1.5.1 Twisted involutions

Recall the set $\mathcal{I}$ of twisted involutions and the Richardson-Springer map ϕ defined in Subsection 1.3. These play an important role in the combinatorial description of $K\backslash G/B$ in some of the cases we consider — namely, the three non-equal rank cases in type A, these being $K = SO(2n+1, \mathbb{C})$, $K = SO(2n, \mathbb{C})$, and $K = Sp(2n, \mathbb{C})$. (When K is the special orthogonal group, the analysis differs depending on whether the rank of G is even or odd, so we treat these as separate cases.)

At least in these cases, the weak ordering on $K\backslash G/B$ can be deduced combinatorially from an analogous "weak Bruhat ordering" on $\mathcal{I}$. To describe this, we must make a few more definitions. First, define the "twisted action" of (the group) W on (the set) W by

$$a * w = aw\theta(a)^{-1}.$$

One checks easily that $\mathcal{I}$ is stable under the twisted action, whereby we have a W-action on $\mathcal{I}$.

Next, we define a monoid $M = M(W)$ associated to the Weyl group W. As a set, the elements of M are symbols $m(w)$, one for each $w \in W$. The multiplication on M is defined as follows: Given $w \in W$ and $s \in S$ a simple reflection,

$$m(s)m(w) = \begin{cases} m(sw) & \text{if } l(sw) > l(w), \\ m(w) & \text{otherwise.} \end{cases}$$