with

$$\tilde{q} \;=\; 1 - \frac{2}{DN + n}. \tag{94}$$

Equations (91) and (93) are similar to Eqs. (16) and (29), respectively, although the ranges of conceivable q values are not necessarily the same.

2. *Energy and entropy*

The χ^2-distribution in Eq. (83) is given *ad hoc* [30]-[33]. Several first-principles approaches have been proposed to determine the optimum distribution of β in the superstatistics [34]-[37]. Ref.[37] considered the local energy $E(\beta) = \langle H \rangle_{\Pi}$ and entropy $S(\beta) = -\langle \ln \Pi \rangle_{\Pi}$ for local equilibrium states with an inverse temperature β, and then obtained the energy and entropy of the system given by $E = \langle E(\beta) \rangle_{\beta}$ and $S = \langle S(\beta) \rangle_{\beta}$, where $\langle \cdot \rangle_{\Pi}$ and $\langle \cdot \rangle_{\beta}$ express averages over $\Pi(\beta)$ and $f(\beta)$, respectively. Equations (36) and (44) for ideal gases lead to

$$E(\beta) \;=\; \frac{DN}{2\beta}, \tag{95}$$

$$S(\beta) \;=\; k_B N \left[\left(\frac{D}{2}\right) \ln\left(\frac{2\pi m}{h^2 \beta}\right) + \ln \sigma + \left(\frac{D}{2} + 1\right) \right], \tag{96}$$

from which we obtain

$$E \;=\; \frac{DN}{2\bar{\beta}} \left(\frac{n}{n-2}\right), \tag{97}$$

$$S \;=\; k_B N \left[\left(\frac{D}{2}\right) \ln\left(\frac{2\pi m}{h^2 \bar{\beta}}\right) + \ln \sigma + \left(\frac{D}{2} + 1\right) \right] - \frac{k_B DN}{2} \left[\psi\left(\frac{n}{2}\right) - \ln\left(\frac{n}{2}\right) \right], \tag{98}$$

$\psi(x)$ standing for the poli-gamma function. Similarly from Eqs. (62) and (68) for harmonic oscillators, we obtain

$$E(\beta) \;=\; \frac{DN}{\beta}, \tag{99}$$

$$S(\beta) \;=\; k_B DN \left[\ln\left(\frac{2\pi}{h\omega\beta}\right) + 1 \right], \tag{100}$$

and then

$$E \;=\; \frac{DN}{\bar{\beta}} \left(\frac{n}{n-2}\right), \tag{101}$$

$$S \;=\; k_B DN \left[\ln\left(\frac{2\pi}{h\omega\bar{\beta}}\right) + 1 \right] - k_B DN \left[\psi\left(\frac{n}{2}\right) - \ln\left(\frac{n}{2}\right) \right]. \tag{102}$$