3.2 Cascade of Almost Periodic Solutions

We say that system (1.1) is chaotic through a cascade of almost periodic solutions if for each periodic solution $x(t)$ of (3.5), system (1.1) possesses an almost periodic solution.

One can verify by using the results of [67] that if $x(t)$ is a periodic solution of (3.5), then the function $G(t, y) + H(x(t))$ in (1.1) is almost periodic in t uniformly for $y \in \mathbb{R}^n$ and the unique bounded on $\mathbb{R}$ solution $\phi_{x(t)}(t)$ of (1.1) is almost periodic. On the other hand, if $x_1(t)$ and $x_2(t)$ are two different periodic solutions of (3.5), then the corresponding almost periodic solutions $\phi_{x_1(t)}(t)$ and $\phi_{x_2(t)}(t)$ are different from each other under the condition $(C4)$. Because system (3.5) is chaotic through period-doubling cascade, there exists a cascade of almost periodic solutions in the dynamics of (1.1). Consequently, system (1.1) possesses infinitely many almost periodic solutions in a bounded region. The instability of the existing almost periodic motions is ensured by Lemma 3.1. This result is mentioned in the following theorem.

Theorem 3.1 *Under the conditions* $(C1)-(C5)$, *system (1.1) is chaotic through a cascade of almost periodic solutions.*

A corollary of Theorem 3.1 is as follows.

Corollary 3.1 *Under the conditions* $(C1)-(C5)$, *the coupled system* $(3.5)+(1.1)$ *is chaotic through a cascade of almost periodic solutions.*

In the next section, we will consider the formation of Li-Yorke chaos with infinitely many almost periodic motions.

4 Li-Yorke Chaos with Infinitely Many Almost Periodic Motions

In the original paper [2], chaos with infinitely many periodic solutions, which are separated from the elements of a scrambled set, was introduced. We modify the Li-Yorke definition of chaos by replacing periodic motions by almost periodic ones, and prove its presence in system (1.1) rigorously.

In opposition to the descriptions of regular motions such as periodic, quasi-periodic and almost periodic motions, one encounters with interaction of motions in order to describe the Li-Yorke chaos. Therefore, we need to introduce the concept of chaotic sets of functions.

Let Λ be a compact subset of $\mathbb{R}^m$, and consider the set of uniformly bounded functions $\mathscr{A}$ whose elements are of the form $x(t) : \mathbb{R} \to \Lambda$. We suppose that $\mathscr{A}$ is an equicontinuous family on $\mathbb{R}$. In this section, the perturbation function $x(t)$ in system (1.1) will be provided from the elements of the collection $\mathscr{A}$.

A couple of functions $(x(t), \overline{x}(t)) \in \mathscr{A} \times \mathscr{A}$ is called proximal if for an arbitrary small number $\epsilon > 0$ and an arbitrary large number $E > 0$ there exists an interval $J \subset \mathbb{R}$ with a length no less than E such