For an "arbitrary" $\Lambda(r)$, this integral is not likely to be finite, given the asymptotic behavior of $\Lambda(r)$. For certain choices of $\Phi(r)$, however, $\ell(r)$ does exist, thereby producing a traversable wormhole. Examples are the exact solutions for $\Lambda(r)$ obtained by choosing the following redshift functions: $\Phi(r) \equiv$ constant [6] and $\Phi(r) = \ln\frac{k}{r}$ [3, 10].

Next, let us suppose that $\Lambda_1(r)$ is one of the above exact solutions [corresponding to some $\Phi(r)$], so that

$$\ell_1(r) = \int_{r_0}^{r} e^{\Lambda_1(r')} dr'$$

is finite. Given that $\Phi'(r)$ is finite by assumption, we deduce from Eq. (5) that

$$q_1(r) = \Lambda_1'(r) + \frac{1}{K}\frac{1}{2r}\left(e^{2\Lambda_1(r)} - 1\right)(K-1) \qquad (7)$$

is also finite. To generate a new solution, we let $\epsilon = \epsilon(r)$ be a function with a continuous derivative satisfying the conditions $\epsilon(r_0) = \epsilon'(r_0) = 0$ and $\epsilon(r) \to 0$ as $r \to \infty$. Define $\Lambda_2(r) = \Lambda_1(r) + \epsilon(r)$. Then by Eq. (7)

$$q_2(r) = \Lambda_2'(r) + \frac{1}{K}\frac{1}{2r}\left(e^{2\Lambda_2(r)} - 1\right)(K-1) \qquad (8)$$

is finite. Now consider the equation

$$\Lambda_2'(r) = \frac{1}{K}\left[-[\Phi'(r) + \eta'(r)] - \frac{1}{2r}\left(e^{2\Lambda_2(r)} - 1\right)(K-1)\right] \qquad (9)$$

for some finite differentiable function $\eta = \eta(r)$. Writing Eq. (9) in the form

$$\Lambda_1'(r) + \epsilon'(r) = \frac{1}{K}\left[-[\Phi'(r) + \eta'(r)] - \frac{1}{2r}\left(e^{2\Lambda_1(r)+2\epsilon(r)} - 1\right)(K-1)\right], \qquad (10)$$

we see that, by comparison to Eq. (5), $\eta'(r)$ must satisfy the conditions $\eta'(r_0) = 0$ and $\eta(r) \to 0$ as $r \to \infty$, thereby retaining the asymptotic flatness. The function $\eta'(r)$ is defined only implicitly in Eq. (10), but since we are strictly interested in the *existence* of new solutions, it is sufficient to note that the formal relationship $\eta'(r) = -\Phi'(r) - Kq_2(r)$ implies that $\eta'(r)$ exists and is sectionally continuous, so that $\eta = \eta(r)$ exists, as well.

We conclude that $\Lambda_2(r)$ is a solution to Eq. (5) corresponding to some finite redshift function $\Phi(r) + \eta(r)$. Moreover,

$$\ell_2(r) = \int_{r_0}^{r} e^{\Lambda_2(r')} dr' \qquad (11)$$

is finite since $\epsilon(r_0) = 0$. The result is the existence of an entire family of solutions, one for each $\epsilon = \epsilon(r)$.

Similar arguments can be used to obtain new classes of solutions from exact solutions of wormholes supported by Chaplygin and generalized Chaplygin gas [7], modified Chaplygin gas [2], and even van der Waals quintessence [8]. For the case of a generalized Chaplygin gas, the EoS is $p = -A/\rho^\alpha$, $0 < \alpha \leq 1$, and where A is a constant [7]. After substituting and solving for $\Lambda'(r)$, we get [4]

$$\Lambda'(r) = \frac{\frac{1}{2}(8\pi)^{1+1/\alpha} r (r^2)^{1/\alpha} A^{1/\alpha} e^{2\Lambda(r)}}{\left[1 - e^{-2\Lambda(r)} - 2re^{-2\Lambda(r)}\Phi'(r)\right]^{1/\alpha}} - \frac{1}{2r}\left(e^{2\Lambda(r)} - 1\right). \qquad (12)$$

According to Ref. [7], there is an exact solution for $\Phi' \equiv 0$, making $\ell(r)$ finite. In analogous manner, we now let $\Lambda_1(r)$ be an exact solution of Eq. (12). Then $\Lambda_2(r) + \epsilon(r)$ is a new solution, corresponding to some $\Phi(r) + \eta(r)$. (Assume that $\epsilon(r)$ and $\eta(r)$ satisfy the same conditions as before.) The resulting proper distance $\ell_2(r)$ is again finite.

An example that illustrates the problem of generalizing an exact solution even better comes from the modified Chaplygin gas model, whose EoS is $p = A\rho - B/\rho^\alpha$, $0 < \alpha \leq 1$, and where A and B are constants. This time we begin with $\Lambda(r)$ and determine $\Phi(r)$.

According to Ref. [2], one may choose $b(r) = r_0 + d(r - r_0)$, where d must be less than unity to meet the flare-out condition. The shape function $b(r)$ determines $\Lambda(r)$, as well as $\Phi(r)$ [2]:

$$\begin{aligned}\Phi(r) = \phi_0 &+ \frac{1 + Ad}{2(1-d)}\ln|r - r_0| - \frac{1}{2}\ln r \\ &- \frac{32\pi^2 B}{d(1-d)} r_0^4 \ln|r - r_0| \\ &- \frac{32\pi^2 B}{d(1-d)}\left[\frac{1}{4}(r-r_0)^4 + 4r_0^3(r-r_0) \right. \\ &\left. + 3r_0^2(r-r_0)^2 + \frac{4}{3}r_0(r-r_0)^3\right]. \end{aligned} \qquad (13)$$

This $\Phi(r)$ is not finite unless A has the (corrected) value

$$A = \frac{1}{d^2}(64\pi^2 B r_0^4 - d). \qquad (14)$$

Remark: Since A is part of the EoS, it would be more realistic to state the condition as follows: determine d $(0 < d < 1)$, if it exists, so that Eq. (14) is satisfied.

For such a choice of d, we find that

$$\begin{aligned}\Phi(r) = \phi_0 - \frac{1}{2}\ln r - \frac{32\pi^2 B}{d(1-d)}&\left[\frac{1}{4}(r-r_0)^4 \right. \\ &\left. + 4r_0^3(r-r_0) + 3r_0^2(r-r_0)^2 + \frac{4}{3}r_0(r-r_0)^3\right], \end{aligned} \qquad (15)$$

which has a well defined and continuous derivative. If we now substitute Eqs. (2) and (3) in the EoS and solve for