For item 4, let $z_0 \in \Re^q$ be a center point sampled from ν. When H_0 is false, ν-almost surely $KS(z_0) = c > 0$. By Lebesgue's density theorem ν-almost surely there exists an ϵ such that if $r < \epsilon$ then at least half of the ball $B_q(z_0, r)$ is within the support S. Since F_1 and F_2 are continuous, $KS(z)$ is a continuous function of z. Therefore, there exists an $\epsilon' < \epsilon$ such that $KS(z) > c/2$ for all $z \in B_q(z_0, \epsilon') \cap S$. Similar arguments to those for item 2 show that ν-almost surely for any $z_i \in S \cap B_q(z_0, \epsilon')$,

$$Pr(KS_N(z_i) < c/4) < 4e^{-Nc^2/32}. \tag{3.2}$$

Therefore, $Pr(\cup_{z_i \in S \cap B_q(z_0,\epsilon')} KS_N(z_i) < c/4) < 4Me^{-Nc^2/32}$. Since ν-almost surely $\mathrm{pr}\{Z \in S \cap B_q(z_0, \epsilon')\} > 0$, then ν-almost surely with probability going to one $T1$ is $O(M)$, as long as $M = o(e^N)$. On the other hand when H_0 is true, $E(KS_N(z)) = O(1/\sqrt{N})$, see for example Marsaglia et al. (1983). Therefore, $E(T1) = O(M/\sqrt{N})$, and by Markov's inequality the permutation test based on $T1$ will have ν-almost surely power increasing to one as the sample size increases. For the test based on T_2, from equations (3.2) and (3.1) it follows that for N large enough $p_i < 4e^{-Nc^2/32}$ for $z_i \in S \cap B_q(z_0, \epsilon')$, $i = 1, \ldots, M$. Therefore, for N large enough $-2\sum_{i=1}^M \log p_i$ is greater than $O(NM)\mathrm{pr}\{Z \in S \cap B_q(z_0, \epsilon')\}$. On the other hand, when H_0 is true P_i is uniformly distributed, so $E(-2\sum_{i=1}^M \log P_i)$ is $O(M)$. By Markov's inequality the permutation test based on $-2\sum_{i=1}^M \log p_i$ will have ν-almost surely power increasing to one as the sample size increases. $\square$

The test statistics $S1$ and $T1/M$ converge to meaningful population quantities,

$$\lim_{N,M\to\infty} S_1 = \lim_{M\to\infty} \max_{z_1,\ldots,z_M} KS(z) = \sup_{z\in S} KS(z),$$
$$\lim_{N,M\to\infty} T_1/M = \lim_{M\to\infty} \sum_{i=1}^M KS(z_i)/M = E\{KS(Z)\}, \tag{3.3}$$

where the expectation is over the distribution of the center point Z.