A. The Hamiltonian

Let us consider the non-Hermitian FFA model defined by the following relations for the energy dispersion $E(k)$ and spectral coupling $v(k)$

$$E(k) = -2\kappa_0 \cos k \ , \ v(k) = -\sqrt{\frac{2}{\pi}}\kappa_a \sin k \tag{47}$$

where κ_0, κ_a are two real-valued positive constants and $0 \leq k \leq \pi$. The Hermitian limit of this model, attained by assuming $\mathrm{Im}(E_a) = 0$, is a special case of the FFA model previously investigated in Ref.[36], which is exactly solvable (see also [29]). Note that the continuous spectrum of H spans the band (E_1, E_2), with $E_2 = -E_1 = 2\kappa_0$. The density of states for this model is given by

$$\rho(E) = \left(\frac{\partial E}{\partial k}\right)^{-1} = \begin{cases} \frac{1}{\sqrt{4\kappa_0^2 - E^2}} & -2\kappa_0 < E < 2\kappa_0 \\ 0 & |E| > 2\kappa_0 \end{cases} \tag{48}$$

which shows van-Hove singularities at the band edges, whereas the positive spectral function $V(E)$, defined by $V(E) = \rho(E)|v(E)|^2$, reads

$$V(E) = \begin{cases} \frac{\kappa_a^2}{\pi\kappa_0}\sqrt{1-\left(\frac{E}{2\kappa_0}\right)^2} & -2\kappa_0 < E < 2\kappa_0 \\ 0 & |E| > 2\kappa_0 \end{cases} \tag{49}$$

which is non-singular. Substitution of Eq.(49) into Eq.(11) yields the following expression for the self-energy $\Sigma(z)$ [46]

$$\Sigma(z) = -i\frac{\kappa_a^2}{2\kappa_0^2}\left(\sqrt{4\kappa_0^2 - z^2} + iz\right) \tag{50}$$

and thus [see Eq.(12)]

$$\Delta(\mathcal{E}) = \mathrm{Re}\left(\Sigma(z = \mathcal{E} \pm i0^+)\right) = \tag{51}$$

$$= \begin{cases} \frac{\kappa_a^2}{2\kappa_0^2}\left(\mathcal{E} + \sqrt{\mathcal{E}^2 - 4\kappa_0^2}\right) & \mathcal{E} < -2\kappa_0 \\ \frac{\kappa_a^2}{2\kappa_0^2}\mathcal{E} & -2\kappa_0 \leq \mathcal{E} \leq 2\kappa_0 \\ \frac{\kappa_a^2}{2\kappa_0^2}\left(\mathcal{E} - \sqrt{\mathcal{E}^2 - 4\kappa_0^2}\right) & \mathcal{E} > 2\kappa_0 \end{cases}$$

The condition for the non-Hermitian Hamiltonian to possess a real-valued spectrum (i.e. to avoid complex-valued energies arising from bound states outside the continuum) is derived in Appendix B. Precisely, let $\xi_{1,2}$ be the two roots of the second-order algebraic equation

$$\xi^2 + \frac{E_a}{\kappa_0}\xi + 1 - (\kappa_a/\kappa_0)^2 = 0. \tag{52}$$

Then the Hamiltonian H has a real-valued energy spectrum if and only if $|\xi_{1,2}| \leq 1$. Figure 2 shows the domain in the plane $(\mathrm{Im}(E_a)/\kappa_0, \kappa_a/\kappa_0)$ where H has a purely continuous energy spectrum for a few increasing values of the ratio $|\mathrm{Re}(E_a)/\kappa_0|$. The domain lies in the sector

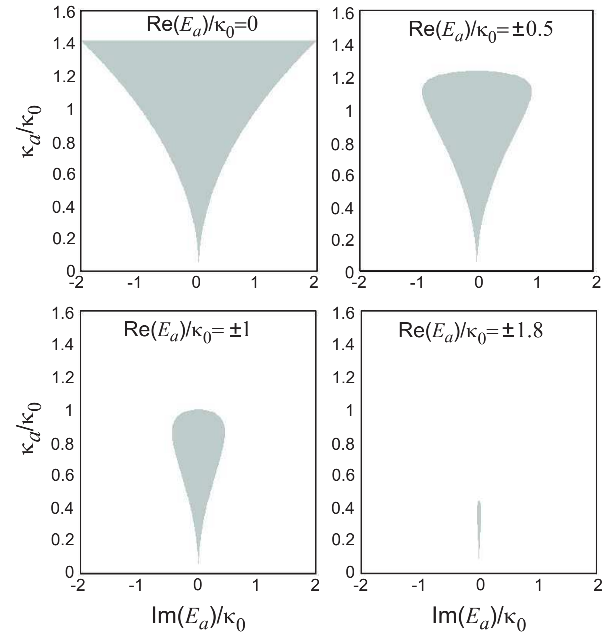

FIG. 2: Domains of non-existence of bound states for the Hamiltonian H in the $(\mathrm{Im}(E_a)/\kappa_0, \kappa_a/\kappa_0)$ plane (shaded regions) for increasing values of the ratio $|\mathrm{Re}(E_a)|/\kappa_0$. For a non-Hermitian Hamiltonian, i.e. $\mathrm{Im}(E_a) \neq 0$, in the shaded regions the energy spectrum of H is real-valued and purely continuous. Spectral singularities occur at the boundary of the shaded regions.

$\kappa_a/\kappa_0 \leq \sqrt{2}$ and shrinks toward $\mathrm{Im}(E_a)/\kappa_0 = \kappa_a/\kappa_0 = 0$ as $|\mathrm{Re}(E_a)/\kappa_0| \to 2^-$. For $|\mathrm{Re}(E_a)/\kappa_0| \leq 2$, bound states do exist for any value of κ_a/κ_0 and $\mathrm{Im}(E_a)/\kappa_0$. The wider domain is attained for $\mathrm{Re}(E_a) = 0$. In particular, for $\mathrm{Re}(E_a) = 0$ and $\kappa_a/\kappa_0 = \sqrt{2}$, from Eq.(52) it follows that H has a real-valued energy spectrum provided that

$$-2\kappa_0 < \mathrm{Im}(E_a) < 2\kappa_0. \tag{53}$$

Let us now consider the occurrence of spectral singularities. According to Eqs.(19) and (20) and using Eqs.(49) and (51), a spectral singularity at energy $\mathcal{E} = \mathcal{E}_0$, inside the interval $(-2\kappa_0, 2\kappa_0)$, is found provided that the following two equations are simultaneously satisfied

$$\mathrm{Im}(E_a) = \pm\frac{\kappa_a^2}{\kappa_0}\sqrt{1-\left(\frac{\mathcal{E}_0}{2\kappa_0}\right)^2} \tag{54}$$

$$\mathrm{Re}(E_a) = \left(1 - \frac{\kappa_a^2}{2\kappa_0^2}\right)\mathcal{E}_0. \tag{55}$$

For arbitrarily given values of E_a, κ_a and κ_0, the above conditions are generally not satisfied [nowhere for $\mathcal{E}_0$ in the range $(-2\kappa_0, 2\kappa_0)$], i.e. the non-Hermitian FFA Hamiltonian is generally diagonalizable. Spectral singularities appear solely when a constraint among $\mathrm{Re}(E_a)/\kappa_0$, $\mathrm{Im}(E_a)/\kappa_0$ and κ_a/κ_0 is satisfied. Let us first assume κ_a/κ_0 strictly smaller that $\sqrt{2}$. In this case, a single spectral singularity, at the energy $\mathcal{E}_0 =$