$$
\begin{aligned}
&P_S^q \xrightarrow[q\to\infty]{} 0 \\
&P_{\max}(W) \xrightarrow[q\to\infty]{} \frac{1}{e}.
\end{aligned} \tag{22}
$$

What property of the simple W state makes it violate the validity of the approximation? Note that for each qubit, the GHZ and balanced generalized W states have an equal number of zeros and ones across all the basis states. The simple W state obviously does not have this property. To understand why this is important, let us reconsider the maximization problem of finding the nearest product state $|\varphi\rangle$.

We can explore the possible product states by taking a small variation around some basis state $|k_0\rangle$. For two binary numbers k_1 and k_2, let us denote the *Hamming distance* between them, which is the number of bits they differ on, as $d(k_1, k_2)$. We can then divide the set S to disjoint subsets according to the Hamming distance from k_0:

$$
\begin{aligned}
&S = S_0 \uplus \ldots \uplus S_q \\
&S_m = \{k \in S \ : \ d(k, k_0) = m\}\,.
\end{aligned} \tag{23}
$$

Without loss of generality we can fix $k_0 = 0$ (this can be arranged by applying local NOT gates which do not affect the entanglement nor the Hamming distances). The Hamming distance $d(k, k_0)$ is then equal to the number of ones in k, and a small variation around k_0 means that the x_m's are small. The terms in Eq. (13) can then be grouped according to the subsets of S:

$$
f_S^q(x_1, \ldots, x_q) = \frac{1}{\sqrt{|S|}} \sum_{n=0}^{q} \sum_{\substack{j_1 \ldots j_q \in S_n \\ j_{m_1}, \ldots, j_{m_n} = 1}} \sqrt{x_{m_1}} \cdot \cdots \cdot \sqrt{x_{m_n}} \prod_{m \neq m_i} \sqrt{1 - x_m}. \tag{24}
$$

The nth term in this expansion has n multiplicands of the form $\sqrt{x_m}$, so it is dominant in respect to the $(n+1)$th term. Through this expansion we see that the nearest product state is in the close surrounding of $|k_0\rangle$ only if S contains a lot of terms within a small Hamming distance from k_0. In the case of the simple W state, all the terms in S have a Hamming distance of 1 from $k_0 = 0$, thus maximizing the term $n = 1$ in the expansion. This shows that a large value of the function f_S^q can be obtained in the proximity of the state $|0\rangle$, and indeed this is the case.

The case of the balanced generalized W state $|\phi(n, 2n)\rangle$ is different. In this case, for each basis state, there are n^2 basis states at a Hamming distance of 2, and one basis state at