as

$$S^\alpha = \frac{1}{2\sqrt{2}} \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^\dagger & -\boldsymbol{A}^T \end{pmatrix}, \quad X^a = \frac{1}{2\sqrt{2}} \begin{pmatrix} \boldsymbol{C} & \boldsymbol{D} \\ \boldsymbol{D}^\dagger & \boldsymbol{C}^T \end{pmatrix}, \tag{B.1}$$

where $\boldsymbol{A}$ is hermitian, $\boldsymbol{C}$ is hermitian and traceless, $\boldsymbol{B}^T = \boldsymbol{B}$ and $\boldsymbol{D}^T = -\boldsymbol{D}$. The $\{S\}$ are also $Sp(4)$ generators since they obey Eq. (2.1). We define

$$S^\alpha = \frac{1}{2\sqrt{2}} \begin{pmatrix} \tau^\alpha & \mathbf{0} \\ \mathbf{0} & -(\tau^\alpha)^T \end{pmatrix}, \quad (\alpha = 1, 2, 3, 4). \tag{B.2}$$

For $\alpha = 1, 2, 3$, we have the standard Pauli matrices, while for $\alpha = 4$ we define $\tau^4 = \mathbf{1}$. These are simply the generators for $SU(2) \times U(1)$. For $\alpha = 5, ..., 10$

$$S^\alpha = \frac{1}{2\sqrt{2}} \begin{pmatrix} \mathbf{0} & B^\alpha \\ (B^\alpha)^\dagger & \mathbf{0} \end{pmatrix}, \quad (\alpha = 5, ..., 10), \tag{B.3}$$

and

$$B^5 = \mathbf{1}_2,\ B^6 = i\mathbf{1}_2,\ B^7 = \tau^3,\ B^8 = i\tau^3,\ B^9 = \tau^1,\ B^{10} = i\tau^1. \tag{B.4}$$

The five broken generators $\{X\}$ are

$$X^a = \frac{1}{2\sqrt{2}} \begin{pmatrix} \tau^a & \mathbf{0} \\ \mathbf{0} & (\tau^a)^T \end{pmatrix}, \quad (a = 1, 2, 3), \tag{B.5}$$

where τ^a are the standard Pauli matrices. For $a = 4, 5$

$$X^a = \frac{1}{2\sqrt{2}} \begin{pmatrix} \mathbf{0} & D^a \\ (D^a)^\dagger & \mathbf{0} \end{pmatrix}, \quad (a = 4, 5), \tag{B.6}$$

and

$$D^4 = \tau^2,\ D^5 = i\tau^2. \tag{B.7}$$

The generators are normalized as follows:

$$\mathrm{tr}(S^\alpha S^\beta) = \frac{1}{2}\delta^{\alpha\beta}, \quad \mathrm{tr}(X^a X^b) = \frac{1}{2}\delta^{ab}, \quad \mathrm{tr}(S^\alpha X^a) = 0. \tag{B.8}$$

Appendix C: $\mathcal{O}(p^4)$ HLS Lagrangian

In this appendix, we present a complete list of the $\mathcal{O}(p^4)$ HLS Lagrangian for general N_f and $N_C = 2$, following Refs. [11, 17]. For the construction, we need the building blocks

$$\hat{\mathcal{V}}_{\mu\nu} = \frac{1}{2}[\xi_R \bar{\Sigma} G^T_{\mu\nu} \bar{\Sigma} \xi^\dagger_R + \xi_L G_{\mu\nu} \xi^\dagger_L], \quad \hat{\mathcal{A}}_{\mu\nu} = \frac{1}{2}[\xi_R \bar{\Sigma} G^T_{\mu\nu} \bar{\Sigma} \xi^\dagger_R - \xi_L G_{\mu\nu} \xi^\dagger_L], \tag{C.1}$$