A Proof of Remark 3.1

Suppose Assumption 2.2 is satisfied, $\lim_{t\to 1}\Xi(t)=\infty$ and conditions (3.3) and (3.4) hold. Then L'Hôpital rule will give (notice that due to (3.4) and continuity of $\sigma_z(t)$ in the vicinity of 1 we have $\lim_{t\to 1}\left(1+\sigma_z^2(t)\right)<2$)

$$\lim_{t\to 1}\lambda^2(t)\Xi(t)\log\log\left(\Xi(t)\right)=\frac{1}{2}\lim_{t\to 1}\left(1+\sigma_z^2(t)\right)\lim_{t\to 1}\left(\Sigma_z(t)+\sigma^2-t\right)\log\log\left(\Xi(t)\right).$$

Since by L'Hôpital rule we have

$$\lim_{t\to 1}\lambda^2(t)\Xi(t)=0,$$

it follows that

$$\begin{aligned}
0 &\leq \lim_{t\to 1}\lambda^2(t)\Xi(t)\log\log\left(\Xi(t)\right)\leq\frac{1}{2}\lim_{t\to 1}\left(1+\sigma_z^2(t)\right)\lim_{t\to 1}\left(\Sigma_z(t)+\sigma^2-t\right)\log\log\left(\lambda^{-2}(t)\right)\\
&= \frac{1}{2}\lim_{t\to 1}\left(1+\sigma_z^2(t)\right)\lim_{t\to 1}\left(\Sigma_z(t)+\sigma^2-t\right)\log\left(\int_0^t\frac{1}{\Sigma_z(s)+\sigma^2-s}ds\right)\\
&= \frac{1}{2}\lim_{t\to 1}\left(1+\sigma_z^2(t)\right)\lim_{t\to 1}\frac{\log(f(t))}{f'(t)},
\end{aligned}$$

where $f(t)=\int_0^t\frac{1}{\Sigma_z(s)+\sigma^2-s}ds$ and $\lim_{t\to 1}f(t)=\infty$. Since $\lim_{x\to\infty}\frac{\log(x)}{x^\alpha}=0$, for any $\alpha>0$ we need to show that

$$\limsup_{t\to 1}\frac{f^\alpha(t)}{f'(t)}<\infty \tag{A.14}$$

for some $\alpha>0$ to establish (3.6).

Consider any $\alpha\in(0,1)$ and denote by

$$0<g(t)=\frac{f^\alpha(t)}{f'(t)},$$

then for $t\geq t^*$ we have

$$f^{1-\alpha}(t)=(1-\alpha)\int_{t^*}^t\frac{1}{g(s)}ds+c$$

where c is some positive constant. Due to this expression and since $\lim_{t\to 1}f(t)=\infty$, $\alpha<1$ and,