in the absence of vaccination. For such systems

$$
\begin{aligned}
&s_{\rm ext}[\xi(t)] = s_{\rm ext}^{(0)} + s_{\rm ext}^{(1)}[\xi(t)], \qquad (14)\\
&s_{\rm ext}^{(1)}[\xi(t)] = \min_{t_0}\int_{-\infty}^{\infty} dt\chi(t-t_0)\xi(t),\\
&\chi(t) = -H^{(1)}\big(\mathbf{x}_{\rm opt}^{(0)}(t), \mathbf{p}_{\rm opt}^{(0)}(t)\big)\left|w^{(1)}(\mathbf{x}_A, \mathbf{r}_\xi)\right|^{-1}.
\end{aligned}
$$

The quantity $\chi(t)$ is called logarithmic susceptibility [13, 20, 34, 35]; it gives the change of the logarithm of the extinction rate, which is linear in the vaccination rate for moderately low vaccination rate.

The minimization over $t_0$ in Eq. (14) accounts for lifting the time-translational invariance of the optimal extinction paths mentioned earlier. For $\xi(t) \equiv 0$, extinction can occur at any time ($t_r \ll t \ll W^{-1}$) with rate $W_e$. Periodic vaccination synchronizes extinction events; it periodically modulates the extinction rate, and the modulation is exponentially strong for $N|s_{\rm ext}^{(1)}| \gg 1$ (see below). Formally, in a modulated system there is only one optimal extinction path per period, as explained in Sec. II, which is here reflected in the minimization over $t_0$. This optimal path minimizes the disease extinction barrier $\mathcal{Q} = Ns_{\rm ext}$ [19, 20, 34, 35]. Equation (14) is closely related to the Mel'nikov theorem for dynamical systems [20, 36].

The constraint for minimizing the action over $\xi(t)$ in Eq. (13) has a form of an integral over the vaccination period $T$. It is therefore convenient to write action $s_{\rm ext}^{(1)}$ also in the form of such an integral,

$$
\begin{aligned}
&s_{\rm ext}^{(1)}[\xi(t)] = \min_{t_0}\int_0^T dt\xi(t)\chi_T(t-t_0),\\
&\chi_T(t) = \sum_{n=-\infty}^{\infty}\chi(t+nT). \qquad (15)
\end{aligned}
$$

The function $\chi_T(t)$ is obtained by superimposing the parts of $\chi(t)$ which differ by $T$. By construction, $\chi_T(t)$ is periodic in time $t$.

#### 2. Temporal shape of optimal vaccination

To find the optimal shape of vaccination rate $\xi(t)$ we first minimize the time integral in the variational problem Eqs. (13) – (15) with respect to $\xi(t)$ for a given $t_0$. Since $\xi(t) \geq 0$, it is convenient to perform the minimization with respect to $\xi^{1/2}(t)$ rather than $\xi(t)$. The minimization shows that $\xi^{1/2}(t) \neq 0$ only for $t = t_\lambda$, where $t_\lambda$ is given by equation $\chi_T(t_\lambda - t_0) = -\lambda/T$. From the constraint on the period-averaged $\xi(t)$ we then have

$$\xi(t) = \Xi T\sum_n \delta(t - t_\lambda + nT). \qquad (16)$$

Substituting this expression into the functional $\tilde{s}_{\rm ext}$ and minimizing with respect to $t_0$, we obtain the action in a simple explicit form

$$
\begin{aligned}
&s_{\rm ext} = \min \tilde{s}_{\rm ext} = s_{\rm ext}^{(0)} + s_{\rm ext}^{(1)},\\
&s_{\rm ext}^{(1)} = \Xi T \min_{0\leq t<T}\chi_T(t). \qquad (17)
\end{aligned}
$$

Alternatively, this expression can be rewritten in terms of the Fourier transform of the logarithmic susceptibility:

$$
\begin{aligned}
&s_{\rm ext}^{(1)} = \Xi \min_t \sum\nolimits_n \tilde{\chi}(n\Omega)\exp[in\Omega t], \qquad (18)\\
&\tilde{\chi}(\omega) = \int_{-\infty}^{\infty} dt\chi(t)\exp(i\omega t),
\end{aligned}
$$

where $\Omega = 2\pi/T$ is the cyclic frequency of vaccination.

