Hence, $\lim_{n\to\infty} \rho(\psi_{t_n}, \psi) = 0.$

Next, we will show the existence of a positive number $\overline{\epsilon}_0$ such that $\rho(\psi_{t_n+\tau_n}, \psi_{\tau_n}) \geq \overline{\epsilon}_0$ for each $n \in \mathbb{N}$. Denote $\beta = \min\{1, L_1\}$. For each $k \in \mathbb{N}$, we have that

$$\begin{aligned}
\rho_k(\psi_{t_n+\tau_n}, \psi_{\tau_n}) &= \min\left\{1, \sup_{s\in[-k,k]} \|h(\phi(t_n+\tau_n+s)) - h(\phi(\tau_n+s))\| \right\} \\
&\geq \min\left\{1, L_1 \sup_{s\in[-k,k]} \|\phi(t_n+\tau_n+s) - \phi(\tau_n+s)\| \right\} \\
&\geq \beta\rho_k(\phi_{t_n+\tau_n}, \phi_{\tau_n}).
\end{aligned}$$

Thus, the inequality

$$\rho(\psi_{t_n+\tau_n}, \psi_{\tau_n}) \geq \beta\rho(\phi_{t_n+\tau_n}, \phi_{\tau_n}) \geq \overline{\epsilon}_0$$

holds for each $n \in \mathbb{N}$, where $\overline{\epsilon}_0 = \beta\epsilon_0$. Consequently, the function $\psi(t)$ is unpredictable. □

A corollary of Theorem 5.2 is as follows.

Corollary 5.1 *If $\phi : \mathbb{R} \to \mathscr{H}$ is an unpredictable function, where $\mathscr{H}$ is a bounded subset of $\mathbb{R}^p$, then the function $\psi : \mathbb{R} \to \mathbb{R}^p$ defined as $\psi(t) = P\phi(t)$, where P is a constant, nonsingular, $p \times p$ matrix, is also an unpredictable function.*

Proof. The function $h : \mathscr{H} \to \mathbb{R}^p$ defined as $h(u) = Pu$ satisfies the inequality

$$L_1 \|u_1 - u_2\| \leq \|h(u_1) - h(u_2)\| \leq L_2 \|u_1 - u_2\|,$$

for $u_1, u_2 \in \mathscr{H}$ with $L_1 = 1/\left\|P^{-1}\right\|$ and $L_2 = \|P\|$. Therefore, by Theorem 5.2, the function $\psi(t)$ is unpredictable. □

In the next section, the existence of Poincaré chaos in the dynamics of differential equations will be presented.

6 Unpredictable solutions of differential equations

Consider the differential equation

$$x'(t) = -\frac{3}{2}x(t) + \nu(t), \tag{6.9}$$

where the function $\nu(t)$ is defined as

$$\nu(t) = \begin{cases} 0.7, & \text{if} \quad \zeta_{2j} < t \leq \zeta_{2j+1}, \ j \in \mathbb{Z}, \\ -0.4, & \text{if} \quad \zeta_{2j-1} < t \leq \zeta_{2j}, \ j \in \mathbb{Z}. \end{cases} \tag{6.10}$$