$$\frac{a^2\delta Q_d^{0I}}{\rho_d} \approx -3\mathcal{H}(\xi_1\Delta_m r + \xi_2\Delta_d),$$

where $r = \rho_m/\rho_d$.

It is useful to rewrite these equations in the real space,

$$\begin{aligned}
&\Delta_m' + \nabla_{\bar{x}} \cdot V_m = 3\mathcal{H}\xi_2(\Delta_d - \Delta_m)/r \quad , \\
&V_m' + \mathcal{H}V_m = -\nabla_{\bar{x}}\Psi - 3\mathcal{H}(\xi_1 + \xi_2/r)V_m \quad ;
\end{aligned} \tag{5}$$

$$\begin{aligned}
&\Delta_d' + (1+w)\nabla_{\bar{x}} \cdot V_d = 3\mathcal{H}(w - C_e^2)\Delta_d + 3\mathcal{H}\xi_1 r(\Delta_d - \Delta_m) \quad , \\
&V_d' + \mathcal{H}V_d = -\nabla_{\bar{x}}\Psi - \frac{C_e^2}{1+w}\nabla_{\bar{x}}\Delta_d - \frac{w'}{1+w}V_d + 3\mathcal{H}\left\{(w - C_a^2) + \frac{1+w-C_a^2}{1+w}(\xi_1 r + \xi_2)\right\}Vd \quad ;
\end{aligned} \tag{6}$$

where $\bar{x}$ refers to the conformal coordinates.

Defining $\sigma_m = \delta\rho_m$, $\sigma_d = \delta\rho_d$, and assuming that the EOS of DE is constant $w' = 0$, we can change Eqs.(5, 6) into,

$$\begin{aligned}
&\dot{\sigma}_m + 3H\sigma_m + \nabla_x(\rho_m V_m) = 3H(\xi_1\sigma_m + \xi_2\sigma_d), \\
&\frac{\partial}{\partial t}(aV_m) = -\nabla_x(a\Psi) - 3H(\xi_1 + \xi_2/r)(aV_m);
\end{aligned} \tag{7}$$

$$\begin{aligned}
&\dot{\sigma}_d + 3H(1 + C_e^2)\sigma_d + (1+w)\nabla_x(\rho_d V_d) = -3H(\xi_1\sigma_m + \xi_2\sigma_d), \\
&\frac{\partial}{\partial t}(aV_d) = -\nabla_x(a\Psi) - \frac{C_e^2}{1+w}\nabla_x \cdot (a\Delta_d) + 3H\left[(w - c_a^2) + \frac{1+w-c_a^2}{1+w}(\xi_1 r + \xi_2)\right](aV_d);
\end{aligned} \tag{8}$$

where $\nabla_x = \frac{1}{a}\nabla_{\bar{x}}$. The dot denotes the derivative with respect to the cosmic time. Ψ indicates the peculiar potential, which can be decomposed into $\Psi = \psi_m + \psi_d$, satisfying the Poisson equation [26],

$$\nabla_\lambda^2\psi_\lambda = 4\pi G(1 + 3w_\lambda)\sigma_\lambda, \tag{9}$$

where σ_λ represents the inhomogeneous fluctuation field and the subscript "λ" denotes DM or DE, respectively. We have included the correction from General Relativity. In a homogeneous and isotropic background $< \psi_\lambda >= 0$, since $< \sigma_\lambda >= 0$. For DE and DM, their peculiar potentials read [26]

$$\psi_m = -4\pi G \int dV' \frac{\sigma_m}{\mid x - x' \mid}, \tag{10}$$

$$\psi_d = -4\pi G \int dV' \frac{(1+3w)\sigma_d}{\mid x - x' \mid}. \tag{11}$$

III. DERIVATION OF THE LAYZER-IRVINE EQUATION

In this section we derive the Layer-Irvine equation [38] when there is an interaction between DE and DM. Layzer-Irvine equation describes how a collapsing system reaches a state of dynamical equilibrium in an expanding universe.