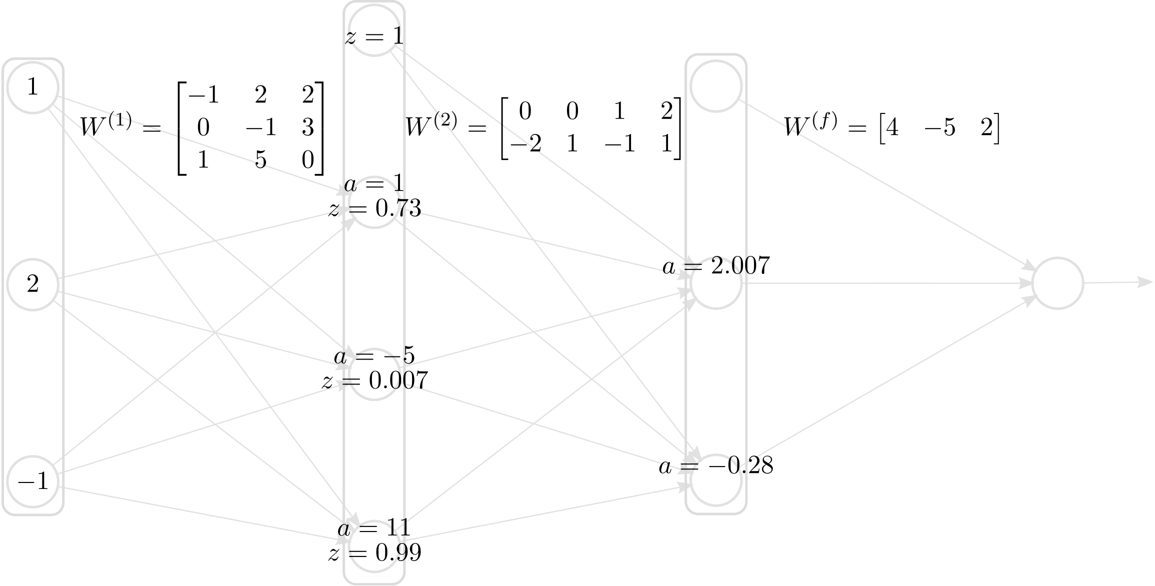

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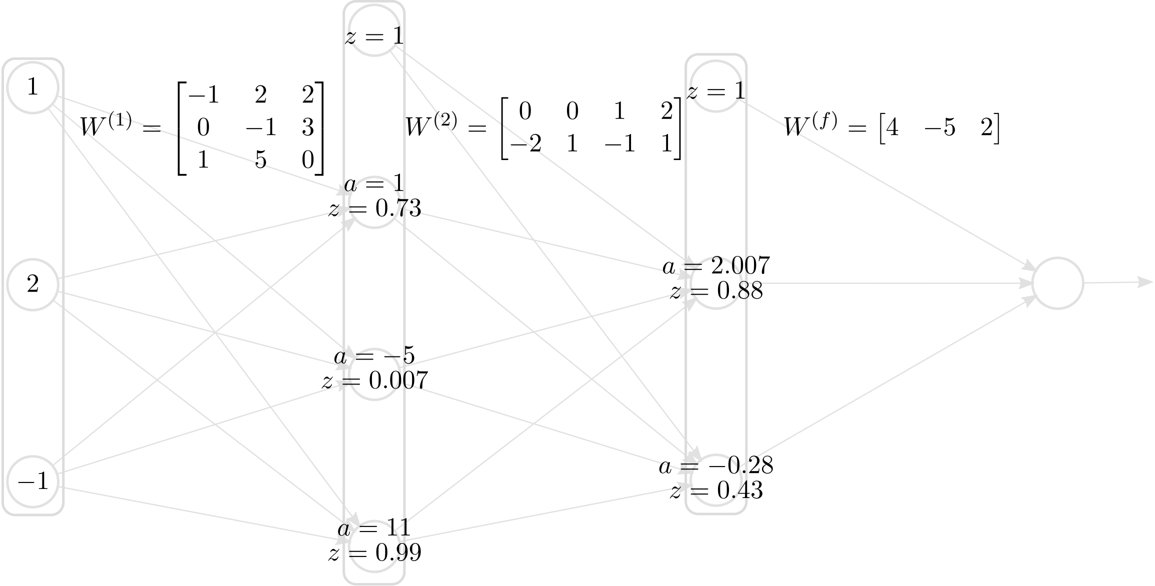

