Putting these three equations together, we finally obtain

$$\begin{aligned}
\mathcal{P}\left(\mathcal{L}^{(1)}\left[\mathcal{L}^{(0)}\right]^{-1}\mathcal{L}^{(1)}(\rho_{\rm ss}^{(0)})\right) &= \sum_{\mathbf{p},\alpha,\beta}\frac{\Omega^2}{2}\left(\mathrm{i}\eta_\beta\,\hat{k}_{\mathrm{L}\beta}-\frac{d_\beta(\mathbf{k}_\mathrm{L}+\mathbf{p})-d_\beta^*(\mathbf{k}_\mathrm{L})}{c^*(\mathbf{k}_\mathrm{L})}\right)\left(\mathrm{i}\eta_\alpha\,\hat{k}_{\mathrm{L}\alpha}-\frac{d_\alpha(\mathbf{k}_\mathrm{L})-d_\alpha(\mathbf{p}+\mathbf{k}_\mathrm{L})}{c(\mathbf{k}_\mathrm{L})}\right)\\
&\times\left(\frac{b_{\mathbf{p}\beta}^\dagger b_{\mathbf{p}\alpha}\rho_{\rm ss}^{(0)}-b_{\mathbf{p}\alpha}\,\rho_{\rm ss}^{(0)}\,b_{\mathbf{p}\beta}^\dagger}{-c(\mathbf{p}+\mathbf{k}_\mathrm{L})+2\mathrm{i}\nu_\alpha}+\frac{b_{-\mathbf{p}\beta}b_{-\mathbf{p}\alpha}^\dagger\,\rho_{\rm ss}^{(0)}-b_{-\mathbf{p}\alpha}^\dagger\,\rho_{\rm ss}^{(0)}\,b_{-\mathbf{p}\beta}}{-c(\mathbf{p}+\mathbf{k}_\mathrm{L})-2\mathrm{i}\nu_\alpha}\right)\\
&+\sum_{\alpha,\beta}\frac{\mathrm{i}N\Omega^4}{\nu_\alpha|c(\mathbf{k}_\mathrm{L})|^4}\left(\mathrm{Im}\,d_\beta(\mathbf{k}_\mathrm{L})-\eta_\beta\,\mathrm{Re}\,c(\mathbf{k}_\mathrm{L})\,\hat{k}_{\mathrm{L}\beta}\right)\left(\mathrm{Im}\,d_\alpha(\mathbf{k}_\mathrm{L})-\eta_\alpha\,\mathrm{Re}\,c(\mathbf{k}_\mathrm{L})\,\hat{k}_{\mathrm{L}\alpha}\right)\\
&\times\left(-b_{\mathbf{0}\alpha}\rho_{\rm ss}^{(0)}\,b_{\mathbf{0}\beta}^\dagger+b_{\mathbf{0}\beta}^\dagger b_{\mathbf{0}\alpha}\,\rho_{\rm ss}^{(0)}+b_{\mathbf{0}\alpha}^\dagger\,\rho_{\rm ss}^{(0)}\,b_{\mathbf{0}\beta}-b_{\mathbf{0}\beta}b_{\mathbf{0}\alpha}^\dagger\,\rho_{\rm ss}^{(0)}\right)+\mathrm{H.c.} \qquad \text{(A.29)}
\end{aligned}$$

Again, α and β are only independent in the isotropic case; in the fully anisotropic case we must have $\alpha=\beta$.

6. The heating and cooling rates of the different phonon modes

Substituting Eqs. (A.20) and (A.29) into Eq. (A.18), and collecting terms acting on the same phonon mode by relabelling $-\mathbf{p}\to\mathbf{p}$ where necessary, we obtain for the phonon steady state

