where $D_2(y, \widehat{z}, \mu)$ is the derivative (16) of the score function. The expectation in (34) is taken over

$$(z, \widehat{z}) \sim \mathcal{N}(0, P_z(\mu)), \tag{35}$$

where $P_z(\mu)$ is the covariance matrix

$$P_z(\mu) = \left(\begin{array}{cc} \mu^z_{\rm init} & \mu^z_{\rm init} - \mu \\ \mu^z_{\rm init} - \mu & \mu^z_{\rm init} - \mu \end{array} \right), \tag{36}$$

and the conditional distribution of y given z is given by $p_{Y|Z}(y|z)$.

Now consider the recursion

$$\begin{aligned} \mu^q(t) &= \overline{\mathcal{E}}_{\rm out}(\mu^z(t)), & (37a)\\ \mu^x(t+1,s) &= \overline{\mathcal{E}}_{\rm in}(\mu^q(t),s), & (37b)\\ \mu^z(t+1) &= \beta\overline{\mathcal{E}}_{\rm in}(\mu^q(t)), & (37c) \end{aligned}$$

defined for $t \geq 1$. We can also write (37) with the single equation

$$\mu^z(t+1) = \beta\overline{\mathcal{E}}_{\rm in}\left[\overline{\mathcal{E}}_{\rm out}(\mu^z(t))\right]. \tag{38}$$

In [14], the equations (37) (or the single equation version (38)) are called the *state evolution* equations for BP as they describe the evolution of the error variances.

We consider two possible initial conditions for this recursion: one low value and one high value. The low sequence will be initialized with $\mu^z(1) = \mu^z_{\rm lo}(1) = 0$, and the high sequence will be initialized with $\mu^z(1) = \mu^z_{\rm hi}(1) = \mu^z_{\rm init}$ in (33). We will use the subscripts as in $\mu^z_{\rm lo}(t)$ and $\mu^z_{\rm hi}(t)$ to differentiate between the two sequences.

Now, for $t \in \mathbb{Z}^+$, let $\theta^x(t)$ be the random vector

$$\theta^x(t) = \left(x, s, F_{\rm in}(q,\mu), \mathcal{E}_{\rm in}(q,\mu)\right), \tag{39}$$

where $x \sim p_X(x)$, $s \sim p_S(s)$, q is distributed as (14), and $\mu = \mu^q(t-1)/s$ with $\mu^q(t-1)$ being the (deterministic) quantity in the state evolution (SE) equation (37). To initialize, let

$$\theta^x(1) = \left(x, s, \widehat{x}_{\rm init}, \mu^x_{\rm init}\right), \tag{40}$$

where $\widehat{x}_{\rm init}$ and $\mu^x_{\rm init}$ are the mean and variance of the prior of $p_X(x)$. We will see below that when we use $\mu^q(t) = \mu^q_{\rm hi}(t)$, the SE output with the "high" initial condition $\theta^x(t)$ describes the limit of the random vector $\theta^x_{i\leftarrow j}(n,d,t)$ in (31). When $\mu^q(t) = \mu^q_{\rm lo}(t)$, $\theta^x(t)$ is the limit of a related quantity for a certain "genie-aided" algorithm (see Appendix B).

Also, for all $t \in \mathbb{Z}^+$, define the random vector

$$\theta^z(t) = (z, \widehat{z}, \mu^z(t)), \tag{41}$$

where $\mu^z(t)$ is the output of the state evolution equations (37) and $(z, \widehat{z}) \sim \mathcal{N}(0, P_z(\mu^z(t)))$.