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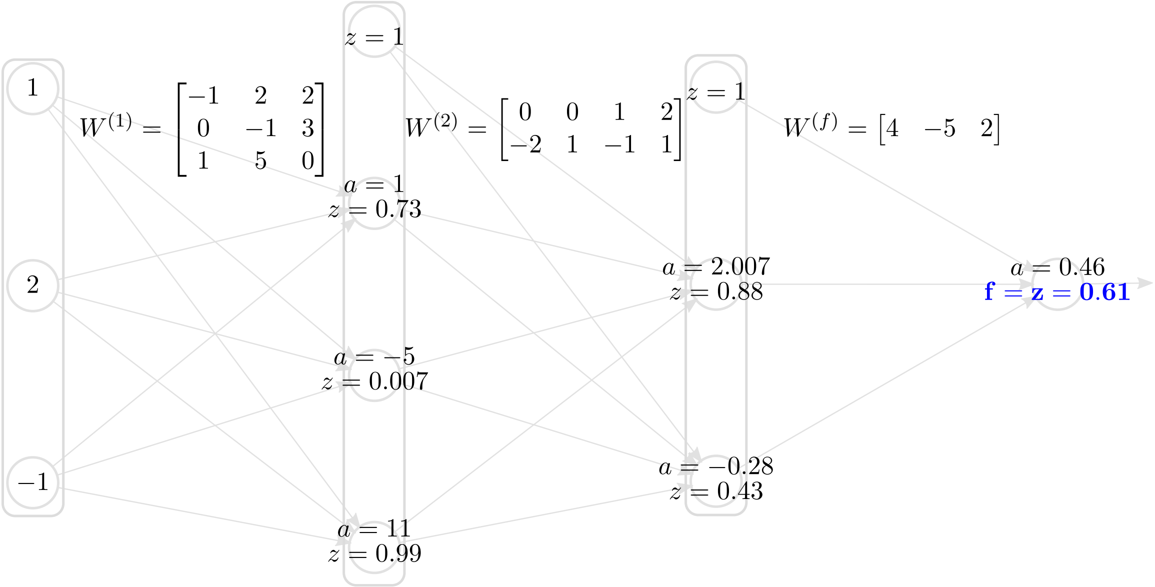

$$\mathbf{f = z = 0.61}$$

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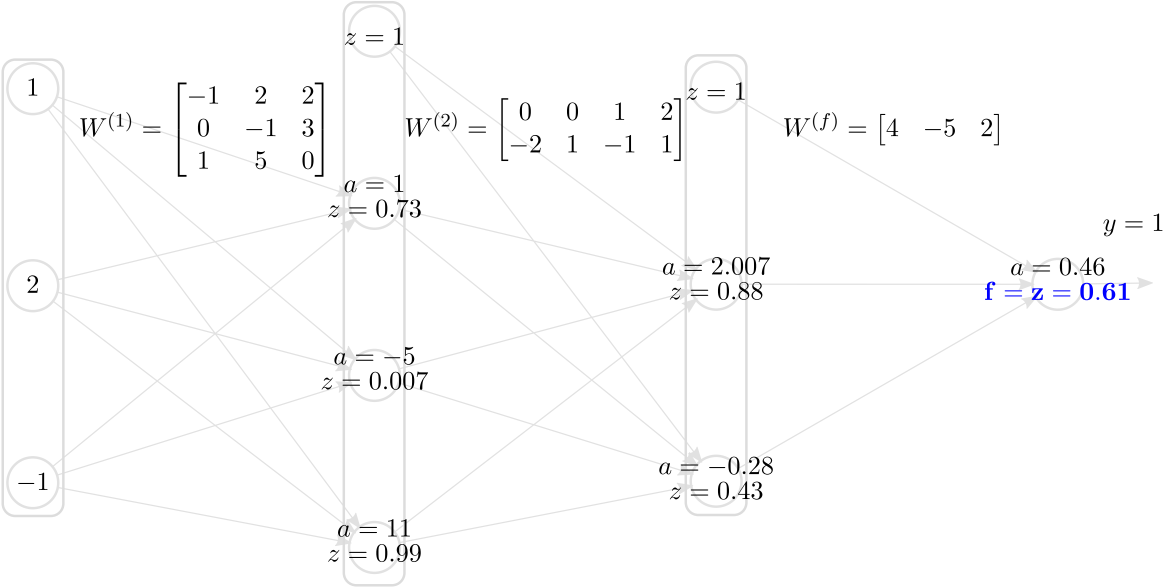
Layered Neural Network Example, backward propagation



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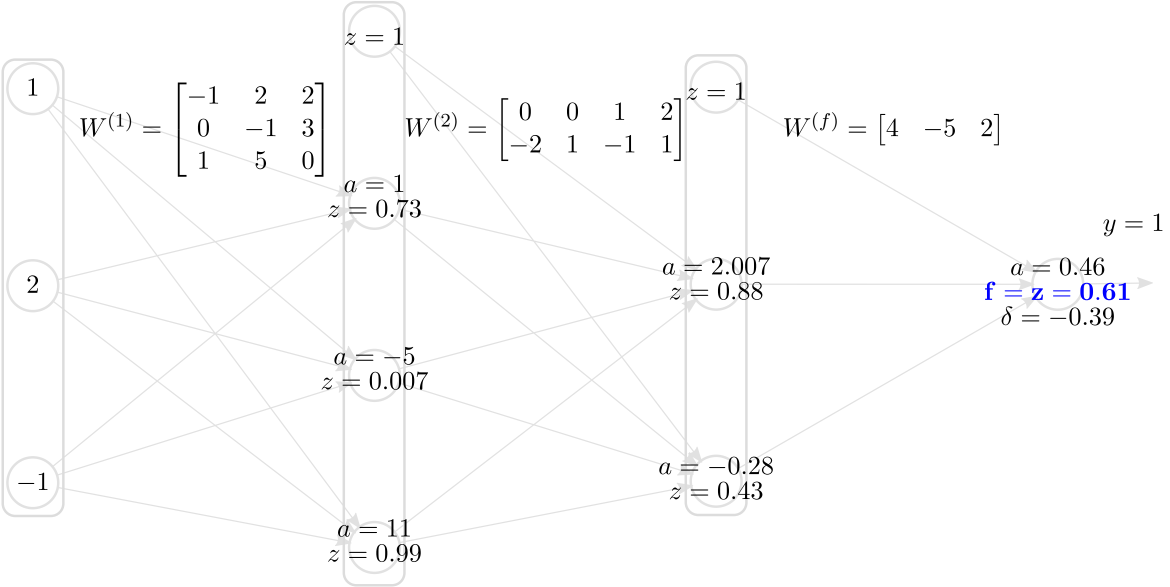

