two components of the total magnetic field: $\frac{|\mathbf{B}^{ext}|}{|\mathbf{B}^{self}|} \sim O\left(\varepsilon_M^k\right)$, with $k \geq 1$. This means that the self-field is the dominant component: the magnetic field is primarily self-generated. Also, the overall magnetic field is assumed to be slowly varying in time, i.e., to be of the form $\mathbf{B}(\mathbf{r}, \varepsilon_M t)$, while $\mathbf{B}^{self}$ and $\mathbf{B}^{ext}$ are defined as

$$\mathbf{B}^{self} = I(\mathbf{r}, \varepsilon_M t)\nabla\varphi + \nabla\psi_p(\mathbf{r}, \varepsilon_M t) \times \nabla\varphi, \quad (2)$$
$$\mathbf{B}^{ext} = \nabla\psi_D(\mathbf{r}, \varepsilon_M t) \times \nabla\varphi, \quad (3)$$

where $\mathbf{B}_T \equiv I(\mathbf{r}, \varepsilon_M t)\nabla\varphi$ and $\mathbf{B}_P \equiv \nabla\psi_p(\mathbf{r}, \varepsilon_M t) \times \nabla\varphi$ are the toroidal and poloidal components of the self-field, with $(\psi_p, \varphi, \vartheta)$ defining locally a set of magnetic coordinates. Moreover, the external magnetic field $\mathbf{B}^{ext}$ is assumed to be purely poloidal and defined in terms of the vacuum potential $\psi_D(\mathbf{r}, \varepsilon_M t)$. In particular, we notice here that for typical astrophysical applications of interest, the function $\psi_D(\mathbf{r}, \varepsilon_M t)$ can be conveniently identified with the flux function of a dipolar magnetic field. It follows that the magnetic field can also be written in the form

$$\mathbf{B} = I(\mathbf{r}, \varepsilon_M t)\nabla\varphi + \nabla\psi(\mathbf{r}, \varepsilon_M t) \times \nabla\varphi, \quad (4)$$

where the function $\psi(\mathbf{r}, \varepsilon_M t)$ is defined as $\psi(\mathbf{r}, \varepsilon_M t) \equiv \psi_p(\mathbf{r}, \varepsilon_M t) + \psi_D(\mathbf{r}, \varepsilon_M t)$, and $(\psi, \varphi, \vartheta)$ define a set of local magnetic coordinates, as implied by the equation $\mathbf{B} \cdot \nabla\psi = 0$ which is identically satisfied. In addition, it is assumed that the charged particles of the plasma are subject to the action of *effective EM potentials* $\left\{\Phi_s^{eff}(\mathbf{r}, \varepsilon_M t), \mathbf{A}(\mathbf{r}, \varepsilon_M t)\right\}$, where $\mathbf{A}(\mathbf{r}, \varepsilon_M t)$ is the vector potential corresponding to the magnetic field of Eq.(4), while $\Phi_s^{eff}(\mathbf{r}, \varepsilon_M t)$ is given by

$$\Phi_s^{eff}(\mathbf{r}, \varepsilon_M t) = \Phi(\mathbf{r}, \varepsilon_M t) + \frac{M_s}{Z_s e}\Phi_G(\mathbf{r}, \varepsilon_M t), \quad (5)$$

with $\Phi_s^{eff}(\mathbf{r}, \varepsilon_M t)$, $\Phi(\mathbf{r}, \varepsilon_M t)$ and $\Phi_G(\mathbf{r}, \varepsilon_M t)$ denoting the *effective* electrostatic potential and the electrostatic and generalized gravitational potentials (the latter, in principle, being produced both by the central object and the accretion disc). Finally, both the equilibrium *effective electric field* $\mathbf{E}_s^{eff}$, generated by the combined action of the effective EM potentials and defined as

$$\mathbf{E}_s^{eff} \equiv -\nabla\Phi_s^{eff}, \quad (6)$$

and the magnetic field $\mathbf{B}$ are also assumed to be axisymmetric.

