effective action for the neutron that appears in Eq. (4). The Minkowski space electric field we denote by $\vec{E}_M$. Without subtracting the magnetic moment contribution, the electric dipole operator picks up an additional contribution

$$\vec{p}_E = \frac{\partial H}{\partial \vec{E}_M} = -4\pi\alpha_E\,\vec{E}_M - \frac{\mu}{2M}\vec{K}, \tag{A1}$$

that does not vanish when the electric field is turned off. This extra contribution is a motion-induced effect as can be seen from neutron matrix elements. For a neutron moving non-relativistically,

$$\langle \vec{p}_E \rangle \equiv \langle N(\vec{v})|\,\vec{p}_E\,|N(\vec{v})\rangle = -4\pi\alpha_E\,\vec{E}_M + \frac{\mu\langle\vec{\sigma}\rangle}{2M} \times \vec{v} + \ldots, \tag{A2}$$

we see that the additional term corresponds to a motion-induced dipole moment [43]. In the external field, the electric dipole moment contributes to the total energy in the form

$$\langle E\,\rangle = \vec{E}_M \cdot \int_0^{E_M} \langle \vec{p}_{E'}\rangle\, dE'_M = -\frac{1}{2}4\pi\alpha_E\vec{E}_M^{\,2} + \frac{\mu\langle\vec{\sigma}\rangle}{2M}\cdot\left(\vec{v}\times\vec{E}_M\right). \tag{A3}$$

The second term is readily identified as the interaction energy of the magnetic moment with magnetic field seen in the neutron's rest frame: $\langle\vec{m}\rangle\cdot\vec{B}$, with $\vec{m} = \frac{\mu\vec{\sigma}}{2M}$, and $\vec{B} = \vec{v}\times\vec{E}_M$. This explains why one sees a motion-induced electric dipole moment in the frame in which neutron moves with velocity $\vec{v}$.

Without neutron motion, contributions from the motion-induced electric dipole moment naïvely vanish. At second order, however, the motion-induced dipole can interact with itself via neutron propagation. We must employ a limiting procedure to handle the nucleon pole. In a quantum mechanical notation, the shift due to an intermediate-state neutron with energy $k_0 = M + \frac{1}{2}M\vec{v}^{\,2}$ has the form

$$\Delta E = \left\langle N(\vec{0})\middle|\frac{\mu}{2M}\vec{K}\cdot\vec{E}_M\middle|N(\vec{v})\right\rangle\frac{1}{M-k_0}\left\langle N(\vec{v})\middle|\frac{\mu}{2M}\vec{K}\cdot\vec{E}_M\middle|N(\vec{0})\right\rangle. \tag{A4}$$

The off-diagonal matrix elements in the non-relativistic limit evaluate to half the value of the diagonal matrix elements in that limit. We can thus write the energy shift as

$$\Delta E = \frac{1}{4}\vec{m}\cdot\left(\vec{v}\times\vec{E}_M\right)\frac{1}{0-\frac{1}{2}M\vec{v}^{\,2}}\,\vec{m}\cdot\left(\vec{v}\times\vec{E}_M\right). \tag{A5}$$

In the limit of zero velocity, a non-vanishing contribution from the motion-induced electric dipole moment emerges. This contribution is the same as that derived in Eq. (6).

Appendix B: Analysis of Unpolarized Neutron Correlation Functions

Here we present the analysis of unpolarized neutron correlation functions. On each configuration, for each value of the electric field strength, we form the unpolarized, source-averaged lattice correlation function, $\overline{g}(x_4, n)_i$. Parity invariance is enforced by taking the geometric mean of correlators obtained for a given field value and its negative. Specifically we form

$$\overline{\mathfrak{g}}(x_4, n)_i = \sqrt{\overline{g}(x_4, n)_i\,\overline{g}(x_4, -n)_i}, \tag{B1}$$