Theorem 1: Consider the relaxed BP algorithm under the large sparse limit model above with transform matrix Φ and indices i and j satisfying Assumption 1 for some *fixed* iteration number t. Then:

(a) The random vectors in (31) converge in distribution as follows:

$$\begin{aligned} \lim_{d\rightarrow\infty}\lim_{n\rightarrow\infty} \theta^x_{i\leftarrow j}(n,d,t) &= \theta^x(t) & (42a)\\ \lim_{d\rightarrow\infty}\lim_{n\rightarrow\infty} \theta^x_{j}(n,d,t) &= \theta^x(t) & (42b)\\ \lim_{d\rightarrow\infty}\lim_{n\rightarrow\infty} \theta^z_{i\leftarrow j}(n,d,t) &= \theta^z(t) & (42c)\\ \lim_{d\rightarrow\infty}\lim_{n\rightarrow\infty} \theta^z_{i}(n,d,t) &= \theta^z(t), & (42d) \end{aligned}$$

where the random vectors $\theta^x(t)$ and $\theta^z(t)$ are defined as above with $\mu^q(t) = \mu^q_{\rm hi}(t)$ and $\mu^z(t) = \mu^z_{\rm hi}(t)$.

(b) The error variances satisfy the limits

$$\lim_{d\rightarrow\infty}\lim_{n\rightarrow\infty} \mathbf{E}\left[|x_j - \widehat{x}_j(t)|^2 | s_j = s\right] = \mu^x_{\rm hi}(t,s), \tag{43a}$$

$$\lim_{d\rightarrow\infty}\lim_{n\rightarrow\infty} \mathbf{E}\left[|z_i - \widehat{z}_i(t)|^2\right] = \mu^z_{\rm hi}(t), \tag{43b}$$

where $\mu^x_{\rm hi}(t,s)$ and $\mu^z_{\rm hi}(t)$ are the output of the SE equations (37) with the "hi" initial condition.

(c) The minimum conditional error variance of x_j and z_i given Φ and $\mathbf{y}$ satisfy the asymptotic lower bounds

$$\lim_{d\rightarrow\infty}\lim_{n\rightarrow\infty} \mathbf{E}\left[\mathbf{var}(x_j|\mathbf{y},\Phi) | s_j = s\right] \geq \mu^x_{\rm lo}(t,s), \tag{44a}$$

$$\lim_{d\rightarrow\infty}\lim_{n\rightarrow\infty} \mathbf{E}\left[\mathbf{var}(z_i|\mathbf{y},\Phi)\right] \geq \mu^z_{\rm lo}(t), \tag{44b}$$

where $\mu^x_{\rm lo}(t,s)$ and $\mu^z_{\rm lo}(t)$ are the output of the SE equations (37) with the "lo" initial condition.

Proof: See Appendices E and F. ■

The performance bounds in parts (a) and (b) are largely identical to the results in [14] except that they apply to relaxed BP instead of BP. This is our main result: in the large sparse limit model, relaxed BP and standard BP have the identical asymptotic behavior. The lower bound in part (c) of the theorem is also very close to results in [14] and just repeated here for completeness.

Part (a) of the theorem provides a simple scalar characterization for this asymptotic behavior. Specifically, using the definition of $\theta^x(t)$ in (39), Theorem 1 shows that the componentwise behavior of the relaxed BP follows a *scalar equivalent model* as shown in Fig. 3: The component x_j is first corrupted by Gaussian noise yielding a noisy component q_j. The relaxed BP estimate $\widehat{x}_j(t)$ then behaves identically to the optimal scalar MMSE estimate of x_j from the AWGN measurement q_j. From this scalar equivalent joint distribution of the components and their estimates, one can compute any componentwise separable performance metric such as mean-squared error or probability of detection.

The effective Gaussian noise levels in the scalar models are described by $\mu^z_{\rm hi}(t)$ and $\mu^q_{\rm hi}(t)$ from the state evolution equations (37). Since the state evolution equations can be evaluated easily with numerical integration, Theorem 1 thus provides a simple, computationally-tractable method for exactly characterizing the performance of the relaxed BP algorithm.

Part (b) shows that the SE outputs $\mu^x_{\rm hi}(t,s)$ and $\mu^z_{\rm hi}(t)$ respectively describe the asymptotic estimation error on the components x_j and prediction error on the outputs z_i. Part (c) provides corresponding lower bounds on these error variances for any estimator.