We are interested in the solution for which $s_{\rm ext}^{(1)}$ is negative, which requires $\min\chi_T(t) < 0$. Only in this case will vaccination reduce the barrier for disease extinction and increase the disease extinction rate. The barrier reduction due to vaccination, $\mathcal{Q}^{(1)} = Ns_{\rm ext}^{(1)} \propto N\Xi$, becomes large for $N \gg 1$ even if the average vaccination rate $\Xi$ is small. Thus, for not too small vaccination rates, where the eikonal approximation is valid [20, 34], the effect of vaccination on the disease extinction rate is exponentially strong.

The expression for the action change $s_{\rm ext}^{(1)}$, Eq. (17), can be also obtained in a more intuitive way. Indeed, since $\xi(t)$ is non-negative, it follows from Eq. (15) that

$$s_{\rm ext}^{(1)}[\xi(t)] \geq \min_t \chi_T(t)\int_0^T dt\xi(t) = \Xi T\min_t\chi_T(t). \qquad (19)$$

The minimum is provided by $\xi(t) = \Xi T\sum_n \delta(t - t_{min} + nT)$. Formally, $t_{min}$ is the instance of time where $\chi_T(t)$ is minimal. In fact, it is the optimal path that adjusts to the vaccination pulses so as to increase the probability of disease extinction. This provides the mechanism of synchronization by vaccination. Equation (19) immediately leads to Eqs. (16) and (17) with $t_\lambda$ replaced by $t_{min}$.

In addition to the constraint on the average vaccination rate, there may be an upper limit on the instantaneous vaccination rate, which is imposed by condition $w(\mathbf{x}, \mathbf{r}) = w^{(0)}(\mathbf{x}, \mathbf{r}_\xi) + \xi(t)w^{(1)}(\mathbf{x}, \mathbf{r}_\xi) \geq 0$. In the case where $w^{(1)}(\mathbf{x}, \mathbf{r}_\xi, t) < 0$, as for vaccination of newly arrived susceptibles, this condition is met provided $\xi_0(t) \leq \xi_{0m} \equiv \min\{w^{(0)}(\mathbf{x}, \mathbf{r}_\xi)/\left|w^{(1)}(\mathbf{x}, \mathbf{r}_\xi)\right|\}$. In this case the optimal vaccination protocol changes.

The new protocol can be found from the variational problem (13) by changing from $\xi(t) \equiv \xi_0(t)\left|w^{(1)}(\mathbf{x}_A, \mathbf{r}_\xi)\right|$ to an auxiliary function $\eta(t)$ such that $\xi_0(t) = \xi_{0m}[1 + \eta^2(t)]^{-1}$, and then finding the minimum of $\tilde{s}_{\rm ext}$ with respect to $\eta(t)$. This choice satisfies the constraints $0 \leq \xi_0(t) \leq \xi_{0m}$. Variation with respect to $\eta(t)$ shows that $\tilde{s}_{\rm ext}$ has an extremum for $\eta(t) = 0$ or $\eta(t) = \infty$ for $t \neq t_\lambda$, where $t_\lambda$ is given by equation $\chi_T(t_\lambda - t_0) = -\lambda/T$. The value of $\eta(t)$ at the isolated instances $t = t_\lambda$ is arbitrary. Only regions where $\eta(t) = 0$, so that $\xi_0(t) = \xi_{0m}$, contribute to $\tilde{s}_{\rm ext}$. Obviously, $\tilde{s}_{\rm ext}$ is minimal when $\xi_0(t) = \xi_{0m}$ for $|t - t_{min}| \leq \Delta t/2$, where $t_{min}$ is the