there are two sets of first order constraints, obtained from Eqs. (4) and defined as

$$\begin{aligned} \mathcal{C}_{iab} &\equiv \partial_i g_{ab} - D_{iab} = 0 \ , \\ \mathcal{C}_{ijab} &\equiv \partial_i D_{jab} - \partial_j D_{iab} = 0 \ , \end{aligned} \tag{8}$$

which were introduced when performing the reduction to first order [2, 31]. On the other hand, there are the Hamiltonian and momentum constraints, that in the Generalized Harmonic formulation show up in terms of a four-vector $Z_a$, which is defined as

$$2Z^a \equiv -\Gamma_a - H^a(t, x^i) \, . \tag{9}$$

It can be shown that the Hamiltonian and momentum constraints are satisfied if $Z^a = \partial_t Z^a = 0$ [32]. In order to dynamically control the violation of the constraints, we have included certain terms proportional to these constraints (8)–(9). These additional terms depend on free parameters $\sigma_0$ and $\sigma_1$, allowing one to dynamically damp constraint—including the Hamiltonian, momentum, and first-order constraints (8)—violating modes on a time scale proportional to $-\sigma_i$ ([31, 33]).

We evolve the gravitational field equations shown in Eqs. (5-7). These equations rely on the computation of the 4-dimensional Christoffel symbols from the metric $g_{ab}$

$$\Gamma_{abc} = \frac{1}{2} \left( D_{bca} + D_{cba} - D_{abc} \right) \, . \tag{10}$$

While we evolve the $D_{iab}$ functions, the set $D_{0ab}$ are not evolved, but are calculated from evolved quantities as

$$D_{0ab} = -\alpha Q_{ab} + \beta^k D_{kab} \, . \tag{11}$$

This description suffices to explain the gravitational evolution, and the following section describes the evolution of the matter. However, we note here that the MHD equations are written in the standard 3+1 decomposition of spacetime and thus require the spatial metric $h_{ij}$, the lapse $\alpha$, shift $\beta^i$, and ADM extrinsic curvature, $K_{ij}$. These quantities can be written in terms of our evolved fields using

$$\begin{aligned} h_{ij} &= g_{ij} \ , \quad \alpha = \sqrt{-1/g^{00}} \ , \quad \beta^i = \gamma^{ij} g_{0j} \ , \\ K_{ij} &= \frac{1}{2} Q_{ij} + \frac{1}{\alpha} (D_{(ij)0} - \beta^k D_{(ij)k}). \end{aligned} \tag{12}$$

Conversely, the Hamiltonian and momentum constraints are usually written in terms of spatial derivatives of the metric $D_{kij}$ and the extrinsic curvature $K_{ij}$. In fact, we use these 3+1 quantities (and similar expressions for their derivatives) to calculate the residuals of the Hamiltonian and momentum constraints expressed in their standard form.

### B. MHD equations

We now briefly introduce the perfect fluid equations. Additional information can be found in our previous work [27, 28] as well as in topical review articles [34, 35].

The stress-energy tensor for the perfect fluid in the presence of a Maxwell field is given by

$$\begin{aligned} T_{ab} &= [\rho_o (1+\epsilon) + P] \, u_a u_b + P g_{ab} \\ &+ \ F_a{}^c F_{bc} - \frac{1}{4} g_{ab} F^{cd} F_{cd} \ . \end{aligned} \tag{13}$$

The fluid is described by rest mass density $\rho_o$, the specific internal energy density $\epsilon$, the isotropic pressure $P$ and the four velocity of the fluid $u^a$. With these quantities we can construct the enthalpy

$$h_e = \rho_o + \rho_o \epsilon + P, \tag{14}$$

and construct the standard spatial coordinate velocity of the fluid $v^i$ as

$$W \equiv -n^a u_a, \qquad v^i \equiv \frac{1}{W} \, h^i{}_j u^j, \tag{15}$$

where $W$ is the Lorentz factor between the fluid frame and the fiducial ADM observers.

The Maxwell tensor $F_{ab}$ can be written as

$$F^{ab} = n^a E^b - n^b E^a + \epsilon^{abcd} \ B_c \ n_d, \tag{16}$$

where $E^a$ and $B^a$ are the electric and magnetic fields measured by a “normal” observer $n^a$. Consequently, both fields are purely spatial, i.e., $E^a n_a = B^a n_a = 0$.

The evolution of the magnetized fluid is described by different sets of conservation laws. The magnetic field, in the ideal MHD limit, follows the Maxwell equation

$$\nabla_\mu ({}^*F^{\mu\nu} + g^{\mu\nu} \Psi) \ = \ \kappa n^\nu \Psi \, , \tag{17}$$

where ${}^*F^{ab} \equiv \epsilon^{abcd} F_{cd}/2$ is the dual of the Maxwell tensor and we have introduced a real scalar field $\Psi$ to control the divergence constraint. This technique is known as divergence cleaning [36] and allows for a convenient way to control the constraint violation by inducing a damped wave equation for the scalar field $\Psi$. The other Maxwell equation, in the ideal MHD limit, only gives the definition for the current density, since the electric field is given in terms of the velocity of the fluid and the magnetic field, that is,

$$E_i = -\epsilon_{ijk} v^k B^k \, . \tag{18}$$

Conservation of the stress energy tensor in Eq. (13),

$$\nabla_a T^{ab} = 0, \tag{19}$$

provides 4 evolution equations for the fluid variables, namely the velocity and the internal energy. Conservation of the baryon number

$$\nabla_a (\rho_o u^a) = 0 \tag{20}$$

leads to the evolution equation of the rest mass density $\rho_o$. Closure of the equations is achieved by introducing an equation of state (EOS) relating the pressure with