$$
\begin{aligned}
&D_iD_j\left(\overset{(2)}{\Psi}-\overset{(2)}{\Phi}\right)+\left\{\left(-\Delta+2\partial_\eta^2+4\mathcal{H}\partial_\eta-2K\right)\overset{(2)}{\Psi}+\left(2\mathcal{H}\partial_\eta+2\partial_\eta\mathcal{H}+4\mathcal{H}^2+\Delta+2K\right)\overset{(2)}{\Phi}\right\}\gamma_{ij}\\
&-\frac{1}{a^2}\partial_\eta\left(a^2D_{(i}\overset{(2)}{\nu}_{j)}\right)+\frac{1}{2}\left(\partial_\eta^2+2\mathcal{H}\partial_\eta+2K-\Delta\right)\overset{(2)}{\chi}_{ij}-8\pi G\left(\partial_\eta\varphi\partial_\eta\varphi_2-a^2\varphi_2\frac{\partial V}{\partial\varphi}(\varphi)\right)\gamma_{ij}=\Gamma_{ij}, \qquad (6.22)
\end{aligned}
$$

where Γ_0, Γ_i Γ_{ij} are the collection of the quadratic term of the first-order perturbations as follows:

$$
\begin{aligned}
\Gamma_0 &:= 4\pi G\left((\partial_\eta\varphi_1)^2+D_i\varphi_1D^i\varphi_1+a^2(\varphi_1)^2\frac{\partial^2V}{\partial\varphi^2}\right)-4\partial_\eta\mathcal{H}\left(\overset{(1)}{\Phi}\right)^2-2\overset{(1)}{\Phi}\partial_\eta^2\overset{(1)}{\Phi}-3D_k\overset{(1)}{\Phi}D^k\overset{(1)}{\Phi}-10\overset{(1)}{\Phi}\Delta\overset{(1)}{\Phi}\\
&\quad-3\left(\partial_\eta\overset{(1)}{\Phi}\right)^2-16K\left(\overset{(1)}{\Phi}\right)^2-8\mathcal{H}^2\left(\overset{(1)}{\Phi}\right)^2+D_lD_k\overset{(1)}{\Phi}\overset{(1)}{\chi}{}^{lk}+\frac{1}{8}\partial_\eta\overset{(1)}{\chi}_{lk}\partial_\eta\overset{(1)}{\chi}{}^{kl}+\mathcal{H}\overset{(1)}{\chi}_{kl}\partial_\eta\overset{(1)}{\chi}{}^{lk}\\
&\quad-\frac{3}{8}D_k\overset{(1)}{\chi}_{lm}D^k\overset{(1)}{\chi}{}^{ml}+\frac{1}{4}D_k\overset{(1)}{\chi}_{lm}D^l\overset{(1)}{\chi}{}^{mk}-\frac{1}{2}\overset{(1)}{\chi}{}^{lm}\Delta\overset{(1)}{\chi}_{lm}+\frac{1}{2}K\overset{(1)}{\chi}_{lm}\overset{(1)}{\chi}{}^{lm}; \qquad (6.23)\\
\Gamma_i &:= 16\pi G\partial_\eta\varphi_1D_i\varphi_1-4\partial_\eta\overset{(1)}{\Phi}D_i\overset{(1)}{\Phi}+8\mathcal{H}\overset{(1)}{\Phi}D_i\overset{(1)}{\Phi}-8\overset{(1)}{\Phi}\partial_\eta D_i\overset{(1)}{\Phi}+2D^j\overset{(1)}{\Phi}\partial_\eta\overset{(1)}{\chi}_{ji}-2\partial_\eta D^j\overset{(1)}{\Phi}\overset{(1)}{\chi}_{ij}\\
&\quad-\frac{1}{2}\partial_\eta\overset{(1)}{\chi}_{jk}D_i\overset{(1)}{\chi}{}^{kj}-\overset{(1)}{\chi}_{kl}\partial_\eta D_i\overset{(1)}{\chi}{}^{lk}+\overset{(1)}{\chi}{}^{kl}\partial_\eta D_k\overset{(1)}{\chi}_{il}; \qquad (6.24)\\
\Gamma_{ij} &:= 16\pi GD_i\varphi_1D_j\varphi_1+8\pi G\left\{(\partial_\eta\varphi_1)^2-D_l\varphi_1D^l\varphi_1-a^2(\varphi_1)^2\frac{\partial^2V}{\partial\varphi^2}\right\}\gamma_{ij}-4D_i\overset{(1)}{\Phi}D_j\overset{(1)}{\Phi}-8\overset{(1)}{\Phi}D_iD_j\overset{(1)}{\Phi}\\
&\quad+\left(6D_k\overset{(1)}{\Phi}D^k\overset{(1)}{\Phi}+4\overset{(1)}{\Phi}\Delta\overset{(1)}{\Phi}+2\left(\partial_\eta\overset{(1)}{\Phi}\right)^2+8\partial_\eta\mathcal{H}\left(\overset{(1)}{\Phi}\right)^2+16\mathcal{H}^2\left(\overset{(1)}{\Phi}\right)^2+16\mathcal{H}\overset{(1)}{\Phi}\partial_\eta\overset{(1)}{\Phi}-4\overset{(1)}{\Phi}\partial_\eta^2\overset{(1)}{\Phi}\right)\gamma_{ij}\\
&\quad-4\mathcal{H}\partial_\eta\overset{(1)}{\Phi}\overset{(1)}{\chi}_{ij}-2\partial_\eta^2\overset{(1)}{\Phi}\overset{(1)}{\chi}_{ij}-4D^k\overset{(1)}{\Phi}D_{(i}\overset{(1)}{\chi}_{j)k}+4D^k\overset{(1)}{\Phi}D_k\overset{(1)}{\chi}_{ij}-8K\overset{(1)}{\Phi}\overset{(1)}{\chi}_{ij}+4\overset{(1)}{\Phi}\Delta\overset{(1)}{\chi}_{ij}-4D^kD_{(i}\overset{(1)}{\Phi}\overset{(1)}{\chi}_{j)k}\\
&\quad+2\Delta\overset{(1)}{\Phi}\overset{(1)}{\chi}_{ij}+2D_lD_k\overset{(1)}{\Phi}\overset{(1)}{\chi}{}^{lk}\gamma_{ij}+\partial_\eta\overset{(1)}{\chi}_{ik}\partial_\eta\overset{(1)}{\chi}_j{}^k-D^k\overset{(1)}{\chi}_{il}D_k\overset{(1)}{\chi}_j{}^l+D^k\overset{(1)}{\chi}_{il}D^l\overset{(1)}{\chi}_{jk}-\frac{1}{2}D_i\overset{(1)}{\chi}{}^{lk}D_j\overset{(1)}{\chi}_{lk}\\
&\quad-\overset{(1)}{\chi}_{lm}D_iD_j\overset{(1)}{\chi}{}^{ml}+2\overset{(1)}{\chi}{}^{lm}D_lD_{(i}\overset{(1)}{\chi}_{j)m}-\overset{(1)}{\chi}{}^{lm}D_mD_l\overset{(1)}{\chi}_{ij}\\
&\quad-\frac{1}{4}\left(3\partial_\eta\overset{(1)}{\chi}_{lk}\partial_\eta\overset{(1)}{\chi}{}^{kl}-3D_k\overset{(1)}{\chi}_{lm}D^k\overset{(1)}{\chi}{}^{ml}+2D_k\overset{(1)}{\chi}_{lm}D^l\overset{(1)}{\chi}{}^{mk}-4K\overset{(1)}{\chi}_{lm}\overset{(1)}{\chi}{}^{lm}\right)\gamma_{ij}. \qquad (6.25)
\end{aligned}
$$

