A. Interpolating fields

The first non-trivial question is to look for the interpolating QCD operator which triggers the $J^{PC} = 0^{++}$ isoscalar states from the vacuum. On the lattice this has been done in a variety of ways, mainly involving quark operators (such as $\bar{q}q$, see also [48]). A clear candidate, often used for the gluonium, is given by the trace of the energy-momentum tensor, $\Theta^{\mu\nu}$, which satisfies the trace anomaly equation [49],

$$\begin{aligned} \partial^\mu D_\mu &= \Theta^\mu_\mu \equiv \Theta \\ &= \frac{\beta(\alpha)}{2\alpha} G^{\mu\nu a} G^a_{\mu\nu} + \sum_q m_q \left[1 + \gamma_m(\alpha)\right] \bar{q}q. \end{aligned} \tag{10}$$

Here $\beta(\alpha) = \mu^2 d\alpha/d\mu^2$ denotes the beta function, α is the running coupling constant, $\gamma_m(\alpha) = d\log m/d\log\mu^2$ is the anomalous dimension of the current quark mass m, the symbol D_μ denotes the dilatation current, and $G^a_{\mu\nu}$ is the field strength tensor of the gluon field. The operator Θ, besides having the $J^{PC} = 0^{++}$ quantum numbers, is renorm-invariant. In addition, in the chiral limit of massless quarks, $m_q \to 0$, it is $SU_R(2) \otimes SU_L(2)$ chirally invariant. This is different from the usual $\bar{q}q$ interpolating field, which is not chirally invariant and mixes under the chiral transformations with the $\bar{q}i\gamma_5\vec{\tau}q$ operator. Actually, under the operation $q \to \gamma_5 q$, the operator Θ is chirally even while $\bar{q}q$ is chirally odd.

In perturbation theory one has

$$\begin{aligned} \beta(\alpha) &= -\alpha\left[\beta_0\left(\frac{\alpha}{4\pi}\right) + \beta_1\left(\frac{\alpha}{4\pi}\right)^2 + \dots\right], \\ \gamma_m(\alpha) &= \frac{\alpha}{4\pi} + \dots, \end{aligned} \tag{11}$$

where

$$\begin{aligned} \beta_0 &= \frac{11}{3}N_c - \frac{2}{3}N_f, \\ \beta_1 &= \frac{34}{3}N_c^2 - \frac{13}{3}N_f N_c + \frac{N_f}{N_c}, \end{aligned} \tag{12}$$

and N_f denotes the number of active flavors. We recall that in the large-N_c limit $\alpha \sim 1/N_c$ and $G^2 \sim (N_c^2 - 1)$, *i.e.*, is proportional to the number of gluons. Hence, in Eq. (10) there is a contribution scaling as N_c^2 and flavor-independent, as well as another contribution, subleading in N_c and scaling as $N_c N_f$ (the higher orders in, for instance, the $\overline{MS}$ scheme, generate the N_f^2 terms which are dependent on the renormalization scheme). In the chiral limit we only have the gluonic operator contribution to Eq. (10), however, still some information on the quark degrees of freedom remains via the β-function. Thus, in the chiral limit we may distinguish between the gluonic and quark contributions by the large-N_c and N_f scaling behavior,

$$\Theta_g \sim N_c^2 \qquad \Theta_q \sim N_c N_f. \tag{13}$$

B. Two-point correlations

For a scalar-isoscalar particle $|n\rangle$ we have a non-vanishing matrix element to the vacuum for the dilatation current,

$$\langle 0|D^\mu|n\rangle = iq^\mu f_n, \tag{14}$$

and hence

$$\langle 0|\partial^\mu D_\mu|n\rangle = \langle 0|\Theta|n\rangle = m_n^2 f_n \tag{15}$$

for the on-shell particles. From now on the mass of the scalar-isoscalar states is denoted as m_n. The two-point correlation function of the energy-momentum tensor reads

$$\Pi^{\mu\nu;\alpha\beta}_{\Theta\Theta}(q) = i\int d^4x e^{iq\cdot x}\langle 0|T\left\{\Theta^{\mu\nu}(x)\Theta^{\alpha\beta}(0)\right\}|0\rangle. \tag{16}$$

The energy momentum tensor is conserved, therefore

$$\langle 0|\Theta^{\mu\nu}|n\rangle = \frac{1}{3}f_n(g^{\mu\nu}q^2 - q^\mu q^\nu), \tag{17}$$

where the factor of 1/3 complies with Eq. (15). After inserting a complete set of states we find

$$\Pi^{\mu\nu;\alpha\beta}_{\Theta\Theta}(q) = \frac{1}{9}(g^{\mu\nu}q^2 - q^\mu q^\nu)(g^{\alpha\beta}q^2 - q^\alpha q^\beta)\Pi_{\Theta\Theta}(q), \tag{18}$$

with

$$\begin{aligned} \Pi_{\Theta\Theta}(q) &= i\int d^4x e^{iq\cdot x}\langle 0|T\left\{\Theta(x)\Theta(0)\right\}|0\rangle \\ &= \sum_n \frac{f_n^2 q^4}{m_n^2 - q^2} + \text{c.t.}, \end{aligned} \tag{19}$$

where c.t. stands for the counterterms. According to the N_c counting rules, the leading part of $\Pi_{\Theta\Theta}$ scales as N_c^2, and the next-to-leading part as $N_c N_f$.

The sum in Eq. (19) runs over *all* $J^{PC} = 0^{++}$ isoscalar states, *i.e.* both mesons (m) and glueballs (g). The difference is, however, in the scaling of the coupling constants f_n with N_c. If all states are contributing and no cancellations occur, then the glueball must have a larger coupling, $f_n^2 \sim N_c^2$, whereas the $\bar{q}q$-meson has $f_n^2 \sim N_c N_f$:

$$\begin{aligned} f_n &\sim N_c \qquad \text{(glueball)}, \\ f_n &\sim \sqrt{N_c} \qquad \text{(meson)}. \end{aligned} \tag{20}$$

The Operator Product Expansion (OPE) in QCD yields [50–54]

$$\Pi(q^2) = q^4\left[C_0 \log q^2 + \sum_n \frac{C_{2n}}{q^{2n}}\right], \tag{21}$$