points of the circle, then

$$\det g^q_{\mu\nu}(P) = \lim_{\rho\to 0} \frac{(d^q_m d^q_M)^2}{\rho^4}. \tag{50}$$

The computation of the determinant is done simultaneously with the calculation of the curvature, which also requires the walk on the same circle. The numerical calculations give a direct confirmation that $F^q(P)$=1 for all points of the sphere.

VII. QUANTUM GEOMETRY FOR TWO FLUXES ADDED

In this case we have 4 anyons and the zero modes space is 2-dimensional. We will keep 3 anyons fixed and move the forth one along different braiding paths.

For the Non-Abelian case, there is no simple Stokes theorem,[33,34] which means the monodromy can not be simply computed from the curvature as we did for the previous case. Even so, mapping the curvature provides a clear picture of the non-comutative and topological properties of the states.

The parameter space remains 2-dimensional. As before, a point in this parameter space indicates the position of the mobile anyon. Thus $dF = \hat{F} dw^1 \wedge dw^2$, but $\hat{F}$ is now a 2×2 matrix. We compute $\hat{F}(P)$ using the same algorithm (see Eq. 48). Using the Pauli's matrices, σ_i, i=1, 2, 3, $\hat{F}(P)$ can be uniquely decomposed as:

$$\hat{F}(P) = f_0(P) + \mathbf{f}(P)\cdot\boldsymbol{\sigma}, \tag{51}$$

where $f_0(P) = \frac{1}{2}\mathrm{Tr}\hat{F}(P)$ and $\mathbf{f}(P)$ is a 3-component vector. We will refer to f_0 as the Abelian and to $\mathbf{f}\cdot\boldsymbol{\sigma}$ as the non-Abelian part of the curvature. It is important to notice how different quantities behave when changing the gauge, i.e. the basis in the 2-dimensional zero modes space. We have: $f_0(P)$ is gauge independent; the magnitude of $\mathbf{f}(P)$ is gauge independent; the orientation of $\mathbf{f}(P)$ is gauge dependent.

Fig. 9 shows plots of $\mathrm{Tr}\hat{F}(P)$ for different system sizes. For each size, we show $\mathrm{Tr}\hat{F}(P)$ calculated with the standard and with the quantum metric tensor. The numerics confirm again the theoretical prediction that $\mathrm{Tr}\hat{F}(P)$=1.

To demonstrate that the braid group is non-comutative, we need first to show that $\mathbf{f}(P)$ is non-zero. This, however, is not enough. We need also to rule out the existence of a particular gauge in which the adiabatic connection becomes diagonal at every point of the parameter space. If such a gauge exists, the fiber bundle of the zero-modes degenerates into a trivial $U(1)\times U(1)$ fiber bundle, in which case there will be two 1-dimensional fibers that do not mix during the adiabatic braiding. Consequently, all monodromies will take a diagonal form in this gauge and they will commute with each other.

As already mentioned, the magnitude of $\mathbf{f}(P)$ is gauge independent. Thus we can see if $\mathbf{f}(P)$ is zero or not by simply plotting its magnitude, which is shown in Fig. 10 for different sizes. The graph clearly demonstrates that $\mathbf{f}(P)$ is non-zero and it appears to be concentrated near the positions of the fixed anyons. We will further discuss Fig. 10 in the next Section.

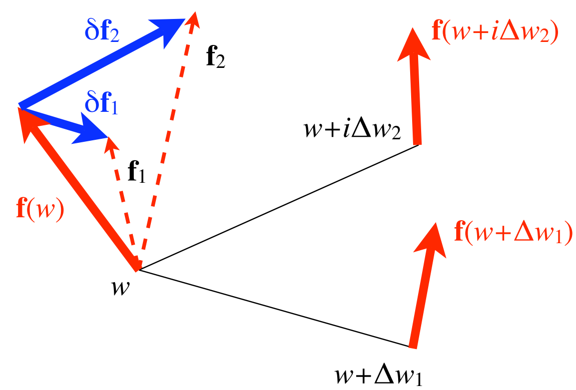

FIG. 11: (Color online) A diagram of the parallel transport used in the calculation of the Twist density ρ_{TW}.

Next, we introduce a scalar function which we call the Twist density ρ_{TW}, which gives a measure of how much are the fibers twisted during the adiabatic parallel transport. We start the construction from the following 2-form:

$$d\hat{\rho} = [D_\mu \hat{F}, D_\nu \hat{F}]\ dw_\mu \wedge dw_\nu, \tag{52}$$

where D_μ denotes the covariant derivative corresponding to the adiabatic connection and $[,]$ denotes the usual commutator. This form is invariant to coordinate transformations, thus well defined. The coefficients of this form are 2×2 matrices. In the special case of a two dimensional parameter space, the form reduces to:

$$d\hat{\rho} = [D_1 \hat{F}, D_2 \hat{F}]\ dw_1 \wedge dw_2, \tag{53}$$

Based on the above observations, we construct the following 2-form,

$$d\rho_{\mathrm{TW}} \equiv \sqrt{\det[D_1 \hat{F}, D_2 \hat{F}]}\ dw_1 \wedge dw_1, \tag{54}$$

whose coefficient is a pseudo-scalar function. Since $d\hat{\rho}$ is invariant to coordinate transformations, $d\rho_{\mathrm{TW}}$ is also invariant and hence well defined. The density

$$\rho_{\mathrm{TW}} = \sqrt{\det[D_1 \hat{F}, D_2 \hat{F}]} \tag{55}$$

is gauge invariant and it is identically zero for trivial fiber bundles, in particular for $U(1)\times U(1)$ fiber bundle over our parameter space. Thus, if we show that ρ_{TW} is non-zero, that will be equivalent to demonstrating that the zero modes fiber bundle is non-trivial.

Let us now give the physical interpretation of our construction. For this we consider three points on the sphere: w, $w+\Delta w_1$ and $w+i\Delta w_2$, as in Fig. 11. Assume that we