HL theory, due to the restricted diffeomorphisms, these quantities become tensors, and can be easily used to construct various scalars. If any of such scalars is singular, such a singularity cannot be limited by the restricted coordinate transformations (1.3). Then, we may say that the spacetime is singular. It is exactly in this vein that we study singularities in the HL theory. In particular, we first generalize the definitions of scalar, non-scalar and coordinate singularities in GR to the HL theory in Sec. II, and then in Sec. III we study two representative families of spherical static solutions of the HL theory, and identify scalar curvature singularities using the three quantities K, $K_{ij}K^{ij}$ and R. In Sec. IV, we present our main conclusions and remarks. There is also an Appendix, in which we show explicitly that the second class of the LMP solutions written in the ADM frame with projectability condition in general satisfy the HL equations coupled with an anisotropic fluid with heat flow.

Before proceeding further, we would like to note that black holes in GR for asymptotically-flat spacetimes are well-defined [31]. However, how to generalize such definitions to more general spacetimes is still an open question [32, 33]. The problem in the HL theory becomes more complicated [26, 34], partially because of the fact that particles in the HL theory can have non-standard dispersion relations, and therefore no uniform maximal speed exists. As a result, the notion of a horizon is observer-depedent.

II. SINGULARITIES IN HL THEORY

In GR, there are powerful Hawking-Penrose theorems [31], from which one can see that spacetimes with quite "physically reasonable" conditions are singular. Although the theorems did not tell the nature of the singularities, Penrose's cosmic censorship conjecture states that those formed from gravitational collapse in a "physically reasonable" situation are always covered by horizons [35].

To study further the nature of singularities in GR, Ellis and Schmidt divided them into two different kinds, *spacetime curvature singularities* and *coordinate singularities* [36]. The former is real and cannot be made disappear by any Lorentz transformations (1.10), while the latter is coordinate-depedent, and can be made disappear by proper Lorentz transformations. Spacetime curvature singularities are further divided into two sub-classes, *scalar curvature singularities* and *non-scalar curvature singularities*. If any of the scalars constructed from the 4-dimensional Riemann tensor $R^{\sigma}_{\mu\nu\lambda}$ and its derivatives is singular, then the spacetime is said singular, and the corresponding singularity is a scalar one. If none of these scalars is singular, spacetimes can be still singular. In particular, tidal forces and/or distortions (which are the double integrals of the tidal forces), experienced by an observer, may become infinitely large [37]. These kinds of singularities are usually referred to as non-scalar curvature singularities.

To generalize these definitions to the HL theory, as mentioned in the Introduction, both the extrinsic curvature K_{ij} and the 3-dimensional Riemann tensor R^i_{jkl} can be used to construct gauge-invariant quantities, as now they are all tensors under the restricted transformations (1.3). Then, we can see that now there are three kinds of scalars: one is constructed from K_{ij} and its derivatives; one is from the 3-dimensional Riemann tensor R^i_{jkl} and its derivatives; and the other is the mixture of K_{ij}, R^i_{jkl} and their derivatives. Therefore, we may define a scalar curvature singularity in the HL theory as the one where any of the scalars constructed from K_{ij}, R^i_{jkl} and their derivatives is singular. A non-scalar curvature singularity is the one where none of these scalars is singular, but some other physical quantities, such as tidal forces and distortions experienced by observers, become unbounded. Coordinate singularities are the ones that can be limited by the restricted coordinate transformations (1.3).

Several comments now are in order. The scalars constructed from the 3-dimensional tensors K_{ij} and R^i_{jkl} and their derivatives include all scalars constructed from the 4-dimensional $R^{\sigma}_{\mu\nu\lambda}$ and its derivatives. Thus, according to the above definitions, all scalar singularities under the general Lorentz transformations (1.10) are also scalar singularities under the restricted transformations (1.3), but not the other way around. In this sense, scalar singularities in the HL theory are more general than those in GR. One simple example is the anti-de Sitter Schwarzschild solutions, which are also solutions of the SVW generalization with $\xi = 0$, as in this case the 3-dimensional Ricci tensor R_{ij} vanishes identically, and the contributions of high order derivatives of curvature to the potential $\mathcal{L}_V$ are zero, as can be seen from Eq. (1.8). However, as shown in the next section, the corresponding two scalars K and $K_{ij}K^{ij}$ all become singular at $r = (3M/|\Lambda|)^{1/3}$. This singularity is absent in GR [31].

In 3-dimensional space, the Weyl tensor vanishes identically, and the Riemann tensor is determined algebraically by the curvature scalar and the Ricci tensor:

$$
\begin{aligned}
R_{ijkl} &= g_{ik}R_{jl} + g_{jl}R_{ik} - g_{jk}R_{il} - g_{il}R_{jk} \\
&\quad -\frac{1}{2}\big(g_{ik}g_{jl} - g_{il}g_{jk}\big)R. \qquad (2.1)
\end{aligned}
$$

Therefore, the singular behavior of the scalars made of the 3-dimensional Riemann tensor R^i_{jkl} may well be represented by the 3-dimensional curvature scalar R.

III. SINGULARITIES IN SPHERICAL STATIC SPACETIMES

The metric of general spherically symmetric static spacetimes that preserve the ADM form of Eq. (1.2) with the projectability condition can be cast in the form [38],

$$
ds^2 = -dt^2 + e^{2\nu}\left(dr + e^{\mu-\nu}dt\right)^2 + r^2 d\Omega^2, \quad (3.1)
$$