Note that we have set the constant of integration to zero so that the electric field at spatial infinity vanishes.

The charge within some interval (a,b) is given by Gauss' law

$$\begin{aligned} q &= F_{01}(t,x=b) - F_{01}(t,x=a) \\ &= g[\phi(t,x=b) - \phi(t,x=a)] \end{aligned} \tag{11}$$

Inserting (9) into the scalar field equation, (5), gives

$$\begin{aligned} (\partial^2 + m^2)\phi &= -m\bar{F} \\ &= -mQ(\Theta(x+L/2) - \Theta(x-L/2)) \end{aligned} \tag{12}$$

where the effective mass of the scalar field is given by the coupling constant,

$$m = g \tag{13}$$

Hence our problem reduces to solving the Klein-Gordon equation for a scalar field of mass m, sourced by the "electric field", $E_1 = F_{01} = Q$, within the capacitor. The initial condition at $t=0$ is given by the requirement that no fermions be present, or in terms of the bosonic variables,

$$\phi(t=0,x) = 0 = \dot{\phi}(t=0,x) \tag{14}$$

Before we evolve the equations, however, it is interesting to find static solutions into which the system can evolve asymptotically.

A. Static Solution

In the asymptotic future, $t \to \infty$, we expect the ϕ solution to be simply the static solution to (12). Since the Klein-Gordon equation (12) is linear, we may first solve it with $\bar{F} = (Q/2)(\Theta(x) - \Theta(-x)) = (Q/2)\mathrm{sgn}(x)$ *i.e.* due to a single point charge at the origin. The static solution to the present problem would then follow using appropriate linear superposition. For now,

$$(\partial^2 + m^2)\phi = -m\frac{Q}{2}\mathrm{sgn}(x) \tag{15}$$

Using the integral representation of the step-function and that of the retarded Green function, $G_{\mathrm{r}}(x-y)$,

$$\Theta(x) = \int \frac{dk}{2\pi i}\frac{e^{ikx}}{k - i0^+}, \tag{16}$$

$$G_{\mathrm{r}}(x-y) = \int \frac{d^2k}{(2\pi)^2}\frac{e^{ik\cdot(x-y)}}{-(k_0 - i0^+)^2 + (k_1)^2 + m^2} \tag{17}$$

provides us with the integral representation

$$\begin{aligned} \phi(x) &= -m\frac{Q}{2}\int d^2y\, G_{\mathrm{r}}(x-y)(\Theta(y) - \Theta(-y)) \\ &= -mQ\int \frac{dk}{2\pi}\frac{\sin(kx)}{k(k^2+m^2)} \end{aligned} \tag{18}$$

The integral in (18) may be evaluated by performing a partial fraction decomposition of the denominator $k(k^2+m^2)$ and converting the resulting three subintegrals into appropriate contour integrals, which may then be computed straightforwardly. The answer is

$$\phi_0(x) = -\frac{Q}{2m}\mathrm{sgn}(x)\left(1 - e^{-m|x|}\right) \tag{19}$$

The static solution to (12) is therefore

$$\begin{aligned} \phi_{\mathrm{s}}(x) &= -\phi_0(x-L/2) + \phi_0(x+L/2) \\ &= -\frac{Q}{m} \times \begin{cases} e^{-m|x|}\sinh(mL/2), & |x| > L/2 \\ 1 - e^{-mL/2}\cosh(mx), & |x| < L/2 \end{cases} \end{aligned} \tag{20}$$

B. Dynamical Solution

The solution to (12) we are seeking must satisfy the initial conditions in (14). To obtain this dynamical solution, we add a homogeneous solution, ϕ_{h}, obeying $(\partial^2 + m^2)\phi_{\mathrm{h}} = 0$, to the static solution ϕ_{s} such that the initial conditions are satisfied. Again, it helps to first solve the problem with a single charge. Then we have to solve (15) with the initial conditions corresponding to (14). From the conditions in (14), we observe that the integral representation of the solution is

$$\begin{aligned} \bar{\phi}(t,x) &= \phi_{\mathrm{s}}(x) + \phi_{\mathrm{h}}(t,x) \\ &= -mQ\int \frac{dk}{2\pi}\frac{\sin(kx)}{k(k^2+m^2)} \\ &\quad \times \left(1 - \cos\left(t\sqrt{k^2+m^2}\right)\right) \end{aligned} \tag{21}$$

The solution to (12) with the same initial conditions is thus

$$\begin{aligned} \phi(t,x) &= -\bar{\phi}(t,x-L/2) + \bar{\phi}(t,x+L/2) \\ &= -2mQ\int \frac{dk}{2\pi}\frac{\cos(kx)\sin(kL/2)}{k(k^2+m^2)} \\ &\quad \times \left(1 - \cos\left(t\sqrt{k^2+m^2}\right)\right) \end{aligned} \tag{22}$$

From this, we can extract the flux of energy passing through a given spatial point x and integrate over all time to get the total energy radiated. It is

$$\begin{aligned} \mathcal{F}(x) &= \int_0^\infty dt\, T^{01} = -\int_0^\infty dt\, \partial_0\phi\partial_1\phi \\ &= -(2mQ)^2\int \frac{dk}{2\pi}\int \frac{dp}{2\pi}\frac{\sin(kL/2)\cos(kx)}{k(k^2+m^2)} \\ &\quad \times \frac{\sin(pL/2)\sin(px)}{k^2-p^2} \end{aligned} \tag{23}$$

We first use

$$\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\frac{\exp(ikx)}{k^2-a^2} = -\frac{1}{2}\frac{\sin(a|x|)}{a}, \tag{24}$$