growing term being $\sim t^{\rho}$ where ρ is the (maximal) difference between the algebraic and geometric multiplicity of defective eigenvalues. In our case, $\rho = 1$ because the maximal algebraic multiplicity of any eigenvalue of $\mathcal{H}(q)$ is 2; moreover, since $\mathcal{H}(q)$ may have defective eigenvalues only for $q = 0$ or $q = -\pi/a$, secular growing terms may appear solely when the wave number k of the exciting plane wave is an integer multiple of $k_B/2$. The absence of secular growing terms in $c_l(t)$ for any excitation wave number $k = nk_B/2$ ($n = 0, \pm 1, \pm 2, ...$) implies that the matrix $\mathcal{J}$ is diagonal, which ensures the lack of defective eigenvalues of $\mathcal{H}$ and thus of spectral singularities of H according to Theorem 1.

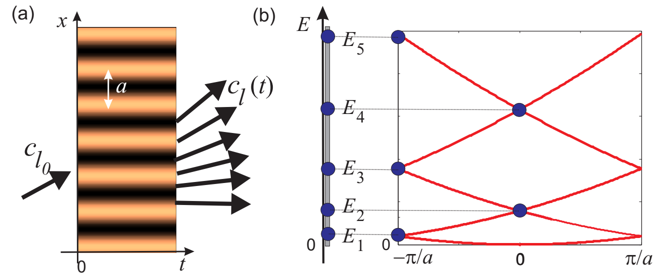

FIG. 1: (color online) (a) Schematic of Bragg scattering of a plane wave off a complex crystal. (b) Band structure of the the complex crystal $V(x) = V_0 \exp(ik_Bx)$. The circles mark the spectral singularities inside the continuous spectrum.

III. BRAGG SCATTERING IN $\mathcal{PT}$ COMPLEX CRYSTALS AND WAVE PACKET DYNAMICS

Let us specialize the previous results to the case of a $\mathcal{PT}$-symmetric lattice, for which $V(-x) = V^*(x)$. Let $V_R(x)$ and $\lambda V_I(x)$ be the real and imaginary parts of the potential, respectively, where $\lambda \geq 0$ measures the anti-Hermitian strength of H. The spectrum of H is real for $\lambda \leq \lambda_c$, where $\lambda_c \geq 0$ defines the symmetry breaking point. According to the previous analysis, complex-conjugate pairs of eigenvalues for $\mathcal{H}(q)$ should appear as λ is increased from below to above λ_c. The typical scenario that describes symmetry breaking in a finite-dimensional $\mathcal{PT}$ matrix is the appearance of an exceptional point via the merging of two real eigenvalues into a single real and defective eigenvalue at $\lambda = \lambda_c$ (see, for instance, [7, 29]). We note that the existence of such a branching point for a certain class of non-hermitian matrices as a control parameter is varied was proven in a rather generally way in Ref.[30] (see also [7]). We may thus conjecture that for a $\mathcal{PT}$ complex crystal symmetry-breaking is accompanied by the appearance of spectral singularity, which arise from defective eigenvalues of $\mathcal{H}$ at $q = 0$ or $q = -\pi/a$. This scenario is in agreement with numerical or analytical results obtained from band computation of specific complex periodic potentials (see, for instance, [6, 8]).

As an example, let us consider the $\mathcal{PT}$ crystal defined by

$$V_R(x) = V_0 \cos(2\pi x/a) \ , \ \ V_I(x) = V_0 \sin(2\pi x/a), \quad (5)$$

which has been recently considered to highlight unusual diffraction and transport properties of complex optical lattices [6, 25]. In this case, $\lambda_c = 1$ [6] and at the symmetry breaking point one has $V(x) = V_0 \exp(ik_Bx)$, a potential which is amenable for an analytical study [20, 22]. For this potential, $\mathcal{H}$ has a block diagonal form, namely $\mathcal{H}_{n,m} = (n + k_Bq)^2\delta_{n,m} + V_0\delta_{m,n+1}$, and its eigenvalues are simply the elements on the main diagonal, i.e. $E_\alpha(q) = (q + \alpha k_B)^2$ ($\alpha = 0, \pm 1, \pm 2, ...$). This means that, as previously noticed [6, 22, 25], the band structure of the $\mathcal{PT}$ potential $V = V_0 \exp(ik_Bx)$ coincides with the free-particle energy dispersion curve $E = k^2$, periodically folded inside the first Brillouin zone [see Fig.1(b)]. The eigenvalues are distinct for $q \neq 0, -\pi/a$. At the crossings of the folded parabolas of Fig.1(b), i.e. at $q = 0$ and at $q = -\pi/a$, one has $E_{-\alpha}(q) = E_\alpha(q)$ and $E_{1-\alpha}(q) = E_\alpha(q)$, respectively, i.e. the eigenvalues coalesce in pairs and become defective. Therefore, the continuous spectrum of H, $E \geq 0$, contains a sequence of spectral singularities at $E_n = (nk_B/2)^2$, $n = 1, 2, 3, ...$ [see Fig.1(b)], which spoil the completeness of the Bloch-Floquet eigenfunctions. The defective nature of degenerate eigenvalues at $q = 0$ and $q = -\pi/a$ can be readily proven by direct calculation of the eigenvectors $\mathbf{w}^{(\alpha)}$ of $\mathcal{H}$. Note that, as wave scattering from complex potential *barriers* enables a finite number of spectral singularities in the continuous spectrum [12, 17], in our example the number of spectral singularities is countable but infinite. According to Theorem 3, a secular growth of Bragg diffraction pattern for a plane wave that excites the crystal at normal incidence (or tilted by an angle which is an integer multiple of the Bragg angle) provides a distinctive signature of the appearance of spectral singularities at the $\mathcal{PT}$ symmetry-breaking transition point $\lambda = \lambda_c = 1$. However, in any experimental setting aimed to observe such a secular growth, the wave that excites the crystal is always spatially limited or truncated, and it is thus of major relevance to investigate the impact of spectral singularities on the evolution of a wave packet with a broadened angular spectrum, an issue which was not considered in previous works by Berry. Here we investigate the Bragg diffraction of a wave packet with a broadened momentum distribution, $\psi(x, 0) = \int dk F(k) \exp(ikx)$, and show that spectral broadening leads to a *saturation* of the secular growth of scattered waves. Such a saturation behavior is basically due to the fact that spectral singularities are of measure zero (they are a countable set of points embedded in the continuous energy spectrum $E \geq 0$). For the sake of simplicity, we consider a shallow lattice and a wave packet with a narrow spectrum $F(k)$ centered at $k = -k_B/2$ of width $\Delta k \ll k_B$. Following the same lines detailed in the proof of Theorem 3, one can show that the diffraction pattern $\psi(x, t)$ can be written as the interference of different wave packets describing