for all large $s > 0$. Now, we absorb the first term on the right-hand side into the left-hand side by choosing $s > 0$ large, we get

$$\int_{\Omega_0} \chi_0^2 \left(|y - y_0|\right) |f|^2 \, e^{2s\varphi(x,y,0)} dxdy \leq CM^2 e^{2s\kappa_1} + Ce^{cs} d^2$$

for all large $s > 0$.

Replacing the integration domain on the left-hand side by $D \times \left\{ y; \;\; |y - y_0| < \dfrac{L}{\rho} \right\} \subset \Omega_0$ and using the facts that $\chi_0 \left(|y - y_0|\right) = 1$ in $D \times \left\{ y; \;\; |y - y_0| < \dfrac{L}{\rho} \right\}$ and

$$e^{2s\varphi(x,y,0)} = \exp\left(2se^{\gamma\psi(x,y,0)}\right) > \exp\left(2se^{\gamma\epsilon}\right) = e^{2s\kappa_2},$$

we obtain,

$$e^{2s\kappa_2} \int_{D \times \left\{ y; \; |y - y_0| < \frac{L}{\rho} \right\}} |f|^2 \, dxdy \leq CM^2 e^{2s\kappa_1} + Ce^{cs} d^2$$

for all $s \geq s_0$, where s_0 is some constant. Since $\kappa_2 > \kappa_1$, the last inequality implies

$$\int_{D \times \left\{ y; \; |y - y_0| < \frac{L}{\rho} \right\}} |f|^2 \, dxdy \leq CM^2 e^{-2s\kappa} + Ce^{cs} d^2 \tag{4.23}$$

for all $s \geq s_0$, where $\kappa = \kappa_2 - \kappa_1 > 0$. We separately consider the two cases:

Case 1. Let $M \geq d$. Choosing $s \geq 0$ such that

$$M^2 e^{-2s\kappa} = e^{Cs} d^2, \text{ that is, } s = \frac{2}{C + 2\kappa} \log \frac{M}{d} \geq 0,$$

we obtain

$$\int_{D \times \left\{ y; \; |y - y_0| < \frac{L}{\rho} \right\}} |f|^2 \, dxdy \leq 2M^{\frac{2C}{C+2\kappa}} d^{\frac{4\kappa}{C+2\kappa}}.$$

Case 2. Let $M < d$. Then setting $s = 0$ in (4.23) we have

$$\int_{D \times \left\{ y; \; |y - y_0| < \frac{L}{\rho} \right\}} |f|^2 \, dxdy \leq 2Cd^2.$$