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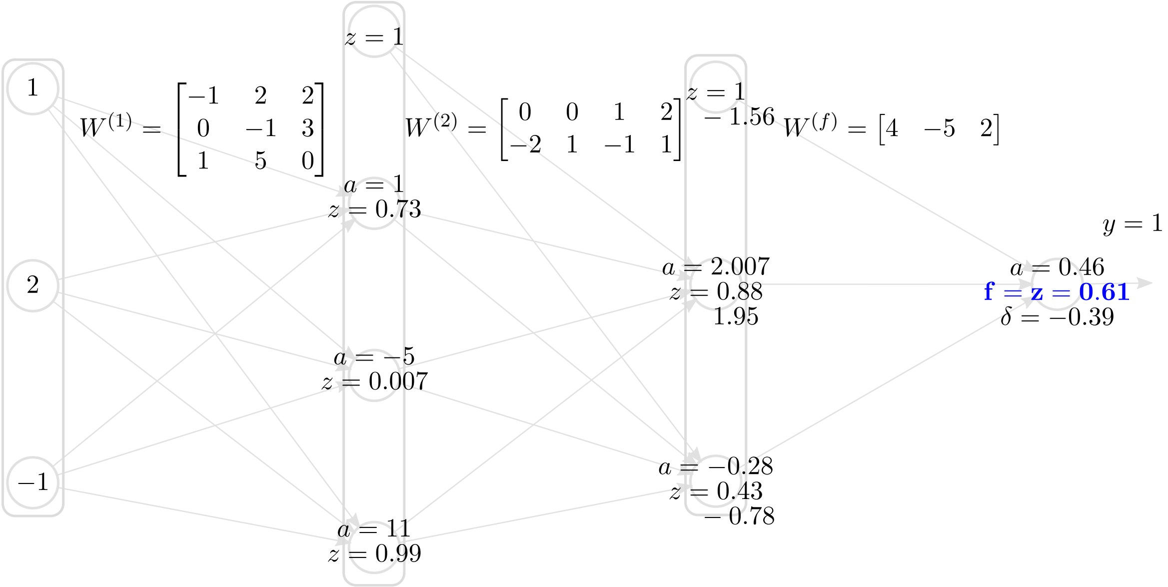


1

2

-1

$$W^{(1)} = \begin{bmatrix} -1 & 2 & 2 \\ 0 & -1 & 3 \\ 1 & 5 & 0 \end{bmatrix}$$

$z = 1$

$a = 1$
$z = 0.73$

$a = -5$
$z = 0.007$

$a = 11$
$z = 0.99$

$$W^{(2)} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ -2 & 1 & -1 & 1 \end{bmatrix}$$

