I. INTRODUCTION

Understanding the phase diagram of quantum chromodynamics (QCD) as a function of temperature T and baryon chemical potential μ is an active area of research. Although much is known about the physics at $\mu = 0$ from lattice QCD calculations [1, 2], there is controversy of what might occur at non-zero μ and small values of T [3–5]. Due to the sign problem, which arises in all current formulations of lattice QCD at non-zero μ, it is impossible to perform first principles calculations to settle the controversy today. Most of our knowledge of the (T, μ) phase diagram of QCD is based on models that are motivated from universality and solved using mean field theory. Can at least some of these models be studied from first principles? For example, recently a Landau-Ginzburg approach was used to uncover parts of the phase diagram of QCD where the low energy physics is described by bosonic excitations [6]. In these regions it should be possible to construct bosonic effective field theory models that share the same symmetries, low energy physics and possibly the phase transitions as QCD. It would be interesting to study these models from first principles. Unfortunately, sign problems also arise in bosonic field theories in the presence of a chemical potential when formulated in the conventional approach. For this reason not many first principles studies of field theories with a chemical potential exist. However, many of these sign problems are solvable today and thus allow us to explore the physics of a chemical potential from first principles. It may be useful to study these simpler field theories before attempting to study QCD.

One of the simplest examples of a relativistic bosonic field theory is the classical non-linear $O(2)$ sigma model on a cubic lattice which has been studied extensively in the context of superfluid transitions using the efficient Wolff cluster algorithm [7]. The phase transition is between two phases: an $O(2)$ symmetric phase and a phase where the symmetry is spontaneously broken. Close to the phase transition the low energy physics is described by an interacting quantum field theory of massive charged bosons in the symmetric phase and of massless Goldstone bosons in the broken phase. At the critical point the low energy physics is scale invariant and the critical behavior belongs to the three dimensional XY universality class.

Since the model contains an exact $O(2)$ global symmetry, one can also introduce a chemical potential μ that couples to the corresponding conserved charge. This chemical potential helps one study the “condensed matter” composed of the fundamental boson present in the theory. When $\mu \neq 0$, the action in the conventional formulation becomes complex and Monte Carlo algorithms suffer