1. *Fluctuations in de Sitter*

By solving the Klein-Gordon equation for a light scalar field in a conformal metric: $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = a^2(\tau, x)(d\tau^2 - dx^2)$, one can find the plane wave solution, $\phi(\mathbf{x}, \tau) = \frac{1}{(2\pi)^{3/2}} \int d^3k \left(\phi_k(\tau) e^{ik\cdot\mathbf{x}} + \text{h.c.}\right)$, for the mode function: [177, 185–191]:

$$\begin{aligned} \phi_k(\tau) &= \left(\frac{\pi}{4}\right)^{1/2} H|\tau|^{3/2} \left(c_1 H_\nu^{(1)}(k\tau) + c_2 H_\nu^{(2)}(k\tau)\right) , \\ \tau &= -H^{-1}e^{-Ht} , \quad \text{and} \quad \nu^2 = \frac{9}{4} - \frac{m^2}{H^2} , \end{aligned} \tag{10}$$

where m is the mass of the scalar field, $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are the Hankel functions and c_1, c_2 are constants. By using a point splitting regularization scheme, it is possible to obtain a Bunch-Davies vacuum for a de Sitter background which actually corresponds to taking $c_1 = 0$, and $c_2 = 1$.

Generically, in a de Sitter phase, the main contribution to the two point correlation function comes from the long wavelength modes; $k|\tau| \ll 1$ or $k \ll H\exp(Ht)$, determined by the Hubble expansion rate [177, 188].

$$\langle\phi^2\rangle \approx \frac{1}{(2\pi)^3} \int_H^{He^{Ht}} d^3k |\phi_k|^2 . \tag{11}$$

The integration yields an indefinite increase in the variance with time

$$\langle\phi^2\rangle \approx \frac{H^3}{4\pi^2} t . \tag{12}$$

This result can also be obtained by considering the Brownian motion of the scalar field [146, 192–195]. For a massive field with $m \ll H$, and $\nu \neq 3/2$, one does not obtain an indefinite growth of the variance of the long wavelength fluctuations, but [178, 186–188, 196]:

$$\langle\phi^2\rangle = \frac{3H^4}{8\pi^2 m^2} \left(1 - e^{-(2m^2/3H^2)t}\right) . \tag{13}$$

In the limiting case when $m \to H$, the variance goes as $\langle\phi^2\rangle \approx H^2$. In the limit $m \gg H$, the variance goes as $\langle\phi^2\rangle \approx (H^3/12\pi^2 m)$. Only in a massless case $\langle\phi^2\rangle$ can be treated as a homogeneous background field with a long wavelength mode.

2. *Adiabatic perturbations and the Sachs-Wolfe effect*

Let us consider small inhomogeneities, $\phi(\mathbf{x}, t) = \phi(t) + \delta\phi(\mathbf{x}, t)$, such that $\delta\phi \ll \phi$. Perturbations in matter densities automatically induce perturbations in the background