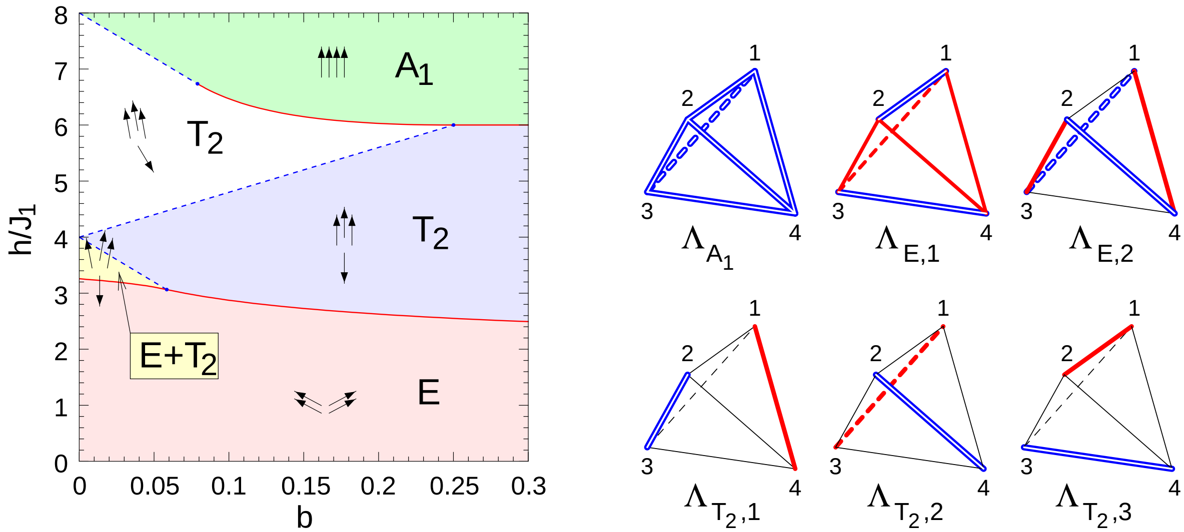

FIG. 3: (Color online) (Left panel) Gound–state phase diagram of a single classical tetrahedron as a function of magnetic field h and dimensionless coupling constant b, taken from Ref. [2]. Solid (red) lines denote first and dashed (blue) lines second order transitions. Spin configurations and relevant irreducible representations (irreps) are shown in each case. (Right panel) Symmetries of the A_1, E, and T_2 irreps of the tetrahedral group $\mathcal{T}_d$ used to classify different states. Solid (red) lines have negative weight; hollow (blue) lines have positive weight. Thin (black) lines have zero weight. See also Eq. (7).

"turned on", focusing on the half–magnetization plateau for $h \approx 4$. For small $|J_3|$, the system now exhibits two characteristic temperature scales — an upper temperature $T^* \approx b$ at which the gap protecting the magnetization plateau opens, and a lower temperature $T_N \approx \mathcal{O}(|J_3|)$ at which the system exhibits long–range magnetic order. This is contrasted with the situation for $h = 0$, where the system also exhibits two characteristic temperature scales, but these correspond to successive phase transitions : a nematic transition at $T_Q \sim b$ and a Néel ordering at $T_N \sim O(|J_3|)$. We discuss the nature of these transitions for $J_3 \to 0$, identifying a line of first–order *multifurcative* points at $J_3 = 0$. And, for $J_3 = 0$, we identify an unusual continuous transition from the coulombic plateau liquid to the vector-multipole phase. At low temperatures this transition appears to have mean field character.

Finally, in Sec. V we conclude with a discussion of the broader implications of these results.

II. DEGENERACIES IN FINITE MAGNETIC FIELD

A. Geometrical arguments

The pyrochlore lattice (Fig. 1) is the simplest example of a three–dimensional (3D) network of corner sharing complete graphs. Its elementary building block is the tetrahedron, in which every site is connected to every other site, i.e. the tetrahedron is a complete graph of order four. Tetrahedra in the pyrochlore lattice can be divided into A and B sublattices, with each lattice site shared between an A- and a B-sublattice tetrahedron. The centres of the two types of tetrahedra together form a (bipartite) diamond lattice.[63] The overall symmetry of the lattice is cubic.

As such, the pyrochlore lattice is a natural 3D analogue of the 2D kagome lattice, a corner sharing network of triangles (complete graphs of order three). In fact the [111] planes of the pyrochlore lattice are alternate kagome and triangular lattices, composed of the triangular "bases" of tetrahedra and their "points", respectively. Much of the unusual physics of the kagome lattice also extends to its higher dimensional cousin.

Lattices composed of complete graphs have the special property that bilinear quantities on nearest neighbor bonds can be recast as a sum of squares. Thus for $b = 0$ the Hamiltonian Eq. (1) can be written

$$\mathcal{H} = 4 \sum_{\text{tetra}} \left(\mathbf{M} - \frac{\mathbf{h}}{8} \right)^2 - \frac{h^2}{16} + \text{const.}\,, \tag{3}$$

where the sum runs over tetrahedra, and

$$\mathbf{M} = \frac{1}{4}(\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4) \tag{4}$$

is the magnetization (per site) of a given tetrahedron. For $h = 0$, a simple classical counting argument shows that two of the eight angles needed to determine the orientation of the four spins in any given tetrahedron remain undetermined. Nearest neighbor interactions do not select one unique ground state on the pyrochlore lattice but rather the entire *manifold* of states for which $|\mathbf{M}| = 0$ in *each* tetrahedron. Thus at $T = 0$, the system is disordered. For fields $h < h_{\rm sat} = 8$ this conclusion is unaltered by the presence of magnetic field. In this case the manifold of ground states is determined by the condition $\mathbf{M} = \mathbf{h}/8$ in each tetrahedron, and the magnetization is linear in h up to the saturation field $h_{\rm sat} = 8$. (We recall that magnetic field is measured in units of J_1, so that in fact $h_{\rm sat} = 8J_1$.)