$z = 1$
-1.56

$a = 2.007$
$z = 0.88$
1.95

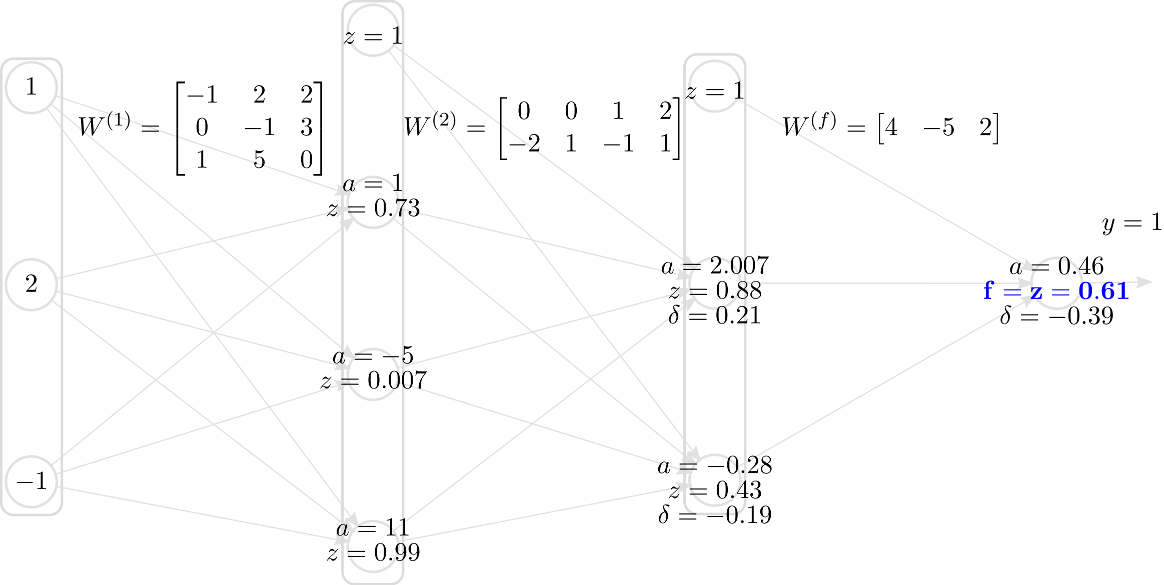
$a = -0.28$
$z = 0.43$
-0.78

$$W^{(f)} = \begin{bmatrix} 4 & -5 & 2 \end{bmatrix}$$

$y = 1$

$a = 0.46$
$\mathbf{f = z = 0.61}$
$\delta = -0.39$

Layered Neural Network Example, backward propagation



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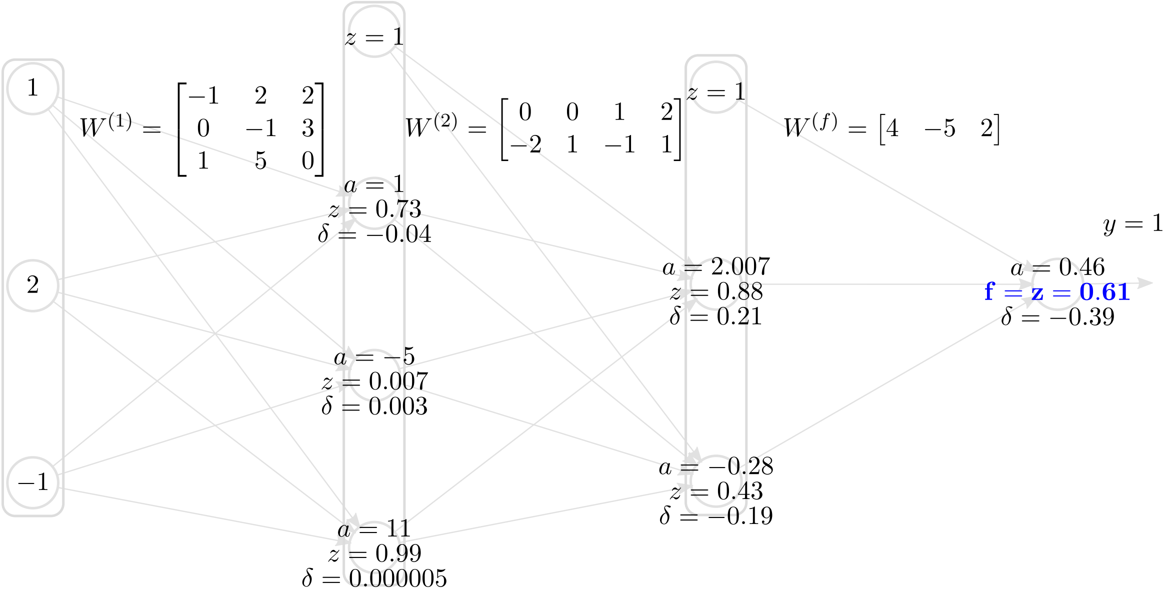

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$z = 1$

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$\delta = 0.003$

$a = 11$
$z = 0.99$
$\delta = 0.000005$

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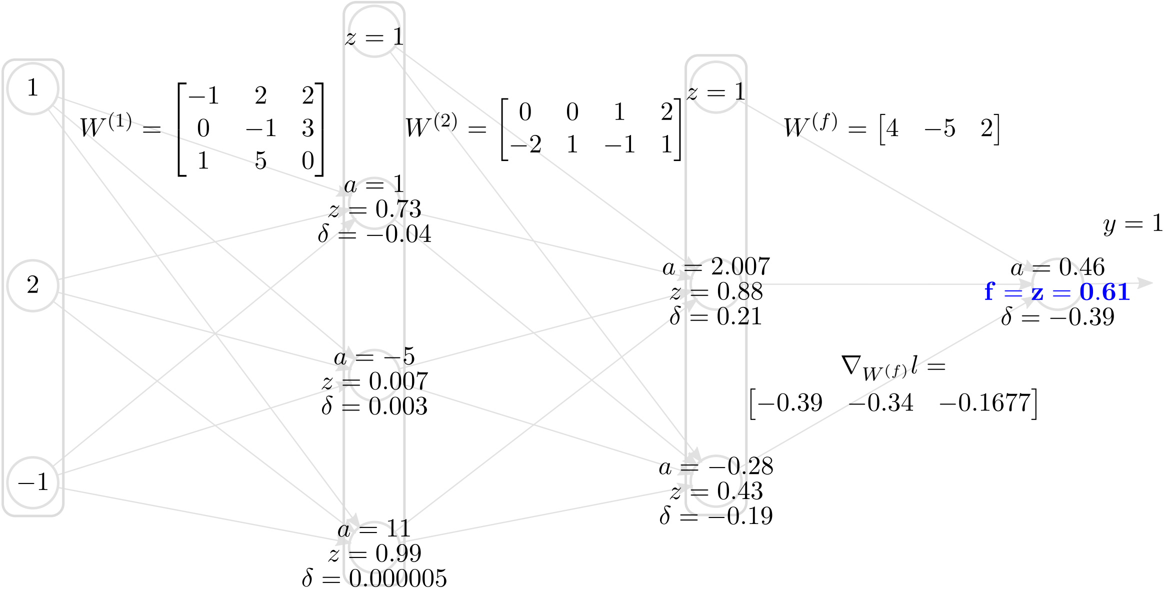
Layered Neural Network Example, backward propagation



