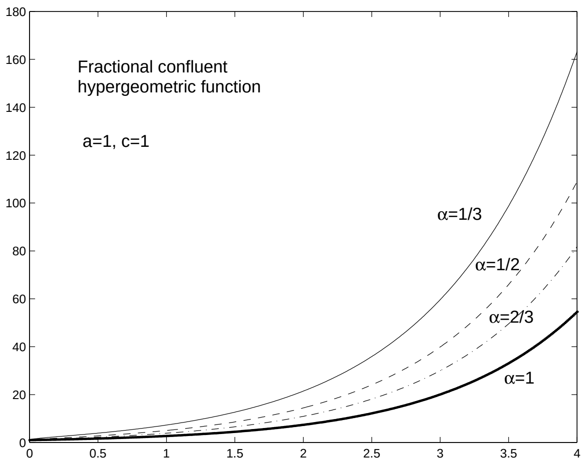

FIG. 6: Fractional confluent hypergeometric function.

V. FRACTIONAL HYPERGEOMETRIC FUNCTION

There are other types of special functions in mathematical physics. A most famous one is the hypergeometric function [12–14]. In this section, we will try to define a fractional generalization of the hypergeometric functions.

Let's first consider the generalization of the confluent hypergeometric differential equation:

$$z^{\alpha}(D^{\alpha})^2 y + (c - z^{\alpha})D^{\alpha} y - ay = 0. \tag{36}$$

Here a and c are complex parameters. When $\alpha = 1$, this is the ordinary confluent hypergeometric equation.

Introducing the fractional Taylor series

$$y(z) = \sum_{k=0}^{\infty} c_k z^{\alpha \cdot k}, \tag{37}$$

and substituting, we get the ratio of successive coefficients

$$\frac{c_{k+1} \cdot \Gamma(k\alpha + \alpha + 1)}{c_k \cdot \Gamma(k\alpha + 1)} = \frac{a + \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha-\alpha+1)}}{c + \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha-\alpha+1)}}, \tag{38}$$

$$\frac{c_1 \cdot \Gamma(\alpha + 1)}{c_0} = \frac{a}{c}. \tag{39}$$

Thus we get a solution of the above differential equation,

$$y(z) = \sum_{k=0}^{\infty} \frac{(a)_k^{\alpha}}{(c)_k^{\alpha}} \frac{1}{\Gamma(k\alpha + 1)} z^{\alpha \cdot k}. \tag{40}$$

Here $(a)_k^{\alpha}$ is defined as

$$\begin{aligned} (a)_0^{\alpha} &= 1, \qquad (a)_1^{\alpha} = a, \\ (a)_k^{\alpha} &= (a)_1^{\alpha}\left(a + \frac{\Gamma(\alpha+1)}{\Gamma(1)}\right)...\left(a + \frac{\Gamma(k\alpha - \alpha + 1)}{\Gamma(k\alpha - 2\alpha + 1)}\right), \\ & \qquad k \geq 2. \end{aligned} \tag{41}$$

This can be seen as a fractional generalization of the rising factorial

$$(a)_k = a(a+1)...(a+k-1). \tag{42}$$

And the series (40) can be seen as a fractional generalization the confluent hypergeometric function. If $\alpha = 1$, it is exactly the confluent hypergeometric function. Profiles of this series (40) with different values of α are displayed in Fig. 6.

For the fractional Gauss hypergeometric function, consider the following series

$$y(z) = \sum_{k=0}^{\infty} \frac{(a)_k^{\alpha}(b)_k^{\alpha}}{(c)_k^{\alpha}} \frac{1}{\Gamma(k\alpha + 1)} z^{\alpha \cdot k}, \tag{43}$$

which reduces to the Gauss hypergeometric series when $\alpha = 1$.

The ratio of successive coefficients is

$$\frac{c_{k+1} \cdot \Gamma(k\alpha + \alpha + 1)}{c_k \cdot \Gamma(k\alpha + 1)} = \frac{\left(a + \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha-\alpha+1)}\right)\left(b + \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha-\alpha+1)}\right)}{\left(c + \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha-\alpha+1)}\right)}, \tag{44}$$

or

$$\begin{aligned} & c_{k+1} \cdot c\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha-\alpha+1)} + c_{k+1} \cdot \frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha-\alpha+1)} \\ &= c_k \cdot ab + c_k \cdot (a+b)\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha-\alpha+1)} + c_k \cdot \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha-\alpha+1)}\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha-\alpha+1)}. \end{aligned} \tag{45}$$

Since

$$y(z) = \sum_{k=0}^{\infty} c_k z^{\alpha \cdot k}, \tag{46}$$

$$z^{\alpha} D^{\alpha} y(z) = \sum_{k=1}^{\infty} c_k \frac{\Gamma(k\alpha + 1)}{\Gamma(k\alpha - \alpha + 1)} z^{\alpha \cdot k}, \tag{47}$$