On the other hand, if $\lim_{t\to 1}\int_0^t \frac{1+\sigma_z^2(s)}{\lambda^2(s)}ds = \infty$, consider the process

$$X_t = \int_0^t \frac{1}{\lambda(s)}dB_s^2 - \int_0^t \frac{\sigma_z(s)}{\lambda(s)}dB_s^1,$$

and a change of time $\tau(t)$ given by

$$\int_0^{\tau(t)} \frac{1+\sigma_z^2(s)}{\lambda^2(s)}ds = t.$$

Then, $W_s = X_{\tau(s)}$ is a Brownian motion. Hence, we can use the law of iterated logarithm to get

$$\begin{aligned}\limsup_{s\to\infty} \frac{W_s}{\sqrt{2s\log\log s}} &= 1\\ \liminf_{s\to\infty} \frac{W_s}{\sqrt{2s\log\log s}} &= -1\end{aligned}$$

or, in the original time,

$$\begin{aligned}\limsup_{t\to 1} \frac{X_t}{\sqrt{2\Xi(t)\log\log(\Xi(t))}} &= 1\\ \liminf_{t\to 1} \frac{X_t}{\sqrt{2\Xi(t)\log\log(\Xi(t))}} &= -1\end{aligned}$$

where $\Xi(t) = \int_0^t \frac{1+\sigma_z^2(s)}{\lambda^2(s)}ds$. Since, due to the Assumptions 2.2, 3.1 and 3.2, we have

$$\lim_{t\to 1} \lambda^2(t)\Xi(t)\log\log(\Xi(t)) = 0,$$

it follows that $\lim_{t\to 1}\lambda(t)X_t = 0$, therefore $Y_1 = Z_1$. ■

With this lemma at hand, establishing that the pair (H^*, θ^*) given in the Theorem 3.1 is indeed an equilibrium is straightforward, as the following proposition demonstrates.

Proposition 3.2 *Suppose that Assumptions 2.1, 2.2, 3.1 and 3.2 are satisfied. Then the pair (H^*, θ^*), where $H^*(y,t)$ satisfies the partial differential equation (PDE) (3.7) with terminal condition (3.8), and the process θ_t^* is given by (3.9), is an equilibrium.*