all users in $\mathcal{U}^{(b)}$. Again invoking Proposition 1 we can explicitly detail $\underline{h}(Z_b, b)$, as

$$\underline{h}(Z_b, b) = \begin{cases} \frac{R^{\min}_{\breve{k}-1} - \tilde{Z}_b R_{\breve{k}-1,1}}{R_{\breve{k}-1,b}} + \sum_{j \in \mathcal{U}^{(b)}: j < \breve{k}-1} \frac{R^{\min}_j}{R_{j,b}}, & \breve{k} - 1 \in \mathcal{U}^{(b)} \\ 0, & \text{Else} \end{cases}$$

$$\breve{k} = \min \left\{ k : k \in \mathcal{U}^{(b)} \ \& \sum_{j \in \mathcal{U}^{(b)}: j \geq k} \frac{R^{\min}_j}{R_{j,1}} \leq Z_b \right\}, \ \tilde{Z}_b = Z_b - \sum_{j \in \mathcal{U}^{(b)}: j \geq \breve{k}} \frac{R^{\min}_j}{R_{j,1}} \tag{12}$$

For any given $\Gamma_b$, we let $S(Z_b, \Gamma_b, b), \ b \in \mathcal{B}'_1, Z_b \geq \bar{h}(\Gamma_b, b)$ denote the slope of the function $\hat{O}(Z_b, \Gamma_b, b)$ at $Z_b$. In particular, $S(Z_b, \Gamma_b, b) = \lim_{\delta \to 0_+} \frac{\hat{O}(Z_b + \delta, \Gamma_b, b) - \hat{O}(Z_b, \Gamma_b, b)}{\delta}$. Henceforth, without loss of generality, we assume $\underline{h}(1, b) \leq 1 \ \& \ \bar{h}(1, b) \leq 1, \ \forall \, b \in \mathcal{B}'_1$.

**Proposition 2.** *For any fixed* $\Gamma_b \geq \underline{h}(1, b)$, $\hat{O}(Z_b, \Gamma_b, b)$ *is continuous, non-decreasing, piecewise linear and concave in* $Z_b \in [\bar{h}(\Gamma_b, b), 1]$. *For any fixed* $Z_b \geq \bar{h}(1, b)$, $\hat{O}(Z_b, \Gamma_b, b)$ *is continuous, non-decreasing, piecewise linear and concave in* $\Gamma_b \in [\underline{h}(Z_b, b), 1]$.

*Proof.* We only prove the first claim since proof for the second one follows along similar lines. The continuity and non-decreasing properties are straightforward to verify. It can be shown that the conditions stated in Proposition 1 provide necessary and sufficient conditions to determine an optimal set of allocation fractions for the problem in (9). To verify the other two properties, we start at $Z_b = \bar{h}(\Gamma_b, b)$. Then, if $\bar{h}(\Gamma_b, b) = 0$, the slack at the pico TP $b$, $\Gamma_b - \sum_{j \in \mathcal{U}^{(b)}} \frac{R^{\min}_j}{R_{j,b}}$ must be distributed among users in the decreasing order $\{w_k R_{k,b}\}_{k \in \mathcal{U}^{(b)}}$ subject to their respective maximum rate limits (cf. slack ordering in Proposition 1). On the other hand, when $\bar{h}(\Gamma_b, b) > 0$ there is no slack at the pico TP for this $Z_b$. The next key observation we use is the one in Proposition 1 pertaining to the order in which macro resources are assigned to users in $\mathcal{U}^{(b)}$. Following our labelling, we see that when $Z_b = \bar{h}(\Gamma_b, b) > 0$ either user $\tilde{k} + 1$ (when $\Xi_b > 0$) or user $\tilde{k}$ (when $\Xi_b = 0$) is the user with the largest index in $\mathcal{U}^{(b)}$ to be assigned a positive resource by TP $b$. Let user $k'$ be this user so that users $k \in \mathcal{U}^{(b)} : k < k'$ are assigned resource only by TP $b$ at this $Z_b = \bar{h}(\Gamma_b, b)$. The slope of $\hat{O}(Z_b, \Gamma_b, b)$ can be determined as $S(\bar{h}(\Gamma_b, b), \Gamma_b, b) = \max \left\{ \max \left\{ \frac{w_k R_{k,b} R_{k',1}}{R_{k',b}} : k \in \mathcal{U}^{(b)} \ \& \ k < k' \right\}, \max\{w_k R_{k,1} : k \in \mathcal{U}^{(b)} \ \& \ k \geq k'\} \right\}$. Then, as $Z_b$ is increased to $Z_b + \delta$, for any arbitrarily small $\delta > 0$, the slack is put to the user yielding the slope $S(\bar{h}(\Gamma_b, b), \Gamma_b, b)$ (i.e., offering the maximum bang-per-buck). If such a user is some $\hat{k} \geq k'$