hence so is $g_\lambda(z;\mu)$. According to Lemma 6, each element in the set $S_{\mu_0}\backslash\{0\}$ is a zero of the function $g_\lambda(z;\mu_0)$.

Next we show that for any bounded interval L of $\mathbb{R}$, $S_{\mu_0}\bigcap L$ is a finite set. Suppose, to the contrary, $S_{\mu_0}\bigcap L$ is infinite, then it has a limit point in $\mathbb{R}$ by the Bolzano-Weierstrass Theorem [18] and hence, $g_\lambda(z;\mu_0)=0$ on the whole complex plane $\mathbb{C}$ by the Identity Theorem [20]. Then, by (16), (54) and (57), for every $x\in\mathbb{R}$,

$$\int_{-\infty}^{\infty}\phi(y-x)r(y)\mathrm{d}y=0 \tag{69}$$

where

$$r(y)=\log p_Y(y;\mu_0)+\lambda y^2+c \tag{70}$$

and $c=\frac{1}{2}\log(2\pi e)+I(\mu_0)-\frac{q}{1-q}d(0)-\lambda(\gamma+1)$ is a constant.

As in the proof of Lemma 2, there exist $a,b\in\mathbb{R}$ such that $|\log p_Y(y;\mu_0)|\le\frac{1}{2}y^2+ay+b$. As a result, there exist some $\alpha,\beta>0$ such that $|r(y)|\le\alpha y^2+\beta$. Since the convolution of $r(y)$ and the Gaussian density is equal to the zero function by (69), $r(y)$ must be the zero function according to [10, Corollary 9]. This requires the capacity-achieving output distribution $p_Y(y;\mu_0)$ be Gaussian, which cannot be true unless X is Gaussian, which contradicts the assumption that X has a probability mass at 0. Therefore, $S_{\mu_0}\bigcap L$ must be a finite set for any bounded interval L, which further implies that S_{μ_0} is at most countable.

Finally, we show that S_{μ_0} is countably infinite. Suppose, to the contrary, $S_{\mu_0}=\{x_i\}_{i=1}^N$ is a finite set with $\mu_0(\{x_i\})=p_i$ and $|x_i|\le B_1$ for all $i=1,2,\ldots,N$. For any $y>B_1$,

$$p_Y(y;\mu_0)=\sum_{i=1}^N p_i\phi(y-x_i)\le e^{-\frac{(y-B_1)^2}{2}}\ . \tag{71}$$

For any $\epsilon>0$, choose $B_2>0$ such that $\int_{-B_2}^{B_2}\phi(x)\mathrm{d}x>1-\epsilon$. By (16), (54), (57) and (58), for any