implicitly indicates the association of u to the Macro TP m where $b \in \mathcal{B}_m$. Without loss of generality we suppose that only a tuple (u,b) for any $u \in \mathcal{U}$ & $b \in \mathcal{B}_m, m \in \mathcal{M}$ for which $R_{u,m} + R_{u,b} \geq R_u^{\min}$ is included in $\underline{\Omega}$. This is because any tuple not satisfying this assumtion will never be selected as its minimum rate cannot be met even when the assigned macro and the pico TPs fully allocate their resources to that user. Let $\underline{\Omega}^{(m)} = \{(u,b) \in \underline{\Omega} : b \in \mathcal{B}_m\}$ denote all possible associations to any pico TP in $\mathcal{B}_m$, the set of pico TPs assigned to macro TP m, and let $\underline{\Omega}_{(u')} = \{(u,b) \in \underline{\Omega} : u = u'\}$ denote all possible associations of a user $u' \in \mathcal{U}$. Define a family of sets $\underline{\boldsymbol{\mathcal{I}}}$ as as the one which includes each subset of $\underline{\Omega}$ such that the tuples in that subset have mutually distinct users. Formally, $\underline{\mathcal{A}} \subseteq \underline{\Omega} : |\underline{\mathcal{A}} \cap \underline{\Omega}_{(u)}| \leq 1 \; \forall \; u \in \mathcal{U} \Leftrightarrow \underline{\mathcal{A}} \in \underline{\boldsymbol{\mathcal{I}}}$. Further, define a family, $\underline{\boldsymbol{\mathcal{J}}}$, contained in $\underline{\boldsymbol{\mathcal{I}}}$ that comprises of each member of $\underline{\boldsymbol{\mathcal{I}}}$ for which (1) is feasible. Using the definitions given in the appendix, we see that while $\underline{\boldsymbol{\mathcal{I}}}$ defines a matroid, $\underline{\boldsymbol{\mathcal{J}}}$ is a downward closed family but need not satisfy the exchange property and hence need not define a matroid. Next, we define a non-negative set function on $\underline{\boldsymbol{\mathcal{J}}}$, $f^{\text{wsr}} : \underline{\boldsymbol{\mathcal{J}}} \to I\!R_+$ such that it is normalized, i.e., $f(\phi) = 0$, and for any non-empty set $\underline{\mathcal{G}} \in \underline{\boldsymbol{\mathcal{J}}}$, we have

$$f^{\text{wsr}}(\underline{\mathcal{G}}) = \sum_{m \in \mathcal{M}} f_m^{\text{wsr}}(\underline{\mathcal{G}} \cap \underline{\Omega}^{(m)}). \tag{22}$$

Each $f_m^{\text{wsr}} : \underline{\boldsymbol{\mathcal{J}}}^{(m)} \to I\!R_+$ in (22) is a normalized non-negative set function that is defined on the family $\underline{\boldsymbol{\mathcal{J}}}^{(m)}$ which comprises of each member of $\underline{\boldsymbol{\mathcal{J}}}$ that is contained in $\underline{\Omega}^{(m)}$, as follows. For any set $\underline{\mathcal{A}} \in \underline{\boldsymbol{\mathcal{J}}}^{(m)}$, we define $f_m^{\text{wsr}}(\underline{\mathcal{A}}) = \hat{O}(1, \mathbf{1})$, where $\hat{O}(1, \mathbf{1})$ is computed as described in Algorithm I in Section III-A for the macro TP m and the set of pico TPs $\mathcal{B}_m$ assigned to it, using unit budgets and the given association in $\underline{\mathcal{A}}$. We recall that a simple necessary and sufficient condition to determine feasibility of the minimum rates for the given association and budgets is provided in Proposition 3. With these definitions in hand, can re-formulate the problem in (1) as the following constrained set function maximization problem.

$$\max_{\underline{\mathcal{G}} \in \underline{\boldsymbol{\mathcal{J}}}} \{f^{\text{wsr}}(\underline{\mathcal{G}})\} \tag{23}$$

We offer our first main result that characterizes $f^{\text{wsr}}(.)$.

Theorem 2. *The set function $f^{\text{wsr}}(.)$ is a normalized non-negative submodular set function and*