$$\frac{\partial P}{\partial a_{ij}} = (-1)^{i+j} P_{ij}(\Lambda) + \frac{\partial P}{\partial \Lambda}\frac{\partial \Lambda}{\partial a_{ij}} \ , \tag{9}$$

where $P_{ij}$ is the characteristic polynomial associated with the minor of $A$ after removing line $i$ and column $j$. Furthermore

$$\frac{\partial P}{\partial D} = -1 + \frac{\partial P}{\partial \Lambda}\frac{\partial \Lambda}{\partial D} \ , \tag{10}$$

From these equations it follows that

$$\frac{\partial P}{\partial a_{ij}} = (-1)^{i+j} P_{ij}\frac{\partial \Lambda}{\partial D} \ , \tag{11}$$

Now, since $A$ is irreducible and non-negative, the largest eigenvalue of $A$ is larger or equal than the largest eigenvalue of any submatrix of $A$ (Frobenius Theorem Gantmacher (2000); Debreu and Hestrein (1953)). The latter result implies that $P_{ij}(\Lambda) \geq 0$. To show that $P_{ij} < 1$ we need to inspect the precise form of $P_{ij}$. For $i = j = 1$ we obtain

$$P_{11}(\Lambda) = \Lambda(\Lambda^3 - (A_{\mathcal{G}} + A_{\mathcal{P}})\Lambda^2 + A_{\mathcal{G}}A_{\mathcal{P}}\Lambda - C \ . \tag{12}$$

Since $\Lambda \geq A_G$ (Frobenius Theorem) we have that

$$\begin{aligned} P_{11}(\Lambda) &\leq \Lambda(\Lambda^3 - (A_{\mathcal{G}} + A_{\mathcal{P}})\Lambda A_{\mathcal{G}} + A_{\mathcal{G}}A_{\mathcal{P}}\Lambda - C) \\ &\leq \Lambda(\Lambda^3 - A_{\mathcal{G}}^2\Lambda - C) \\ &\leq \Lambda^4 \ . \end{aligned} \tag{13}$$

For $\Lambda < 1$ we finally obtain that $P_{11}(\Lambda) \leq 1$ and therefore