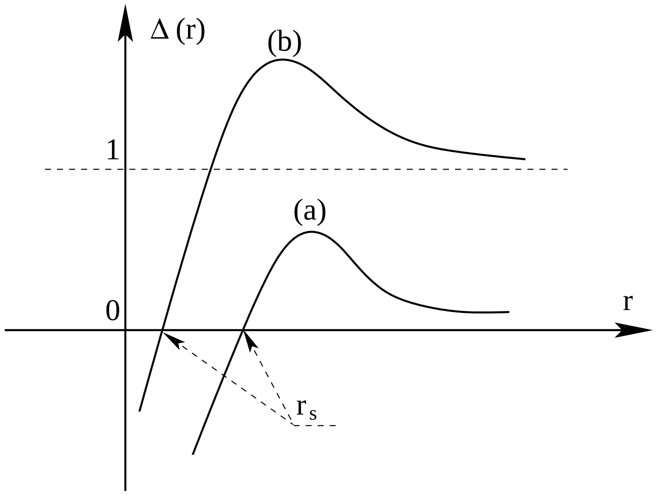

FIG. 6: The function $\Delta(x)$ defined by Eq. (3.22) for the α_- branch with (a) $\lambda = 3,\ \alpha > 0$; and (b) $\lambda > 3$ for any α.

IV. CONCLUSIONS

In this paper, we have studied singularities in the HL theory, and classified them into three different kinds, the scalar, non-scalar, and coordinate singularities, following the classification given in GR [36]. Due to the restricted diffeomorphisms (1.3), the number of the scalars that can be constructed from the extrinsic curvature tensor K_{ij}, the 3-dimensional Riemann tensor R^i_{jkl} and their derivatives is much larger than that constructed from the 4-dimensional Riemann tensor $R^{\sigma}_{\mu\nu\lambda}$ and its derivatives. The latter is invariant under the general Lorentz transformations (1.10). As a result, even for the same spacetime, it may be singular in the HL theory, but not singular in GR. One simple example is the anti-de Sitter Schwarzschild solution written in the ADM form (3.1). This solution is a solution of both HL theory and GR. However, in the HL theory, there are two scalar singularities located, respectively, at the origin $r = 0$ and $r = (3M/|\Lambda|)^{1/3}$, as the two scalars K and $K_{ij}K^{ij}$ become singular at these points. It is well-known that the scalar singularity at $r = (3M/|\Lambda|)^{1/3}$ is absent in GR. This is because both K and $K_{ij}K^{ij}$ are not scalars under the general Lorentz transformations (1.10). Thus, even they are singular at this point, it only represents a coordinate singularity, and all scalars constructed from the 4-dimensional Riemann tensor and its derivatives are finite. On the other hand, due to the restricted transformations (1.3), K and $K_{ij}K^{ij}$ are scalars in the HL theory, and once they are singular, the resulting singularity cannot be transferred away by the restricted transformations. As a result, in the HL theory it represents a real spacetime singularity.

With the above in mind, we have studied the LMP solutions [6], and found that their singularity behavior in the orthogonal frame defined by (3.9) is different from that in the ADM frame defined by (3.1) for the second class of the LMP solutions. In particular, in the orthogonal frame, K_{ij} vanishes, so do K and $K_{ij}K^{ij}$, while the 3-dimensional Ricci scalar R [cf. Eq. (3.18)] can be singular at the origin or infinity, depending on the choice of the parameter λ. However, in the ADM frame at least one of the three scalars K, $K_{ij}K^{ij}$ and R is always singular at two different points, either $r = 0$ and $r = r_s > 0$, or $r = r_1$ and $r = r_2$ with $r_2 > r_1 > 0$, or $r = r_s > 0$ and $r = \infty$, depending on the choice of the free parameter λ, where r_s is a finite non-zero positive constant. This different singular behavior originates from the fact that the two frames are related by the coordinate transformations (3.11), which is not allowed by the foliation-preserving diffeomorphisms (1.3), or in other words, K, $K_{ij}K^{ij}$ and R are not scalars under such transformations. In fact, in the framework of the HL theory, the two frames actually represent two different HL theories, one is with the projectability condition, while the other is without. In particular, the second class of the LMP solutions in the orthogonal frame satisfy the vacuum HL equations, while in the ADM frame they satisfy the HL equations coupled with an anisotropic fluid with heat flow, as shown explicitly in the Appendix.

Our above results show clearly that the problem of singularities in the HL theory is a very delicate problem, due to the restricted diffeomorphisms (1.3), which preserve the ADM foliations (1.2). Further investigations are needed, in particular, in terms of the strength of these singularities. In the examples studied in this paper, all singularities indicated by the two scalars K and $K_{ij}K^{ij}$ at $r = r_s > 0$, including that of the anti-de Sitter Schwarzschild solution, seems weak in the sense of tidal forces and distortions experiencing by observers. Therefore, it is not clear whether or not the spacetime is extendable across such a singularity [37].

Finally, we would like to note that the ADM form (3.1) can be considered as a particular case of the HL theory without projectability condition. Therefore, restricting ourselves only to the HL theory without projectability condition does not solve the singularity problem occurring at $r = r_s$.

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Appendix: The Generalized LMP Solutions with Projectability Condition

The second class of the LMP solutions with projectability condition takes the form of Eq. (3.1) with μ and ν given by Eq. (3.21). Unlike the first class, this one does not satisfy the HL vacuum equations with projectability condition, due to the fact that HL actions