Therefore

$$\sum_\lambda |\mathcal{M}_\beta| = \frac{4(Qe)^2}{\mathbf{k}^2}. \tag{E.25}$$

### E.2.3 Soft Photon Approximation, Massless Dijet

Compute electron (mass $m = 0$) beta decay starting with Eq. (F.84),

$$i\mathcal{M} = \bar{u}(p)\left[\mathcal{M}_0(p,P)\right]u(P)\cdot\left[Qe\left(\frac{p\cdot\epsilon^*}{p\cdot k} - \frac{P\cdot\epsilon^*}{P\cdot k}\right)\right]. \tag{E.26}$$

We will want to exploit the replacement $\sum \epsilon_\mu \epsilon^*_\nu \to -g_{\mu\nu}$ so we will have to preserve the Ward identity (charge conservation),

$$k_\mu\left(\frac{p^\mu}{k\cdot p} - \frac{P^\mu}{k\cdot P}\right) = 0. \tag{E.27}$$

To do so we will have to keep the second term explicitly with $P_M \simeq [M, M, 0, 0]$ being the momentum of the “proton” after (as well as before) beta emission. Then (essentially Eq. (F.86)),

$$\begin{aligned}\sum_\lambda |\mathcal{M}|^2 &= (Qe)^2(-g_{\mu\nu})\left(\frac{p^\mu}{k\cdot p} - \frac{P_M^\mu}{k\cdot P_M}\right)\left(\frac{p^\nu}{k\cdot p} - \frac{P_M^\nu}{k\cdot P_M}\right)\\ &= (Qe)^2\left(\frac{2p\cdot P_M}{(k\cdot p)(k\cdot P_M)} - \frac{M^2}{(k\cdot P_M)^2} - \frac{m^2}{(k\cdot p)^2}\right)\end{aligned} \tag{E.28}$$