In the analytical derivation of the coarse-grained Poisson operator, several approximations that are related to the difference in length scales (3.2) are used. The crossover in length scale from the more microscopic Cahn-Hilliard level to the more macroscopic Doi-Ohta level is performed through the smoothing function $\chi(\mathbf{r}_1-\mathbf{r}_2)$. While the details of the derivation for some of the elements of $L^{(2)}$ are given in the Appendix A, here we give their final expressions

$$
\begin{aligned}
L^{(2)}_{\rho\mathbf{g}}(\mathbf{r}_1,\mathbf{r}_2) &= \rho(\mathbf{r}_2)\,\frac{\partial\chi}{\partial\mathbf{r}_2}. && (3.14a)\\
L^{(2)}_{\mathbf{gg}}(\mathbf{r}_1,\mathbf{r}_2) &= \mathbf{g}\left(\mathbf{r}_2\right)\frac{\partial\chi}{\partial\mathbf{r}_2} - \frac{\partial\chi}{\partial\mathbf{r}_1}\mathbf{g}\left(\mathbf{r}_1\right), && (3.14b)\\
L^{(2)}_{\epsilon g}(\mathbf{r}_1,\mathbf{r}_2) &= \epsilon\left(\mathbf{r}_2\right)\frac{\partial\chi}{\partial\mathbf{r}_2} + p\left(\mathbf{r}_1\right)\frac{\partial\chi}{\partial\mathbf{r}_2}, && (3.14c)\\
L^{(2)}_{P\mathbf{g},\beta}(\mathbf{r}_1,\mathbf{r}_2) &= \frac{\partial}{\partial r_{2\alpha}}\left[\left(2p_{\alpha\beta}(\mathbf{r}_2) - \frac{1}{3}P(\mathbf{r}_2)\delta_{\alpha\beta}\right)\chi\right] + P(\mathbf{r}_2)\frac{\partial\chi}{\partial r_{2\beta}}, && (3.14d)\\
L^{(2)}_{\mathbf{pg},\alpha\beta\gamma}(\mathbf{r}_1,\mathbf{r}_2) &= \frac{\partial}{\partial r_{2\alpha}}\left[\left(p_{\beta\gamma}(\mathbf{r}_2) + \frac{1}{3}P(\mathbf{r}_2)\delta_{\beta\gamma}\right)\chi\right]\\
&\quad + \frac{\partial}{\partial r_{2\beta}}\left[\left(p_{\alpha\gamma}(\mathbf{r}_2) + \frac{1}{3}P(\mathbf{r}_2)\delta_{\alpha\gamma}\right)\chi\right]\\
&\quad - \frac{2}{3}\delta_{\alpha\beta}\frac{\partial}{\partial r_{2\nu}}\left[\left(p_{\nu\gamma}(\mathbf{r}_2) + \frac{1}{3}P(\mathbf{r}_2)\delta_{\nu\gamma}\right)\chi\right]\\
&\quad - \left(\frac{\partial}{\partial r_{2\gamma}}p_{\alpha\beta}(\mathbf{r}_2)\right)\chi, && (3.14e)\\
L^{(2)}_{l\mathbf{g},\beta}(\mathbf{r}_1,\mathbf{r}_2) &= -\frac{l(\mathbf{r}_1)}{P(\mathbf{r}_1)}p_{\alpha\beta}(\mathbf{r}_1)\frac{\partial\chi}{\partial r_{2\alpha}} - \frac{1}{3}l(\mathbf{r}_1)\frac{\partial\chi}{\partial r_{2\beta}} - \frac{\partial l(\mathbf{r}_1)}{\partial r_{1\beta}}\chi. && (3.14f)
\end{aligned}
$$

where $\chi=\chi(\mathbf{r}_1-\mathbf{r}_2)$. With this, we obtained all the elements of the coarse-grained Poisson operator $L^{(2)}$, since the elements $L^{(2)}_{\mathbf{g}P}$, $L^{(2)}_{\mathbf{gp}}$, and $L^{(2)}_{\mathbf{g}l}$ are obtained from the antisymmetry condition for $L^{(2)}$. We note the occurence of the smoothing function $\chi(\mathbf{r}_1-\mathbf{r}_2)$ in the Poisson operator $L^{(2)}$. Indeed, when looking at the elements of $L^{(2)}$ which involve only the hydrodynamic fields ρ, $\mathbf{g}$, and ϵ, we see that they differ from the appropriate elements of $L^{(1)}$ in (2.6) only in locality, i.e., instead of the Dirac delta function, we rather have a smoothing function $\chi(\mathbf{r}_1-\mathbf{r}_2)$. While the original Cahn-Hilliard model is local in space, the coarse-grained model, instead, takes into account the whole volume element $v(\mathbf{r}_1)$ of the smoothing function $\chi(\mathbf{r}_1-\mathbf{r}_2)$. By assumption of the length scales comparison (3.2), this smoothing function behaves simply as a delta function on the Doi-Ohta level due to the difference in length scales. Alternatively, the expressions for the elements of the Poisson matrix (3.14d)-(3.14f) could be also obtained based only on the mathematical character of