2 Preliminaries

Throughout the paper, we will make use of the usual Euclidean norm for vectors and the norm induced by the Euclidean norm for matrices.

Let (X, d) be a metric space, and $\mathbb{T}$ refer to either the set of real numbers or the set of integers. Suppose that $\pi : \mathbb{T} \times X \to X$ is a flow on X, i.e., $\pi(0, \sigma) = \sigma$ for all $\sigma \in X$, $\pi(t, \sigma)$ is continuous in the pair of variables t and σ, and $\pi(t_1, \pi(t_2, \sigma)) = \pi(t_1 + t_2, \sigma)$ for all t_1, $t_2 \in \mathbb{T}$ and $\sigma \in X$ [16]. We modified the Poisson stable points to unpredictable points in paper [10] as follows.

Definition 2.1 *A point $\sigma \in X$ and the trajectory through it are unpredictable if there exist a positive number ϵ_0 (the unpredictability constant) and sequences $\{t_k\}$ and $\{\tau_k\}$, both of which diverge to infinity, such that $\lim_{k\to\infty} \pi(t_k, \sigma) = \sigma$ and $d[\pi(t_k + \tau_k, \sigma), \pi(\tau_k, \sigma)] \geq \epsilon_0$ for each $k \in \mathbb{N}$.*

To develop the row of periodic, quasiperiodic, and almost periodic oscillations to a new one, we specified in [11]-[13] unpredictability for functions as points of a dynamics.

Definition 2.2 *A uniformly continuous and bounded function $\varphi : \mathbb{R} \to \mathbb{R}^p$ is unpredictable if there exist positive numbers ϵ_0, δ, and sequences $\{t_k\}$, $\{\tau_k\}$, both of which diverge to infinity, such that $\|\varphi(t + t_k) - \varphi(t)\| \to 0$ as $k \to \infty$ uniformly on compact subsets of $\mathbb{R}$, and $\|\varphi(t + t_k) - \varphi(t)\| \geq \epsilon_0$ for each $t \in [\tau_k - \delta, \tau_k + \delta]$ and $n \in \mathbb{N}$.*

The last definition can be considered as a more restrictive version of the next two, which will be useful in the future for applications of functional analysis methods in the theory of differential equations.

Definition 2.3 *A continuous and bounded function $\varphi : \mathbb{R} \to \mathbb{R}^p$ is unpredictable if there exist positive numbers ϵ_0, δ, and sequences $\{t_k\}$, $\{\tau_k\}$, both of which diverge to infinity, such that $\|\varphi(t + t_k) - \varphi(t)\| \to 0$ as $k \to \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\|\varphi(t_k + \tau_k) - \varphi(\tau_k)\| \geq \epsilon_0$ for each $k \in \mathbb{N}$.*

Definition 2.4 *A continuous and bounded function $\varphi : \mathbb{R} \to \mathbb{R}^p$ is unpredictable if there exist positive numbers ϵ_0, δ, and sequences $\{t_k\}$, $\{\tau_k\}$, both of which diverge to infinity, such that $\|\varphi(t_k) - \varphi(0)\| \to 0$ as $k \to \infty$ and $\|\varphi(t_k + \tau_k) - \varphi(\tau_k)\| \geq \epsilon_0$ for each $k \in \mathbb{N}$.*

The following definition of an unpredictable sequence was first mentioned in paper [13] as an analogue for Definition 2.2.

Definition 2.5 *A bounded sequence $\{\varphi_n\}$, $n \in \mathbb{Z}$, in $\mathbb{R}^p$ is called unpredictable if there exist a positive number ϵ_0 and sequences $\{\zeta_k\}$, $\{\eta_k\}$, $k \in \mathbb{N}$, of positive integers, both of which diverge to infinity, such that $\|\varphi_{n+\zeta_k} - \varphi_n\| \to 0$ uniformly as $k \to \infty$ for each n in bounded intervals of integers and $\|\varphi_{\zeta_k+\eta_k} - \varphi_{\eta_k}\| \geq \epsilon_0$ for each $k \in \mathbb{N}$.*