given operation $\mathcal{E}$ occupies a central place in the thermodynamics of quantum information processing. This article mainly contains results on this problem. A complete characterization could be provided only for complete erasures and extreme operations. The characterization of HTOs of generic non-extremal quantum operations is still an open problem.

The set of HTOs associated with a given quantum operation might be considered as a means of describing the disturbance caused by the operation on the environment. This description is provided through a set of operators that has a clear operational interpretation. This points out the main theoretical interest for this problem. In addition to this, both the results obtained and the techniques used in this article can be useful in the implementation of non-unitary operations in future quantum computers. For example, the minimization of the average heat emission during an erasure operation requires a careful construction of the transformation on the extended system of the device and the bath. The constructions used in this article will be useful for such design problems.

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**Appendix A: Adjusting Bath Levels in the Proof of Theorem 3**

It is convenient to think of the subsystem Hamiltonians to be dependent on a continuous parameter $\nu$. For this purpose, consider $d$ continuous functions $\epsilon_i(\nu)$ $(i = 0, 1, \cdots, d-1)$ of a real parameter $\nu \in [0, 1]$, which will be used for energy eigenvalues of subsystems. They are chosen to satisfy the following properties:

(i) The zeroth level has zero energy, $\epsilon_0(\nu) = 0$, for all $\nu$.

(ii) The energies of all excited levels diverge to $+\infty$ as $\nu$ approaches to 1 (i.e., $\lim_{\nu\to 1} \epsilon_i(\nu) = \infty$ for any $i \geq 1$.)

(iii) The energy spectrum at $\nu = 0$ is identical with the eigenvalues of $Q$ up to a constant shift as $\epsilon_i(0) = q_i - q_0$ for all $i$.

Let $\{\nu_k\}$ be an infinite, increasing sequence starting from 0 and converging to 1, i.e., $\nu_1 = 0 \leq \nu_2 \leq \nu_3 \leq \cdots$ and $\lim_{k\to\infty} \nu_k = 1$. The energy levels of $\mathrm{X}_k$ will be taken as $\epsilon_i(\nu_k)$.