Let us introduce the function $w(t) = e^{\alpha t}\|\phi_{x,\zeta}(t) - \phi_{\widetilde{x},\zeta}(t)\|$. Inequality (7.21) implies that

$$w(t) \leq \frac{2K(M_f + M_h) - KL_1\gamma\epsilon}{\alpha}e^{\alpha R} + \frac{KL_1\gamma\epsilon}{\alpha}e^{\alpha t} + KL_f\int_R^t w(s)ds.$$

Applying Lemma 2.2 [40] to the last inequality we obtain that

$$w(t) \leq \frac{KL_1\gamma\epsilon}{\alpha - KL_f}e^{\alpha t}\left(1 - e^{(KL_f-\alpha)(t-R)}\right) + \frac{2K(M_f + M_h)}{\alpha}e^{KL_f t}e^{-(KL_f-\alpha)R}.$$

Therefore, we have for $t \in [R, R + E_0]$ that

$$\|\phi_{x,\zeta}(t) - \phi_{\widetilde{x},\zeta}(t)\| < \frac{KL_1\gamma\epsilon}{\alpha} + \frac{2K(M_f + M_h)}{\alpha}e^{(KL_f-\alpha)(t-R)}.$$

Since the number E is sufficiently large such that $E \geq \dfrac{2}{\alpha - KL_f}\ln\left(\dfrac{1}{\gamma\epsilon}\right)$, if $t \in [R+E/2, R+E_0]$, then $e^{(KL_f-\alpha)(t-R)} \leq \gamma\epsilon$. Thus,

$$\|\phi_{x,\zeta}(t) - \phi_{\widetilde{x},\zeta}(t)\| < \left[\frac{KL_1}{\alpha - KL_f} + \frac{2K(M_f + M_h)}{\alpha}\right]\gamma\epsilon \leq \epsilon$$

for $t \in [R + E/2, R + E_0]$. It is worth noting that the interval $[R + E/2, R + E_0]$ has a length no less than $E/2$. Consequently, the couple $(\phi_{x,\zeta}(t), \phi_{\widetilde{x},\zeta}(t)) \in \mathscr{B}_\zeta \times \mathscr{B}_\zeta$ is proximal for any sequence $\zeta \in \Theta$. $\square$

Proof of Lemma 4.2

Since the couple $(x(t), \widetilde{x}(t)) \in \mathscr{A} \times \mathscr{A}$ is frequently $(\epsilon_0, \Delta)-$separated, there exist infinitely many disjoint intervals I_k, $k \in \mathbb{N}$, with lengths no less than Δ such that $\|x(t) - \widetilde{x}(t)\| > \epsilon_0$ for each t from these intervals. According to condition $(C6)$ the set of functions $\mathscr{A}$ is an equicontinuous family on $\mathbb{R}$. Therefore, using the uniform continuity of the function $g : \Lambda \times \Lambda \to \mathbb{R}^n$ defined as $g(x_1, x_2) = h(x_1) - h(x_2)$, one can confirm that the family

$$\mathcal{U} = \{h(x(t)) - h(\widetilde{x}(t)) : \ x(t) \in \mathscr{A}, \ \widetilde{x}(t) \in \mathscr{A}\}$$

is also equicontinuous on $\mathbb{R}$. Suppose that $h(x) = (h_1(x), h_2(x), \ldots, h_n(x))$, where each h_j, $j = 1, 2, \ldots, n$, is a real valued function. In accordance with the equicontinuity of the family $\mathcal{U}$, there exists a positive number $\tau < \Delta$, which does not depend on $x(t)$ and $\widetilde{x}(t)$, such that for any $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < \tau$ we have

$$|(h_j(x(t_1)) - h_j(\widetilde{x}(t_1))) - (h_j(x(t_2)) - h_j(\widetilde{x}(t_2)))| < \frac{L_2\epsilon_0}{2\sqrt{n}} \tag{7.22}$$

for each $j = 1, 2, \ldots, n$.

Fix an arbitrary natural number k. Let us denote by s_k be the midpoint of the interval I_k, and set