$$\begin{aligned}
0 &= \frac{1}{2}\sum_{\mathbf{p},\alpha,\beta}\Big[M_{\mathrm{h}\alpha\beta}(\mathbf{p})(b_{\mathbf{p}\alpha}^\dagger\,\rho_{\rm ss}^{(0)}\,b_{\mathbf{p}\beta}-b_{\mathbf{p}\beta}b_{\mathbf{p}\alpha}^\dagger\,\rho_{\rm ss}^{(0)})+M_{\mathrm{h}\alpha\beta}^*(\mathbf{p})(b_{\mathbf{p}\beta}^\dagger\,\rho_{\rm ss}^{(0)}\,b_{\mathbf{p}\alpha}-\rho_{\rm ss}^{(0)}\,b_{\mathbf{p}\alpha}b_{\mathbf{p}\beta}^\dagger)\\
&+M_{\mathrm{c}\alpha\beta}(\mathbf{p})(b_{\mathbf{p}\alpha}\,\rho_{\rm ss}^{(0)}\,b_{\mathbf{p}\beta}^\dagger-b_{\mathbf{p}\beta}^\dagger b_{\mathbf{p}\alpha}\,\rho_{\rm ss}^{(0)})+M_{\mathrm{c}\alpha\beta}^*(\mathbf{p})(b_{\mathbf{p}\beta}\,\rho_{\rm ss}^{(0)}\,b_{\mathbf{p}\alpha}^\dagger-\rho_{\rm ss}^{(0)}\,b_{\mathbf{p}\alpha}^\dagger b_{\mathbf{p}\beta})\Big]\,. \qquad \text{(A.30)}
\end{aligned}$$

As this does not contain atomic operators, we can cancel the atomic part $|\tilde{0}\rangle\langle\tilde{0}|$, which replaces $\rho_{\rm ss}^{(0)}$ by $\rho_{\rm phn}^{(0)}$. It is also quadratic in phonon operators and diagonal in phonon wavevector.

For $\mathbf{p}\neq\mathbf{0}$, the coefficients $M_{\mathrm{h}\alpha\beta}(\mathbf{p})$ and $M_{\mathrm{c}\alpha\beta}(\mathbf{p})$ are given by

$$\begin{aligned}
M_{\mathrm{h}\alpha\beta}(\mathbf{p}) &= -\frac{\Omega^2}{c(\mathbf{k}_\mathrm{L}-\mathbf{p})+2\mathrm{i}\nu_\alpha}\left(\mathrm{i}\eta_\beta\,\hat{k}_{\mathrm{L}\beta}-\frac{d_\beta(\mathbf{k}_\mathrm{L}-\mathbf{p})-d_\beta^*(\mathbf{k}_\mathrm{L})}{c^*(\mathbf{k}_\mathrm{L})}\right)\left(\mathrm{i}\eta_\alpha\,\hat{k}_{\mathrm{L}\alpha}-\frac{d_\alpha(\mathbf{k}_\mathrm{L})-d_\alpha(\mathbf{k}_\mathrm{L}-\mathbf{p})}{c(\mathbf{k}_\mathrm{L})}\right)\\
&-\frac{\Omega^2}{|c(\mathbf{k}_\mathrm{L})|^2}[-\mathrm{i}\mathrm{Im}\,c(\mathbf{k}_\mathrm{L})\,\eta_\alpha\eta_\beta\,\hat{k}_{\mathrm{L}\alpha}\hat{k}_{\mathrm{L}\beta}-2\mathrm{i}\mathrm{Im}\,e_{\alpha\beta}(\mathbf{k}_\mathrm{L})+2e_{\alpha\beta}(\mathbf{k}_\mathrm{L}-\mathbf{p})],\\
M_{\mathrm{c}\alpha\beta}(\mathbf{p}) &= -\frac{\Omega^2}{c(\mathbf{k}_\mathrm{L}+\mathbf{p})-2\mathrm{i}\nu_\alpha}\left(\mathrm{i}\eta_\beta\,\hat{k}_{\mathrm{L}\beta}-\frac{d_\beta(\mathbf{k}_\mathrm{L}+\mathbf{p})-d_\beta^*(\mathbf{k}_\mathrm{L})}{c^*(\mathbf{k}_\mathrm{L})}\right)\left(\mathrm{i}\eta_\alpha\,\hat{k}_{\mathrm{L}\alpha}-\frac{d_\alpha(\mathbf{k}_\mathrm{L})-d_\alpha(\mathbf{k}_\mathrm{L}+\mathbf{p})}{c(\mathbf{k}_\mathrm{L})}\right)\\
&-\frac{\Omega^2}{|c(\mathbf{k}_\mathrm{L})|^2}[-\mathrm{i}\mathrm{Im}\,c(\mathbf{k}_\mathrm{L})\,\eta_\alpha\eta_\beta\,\hat{k}_{\mathrm{L}\alpha}\hat{k}_{\mathrm{L}\beta}-2\mathrm{i}\mathrm{Im}\,e_{\alpha\beta}(\mathbf{k}_\mathrm{L})+2e_{\alpha\beta}(\mathbf{k}_\mathrm{L}+\mathbf{p})]\,. \qquad \text{(A.31)}
\end{aligned}$$

