$$
i\frac{\partial}{\partial t}\Psi(x,y,t) = \Bigg\{ -\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) + V\left(\cos^2(x)+\cos^2(y)\right) - F_{ex}x + v_{so}\begin{bmatrix} 0 & -i\frac{\partial}{\partial x}-\frac{\partial}{\partial y} \\ -i\frac{\partial}{\partial x}+\frac{\partial}{\partial y} & 0 \end{bmatrix}
+ \begin{bmatrix} g_{11}n_1(x,y,t)+g_{12}n_2(x,y,t) & 0 \\ 0 & g_{12}n_1(x,y,t)+g_{22}n_2(x,y,t) \end{bmatrix} \Bigg\}\Psi(x,y,t). \tag{18}
$$

Here,

$$
\Psi(x,y,t) = \begin{bmatrix} \langle x,y|\langle u_1|\Psi(t)\rangle \\ \langle x,y|\langle u_2|\Psi(t)\rangle \end{bmatrix} = \begin{bmatrix} \psi_1(x,y,t) \\ \psi_2(x,y,t) \end{bmatrix} \tag{19}
$$

is a two-component spinor wave function of the dark states, $n_i(x,y,t) = |\psi_i(x,y,t)|^2$, $F_{ex}$ the external longitudinal force, and $g_{11}$, $g_{22}$ and $g_{12}$ are the effective $s$-wave scattering amplitudes related to the scattering lengths $a_i$ as $g_i = N4\pi\hbar^2 a_i/(3E_R\sqrt{2\pi}l_z m)$, where $l_z$ is the width of the single site wave function in the $z$-direction. This definition of $g_i$ implies that the wave function has been normalized as $\int|\Psi(x,y,t)|^2\,dxdy = 1$. In the basis of dark states, the scattering lengths are effective ones different from the regular hyperfine scattering lengths. Nevertheless, the ratio between the two scattering amplitudes should approximately be the same regardless of basis, and, in particular, for a spinor condensate such as that of $^{87}$Rb one has $g_{11} \approx g_{22} \approx g_{12}$ [59]. We will restrict the analysis to repulsive interactions, $g_{12}$, $g_{22}$, $g_{12} > 0$, and we will furthermore assume $g_{11} = g_{22} = g_{12} \equiv g$ [59].

In the absence of a lattice, the SO coupling gives rise to a non-Abelian Mead-Berry curvature [34–36], which results in a transverse splitting of spin components, the intrinsic SHE. However, in Figs. 2 and 3 the motion along the longitudinal $x$-direction was either constant or monotonously accelerating, while for a weakly tilted lattice, where BOs dominate the longitudinal motion, it is neither clear how the transverse spin current nor how the trembling Zitterbewegung motion will be manifested. For instance, will there be sufficient time for a transverse spin current to be established within a single Bloch period? In Ref. [60], the intrinsic anomalous Hall effect was studied in a single particle system confined by an harmonic potential and hence the motion was as well oscillatory as in the case of BOs. The transverse current appeared as a rotation on top of the harmonic oscillatory motion, i.e., in polar coordinates $(r,\varphi)$ the gauge potential causes a non-zero current in the $\varphi$-direction. Such current would indeed be zero for an SO coupling possessing an Abelian structure rather than non-Abelian, e.g. for couplings on the form $v_{so}(\hat{\sigma}_\alpha\hat{p}_x + \hat{\sigma}_\alpha\hat{p}_y)$ with $\alpha = x, y, z$. It is clear that a corresponding rotational current cannot be strictly encountered in the present system since exact polar symmetry is broken by the tilting of the lattice and by the lattice in itself.

The time evolution of the system is solved by employing the split-operator approach [61]. The method relies on factorizing the time-evolution operator into a spatial part and a momentum part valid in the limit of infinitely small time-step propagation. One convenient feature of this method is that the wave function is in principle obtained at any instant of time in both position $\Psi(x,y,t)$ and momentum $\tilde{\Psi}(p_x,p_y,t)$ space. Furthermore, for a system bounded from below, the approach is capable of giving an approximate ground state of the system by simply propagating it sufficiently long in imaginary time.

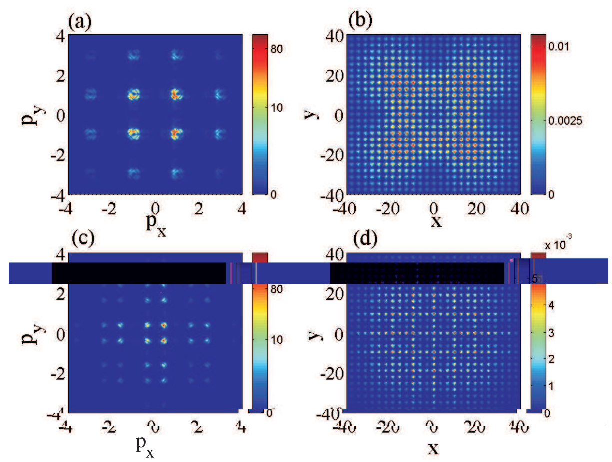


FIG. 5: (Color online) Absolute value of the ground state wave function in momentum space (a) and (c), and in position space (b) and (d). The dimensionless parameters are $V = 5$, $F_{ex} = 0$, and $g = 1$ for all plots, $v_{so} = 2$ in the upper plots (a) and (b), and $v_{so} = 1$ in the lower plots (c) and (d). Thus, the lower plots correspond to the dispersions depicted in Fig. 4.

Our numerical considerations will focus on propagation of an initial (approximated) ground state for the untilted ($F_{ex} = 0$) lattice. That is, we consider the ground state of a spin-orbit coupled BEC in an optical lattice, and then we introduce a quantum quench by suddenly switch on a linear force in the $x$-direction. Hence, we begin by propagating an ansatz function in imaginary time with zero extrinsic force $F_{ex} = 0$. When convergence of the imaginary time evolution has been reached, we switch to real time propagation, including the extrinsic force by adding the potential term $-F_{ex}\hat{x}$ to the Hamiltonian. As predicted by the acceleration theorem, for short times the weak forces $F_{ex}$ will induce a linear increase in the quasi-momentum $k_x$. In contrast to BOs in a regular square