entropy S than the homogeneous phase (see Fig. 15) and it is unstable. For $\Lambda > \Lambda_c^*$, it has a higher entropy than the homogeneous phase (see Fig. 15). However, it is expected to be unstable or, possibly, metastable. Secondary inhomogeneous branches appear for smaller values of the energy but they have a lower value of entropy S and they are unstable. The homogeneous branch is metastable for $\Lambda < \Lambda_c^*$. These conclusions are motivated by two arguments: (i) in the Newtonian model in $d = 3$, we know that there is no fully stable equilibrium state in MCE. The system undergoes a *gravothermal catastrophe* [13, 14]. There is no global maximum of entropy S at fixed mass and energy because we can make it diverge by creating a binary star surrounded by a hot halo [22, 31]. In the modified Newtonian model, the same argument applies. Since we know that the homogeneous branch is stable for $\Lambda < \Lambda_c^*$, then it can only be metastable.

There is no strict caloric curve since there is no fully stable states (global maxima). But there is a physical caloric curve made of metastable states (local maxima) denoted (M) in Fig. 21. The unstable states (saddle points) are denoted (U). Here, the microcanonical and canonical ensembles, regarding the metastable states, are equivalent unlike in the Newtonian case. This is because the homogeneous branch and the inhomogeneous branch connect each other at a *single* point at $\lambda = 1$ by making a cusp (see inset in Fig. 21) while the Newtonian series of equilibria is smooth and presents *two* distinct turning points of temperature and energy (denoted CE and MCE in Fig. 8 of [31]) separated by a region of negative specific heats.

In conclusion, if we take metastable states into account, the system displays a zeroth order phase transition in CE and MCE corresponding to a discontinuity of entropy or free energy. They are associated with an isothermal collapse or a gravothermal catastrophe respectively.

Remark: There is no natural external parameter in the modified Newtonian model. However, the dimension of space d could play the role of an effective external parameter. The preceding results predict the existence of a critical dimension d_c between 1 and 2 at which the phase transition passes from second order ($d < d_c$) to first order ($d > d_c$). However, this transition turns out to occur in a very small range of parameters since we find that the critical dimension d_c is between 1 and $d = 1.00001$ and the concerned range of energies and temperatures is extremely narrow. We have not investigated this transition in detail since the dimension of space is not a physical (tunable) parameter. Furthermore, in the next model, we have an external parameter μ played by screening length that is more relevant.

IV. THE SCREENED NEWTONIAN MODEL

In this section, we discuss phase transitions that appear in the screened Newtonian model corresponding to an attractive Yukawa potential.

A. Physical motivation of the model

We consider a system of particles interacting via the potential $\Phi(\mathbf{r}, t)$ that is solution of the screened Poisson equation

$$\Delta\Phi - k_0^2\Phi = S_d G\rho, \tag{64}$$

where k_0 is the inverse of the screening length. At statistical equilibrium, the density is given by the Boltzmann distribution

$$\rho = Ae^{-\beta m\Phi}. \tag{65}$$

We assume that the system is confined in a finite domain (box) and we impose the Neumann boundary conditions

$$\nabla\Phi\cdot\mathbf{n} = 0, \qquad \nabla\rho\cdot\mathbf{n} = 0, \tag{66}$$

where $\mathbf{n}$ is a unit vector normal to the boundary of the box (the explicit expression of the potential in $d = 1$ is given in Appendix B). This model admits spatially homogeneous solutions ($\rho = \rho_0$ and $\Phi = \Phi_0$ with $-k_0^2\Phi_0 = S_d G\rho_0$) at any temperature. It also admits spatially inhomogeneous solutions at sufficiently low temperatures. We shall study this model in arbitrary dimensions of space d with explicit computations for $d = 1, 2, 3$. This model has different physical applications:

(i) It describes a system of particles interacting via a screened attractive (Newtonian) potential.

(ii) By a proper reinterpretation of the parameters, the field equation (64) describes the relation between the concentration of the chemical and the density of bacteria in the Keller-Segel model (37) where the degradation of the chemical reduces the range of the interaction. In that case, the boundary conditions are of the form (66). Furthermore, the relevant ensemble is the CE since the KS model has a canonical structure. This model has been studied by Childress & Percus [54] in $d = 1$ using an approach different from the one we are going to develop.

For the sake of generality, we shall study this model in the microcanonical and canonical ensembles in any dimension of space.

B. The screened Emden equation

In the screened Newtonian model, the equilibrium density profile is given by the Boltzmann distribution (65) coupled to the screened Poisson equation (64). As in Sec. III B, we look for spherically symmetric solutions. Introducing the central density $\rho_0 = \rho(0)$, the central potential $\Phi_0 = \Phi(0)$, the new field $\psi = \beta m(\Phi - \Phi_0)$ and the scaled distance $\xi = (S_d G\beta m\rho_0)^{1/2} r$ the Boltzmann distribution can be rewritten

$$\rho = \rho_0 e^{-\psi(\xi)}. \tag{67}$$