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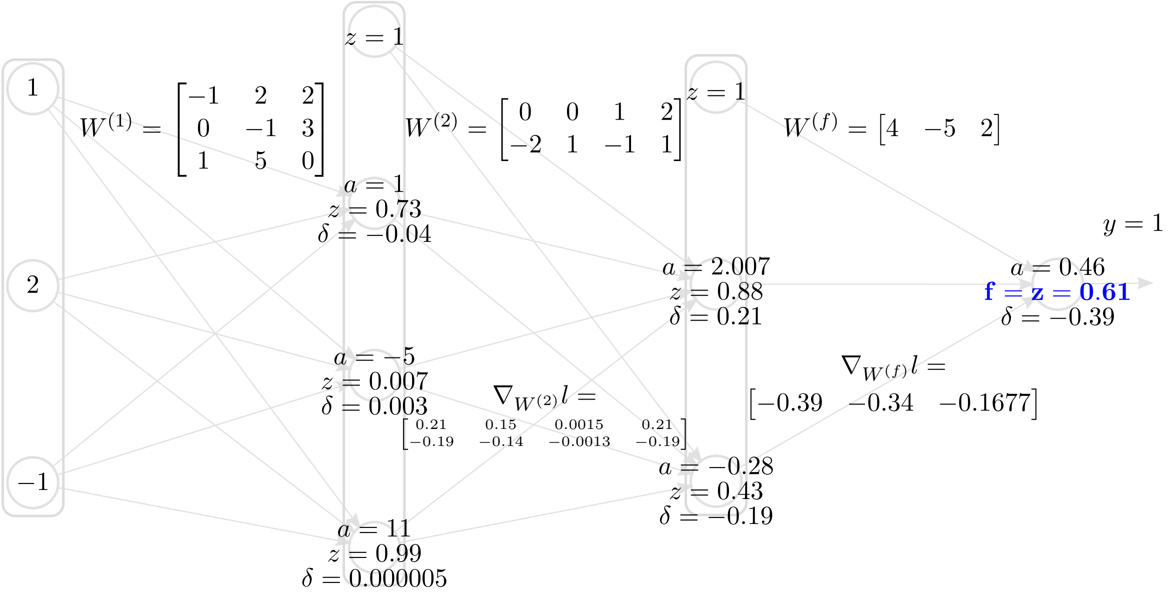
$$\nabla_{W^{(f)}} l = \begin{bmatrix} -0.39 & -0.34 & -0.1677 \end{bmatrix}$$

Layered Neural Network Example, backward propagation



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$$W^{(2)} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ -2 & 1 & -1 & 1 \end{bmatrix}$$

$$\nabla_{W^{(2)}} l = \begin{bmatrix} 0.21 & 0.15 & 0.0015 & 0.21 \\ -0.19 & -0.14 & -0.0013 & -0.19 \end{bmatrix}$$

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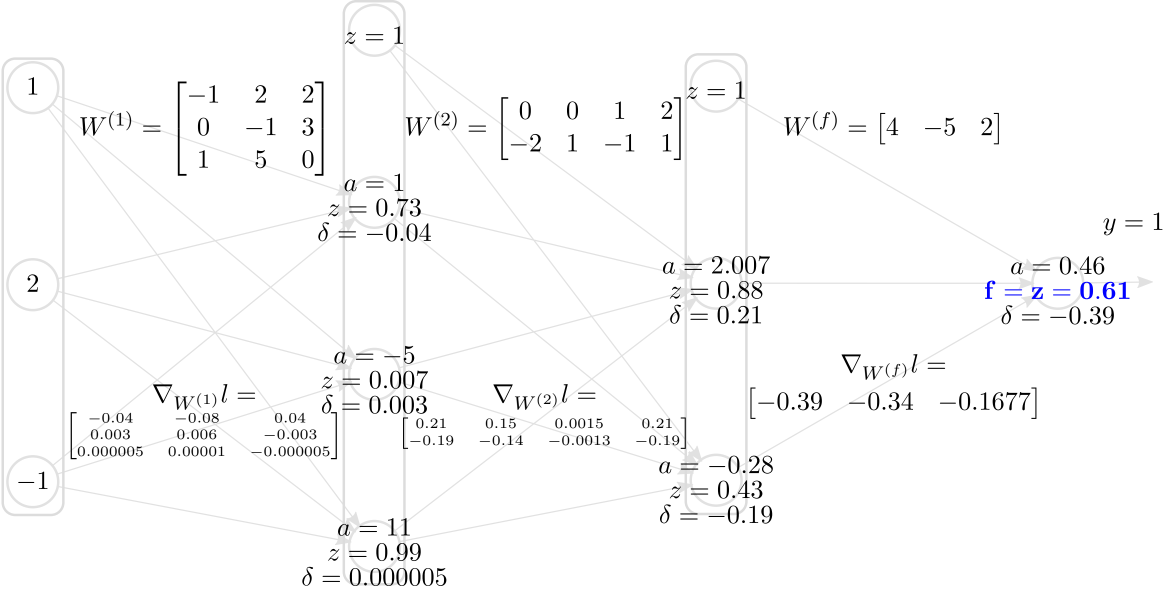
Layered Neural Network Example, backward propagation



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$$\nabla_{W^{(1)}} l = \begin{bmatrix} -0.04 & -0.08 & 0.04 \\ 0.003 & 0.006 & -0.003 \\ 0.000005 & 0.00001 & -0.000005 \end{bmatrix}$$

$$\nabla_{W^{(2)}} l = \begin{bmatrix} 0.21 & 0.15 & 0.0015 & 0.21 \\ -0.19 & -0.14 & -0.0013 & -0.19 \end{bmatrix}$$

