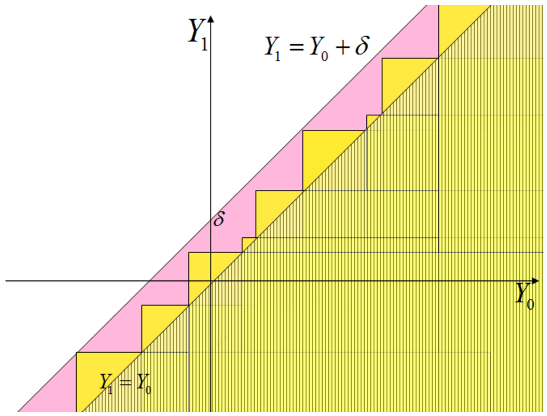

Figure 4: Improved lower bound on the DTE under MTR

where

$$F^L(y_0, y_1) = \max\left\{0, F_0(a_0) + F_1(a_1) - 1, \theta - (F_0(a_0) - F_0(y_0))^+ - (F_1(a_1) - F_1(y_1))^+\right\},$$
$$F^L(y_0, y_1) = \min\left\{F_0(y_0), F_1(y_1), \theta + (F_0(y_0) - F_0(a_0))^+ + (F_1(y_1) - F_1(a_1))^+\right\}.$$

Suppose that marginal distributions F_0 and F_1 are fixed. Lemma 4 shows that sharp bounds on the joint distribution improve when the values of the joint distribution are known at some fixed points. Note that $P(Y_1 \geq Y_0) = 1$ if and only if $F(y, y) = F_1(y)$ for all $y \in \mathbb{R}$. As illustrated in Figure 3,

$$P(Y_0 > Y_1) = P\left[\underset{y \in \mathbb{R}}{\cup} \{Y_0 > y, Y_1 < y\}\right].$$

Therefore,

$$\begin{aligned}
& P(Y_1 \geq Y_0) = 1 \\
\iff & P(Y_0 > Y_1) = 0 \\
\iff & P(Y_0 > y, Y_1 < y) = 0 \text{ for all } y \in \mathbb{R} \\
\iff & F(y, y) = F_1(y), \text{ for all } y \in \mathbb{R}.
\end{aligned}$$

Since for each $y \in \mathbb{R}$ the value of $F(y, y)$ is known from the fixed marginal distribution F_1 under MTR, sharp bounds on the joint distribution can be derived by taking the intersection of the bounds under the restriction $F(y, y) = F_1(y)$ over all $y \in \mathbb{R}$. Technical details are presented in Appendix.