A. Thermal Equilibrium

In the long time limit, and in the presence of a binding potential field, e.g. harmonic field or particles in a box, an equilibrium is reached. Then initial conditions do not play a role. For example $\langle P_{LR}(x_T)\rangle = P_R^{eq}(x_T)$ with

$$P_R^{\rm eq}(x_T) = \frac{1}{Z}\int_{x_T}^{\overline{L}} \exp\left(-\frac{V(x)}{k_b T}\right) {\rm d}x \qquad (16)$$

where Z is the normalizing partition function

$$Z = \int_{-\overline{L}}^{\overline{L}} \exp\left[-\frac{V(x)}{k_b T}\right] {\rm d}x. \qquad (17)$$

Similarly

$$P_L^{\rm eq}(x_T) = \frac{1}{Z}\int_{-\overline{L}}^{x_T} \exp\left(-\frac{V(x)}{k_b T}\right) {\rm d}x. \qquad (18)$$

In Eqs. (16,18) we used the steady state solution, $\lim_{t\to\infty} g(x, x_0, t) = \exp[-V(x)/k_b T]/Z$, which is Boltzmann's distribution suited for a system in thermal equilibrium. Using Eq. (15) the position PDF of the tagged particle is $\lim_{t\to\infty} P(x_T) = P^{\rm eq}(x_T)$

$$P^{\rm eq}(x_T) \sim C \exp\left\{-\frac{N\left[\langle P_R^{eq}(x_T)\rangle - \langle P_L^{eq}(x_T)\rangle\right]^2}{4\langle P_L^{eq}(x_T)\rangle\langle P_R^{eq}(x_T)\rangle}\right\}. \qquad (19)$$

If the potential is symmetric $V(x) = V(-x)$, e.g. particles in a box or harmonic field, we have for not too large x_T, $P_L^{eq}(x_T) \simeq 1/2$, $P_R^{eq}(x_T) \simeq 1/2$ hence from Eq. (15)

$$P^{\rm eq}(x_T) \sim C \exp\left\{-N\left[\langle P_R^{eq}(x_T)\rangle - \langle P_L(x_T)\rangle\right]^2\right\}. \quad (20)$$

Expanding the expression in the exponent in x_T (since N is large) we find using Eqs. (16,18,19)

$$P^{\rm eq}(x_T) \sim \frac{2\sqrt{N}}{\sqrt{\pi}Z} \exp\left[-\frac{4N}{Z^2}(x_T)^2\right] \qquad (21)$$

where with out loss of generality we assigned $V(x=0) = 0$. Hence the standard deviation is

$$\langle (x_T)^2\rangle \sim \frac{Z^2}{8N}. \qquad (22)$$

The same expression is found in Appendix A using the many body Boltzmann distribution, and integrating over all the particles except the tagged particle.

B. Simple Illustration

We now consider the situation of particles free of a force $F(x) = 0$ with open boundary conditions $\overline{L} \to \infty$ where initially all the particles are on the vicinity of the origin. More precisely, the tagged particle is initially situated at $x_T = 0$, N particles to its right on $\epsilon \to 0^+$ and N particles on $-\epsilon$. This problem was solved already by Aslangul [26], and here we recover the known result using our formulas. The Green function $g(x, x_0, t)$ of a free particle is

$$g(x, x_0, t) = \frac{\exp\left[-\frac{(x-x_0)^2}{4Dt}\right]}{\sqrt{4\pi D t}} \qquad (23)$$

where as mentioned D is the diffusion coefficient of the free particle. With the specified initial conditions we have

$$\langle P_{RL}(x_T)\rangle = \lim_{\epsilon\to 0}\int_0^\infty \delta(x_0-\epsilon)\int_{-\infty}^{x_T} \frac{\exp\left[-\frac{(x-x_0)^2}{4Dt}\right]}{\sqrt{4\pi D t}} {\rm d}x{\rm d}x_0 =$$

$$\frac{1}{2} + \int_0^{x_T} \frac{e^{-\frac{x^2}{4Dt}}}{\sqrt{4\pi D t}} {\rm d}x. \qquad (24)$$

Similarly

$$\langle P_{LR}(x_t)\rangle = \frac{1}{2} - \int_0^{x_T} \frac{e^{-\frac{x^2}{4Dt}}}{\sqrt{4\pi D t}} {\rm d}x. \qquad (25)$$

When $x_T \ll \sqrt{2Dt}$ we Taylor expand in x_T to find $(\langle P_{LR}\rangle - \langle P_{RL}\rangle)^2 \sim (x_T)^2/\pi D t$, using Eq. (15) we recover the result in [26]

$$P(x_T) \sim \frac{1}{\pi}\sqrt{\frac{N}{Dt}} \exp\left[-\frac{N(x_T)^2}{\pi D t}\right], \qquad (26)$$

hence

$$\langle (x_T)^2\rangle \sim \frac{\pi D t}{2N}. \qquad (27)$$

The diffusion is normal, in the sense that the mean square displacement increases linearly in time. However the diffusion of the tagged particle is slowed down compared with a free particle, by a factor of $1/N$ which is due to the collisions with all other Brownian particles in the system. Clearly the approximation breaks down if one is interested in the tails of $P(x_T)$, since we used $x_T \ll \sqrt{2Dt}$. Though clearly when N is large, the probability of finding such a particle is extremely small (i.e. use Eq. (26) $P(x_T = \sqrt{2Dt}) \sim \exp(-N2/\pi)$).

C. Formula for $\langle (x_T)^2\rangle$

We now consider symmetric potential fields $V(x) = V(-x)$, and symmetric initial conditions. The latter means that the density of particles at time $t = 0$ to the left of the tagged particle, i.e. those residing in $x_0 < 0$, is the same as for those residing to the right, $f_R(x_0) = f_L(-x_0)$ (e.g. uniform initial conditions). In this case the subscript R and L is redundant and we use $f(x_0) = f_R(x_0) = f_L(-x_0)$, where $f(x_0) = 0$ if