for $i = 1, \cdots, r$. The logical operators on $U_E |\psi\rangle$ are

$$\bar{Z}_1 = U_E Z_{r+1} U_E^\dagger, \cdots, \bar{Z}_k = U_E Z_n U_E^\dagger,$$

$$\bar{X}_1 = U_E X_{r+1} U_E^\dagger, \cdots, \bar{X}_k = U_E X_n U_E^\dagger.$$

Note that the logical operators commute with the stabilizers, and the normalizer group of $\mathcal{S}$ is

$$\mathcal{N}(\mathcal{S}) = \langle g_1, g_2, \cdots, g_r, \bar{Z}_1, \bar{Z}_2, \cdots, \bar{Z}_k, \bar{X}_1, \bar{X}_2, \cdots, \bar{X}_k \rangle$$

with dimension $2n - r = r + 2k$.

Given a check matrix H of a stabilizer group, Wilde gave an algorithm [12] to find an encoding circuit for this quantum stabilizer code. This algorithm applies a series of CNOT gates, Hadamard gates, Phase gates, SWAP gates, and row operations to the check matrix H such that H takes the form (1). This process is like performing Gaussian elimination on a matrix, but using CNOT gates, Hadamard gates, Phase gates, and SWAP gates, in addition to the elementary row operations of Gaussian elimination. There are two types of elementary row operations over the binary field: adding one row to another, which corresponds to multiplying an operator by another; and exchanging two rows, which corresponds to relabeling two generators. Performing these row operations does not change the row space and hence the error-correcting ability of the codes. The effects of these gate operations on the entries in the check matrix are as follows:

1. A CNOT gate from qubit i to qubit j adds column i to column j in H_X and adds column j to column i in H_Z.
2. A Hadamard gate on qubit i swaps column i in H_Z with column i in H_X.
3. A Phase gate on qubit i adds column i in H_X to column i in H_Z.
4. Three CNOT gates implement a SWAP gate. The effect of a SWAP gate on qubits i