C. Influence of detuning and pure dephasing

In Subsec. A, we have evidenced that a necessary condition for lasing is to achieve the good cavity regime. In the previous Sections we have seen that pure dephasing strongly influences the effective atom-cavity coupling rate. As a consequence, one expects that it should affect the lasing conditions of the system as well. When considering the resonant case of Fig. 4, we see that increasing γ^* lowers the effective coupling $R(\delta, \gamma^*)$, up to the point where the lasing criteria are completely lost (e.g., curves for $\gamma^* = 40g$ in Figs. 4a and b). In particular, the clamping of the autocorrelation function to the Poissonian value $g^2(0) = 1$ disappears, and the emitted field continuously evolves from antibunched to thermal without showing any coherent character. Loss of the lasing criteria appears for a value of the pure dephasing rate $\gamma^* \sim 20g$. As evidenced in Fig.4c, this corresponds to the transition from the good to the bad cavity regime, confirming our previous intuition. By the very same mechanism, lasing can also be lost by increasing the atom-cavity detuning, as it appears by plotting the same quantities in Fig. 5. Starting from $\delta = 0$, the switching from good to bad cavity regime happens for $\delta \sim 2g$, which again yields the disappearance of any lasing signature.

On the other hand, we have seen in Sec. II that if the atom and the cavity are detuned, increasing pure dephasing can even increase the effective coupling between the two systems. This induces a transition from the bad cavity to the good cavity regime, and it allows to recover the lasing conditions. This result is shown in Fig. 6c for a typical pure dephasing rate $\gamma^* \sim 2g$; in particular, one recovers a clear clamping of the autocorrelation function to $g^2(0) \sim 1$, as in Fig. 4b for $\delta = 0$ and $\gamma^* = 0$. In other words, under such conditions it turns out that pure dephasing compensates for atom-cavity detuning. This effect, which in the previous Section was responsible for an improvement of the single photon source figure of merit (for system parameters in the bad cavity regime), leads here to a recovery of the lasing signatures. Again, pure dephasing appears as a valuable resource for solid-state nanophotonic devices.

Finally, we discuss now the interest of pure dephasing in the frame of conventional lasers. If one is just interested in efficient energy conversion, lasers involving a high number of emitters are naturally to be preferred over the single emitter device, as they are less subject to saturation and quenching. Nevertheless, the criterium of high β is challenging to realize