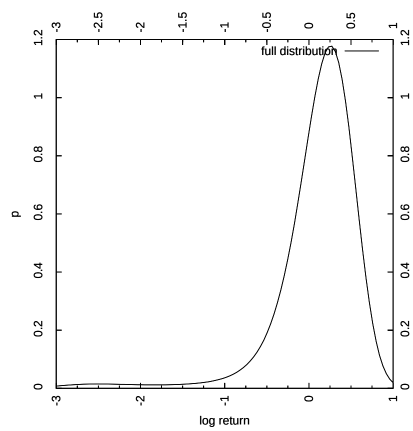


Figure 4: The full distribution contains the risk in the left tail. This balances with the peak above the risk-free rate to give an expected return closer or equal to the risk-free rate.

equal to the risk-free rate and given values for the second, third, and fourth moment. Here we see that the "risk-premium" is really compensated by a fat left-hand tail. The expected value contains both — the risk-reward, if risks do not materialize, and the risk-penalty, if risks do materialize.

Hence we argue that "real distributions" (in particular log-normal with a drift higher than the risk-free rate) are the wrong way to incorporate risk-reward, as they incorporate a risk premium but do not account sufficiently for the downside-risk.

# 4 Connection to Classical Pricing Ideas

So given these concerns, why is Black-Scholes doing a reasonable[9] job? To answer this question let us first note a number of facts.

Firstly, one can argue that due to scale-invariance (as the actual value of a share is meaning less — it scales with the shares in issue) one should look at distributions over log-prices and hence log-returns.

Secondly, the log-normal probability distribution is a normal distribution for log-prices, and hence it is the *maximum-entropy* distribution for *known* variance and mean. This means the distribution makes the "least assumptions" besides what is known (the first two moments). So whether the actual log-returns "are normal" or not is not the question — if we have only given the first two moments as a description of our beliefs then the normal distribution will be the right choice.

Thirdly, if we have no unique extra information (and historic data is available to everyone) then we might want to go with the "markets belief" of an expected return as priced in by the futures.

Given above, the Black-Scholes pricing formulas can easily be derived as the present value of the expected pay-off under a log-normal probability distribution with a mean value of the risk-free rate (see appendix B). This derivation is in fact considerably simpler than the stochastic differential equation approach[10] — for example there is no need to invoke Itô's lemma[11].

Of course, that is not to say that we couldn't do better. Firstly, we might not be certain of the standard deviation, in which case our result incorporates a wrong sense of certainty in a parameter — see section 5.1 below, where we show that an uncertain second moment leads to a skew. Secondly, we might have more information about the probability distribution, *e.g.*, higher moments. Not using such information will of course be suboptimal.

[9] And reasonable does not mean great. The observed skew shows that the Black-Scholes pricing equations are insufficient. See section 5.1, which addresses these issues. See [3] for interesting comments on the history and relevance of Black-Scholes.

[10] It is interesting though that the Greens-function to the Black-Scholes partial differential equation is nothing else but the log-normal distribution with the risk-free rate as mean. So the whole effect of the detour over the stochastic pde is to eliminate the drift.

[11] The equivalent of Itô's lemma in a probabilistic treatment is trivial: If the probability distribution for $\ln(x)$ is a Gaussian with mean $\nu$ and standard deviation $\sigma$ then the expected value of $x$ is $e^{\nu+\sigma^2/2}$. If the expected value of $x$ is known to be $e^r$ then we have to set $\nu = r - \sigma^2/2$.