$\gamma(t)$ near the end of the cooling procedure is exponential rather than algebraic

$$1-\gamma(t)\simeq 2\exp\left\{-\frac{4/T_0}{1-t/\tau}\right\}\quad\text{when}\quad t\to\tau$$

This suggests to analyze a family of cooling laws with an asymptotic behavior of $\gamma(t)$ of the form

$$1-\gamma(t)\simeq B\exp\left\{-\frac{b}{(1-t/\tau)^\beta}\right\}\quad\text{when}\quad t\to\tau \tag{14}$$

with arbitrary positive parameters B, b, β. In this situation, the asymptotic behavior of the final density of domain walls is

$$\rho(\tau)\simeq\frac{1}{4}\sqrt{\frac{\pi}{\tau}}\left(\frac{\ln\tau}{b}\right)^{\frac{1}{2\beta}} \tag{15}$$

This is a remarkably universal asymptotic: The leading $\tau^{-1/2}$ algebraic factor does not depend on the parameters B, b, β and the logarithmic prefactor depends only on one parameter β.

We now turn to derivations of the announced results.

III. SPECIAL COOLING SCHEME

In this section we consider the special cooling procedure (6). First, we employ an exact analysis and then turn to the asymptotic behavior of the final density of domain walls.

A. Exact Analysis

To handle an infinite set of ordinary differential equations (8) we recast it into a single differential equation using a generating function technique. Multiplying (8) by z^k and summing over all $k\geq 1$ we find that the generating function

$$G(z,t)=\sum_{k\geq 1}G_k(t)\,z^k \tag{16}$$

satisfies

$$\frac{\partial G}{\partial t}=\left[-2+\frac{t}{\tau}\left(z+z^{-1}\right)\right]G+\frac{t}{\tau}\left[z-G_1(t)\right] \tag{17}$$

Solving (17) subject to $G(z,t=0)=0$, we get

$$G=\int_0^t dt'\,\frac{t'}{\tau}\left[z-G_1(t')\right]\exp\left[2(t'-t)+\eta\,\frac{z+z^{-1}}{2}\right]$$

where $\eta=(t^2-t'^2)/\tau$. Recalling that the exponential factor $\exp\left[\eta\,\frac{z+z^{-1}}{2}\right]$ is the generating function for the modified Bessel functions

$$\exp\left[\eta\,\frac{z+z^{-1}}{2}\right]=\sum_{n=-\infty}^{\infty}z^n\,I_n(\eta)$$

we conclude that

$$G_k(t)=\int_0^t dt'\,\frac{t'}{\tau}\,e^{2(t'-t)}\left[I_{k-1}(\eta)-G_1(t')I_k(\eta)\right] \tag{18}$$

Thus to determine G_1 we need to solve an integral equation

$$G_1(t)=\int_0^t dt'\,\frac{t'}{\tau}\,e^{2(t'-t)}\left[I_0(\eta)-G_1(t')I_1(\eta)\right] \tag{19}$$

Higher correlation functions G_k with $k\geq 2$ are then expressed via G_1 and (18).

The above results are exact. Now we turn to the most interesting case of slow cooling, $\tau\gg 1$, and compute the final density of domain walls.

B. Final Density of Domain Walls

Using (4), we re-write the integral equation (19) in terms of the density of domain walls:

$$\begin{aligned}1\;&-\;\int_0^t dt'\,\frac{t'}{\tau}\,e^{2(t'-t)}\left[I_0(\eta)-I_1(\eta)\right]\\ &=\;2\rho(t)+2\int_0^t dt'\,\frac{t'}{\tau}\,e^{2(t'-t)}\,I_1(\eta)\,\rho(t')\end{aligned} \tag{20}$$

We are mostly interested in the final density $\rho(\tau)$. Hence we specialize (20) to $t=\tau$. It is also convenient to change the integration variable, $t'\to\eta=(\tau^2-t'^2)/\tau$. We arrive at the integral equation

$$\begin{aligned}1\;&-\;\frac{1}{2}\int_0^\tau d\eta\,\mathcal{E}(\eta,\tau)\left[I_0(\eta)-I_1(\eta)\right]\\ &=\;2\rho(\tau)+\int_0^\tau d\eta\,\mathcal{E}(\eta,\tau)\,I_1(\eta)\,\rho\left(\tau\sqrt{1-\eta/\tau}\right)\end{aligned} \tag{21}$$

where we used the shorthand notation

$$\mathcal{E}(\eta,\tau)\equiv\exp\left[2\tau\left(\sqrt{1-\eta/\tau}-1\right)\right] \tag{22}$$

The chief contribution to the integral on the right-hand side of (21) is gathered when $\eta\sim\sqrt{\tau}$ [see Eq. (26) below]. Therefore we can replace $\rho\left(\tau\sqrt{1-\eta/\tau}\right)$ by $\rho(\tau)$ and the exponential factor $\mathcal{E}(\eta,\tau)$ by

$$\mathcal{E}(\eta,\tau)=e^{-\eta-\eta^2/4\tau} \tag{23}$$

as it follows by expanding the term inside the square brackets in Eq. (22). Thus the integral on the right-hand side of (21) becomes

$$\rho(\tau)\int_0^\tau d\eta\,e^{-\eta-\eta^2/4\tau}\,I_1(\eta) \tag{24}$$

We can further use the asymptotic formula for the Bessel function,

$$I_1(\eta)\simeq\frac{e^\eta}{\sqrt{2\pi\eta}}\quad\text{when}\quad\eta\gg 1, \tag{25}$$