just happen to have their partners that are strictly equivalent to real potentials after being exposed to some supersymmetric quantum mechanical treatment [11] or integral, Fourier-like transformation [12]. Jones and Mateo [4] have, moreover, used a Darboux-type similarity transformation and have shown that for the Bender's and Boettcher's [1] non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian $H = p^2 - g(ix)^N$; $N = 4$, there exists an equivalent Hermitian Hamiltonian $h = \sigma^{-1} H \sigma$; $\sigma = \exp(Q/2)$, where σ is Hermitian and positive definite. Similar proposal was carried out by Bender et al. [3]. For more details the reader is advised to refer to [3,4]. In our current methodical proposal, we try to have our input in this direction and fill this gap partially, at least.

Through the forthcoming proposition (in section 2) or through a similarity transformation (in section 3) with a metric operator η (defined in (21) below) we report that for every non-Hermitian complex $\mathcal{PT}$-symmetric Hamiltonian (with positive mass $m = m_+ = +|m|$) there exists a Hermitian partner Hamiltonian (with negative mass $m = m_- = -|m|$) in Hilbert space $L^2(\mathbb{R}) = \mathcal{H}$. In section 3, we also discuss isospectrality and orthonormalization conditions associated with both the Hermitian partner (not necessarily $\mathcal{PT}$-symmetric) and the non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians. An obvious correspondence is constructed, therein. This has not been discussed elsewhere, to the best of our knowledge. We give our concluding remarks in section 4.

2 A transformation toy: $x \longrightarrow \pm iy$; $x, y \in \mathbb{R}$

In connection with an over simplified transformation toy $x \longrightarrow \pm iy \in i\mathbb{R}$; $x, y \in \mathbb{R}$ ($x \longrightarrow \pm iy$ to be understood as $x \longrightarrow +iy$ and/or $x \longrightarrow -iy$), t' Hooft and Nobbenhuis [44] have used a complex space-time symmetry transformation

$$x \longrightarrow iy \Longleftrightarrow p_x \rightarrow -ip_y \,;\; x, y \in \mathbb{R},$$