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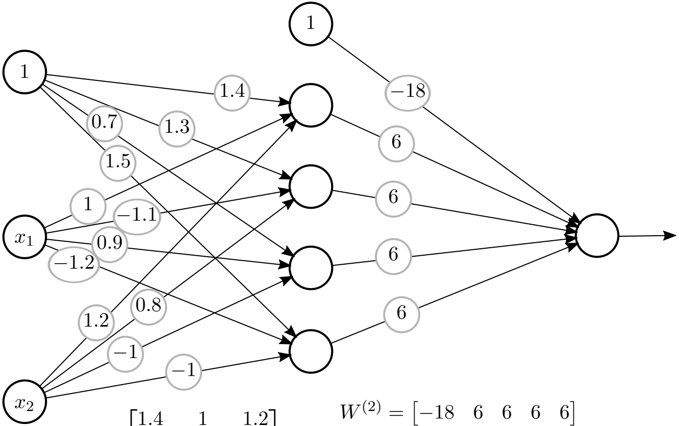
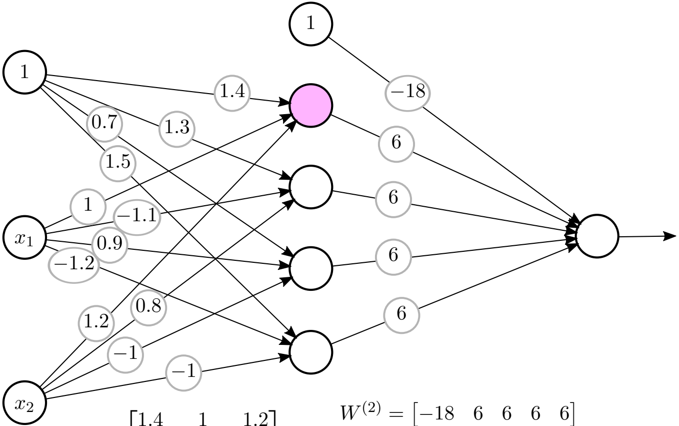
Function Approximation



$$W^{(1)} = \begin{bmatrix} 1.4 & 1 & 1.2 \\ 1.3 & -1.1 & 0.8 \\ 0.7 & 0.9 & -1 \\ 1.5 & -1.2 & -1 \end{bmatrix}$$

$$W^{(2)} = \begin{bmatrix} -18 & 6 & 6 & 6 & 6 \end{bmatrix}$$

Function Approximation



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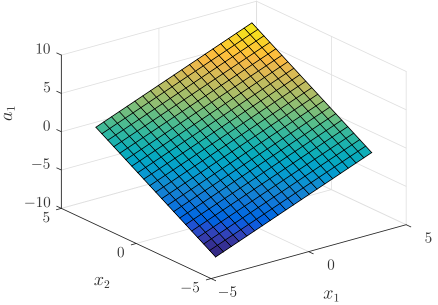

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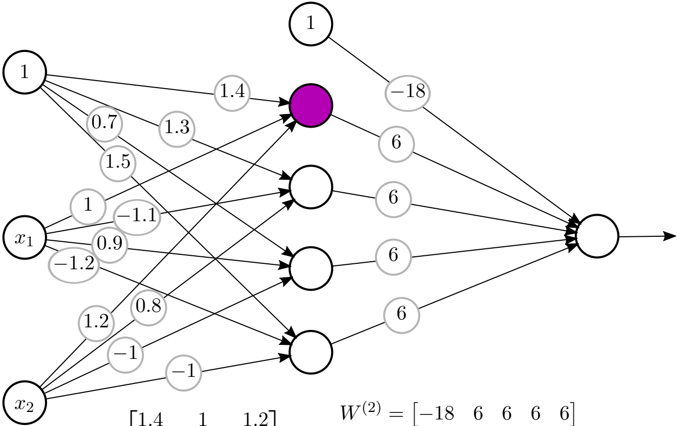

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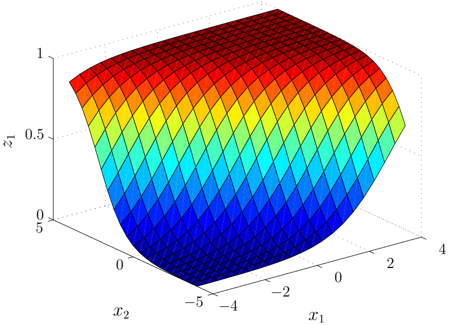

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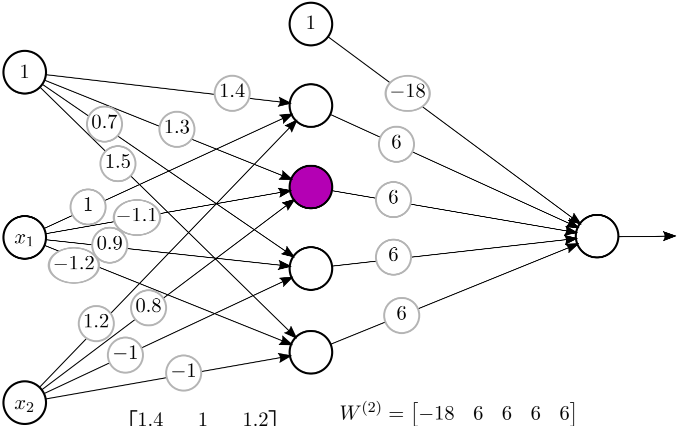

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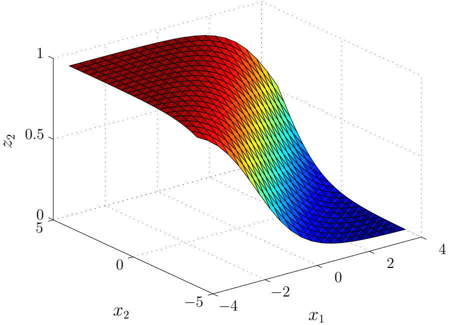

