Theorem 6 *(Convex KKM Theorem) Let E be a t.v.s., $\emptyset \neq X \subseteq Y \subseteq E$ with Y convex. If $\Gamma : X \longrightarrow 2^Y$ is a set-valued map verifying:*

(i) Γ is a KKM map;

(ii) all values of Γ are non-empty, closed and convex.

Then, the family $\{\Gamma(x)\}_{x\in X}$ has the finite intersection property.

If in addition, there exists a non-empty subset X_0 of X contained in a convex compact subset D of Y such that $\bigcap_{x\in X_0}\Gamma(x)$ is compact, then $\bigcap_{x\in X}\Gamma(x) \neq \varnothing$.

Proof. We prove that Proposition 3 is equivalent to Theorem 6.

($\Longrightarrow$) Let $\Gamma : X \longrightarrow 2^Y$ be a KKM map with closed convex values. We show by induction on n that $conv(\{x_1, \ldots, x_n\}) \cap \bigcap_{i=1}^n \Gamma(x_i) \neq \emptyset$, for any finite subset $\{x_1, \ldots, x_n\}$ of X.

When $n = 1$, $x_1 = conv(\{x_1\}) \subset \Gamma(x_1)$.

Assume that the conclusion holds true for any set with $n = k$ elements, and let $n = k+1$. Put $C = conv(\{x_1, \ldots, x_n\})$ and $C_i = \Gamma(x_i) \cap C$. Since Γ is KKM, $C \subseteq \bigcup_{i=1}^n \Gamma(x_i)$ which implies $C = \bigcup_{i=1}^n (\Gamma(x_i) \cap C) = \bigcup_{i=1}^n C_i$, a convex set. By the induction hypothesis, for each i, we have $conv(\{x_1, \ldots, \hat{x}_i, \ldots, x_n\}) \cap \bigcap_{j=1, j\neq i}^n \Gamma(x_j) \neq \emptyset$. Proposition 4 implies that $\bigcap_{i=1}^n (\Gamma(x_i) \cap C) \neq \emptyset$, i.e., $\bigcap_{i=1}^n \Gamma(x_i) \neq \emptyset$.

($\Longleftarrow$) Assume $C_1, \ldots, C_n, C = \bigcup_{i=1}^n C$ are closed convex sets in a topological vector space satisfying hypotheses (i) and (ii) of Proposition 3 above.

For each j, let $x_j \in \bigcap_{i=1, i\neq j}^n C_i$ and consider $X = \{x_j\}_{j=1}^n$. The set C being convex, $conv(X) \subseteq C$ and for all j, i with $j \neq i$, $x_j \in C_i$, which implies that $A_i = conv(\{x_j\}_{j=1, j\neq i}^n) \subset C_i$. Define $\Gamma : X \longrightarrow 2^C$ by $\Gamma(x_i) := C_i$ for each $i = 1, ..., n$. The values of Γ are clearly closed and convex. Also, $conv(X) \subseteq C = \bigcup_{i=1}^n (C_i \cap C) = \bigcup_{i=1}^n \Gamma(x_i)$, and for each $\{x_{i_1}, \ldots, x_{i_k}\} \subset X$, we have $conv(\{x_{i_1}, \ldots, x_{i_k}\}) \subset A_{i_j} \subset C_{i_j} = \Gamma(x_{i_j})$ for some $j \neq 1, \ldots, k$. Hence