cations:

$$\begin{aligned}
\widehat{\xi \star \eta} &= \sum_y \widehat{\xi(-y)}\tau_y \eta \\
&= \sum_x \sum_y \xi(-y)\eta(x+y)u^x \\
&= \sum_{k=x+y} \sum_{l=-y} \xi(l)\eta(k)u^{k+l} \\
&= \sum_l \xi(l)u^l \sum_k \eta(k)u^k \\
&= \hat{\xi} \cdot \hat{\eta}.
\end{aligned}$$

Because of this property, we refer to (5) as an algebraic Fourier transform. We get

$$\begin{aligned}
\begin{pmatrix} \hat{\xi}_X^{t+1}(u) \\ \hat{\xi}_Z^{t+1}(u) \end{pmatrix} &= \begin{pmatrix} \hat{\mathbf{t}}_{X\to X}(u) & \hat{\mathbf{t}}_{Z\to X}(u) \\ \hat{\mathbf{t}}_{X\to Z}(u) & \hat{\mathbf{t}}_{Z\to Z}(u) \end{pmatrix} \cdot \begin{pmatrix} \hat{\xi}_X^{t}(u) \\ \hat{\xi}_Z^{t}(u) \end{pmatrix} \\
&= \begin{pmatrix} \hat{\mathbf{t}}_{X\to X}(u) \cdot \hat{\xi}_X^t(u) + \hat{\mathbf{t}}_{Z\to X}(u) \cdot \hat{\xi}_Z^t(u) \\ \hat{\mathbf{t}}_{X\to Z}(u) \cdot \hat{\xi}_X^t(u) + \hat{\mathbf{t}}_{Z\to Z}(u) \cdot \hat{\xi}_Z^t(u) \end{pmatrix}.
\end{aligned}$$

In the following, we will omit the hat "ˆ" and the variable u for the sake of a short notation. Furthermore, we will replace ξ_X by ξ_+, ξ_Z by ξ_-, and $\mathbf{t}_{X\to X}$ by $\mathbf{t}_{11}$ etc. to be consistent with the notation introduced in [1]. We then have

$$\xi^{t+1} = \mathbf{t}\xi^t = \begin{pmatrix} \mathbf{t}_{11} & \mathbf{t}_{12} \\ \mathbf{t}_{21} & \mathbf{t}_{22} \end{pmatrix} \begin{pmatrix} \xi_+^t \\ \xi_-^t \end{pmatrix}. \tag{6}$$

Our example glider CQCA now looks like this:

$$\mathbf{t}_G = \begin{pmatrix} 0 & 1 \\ 1 & u^{-1}+u \end{pmatrix}. \tag{7}$$

We have already seen that the CQCAs have to fulfill certain conditions, namely the local rule has to be a translation-invariant homomorphism which maps Pauli products to Pauli products. Furthermore, it has to obey commutation relations with its translates. We have to translate these conditions to the polynomial picture. The matrix we use does not have any dependence on the position on the chain, so translation invariance is already included in this formulation. The commutation relations are encoded in the symplectic form, as we can see from (3). For T to conserve the commutation relations, the corresponding classical automaton has to conserve the symplectic form. This is why the classical CAs that correspond to CQCAs are called symplectic cellular automata (SCAs). In [1] it was proven that a 2×2 matrix $\mathbf{t}$ with Laurent-polynomial entries is a SCA if and only if it fulfills the following conditions:

1. $\det(\mathbf{t}) = u^{2a}$, $a \in \mathbb{Z}$;
2. all entries $\mathbf{t}_{ij}$ are symmetric polynomials centered around the same (but arbitrary) lattice point a;
3. the entries $\mathbf{t}_{1j}$, $\mathbf{t}_{2j}$ of both column vectors, which are the pictures of $(1,0)$ and $(0,1)$, are coprime.

