manifold $\mathcal{N} = \mathcal{M} \times \mathbb{R}$, where $4 = \dim \mathcal{M}$ and $\lambda \in \mathbb{R}$. The background spacetime $\mathcal{M}_0 = \mathcal{N}|_{\lambda=0}$ and the physical spacetime $\mathcal{M} = \mathcal{M}_\lambda = \mathcal{N}|_{\mathbb{R}=\lambda}$ are also submanifolds embedded in the extended manifold $\mathcal{N}$. Each point on $\mathcal{N}$ is identified by a pair (p, λ), where $p \in \mathcal{M}_\lambda$, and each point in $\mathcal{M}_0 \subset \mathcal{N}$ is identified by $\lambda = 0$.

Through this construction, the manifold $\mathcal{N}$ is foliated by four-dimensional submanifolds $\mathcal{M}_\lambda$ of each λ, and these are diffeomorphic to $\mathcal{M}$ and $\mathcal{M}_0$. The manifold $\mathcal{N}$ has a natural differentiable structure consisting of the direct product of $\mathcal{M}$ and $\mathbb{R}$. Further, the perturbed spacetimes $\mathcal{M}_\lambda$ for each λ must have the same differential structure with this construction. In other words, we require that perturbations be continuous in the sense that $\mathcal{M}$ and $\mathcal{M}_0$ are connected by a continuous curve within the extended manifold $\mathcal{N}$. Hence, the changes of the differential structure resulting from the perturbation, for example the formation of singularities and singular perturbations in the sense of fluid mechanics, are excluded from consideration.

Let us consider the set of field equations

$$\mathcal{E}[Q_\lambda] = 0 \tag{2.6}$$

on the physical spacetime $\mathcal{M}_\lambda$ for the physical variables Q_λ on $\mathcal{M}_\lambda$. The field equation (2.6) formally represents the Einstein equation for the metric on $\mathcal{M}_\lambda$ and the equations for matter fields on $\mathcal{M}_\lambda$. If a tensor field Q_λ is given on each $\mathcal{M}_\lambda$, Q_λ is automatically extended to a tensor field on $\mathcal{N}$ by $Q(p, \lambda) := Q_\lambda(p)$, where $p \in \mathcal{M}_\lambda$. In this extension, the field equation (2.6) is regarded as an equation on the extended manifold $\mathcal{N}$. Thus, we have extended an arbitrary tensor field and the field equations (2.6) on each $\mathcal{M}_\lambda$ to those on the extended manifold $\mathcal{N}$.

Tensor fields on $\mathcal{N}$ obtained through the above construction are necessarily "tangent" to each $\mathcal{M}_\lambda$. To consider the basis of the tangent space of $\mathcal{N}$, we introduce the normal form and its dual, which are normal to each $\mathcal{M}_\lambda$ in $\mathcal{N}$. These are denoted by $(d\lambda)_a$ and $(\partial/\partial\lambda)^a$, respectively, and they satisfy $(d\lambda)_a(\partial/\partial\lambda)^a = 1$. The form $(d\lambda)_a$ and its dual, $(\partial/\partial\lambda)^a$, are normal to any tensor field extended from the tangent space on each $\mathcal{M}_\lambda$ through the above construction. The set consisting of $(d\lambda)_a$, $(\partial/\partial\lambda)^a$ and the basis of the tangent space on each $\mathcal{M}_\lambda$ is regarded as the basis of the tangent space of $\mathcal{N}$.

Now, we define the perturbation of an arbitrary tensor field Q. We compare Q on $\mathcal{M}_\lambda$ with Q_0 on $\mathcal{M}_0$, and it is necessary to identify the points of $\mathcal{M}_\lambda$ with those of $\mathcal{M}_0$ as mentioned above. This point identification map is the gauge choice of the second kind as mentioned above. The gauge choice is made by assigning a diffeomorphism $\mathcal{X}_\lambda : \mathcal{N} \to \mathcal{N}$ such that $\mathcal{X}_\lambda : \mathcal{M}_0 \to \mathcal{M}_\lambda$. Following the paper of Bruni et al.[10], we introduce a gauge choice $\mathcal{X}_\lambda$ as an one-parameter groups of diffeomorphisms, i.e., an exponential map, for simplicity. We denote the generator of this exponential map by ${}_{\mathcal{X}}\eta^a$. This generator ${}_{\mathcal{X}}\eta^a$ is decomposed by the basis on $\mathcal{N}$ which are constructed above. Although the generator ${}_{\mathcal{X}}\eta^a$ should satisfy some appropriate properties[8], the arbitrariness of the gauge choice $\mathcal{X}_\lambda$ is represented by the tangential component of the generator ${}_{\mathcal{X}}\eta^a$ to $\mathcal{M}_\lambda$.

