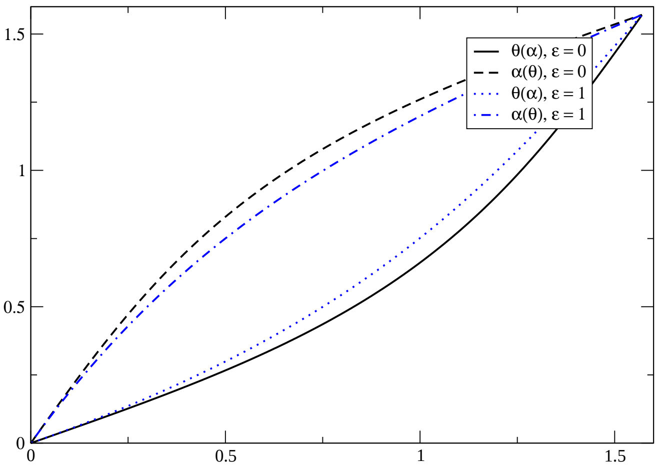

FIG. 2: (color online) The angle θ as a function of the launch angle α for two different values of ε and their inverses.

in the review of referencei [2], is absent. Even when it is possible to write the locus in terms of $y_m(x_m)$ this form does not shows the formal elegance of relation (13).

Now we return to equation (10) since we need to solve explicitly it in order to have the function $\alpha(\theta_m)$. This task is not trivial since even when we approximate the rhs in expression (10) up to first order in ε,

$$\tan\theta_m = \frac{\tan\alpha}{2}(1 + \frac{\varepsilon}{3}\sin\alpha), \tag{16}$$

the inversion is not easy. A way to do the inversion is to expand in a Taylor series the rhs and then invert the series term by term.[9] Using *Mathematica* to perform this procedure up to $\mathcal{O}(18)$, we obtain as a result

$$\begin{aligned}\alpha(\theta) \approx\ & \arctan(2\tan\theta) - \tfrac{1}{3}\varepsilon(2\tan\theta)^2 + \tfrac{2}{9}\varepsilon^2(2\tan\theta)^3 \\ & + \tfrac{1}{54}(27\varepsilon - 10\varepsilon^3)(2\tan\theta))^4 + \cdots\end{aligned} \tag{17}$$

The ε-independent terms had been resumated to yield $\arctan(2\tan\theta)$. However, the series does not converge for values in the argument larger than 1. The reason is the small convergence ratio for the Taylor expansion of $\arcsin(\cdot)$.

An easier way to perform the inversion is to evaluate $\theta_m(\alpha)$ using Eq. (10) and plot the points $(\theta_m(\alpha), \alpha)$, the result is shown in figure 2. The result is in agreement with the plot of Eq. (17) up to its convergence ratio and it is not shown. Notice that this method is exact in the sense that we can obtain as many pair of numbers as we need, a function is, finally, a relation one to one between two sets of real numbers. Another result is to obtain the derivative $d\alpha/d\theta$, since it shall be needed in the following sections. To this end, we note that both functions increase monotonically and their derivatives are not zero, except at the interval end. Hence, we can use the inverse function theorem in order to obtain

$$\frac{d\alpha}{d\theta} = \frac{1}{\frac{d\theta}{d\alpha}}. \tag{18}$$

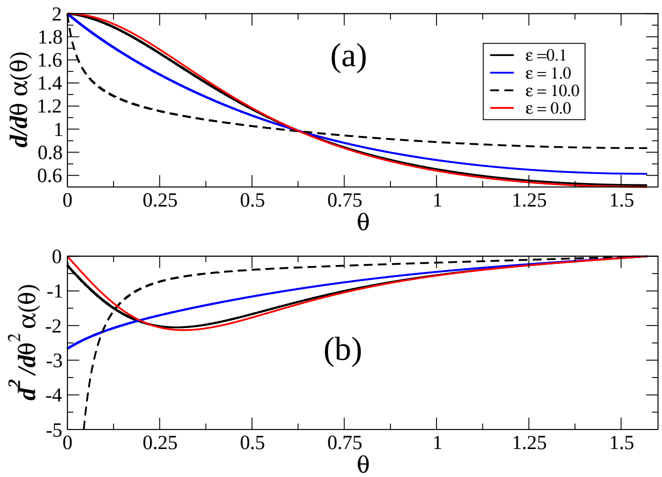

FIG. 3: (color online) (a) First and (b) second derivatives of α as function of θ for various values of parameter ε. Note that major changes occur for $\theta < \pi/4$.

The result is shown in Fig. 3(a) as well as the second derivative in Fig. 3(b). The second derivative is calculated using an approximation to the slope to the function previously calculated and using 10000 points in the interval $[0, \pi/2]$. A smaller number of points could be considered.

IV. THE CURVATURE OF $\mathcal{C}_m$.

A. Polar angle parameterization.

In the drag-free situation, $\mathcal{C}_m$ is an ellipse and its description is well know, however, in the presence of linear drag this is not the case. We do not expect that the locus could be a conic section and henceforth we need to characterize it. It is usual to consider curvature, radius of curvature or the length of arc in order to characterize a locus. In the present case we consider the curvature of $\mathcal{C}_m$ in both parameterizations, first with the polar angle θ_m and secondly with the launch angle α. We left the calculus of the length of arc to a posterior work, since the calculations became increasingly complex and the goal of the present section is to start the understanding of $\mathcal{C}_m$ and to illustrate the way it can be done using the Lambert W function. Here and in the rest of the section we drop, for clearness, the subindex m in $\tilde{r}_m$ and θ_m.

The corresponding formula for the curvature K for polar coordinates is [10]

$$K = \frac{1}{\rho}\frac{\tilde{r}^2 + 2\tilde{r}_\theta^2 - \tilde{r}\tilde{r}_{\theta\theta}}{(\tilde{r}^2 + \tilde{r}_\theta^2)^{3/2}}, \tag{19}$$

in order to use Eq. (13). Here the subindex θ corresponds a derivative respect to that variable.

A direct calculation on the drag-free $\tilde{r}(\theta)$ of Eq.(15)