$f^3_{R\,n}$ must be constant; then, using also eq. (31) we get:

$$f^3_{L\,0} = f^3_{R\,0} \equiv f_0, \tag{33}$$

where f_0 is a constant. The massless mode has to be identified with the photon $\Rightarrow N^{(0)}_\mu \equiv A_\mu$; since it is the only massless mode in the spectrum we have that the symmetry of the vacuum is, correctly, just $U(1)_{e.m.}$. The "charged" and "neutral" labels we have given to the three sectors refer to their transformation properties with respect to this unbroken symmetry.

As in the case of the left charged sector, the BCs at $y = \pi R$ in this case explicitly contain the mass of the n^{th} mode, so that again the basis wave-functions $f^3_{L\,n}$ and $f^3_{R\,n}$ have nonstandard orthogonality properties. The correct relations are:

$$\left(f^3_{L\,n}, f^3_{L\,m}\right)_{\tilde{g}} = (N^L_m)^2 \delta_{mn}, \quad \left(f^3_{R\,n}, f^3_{R\,m}\right)_{\tilde{g}'} = (N^R_m)^2 \delta_{mn}, \tag{34}$$

where $(\cdot,\cdot)_{\tilde{g}'}$ is defined in a way analogous to $(\cdot,\cdot)_{\tilde{g}}$ (eq. (20)). Completeness relations similar to that in eq. (21) also hold. Note that it is not possible to set both N^L_n and N^R_n to 1. In fact, since they obey the same differential equation (29) and the same BC at $y = 0$ (30), $f^3_{L\,n}$ and $f^3_{R\,n}$ are proportional to each other:

$$f^3_{L\,n} = K_n f^3_{R\,n}, \tag{35}$$

and the constants K_n are fixed by the BCs at $y = \pi R$ (31). To get, also in this case, canonically normalized kinetic terms we have to set:

$$(N^L_n)^2 + (N^R_m)^2 = 1; \tag{36}$$

the ratio $(N^L_n)/(N^R_n)$ will be fixed by the value of K_n and by eq. (34). In particular, for the massless mode it is easy to get

$$\frac{1}{f_0^2} = \frac{2\pi R}{g_5^2} + \frac{1}{\tilde{g}^2} + \frac{1}{\tilde{g}'^2}. \tag{37}$$