The pull-back $\mathcal{X}_\lambda^* Q$, which is induced by the exponential map $\mathcal{X}_\lambda$, maps a tensor field Q on the physical manifold $\mathcal{M}_\lambda$ to a tensor field $\mathcal{X}_\lambda^* Q$ on the background spacetime. In terms of this generator ${}_{\mathcal{X}}\eta^a$, the pull-back $\mathcal{X}_\lambda^* Q$ is represented by the Taylor expansion

$$\begin{aligned} Q(r) &= Q(\mathcal{X}_\lambda(p)) = \mathcal{X}_\lambda^* Q(p) \\ &= Q(p) + \lambda \left.\pounds_{{}_{\mathcal{X}}\eta} Q\right|_p + \frac{1}{2}\lambda^2 \left.\pounds_{{}_{\mathcal{X}}\eta}^2 Q\right|_p \\ &\quad + O(\lambda^3), \end{aligned} \tag{2.7}$$

where $r = \mathcal{X}_\lambda(p) \in \mathcal{M}_\lambda$. Because $p \in \mathcal{M}_0$, we may regard the equation

$$\begin{aligned} \mathcal{X}_\lambda^* Q(p) &= Q_0(p) + \lambda \left.\pounds_{{}_{\mathcal{X}}\eta} Q\right|_{\mathcal{M}_0}(p) + \frac{1}{2}\lambda^2 \left.\pounds_{{}_{\mathcal{X}}\eta}^2 Q\right|_{\mathcal{M}_0}(p) \\ &\quad + O(\lambda^3) \end{aligned} \tag{2.8}$$

as an equation on the background spacetime $\mathcal{M}_0$, where $Q_0 = Q|_{\mathcal{M}_0}$ is the background value of the physical variable of Q. Once the definition of the pull-back of the gauge choice $\mathcal{X}_\lambda$ is given, the first- and the second-order perturbations ${}^{(1)}_{\mathcal{X}}Q$ and ${}^{(2)}_{\mathcal{X}}Q$ of a tensor field Q under the gauge choice $\mathcal{X}_\lambda$ are simply given by the expansion

$$\mathcal{X}_\lambda^* Q_\lambda|_{\mathcal{M}_0} = Q_0 + \lambda {}^{(1)}_{\mathcal{X}}Q + \frac{1}{2}\lambda^2 {}^{(2)}_{\mathcal{X}}Q + O(\lambda^3) \tag{2.9}$$

with respect to the infinitesimal parameter λ. Comparing Eqs. (2.8) and (2.9), we define the first- and the second-order perturbations of a physical variable Q_λ under the gauge choice $\mathcal{X}_\lambda$ by

$${}^{(1)}_{\mathcal{X}}Q := \left.\pounds_{{}_{\mathcal{X}}\eta} Q\right|_{\mathcal{M}_0}, \quad {}^{(2)}_{\mathcal{X}}Q := \left.\pounds_{{}_{\mathcal{X}}\eta}^2 Q\right|_{\mathcal{M}_0}. \tag{2.10}$$

We note that all variables in Eq. (2.9) are defined on $\mathcal{M}_0$.

Now, we consider two *different gauge choices* based on the above understanding of the second kind gauge choice. Suppose that $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$ are two exponential maps with the generators ${}_{\mathcal{X}}\eta^a$ and ${}_{\mathcal{Y}}\eta^a$ on $\mathcal{N}$, respectively. In other words, $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$ are two gauge choices (see Fig. 2). Then, the integral curves of each ${}_{\mathcal{X}}\eta^a$ and ${}_{\mathcal{Y}}\eta^a$ in $\mathcal{N}$ are the orbits of the actions of the gauge choices $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$, respectively. Since we choose the generators ${}_{\mathcal{X}}\eta^a$ and ${}_{\mathcal{Y}}\eta^a$ so that these are transverse to each $\mathcal{M}_\lambda$ everywhere on $\mathcal{N}$, the integral curves of these vector fields intersect with each $\mathcal{M}_\lambda$. Therefore, points lying on the same integral curve of either of the two are to be regarded as *the same point* within the respective gauges. When these curves are not identical, i.e., the tangential components to each $\mathcal{M}_\lambda$ of ${}_{\mathcal{X}}\eta^a$ and ${}_{\mathcal{Y}}\eta^a$ are different, these point identification maps $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$ are regarded as *two different gauge choices*.

We next introduce the concept of *gauge invariance*. In particular, in this paper, we consider the concept of *order by order gauge invariance*[12]. Suppose that $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$