What we will do is to solve a problem more relevant to finding the zeroth order contribution than bremsstrahlung radiation: beta decay radiation (pair production).

F.4 Pair Production

F.4.1 Classical

We now take as our current a (charge) pair production

$$j^\mu(x) = e\int d\tau \left[\frac{dy_1^\mu(\tau)}{d\tau}\delta^{(4)}\big((x-y_1(\tau)\big) - \frac{dy_2^\mu(\tau)}{d\tau}\delta^{(4)}\big((x-y_2(\tau)\big)\right], \quad \text{(F.100)}$$

where we have

$$y_{1,2}^\mu(\tau) = \begin{cases} 0, & \tau < 0 \\ \frac{p_{1,2}^\mu}{m_{1,2}}\tau, & \tau > 0 \end{cases}; \quad \text{(F.101)}$$

we have allowed ourselves the freedom to have both difference masses and different momenta for the two (opposite) charges.

The Fourier transform of our current is

$$\begin{aligned} \tilde{j}^\mu(k) &= \int d^4x e^{ik\cdot x} j^\mu(x) \\ &= e\lim_{\epsilon\to 0^+}\int_0^\infty d\tau \left[\frac{p_1^\mu}{m_1}e^{\tau\left(-\epsilon+i\frac{k\cdot p_1}{m_1}\right)} - \frac{p_2^\mu}{m_2}e^{\tau\left(-\epsilon+i\frac{k\cdot p_2}{m_2}\right)}\right] \\ &= ie\left[\frac{p_1}{k\cdot p_1+i\epsilon} - \frac{p_2}{k\cdot p_2+i\epsilon}\right]. \end{aligned} \quad \text{(F.102)}$$

The only difference classically from the bremsstrahlung result is that the