the same basic factorized structure as in Eq. (2), involving distinct (though process-dependent) TMD PDFs and FFs for all external hadrons. The TMD PDFs and FFs have the usual structure of a pair of field operators and a Wilson line with an expectation value corresponding to a specific external hadron state. The only difference from the standard case is that they are equipped with non-standard and potentially complex Wilson line structures. In particular, Eq. (3) contains no matrix element of the form $\langle H_1 H_2 | \cdots | H_1 H_2 \rangle$. So, by *generalized* TMD-factorization we mean that a TMD-factorization formula is recovered simply by replacing the Wilson lines in the definitions of the correlation functions by non-standard ones, which may be different for each hard subprocess and for each way of routing color through the hard part. The TMD PDFs needed in a generalized TMD-factorization formula for Eq. (1) could be totally different from the ones parameterized in, e.g. DIS and DY.

A conjectured TMD-factorization of the form of Eq. (3) is a basic assumption in a number of recent studies [11–16]. The minimal Wilson line structures needed for Eq. (3) can be determined by considering a single extra gluon at a time, radiated from each of the external hadrons and attaching everywhere in the hard subprocess. The resulting process-dependent gauge invariant correlation functions have been tabulated in Refs. [11, 12]. These correlation functions have also been used to calculate physical observables such as weighted spin asymmetries [13, 14].

Collins and Qiu [17] verified explicitly that *standard* TMD-factorization fails in a sample calculation of a single spin asymmetry (SSA). That is, they showed in an explicit calculation that the process-dependence of the Wilson line structures observed in Ref. [9] indeed corresponds to non-universality for the TMD PDFs. For their calculation, they used a model Abelian theory and calculated the effect of a single extra gluon. An explicit illustration of the violation of standard TMD-factorization was also given for unpolarized scattering in Ref. [18], again using a model Abelian gauge theory. The two-gluon example for unpolarized scattering is important as it directly illustrates that standard TMD-factorization cannot generally be recovered by rescaling the hard part with a constant color factor. (Compare this with the procedure of Refs. [19].) In Ref. [20], it was shown explicitly that the observed breakdown of standard TMD-factorization described in Refs. [17, 18] is consistent with the generalized TMD-factorization proposed in Refs. [11, 12, 15], again within the Abelian theory.

However, all the cases studied so far have only considered graphs with extra gluons radiated from *one* of the hadrons at a time. What is missing is a treatment of non-Abelian gluons radiated from different hadrons simultaneously. If a generalized TMD-factorization approach is possible, then extra gluons radiated from all hadrons simultaneously must be shown to eikonalize and factorize after a sum over graphs. Given the complex color structures that arise in a non-Abelian gauge theory, it is unclear that such a procedure is possible in real QCD.

The purpose of this paper is to show explicitly that even generalized TMD-factorization breaks down in a non-Abelian gauge theory at the level of two extra gluons. In other words, the violation of standard TMD-factorization, already found in previous work, cannot be dealt with simply by replacing the Wilson lines in the standard correlation functions by more complicated ones and summing over different subprocesses and color structures as in Eq. (3).

As seen in Ref. [17], the basic reasons for a breakdown of *standard* TMD-factorization are illustrated most directly in a calculation of an SSA with a single extra gluon. We will find analogously that the breakdown of *generalized* TMD-factorization in a non-Abelian gauge theory is most easily illustrated in a calculation of a double Sivers effect in a double transverse spin asymmetry (DSA). As in Refs. [17, 18] we will use a model field theory to describe the quarks, spectators and hadrons. A proper counterexample to generalized TMD-factorization must verify that terms which violate generalized TMD-factorization graph-by-graph do not cancel in a sum over graphs. This is most easily done in a simple spectator model that restricts the number of relevant Feynman graphs.

In Sect. II we discuss the particular model and describe the procedure for deriving a violation of generalized TMD-factorization. In Sect. III we review the steps for factorization with one extra gluon. We explicitly review the breakdown of standard TMD-factorization for two extra gluons from one hadron in Sect. IV. In Sect. V we discuss the generalized TMD-factorization formula that is required to recover a factorized structure. In Sect. VI we demonstrate that the generalized TMD-factorization formula is inconsistent with having extra gluons radiated from both hadrons simultaneously. We end with concluding remarks in Sect. VII.

II. SETUP

A simple model field theory provides a direct illustration of why factorization fails in a gauge theory, while avoiding the complications of dealing with a large number of Feynman graphs. We will continue to use the model field theory of Refs. [17, 18, 21], though with a few important differences. The hadrons continue to correspond to different flavors. The "quarks" continue to correspond to scalar fields ϕ_f, while the "hadron" fields H_f and the spectator "diquarks" ψ_f are Dirac spinors. The subscript $f = 1, 2$ labels flavor. The main difference from Refs. [17, 18, 21] which we will introduce is that the gauge field will be the massless $\mathrm{SU}(N_c)$ non-Abelian gauge field. (In QCD $N_c = 3$.) By contrast, Refs. [17, 18] used a massive Abelian gauge field that coupled with different charges, g_1 and g_2, to the quarks and diquarks in hadrons H_1 and H_2. In this paper, the non-Abelian