Upon dividing both sides by PQ, (36) becomes

$$(1/P)(\partial_u)^2 P + (1/Q)(\partial_v)^2 Q = (k^2 f^2/4)[2\cosh(2u) - 2\cos(2v)], \tag{37}$$

from which it follows that

$$(1/P)(\partial_u)^2 P - (k^2 f^2/4)[2\cosh(2u)] = -(1/Q)(\partial_v)^2 Q - (k^2 f^2/4)[2\cos(2v)]. \tag{38}$$

Therefore, there is a common separation constant a such that

$$(1/P)(\partial_u)^2 P - (k^2 f^2/4)[2\cosh(2u)] = a \tag{39a}$$

and

$$-(1/Q)(\partial_v)^2 Q - (k^2 f^2/4)[2\cos(2v)] = a. \tag{39b}$$

Correspondingly, P and Q must satisfy the ordinary linear differential equations

$$d^2P/du^2 - [a - 2q\cosh(2u)]P = 0, \tag{40a}$$

$$d^2Q/dv^2 + [a - 2q\cos(2v)]Q = 0, \tag{40b}$$

where

$$q = -k^2 f^2/4. \tag{40c}$$

Equation (40b) for Q is called the *Mathieu* equation, and Equation (40a) for P is called the *modified Mathieu* equation. For our purposes, we will need solutions $Q(v)$ of (40b) that are periodic with period 2π. Such solutions exist only for certain *characteristic values* of the separation constant a. These values are denoted $a_n(q)$ for $n = 0, 1, 2, 3, \cdots$ and $b_n(q)$ for $n = 1, 2, 3, \cdots$. The solutions associated with the separation constants $a = a_n(q)$ are denoted $\mathrm{ce}_n(v, q)$. They are even functions of v and, in the small q limit, are proportional to the functions $\cos(nv)$. The solutions associated with the separation constants $a = b_n(q)$ are denoted $\mathrm{se}_n(v, q)$. They are odd functions of v and, in the small q limit, are proportional to the functions $\sin(nv)$. The functions $\mathrm{ce}_n(v, q)$ and $\mathrm{se}_n(v, q)$ form a complete orthogonal set over the interval $v \in [0, 2\pi]$ and are normalized so that

$$\int_0^{2\pi} dv\ \mathrm{ce}_m(v,q)\ \mathrm{ce}_n(v,q) = \pi\delta_{mn}, \tag{41a}$$

$$\int_0^{2\pi} dv\ \mathrm{se}_m(v,q)\ \mathrm{se}_n(v,q) = \pi\delta_{mn}, \tag{41b}$$

$$\int_0^{2\pi} dv\ \mathrm{ce}_m(v,q)\ \mathrm{se}_n(v,q) = 0. \tag{41c}$$

With regard to the solutions of the modified Mathieu equation, note that (40b) is transformed into (40a) under $v \to iu$. As a result, corresponding (real-valued) solutions to (40a) are defined by $\mathrm{Ce}_n(u, q) = \mathrm{ce}_n(iu, q)$ and $\mathrm{Se}_n(u, q) = -i\,\mathrm{se}_m(iu, q)$. We refer the reader to [3, 4] and [8] for a detailed treatment of the Mathieu functions and their properties.

4. Elliptic Cylinder Harmonic Expansion and On-Axis Gradients

The stage is now set to describe the expansion of any harmonic function ψ in terms of Mathieu functions. The general harmonic function that is analytic in x and y near the origin can be written in the coordinates (24) in the form

$$\psi(u,v,z) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dk\ c_n(k) e^{ikz} \mathrm{Ce}_n(u,q)\ \mathrm{ce}_n(v,q) + \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} dk\ s_n(k) e^{ikz} \mathrm{Se}_n(u,q)\ \mathrm{se}_n(v,q) \tag{42}$$

where the functions $c_n(k)$ and $s_n(k)$ are arbitrary. We will call (42) an *elliptic cylinder harmonic* expansion.

To exploit this expansion, suppose the magnetic field $\boldsymbol{B}(x, y, z)$ is interpolated onto the surface $u = U$ of an elliptic cylinder using values at the grid points near the surface. See Fig. 5. Let us employ the notation $\boldsymbol{B}(x, y, z) = \boldsymbol{B}(u, v, z)$ so that the magnetic field on the surface can be written as $\boldsymbol{B}(U, v, z)$. Next, from the values on the surface, compute $B_u(U, v, z)$, the component of $\boldsymbol{B}(x, y, z)$ *normal* to the surface. Our aim will be to determine the on-axis gradients from a knowledge of $B_u(U, v, z)$. At this point we note that the functions $\exp(ikz)\mathrm{se}_n(v, q)$ and $\exp(ikz)\mathrm{ce}_n(v, q)$ form a complete set over the surface of the elliptical cylinder.

Let us begin by solving (32a) for $(\partial\psi/\partial u)$. We find, using (26), the result,

$$\begin{aligned}(\partial\psi/\partial u) &= f[\cosh^2(u) - \cos^2(v)]^{1/2} B_u \\ &= f(\sinh u \cos v) B_x + f(\cosh u \sin v) B_y.\end{aligned} \tag{43}$$

We see that the right side of (43) is a well-behaved function $F(u, v, z)$ whose values are known for $u = U$,

$$F(U,v,z) = f(\sinh U \cos v) B_x(U,v,z) + f(\cosh U \sin v) B_y(U,v,z). \tag{44}$$

Moreover, using the representation (42) in (43) and (44), we may also write

$$F(U,v,z) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dk\ c_n(k) e^{ikz} \mathrm{Ce}_n'(U,q)\ \mathrm{ce}_n(v,q) + \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} dk\ s_n(k) e^{ikz} \mathrm{Se}_n'(U,q)\ \mathrm{se}_n(v,q). \tag{45}$$

Next multiply both sides of (45) by $\exp(-ik'z)$ and integrate over z. So doing gives the result

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dz\ e^{-ikz} F(U,v,z) = \sum_{n=0}^{\infty} c_n(k) \mathrm{Ce}_n'(U,q)\ \mathrm{ce}_n(v,q) + \sum_{n=1}^{\infty} s_n(k) \mathrm{Se}_n'(U,q)\ \mathrm{se}_n(v,q). \tag{46}$$