Now $l \equiv 3\ (4)$, so $q_1(i)q_{-1}(i) = 4m_p(i) = 4(1 + i + ... + i^l) = 4 \cdot ((1 + i - 1 - i) + (1 + i - 1 - i) + ... + (1 + i - 1)) = 4i$

So $f(i)^2 - p^* g(i)^2 = f(i)^2 + pg(i)^2 = 4i$, and so $y_2^2(1+i^*)^2 + p\xi_2(1-i^*)^2 = 2y_2^2 i^* - 2p\xi_2^2 i^* = 4i$ or, dividing by $2i^* = \pm 2i$,

$$y_2^2 - p\xi_2^2 = \pm 2$$

$$\Rightarrow \quad (y_2 + \sqrt{p}\xi_2)^2(y_2 - \sqrt{p}\xi_2)^2 = 4$$

Now y_2, ξ_2 are odd, else $y_2^2 - p\xi_2^2 \equiv y_2^2 + \xi_2^2 \equiv 0 \not\equiv \pm 2\ (4)$. So the coefficients of $(y_2 + \sqrt{p}\xi_2)^2 = (y_2^2 + p\xi_2^2) + 2y_2\xi_2\sqrt{p}$ are even. We can thus write $a = \frac{(y_2^2 + p\xi_2^2)}{2}$, $b = y_2\xi_2 \in \mathbb{Z}$ and get

$$a^2 - pb^2 = \frac{(y_2 + \sqrt{p}\xi_2)^2(y_2 - \sqrt{p}\xi_2)^2}{2 \cdot 2} = \frac{4}{4} = 1$$

This solves the equation, where

$$(a, b) \quad = \quad \left(\frac{i^*}{4}(pg(i)^2 - f(i)^2)\ ,\ \frac{1}{2}g(i)f(i)\right)$$

where we can directly compute $f(i)$ and $g(i)$

To apply this method to the general case of Pell's Equation (where d is square-free but not necessarily prime), since d is square-free, it can be written as $d = \prod_{k=1}^{r} p_k$ where the p_k's are rational primes. So it suffices to study the case where $d = pq$ for primes p and q and deduce the general case by induction. We will not describe said case in depth here since this paper mainly focuses on prime cyclotomic fields, but we remark that taking $\mathbb{Q}(\zeta_{pq})$, $m_{pq}(x) = m_p(x)m_q(x)\frac{(x^{pq}-1)/(x-1)}{((x^p-1)/(x-1))\cdot((x^q-1)/(x-q))} = \frac{(x^{pq}-1)(x-1)}{(x^p-1)(x^q-1)}$ which can be shown to be irreducible by a similar method as the simple proof for showing that $\sum_{k=0}^{p-1} x^k$ is the minimal polynomial of ζ_p in $\mathbb{Z}[x]$. Following the same reasoning as in the case where $d = p$, we can write $4m_{pq}(x) = f(x)^2 \pm pqg(x)^2$ where $f(x), g(x) \in \mathbb{Z}$. The rest of the problem is solved in a similar fashion as well.

Using some interesting approximation methods and quadratic number fields, Ireland & Rosen [5] show that $x^2 - dy^2 = 1$ has *infinitely many solutions* for any square-free integer