For $\mathbf{p}=\mathbf{0}$, there appear to be additional terms

$$\begin{aligned}
M_{\mathrm{h}\alpha\beta}(\mathbf{0}) &= \frac{\eta_\alpha\,\hat{k}_{\mathrm{L}\alpha}\Omega^2}{c(\mathbf{k}_\mathrm{L})+2\mathrm{i}\nu_\alpha}\left(\eta_\beta\,\hat{k}_{\mathrm{L}\beta}-\frac{2\mathrm{Im}\,d_\beta(\mathbf{k}_\mathrm{L})}{c^*(\mathbf{k}_\mathrm{L})}\right)-\frac{\Omega^2}{|c(\mathbf{k}_\mathrm{L})|^2}[-\mathrm{i}\,\mathrm{Im}\,c(\mathbf{k}_\mathrm{L})\,\eta_\alpha\eta_\beta\,\hat{k}_{\mathrm{L}\alpha}\hat{k}_{\mathrm{L}\beta}+2\mathrm{Re}\,e_{\alpha\beta}(\mathbf{k}_\mathrm{L})]\\
&-\frac{\mathrm{i}N\Omega^4}{\nu_\alpha|c(\mathbf{k}_\mathrm{L})|^4}\left(\mathrm{Im}\,d_\beta(\mathbf{k}_\mathrm{L})-\eta_\beta\,\mathrm{Re}\,c(\mathbf{k}_\mathrm{L})\,\hat{k}_{\mathrm{L}\beta}\right)\left(\mathrm{Im}\,d_\alpha(\mathbf{k}_\mathrm{L})-\eta_\alpha\,\mathrm{Re}\,c(\mathbf{k}_\mathrm{L})\,\hat{k}_{\mathrm{L}\alpha}\right)\,,\\
M_{\mathrm{c}\alpha\beta}(\mathbf{0}) &= \frac{\eta_\alpha\,\hat{k}_{\mathrm{L}\alpha}\Omega^2}{c(\mathbf{k}_\mathrm{L})-2\mathrm{i}\nu_\alpha}\left(\eta_\beta\,\hat{k}_{\mathrm{L}\beta}-\frac{2\mathrm{Im}\,d_\beta(\mathbf{k}_\mathrm{L})}{c^*(\mathbf{k}_\mathrm{L})}\right)-\frac{\Omega^2}{|c(\mathbf{k}_\mathrm{L})|^2}[-\mathrm{i}\,\mathrm{Im}\,c(\mathbf{k}_\mathrm{L})\,\eta_\alpha\eta_\beta\,\hat{k}_{\mathrm{L}\alpha}\hat{k}_{\mathrm{L}\beta}+2\mathrm{Re}\,e_{\alpha\beta}(\mathbf{k}_\mathrm{L})]\\
&+\frac{\mathrm{i}N\Omega^4}{\nu_\alpha|c(\mathbf{k}_\mathrm{L})|^4}\left(\mathrm{Im}\,d_\beta(\mathbf{k}_\mathrm{L})-\eta_\beta\,\mathrm{Re}\,c(\mathbf{k}_\mathrm{L})\,\hat{k}_{\mathrm{L}\beta}\right)\left(\mathrm{Im}\,d_\alpha(\mathbf{k}_\mathrm{L})-\eta_\alpha\,\mathrm{Re}\,c(\mathbf{k}_\mathrm{L})\,\hat{k}_{\mathrm{L}\alpha}\right)\,. \qquad \text{(A.32)}
\end{aligned}$$

However, these extra terms (the last line of each equation) are pure imaginary, symmetric in α,β and of opposite signs in $M_{\mathrm{h}\alpha\beta}$ and $M_{\mathrm{c}\alpha\beta}$. As $b_{\mathbf{p}\alpha}b_{\mathbf{p}\beta}^\dagger-b_{\mathbf{p}\beta}^\dagger b_{\mathbf{p}\alpha}=\delta_{\alpha\beta}$ is a number so commutes with ρ, such terms cancel out in Eq. (A.30), so Eq. (A.31) can also be used for $\mathbf{p}=\mathbf{0}$.