problem wherein the null hypothesis relies on unknown functions R_0 and S_0 indexed by the data-generating distribution P_0.

Other investigators have proposed omnibus tests of hypotheses of the form $\mathcal{H}_0$ versus $\mathcal{H}_1$ in the literature. In the setting of Example 1 above, the work presented in Racine et al. (2006) and Lavergne et al. (2015) is particularly relevant. The null hypothesis of interest in these papers consists of the equality $E_{P_0}(Y \mid A, W) = E_{P_0}(Y \mid W)$ holding almost surely. If individuals have a nontrivial probability of receiving treatment in all strata of covariates, this null hypothesis is equivalent to $\mathcal{H}_0$. In both these papers, kernel smoothing is used to estimate the required regression functions. Therefore, key smoothness assumptions are needed for their methods to yield valid conclusions. The method we present does not hinge on any particular class of estimators and therefore does not rely on this condition.

To develop our approach, we use techniques from the higher-order pathwise differentiability literature (see, e.g., Pfanzagl, 1985; Robins et al., 2008; van der Vaart, 2014; Carone et al., 2014). Despite the elegance of the theory presented by these various authors, it has been unclear whether these higher-order methods are truly useful in infinite-dimensional models since most functionals of interest fail to be even second-order pathwise differentiable in such models. This is especially troublesome in problems in which under the null the first-order derivative of the parameter of interest (in an appropriately defined sense) vanishes, since then there seems to be no theoretical basis for adjusting parameter estimates to recover parametric rate asymptotic behavior. At first glance, the MMD parameter seems to provide one such disappointing example, since its first-order derivative indeed vanishes under the null. The latter fact is a common feature of problems wherein the null value of the parameter is on the boundary of the parameter space. It is also not an entirely surprising phenomenon, at least heuristically, since the MMD achieves its minimum of zero under the null hypothesis. Nevertheless, we are able to show that this parameter is indeed second-order pathwise differentiable under the null hypothesis – this is a rare finding in infinite-dimensional models. As such, we can employ techniques from the recent higher-order pathwise differentiability