3 Proposed test: formulation and inference under the null

3.1 Formulation of test

We begin by constructing an estimator of ψ_0 from which a test can then be devised. Using the fact that $\Psi(P) = P^2\Gamma_P$, as implied by (3), we note that if Γ_0 were known, the U-statistic $\mathbb{U}_n\Gamma_0$ would be a natural estimator of ψ_0, where $\mathbb{U}_n$ denotes the empirical measure that places equal probability mass on each of the $n(n-1)$ points (O_i, O_j) with $i \neq j$. In practice, Γ_0 is unknown and must be estimated. This leads to the estimator $\psi_n \triangleq \mathbb{U}_n\Gamma_n$, where we write $\Gamma_n \triangleq \Gamma_{\hat{P}_n}$ for some estimator $\hat{P}_n$ of P_0 based on the available data. Since a large value of ψ_n is inconsistent with $\mathcal{H}_0$, we will reject $\mathcal{H}_0$ if and only if $\psi_n > c_n$ for some appropriately chosen cutoff c_n.

In the nonparametric model considered, it may be necessary, or at the very least desirable, to utilize a data-adaptive estimator $\hat{P}_n$ of P_0 when constructing Γ_n. Studying the large-sample properties of ψ_n may then seem particularly daunting since at first glance we may be led to believe that the behavior of $\psi_n - \psi_0$ is dominated by $P_0^2\,(\Gamma_n - \Gamma_0)$. However, this is not the case. As we will see, under some conditions, $\psi_n - \psi_0$ will approximately behave like $(\mathbb{U}_n - P_0^2)\,\Gamma_0$. Thus, there will be no contribution of $\hat{P}_n$ to the asymptotic behavior of $\psi_n - \psi_0$. Though this result may seem counterintuitive, it arises because $\Psi(P)$ can be expressed as $P^2\Gamma_P$ with Γ_P a second-order gradient (or rather an extension thereof) up to a proportionality constant. More concretely, this surprising finding is a direct consequence of (3).

As further support that ψ_n is a natural test statistic, even when a data-adaptive estimator $\hat{P}_n$ of P_0 has been used, we note that ψ_n could also have been derived using a second-order one-step Newton-Raphson construction, as described in Robins et al. (2008). The latter is