III. FIRST INTEGRALS OF MOTION AND GUIDING-CENTER ADIABATIC INVARIANTS

In the present formulation, assuming axi-symmetry and stationary EM and gravitational fields, the exact first integrals of motion can be immediately recovered from the symmetry properties of the single charged particle Lagrangian function $\mathscr{L}$. In particular, these are the total particle energy

$$E_s = \frac{M_s}{2}v^2 + Z_s e\Phi_s^{eff}(\mathbf{r}), \quad (7)$$

and the canonical momentum $p_{\varphi s}$ (conjugate to the ignorable toroidal angle φ)

$$p_{\varphi s} = M_s R\mathbf{v} \cdot \mathbf{e}_\varphi + \frac{Z_s e}{c}\psi \equiv \frac{Z_s e}{c}\psi_{*s}. \quad (8)$$

Gyrokinetic theory allows one to derive the adiabatic invariants of the system [28, 29]; by construction, these are quantities conserved only in an asymptotic sense, i.e., only to a prescribed order of accuracy. As is well known, gyrokinetic theory is a basic prerequisite for the investigation both of kinetic instabilities (see for example [32–34]) and of equilibrium flows occurring in magnetized plasmas [24, 35–38]. For astrophysical plasmas close to compact objects, this generally involves the treatment of strong gravitational fields which needs to be based on a covariant formulation (see [39–42]). However, for non-relativistic plasmas (in the sense already discussed), the appropriate formulation can also be directly recovered via a suitable reformulation of the standard (non-relativistic) theory for magnetically confined laboratory plasmas [29–31, 43–49]. In connection with this, consider again the Lagrangian function $\mathscr{L}$ of charged particle dynamics. By performing a gyrokinetic transformation of $\mathscr{L}$, accurate to the prescribed order in ε_M, it follows that - by construction - the transformed Lagrangian $\mathscr{L}'$ becomes independent of the guiding-center gyrophase angle ϕ'. Therefore, by construction, the canonical momentum $p'_{\phi' s} = \partial\mathscr{L}'/\partial\dot{\phi}'$, as well as the related magnetic moment defined as $m'_s \equiv \frac{Z_s e}{M_s c}p'_{\phi' s}$, are adiabatic invariants. As shown by Kruskal (1962 [50]) it is always possible to determine $\mathscr{L}'$ so that m'_s is an adiabatic invariant of arbitrary order in ε_M, in the sense that $\frac{1}{\Omega'_{cs}}\frac{d}{dt}\ln m'_s = 0 + O(\varepsilon_M^{n+1})$, where $\Omega'_{cs} = Z_s eB'/M_s c$ denotes the Larmor frequency evaluated at the guiding-center and the integer n depends on the approximation used in the perturbation theory to evaluate m'_s. In addition, the guiding-center invariants corresponding to E_s and ψ_{*s} (denoted as E'_s and ψ'_{*s} respectively) can also be given in terms of $\mathscr{L}'$. These are also, by definition, manifestly independent of ϕ'.,

This basic property of the magnetic moment m'_s is essential in the subsequent developments. Indeed, we shall prove that it allows the effects of temperature anisotropy to be included in the asymptotic stationary solution.

Let us now define the concept of *gyrokinetic* and *equilibrium* KDFs.

Def. - Gyrokinetic KDF (GK KDF)

A generic KDF $f_s(\mathbf{r}, \mathbf{v}, t)$ *will be referred to as gyrokinetic if its Lagrangian time-derivative* $\frac{d}{dt}f_s(\mathbf{r}, \mathbf{v}, t)$ *is independent of the gyrophase angle* ϕ' *evaluated the guiding-center position when its state* $\mathbf{x} = (\mathbf{r}, \mathbf{v})$ *is espressed as a function of an arbitrary gyrokinetic state* $\mathbf{z}' = (\mathbf{y}', \phi')$. *More generally, in the following* $f_s(\mathbf{r}, \mathbf{v}, t)$