F_i, $i \in \{1,2\}$. We show that if the distribution of Y differs across categories, then so does the distribution of the distance of Y from almost every point z. Therefore, any univariate consistent two-sample test on the distances from z results in a consistent test of the equality of the multivariate distributions F_1 and F_2, for almost every z. It is straightforward to generalize these results to $K > 2$ categories.

Theorem 2.1. *If $H_0 : F_1 = F_2$ is false, then for every $z \in \Re^q$, apart from at most a set of Lebesgue measure 0, there exists an $r > 0$ such that $F'_{1z}(r) \neq F'_{2z}(r)$.*

Proof. Suppose by contradiction, that there is a set $\Gamma \subseteq \Re^q$ with positive Lebesgue measure, such that for all $z \in \Gamma$, $F'_{1z}(r) = F'_{2z}(r)$ for all $r > 0$. It follows that $\int_{y \in B_q(z,r)} dF_1(y) - \int_{y \in B_q(z,r)} dF_2(y) = 0$ for all $r > 0$ and $z \in \Gamma$. Since $|F_1 - F_2| \leq 1$, clearly $F_1 - F_2$ is of at most exponential-quadratic growth. Moreover, the only real analytic function that vanishes on Γ is the zero function, since Γ has positive Lebesgue measure. Therefore, it follows from Proposition 2.1 that $F_1 - F_2 = 0$, thus contradicting the fact that H_0 is false. $\square$

Corollary 2.1. *For every $z \in \Re^q$, apart from at most a set of Lebesgue measure 0, a consistent two-sample univariate test of the null hypothesis $H'_0 : F'_{1z} = F'_{2z}$ will reject $H_0 : F_1 = F_2$ with a power increasing to one as the sample size increases.*

Proof. If $H_0 : F_1 = F_2$ is false, then Theorem 2.1 guarantees that for every z, apart from at most a set of Lebesgue measure 0, the null univariate hypothesis, $H'_0 : F'_{1z} = F'_{2z}$, is false. Since for such a z the asymptotic power of a false null univariate hypothesis will be one for any consistent two-sample univariate test, the power of the multivariate test will be one. $\square$

For the multivariate test of independence, let $X \in R^p$ and $Y \in \Re^q$ be two random vectors with marginal distributions F_X and F_Y, respectively, and with joint distribution F_{XY}. For $z = (z_x, z_y), z_x \in \Re^p, z_y \in \Re^q$, let F'_{XYz} be the joint distribution of