Negative spatial curvature: hyperbolic models

In hyperbolic models or regions containing a center, $\Gamma > 0$ and $\delta_i^{(k)} \geq -2/3$ hold everywhere (conditions (61) and (68)), therefore, as a consequence of (37) and (B1), we have:

$$k_q < 0 \quad \Rightarrow \quad k < 0, \tag{B2}$$

Hence, since all regular hyperbolic models or regions comply with $k_q < 0$, then all these models or regions have also negative local spatial curvature. As explain further below, the converse is not true, as local spatial curvature can be negative in certain elliptic regions in which $k_q \geq 0$.

Elliptic models and positive spatial curvature.

The relation between k_q and k is more complicated in elliptic models or regions, since standard regularity (conditions (61) and (72)) do not place a lower bound on $\delta_i^{(k)}$. Hence, the possibility that $\delta_i^{(k)} < -2/3$ occurs cannot be ruled out, and so $k < 0$ can happen in regions where $k_q > 0$. It is straightforward to show from (62) that $\delta_i^{(k)} \leq -2/3$ can only occur in a regular elliptic model or region ($\Gamma > 0$) if $E' = 0$ for some $0 < r_{\rm tv} < r$ in a domain $\vartheta(r)$. As a consequence, we have $\delta_i^{(k)} > -2/3$ for all regular elliptic models in which $E' \leq 0$ holds for all r (with $E' = 0$ only at the symmetry center), and so (B1) implies in this case that $k_q > 0 \Rightarrow k > 0$ everywhere.

If there is a zero of E' at some $r = r^*$, then $\delta_i^{(k)} < -2/3$ will hold in some regions without violating regularity conditions. Since $-1 \leq E \leq 0$ and $E(0) = 0$, then for $r \approx 0$ we must have $E' < 0$. Thus, the only possible configuration is: $E' \leq 0$ and $\delta_i^{(k)} > -2/3$ for $0 \leq r < r^*$, with $E' \geq 0$ and $\delta_i^{(k)} < -2/3$ for $r > r^*$. Further insight into this situation comes from rewriting (13) as

$$E' = RR'\left(k_q - 3k\right) = -4RR'\left[k_q + \frac{k_q'/k_q}{2R'/R}\right]. \tag{B3}$$

If the regularity condition (A7) holds, then $R'/\sqrt{1+E} > 0$, and so for $E'(r^*) = 0$ to occur the local curvature k must decrease sufficiently to reach $k(r^*) = k_q(r^*)/3$, hence $E' > 0$ (or $\delta_i^{(k)} < -2/3$) leads to further decreasing of k, so that $k(r^*) < k_q(r^*)/3$ holds for $r > r^*$. A situation in which $k < 0$ occurs with $k_q > 0$ can easily be conceived: since k' and k_q' are both monotonously negative, we have a curvature clump and so $k_q > k$, thus, if k_q decays to zero sufficiently fast k might become negative. This happens in the elliptic side of the mixed elliptic/hyperbolic configuration examined in section XIV D.

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