Take a change $z = r_+/r$, Eqs. (3.4) and (3.5) can be rewritten as

$$\psi'' + \frac{f'}{f}\psi' + \frac{r_+^2}{z^4}\left(\frac{\phi^2}{f^2} - \frac{m^2}{f}\right)\psi = 0\,, \tag{3.11}$$

$$\phi'' - \frac{r_+^2}{z^4}\frac{2\psi^2}{f}\phi = 0\,, \tag{3.12}$$

where the prime denotes differentiation with z. Regularity of the functions at the horizon $z = 1$ requires

$$\begin{aligned} \psi(1) &= \frac{3}{2m^2L^2}\psi'(1)\,, \\ \phi(1) &= 0\,. \end{aligned} \tag{3.13}$$

And near the boundary $z = 0$ we have

$$\begin{aligned} \psi &= C_- z^{\lambda_-} + C_+ z^{\lambda_+}\,, \\ \phi &= \mu - \frac{\rho}{r_+}z\,. \end{aligned} \tag{3.14}$$

We will set $C_- = 0$ and fix ρ in the following discussion. With the help of the regular horizon boundary condition (3.13), the leading order approximate solutions near the horizon, $z = 1$, for the Eqs. (3.11) and (3.12) can be expressed as

$$\begin{aligned} \psi(z) &= a\left\{1 + \frac{2m^2L^2}{3}(1-z)\right. \\ &\quad \left. + \frac{L^2}{36}\left[(3 + 9\epsilon^2 + 4m^2L^2)m^2 - \frac{4L^2b^2}{r_+^2}\right](1-z)^2 + \cdots\right\}, \end{aligned} \tag{3.15}$$

$$\phi(z) = b\left[(1-z) + \frac{2L^2a^2}{3}(1-z)^2 + \cdots\right], \tag{3.16}$$

where $a \equiv \psi(1)$ and $b \equiv -\phi'(1)$ with $a,\ b > 0$ which makes $\psi(z)$ and $\phi(z)$ positive near the horizon. Matching smoothly the solutions (3.15), (3.16) with (3.14) at an intermediate point z_m with $0 < z_m < 1$, we have

$$C_+ = \frac{6 + 2m^2L^2(1-z_m)}{3[2z_m + \lambda_+(1-z_m)]z_m^{\lambda_+-1}}\,a\,, \tag{3.17}$$

$$b = \frac{r_+}{2L^2}\sqrt{\frac{A}{[\lambda_+ - (\lambda_+ - 2)z_m](1-z_m)}} \equiv \frac{\tilde{b}r_+}{L^2}\,, \tag{3.18}$$

$$a^2 = \frac{3}{4L^2(1-z_m)}\left(\frac{\rho}{br_+}\right)\left(1 - \frac{br_+}{\rho}\right)\,, \tag{3.19}$$

with

$$\begin{aligned} A &= 4L^4m^4(1-z_m)[\lambda_+ - (\lambda_+ - 2)z_m] + 36\lambda_+ \\ &\quad + 3L^2m^2[(1+3\epsilon^2)(\lambda_+ - 2)z_m^2 - 2(5+3\epsilon^2)(\lambda_+ - 1)z_m + 3(3+\epsilon^2)\lambda_+]\,. \end{aligned} \tag{3.20}$$