$$\beta(t) = \sum_{k=1}^{q} V_k \psi_k(t), \tag{8}$$

where $\{\phi_k(t)\}$ and $\{\psi_k(t)\}$ are orthonormal functions in $\mathscr{L}^2(I)$, the U_ks are uncorrelated with $\mathrm{E}(U_k) = 0$ and $\mathrm{var}(U_k) = \gamma_k$, and the V_ks are uncorrelated with $\mathrm{E}(V_k) = 0$ and $\mathrm{var}(V_k) = \lambda_k$. Without loss of generality we assume $\gamma_1 \geq \cdots \geq \gamma_p > 0$ and $\lambda_1 \geq \cdots \geq \lambda_q > 0$.

From (6), (7) and (8) it follows that the total variance of $z_{ij}(t)$, defined as $\mathrm{E}(\|z_{ij} - \mu\|^2)$ with $\|\cdot\|$ the usual $\mathscr{L}^2$-norm, can be decomposed as $\mathrm{E}(\|\alpha\|^2) + \mathrm{E}(\|\beta\|^2)$, where $\mathrm{E}(\|\alpha\|^2) = \sum_{k=1}^{p} \gamma_k$ is the main-factor variance and $\mathrm{E}(\|\beta\|^2) = \sum_{k=1}^{q} \lambda_k$ is the residual-factor variance. The ratio

$$h_z = \frac{\sum_{k=1}^{p} \gamma_k}{\sum_{k=1}^{p} \gamma_k + \sum_{k=1}^{q} \lambda_k} \tag{9}$$

is then the proportion of amplitude variability explained by the main factor. In Section 4 we will derive asymptotic confidence intervals for h_z.

The mean function $\mu(t)$ and the Karhunen–Loève components $\{\phi_k(t)\}$ and $\{\psi_k(t)\}$ are functional parameters that must be estimated from the data, using for instance semiparametric spline models. Let $\mathbf{b}(t) = (b_1(t), \ldots, b_s(t))^T$ be a spline basis in $\mathscr{L}^2(I)$ (for simplicity we will use the same spline basis for all functional parameters, but this is not strictly necessary); then we assume $\mu(t) = \mathbf{b}(t)^T \mathbf{m}$, $\phi_k(t) = \mathbf{b}(t)^T \mathbf{c}_k$, and $\psi_k(t) = \mathbf{b}(t)^T \mathbf{d}_k$, for parameters $\mathbf{m}$, $\mathbf{c}_k$ and $\mathbf{d}_k$ in $\mathbb{R}^s$. Let $\mathbf{C} = [\mathbf{c}_1, \ldots, \mathbf{c}_p] \in \mathbb{R}^{s \times p}$, $\mathbf{D} = [\mathbf{d}_1, \ldots, \mathbf{d}_q] \in \mathbb{R}^{s \times q}$ and $\mathbf{J} = \int_a^b \mathbf{b}(t)\mathbf{b}(t)^T dt \in \mathbb{R}^{s \times s}$. The orthogonality conditions on the ϕ_ks and the ψ_ks translate into the conditions $\mathbf{C}^T \mathbf{J} \mathbf{C} = \mathbf{I}_p$ and $\mathbf{D}^T \mathbf{J} \mathbf{D} = \mathbf{I}_q$ for $\mathbf{C}$ and $\mathbf{D}$. Regarding the U_ks and V_ks in (7) and (8), we assume that $\mathbf{U} = (U_1, \ldots, U_p)^T$ follows a multivariate $N(\mathbf{0}, \mathbf{\Gamma})$ distribution with $\mathbf{\Gamma} = \mathrm{diag}(\gamma_1, \ldots, \gamma_p)$ and that $\mathbf{V} = (V_1, \ldots, V_q)^T$ follows a multivariate $N(\mathbf{0}, \mathbf{\Lambda})$ distribution with $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_q)$. To summarize, the warped process (6) is parameterized by $\mathbf{m}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$.

For the warping functions $w_{ij}(t)$ we cannot simply assume an additive model like (6)