Non-Universal Extinction Transition for Boundary Active Site

S. Burov[1] and D. A. Kessler[1]

[1]*Department of Physics, Bar Ilan University, Ramat-Gan 52900, Israel*

We present a generalized model of a diffusion-reaction system where the reaction occurs only on the boundary. This model reduces to that of Barato and Hinrichsen when the occupancy of the boundary site is restricted to zero or one. In the limit when there is no restriction on the occupancy of the boundary site, the model reduces to an age dependent Galton-Watson branching process and admits an analytic solution. The model displays a boundary-induced phase transition into an absorbing state with rational critical exponents and exhibits aging at criticality below a certain fractal dimension of the diffusion process. Surprisingly the behavior in the critical regime for intermediate occupancy restriction N varies with N. In fact, by varying the lifetime of the active boundary particle or the diffusion coefficient in the bulk, the critical exponents can be continuously modified.

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Nonequilibrium phase transitions can differ significantly from their equilibrium cousins, and are a subject of continuing interest. Recently, Barato and Hinrichsen [1, 2] (BH) studied a reaction-diffusion model apparently exhibiting a new universality class of nonequilibrium phase transitions, boundary induced phase transitions. The model they studied was a variant of a model introduced earlier by Deloubrière and van Wijland [3], who however did not discover the novel scaling behavior. In this letter we generalize the model and solve it analytically in a certain limit. Our model exhibits a phase transition and at criticality we find the presence of aging [4, 5] , a nonequilibrium property observed in such diverse systems as spin-glasses [4], gels [6] and turbulence [7]. Furthermore a critical fractional dimension for the diffusion process is naturally obtained in the context of our model. Interestingly we find that the critical exponents of the model vary continuously with the parameters of the model. We also note that our toy model is interesting in the context of catalytic reactions and biological situations when the reaction occurs on a specific membrane, such as a boundary of a cell.

Our model is defined as follows: As in the BH model, a particle starting on the boundary follows a birth-death process. It produces an offspring (next to the boundary site) with probability p ($A \xrightarrow{p} A + O$) and dies with probability $1-p$ ($A \xrightarrow{1-p} \varnothing$). The particle A continues to reproduce until it dies. The offspring O diffuse freely in the bulk since their birth and up to the time they reach the boundary, and then they start to reproduce again according to the same death-birth process. By varying the rate of offspring production, one expects to reach extinction for $p \to 0$ (absorbing state) and a growing or stable population for $p \to 1$, intuitively one expects a transition between those two states as p grows from 0 to 1. In the BH model, there is a constraint that only one particle can exist on the boundary site at any give time. Any other particle trying to enter the boundary site dies immediately. We generalize this to allow up to N particles to coexist simultaneously on the boundary. As shown by BH, the role of the bulk diffusion is just to generate a probability distribution $\psi(t)$ for the arrival of the newly born particle to the boundary site, where it can, if the boundary site is not fully occupied, reproduce some number of times, setting off new processes and then die. Introducing constraints on the occupancy of the bulk sites has no effect on the global dynamics, due to the indistinguishability of different particles [8]. By totally relaxing the constraint on the boundary occupancy, i.e., $N \to \infty$, we are capable of completely solving the model since technically the unrestricted model is a random branching, or Galton-Watson, process.

In the unrestricted case we can make another simplification, since the time to die is governed by a short-range exponential distribution, we can, without doing any harm to the model, consider all births as happening simultaneously. In this view, the particle generates k children, governed by a geometrical distribution, $p_k = (1-p)p^k$, and then immediately dies. The basic technique employed in the analysis is the use of the generating functional [9] and parallels the solution of the standard age-dependent branching process [10]. Let us denote by $Z(t, t+s)$ the number of particles that been observed on the boundary during the time interval $(t, t+s)$. We define a generating function for the random variable $Z(t, t+s)$ as the average of $s^{Z(t,t+s)}$ over the distribution of $Z(t, t+s)$

$$G_{(t,t+s)}(s) = \sum_{Z(t,t+s)=0}^{\infty} \mathrm{P}(Z(t, t+s))s^{Z(t,t+s)}, \qquad (1)$$

where $\mathrm{P}(Z(t, t+s))$ denotes the probability to observe $Z(t, t+s)$ particles on the boundary in the specified time interval. The generating function for the number of offsprings is $G_O(s) = \langle s^k \rangle = (1-p)/(1-ps)$. We set the initial conditions such that at $t = 0$ a single particle was injected into the bulk and it has the probability distribution $\psi(t)$ to return to the boundary at time $t = t_R$. By conditioning on the outcome of the returning time