Here, we used Eqs. (4.8), (5.12), (5.14), (5.16) and (5.18).

The tensor part of Eq. (6.22) is given by

$$
\begin{aligned}
\left(\partial_\eta^2+2\mathcal{H}\partial_\eta+2K-\Delta\right)\overset{(2)}{\chi}_{ij} &= 2\Gamma_{ij}-\frac{2}{3}\gamma_{ij}\Gamma_k{}^k-3\left(D_iD_j-\frac{1}{3}\gamma_{ij}\Delta\right)(\Delta+3K)^{-1}\left(\Delta^{-1}D^kD_l\Gamma_k{}^l-\frac{1}{3}\Gamma_k{}^k\right)\\
&\quad+4\left\{D_{(i}(\Delta+2K)^{-1}D_{j)}\Delta^{-1}D^lD_k\Gamma_l{}^k-D_{(i}(\Delta+2K)^{-1}D^k\Gamma_{j)k}\right\}. \qquad (6.26)
\end{aligned}
$$

This tensor mode is also called the second-order gravitational waves.

Further, the vector part of Eqs. (6.21) and (6.22) yields the initial value constraint and the evolution equation of the vector mode $\overset{(2)}{\nu}_j$:

$$
\overset{(2)}{\nu}_i=\frac{2}{\Delta+2K}\left\{D_i\Delta^{-1}D^k\Gamma_k-\Gamma_i\right\}, \quad \partial_\eta\left(a^2\overset{(2)}{\nu}_i\right)=\frac{2a^2}{\Delta+2K}\left\{D_i\Delta^{-1}D^kD_l\Gamma_k{}^l-D_k\Gamma_i{}^k\right\}. \qquad (6.27)
$$