satisfies a quadratic inequality and is bounded above by M:

$$\begin{aligned}
& \alpha||x||^2 \leq a(x,x) \leq C\|x\|\|y_0\| + \|\ell\|\|x\| + \|\ell\|\|y_0\| \\
\Longrightarrow \quad & \alpha||x||^2 - (C\|y_0\| + \|\ell\|)\|x\| - \|\ell\|\|y_0\| \leq 0 \\
\Longleftrightarrow \quad & ||x||^2 - \beta\|x\| - \gamma \leq 0 \\
\Longrightarrow \quad & \|x\| \leq \tfrac{1}{2}(\beta + \sqrt{\beta^2 + 4\gamma}) = M.
\end{aligned}$$

Consequently, if $x \in X, \|x\| > M$, then $f(x, y_0) > 0$ and the compactness condition in Theorem 14 is satisfied. Since $f(x,x) = 0$ for any $x \in X$, (A) of Theorem 14 is impossible, and (B) holds, i.e., $f(\bar{x}, y) = a(\bar{x}, \bar{x}-y) - \ell(\bar{x}) + \ell(y) \leq 0$ for some $\bar{x} \in X$ and all $y \in X$ and the proof of the existence is complete.

The uniqueness follows at once from the bilinearity and the coercivity of the form a as follows: if $a(\bar{x}_i, \bar{x}_i - y) - \ell(\bar{x}_i) + \ell(y) \leq 0$ for two elements $\bar{x}_i \in X, i = 1, 2$, and all $y \in X$, then adding $a(\bar{x}_1, \bar{x}_1 - \bar{x}_2) \leq \ell(\bar{x}_1) - \ell(\bar{x}_2)$ to $a(\bar{x}_2, \bar{x}_2 - \bar{x}_1) \leq \ell(\bar{x}_2) - \ell(\bar{x}_1)$ gives $0 \leq \alpha\|\bar{x}_1 - \bar{x}_2\|^2 \leq a(\bar{x}_1 - \bar{x}_2, \bar{x}_1 - \bar{x}_2) \leq 0$, i.e., $\bar{x}_1 = \bar{x}_2$. ■

The coincidence $(\mathcal{N}, \mathcal{N}^{-1})$ (Theorem 12) can be expressed in analytical terms as a second alternative for nonlinear systems of inequalities as follows:

Theorem 17 *Let X and Y be two convex subsets of topological vector spaces and let $f, g : X \times Y \longrightarrow \mathbb{R}$ be two functions satisfying:*

(i) $f(x,y) \leq g(x,y)$ for all $(x,y) \in X \times Y$;

(ii) $x \mapsto f(x,y)$ is quasiconcave on X, for each fixed $y \in Y$;

(iii) $y \mapsto f(x,y)$ is l.s.c. and quasiconvex on Y, for each fixed $x \in X$;

(iv) $x \mapsto g(x,y)$ is u.s.c. and quasiconcave on X, for each fixed $x \in X$;

(v) $y \mapsto g(x,y)$ is quasiconvex on Y, for each fixed $x \in X$.

(vi) Given $\lambda \in \mathbb{R}$ arbitrary, assume that either Y is compact, or X is compact, or there exist a compact subset K of X and a convex compact subset C of Y such that for any $x \in X \setminus K$ there exists $y \in C$ with $g(x,y) < \lambda$.