It can be shown that $\lim_{N\to\infty} \frac{d_N}{N!} = e^{-1}$ and the limit is approached quite quickly. Thus, the number of derangements is in general very large for large N. Performing optimization over this large combinatorial set of derangements in our problem is a formidable task. For example, in our fair switching problem, we want to maximize the system throughput by scheduling over a subset of derangements. It would be nice if for our problem, the optimal solution is not very sensitive to the particular selection of derangements. In Part B, we will formalize the concept of "condensed derangement sets".

B. Condensed Derangement Set

Definition 1: A set of $N-1$ derangements, $\boldsymbol{D}_1$, $\boldsymbol{D}_2$, $\cdots$, $\boldsymbol{D}_{N-1}$, is said to be a *condensed derangement set* if

$$\sum_{n=1}^{N-1} \boldsymbol{D}_n = \boldsymbol{J} - \boldsymbol{I}, \tag{15}$$

where $\boldsymbol{J}$ is a matrix with all "1" elements, and $\boldsymbol{I}$ is the identity matrix.

With the help of a computer program, we obtain all the four condensed derangement sets for $N = 4$, i.e., $\mathbb{Q}_1 = \{\boldsymbol{P}_1, \boldsymbol{P}_5, \boldsymbol{P}_9\}$, $\mathbb{Q}_2 = \{\boldsymbol{P}_1, \boldsymbol{P}_6, \boldsymbol{P}_8\}$, $\mathbb{Q}_3 = \{\boldsymbol{P}_2, \boldsymbol{P}_4, \boldsymbol{P}_9\}$, and $\mathbb{Q}_4 = \{\boldsymbol{P}_3, \boldsymbol{P}_5, \boldsymbol{P}_7\}$. Furthermore, there are $d_5 = 44$ derangements for $N = 5$ and the number of condensed derangement sets is 56.

In fair switching, we want to switch an equal amount of traffic from any station i to any station j, $i \neq j$. This can be achieved by scheduling the derangements in the condensed derangement set in a weighted round-robin manner (as detailed in "Approach to Problem 2" below). Given a condensed set, the scheduling to achieve fair switching is rather simple. However, unless proven otherwise, different condensed sets may potentially yield solutions of different performance. And the number of condensed derangement sets could be huge for large N. We define a problem as follows.

Problem Definition 2: Suppose that we want to send equal amounts of traffic from S_i to S_j $\forall$ $i \neq j$. Which condensed derangement sets should be used to schedule transmissions? Does it matter?

Approach to Problem 2: The derangements in a condensed derangement set are the building blocks for scheduling. For example, in a complete round transmissions, we may schedule derangement $\boldsymbol{D}_n$ for k_n time slots. Then the length of the complete round transmissions will be $\sum_{n=1}^{N-1} k_n$.

Consider the case of $N = 4$. There are four condensed derangement sets. The question is which condensed derangement set will result in the highest throughput. To answer this question, we can approach the problem as follows.