where $\mu = \mu(r),\ \nu = \nu(r)$. Then, we have

$$\begin{aligned} R &= \frac{2e^{-2\nu}}{r^2}\Big[2r\nu' - \big(1-e^{2\nu}\big)\Big],\\ K &= e^{\mu-\nu}\left(\mu' + \frac{2}{r^2}\right),\\ K_{ij}K^{ij} &= e^{2(\mu-\nu)}\left(\mu'^2 + \frac{2}{r^2}\right), \end{aligned} \tag{3.2}$$

where $\nu' \equiv d\nu/dr$, etc. It is interesting to note that for the metric (3.1) we have

$$C_{ij} = 0 = \epsilon^{ijk}R_{il}\nabla_j R^l_k, \tag{3.3}$$

where C_{ij} is the Cotton tensor, defined as

$$C_{ij} = \epsilon^{ikl}\nabla_k\left(R^j_l - \frac{1}{4}R\delta^j_l\right). \tag{3.4}$$

As a result, the HL theory with detailed balance [1]

$$\begin{aligned} S_{HLd} &= \int dtdx^3 N\sqrt{g}\Bigg\{\frac{2}{\kappa^2}\Big[K_{ij}K^{ij} - \lambda K^2\Big]\\ &\quad + \frac{\kappa^2\mu^2\big(\Lambda_W R - 3\Lambda_W^2\big)}{8(1-3\lambda)} + \frac{\kappa^2\mu^2\big(1-4\lambda\big)R^2}{32(1-3\lambda)}\\ &\quad - \frac{\kappa^2\mu^2}{8}R_{ij}R^{ij} + \frac{\kappa^2\mu}{2w^2}\epsilon^{ijk}R_{il}\nabla_j R^l_k\\ &\quad - \frac{\kappa^2}{2w^2}C_{ij}C^{ij}\Bigg\}, \end{aligned} \tag{3.5}$$

can be effectively considered as a particular case of the general SVW action (1.7) with

$$\begin{aligned} G &= \frac{\kappa^2}{32\pi c^2},\quad c^2 = \frac{\kappa^4\mu^2\Lambda_W}{16(1-3\lambda)},\quad \Lambda = \frac{3\Lambda_W}{2},\\ g_2 &= \frac{4\lambda-1}{4\Lambda_W}\zeta^2,\quad g_3 = \frac{1-3\lambda}{\Lambda_W}\zeta^2,\\ \xi &= 1-\lambda,\quad g_4 = g_5 = ... = g_8 = 0. \end{aligned} \tag{3.6}$$

It should be noted that these relations are valid only when the conditions of Eq. (3.3) hold. In general, these two terms do not vanish and violate parity, while the SVW action always preserves it. It must not be confused with the parameter μ used in the action (3.5) and the metric coefficient used in (3.1).

To study singularities in the HL theory, in the rest of this paper we shall restrict ourselves to two representative cases, the (anti-) de Sitter Schwarzschild solutions and the solutions found by Lu, Mei and Pope (LMP) [6].

A. (Anti-) de Sitter Schwarzschild Solutions

The (anti-) de Sitter Schwarzschild solutions are given by [38]

$$\mu = \frac{1}{2}\ln\left(\frac{M}{r} + \frac{\Lambda}{3}r^2\right),\quad \nu = 0. \tag{3.7}$$

When $\Lambda > 0$, it represents the de Sitter Schwarzschild solutions, and when $\Lambda < 0$ it represents the anti-de Sitter Schwarzschild solutions. As mentioned previously, they are also solutions of the SVW generalization with $\xi = 0$. Inserting the above into Eq.(3.2), we find that

$$\begin{aligned} R &= 0,\\ K &= \left(\frac{3M+\Lambda r^3}{12r^3}\right)^{1/2}\left(\frac{4}{r} - \frac{3M-2\Lambda r^3}{3M+\Lambda r^3}\right),\\ K_{ij}K^{ij} &= \frac{3M+\Lambda r^3}{12r^3}\left[8 + \left(\frac{3M-2\Lambda r^3}{3M+\Lambda r^3}\right)^2\right]. \end{aligned} \tag{3.8}$$

Clearly, when $\Lambda \geq 0$, K and $K_{ij}K^{ij}$ are singular only at the center $r = 0$. However, when $\Lambda < 0$, they are also singular at $r = r_\Lambda \equiv (3M/|\Lambda|)^{1/3}$. In contrast to GR, this singularity is a scalar one, and cannot be removed by any coordinate transformations given by Eq. (1.3).

In GR, the (anti-) de Sitter Schwarzschild solutions are usually given in the orthogonal gauge,

$$ds^2 = -e^{2\Psi(r)}d\tau^2 + e^{2\Phi(r)}dr^2 + r^2 d\Omega^2, \tag{3.9}$$

with

$$\Psi = -\Phi = \frac{1}{2}\ln\left(1 - \frac{2M}{r} + \frac{1}{3}\Lambda r^2\right). \tag{3.10}$$

Clearly, the metric (3.9) does not satisfy the projectability condition, and its coefficient g_{rr} is singular at $e^{2\Phi(r_{EH})} = 0$. But, in GR this is a coordinate singularity, and all scalars made of the 4-dimensional Riemann tensor and its derivatives are finite at $r = r_{EH}$.

It is interesting to note that in the orthogonal gauge (3.9), K and $K_{ij}K^{ij}$ all vanish, as can be seen from Eqs. (1.9), while the 3-dimensional Ricci scalar is given by $R_{orth} = -2\Lambda$, where R_{orth} denotes the quantity calculated in the orthogonal gauge (3.9).

In [38], it was showed explicitly that metric (3.9) is related to metric (3.1) by the coordinate transformations,

$$\tau = t - \int^r \sqrt{e^{-2\Psi} - 1}\, e^{\Phi}dr, \tag{3.11}$$

under which we have

$$\begin{aligned} \Phi(r) &= \nu(r) - \frac{1}{2}\ln\Big(1 - e^{2\mu}\Big),\\ \Psi(r) &= \frac{1}{2}\ln\Big(1 - e^{2\mu}\Big), \end{aligned} \tag{3.12}$$

or inversely,

$$\mu = \frac{1}{2}\ln\Big(1 - e^{2\Psi}\Big),\quad \nu = \Phi(r) + \Psi(r). \tag{3.13}$$

Note that the coordinate transformations (3.11) are not allowed by the foliation-preserving diffeomorphisms (1.3). In addition, since K, $K_{ij}K^{ij}$ and R are not scalars under these transformations, it explains why we have completely different physical interpretations in the two