required. ■

Theorem 7.1 *Under conditions stated in Theorem 5.1, we have $\|P_{K,n}\|_\infty \lesssim 1$ wpa1 provided the following are satisfied:*

(i) $\| (B'B/n) - E[b^K(X_i)b^K(X_i)']\| = o_p(1)$, *and*

(ii) $\max_{1\leq k\leq K} \left|\frac{\frac{1}{n}\sum_{i=1}^n |b_{Kk}(X_i)| - E[|b_{Kk}(X_i)|]}{E[|b_{Kk}(X_i)|]}\right| = o_p(1)$.

Proof of Theorem 7.1. Condition (ii) $\max_{1\leq k\leq K} \frac{\frac{1}{n}\sum_{i=1}^n |b_{Kk}(X_i)| - E[|b_{Kk}(X_i)|]}{E[|b_{Kk}(X_i)|]} = o_p(1)$ implies

$$\max_{1\leq k\leq K} \frac{1}{n}\sum_{i=1}^n |b_{Kk}(X_i)| \lesssim \max_{1\leq k\leq K} \|b_{Kk}\|_{L^1(X)} \lesssim K^{-1/2} \tag{135}$$

where the final inequality is by the proof of Theorem 5.1. Moreover, $\sup_x \|b^K(x)\|_{\ell^1} \lesssim \sqrt{K}$ by the proof of Theorem 5.1. It follows analogously to Lemma 7.1 that $\|P_{K,n}\|_\infty \lesssim \| (B'B/n)^{-1} \|_\infty$ wpa1 (noting that $B'B/n$ is invertible wpa1 because $\| (B'B/n) - E[b^K(X_i)b^K(X_i)']\| = o_p(1)$ and $\lambda_{K,n} \lesssim 1$).

Condition (i) $\| (B'B/n) - E[b^K(X_i)b^K(X_i)']\| = o_p(1)$ implies (1) $\lambda_{\min}(B'B/n) \gtrsim \lambda_{\min}(E[b^K(X_i)b^K(X_i)'])$, (2) $\lambda_{\max}(B'B/n) \lesssim \lambda_{\max}(E[b^K(X_i)b^K(X_i)'])$, and (3) $\| (B'B/n)^{-1} \| \lesssim \| \left(E[b^K(X_i)b^K(X_i)']\right)^{-1} \|$ all hold wpa1. Moreover, $\lambda_{\min}(E[b^K(X_i)b^K(X_i)']) \gtrsim 1$ and $\lambda_{\max}(E[b^K(X_i)b^K(X_i)']) \lesssim 1$ by the proof of Theorem 5.1. It follows by Lemma 7.2 that $\| (B'B/n)^{-1} \|_{\ell^\infty} \lesssim 1$ wpa1, as required. ■

Proof of Theorem 5.2. Condition (i) of Theorem 7.1 is satisfied because $\lambda_{K,n} \lesssim 1$ and the condition $\|(\widetilde{B}'\widetilde{B}/n) - I_K\| = o_p(1)$ under the conditions on K (see Lemma 2.1 for the i.i.d. case and Lemma 2.2 for the weakly dependent case). Therefore,

$$\begin{aligned} \|(B'B/n) - E[b^K(X_i)b^K(X_i)']\| &\leq [\lambda_{\min}(E[b^K(X_i)b^K(X_i)'])]^{-1}\|(\widetilde{B}'\widetilde{B}/n) - I_K\| &(136)\\ &\lesssim \|(\widetilde{B}'\widetilde{B}/n) - I_K\| = o_p(1)\,. &(137)\end{aligned}$$

It remains to verify condition (ii) of Theorem 7.1. Let $b_{K1} = \varphi^d_{J,0}, \ldots, b_{KK} = \varphi^d_{J,2^J-1}$ with $K = 2^{dJ}$ as in the proof of Theorem 5.1. Similar arguments to the proof of Theorem 5.1 yield the bounds $\|b_{Kk}\|_\infty \lesssim 2^{dJ/2} = \sqrt{K}$ uniformly for $1 \leq k \leq K$. Let $f_X(x)$ denote the density of X. Then by