adiabatic connection):

$$\mathbf{d}a = \frac{1}{v} H(\mathbf{x}) d\ell + i[\mathbf{d}P_{\mathbf{x}}, P_{\mathbf{x}}], \tag{25}$$

in which case U_a can be viewed as the solution of the following differential equation:

$$i\mathbf{d}U_a = \mathbf{d}a\, U_a, \quad U_a(\mathbf{x}_0) = I. \tag{26}$$

Thus, by fixing the speed v, we have eliminated the time. The quantum algorithms will use the adiabatic unitary transformations $U_a(\Gamma)$, $U_a(\Gamma')$, ..., resulted from taking the probes along certain closed paths Γ, Γ', ...

Assume that, for each configuration of the anyons (equal to say that for each $\mathbf{x}$), we chose a basis set (a gauge) $\psi_1(\mathbf{x}), ..., \psi_D(\mathbf{x})$ in the D=2^{n-1} dimensional zero mode space. We use $\vec{\psi}(\mathbf{x})$ to denote the vector of components $\psi_1(\mathbf{x}), ..., \psi_D(\mathbf{x})$. Due to the fundamental property of the adiabatic evolution Eq. (23), there exists a unitary $D\times D$ matrix $\hat{W}(\mathbf{x})$ such that

$$U_a(\mathbf{x})\vec{\psi}(\mathbf{x}_0) = \hat{W}(\mathbf{x})\vec{\psi}(\mathbf{x}). \tag{27}$$

The Wilczeck-Zee connection is given by:[30]

$$\mathbf{d}\hat{A} = i\hat{W}(\mathbf{x})^{-1}\mathbf{d}\hat{W}(\mathbf{x}), \tag{28}$$

which takes the classical form:

$$\mathbf{d}A_{ij}(\mathbf{x}) = -\frac{1}{v} H(\mathbf{x})_{ji} d\ell - i\langle\psi_j(\mathbf{x}), \mathbf{d}\psi_i(\mathbf{x})\rangle, \tag{29}$$

where $H(\mathbf{x})_{ji}$ are the matrix elements of the Hamiltonian in the chosen basis set. Since we are dealing with zero modes, the first term in the right hand side, above, is identically zero.

The $D\times D$ unitary matrix $\hat{W}(\mathbf{x})$ implements the adiabatic evolution in the invariant subspaces $P_{\mathbf{x}}\mathcal{H}_0$ and it can be computed as the unique solution of the differential equation:

$$i\mathbf{d}\hat{W}(\mathbf{x}) = \hat{W}(\mathbf{x})\ \mathbf{d}\hat{A}, W(\mathbf{x}_0) = I \tag{30}$$

which is to be integrated along the braiding path. For a loop Γ that starts and ends at $\mathbf{x}_0$, we solved this equation numerically, by considering a large number of points along the loop, $\mathbf{x}_0$,...,$\mathbf{x}_K$, and constructing the monodromy:

$$\hat{W}_\Gamma = P_{\mathbf{x}_0} P_{\mathbf{x}_1} P_{\mathbf{x}_2} ... P_{\mathbf{x}_K} P_{\mathbf{x}_0}. \tag{31}$$

This amounts to finding the null space of the pinning Hamiltonian for each $\mathbf{x}_k$. If we define $W_\Gamma(k) = P_{\mathbf{x}_0} P_{\mathbf{x}_2} ... P_{\mathbf{x}_k}$, then

$$\hat{W}_\Gamma(k+1) - W_\Gamma(k) = \hat{W}_\Gamma(k)(P_{\mathbf{x}_{k+1}} - P_{\mathbf{x}_k})P_{\mathbf{x}_k} - \hat{W}_\Gamma(k)P_{\mathbf{x}_{k+1}}(P_{\mathbf{x}_{k+1}} - P_{\mathbf{x}_k}), \tag{32}$$

which is the finite difference version of

$$i\mathbf{d}\hat{W}_\Gamma(x) = i\hat{W}_\Gamma(x)[\mathbf{d}P_{\mathbf{x}}, P_{\mathbf{x}}], \tag{33}$$

which is the same as Eq. (30). The numerically calculated $\hat{W}_\Gamma$ matrix becomes a unitary matrix only in the limit when the number of discrete points goes to infinity. To quantify how much does $\hat{W}_\Gamma$ deviate from a unitary matrix, we compute the absolute value of the determinant of $\hat{W}_\Gamma$, which is compared to the value of 1, appropriate for a unitary matrix. In all the calculations presented in this paper, K was chosen large enough so that $|\det \hat{W}_\Gamma|$=0.999 or better. This is a measure of how well converged are our numerical calculations.

B. The Quantum Geometry of the zero modes states

We can endow the zero modes with a curvature. The curvature form associated with the adiabatic connection is given by:

$$d\hat{F} = \left\{\partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu]\right\} dx^\mu \wedge dx^\nu. \tag{34}$$

The explicit expressions of its coefficients are:

$$(\hat{F}_{\mu\nu})_{ij}(\mathbf{x}) = 2\mathrm{Im}\langle\partial_\mu\psi_j(\mathbf{x}), [1 - P_{\mathbf{x}}]\partial_\nu\psi_i(\mathbf{x})\rangle. \tag{35}$$

Besides the adiabatic connection and curvature, we can endow the parameter space with an intrinsic metric tensor, which we will refer to as the quantum metric tensor. First of all, we can introduce the following quantum distance:[31]

$$d^q(\mathbf{x}, \mathbf{x}') = \|P_{\mathbf{x}} - P_{\mathbf{x}'}\|_{HS}, \tag{36}$$

where HS means the Hilbert-Schmidt norm. This distance is at least second order differentiable in the coordinates $\mathbf{x}$ and $\mathbf{x}'$ and for this reason we can generate the quantum metric tensor $g_{\mu\nu}$ via a Taylor expansion:

$$d^q(\mathbf{x}, \mathbf{x} + \delta\mathbf{x}) = \frac{1}{2} g^q_{\mu\nu}(\mathbf{x}) \delta x^\mu \delta x^\nu + o(\delta\mathbf{x}^3). \tag{37}$$

The coefficients of the quantum metric tensor are given by the classical expression:[31]

$$g^q_{\mu\nu}(\mathbf{x}) = 2\mathrm{Re}\sum_i \langle\partial_\mu\psi_i(\mathbf{x}), [1 - P_{\mathbf{x}})]\partial_\nu\psi_i(\mathbf{x})\rangle. \tag{38}$$

In the following, we demonstrate an interesting relation between the adiabatic curvature and the quantum metric tensor. We fix all anyons, except one, in which case the parameter space becomes 2-dimensional. A point in this parameter space describes the position of the itinerant anyon on the sphere. To compute the coefficient of the curvature form at an arbitrarily chosen point P on the sphere, we introduce a local coordinate system by using the complex coordinate $w = 2R\tan\frac{\theta}{2}e^{-i\phi}$, where (θ, ϕ) are the usual spherical parameters when the North pole is fixed at P. We take w_1=Re$\{w\}$ and w_2 =Im$\{w\}$ as the two independent variables (see Fig. 6). In these local coordinate system, the curvature form becomes

$$dF = \hat{F}(w) dw^1 \wedge dw^2, \tag{39}$$