Note that (ii) implies the uniform integrability of the sequence $(X_n)_{n\in\mathbb{N}}$.

We first show (i). Obviously, $G_n(f) \to \wr f \wr_D$ due to (7). Standard arguments such as the monotone convergence theorem yield $G_n(f_n) \to \wr f \wr_D$ if $f_n, f \in \bar{C}^-(S)$ with $\|f_n - f\|_\infty \to 0$. Now noticing that $\boldsymbol{M}^{(n)} \to_{\mathcal{D}} \boldsymbol{\eta}$, the assertion is immediate from the extended continuous mapping theorem, see cf. Billingsley (1968, Theorem 5.5).

Now we proof (ii). Elementary calculations show that for all $n \geq 2$

$$\begin{aligned}
E\left(X_n^2\right) &= \int_{\bar{C}^-(S)} n^2 P\left(n(\boldsymbol{U}-1) > f\right)^2 \, \left(P * \boldsymbol{M}^{(n)}\right)(\mathrm{d}f) \\
&\leq \int_{\bar{C}^-(S)} n^2 P\left(n\left(U_s-1\right) > f(s)\right)^2 \, \left(P * \boldsymbol{M}^{(n)}\right)(\mathrm{d}f) \\
&= E\left(\left(M_s^{(n)}\right)^2\right) = \frac{2n}{n+1} \leq 2.
\end{aligned}$$

□

Corollary 3.2. *Denote by* $M(n) := \sum_{i=1}^n 1_{\left\{\boldsymbol{X}^{(i)} > \max_{1\leq j<i} \boldsymbol{X}^{(j)}\right\}}$ *the number of complete records among* $\boldsymbol{X}^{(1)}, \dots, \boldsymbol{X}^{(n)}$*. Then*

$$\frac{E(M(n))}{\log(n)} \to_{n\to\infty} E\left(\wr \boldsymbol{\eta} \wr_D\right).$$

Proof. The assertion follows from Theorem 3.1 and the fact that $\left(\sum_{i=1}^n \frac{a_i}{i}\right)/\log(n)$ $\to_{n\to\infty} a$, if $(a_n)_{n\in\mathbb{N}}$ is some real-valued sequence with $a_n \to_{n\to\infty} a$. □

The following lemma provides an alternative representation for the extremal concurrence probability. Denote by 1_A the indicator function of some set A, i.e. $1_A(\omega) = 1$, if $\omega \in A$, and $1_A(\omega) = 0$, else.

Lemma 3.3. *Let* $\boldsymbol{\eta} = (\eta_s)_{s\in S}$ *be an SMSP in* $\bar{C}^-(S)$ *with D-norm* $\|\cdot\|_D$ *and generator* $\boldsymbol{Z} = (Z_s)_{s\in S}$*, and* $f \in \bar{E}^-(S)$*. Then*

(i)

$$E\left(\wr \boldsymbol{\eta} \wr_D\right) = E\left(\|1/\boldsymbol{Z}\|_D^{-1} \, 1_{\{\boldsymbol{Z}>0\}}\right).$$