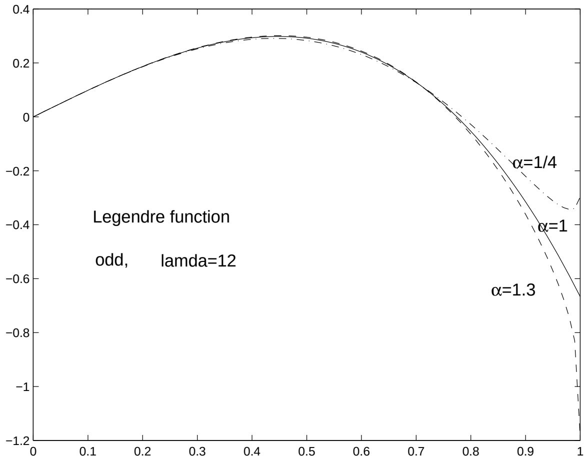

FIG. 3: Legendre odd function. Notice the direct of the curves.

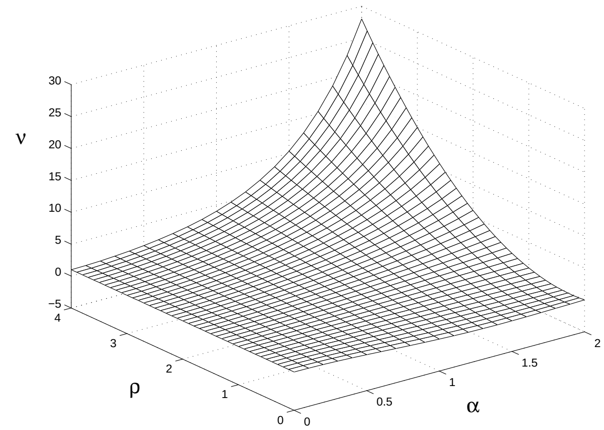

FIG. 4: Parameters of fractional Bessel function. ν as a function of α and ρ.

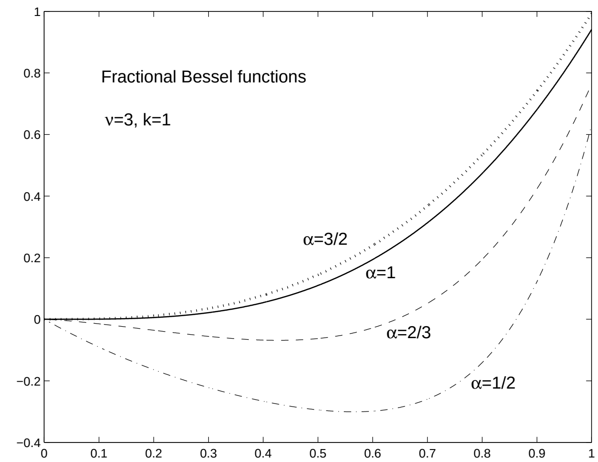

FIG. 5: Fractional Bessel function. The fractional index α shows up in the exponentials of the fractional Bessel series; a small difference from 1 changes the profile largely. For a big α, ρ is small, so the change is suppressed.

Here $\triangle_2$ is the two dimensional Laplacian operator. Using the fractional cylindrical Laplacian operator defined above, this equation becomes

$$\frac{\partial u}{\partial t} = \frac{1}{r^\alpha} D_r^\alpha (r^\alpha D_r^\alpha\ u) + \frac{\Gamma^2(\alpha+1)}{r^{2\alpha}} \frac{\partial^2}{\partial \theta^2} u. \tag{26}$$

By separation of variables $u(t,r,\theta) = R(r)\Theta(\theta)T(t)$, it can be decomposed to

$$T' + a^2 k^2 T = 0, \tag{27}$$

$$\frac{\partial^2 \Theta}{\partial \theta^2} + \nu^2 \Theta = 0, \tag{28}$$

$$\frac{1}{\Gamma^2(\alpha+1)} r^\alpha D_r^\alpha (r^\alpha D_r^\alpha\ R) + k^2 \frac{r^{2\alpha}}{\Gamma^2(\alpha+1)} R - \nu^2 R = 0. \tag{29}$$

The first two equations are simple. The third equation is a fractional generalization of the Bessel equation [11, 12]. It can be solved by fractional series expansion. Since Bessel equation is singular at $r = 0$. We must use such ansatz: $R = r^{\alpha\rho} \sum_{m=0}^{\infty} c_m r^{\alpha m}$. Substitute it into the above equation, we get

$$c_0 \left[\left(\frac{\Gamma(\alpha\rho+1)}{\Gamma(\alpha\rho-\alpha+1)} \right)^2 - \nu^2 \Gamma^2(\alpha+1) \right] = 0, \tag{30}$$

$$c_1 \left[\left(\frac{\Gamma(\alpha\rho+\alpha+1)}{\Gamma(\alpha\rho+1)} \right)^2 - \nu^2 \Gamma^2(\alpha+1) \right] = 0, \tag{31}$$

$$c_m \left[\left(\frac{\Gamma(\alpha\rho+\alpha m+1)}{\Gamma(\alpha\rho+\alpha m-\alpha+1)} \right)^2 - \nu^2 \Gamma^2(\alpha+1) \right] + c_{m-2} k^2 = 0. \tag{32}$$

To have a starting term, $c_0 \neq 0$, so

$$\left[\left(\frac{\Gamma(\alpha\rho+1)}{\Gamma(\alpha\rho-\alpha+1)} \right)^2 - \nu^2 \Gamma^2(\alpha+1) \right] = 0, \tag{33}$$

and $c_1 = 0$.

By the recursive relation, a solution is implied

$$R_\rho(r) = r^{\alpha\rho} \sum_{n=0}^{\infty} (-1)^n d_n k^{2n} r^{\alpha \cdot 2n}, \tag{34}$$

where $d_0 = 1$,

$$d_n = d_{n-1} \frac{1}{\left[\left(\frac{\Gamma(\alpha\rho+\alpha\cdot 2n+1)}{\Gamma(\alpha\rho+\alpha\cdot 2n-\alpha+1)} \right)^2 - \nu^2 \Gamma^2(\alpha+1) \right]}, \tag{35}$$

and ρ satisfies Eq.(33).

Eq.(33) in one-order calculus ($\alpha = 1$) is simply $\rho = \pm\nu$. We meshed in Fig. 4 the surface defined by the equation (33). After solving the equation with $\nu = 3$ for ρ when α varies, we plotted in Fig. 5 $R_\rho(r)$ belonging to different values of α.