In the limit $T/R \to \infty$, when $\mu \to 0$, this equation simplifies to

$$\pi \frac{T}{R} = \ln \frac{4}{\mu}\,. \tag{21}$$

When $z = s$ runs along the real axis, the variable ω runs along the boundary of the rectangle with $s = -1/\sqrt{\mu}, -\sqrt{\mu}, +\sqrt{\mu}, +1/\sqrt{\mu}$ ($\mu < 1$) mapped, respectively, onto the vertices of the rectangle: $(-R/2, T/2), (-R/2, -T/2), (R/2, -T/2), (R/2, T/2)$. The given choice of the argument of the mapping preserves the symmetry $s \to 1/s$.

When z has positive imaginary part, the coordinates

$$X_1(z) = A \,\mathrm{Re}\, F\left(\frac{z}{\sqrt{\mu}}, \mu\right), \qquad X_2(z) = A \,\mathrm{Im}\, F\left(\frac{z}{\sqrt{\mu}}, \mu\right) - \frac{AK\left(\sqrt{1-\mu^2}\right)}{2} \tag{22}$$

take their values inside the rectangle. These coordinates are conformal. For this reason we have

$$x_1(t_*(s)) = A \,\mathrm{Re}\, F\left(\frac{s}{\sqrt{\mu}}, \mu\right), \qquad x_2(t_*(s)) = A \,\mathrm{Im}\, F\left(\frac{s}{\sqrt{\mu}}, \mu\right) - \frac{AK\left(\sqrt{1-\mu^2}\right)}{2}, \tag{23}$$

whose implementation for the function $t_*(s)$ is discussed below.

The boundary contour given by Eq. (23) satisfies Douglas' minimization (see Appendix B, Eq. (B3)). Correspondingly, the Douglas integral

$$\frac{1}{4\pi} \int_{-\infty}^{+\infty} \mathrm{d}s \int_{-\infty}^{+\infty} \mathrm{d}s' \, \frac{\left[x(t_*(s_1)) - x(t_*(s_2))\right]^2}{(s-s')^2} = 2A^2 K(\mu) K\left(\sqrt{1-\mu^2}\right) = RT \tag{24}$$

as it should. We have verified these two equations numerically.

A natural parametrization of the boundary of a rectangle is through $\tau \in S^1$:

$$x_1 = \frac{T}{2}\tan\tau\,, \quad x_2 = -\frac{T}{2} \qquad -\arctan\frac{R}{T} \le \tau \le \arctan\frac{R}{T} \tag{25a}$$

$$x_1 = \frac{R}{2}\,, \quad x_2 = -\frac{R}{2}\cot\tau \qquad \arctan\frac{R}{T} \le \tau \le \pi - \arctan\frac{R}{T} \tag{25b}$$

$$x_1 = \frac{T}{2}\tan\tau\,, \quad x_2 = \frac{T}{2} \qquad \pi - \arctan\frac{R}{T} \le \tau < \pi \tag{25c}$$

and analogously for negative τ. Introducing

$$t = \tan\frac{\tau}{2}, \tag{26}$$

we rewrite Eq. (25) as

$$x_1 = T\frac{t}{1-t^2}\,, \quad x_2 = -\frac{T}{2} \qquad -\frac{\sqrt{T^2+R^2}-T}{R} \le t \le \frac{\sqrt{T^2+R^2}-T}{R} \tag{27a}$$

$$x_1 = \frac{R}{2}\,, \quad x_2 = R\frac{t^2-1}{4t} \qquad \frac{\sqrt{T^2+R^2}-T}{R} \le t \le \frac{\sqrt{T^2+R^2}+T}{R} \tag{27b}$$

$$x_1 = T\frac{t}{t^2-1}\,, \quad x_2 = \frac{T}{2} \qquad \frac{\sqrt{T^2+R^2}+T}{R} \le t < +\infty. \tag{27c}$$