perbolic surfaces, respectively. After rescaling, the general equation for $x'^2(\lambda')$ is found to be:

$$\frac{dx'^2}{d\lambda'} = -2k - \frac{2}{x'^2} - k\,\frac{5}{2}\frac{1}{x'^4} - 2(\frac{59}{24}\zeta(3) + \frac{29}{24})\frac{1}{x'^6}, \text{for } \alpha' > 0 \tag{10}$$

$$\frac{dx'^2}{d\lambda'} = -2k + \frac{2}{x'^2} - k\,\frac{5}{2}\frac{1}{x'^4} + 2(\frac{59}{24}\zeta(3) + \frac{29}{24})\frac{1}{x'^6}, \text{for } \alpha' < 0 \tag{11}$$

Here $k = \pm 1$ stands for spherical (+) or hyperbolic (-) surfaces. The first term in the R.H.S corresponds to unnormalized Ricci flow whereas the subsequent terms are for 2nd, 3rd and 4th order flows, respectively.

A. Geometric flow analysis

1. <u>2nd order flow on S^2</u>

Recall that the Ricci flow on the two–sphere gives rise to an ancient solution. Including the second order term leads to the following proposition.

Proposition III.1 *For 2nd order flow on the canonical two-sphere, with any choice of initial radius and $\alpha' < 0$ we obtain $a'_\infty = 1$. If $\alpha' < 0$ and $a'_0 = 1$ we obtain a soliton, while for $\alpha' > 0$ and with any a'_0, we obtain an ancient solution.*

The fact that for $\alpha' > 0$ we get an ancient solution is shown in Fig.1 (top left). If we consider $\alpha' < 0$, which is essentially backward $2nd$ order flow upto a scaling, we can see that the solution space is divided into two regions namely $a^2 < 1$ and $a^2 > 1$. It is also useful to note that if we choose the initial condition $a_0 = 1$ for $\alpha' < 0$ we get a soliton. The solution of the flow equations for $a^2 > 1$, $a^2 < 1$ with $\alpha' < 0$ are given as:

$$a'(\lambda')^2 \;+\; \ln(a'(\lambda')^2 - 1) - a_0'^2 \;-\; \ln(a_0'^2 - 1) = -2\lambda', \qquad \text{for } \; a'^2(\lambda') > 1 \tag{12}$$

$$a'(\lambda')^2 \;+\; \ln(1 - a'(\lambda')^2) - a_0'^2 \;-\; \ln(1 - a_0'^2) = -2\lambda' \qquad \text{for } \; a'^2(\lambda') < 1 \tag{13}$$

The proof of the proposition is evident through the plots (in Fig.1) and also from the general expressions given above. Here and henceforth, the values chosen while obtaining the graphs are representative. We have also checked things over the full, respective ranges.