Integrating the above equation, it is easy to get the solution for $\rho_{\rm sc}$

$$\rho_{\rm sc} = \rho_0 \left\{ \frac{a_0^{3[\omega_0+1]}}{a^{3[\omega_{\rm sc}(a)+1]}} \right\} \exp\left[-3\int_a^{a_0} da\omega'_{\rm sc}(a)\ln a \right], \tag{11}$$

with ρ_0, a_0, ω_0 denote today's energy density, scale factor and EoS parameter respectively. For $a_0 \gg a_\star$, it's not necessary to consider the quantum effect of those present values (For conservation, we will define the subscript '0' denotes the present value of a physical quantity.). And the prime means the derivation with respect to the scale factor a.

In the above discussion, we defined the semiclassical energy density as $\rho_{\rm sc} = E_{\rm m}/a^3$, but this is not the only definition. Singh has showed us that there are two ways to obtain the semiclassical density [27]. One way is just like the definition we discussed above. The other one is to define a density operator $\hat{\rho}_{\rm q} = \widehat{H_{\rm m}/a^3}$ and then take their eigenvalues. The eigenvalues for $\hat{\rho}_{\rm q}$ can be obtained by considering $\hat{H}_{\rm m}$ and $\widehat{1/a^3}$ [27]

$$\rho_{\rm q} = d_{j,l}(a)E_m(a,\phi) = D_l(q)a^{-3}E_{\rm m}(a,\phi) = D_l(q)\rho_{\rm sc}, \tag{12}$$

in which $d_{j,l} = D_l(q)a^{-3}$ is the eigenvalue for $\widehat{1/a^3}$ for the large j. And D_l is given as [27]

$$\begin{aligned} D_l(q) \;=\; & \left\{ \frac{27|q|^{1-2l/3}}{8l}\left[\frac{1}{l+3}\left((q+1)^{2(l+3)/3} \right.\right.\right. \\ & \left. -|q-1|^{2(l+3)/3} \right) - \frac{2q}{2l+3}\left((q+1)^{2(l+3)/3} \right. \\ & \left.\left.\left. -\mathrm{sgn}(q-1)|q-1|^{2(l+3)/3} \right) \right] \right\}^{\frac{3}{2-2l}}, \end{aligned} \tag{13}$$

where $q = a^2/a_\star^2$, and l is the quantum ambiguity parameters with $0 < l < 1$ [26]. For $a \lesssim a_\star$, $D_l \lesssim 1$, this means $\rho_{\rm sc} \lesssim \rho_{\rm q}$.

Now, we turn to discuss the relationship between the semiclassical energy density and the classical one. The classical energy conservation is

$$\dot{\rho}_{\rm cl} + 3H(\rho_{\rm cl} + P_{\rm cl}) = 0. \tag{14}$$

Considering the varying EoS parameter $\omega_{\rm cl}(a) = \rho_{\rm cl}/P_{\rm cl}$, the above equation can be written as

$$\dot{\rho}_{\rm cl} + 3H\rho_{\rm cl}[1 + \omega_{\rm cl}(a)] = 0. \tag{15}$$

It is easy to obtain the expression for $\rho_{\rm cl}$ from the above equation

$$\rho_{\rm cl} = \rho_0 \left\{ \frac{a_0^{3[\omega_0+1]}}{a^{3[\omega_{\rm cl}(a)+1]}} \right\} \exp\left[-3\int_a^{a_0} da\omega'_{\rm cl}(a)\ln a \right]. \tag{16}$$

Considering the definition of $\rho_{\rm q}$, we can get the eigenvalues for $\widehat{\rho}_{\rm q}$ using the eigenvalues for $\widehat{1/a^3}$, $d_{j,l}$, instead of a^{-3} in Eq.(16). So, the semiclassical expression for $\rho_{\rm q}$ can be written as

$$\begin{aligned} \rho_{\rm q} \;&=\; \rho_0 \left\{ a_0^{3[\omega_0+1]} d_{j,l}^{[\omega_{\rm cl}(a)+1]} \right\} \exp\left[-3\int_a^{a_0} da\omega'_{\rm cl}(a)\ln a \right] \\ &=\; D_l^{\omega_{\rm cl}(a)+1}\rho_{\rm cl}(a). \end{aligned} \tag{17}$$

Remembering the relationship between $\rho_{\rm q}$ and $\rho_{\rm sc}$, which is described by Eq.(12), we can get the relationship between $\rho_{\rm sc}$ and $\rho_{\rm cl}$

$$\rho_{\rm sc} = D_l^{\omega_{\rm cl}(a)}\rho_{\rm cl}. \tag{18}$$

Now we want to get the relationship between $\omega_{\rm sc}(a)$ and $\omega_{\rm cl}(a)$. Differentiating $\ln\rho_{\rm sc}$ with respect to $\ln a$, one can obtain

$$\frac{d\ln\rho_{\rm sc}}{d\ln a} = \ln D_l \frac{d\omega_{\rm cl}(a)}{d\ln a} + \omega_{\rm cl}(a)\frac{d\ln D_l}{d\ln a} + \frac{d\ln\rho_{\rm cl}(a)}{d\ln a}. \tag{19}$$

Note that $\frac{d\ln\rho_{\rm sc}(a)}{d\ln a} = \frac{d\rho_{\rm sc}(a)}{H\rho_{\rm sc}(a)dt} = -3[1+\omega_{\rm sc}(a)]$, and $\frac{d\ln\rho_{\rm cl}(a)}{d\ln a} = \frac{d\rho_{\rm cl}(a)}{H\rho_{\rm cl}(a)dt} = -3[1+\omega_{\rm cl}(a)]$, it is easy to get

$$\omega_{\rm sc}(a) = \omega_{\rm cl}(a)\left[1 - \frac{1}{3}\ln D_l \frac{d\ln|\omega_{\rm cl}(a)|}{d\ln a} - \frac{1}{3}\frac{d\ln D_l}{d\ln a} \right]. \tag{20}$$

Remembering that $\omega_{\rm cl}(a)$ may be a negative value for dark energy fluid, so we need choose the absolute value when we calculate its logarithm, but it dose not change the answer. Considering Eq.(20), it is easy to find that the effective EoS in loop quantum cosmology is possible "-1" crossing. If $\omega_{\rm cl} = $ const., Eq.(20) will still hold, but the second term in the square brackets is zero. So the constant classical EoS will be a varying term in the effective region.

In this section, we get the equation of the effective EoS of dark energy with a varying EoS in loop quantum cosmology. Now, we can turn to consider the thermodynamical properties of dark energy in loop quantum cosmology scenario.

III. THERMODYNAMICS PROPERTIES OF DARK ENERGY

In this section, we will discuss the thermodynamics properties of dark energy in semiclassical regions. This section includes two subsections. In the first subsection, we will discuss the entropy-area relation of apparent horizon in semiclassical scenario. In the next subsection, we will show the temperature, chemical potential and entropy of dark energy fluid inner of the apparent horizon.

A. Modified entropy-area relation

According to the fact that the Einstein equation can be rewritten as the form $dE_{\rm m} = TdS + W_{\rm m}dV$ [28–30],