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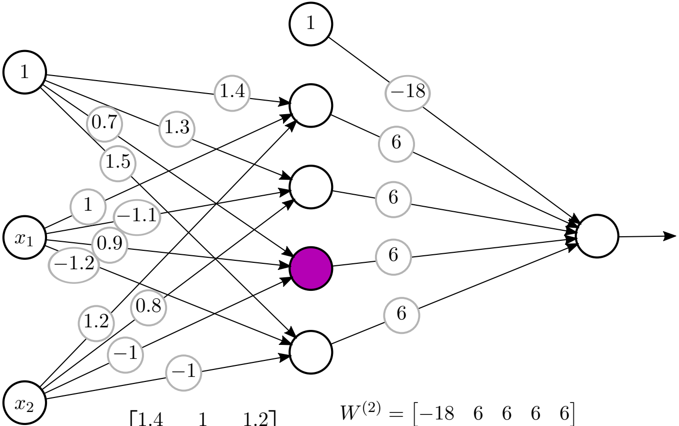

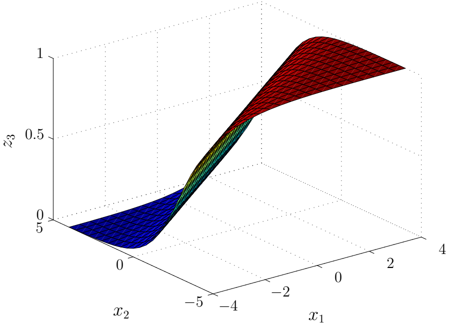

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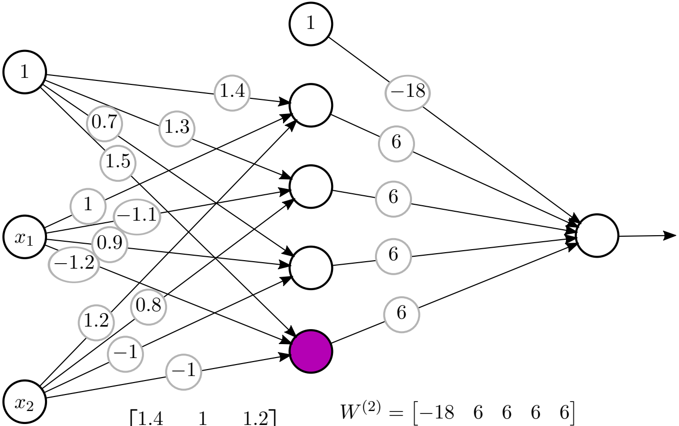

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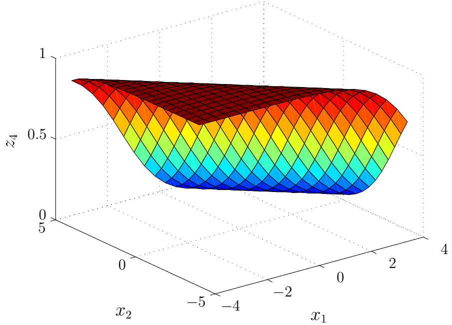

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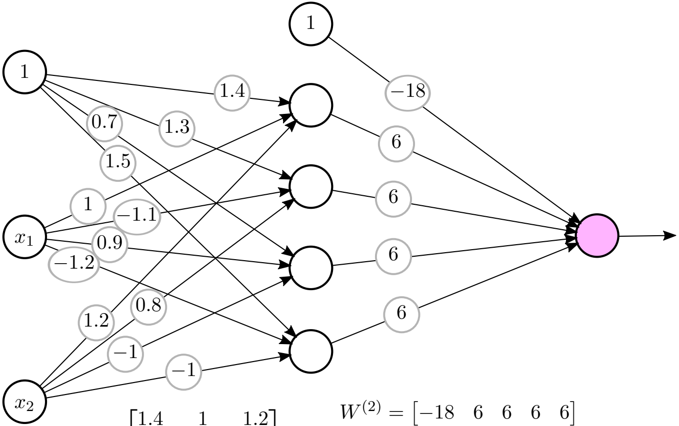

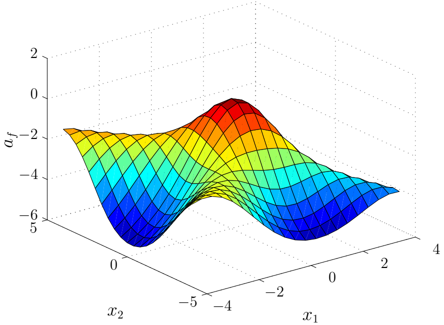

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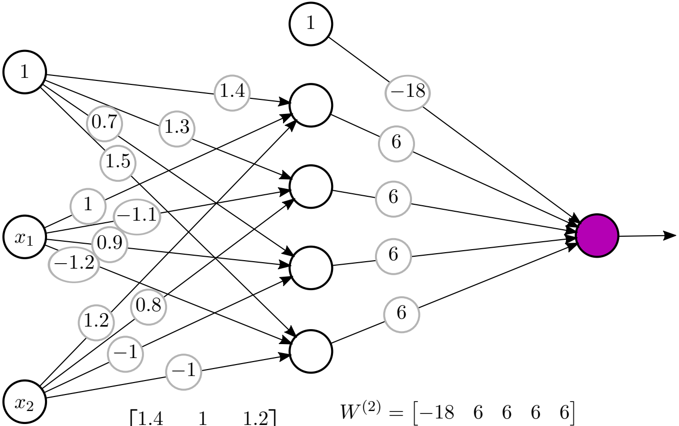

