Theorem 3.2 *Assume G is a semigroup, $I(X,\rho,R)$ is a generalized incidence ring, and $\Phi : \rho \to G$ is a homomorphism. Let S_a be given by equation 5 for each $a \in G$. Then $I(X,\rho,R) = \bigoplus_{a\in G} S_a$ is a G-graded ring if and only if $\operatorname{Im}\Phi$ is finite.*

$$S_a = \{f \in I(X,\rho,R) : f(r) \neq 0 \textit{ implies } \Phi(r) = a \textit{ for all } r \in \rho\} \tag{5}$$

Proof. It is easy to see S_a is an R-submodule for all $a \in G$ and $S_a \cap S_b = \{0\}$ if $b \in G$ and $b \neq a$. We show $S_a S_b \subseteq S_{ab}$ for all $a, b \in G$. Suppose $f \in S_a$, $g \in S_b$ and $(fg)(x,y) \neq 0$ for some $x, y \in X$ with $x\rho y$. By equation 1 there exists $z \in [x,y]$ such that $x\rho z$, $z\rho y$, and $f(x,z)\,g(z,y) \neq 0$. Thus $f(x,z) \neq 0$ and $g(z,y) \neq 0$ which implies $\Phi(x,z) = a$ and $\Phi(z,y) = b$. Moreover, $ab = \Phi(x,y)$ since (x,z,y) is a transitive triple and Φ is a homomorphism. This proves $fg \in S_{ab}$ as desired.

To complete the proof we show $\operatorname{Im}\Phi$ is finite if and only if $I(X,\rho,R) = \bigoplus_{a\in G} S_a$. First assume $\operatorname{Im}\Phi$ is a finite subset of G. Then there is a positive integer m and $a_1, \ldots, a_m \in G$ such that $\operatorname{Im}\Phi = \{a_1, \ldots, a_m\}$. We must prove an arbitrarily chosen $f \in S$ is a sum of finitely many homogeneous elements.

For each $i = 1, \ldots, m$ let $f_i \in I(X,\rho,R)$ be the the function satisfying equation 6 for all $(x,y) \in X$.

$$f_i(x,y) = \begin{cases} f(x,y) & \text{if } x\rho y \text{ in } X \text{ and } \Phi(x,y) = a_i \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

By construction $f_i \in S_{a_i}$ for all $i \leq m$. It is easy to prove f is the sum of finitely many homogeneous elements, $f_1, \ldots, f_m$, as desired.

To prove the other direction assume $S = \bigoplus_{a\in G} S_a$. We choose $h \in I(X,\rho,R)$ so that for all $x, y \in X$ we have $h(x,y) = 1$ if $x\rho y$ and otherwise $h(x,y) = 0$. If