Setting $z = \alpha n\sqrt{p}/4$ and using Stirling's approximation, we have, a.s.,

$$\left(\frac{np}{2}-\frac{\alpha n\sqrt{p}}{4}\right)^{z}\Gamma\left(\frac{np}{2}-z\right)=\left(\frac{np}{2}-\frac{\alpha n\sqrt{p}}{4}\right)^{\frac{np}{2}-\frac{1}{2}}e^{-\frac{np}{2}+\frac{\alpha n\sqrt{p}}{4}}\sqrt{2\pi}\left(1+o\left(1\right)\right)$$

so that

$$\begin{aligned}\mathcal{I}\left(0,p-\alpha\sqrt{p}\right) &< \left(\frac{n-A_2}{2}\right)^{-\frac{np}{2}}\left(\frac{np}{2}-\frac{\alpha n\sqrt{p}}{4}\right)^{\frac{np}{2}-\frac{1}{2}}e^{-\frac{np}{2}+\frac{\alpha n\sqrt{p}}{4}}\sqrt{2\pi}\left(1+o\left(1\right)\right)\\ &< p^{\frac{np}{2}}e^{-\frac{np}{2}}e^{p\left(\frac{A_2}{2}+\frac{A_2^2}{4n}-\frac{\alpha^2 n}{16p}\right)}\left(1+o\left(1\right)\right),\text{ a.s..}\end{aligned}$$

Comparing this to (51), we see that α can be chosen so that

$$\mathcal{I}\left(0,p-\alpha\sqrt{p}\right)=o(1)\mathcal{I}\left(0,\infty\right), \tag{53}$$

a.s.. Combining (52) and (53), we get (50).

Now, letting $\tilde{\theta}_{pj}=\frac{x}{S_p}\theta_{pj}=\frac{x}{S_p}\frac{1}{2c_p}\frac{h_j}{1+h_j}$, note that there exist $\varepsilon>0$ and $\eta>0$ such that $\left\{2\tilde{\theta}_{pj}:h_j\in[0,\sqrt{c}-\delta]\text{ and }x\in\left[p-\alpha\sqrt{p},p+\alpha\sqrt{p}\right]\right\}\subseteq\Theta_{\varepsilon\eta}$ for all sufficiently large p, a.s.. Hence, by (50), and Proposition 2, a.s.,

$$\begin{aligned}\mathcal{I}\left(0,\infty\right) &= \int_{p-\alpha\sqrt{p}}^{p+\alpha\sqrt{p}}x^{\frac{np}{2}-1}e^{-\frac{n}{2}x}e^{p\sum_{j=1}^{r}\left[\tilde{\theta}_{pj}\tilde{v}_{pj}-\frac{1}{2p}\sum_{i=1}^{p}\ln\left(1+2\tilde{\theta}_{pj}\tilde{v}_{pj}-2\tilde{\theta}_{pj}\lambda_{pi}\right)\right]}\\ &\quad\times\left(\prod_{j=1}^{r}\prod_{s=1}^{j}\sqrt{1-4\left(\tilde{\theta}_{pj}\tilde{v}_{pj}\right)\left(\tilde{\theta}_{ps}\tilde{v}_{ps}\right)c_p}+o(1)\right)\mathrm{d}x,\end{aligned} \tag{54}$$

where $o(1)$ is uniform in $h\in[0,\sqrt{c}-\delta]^r$ and $x\in\left[p-\alpha\sqrt{p},p+\alpha\sqrt{p}\right]$.

Expanding $\tilde{\theta}_{pj}\tilde{v}_{pj}-\frac{1}{2p}\sum_{i=1}^{p}\ln\left(1+2\tilde{\theta}_{pj}\tilde{v}_{pj}-2\tilde{\theta}_{pj}\lambda_{pi}\right)$ and $\left(\tilde{\theta}_{pj}\tilde{v}_{pj}\right)\left(\tilde{\theta}_{ps}\tilde{v}_{ps}\right)$ into power series of $\frac{x}{p}-1$, we get

$$\begin{aligned}\mathcal{I}\left(0,\infty\right) &= \int_{p-\alpha\sqrt{p}}^{p+\alpha\sqrt{p}}x^{\frac{np}{2}-1}e^{-\frac{n}{2}x}e^{p\left(B_0+B_1\left(\frac{x}{p}-1\right)+B_2\left(\frac{x}{p}-1\right)^2\right)}\\ &\quad\times\left(\prod_{j=1}^{r}\prod_{s=1}^{j}\sqrt{1-4\left(\theta_{pj}v_{pj}\right)\left(\theta_{ps}v_{ps}\right)c_p}+o(1)\right)\mathrm{d}x,\end{aligned}$$

where B_0,B_1 and B_2 are $O(1)$ uniformly in $h\in[0,\sqrt{c}-\delta]^r$. Further, consider the integral

$$I^{(0)}=\int_{p-\alpha\sqrt{p}}^{p+\alpha\sqrt{p}}x^{\frac{np}{2}-1}e^{-\frac{n}{2}x}e^{p\left(B_1\frac{x}{p}+B_2\left(\frac{x}{p}-1\right)^2\right)}\mathrm{d}x.$$