cosmic time t_i

$$\hat{\Omega}_i = \frac{2m_{qi}}{\mathcal{H}_{qi}^2} = \frac{2m_{qi}}{2m_{qi}-k_{qi}}, \qquad \hat{\Omega}_i - 1 = \frac{k_{qi}}{\mathcal{H}_{qi}^2} = \frac{k_{qi}}{2m_{qi}-k_{qi}}, \tag{80}$$

Notice that the sign of $\hat{\Omega}_i - 1$ in any region of r determines the sign of $\hat{\Omega} - 1$ for all times, which is the same as with k_{qi} and k_q. Hence, $\hat{\Omega} - 1$ behaves as the FLRW Omega factor for spatial curvature, so that $\hat{\Omega}_i - 1$ determines the kinematic class: parabolic if $\hat{\Omega}_i = 1$, elliptic if $\hat{\Omega}_i > 1$ and hyperbolic if $0 < \hat{\Omega}_i < 1$. Also, irrespective of the kinematic class we have for every LTB model or region: $\hat{\Omega} \to 1$ as $L \to 0$ (near the central curvature singularity). For all hyperbolic models or regions $\hat{\Omega}$ is bounded between 0 and 1 for all choices of $\hat{\Omega}_i$, $\mathcal{H}_{qi}$ (or m_{qi}, k_{qi}), hence: $\hat{\Omega} \to 0$ as $L \to \infty$, whereas for elliptic models or regions $\hat{\Omega} \to \infty$ as $L \to L_{\text{max}}$ because $\mathcal{H}_q \to 0$.

A. Analytic solutions in terms of $\hat{\Omega}_i$

It is straightforward to parametrize the analytic solutions in terms of $\hat{\Omega}_i$, $\mathcal{H}_{qi}$. For the parabolic case we have $\hat{\Omega}_i = 1$, and thus we just obtain (46) with $\sqrt{2m_{qi}} = \mathcal{H}_{qi}$. For the hyperbolic and elliptic models or regions we substitute

$$m_{qi} = \mathcal{H}_{qi}^2\,\hat{\Omega}_i, \qquad k_{qi} = \mathcal{H}_{qi}^2\,(\hat{\Omega}_i - 1), \tag{81}$$

in (50) and (55). Since $\hat{\Omega}_i$, $\mathcal{H}_{qi}$ are roughly equivalent to inhomogeneous generalization of fiducial Omega and Hubble factors, parametrizing the analytic solutions with these initial value functions could be more intuitive than doing it with the density and spatial curvature profiles m_{qi}, k_{qi}. The solutions (50) and (55) become

- Hyperbolic models or regions: $\hat{\Omega}_i - 1 \le 0$.

$$c(t - t_i) = \frac{W - W_i}{\mathcal{H}_{qi}}. \tag{82}$$

- Elliptic models or regions: $\hat{\Omega}_i - 1 \ge 0$.

$$\mathcal{H}_{qi}\, c(t - t_i) = \begin{cases} W - W_i \qquad \text{expanding phase} \\ \pi\hat{\Omega}_i(\hat{\Omega}_i - 1)^{-3/2} - W - W_i \\ \qquad \text{collapsing phase} \end{cases} \tag{83}$$

where the functions W and W_i are

Hyperbolic models or regions

$$W = \frac{\left[\hat{\Omega}_i + (1-\hat{\Omega}_i)L\right]^{1/2} L^{1/2}}{1-\hat{\Omega}_i} - \frac{\hat{\Omega}_i}{2(1-\hat{\Omega}_i)^{3/2}}\text{arccosh}\left(\frac{2L}{\hat{\Omega}_i} + 1 - 2L\right), \tag{84a}$$

$$W_i = \frac{1}{1-\hat{\Omega}_i} - \frac{\hat{\Omega}_i}{2(1-\hat{\Omega}_i)^{3/2}}\text{arccosh}\left(\frac{2}{\hat{\Omega}_i} - 1\right). \tag{84b}$$

Elliptic models or regions

$$W = \frac{\hat{\Omega}_i}{2(\hat{\Omega}_i - 1)^{3/2}}\arccos\left(\frac{2L}{\hat{\Omega}_i} + 1 - 2L\right) - \frac{\left[\hat{\Omega}_i - (\hat{\Omega}_i - 1)L\right]^{1/2} L^{1/2}}{\hat{\Omega}_i - 1}, \tag{85a}$$

$$W_i = \frac{\hat{\Omega}_i}{2(\hat{\Omega}_i - 1)^{3/2}}\arccos\left(\frac{2}{\hat{\Omega}_i} - 1\right) - \frac{1}{\hat{\Omega}_i - 1}. \tag{85b}$$

Setting $t = t_{\text{bb}}$ and $L = 0$ in (82) and in the expanding phase of (83) yields the bang time for hyperbolic and elliptic configurations

$$ct_{\text{bb}} = ct_i - \frac{W_i}{\mathcal{H}_{qi}}, \tag{86}$$

while the maximal expansion and collapse times, $t = t_{\text{max}}$ and $t = t_{\text{coll}}$, given by (57) are

$$t_{\text{max}} = ct_{\text{bb}} + \frac{\pi\hat{\Omega}_i}{2\mathcal{H}_{qi}[\hat{\Omega}_i - 1]^{3/2}}, \tag{87a}$$

$$t_{\text{coll}} = ct_{\text{bb}} + \frac{\pi\hat{\Omega}_i}{\mathcal{H}_{qi}[\hat{\Omega}_i - 1]^{3/2}}. \tag{87b}$$

We can also parametrize Γ, as well as the Hellaby–Lake conditions, in terms of $\hat{\Omega}_i$, $\mathcal{H}_{qi}$ and their gradients. For