$$
\begin{aligned}
\{A_i(\vec{x}), A_j(\vec{y})\}^* &= 0, \\
\{A_i(\vec{x}), \pi_j(\vec{y})\}^* &= \delta_{ij}\delta^{(3)}(\vec{x}-\vec{y}), \qquad (3.10)\\
\{\pi_i(\vec{x}), \pi_j(\vec{y})\}^* &= 0,
\end{aligned}
\tag{3.11}
$$

with the following Hamiltonian,

$$
\begin{aligned}
\mathcal{H} = V^{(1)}|_{\Omega=0} &= \frac{1}{2}\pi_i^2 - \frac{1}{2m^2}\pi_i\partial^i\partial_j\pi^j + \frac{1}{4}F_{ij}^2 + \frac{1}{2}m^2A_i^2, \\
&= \frac{1}{2}\pi_i M^i_j \pi^j + \frac{1}{4}F_{ij}^2 + \frac{1}{2}m^2A_i^2,
\end{aligned}
\tag{3.12}
$$

where the phase space metric is

$$
M^i_j = g^i_j - \frac{\partial^i\partial_j}{m^2}\ , \tag{3.13}
$$

which completes the noninvariant analysis.

At this point we are ready to carry out the symplectic gauge-invariant formulation of the Abelian Proca model in order to disclose the gauge symmetry present in the model. To this end, we will extend the symplectic gauge-invariant formalism [28], proposed by three of us in order to unveil, at that time, the gauge symmetry present on the Skyrme model. The basic concept behind the extended symplectic gauge-invariant formalism lives on the extension of the original phase space with the introduction of two arbitrary functions, Ψ and G, where both rely on both the original phase space variables and the WZ variable (θ). The former (Ψ) is introduced into the kinetic sector and, the later (G), within the potential sector of the first-order Lagrangian. The process starts with the computation of Ψ and finishes with the calculation of G.

In order to reformulate the Proca model as a gauge invariant field theory, we will start with the first-order Lagrangian $\mathcal{L}^{(0)}$, given in Eq. (3.2), with the arbitrary terms, given by,

$$
\tilde{\mathcal{L}}^{(0)} = \pi^i\dot{A}_i + \dot{\theta}\Psi - \tilde{V}^{(0)}, \tag{3.14}
$$

with

$$
\tilde{V}^{(0)} = \frac{1}{2}{\pi_i}^2 + \frac{1}{4}F_{ij}^2 + \frac{1}{2}\,m^2\,{A_i}^2 - A_0(\partial_i\pi^i + \frac{1}{2}\,m^2\,A_0) + G, \tag{3.15}
$$

where $\Psi \equiv \Psi(A_i, \pi_i, A_0, \theta)$ and $G \equiv G(A_i, \pi_i, A_0, \theta)$ are the arbitrary functions to be determined. Now, the symplectic fields are $\tilde{\xi}^{(0)}_\alpha = (A_i, \pi_i, A_0, \theta)$ while the symplectic matrix is

$$
f^{(0)} = \begin{pmatrix}
0 & -\delta_{ij} & 0 & \frac{\partial\Psi_y}{\partial A_i^x} \\
\delta_{ji} & 0 & 0 & \frac{\partial\Psi_y}{\partial \pi_i^x} \\
0 & 0 & 0 & \frac{\partial\Psi_y}{\partial A_0^x} \\
-\frac{\partial\Psi_x}{\partial A_j^y} & -\frac{\partial\Psi_x}{\partial \pi_j^y} & -\frac{\partial\Psi_x}{\partial A_0^y} & f_{\theta_x\theta_y}
\end{pmatrix} \delta^{(3)}(\vec{x}-\vec{y}), \tag{3.16}
$$

with

$$
f_{\theta_x\theta_y} = \frac{\partial\Psi_y}{\partial\theta_x} - \frac{\partial\Psi_x}{\partial\theta_y}, \tag{3.17}
$$

where $\theta_x \equiv \theta(x)$, $\theta_y \equiv \theta(y)$, $\Psi_x \equiv \Psi(x)$ and $\Psi_y \equiv \Psi(y)$.

In order to unveil the hidden $U(1)$ gauge symmetry inside the Proca model, the symplectic matrix above must be singular, then, $\Psi \equiv (A_i, \pi_i, \theta)$. As established by the symplectic gauge-invariant formalism, the corresponding zero-mode $\nu^{(0)}(\vec{x})$, identified as being the generator of the symmetry, satisfies the following relation,

$$
\int d^3y\ \nu^{(0)}_\alpha(\vec{x})\ f_{\alpha\beta}(\vec{x}-\vec{y}) = 0, \tag{3.18}
$$

producing a set of equations that allows to determine Ψ explicitly. At this point, it is very important to notice that the extended symplectic gauge-invariant formalism opens up the possibility to disclose the gauge symmetry of the physical model. The zero-mode does not generate a new constraint, however, it determines the arbitrary function Ψ and consequently, obtain the gauge invariant reformulation of the model. We will scrutinize the gauge symmetry related to the following zero-mode,

$$
\bar{\nu}^{(0)} = \begin{pmatrix} \partial_i & 0 & 0 & 1 \end{pmatrix}. \tag{3.19}
$$

Since this zero-mode and the symplectic matrix (3.16) must satisfy the gauge symmetry condition given in Eq. (3.18), a set of equations is obtained and after an integration, Ψ is computed as

$$
\Psi = -\partial_i\pi^i. \tag{3.20}
$$

Hence, the symplectic matrix becomes

$$
f^{(0)} = \begin{pmatrix}
0 & -\delta_{ij} & 0 & 0 \\
\delta_{ji} & 0 & 0 & -\partial_i^y \\
0 & 0 & 0 & 0 \\
0 & \partial_j^x & 0 & 0
\end{pmatrix} \delta^{(3)}(\vec{x}-\vec{y}), \tag{3.21}
$$

which is singular by construction. Due to this, the first-order Lagrangian is

$$
\tilde{\mathcal{L}}^{(0)} = \pi^i\dot{A}_i - \partial_i\pi^i\dot{\theta} - \tilde{V}^{(0)}, \tag{3.22}
$$

with $\tilde{V}^{(0)}$ given in Eq. (3.15).

Now, we start with the second step of the formalism to transform the model into a gauge theory. The zero-mode $\bar{\nu}^{(0)}$ does not produce a constraint when contracted with the gradient of the symplectic potential, namely,

$$
\nu^{(0)}_\alpha \frac{\partial\tilde{V}^{(0)}}{\partial\tilde{\xi}_\alpha} = 0, \tag{3.23}
$$