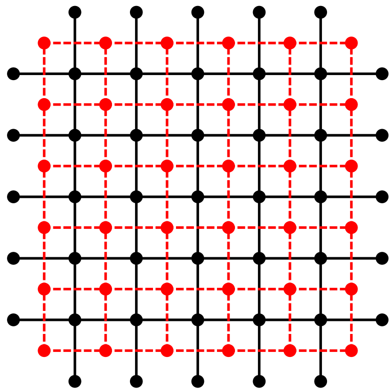

FIG. 7: The lattice with fixed spin boundary conditions (black) and the dual lattice with free boundary conditions (red).

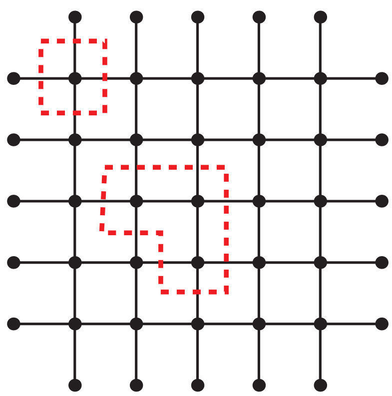

FIG. 8: Graphical representation of a term in the low temperature expansion of the partition function. The spins on sites inside the regions enclosed by dashed red lines point in a direction opposite to that in which all other spins point. The spins on the boundary, which are fixed, are not allowed to reverse.

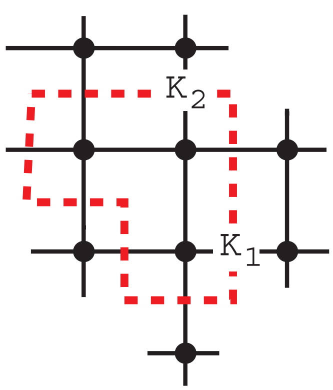

FIG. 9: A region of reversed spins

the portion of the boundary intersects a vertical bond the contribution is e^{-2K_2}.

An important feature of this expansion is that fixed boundary conditions, in which the spins at the periphery of the system are constrained to point up, guarantee that all contributions to the low temperature expansion consist of complete closed boundaries

The next step in the development is to generate a high temperature expansion for the partition function of the spin system on the dual lattice. The spins on this lattice interact via nearest neighbor bonds that are both horizontal and vertical as in the original lattice. The strength of the horizontal bonds is K_1^* and of the vertical bonds is K_2^*. The high temperature expansion is based on the decomposition

$$\begin{aligned} \exp[K\sigma\sigma'] &= \cosh K + \sigma\sigma' \sinh K \\ &= \cosh K \, (1 + \sigma\sigma' \tanh K) \end{aligned} \tag{C1}$$

The expansion is in the second term in parentheses on the last line of (C1). Because there is precisely one such term for each pair of spins and because each spin must appear an even number of times in the expansion, the high temperature expansion is represented graphically by figures like the border shown in Fig. 9, this time with the portions of the figure standing for bonds between spins on the dual lattice, as shown in Fig. 10. The portion is shown in Fig. 10. Each segment of the portion yields a

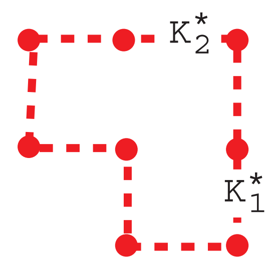

FIG. 10: The portion of the high temperature expansion corresponding to the detail in the low temperature expansion shown in Fig. 9.

factor $\tanh K_1^*$ if it is horizontal and $\tanh K_2^*$. Perfect correspondence is achieved if

$$\tanh K_1^* = \exp[-2K_2] \tag{C2}$$

$$\tanh K_2^* = \exp[-2K_1] \tag{C3}$$

This is the standard set of duality relations.

Here, in order that the high temperature expansion also consist of closed contours, we must allow all spins on the dual lattice to vary without restrictions. That is, the boundary conditions on the dual lattice must be free. This establishes the complete duality between the two versions of the two dimensional Ising model.