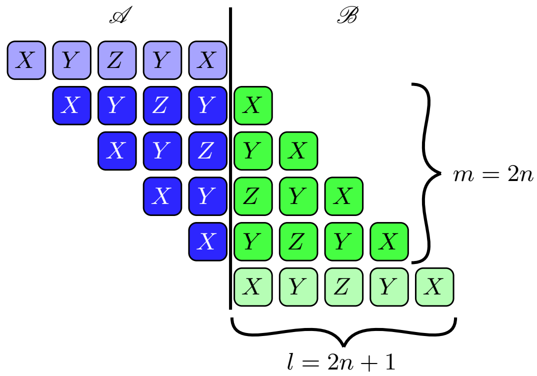

Figure 5: Cut stabilizer generators: it is apparent that for operators of length $l = 2n+1$, $2n$ of them have support on both $\mathscr{A}$ and $\mathscr{B}$.

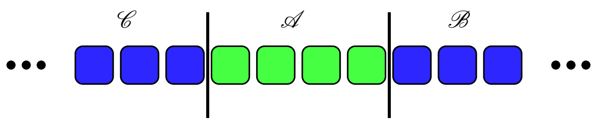

Figure 6: A finite region $\mathscr{A}$ of 4 spins is cut out of the chain leaving two infinite ends $\mathscr{B}$ and $\mathscr{C}$.

Theorem III.4. *Given a pure translation-invariant stabilizer state ω_ξ of stabilizer generator length $2n+1$, a region of length L shares $2n$ maximally entangled qubit pairs with the rest of the chain if $2n \le L$ and L qubits pairs if $2n > L$.*

Proof. The proof works exactly as in the bipartite case. In the case $2n \le L$ the cut stabilizers are only cut on one side. But all stabilizers that are cut on the left-hand side commute with those cut on the right-hand side. Thus, we have two independent cuts of the bipartite case and therefore $2n$ pairs of maximally entangled qubits. In the case $2n > L$ some stabilizers are cut on both sides. We use the same technique to produce the mutually commuting anti-commuting pairs which encode the qubits as in Theorem III.3. We always find L pairs of maximally entangled qubits. $\square$

IV. ENTANGLEMENT GENERATION

Now we come to the generation of entanglement through CQCA action. As we have seen in the previous section, the bipartite entanglement of a pure translation-invariant stabilizer state depends linearly on the length of the generators of the stabilizer group. So, it suggests itself to study the evolution of the length of the stabilizer generators under CQCA action. First, let us define the asymptotic entanglement generation rate from stabilizer states.

Definition IV.1. *The asymptotic entanglement generation rate from stabilizer states for CQCAs is defined as*

$$\frac{\Delta E}{\Delta t}(\omega_\xi) = \lim_{t\to\infty} \frac{1}{t} E(\omega_\xi, t), \tag{11}$$

where $E(\omega_\xi, t)$ is the bipartite entanglement at time t.

Lemma IV.2. *The length $2n+1$ of the minimal stabilizer generators grows asymptotically with*

$$\frac{\Delta n}{\Delta t}(\omega_\xi) = \lim_{t\to\infty} \frac{1}{t}(2n(\omega_\xi, t)+1) = \mathrm{dg}(\mathrm{tr}\,\mathbf{t}) \tag{12}$$

for any centered CQCA T *and any translation-invariant pure stabilizer state ω_ξ.*

Proof. We know that CQCAs map pure translation-invariant stabilizer states onto pure translation-invariant stabilizer states. The image of a state with stabilizer generators $\mathbb{S} = \{\mathbf{w}(\tau_x\xi),\ x \in \mathbb{Z}\}$ under the action of t steps of a CQCA T is a state with stabilizer generators $\mathbb{S}^t = \{\mathbf{w}(\tau_x\mathbf{t}^t\xi),\ x \in \mathbb{Z}\}$. Furthermore, we know that any stabilizer state can be generated from the "all-spins-up" state by a CQCA $\mathbf{b}$. So we have $\mathbb{S}^t = \{\mathbf{w}(\tau_x\mathbf{t}^t\mathbf{b}(0,1)),\ x \in \mathbb{Z}\}$. The length of the stabilizer generator is determined by the highest order of the stabilizer generator polynomials, $\mathrm{dg}(\xi)$. Namely, the stabilizer generator is of length $2\cdot\mathrm{dg}(\xi)+1$. So, we have to calculate $\mathrm{dg}(\mathbf{t}^t\xi) = \mathrm{dg}(\mathbf{t}^t\mathbf{b}\binom{0}{1})$.

For an arbitrary product of CQCAs $\prod_{i=1}^k \mathbf{t}_i$ we can define the series $(a_l)_{1\le l\le k} = \mathrm{dg}(\prod_{i=1}^l \mathbf{t}_i)$. It is subadditive, i.e. $a_{n+m} \le a_n + a_m$, because the concatenation of CQCAs is essentially the multiplication and addition of polynomials, which is subadditive in the exponents. For subadditive series a_n Fekete's lemma [11] states that the limit $\lim_{n\to\infty}\frac{a_n}{n}$ exists. In our case the series is always positive, so the limit is positive and finite. An easy way to determine the limit is to take a subseries, which of course has the same limit. The subseries of the $t = 2^k$th ($k \in \mathbb{N}$) steps is a good candidate, because we can make use of the Cayley-Hamilton theorem to obtain

$$\mathbf{t}^{2^k} = \mathbf{t}(\mathrm{tr}\,\mathbf{t})^{2^k-1} + \mathbb{1}\sum_{i=1}^k (\mathrm{tr}\,\mathbf{t})^{2^k-2^i}.$$

Furthermore, we have

$$\begin{aligned}\mathrm{dg}(\mathbf{t}^{2^k}\mathbf{b}\tbinom{0}{1}) &= \mathrm{dg}(\mathbf{t}\mathbf{b}\tbinom{0}{1}(\mathrm{tr}\,\mathbf{t})^{2^k-1} + \mathbf{b}\tbinom{0}{1}\sum_{i=1}^k(\mathrm{tr}\,\mathbf{t})^{2^k-2^i}) \\ &= a\cdot c(k) + b\cdot d(k),\end{aligned}$$

with $a = \mathbf{t}\mathbf{b}\binom{0}{1}$, $b = \mathbf{b}\binom{0}{1}$, $c(k) = (\mathrm{tr}\,\mathbf{t})^{2^k-1}$, $d(k) = \sum_{i=1}^k(\mathrm{tr}\,\mathbf{t})^{2^k-2^i}$.

Let us first assume that for some k_0 we have $\mathrm{dg}(\mathbf{t}^{2^k}\mathbf{b}\binom{0}{1}) > \mathrm{dg}(b)$. We start by determining a recur-