The Casimir force has precisely has the desired form (1.1). That is, for the one-dimensional Ising model

$$\vartheta_{1d\ \mathrm{per}}(x) = -x\frac{e^{-x}}{1+e^{-x}} \tag{2.13}$$

Note that the factor k_BT plays an essential role in our result for the Casimir force. To the extent that it can be said to exist, the critical point of the one-dimensional Ising model is at $T=0$. Unless we normalize the critical Casimir force in terms of this temperature-dependent factor, we are forced to conclude that it vanishes.

B. Antiperiodic boundary conditions

The second boundary condition we study here is antiperiodic. We enforce this boundary condition by introducing an array of antiferromagnetic bonds in the system. In this case, there are $N-1$ two-by-two transfer matrices of the form (2.2) and one transfer matrix of the form

$$\mathbf{T}_2 = \begin{pmatrix} e^{-\beta J} & e^{\beta J} \\ e^{\beta J} & e^{-\beta J} \end{pmatrix} \tag{2.14}$$

The eigenvector decomposition of the transfer matrix $\mathbf{T}_2$ is

$$\mathbf{T}_2 = |e\rangle 2\cosh(\beta J)\langle e| - |o\rangle 2\sinh(\beta J)\langle o| \tag{2.15}$$

The partition function of this system is the trace of the quantity $\mathbf{T}_1^{N-1}\cdot\mathbf{T}_2$. Making use of (2.5) and (2.6), we have

$$\begin{aligned} \mathcal{Z}_{\mathrm{antiper}} &= \mathrm{Tr}\left(\mathbf{T}_1^{N-1}\cdot\mathbf{T}_2\right) \\ &= (2\cosh\beta J)^N - (2\sinh\beta J)^N \end{aligned} \tag{2.16}$$

This means that the free energy is given by

$$\begin{aligned} \mathcal{F} &= -k_BT\ln\mathcal{Z} \\ &= -Nk_BT\ln(2\cosh\beta J) - k_BT\ln\left[1-(\tanh\beta J)^N\right] \\ &= Nf_B - k_bT\ln\left[1-e^{-N/\xi}\right] \end{aligned} \tag{2.17}$$

The quantity f_B in the last line of (2.17) is the "bulk" free energy of the one-dimensional Ising model. The quantity $\xi = -1/(\ln\tanh\beta J)$ is the correlation length of that model. We then take the negative derivative of the free energy with respect to L, add in f_B, and end up with

$$\begin{aligned} F^{\mathrm{Cas}}_{1d\ \mathrm{antiper}} &= -\frac{\partial\mathcal{F}}{\partial N} + f_B \\ &= \frac{k_BT}{\xi}\frac{e^{-N/\xi}}{1-e^{-N/\xi}} \\ &= \frac{k_BT}{N}\frac{N}{\xi}\frac{e^{-N/\xi}}{1-e^{-N/\xi}} \\ &\equiv \frac{k_BT}{N}\vartheta_{1d\ \mathrm{antiper}}(N/\xi) \end{aligned} \tag{2.18}$$

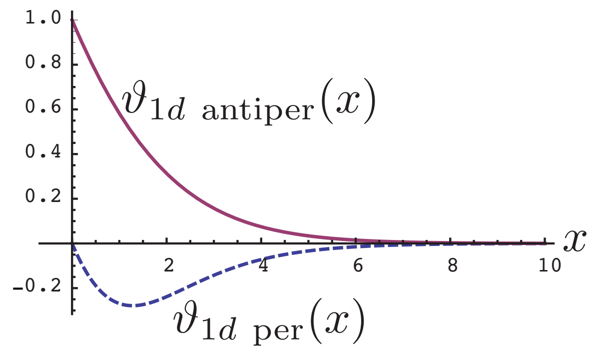

FIG. 1: The universal Casimir force function $\vartheta_{1d\ \mathrm{per}}(x)$, shown as a dashed curve and the function $\vartheta_{1d\ \mathrm{antiper}}(x)$, shown as a solid curve.

Fiture 1 displays universal functions $\vartheta_{1d\ \mathrm{per}}(x)$ and $\vartheta_{1d\ \mathrm{antiper}}(x)$. Two distinct features of this scaling function distinguish it from the case of periodic boundary conditions; it is positive, indicating a repulsive Casimir force, and it does not go to zero at the zero temperature critical point.

C. Free and Fixed Boundary Conditions

In the case of free boundary conditions, the partition function of the N-layer system is given by

$$\begin{aligned} \mathcal{Z}_{\mathrm{free}} &= 2\langle e|\mathbf{T}_1^N|e\rangle \\ &= 2(2\cosh(\beta J))^N \end{aligned} \tag{2.19}$$

Taking the derivative with respect to N of the log of the above result and subtracting the bulk contribution to that derivative, we find perfect cancellation. There is no residual force.

When the spins are fixed at the boundary, the state between which the N^{th} power of the transfer matrix is sandwiched is

$$|f\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.20}$$

and the partition function is given by

$$\begin{aligned} &\mathcal{Z}_{\mathrm{fixed}} \\ &= \langle f|\mathbf{T}_1^N|f\rangle \\ &= \frac{1}{2}\left((2\cosh(\beta J))^N + (2\sinh(\beta J))^N\right) \end{aligned} \tag{2.21}$$

Given the near identity between the final line of (2.21) and what was obtained for the partition function of the system with periodic boundary conditions in (2.7), we find immediately that the critical Casimir force induced by fixed boundary conditions matches the corresponding force in the case of periodic boundary conditions.