We now show the necessity of the stated conditions. Suppose that $\sigma_0 = 0$ and $\mathcal{H}_0$ does not hold. It is easy to verify that

$$\begin{aligned}\tilde{D}_1^{\Psi} \triangleq\ & E_{P_0}\left[e^{-[R_0(O)-R_0(o)]^2}\right] + E_{P_0}\left[e^{-[S_0(O)-S_0(o)]^2}\right] \\ & - E_{P_0}\left[e^{-[R_0(O)-S_0(o)]^2}\right] - E_{P_0}\left[e^{-[R_0(o)-S_0(O)]^2}\right] - \psi_0\end{aligned}$$

is a first-order gradient in the model where R_0 and S_0 are known (possibly an inefficient gradient depending on the form of R and S). Call the variance of this gradient $\tilde{\sigma}_0$. As the model where R_0 and S_0 are known is a submodel of the (locally) nonparametric model, $\tilde{\sigma}_0 \leq \sigma_0$, and hence $\tilde{\sigma}_0 = 0$ and $\tilde{D}_1^{\Psi} \equiv 0$. Now, if $\tilde{\sigma}_0 = 0$ and $\mathcal{H}_0$ does not hold, then A.3 shows that $R_0(O)$ and $S_0(O)$ are degenerate. Finally, $\tilde{D}_1^{\Psi} \equiv 0$ and the degeneracy of $R_0(O)$ and $S_0(O)$ shows that for almost all o,

$$D_1^{\Psi}(o) = 2D^{RS}(o) = 2(s_0 - r_0)\left(D_0^R(o) - D_0^S(o)\right)e^{-[r_0-s_0]^2},$$

where we use r_0 and s_0 to denote the (probability 1) values of $R_0(O)$ and $S_0(O)$. The above is zero almost surely if and only if $D_0^R \equiv D_0^S$. Thus $\sigma_0 = 0$ only if (i) or (ii) holds. $\square$

We give the following lemma before proving Theorem 2. Before giving the lemma, we define the function $\Pi : \mathcal{S} \to \mathbb{R}$. Suppressing the dependence on P_0 and h_1, h_2, for all $V \in \mathcal{S}$ and $t \neq 0$ we define

$$\begin{aligned}\Pi(V) \triangleq 2\int\int \Big[&2(V_0(o_2) - V_0(o_1))\dot{V}_0(o_2)h_1(o_2) + 2(V_0(o_2) - V_0(o_1))^2\dot{V}_0(o_2)^2 \\ &+ h_2(o_2) - \dot{V}_0(o_2)^2 + (V_0(o_2) - V_0(o_1))\ddot{V}_0(o_2)\Big]e^{-[V_0(o_2)-V_0(o_1)]^2}dP_0(o_2)dP_0(o_1).\end{aligned}$$

Lemma A.4. *For any fluctuation submodel consistent with (A.4), $T, U \in \mathcal{S}$ with $T_0(O) \overset{d}{=} U_0(O)$,*