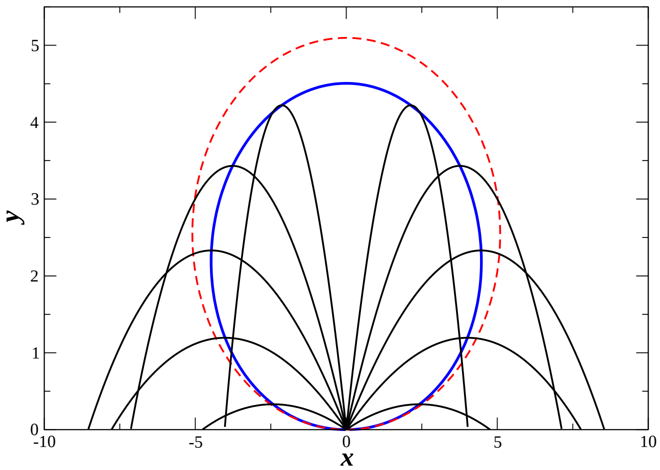

FIG. 1: (color online) Locus $\mathcal{C}_m(\varepsilon)$ formed by the apexes of all the projectile trajectories (continuous line in blue) given by Eqs. (7) and (8) in rectangular coordinates or by Eq. (13), the last one express $\mathcal{C}_m$ in polar coordinates and in term of the Lambert W function. The dashed red line is the ellipse of eccentricity $e = \sqrt{3}/2$ which represents the drag-free case, i.e. $\mathcal{C}_m(0)$. The parameters are $V_0 = 10$ and $\varepsilon = 0.1$.

where we introduce the dimensionless perturbative parameter $\varepsilon \equiv bV_0/g$, the dimensionless length $\rho = V_0^2/g$, and noticing that $\frac{b^2}{g}$ can be expressed as $\frac{\varepsilon^2}{\rho}$. An alternative procedure consists in set the derivative dy/dt to zero to obtain the time of flight to the apex of the trajectory and, evaluate the coordinates at that time. The points (x_m, y_m) conform the locus of apexes $\mathcal{C}_m(\varepsilon)$ for all parabolic trajectories as a function of the launch angle α. In Fig. 1 we plot $\mathcal{C}_m(\varepsilon)$ described by Eqs. (7) and (8), for the drag-free case (in dashed red line) and for $\varepsilon = 0.1$ in continuous blue line. Several projectile trajectories are plotted in thin black lines. The locus of apexes $\mathcal{C}_m(\varepsilon)$ defined by (x_m, y_m) of Eqs. (7) and (8) is described parametrically by the launch angle α and it changes for different values of $\varepsilon \equiv bV_0/g$. In the next section we shall find a description of $\mathcal{C}_m(\varepsilon)$ in terms of polar coordinates and in a closed form using the Lambert W function.

III. THE LOCUS $\mathcal{C}_m$ AS A LAMBERT W FUNCTION

In order to obtain an analytical closed form of the locus we change the variables to polar ones, i.e., $x_m = r_m \cos\theta_m$ and $y_m = r_m \sin\theta_m$. The selection of a description departing from that origin instead of the center or the focus of the ellipse is because the resulting geometrical place is no longer symmetric and the only invariant point is just the launching origin. We substitute the polar forms of x_m and y_m into equations (7) and (8) and rearranging terms it must be expressed as

$$\begin{aligned}\frac{r_m(\theta_m)}{\rho}\cos\theta_m \exp\left(-\varepsilon^2 \sin\theta_m \frac{r_m(\theta_m)}{\rho}\right) \\ = \cos\alpha\sin\alpha\exp(-\varepsilon\sin\alpha).\end{aligned} \tag{9}$$

The lhs depends on r_m and θ_m meanwhile the rhs depends on α, however the last angle is a function of θ_m and reads as

$$\tan\theta_m = \frac{1}{\varepsilon^2}\frac{(\varepsilon\sin\alpha - \ln(1+\varepsilon\sin\alpha))}{\cos\alpha\sin\alpha\frac{1}{1+\varepsilon\sin\alpha}}, \tag{10}$$

by making $\tan\theta_m = y_m/x_m$ from Eqs. (7) and (8).

In order to obtain $\tilde{r}(\theta_m) \equiv r_m(\theta_m)/\rho$ we set

$$f(\alpha(\theta_m)) \equiv \cos\alpha\sin\alpha\exp(-\varepsilon\sin\alpha), \tag{11}$$

since Eq. (10) allows us to have, implicitly, $\alpha(\theta_m)$. We shall return to this point later. Hence, we can write Eq. (9) as

$$-\varepsilon^2\sin\theta_m\,\tilde{r}(\theta_m)\exp\left(-\varepsilon^2\sin\theta_m\,\tilde{r}(\theta_m)\right) = -\varepsilon^2\tan\theta_m\,f(\alpha). \tag{12}$$

Where we multiplied both sides of Eq. (9) by $-\varepsilon^2\sin\theta_m$. Setting $z = -\varepsilon^2\tan\theta_m\,f(\alpha)$ and $W(z) = -\varepsilon^2\sin\theta_m\,\tilde{r}(\theta_m)$ in Eq. (12), it shall have the familiar Lambert W function form, $z = W(z)\exp(W(z))$, from which we can obtain $\tilde{r}$ as

$$\tilde{r}(\theta_m) = -\frac{1}{\varepsilon^2\sin\theta_m}W(-\varepsilon^2\tan\theta_m\,f(\alpha)). \tag{13}$$

It is important to note that the argument of the Lambert function in this equation is negative for all the values $\varepsilon > 0$. $W(x)$ remains real in the range $x \in [-1/e, 0)$ and have the branches denoted by 0 and -1.[2] We select the principal branch, 0, since it is the bounded one, however, for values of $\varepsilon > 1.1$ there is a precision problem since the required argument values are near to $-1/e \equiv -\exp(-1)$. It is important to stress that in Eq. (13) the independent variable is the angle θ and, it constitutes the parameterization of the curve $\mathcal{C}_m$.

The polar expression of $\mathcal{C}_m$ can also be written in terms of the tree function $T(z) = -W(-z)$, giving

$$\tilde{r}(\theta_m) = \frac{1}{\varepsilon^2\sin\theta_m}T(\varepsilon^2\tan\theta_m f(\alpha(\theta_m))). \tag{14}$$

We recover the drag-free result

$$\tilde{r} = 2\frac{\sin\theta_m}{1+3\sin^2\theta_m} \tag{15}$$

when $\varepsilon \to 0$. An explanation of this unfamiliar form of an ellipse is given in appendix A followed by a discussion about the $\varepsilon \to 0$ limit of expression (13) in appendix B.

Formula (13) exhibits the deep relationship between the Lambert W function and the linear drag force projectile problem, since not only the range is given as this function [4, 5]. The problem open the opportunity to study the W function in polar coordinates, that, almost