Appendix: Complete Elliptic integrals and the Nome q expansion

In this appendix we present the definition and some properties of elliptic integrals which are needed for the derivation of our result. The complete elliptic integrals of the first (**K**) and the second (**E**) kind are defined as

$$\mathbf{K}(m) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - m\sin^2\phi}}, \tag{A.1}$$

$$\mathbf{E}(m) = \int_0^{\pi/2} \sqrt{1 - m\sin^2\phi}\, d\phi. \tag{A.2}$$

For our purpose it is useful to note that

$$\int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2\theta_0 - \sin^2\theta}} = \mathbf{K}(\sin^2\theta_0), \tag{A.3}$$

$$\int_0^{\theta_0} \sqrt{\sin^2\theta_0 - \sin^2\theta}\, d\theta = \mathbf{E}(\sin^2\theta_0) - (1 - \sin^2\theta_0)\mathbf{K}(\sin^2\theta_0). \tag{A.4}$$

In order to study the elliptic integrals near the logarithmic singularity, it is convenient to use the q-series, defined as

$$q \equiv \exp[-\pi\mathbf{K}(1-m)/\mathbf{K}(m)] \tag{A.5}$$

$$= \frac{m}{16} + 8\left(\frac{m}{16}\right)^2 + 84\left(\frac{m}{16}\right)^3 + \cdots . \tag{A.6}$$

Inverting, one obtains

$$m = 16(q - 8q^2 + 44q^3 - 192q^4 + \cdots). \tag{A.7}$$

Now that we have $m(q)$ and $q(m)$ as given above, we have the following alternative expansions.

$$\mathbf{K}(m) = \frac{\pi}{2}(1 + 4q + 4q^2 + 4q^4 + \cdots), \tag{A.8}$$

$$\mathbf{E}(m) = \frac{\pi}{2}(1 - 4q + 20q^2 - 64q^3 + \cdots). \tag{A.9}$$

And more importantly,

$$\mathbf{K}(1-m) = -\frac{\ln q}{2}(1 + 4q + 4q^2 + 4q^4 + \cdots), \tag{A.10}$$

$$\mathbf{E}(1-m) = (1 - 4q + 12q^2 - 32q^3 + \cdots) - 4q\ln q\,(1 - 2q + 8q^2 + \cdots). \tag{A.11}$$