A.2 The ADP test

We want to show that if H_0 is false, then for an arbitrary fixed α, $\lim_{N\to\infty} Pr(S^{ADP}_{m\times m} > S^{tab}_{1-\alpha}) = 1$, where $S^{tab}_{1-\alpha}$ denotes the $1-\alpha$ quantile of the null distribution of $S^{ADP}_{m\times m}$. We use $\mathcal{A}, c, \mathcal{A}_1, \mathcal{A}_2, f_1, f_2$ as defined in the beginning of Appendix A of the main text.

For the ADP test, recall that the partitioning is based on selecting $m-1$ points from $1.5, \ldots, N-0.5$ for the partitions of the ranked x-values, and separately for the partitions of the ranked y-values. For a fixed rectangle, we say a grid point $(i+0.5, j+0.5)$ is in the rectangle if the two x-values with ranks i and $i+1$, and the two y-values with ranks j and $j+1$, are in the rectangle, for $(i,j) \in \{1,\ldots,N\}^2$. Let $\Gamma\{(x_1,y_1),\ldots,(x_N,y_N)\}$ be the set of partitions of size m with at least one grid point in $\mathcal{A}_1$ and at least one grid point in $\mathcal{A}_2$. Let N_{ix} be the number of x-coordinates of the grid points in $\mathcal{A}_i, i \in \{1,2\}$, and N_{iy} be the number of y-coordinates of the grid points in $\mathcal{A}_i, i \in \{1,2\}$.

Let $\mathcal{I} \in \Gamma\{(x_1,y_1),\ldots,(x_N,y_N)\}$ define an (arbitrary fixed) ADP partition in Γ. There exist two x-values in $\mathcal{A}_1$ that are separated by a grid point in $\mathcal{I}$, and two x-values in $\mathcal{A}_2$ that are separated by a grid point in $\mathcal{I}$, denote the average of these two x-values by x_1^* and x_2^*. Let y_1^* and y_2^* be similarly defined for the y-values.

Let C be the cell defined by the points $(x_i^*, y_i^*), i = 1,2$. The fraction of observed counts in the cell C is a linear combination of empirical cumulative distribution functions

$$\frac{o_C}{N} = \hat{F}_{XY}(x_1^*, y_1^*) + \hat{F}_{XY}(x_2^*, y_2^*) - \hat{F}_{XY}(x_1^*, y_2^*) - \hat{F}_{XY}(x_2^*, y_1^*),$$

and the expected fraction under the null, is a function of the cumulative marginal