We require that

$$\det(J) = (\partial_{r_*} u)^2 - (\partial_t u)^2 \neq 0 \tag{21}$$

in that part of the manifold which is described by the coordinates $(v, u, \vartheta, \varphi)$. In the next section we will show that the above condition is indeed satisfied. Finally, it is not difficult to verify that the choice

$$\partial_t u + \partial_{r_*} u = -\partial_t v - \partial_{r_*} v, \quad \partial_t u - \partial_{r_*} u = \partial_t v - \partial_{r_*} v$$

is equivalent to the previous one in the sense that both give rise to the same wave equations. Thus, from (20) we can derive the following wave equations:

$$\partial_{tt} u - \partial_{r_* r_*} u = 0, \qquad \partial_{tt} v - \partial_{r_* r_*} v = 0,$$

with the solutions

$$u(t, r_*) = h(r_* + t) + g(r_* - t), \qquad v(t, r_*) = h(r_* + t) - g(r_* - t). \tag{22}$$

Substituting (22) into (14) or (15) gives

$$4 \frac{dh}{dy} \frac{dg}{dz} = F(r_*) \tag{23}$$

with $y := r_* + t$, $z := r_* - t$ whereas (16) gives a trivial identity. Note that (23) fixes the relative signs of the functions h and g since by definition of the function $F(r_*)$ we have $F(r_*) > 0$ for $r > r_+$ and $F(r_*) < 0$ for $r_- < r < r_+$. Moreover, if we substitute (22) into (21) the invertibility condition simplifies to the requirement

$$F(r_*) \neq 0. \tag{24}$$

Clearly, (24) is not satisfied for $r = r_\pm$. This means that on the spheres with radius $r_\pm$ the transformations from spherical coordinates to ones which we are constructing, are not invertible. However, this is not really a problem since our goal is to construct several charts patching different regions of the manifold and by construction we will see that the transfer functions between these charts are always invertible. Finally, if we compute $\partial_{r_*}(23)/(23)$ and $\partial_t(23)/(23)$ with the requirement that $r \neq r_\pm$ we end up with the following equations:

$$\left(\frac{d^2h}{dy^2}\right) / \left(\frac{dh}{dy}\right) + \left(\frac{d^2g}{dz^2}\right) / \left(\frac{dg}{dz}\right) = \frac{1}{F}\frac{dF}{dr_*}, \qquad \left(\frac{d^2h}{dy^2}\right) / \left(\frac{dh}{dy}\right) - \left(\frac{d^2g}{dz^2}\right) / \left(\frac{dg}{dz}\right) = 0.$$