by Chebyshev's inequality. Choose $K_\epsilon = [-a_\epsilon, a_\epsilon]$, then K_ϵ is compact in $\mathbb{R}$ and $\mu(K_\epsilon) \geq 1-\epsilon$ for all $\mu \in \Lambda(\gamma, q)$, thus $\Lambda(\gamma, q)$ is tight.

Let $B_m = \left[-\frac{1}{m}, \frac{1}{m}\right]$ for $m = 1, 2, \ldots$. Let $\{\mu_n\}_{n=1}^\infty$ be a convergent sequence in $\Lambda(\gamma, q)$ with limit μ_0. Since $\mu_n(B_m) \geq q$ for every m, n, we have [15, Section 3.1]

$$q \leq \limsup_{n\to\infty} \mu_n(B_m) \leq \mu_0(B_m), \tag{19}$$

and hence

$$\mu_0(\{0\}) = \mu_0\left(\bigcap_{m=1}^\infty B_m\right) = \lim_{m\to\infty} \mu_0(B_m) \geq q. \tag{20}$$

Moreover, let $f(x) = x^2$ which is continuous and bounded below. By weak convergence [15, Section 3.1], we have

$$\mathsf{E}_{\mu_0}\left\{X^2\right\} = \int f \mathrm{d}\mu_0 \leq \liminf_{n\to\infty} \int f \mathrm{d}\mu_n \leq \gamma. \tag{21}$$

Therefore, $\mu_0 \in \Lambda(\gamma, q)$, i.e., $\Lambda(\gamma, q)$ is closed, and the compactness of $\Lambda(\gamma, q)$ then follows. ■

Since the mutual information $I(\mu)$ is continuous on $\mathcal{P}$ [17, Theorem 9], it must achieve its maximum on the compact set $\Lambda(\gamma, q)$. Hence the capacity-achieving distribution μ_0 exists.

According to [17, Corollary 2], the mutual information $I(\mu)$ is strictly concave. It is easy to see that $\Lambda(\gamma, q)$ is convex. Hence the capacity-achieving distribution μ_0 must be unique.

B. Sufficient and Necessary Conditions

We denote the finite-power set as

$$\Lambda(q) = \cup_{0\leq\gamma<\infty} \Lambda(\gamma, q). \tag{22}$$

Let $\phi(\cdot)$ defined in (14) be extended to the complex plane. The relative entropy $d(x; \mu)$ defined in (16) can be extended to the complex plane $\mathbb{C}$ and has the following property:

Lemma 2: For any $\mu \in \Lambda(q)$ and $z \in \mathbb{C}$,

$$d(z; \mu) = \int_{-\infty}^{\infty} \phi(y-z) \log \frac{\phi(y-z)}{p_Y(y; \mu)} \mathrm{d}y \tag{23}$$