manageable for the envelopes $\bar{D}_R(\cdot) = \sup_{D \in \mathcal{D}_R} |D(\cdot)|$, $\mathbb{E}[\bar{D}_R(\mathbf{z})^2] = O(R)$, and for every $\epsilon > 0$ there is a constant K such that $\mathbb{E}[\bar{D}_R(\mathbf{z})^2 \mathbb{1}\{\bar{D}_R(\mathbf{z}) > K\}] < \epsilon R$.

Condition CRA is similar to, but slightly stronger than, assumptions (iii)-(vii) of the main theorem of Kim and Pollard (1990), to which the reader is referred for a discussion of these assumptions as well as a definition of the term (uniformly) manageable. To be specific, parts (ii)-(iv) and (vi) are identical to their counterparts in Kim and Pollard (1990), part (v) is a locally uniform (with respect to θ near θ_0) version of its counterpart in Kim and Pollard (1990), while (i) can be thought of as replacing the high level condition $\hat{\theta}_n \to_{\mathbb{P}} \theta_0$ with more primitive conditions that imply it for (approximate M-estimators) $\hat{\theta}_n$ satisfying

$$M_n(\hat{\theta}_n) \geq \sup_{\theta \in \mathbf{\Theta}} M_n(\theta) - o_{\mathbb{P}}(n^{-2/3}). \tag{8}$$

In the case of both (i) and (v), the purpose of strengthening the assumptions of Kim and Pollard (1990) is to be able to analyze the bootstrap.

Our main result is the following.

Theorem 1 *Suppose Condition CRA holds and that $\hat{\theta}_n$ satisfies (8). If $\tilde{\mathbf{V}}_n \to_{\mathbb{P}} \mathbf{V}_0$ and if $\tilde{M}_n^*(\tilde{\theta}_n^*) \geq \sup_{\theta \in \mathbf{\Theta}} \tilde{M}_n^*(\theta) - o_{\mathbb{P}}(n^{-2/3})$, then*

$$\sup_{\mathbf{t} \in \mathbb{R}^d} \left| \mathbb{P}^*[\sqrt[3]{n}(\tilde{\theta}_n^* - \hat{\theta}_n) \leq \mathbf{t}] - \mathbb{P}[\sqrt[3]{n}(\hat{\theta}_n - \theta_0) \leq \mathbf{t}] \right| \to_{\mathbb{P}} 0. \tag{9}$$

Under the conditions of the theorem, it follows from Kim and Pollard (1990) that (2) holds, with $\mathcal{G}$ having covariance kernel H. Mimicking the derivation of that result, the proof of the theorem proceeds by establishing the following bootstrap counterpart of (2):

$$\sqrt[3]{n}(\tilde{\theta}_n^* - \hat{\theta}_n) \rightsquigarrow_{\mathbb{P}} \operatorname*{argmax}_{\mathbf{s} \in \mathbb{R}^d} \{\mathcal{Q}(\mathbf{s}) + \mathcal{G}(\mathbf{s})\}. \tag{10}$$

The theorem offers a valid bootstrap-based distributional approximation for $\hat{\theta}_n$. To implement the approximation, only a consistent estimator of $\mathbf{V}_0 = -\partial^2 M(\theta_0)/\partial\theta\partial\theta'$ is needed. A generic