# Supplementary material to: Restricted Thermalization for Two Interacting Atoms in a Multimode Harmonic Waveguide


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This supplementary material contains a detailed derivation of the eigenenergies and eigenstates for a system consisting of two atoms in a circular, transversely harmonic waveguide in the multimode regime.


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The waveguide system [1], as well as the Šeba billiard [2], belongs to the class of problems where an integrable system of Hamiltonian $\hat{H}_0$ is perturbed by a separable rank I interaction $V|\mathcal{L}\rangle\langle\mathcal{R}|$ (see *e.g.* [3]). Eigenstates $|\alpha\rangle$ of the interacting system are solutions of the Schrödinger equation

$$[\hat{H}_0 + V|\mathcal{L}\rangle\langle\mathcal{R}|]|\alpha\rangle = E_\alpha|\alpha\rangle\,, \tag{1}$$

where $E_\alpha$ are the corresponding eigenenergies. Let us expand the eigenstate $|\alpha\rangle$ over the eigenstates $|\vec{n}\rangle$ (of energy $E_{\vec{n}}$) of the non-interacting hamiltonian $\hat{H}_0$. The Schrödinger equation leads to the following equations for the expansion coefficients

$$(E_\alpha - E_{\vec{n}})\langle\vec{n}|\alpha\rangle = \langle\vec{n}|\mathcal{L}\rangle\langle\mathcal{R}|\alpha\rangle \propto \langle\vec{n}|\mathcal{L}\rangle, \tag{2}$$

where the omitted factor in the second equality is independent of $\vec{n}$.

As a result, any eigenfunction $|\alpha\rangle$ functionally coincides with the Green function of the non-interacting Hamiltonian taken at the energy of the eigenstate $E_\alpha$,

$$|\alpha\rangle \propto \sum_{\vec{n}} \frac{|\vec{n}\rangle\langle\vec{n}|\mathcal{L}\rangle}{E_\alpha - E_{\vec{n}}}. \tag{3}$$

The omitted factor is determined by the normalization conditions.

Substitution of this expression to the Schrödinger equation (1) leads to the following eigenenergy equation:

$$\sum_{\vec{n}} \frac{\langle\mathcal{R}|\vec{n}\rangle\langle\vec{n}|\mathcal{L}\rangle}{E_\alpha - E_{\vec{n}}} = \frac{1}{V}. \tag{4}$$

Similar expressions were obtained in Refs. [2, 4] for the case of the Šeba billiard and its generalizations.

In the case of the relative motion of two short-range-interacting atoms in a circular, transversely harmonic waveguide, the derivation uses ideas of the analogous model involving an infinite waveguide [5–7]. The unperturbed Hamiltonian here is given by Eq. (1) in [1] and the Fermi-Huang interaction (see Eq. (2) in [1]) can be written in the separable form [3] with the interaction strength and formfactors given by

$$V = \frac{2\pi\hbar^2 a_s}{\mu}, \quad |\mathcal{L}\rangle = \delta_3(\boldsymbol{r}), \quad \langle\mathcal{R}| = \delta_3(\boldsymbol{r})\frac{\partial}{\partial r}(r\,\cdot). \tag{5}$$

In what follows, we will restrict the Hilbert space to the states of zero $z$-component of the angular momentum and even under the $z \leftrightarrow -z$ reflection; the interaction has no effect on the rest of the Hilbert space. The non-interacting eigenstates are products of the transverse and longitudial wavefunctions. The transverse two-dimensional zero-angular-momentum harmonic wavefunctions,

$$\langle\rho|n\rangle = \frac{1}{\sqrt{\pi}a_\perp} L_n^{(0)}(\xi)\exp(-\xi/2)\,, \tag{6}$$

labeled by the quantum number $n \geq 0$, are expressed in terms of the Legendre polynomials, $L_n^{(0)}(\xi)$, where $\xi = (\rho/a_\perp)^2$ and $a_\perp = (\hbar/\mu\omega_\perp)^{1/2}$ is the transverse oscillator range. The longitudial wavefunctions, labeled by $l \geq 0$, are the symmetric plane waves satisfying periodic conditions with the period $L$:

$$\langle z|k\rangle = (2/L)^{1/2}\cos 2\pi k\zeta, \qquad \langle z|0\rangle = L^{-1/2}, \tag{7}$$

where $\zeta \equiv z/L - 1/2$. The unperturbed spectrum is therefore given by

$$E_{nl} = 2\hbar\omega_\perp n + \hbar^2(2\pi l/L)^2/(2\mu) \tag{8}$$

Substituting the above non-interacting eigenstates and eigenenergies to Eq. (3) and using Eq. (1.445.8) in [8] for summation over $l$, one obtains Eq. (3) in [1] for the interacting eigenstates. The normalization factor $C_\alpha$ is determined by the condition

$$2\pi\int_0^L dz\int_0^\infty \rho^2 d\rho\langle\alpha'|\rho,z\rangle\langle\rho,z|\alpha\rangle = \delta_{\alpha\alpha'}. \tag{9}$$