Appendix A: Matrix elements of the resolvent

In this Appendix we derive the expressions of the matrix elements $\mathcal{G}_{a,a}(z) = \langle a|G(z)a\rangle$, $\mathcal{G}_{k,a}(z) = \langle k|G(z)a\rangle$, $\mathcal{G}_{a,k}(z) = \langle a|G(z)k\rangle$ and $\mathcal{G}_{k,k'}(z) = \langle k|G(z)k'\rangle$ of the resolvent [Eqs.(6-9) given in the text].
From the identity $G(z)(z-H_0-V) = (z-H_0-V)G(z) = \mathcal{I}$ it follows that

$$\begin{aligned}
\langle a|G(z)(z-H_0)a\rangle - \langle a|G(z)Va\rangle &= 1 && \text{(A1)}\\
\langle a|G(z)(z-H_0)k\rangle - \langle a|G(z)Vk\rangle &= 0 && \text{(A2)}\\
\langle k|(z-H_0)G(z)a\rangle - \langle k|VG(z)a\rangle &= 0 && \text{(A3)}\\
\langle k|G(z)(z-H_0)k'\rangle - \langle k|G(z)Vk'\rangle &= \delta(k-k') && \text{(A4)}
\end{aligned}$$

Taking into account that

$$V|a\rangle = \int dk v^*(k)|k\rangle \ , \ V|k\rangle = v(k)|a\rangle \tag{A5}$$

and that $(z-H_0)|a\rangle = (z-E_a)|a\rangle$, $(z-H_0)|k\rangle = (z-E(k))|k\rangle$, Eqs.(A1) and (A2) take the form

$$(z-E_a)\mathcal{G}_{a,a}(z) - \int dk v^*(k)\mathcal{G}_{a,k} = 1 \tag{A6}$$

$$(z-E(k))\mathcal{G}_{a,k}(z) - v(k)\mathcal{G}_{a,a}(z) = 0 \tag{A7}$$

which can be solved for $\mathcal{G}_{a,a}$ and $\mathcal{G}_{a,k}$, yielding Eqs.(6) and (7) given in the text. To calculate $\mathcal{G}_{k,a}(z)$, we use Eq.(A3) and note that $\langle k|(z-H_0)G(z)a\rangle = \langle (z^*-H_0^\dagger)k|G(z)a\rangle = (z-E(k))\mathcal{G}_{k,a}(z)$ and $\langle k|VG(z)a\rangle = \langle Vk|G(z)a\rangle = v^*(k)\mathcal{G}_{a,a}(z)$. This yields $(z-E(k))\mathcal{G}_{k,a}(z) - v^*(k)\mathcal{G}_{a,a}(z) = 0$, which can be solved for $\mathcal{G}_{k,a}(z)$, yielding Eq.(8) given in the text. Finally, the matrix element $\mathcal{G}_{k,k'}(z)$ is obtained from Eq.(A4), which can be written in the form $(z-E(k'))\mathcal{G}_{k,k'}(z) - v(k')\mathcal{G}_{k,a}(z) = \delta(k-k')$, i.e.

$$\mathcal{G}_{k,k'}(z) = \frac{v(k')\mathcal{G}_{k,a}(z)}{z-E(k')} + \frac{\delta(k-k')}{z-E(k')}. \tag{A8}$$

Substitution of Eq.(8) into Eq.(A8) finally yields Eq.(9) given in the text.

Appendix B: Conditions for a real-valued energy spectrum of the non-Hermitian Hamiltonian

In this Appendix we derive the necessary and sufficient conditions that ensure a real-valued energy spectrum for the non-Hermitian FFA Hamiltonian H introduced in Sec.III.A. As shown in Sec.II.B, this condition is equivalent to the vanishing of the point-spectrum of H, i.e. to the absence of bound states. The detailed calculations can be performed following two different, though equivalent, approaches. The first one starts from the representation of H in the $\{|a\rangle, |k\rangle\}$ basis (the Bloch basis), whereas the second approach uses a different decomposition of H, namely on the $\{|a\rangle, |n\rangle\}$ basis, where $|n\rangle$ are the Wannier states introduced in Sec.III.B (the Wannier basis). For the sake of completeness, we present the detailed calculations for both approaches.

1. Bloch-basis representation of H. As shown in Sec.II.B, the absence of bound states of H requires that Eq.(18) does not admit of any solution in the complex z plane. Using the expression (50) of the self-energy $\Sigma(z)$, Eq.(18) takes the form

$$\left(1 - \frac{\kappa_a^2}{2\kappa_0^2}\right) z - E_a = -i\frac{\kappa_a^2}{2\kappa_0^2}\sqrt{4\kappa_0^2 - z^2}. \tag{B1}$$

We can solve Eq.(B1) by introducing, in place of z, the new complex-valued variable μ defined by

$$z = -\kappa_0[\exp(\mu) + \exp(-\mu)] = -2\kappa_0 \cosh\mu. \tag{B2}$$

Without loss of generality, we may assume $\mathrm{Re}(\mu) > 0$. In fact, the function $z(\mu)$ defined by Eq.B(2) is invariant for the inversion $\mu \to -\mu$, so that we may restrict our analysis to the case $\mathrm{Re}(\mu) > 0$. With such a substitution, the square root on the right hand side of Eq.(B1) can be solved analytically, yielding $\pm 2i\kappa_0 \sinh\mu$. Some care should be taken when choosing the right determination (i.e. sign) of the square root [46]. For $\mathrm{Re}(\mu) > 0$, one obtains

$$2\left(1 - \frac{\kappa_a^2}{2\kappa_0^2}\right)\cosh\mu + \frac{E_a}{\kappa_0} = -\frac{\kappa_a^2}{\kappa_0^2}\sinh\mu. \tag{B3}$$

After setting $\xi = \exp(\mu)$, from Eq.(B3) one obtains Eq.(52) given in the text once $\cosh\mu$ and $\sinh\mu$ are expressed in terms of the exponentials $\exp(\pm\mu) = \xi^{\pm 1}$. Therefore, if the two roots $\xi_{1,2}$ of Eq.(52) satisfy the condition $|\xi_{1,2}| \leq 1$, Eq.(B3) does not have roots with $\mathrm{Re}(\mu) > 0$, and hence H does not have bound states.

2. Wannier-basis representation of H. In this approach, we use the tight-binding representation of the Hamiltonian H using the Wannier function basis [Eq.(63)]. Bound states of H correspond in this case to surface states localized near the edge of the truncated lattice of Fig.3. They can be directly determined by looking for a solution to Eqs.(65-67) of the form

$$c_n(t) = \exp[-\mu(n-1) - iEt], \ c_a(t) = A\exp(-iEt) \tag{B4}$$

($n \geq 1$), where E is the energy of the surface state. The constants μ and A, as well as the dependence of E on μ, are readily determined by substituting Eq.(B4) into Eqs.(65-67). One obtains

$$\begin{aligned}
E &= -2\kappa_0 \cosh\mu && \text{(B5)}\\
E &= -\kappa_0 \exp(-\mu) - \kappa_a A && \text{(B6)}\\
EA &= -\kappa_a + E_a A. && \text{(B7)}
\end{aligned}$$

from which the following second-order algebraic equations for $\xi = \exp(\mu)$ is readily obtained

$$\xi^2 + \frac{E_a}{\kappa_0}\xi + 1 - \frac{\kappa_a^2}{\kappa_0^2} = 0 \tag{B8}$$