$(\ell-1)$-dimensional volume of the perimeter, respectively, and $\mathrm{Int}(\sigma)$ its interior.

Lemma 2.5.7. *For any fixed center* $\mathbf{a}$ *in the* ℓ*-simplex* σ *with* $d < \ell \leq p = d + k$*,*

$$\int_{\mathrm{Int}(\sigma)} J_d\phi(\mathbf{x}, \mathbf{a})\, \mathrm{d}\mathcal{L}^\ell(\mathbf{x}) \;\leq\; \mathrm{diameter}(\sigma)\, \mathrm{P}(\sigma).$$

Proof. Consider the $(\ell-1)$-dimensional faces τ_j of σ, with $\mathrm{perimeter}(\sigma) = \{\cup_j \tau_j \,|\, \tau_j \in \sigma,\, \dim(\tau_j) = \ell - 1\}$. Let σ_j denote the ℓ-simplex generated by $\mathbf{a}$ and τ_j. Then

$$\int_{\mathrm{Int}(\sigma)} J_d\phi(\mathbf{x}, \mathbf{a})\, \mathrm{d}\mathcal{L}^\ell(\mathbf{x}) = \sum_j \int_{\mathrm{Int}(\sigma_j)} J_d\phi(\mathbf{x}, \mathbf{a})\, \mathrm{d}\mathcal{L}^\ell(\mathbf{x}).$$

Let $\tau_j(h)$ denote the $(\ell-1)$-simplex formed by the intersection of σ_j and the $(\ell-1)$-hyperplane parallel to τ_j at a distance h from $\mathbf{a}$. Thus, $\tau_j(\hat{h})$ is τ itself. We observe that our bound on $J_d\phi(\mathbf{x}, \mathbf{a})$ is constant in $\tau_j(h)$ for any h. The $(\ell-1)$-dimensional volume of $\tau_j(h)$ is given by

$$\mathrm{V}_{\ell-1}(\tau_j(h)) = \left(\frac{h}{\hat{h}}\right)^{\ell-1} \mathrm{V}_{\ell-1}(\tau_j).$$

Using the bound on $J_d\phi(\mathbf{x}, \mathbf{a})$ from Lemma 2.5.6, and noting that $\mathrm{diameter}(\sigma_j) \leq \mathrm{diameter}(\sigma)\ \forall\, j$, we get

$$\begin{aligned}\int_{\mathrm{Int}(\sigma_j)} J_d\phi(\mathbf{x}, \mathbf{a})\, \mathrm{d}\mathcal{L}^\ell(\mathbf{x}) &\leq \int_0^{\hat{h}} \left(\frac{h}{\hat{h}}\right)^{\ell-1} \mathrm{V}_{\ell-1}(\tau_j) \left(\frac{\hat{h}}{h}\right)^{d} \frac{\mathrm{diameter}(\sigma)}{\hat{h}}\, \mathrm{d}h \\ &= \frac{\mathrm{V}_{\ell-1}(\tau_j)\, \mathrm{diameter}(\sigma)}{\ell - d}.\end{aligned}$$

Summing this quantity over all $\tau_j \in \mathrm{perimeter}(\sigma)$ and replacing $\ell - d \geq 1$ with 1 gives the overall bound. $\square$

We now bound the integral of $J_d\phi(\mathbf{x}, \mathbf{a})$ over centers $\mathbf{a}$ with a fixed $\mathbf{x}$ that we are retracting onto $\mathrm{perimeter}(\sigma)$. Examination of the corresponding proof for the original deformation theorem [29, Section 7.7] shows that symmetry of the cubical mesh plays a very special role, which cannot be duplicated in the case of simplicial complex. In