TABLE II: Fluctuation entropy per site calculated for randomly generated $uuud$ states in an $N = 1024$ cluster. Here $s_{\rm f}^{\rm min}$, $s_{\rm f}^{\rm max}$, and $\langle s_{\rm f}\rangle$ are the lowest, highest, and mean value of the entropy, respectively, and $\Delta s_{\rm f} = s_{\rm f}^{\rm max} - \langle s_{\rm f}\rangle$ measures the deviation of the highest value of entropy from the mean. Statistical errors on all numbers are less than 10^{-6}.

b	$s_{\rm f}^{\rm min}$	$s_{\rm f}^{\rm max}$	$\langle s_{\rm f}\rangle$	$\Delta s_{\rm f}$
0.05	-0.79931	-0.79608	-0.79837	0.00228
0.1	-1.63451	-1.63191	-1.63374	0.00183
0.2	-2.54944	-2.54758	-2.54888	0.00130
0.3	-3.12864	-3.12725	-3.12823	0.00098
0.4	-3.56055	-3.55948	-3.56025	0.00077
0.5	-3.90757	-3.90672	-3.90734	0.00062
0.6	-4.19874	-4.19806	-4.19857	0.00051

form

$$\frac{\mathcal{F}}{N} = \frac{E_0}{N} - T\ln T + \frac{T}{2N}\langle \ln\det \mathcal{M}\rangle_{\Omega_0} - \frac{T}{N}\ln\Omega_0 + \mathcal{O}(T^2), \quad (38)$$

where E_0 is the ground state energy, Ω_0 its degeneracy, and $\langle ...\rangle_{\Omega_0}$ the average over all degenerate ground states.

For a generic ordered phase, Ω_0 is finite, and $\det\mathcal{M}$ takes on the same value for all (symmetry related) ground states. In this case $\ln\Omega_0/N \to 0$ for $N\to\infty$. However for the dimer manifold, $\Omega_0 \approx 1.14^N$, which means that the ground state has a *finite entropy per site*

$$\frac{S_0}{N} \approx \ln 1.14 \approx 0.13\ . \quad (39)$$

In this case, different ground states are not related by simple lattice symmetries and the fluctuation entropy per site

$$s_{\rm f} = -\frac{\ln\det\mathcal{M}}{2N} \quad (40)$$

takes on a range of values.

We have studied the distribution of values of $s_{\rm f}$ within the dimer manifold for a range of values of b, by numerically calculating $\det\mathcal{M}$ for 10000 randomly generated $uuud$ states in a cluster of $N = 1024$ sites ($L = 4$), using a Monte Carlo algorithm based on loop updates of spins. We found that the highest value of $s_{\rm f}$ is achieved by an eight–fold degenerate, 16–sublattice "R-state"[13], in which the four A–sublattice tetrahedra within the 16–site cubic unit cell of the pyrochlore lattice take on all four possible $uuud$ configurations [Fig. 11(c)]. This state has overall cubic symmetry, and is actually observed in the plateau phase of $HgCd_2O_4$[6]. The lowest value of $s_{\rm f}$ is achieved by the four–sublattice order shown in Fig. 11(b). The calculated values of the maximum and minimum values $s_{\rm f}^{\rm max}$ and $s_{\rm f}^{\rm min}$ are listed in Table II together with the mean value $\langle s_{\rm f}\rangle$ and the difference between $s_{\rm f}^{\rm max}$ and the mean $\langle s_{\rm f}\rangle$, $\Delta s_{\rm f}$.

From these results it is immediately clear *why* thermal fluctuations alone fail to select a unique ground state for any value

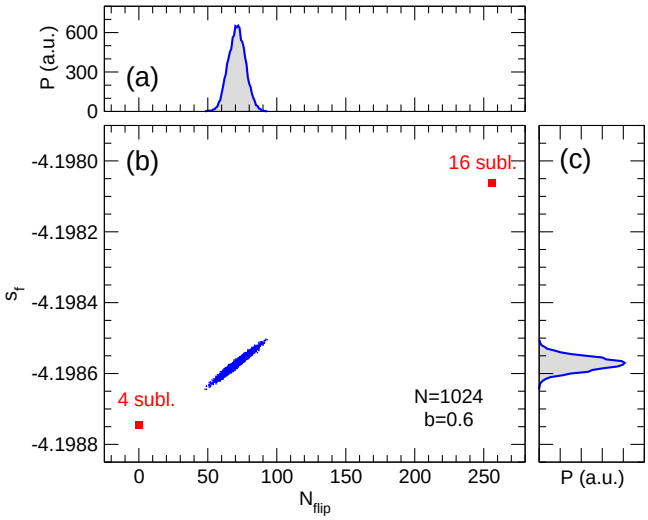

FIG. 15: (Color online) (a) Probability distribution of the flippable hexagons within the dimer manifold. (b) The fluctuation entropy per site $s_{\rm f}$ [Eq. (40)] as a function of the number of "flippable" hexagons. The lower bound $s_{\rm f} = -4.19874$ is set by the four–sublattice state shown in Fig. 11(b), which has no flippable hexagons. The upper bound $s_{\rm f} = -4.19806$ is set by the 16–sublattice state with the maximum number of flippable hexagons [see Fig. 11(c)]. The blue dots represent a sample of 10000 random configurations. (c) Probability distribution of the fluctuation entropy per site $s_{\rm f}$ within the dimer manifold. All results are for a cluster of $N = 1024$ sites with $b = 0.6$.

of b considered in this paper. The fluctuation entropy per site gained by choosing the cubic 16–sublattice state is miserly, for example, $\Delta s_{\rm f} = 0.00183$ for $b = 0.1$ and $\Delta s_{\rm f} = 0.00051$ for $b = 0.6$. These numbers must be compared with the extensive entropy $S_0/N \approx 0.13$ of the liquid phase, *all* of which is lost if the system orders. So for the values of b considered here, thermal fluctuations cannot drive the system to order.

However it is amusing to note that the entropy gain $\Delta s_{\rm f}$ *increases* as b decreases, scaling approximately as $\ln b$, as shown in Table II. This raises the intriguing possibility that b acts as a singular perturbation, and that for sufficiently small b, fluctuations might overcome the extensive entropy $S_0/N \approx 0.13$ of the dimer manifold, driving the system order — even though it is *disordered* for $b = 0$. Such an order-from disorder effect would presumably favor the cubic 16–sublattice R-state, which is also believed to be selected by quantum fluctuations at $T = 0$[12,13,37]. However in the present model, it would occur only for vanishingly low temperatures, and would therefore be extremely difficult to access in simulation. This question remains for future study.

The result above explains why the system does not order at finite temperature, but not *why* the fluctuation entropy favors the 16–sublattice state? We can answer this question by looking at the distribution of the fluctuation entropies $s_{\rm f}$ within the dimer manifold of $uuud$ states. Figure 15 shows the distribution for $b = 0.6$. The $uuud$ states can be broken up into classes of states with a different net flux of an effective magnetic (or, equivalently, electric) field[12,13,29–31,37]. This