we have solved (3.1). Summarizing, we get a solution from

$$(a,b) \;=\; \left(\frac{1}{4}(g(1)^2+\frac{f(1)^2}{p})\ ,\ \frac{f(1)g(1)}{2p}\right)$$
$$\text{where we can directly compute } f(1) \text{ and } g(1)$$

If $p \equiv 5\ (8)$, $y_1^2+3\xi_1^2 \equiv 4\ (8)$. Given that the only quadratic residues modulo 8 are $0,1,4$, we must have $(y_1^2,\xi_1^2) \equiv (1,1),(0,4)$ or $(4,0)\ \ (8)$.

We now use the fact that $8^2 = 2^{2\cdot 3} = 4^3$ and consider $(y_1+\sqrt{p}\xi_1)^3 = (y_1^3+3p\xi_1^2y_1) + \sqrt{p}(p\xi_1^3+3y_1^2\xi_1) = y_2+\sqrt{p}\xi_2$ and see that $y_2^2-p\xi_2^2 = (y_1^2-p\xi_1^2)^3 = -4^3$.

But $y_2 = y_1(y_1^2+3p\xi_1^2) \equiv y_1(y_1^2-\xi_1^2)\ (8)$. $(y_1^2,\xi_1^2) \equiv (1,1)\ (8) \Rightarrow y_2 \equiv 0\ (8)$. $(y_1^2,\xi_1^2) \equiv (0,4)$ or $(4,0)\ (8) \Rightarrow y_2 \equiv 4\cdot 4,\ 0\cdot 4$ or $\pm 2\cdot 4 \equiv 0\ (8)$. So in any case $y_2 \equiv 0\ (8)$.

Similarly $\xi_2 = \xi_1(p\xi_1^2+3y_1^2) \equiv \xi_1(5\xi_1^2+3y_1^2)\ (8)$. $(y_1^2,\xi_1^2) \equiv (1,1)\ (8) \Rightarrow \xi_2 \equiv \xi_2(5+3) \equiv 0\ (8)$. $(y_1^2,\xi_1^2) \equiv (0,4)$ or $(4,0)\ (8) \Rightarrow \xi_2 \equiv \pm 2\cdot 4,\ 0\cdot 4$ or $4\cdot 0 \equiv 0\ (8)$. So in any case $\xi_2 \equiv 0\ (8)$.

So $8 \mid y_2,\xi_2$ and thus, writing $y_3 = \frac{y_2}{8}, \xi_3 = \frac{\xi_2}{8} \in \mathbb{Z}$, we get $(y_3^2-p\xi_3^2) = \frac{-4^3}{8^2} = -1$. As in the case where $p \equiv 1\ (8)$, writing $(y_3 \pm \sqrt{p}\xi_3)^2 = a \pm b\sqrt{p}$, $a,b \in \mathbb{Z}$, $(x,y) = (a,b)$ is a solution of (3.1). Summarizing, we get a solution from

$$(a,b) = \left(\begin{array}{c} \frac{1}{64}((g(1)^3+\frac{3f(1)^2g(1)}{p})^2+p(\frac{f(1)^3}{p^2}+3\frac{g(1)^2f(1)}{p})^2)\ , \\ \frac{1}{32}(g(1)^3+3\frac{f(1)^2g(1)}{p})(\frac{f(1)^3}{p^2}+3\frac{g(1)^2f(1)}{p}) \end{array}\right)$$

Case 2: $p \equiv 3\ (4)$.

Let $l = \frac{p-1}{2}$. $p \equiv 3\ (4) \Rightarrow l$ is odd. We see that $f(x) = \frac{1}{2}(q_1(x)+q_{-1}(x)) = \prod\limits_{\substack{1\le k<p \\ (\frac{k}{p})=1}}(x-\zeta^k) + \prod\limits_{\substack{1\le k<p \\ (\frac{k}{p})=-1}}(x-\zeta^k)$. f is of degree l. We shall find a relation amongst the coefficients of f by comparing $f(\zeta)$ and $f(\overline{\zeta}) = \overline{f(\zeta)}$ (since $f(x) \in \mathbb{Z}[x]$). Trivially $\left(\frac{1}{p}\right) = 1$, so $\prod\limits_{\substack{1\le k<p \\ (\frac{k}{p})=1}}(\zeta-\zeta^k) = 0$ and so $f(\zeta) = \prod\limits_{\substack{1\le k<p \\ (\frac{k}{p})=-1}}(\zeta-\zeta^k)$. Also note that $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = -1$, and so $\left(\frac{k}{p}\right) = -\left(\frac{-k}{p}\right)$ for all $1 \le k \le p-1$. So $f(\zeta) = \prod\limits_{\substack{1\le k<p \\ (\frac{k}{p})=1}}(\zeta-\zeta^{-k})$. By the same line of reasoning, $f(\overline{\zeta}) = f(\zeta^{-1}) =$