the corresponding edge was blue, but as we have just noted, case (2) does not occur when we restrict our attention to involutions which actually correspond to $Sp(2n,\mathbb{C})$-orbits. The upshot is that in the present case, all roots are complex. In particular, there are no blue edges in the weak order graph.

We sum up the discussion as follows:

Proposition 2.4.3. *The weak order on $K\backslash G/B$ corresponds to the order on fixed point-free involutions given inductively as follows: Starting with w_0, for any fixed point-free involution b, and for any s_i such that $l(s_i b) < l(b)$, $b <_i s_i b s_i$ if and only if $s_i b s_i \neq b$. All roots are complex, and hence all edges are black.*

The parametrization of $K\backslash G/B$ by fixed point-free involutions encodes precisely the same linear algebraic description of the orbits in this case as it does in the case of the orthogonal groups. Namely, letting γ denote the symplectic form with isometry group K, if we define $\mathcal{O}_b$ to be

$$\{F_\bullet \in X \mid \operatorname{rank}(\gamma|_{F_i \times F_j}) = r_b(i,j) \text{ for all } i,j\},$$

then $\mathcal{O}_b$ is a single K-orbit on G/B, and the association $b \mapsto \mathcal{O}_b$ defines a bijection between the set of fixed point-free involutions and $K\backslash G/B$.

This can be seen in the same way as in the orthogonal case, and a representative of each orbit may be produced by the same procedure. Because the argument is identical, we omit the details.

2.4.3 Example

We give the details of the computation in the very small case $n = 2$ (so $(G,K) = (SL(4,\mathbb{C}), Sp(4,\mathbb{C}))$). Here, there are 3 fixed point-free involutions, and hence 3 orbits. The involutions are $(1,2)(3,4)$, $(1,3)(2,4)$, and $(1,4)(2,3)$.