to consistently extend the AdS coupling to the UV domain, consistent with pQCD.

## VIII. HOLOGRAPHIC COUPLING IN CONFIGURATION SPACE

In order to obtain modifications to the instantaneous Coulomb potential in configuration space $V(r) = -C_F \alpha_V(r)/r$ from the running coupling, one must transform the coupling defined by the static quark potential $V(q) = -4\pi C_F \alpha_V(q)/q^2$ in the nonrelativistic limit and extract the coefficient of $1/r$ to define the coupling $\alpha_V(r)$ in the $V$ scheme. The couplings are related by the Fourier transform [60]

$$\alpha_V(r) = \frac{2}{\pi}\int_0^\infty dq\, \alpha_V(q) \frac{\sin(qr)}{q}. \tag{17}$$

From (9) we find the expression

$$\alpha_V^{AdS}(r) = C\, \mathrm{erf}(\kappa r) = \frac{2}{\sqrt{\pi}}\, C \int_0^{\kappa r} e^{-t^2} dt, \tag{18}$$

where $C = \alpha_V(Q = 0) = \alpha_V(r \to \infty)$ since $\mathrm{erf}(\mathrm{x} \to +\infty) = 1$. We have written explicitly the normalization at $Q = 0$ in the $V$ scheme since it is not expected to be equal to the normalization in the $g_1$ scheme for the reasons discussed in Sec. VII.

The couplings in the $V$ and $g_1$ schemes are related at leading twist by the CSR: [10]

$$\frac{\alpha_V(Q^2)}{\pi} = \frac{\alpha_{g_1}(Q^{*2})}{\pi} - 1.09\left(\frac{\alpha_{g_1}(Q^{**2})}{\pi}\right)^2 + 25.6\left(\frac{\alpha_{g_1}(Q^{**2})}{\pi}\right)^3 + \cdots, \tag{19}$$

with $Q^* = 1.18\, Q$, $Q^{**} = 2.73\, Q$, and we set $Q^{***} = Q^{**}$. We have verified that this relation numerically holds at least down to $Q^2 = 0.6$ GeV$^2$, as shown in the figure in the Appendix (Fig. 7). In order to transform $\alpha_{g_1}(Q^2)$ into $\alpha_V(Q^2)$ over the full $Q^2$ range, we extrapolate the CSR to the nonperturbative domain. For guidance, we use the fact that QCD is near conformal at very small $Q$; thus, the ratio $\alpha_V/\alpha_{g_1}$ is $Q$ independent. A model for the ratio $\alpha_V(Q)/\alpha_{g_1}(Q)$ is shown in Fig. 4. We apply this ratio to $\alpha^{AdS}_{Modified,g_1}(Q)$, Eq. (11), and then Fourier transform the result using Eq. (17) to obtain $\alpha^{AdS}_{Modified,V}(r)$. We find $C \simeq 2.2$.

### A. Comparison of $V$ and $g_1$ Results

The right panel of Fig. 5 displays $\alpha_V^{AdS}(r)$ (dashed line) and $\alpha_V(r)$ obtained with the same procedure but applied to the JLab data (lower cross-hatched band). Also shown for