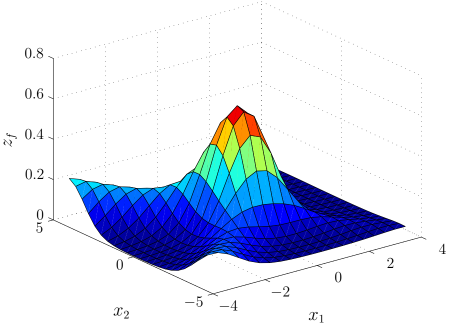

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- Usually err on side of too many

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Start (stochastic) gradient descent:

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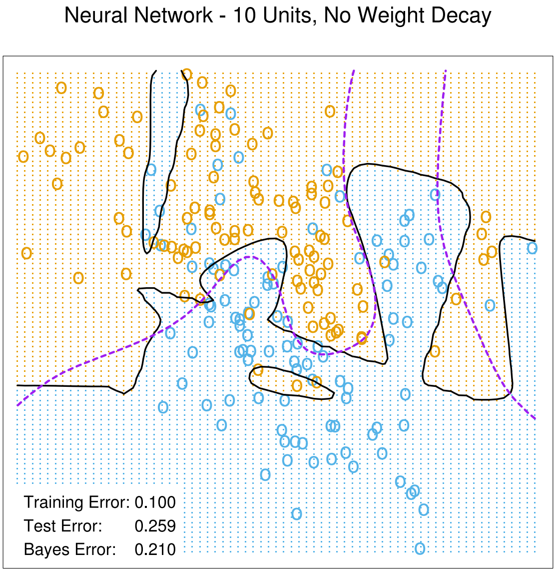
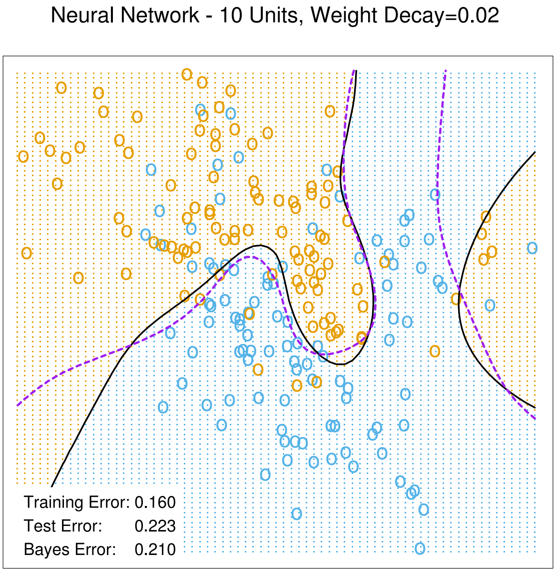
Same as adding $-\eta 2\lambda w_{ij}^{(k)}$ to update of $w_{ij}^{(k)}$

Called “weight decay”

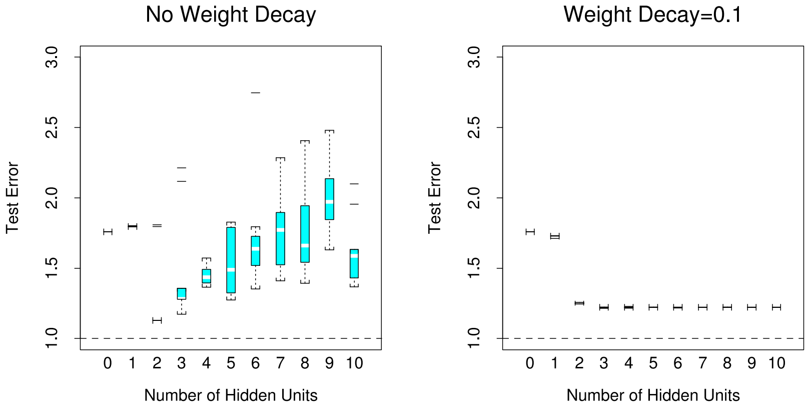
Example


Neural Network - 10 Units, No Weight Decay
Training Error: 0.100
Test Error: 0.259
Bayes Error: 0.210

Example


Neural Network - 10 Units, Weight Decay=0.02
Training Error: 0.160
Test Error: 0.223
Bayes Error: 0.210

Sum of 2 Sigmoids



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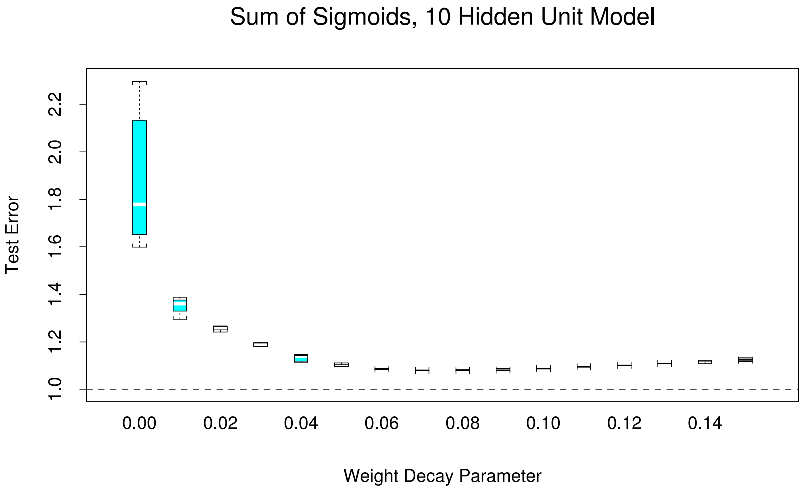

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Either batch (gradient descent) or online (stochastic gradient descent) used.

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Normalize data