Trade-off capacities of the quantum Hadamard channels

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Coding theorems in quantum Shannon theory express the ultimate rates at which a sender can transmit information over a noisy quantum channel. More often than not, the known formulas expressing these transmission rates are intractable, requiring an optimization over an infinite number of uses of the channel. Researchers have rarely found quantum channels with a tractable classical or quantum capacity, but when such a finding occurs, it demonstrates a complete understanding of that channel's capabilities for transmitting classical or quantum information. Here, we show that the three-dimensional capacity region for entanglement-assisted transmission of classical and quantum information is tractable for the Hadamard class of channels. Examples of Hadamard channels include generalized dephasing channels, cloning channels, and the Unruh channel. The generalized dephasing channels and the cloning channels are natural processes that occur in quantum systems through the loss of quantum coherence or stimulated emission, respectively. The Unruh channel is a noisy process that occurs in relativistic quantum information theory as a result of the Unruh effect and bears a strong relationship to the cloning channels. We give exact formulas for the entanglement-assisted classical and quantum communication capacity regions of these channels. The coding strategy for each of these examples is superior to a naïve time-sharing strategy, and we introduce a measure to determine this improvement.

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I. INTRODUCTION

One of the aims of quantum Shannon theory is to characterize the ultimate limits on the transmission of information over a noisy quantum channel. Holevo, Schumacher, and Westmoreland contributed the first seminal result in this direction by providing a lower bound for the ultimate limit of a noisy quantum channel to transmit classical information [1, 2], a result now known as the HSW coding theorem. Lloyd, Shor, and Devetak then contributed increasingly rigorous proofs of the quantum channel coding theorem [3–5], now known as the LSD coding theorem, that provides a lower bound on the ultimate limit for a noisy quantum channel to transmit quantum information. Other expository proofs appeared later, providing insight into the nature of quantum coding [6–10]. Bennett *et al.* [11] and Barnum *et al.* [12] also showed that the capacity of a quantum channel for transmitting quantum information is the same as that channel's capacity for generating shared entanglement between sender and receiver. These three results form the core of the dynamic, single-resource quantum Shannon theory, where a sender exploits a noisy quantum channel to establish a single noiseless resource, namely, classical communication, quantum communication, or shared entanglement, with a receiver.

A formula for the capacity of a channel gives a "single-letter" characterization if the computation of the capacity requires an optimization over only a single use of the channel, and the formula gives a "multi-letter" characterization otherwise. A single-letter characterization implies that the computation of the capacity is tractable for a fixed input dimension of the channel, whereas a multi-letter characterization typically requires an optimization over an infinite number of uses of the channel and is therefore intractable in this case. This "single-letterization" issue does not play a central role in classical information theory for the most basic task of communication over a noisy classical channel, because single-letterization occurs naturally in Shannon's original analysis for classical, memoryless channels [13]. But this issue plays a prominent role in the domain of quantum Shannon theory even for the most basic communication tasks. Our knowledge so far indicates that the computation of the classical capacity is intractable in the general case [14–17], with the same seeming to hold generally for the quantum capacity [18, 19]. These results underscore our incomplete understanding of the nature of quantum information, but they also have the surprising and "uniquely quantum" respective consequences that the strong correlations present in entangled uses of a quantum channel boost the classical capacity and that the degeneracy property of quantum codes can enhance the quantum capacity.

Thus, in hindsight, we might now say that any channel with a single-letter capacity formula is a "rare gem" in quantum Shannon theory, given that the known formulas for capacities generally give multi-letter characterizations. In fact, researchers once conjectured that the HSW formula for the classical capacity would generally give a single-letter characterization [20, 21], until the recent result of Hastings [14]. Researchers have found several examples of these gems for the classical capacity: the identity channel [22], unital qubit channels [23], erasure channels [24], Hadamard channels [25], entanglement-breaking channels [26], depolarizing chan-