

toward the low-energy part of Raman spectrum with the photon energy shift $\omega < \gamma_1/2$, which is determined by the excitation of the electron-hole pairs in the low-energy (degenerate) bands with $vp \ll \gamma_1$. At such low energies, the band structure as well as Landau level structure can be described by the effective two-band Hamiltonian written in the basis of orbitals on the sites $A1$ and $B2$,¹

$$\hat{H}_{\text{eff}} = -\frac{v^2}{\gamma_1} [(p_x^2 - p_y^2) \sigma_x + 2p_x p_y \sigma_y]. \quad (5)$$

To describe the excitation of the low-energy modes corresponding to the transitions between low-energy band states described by \hat{H}_{eff} , we take only the part of \mathcal{R} which acts in that two-dimensional Hilbert space, keep terms in the lowest relevant order in $vp/\gamma_1 \ll 1$ and $\gamma_1/\Omega \ll 1$, and write down an effective amplitude \mathcal{R}_{eff} ,

$$\mathcal{R}_{\text{eff}} \approx \frac{e^2 \hbar^2 v^2}{\epsilon_0 \Omega^2} \left\{ -i\sigma_z (\mathbf{1} \times \tilde{\mathbf{I}}^*)_z + \frac{\gamma_1}{\Omega} [\sigma_x d_y + \sigma_y d_x] \right\}. \quad (6)$$

We point out that the above matrix cannot be obtained within a theory constrained by the two-band approximation, Eq. (5), from the very beginning. Seemingly, one may try to define a contact-interaction-like term due to the terms quadratic in the electron momentum \mathbf{p} in Eq. (5), which carries a prefactor $\frac{e^2 \hbar^2 v^2}{\epsilon_0 \gamma_1 \Omega}$, which may suggest a greater magnitude of scattering than prefactor $\frac{e^2 \hbar^2 v^2}{\epsilon_0 \Omega^2}$ above. However, the scattering amplitude obtained within this model can only be applied to photons with $\Omega < \gamma_1$, which is hardly relevant for Raman spectroscopy since the latter is usually performed with laser beams using $\Omega \sim 1.3 - 2.8 \text{ eV}$.¹⁵⁻²³

The angle-resolved probability of the Raman scattering, $w(\mathbf{q} \approx \mathbf{0})$, determined using Fermi's golden rule and with the help of Eq. (6), is

$$w = \frac{2}{c\pi\hbar^3} \int d\mathbf{p} |\langle f | \mathcal{R}_{\text{eff}} | i \rangle|^2 \times f_i (1 - f_f) \delta(\epsilon_i + \omega - \epsilon_f),$$

where f_i and f_f are filling factors of the initial and final electronic state, respectively, and the spin and valley degeneracies have already been taken into account. This gives³³

$$w \approx \frac{\gamma_1 e^4 \hbar v^2}{c \epsilon_0^2 \Omega^4} \left\{ \Xi_s + \frac{\gamma_1^2}{2\Omega^2} \Xi_o \right\} \theta(\omega - 2\mu); \quad (7)$$

$$\Xi_s = |\mathbf{1} \times \tilde{\mathbf{I}}^*|^2, \quad \Xi_o = 1 + (\mathbf{1} \times \mathbf{I}^*) \cdot (\tilde{\mathbf{I}} \times \tilde{\mathbf{I}}^*).$$

Above, the first term with polarization factor Ξ_s describes the contribution of photons scattered with the same circular polarization as the incoming beam. The second term, with polarization factor Ξ_o , represents the scattered photons with circular polarization opposite to the incoming beam.

In turn, the angle-integrated spectral density of Raman

scattering $g(\omega)$ is

$$g(\omega) = \iint \frac{d\tilde{\mathbf{q}} d\tilde{q}_z}{(2\pi\hbar)^3} w \delta(\tilde{\Omega} - c\sqrt{\tilde{\mathbf{q}}^2 + \tilde{q}_z^2}) \quad (8)$$

$$= 2 \left(\frac{e^2}{4\pi\epsilon_0 \hbar c} \frac{v}{c} \right)^2 \frac{\gamma_1}{\Omega^2} \left\{ 2\Xi_s + \frac{\gamma_1^2}{\Omega^2} \Xi_o \right\} \theta(\omega - 2\mu).$$

Here, the constant spectral density g as a function of ω reflects the parabolicity of the low-energy bands and thus, energy-independent density of states in the bilayer. This is different in monolayer graphene, where $g(\omega) \propto \omega$, reflecting the energy-dependent density of states of electron-hole pairs.²⁹ The characteristic of monolayer graphene crossed polarisation of in/out photons is retained in the case of the bilayer system. Experimentally, constant spectral density g in undoped bilayer graphene is impossible to distinguish from a homogeneous background. However, if the chemical potential μ is not at the neutrality point, then transitions with $\omega < 2\mu$ are essentially blocked. Although new processes, resulting in the creation of the intraband electron-hole pair excitations and very small ω , are possible for $\mu \neq 0$, their contribution carries additional prefactor $v/c \sim \frac{1}{300}$.²⁶ Explicit calculation performed for the monolayer graphene showed that the quantum efficiency of the intraband transitions was of the order of 10^{-15} .²⁹ In contrast, for chemical potential $\mu \sim 50 \text{ meV}$ (corresponding to additional carrier density $n_0 \sim 1.5 \times 10^{12} \text{ cm}^{-2}$), the lost quantum efficiency due to the blocked interband transitions is, according to Eq. (8), $\Delta I \sim 10^{-12}$.

III. INTER-LANDAU-LEVEL TRANSITIONS IN BILAYER GRAPHENE RAMAN

The quantization of electron states into Landau levels gives the Raman spectrum due to the electronic excitations a pronounced structure which can be used to detect their contribution experimentally. We only consider here low-energy Landau levels, as at high energies the Landau level broadening due to, for example, electron-phonon interaction, will smear out the LL spectrum. In strong magnetic fields, low-energy Landau levels are sufficiently described³⁴ by

$$\epsilon_{n\alpha} = \alpha \frac{2\hbar^2 v^2}{\gamma_1 \lambda_B^2} \sqrt{n(n-1)}; \quad (9)$$

$$\Psi_{n\alpha} = \begin{pmatrix} \psi_n \\ 0 \end{pmatrix}, \quad n = 0, 1; \quad \Psi_{n\alpha} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_n \\ \alpha \psi_{n-2} \end{pmatrix}, \quad n \geq 2;$$

where $\lambda_B = \sqrt{\hbar/eB}$ is the magnetic length, n is the Landau level index and $\alpha = +$ denotes the conduction and $\alpha = -$ the valence band. Also, ψ_n is the normalised n -th Landau level wavefunction. In a neutral bilayer, all LLs have additional fourfold degeneracy (two due to the electron spin and two due to the valley). Moreover, levels $n = 0$ and $n = 1$ are degenerate at $\epsilon = 0$ giving rise to an 8-fold degenerate LL. We can project our effective