Furthermore, it was shown that to every CQCA there exists a SCA and an appropriate translation-invariant phase function $\lambda(\xi)$ such that

$$\mathrm{T}[\mathbf{w}(\xi)] = \lambda(\xi)\mathbf{w}(\mathbf{t}\xi), \tag{8}$$

$$\lambda(\xi+\eta) = \lambda(\xi)\lambda(\eta)(-1)^{\xi_+\eta_- - (\mathbf{t}\xi)_+(\mathbf{t}\eta)_-}$$

and $|\lambda(\xi)| = 1\ \forall \xi$ hold. Additionally, $\lambda(\xi)$ is uniquely determined for all ξ by the choice of λ on one site. On the other hand, we can find CQCAs for any given SCA by adding a phase function. Thus, CQCAs and SCAs are equivalent up to a phase. We will therefore only refer to CQCAs, even if we talk about the corresponding SCAs.

The last condition, the homomorphism property, is automatically fulfilled because the choice of the phase function and the conservation of the symplectic form $\sigma(\mathbf{t}\xi, \mathbf{t}\eta) = \sigma(\xi,\eta)$. The multiplication of Pauli matrices is mapped to the addition (modulus two) of phase-space vectors. As our matrices are linear transformations they obey $\mathbf{t}(\xi+\eta) = \mathbf{t}\xi + \mathbf{t}\eta$. This translates to $\mathrm{T}([\mathbf{w}(\xi)\mathbf{w}(\eta)] = \mathrm{T}[\mathbf{w}(\xi)]\mathrm{T}[\mathbf{w}(\eta)]$ via

$$\begin{aligned}
&\mathrm{T}[\mathbf{w}(\xi)\mathbf{w}(\eta)] \\
&= (-1)^{-\eta_+\xi_-}\mathrm{T}[\mathbf{w}(\xi+\eta)] \\
&= \lambda(\xi+\eta)\mathbf{w}(\mathbf{t}(\xi+\eta))(-1)^{-\eta_+\xi_-} \\
&= (-1)^{(\mathbf{t}\eta)_+(\mathbf{t}\xi)_-}\mathbf{w}(\mathbf{t}\xi)\mathbf{w}(\mathbf{t}\eta)\lambda(\xi+\eta)(-1)^{-\eta_+\xi_-} \\
&= \lambda(\xi)^{-1}\lambda(\eta)^{-1}\mathrm{T}[\mathbf{w}(\xi)]\mathrm{T}[\mathbf{w}(\eta)]\lambda(\xi)\lambda(\eta) \\
&\quad \cdot(-1)^{\xi_+\eta_- - (\mathbf{t}\xi)_+(\mathbf{t}\eta)_-}(-1)^{-\eta_+\xi_- + (\mathbf{t}\eta)_+(\mathbf{t}\xi)_-} \\
&= \mathrm{T}[\mathbf{w}(\xi)]\mathrm{T}[\mathbf{w}(\eta)](-1)^{\sigma(\xi,\eta)-\sigma(\mathbf{t}\xi,\mathbf{t}\eta)} \\
&= \mathrm{T}[\mathbf{w}(\xi)]\mathrm{T}[\mathbf{w}(\eta)].
\end{aligned}$$

A very simple CQCA is the shift on the lattice, which has the matrix $u^a\mathbb{1}$. It obviously commutes with all other CQCAs. Thus, we can multiply a CQCA with determinant u^{2a} which has entries that are centered around a by a shift by $-a$ sites to obtain a centered CQCA with determinant 1. From now on we will only consider centered CQCAs.

A nice property of these centered CQCAs, which we will need to prove that the entanglement generation is linear, is that their matrices form a multiplicative group. Multiplying the matrices means concatenating the CQCAs.

2. *Classes of CQCAs*

CQCAs show a variety of time evolutions that can be roughly grouped into three classes. The first class shows periodic behavior, the second class consist of CQCAs that have glider observables, which just move on the lattice as shown in Figure 2, and the last case generates fractal