VI. MODIFIED MODE EQUATION

In the following, we show that when dealing with quasi-bound states not only Maxwell's equations but also the constitutive relations including the frequency dispersion have to be extended to the complex frequency plane. This fact directly leads to a modified mode equation which is capable of describing linear frequency dispersion. This mode equation predicts positive quality factors in agreement with the requirements of positive field energy and causality.

The key observation to start with is that a quasi-bound state

$$\vec{E}_\Omega(t) = \vec{E}_0 e^{-i\Omega t} \tag{21}$$

with complex-valued frequency Ω is not a monochromatic wave as its Fourier decomposition

$$\vec{E}_\Omega(t) = \int_{-\infty}^{\infty} d\omega\, \vec{E}(\omega) e^{-i\omega t} \tag{22}$$

gives nonvanishing $\vec{E}(\omega)$ in a region around $\omega \approx \mathrm{Re}(\Omega)$. With this decomposition we can write the constitutive relation (3) as

$$\vec{D}_\Omega(t) = \int_{-\infty}^{\infty} d\omega\, \varepsilon(\omega) \vec{E}(\omega) e^{-i\omega t} \,. \tag{23}$$

To proceed further, let us fix a real-valued frequency ω_r around which we study electromagnetic modes. We restrict ourselves to modes with sufficiently small $|\mathrm{Im}(\Omega)|$, i.e., not too small Q-factor, and $\mathrm{Re}(\Omega) \approx \omega_\mathrm{r}$. This allows the expansion of the permittivity

$$\varepsilon(\omega) \approx \varepsilon(\omega_\mathrm{r}) + \frac{\partial \varepsilon}{\partial \omega}\Big|_{\omega_\mathrm{r}} (\omega - \omega_\mathrm{r}) \,. \tag{24}$$

Inserting this expansion into Eq. (23) gives

$$\vec{D}_\Omega(t) \;=\; \varepsilon(\omega_\mathrm{r}) \vec{E}_\Omega(t) + \frac{\partial \varepsilon}{\partial \omega}\Big|_{\omega_\mathrm{r}} \left(i \frac{\partial}{\partial t} - \omega_\mathrm{r} \right) \vec{E}_\Omega(t) . \tag{25}$$

Now we exploit the property of quasi-bound states $\frac{\partial}{\partial t} \vec{E}_\Omega = -i\Omega \vec{E}_\Omega$, which can be deduced from Eq. (21) and leads to

$$\vec{D}_\Omega(t) \;=\; \tilde{\varepsilon}(\Omega) \vec{E}_\Omega(t) \,, \tag{26}$$

with modified permittivity

$$\tilde{\varepsilon}(\Omega) = \varepsilon(\omega_\mathrm{r}) + \frac{\partial \varepsilon}{\partial \omega}\Big|_{\omega_\mathrm{r}} (\Omega - \omega_\mathrm{r}) \,. \tag{27}$$

Equation (26) represents an analytic continuation of the permittivity $\varepsilon(\omega)$ to the complex-frequency plane Ω. For this continuation the linearization in Eq. (24) is not needed. In fact, for the modified mode equation that we derive in the following, the extension to any analytic function $\varepsilon(\omega)$ is straightforward. Nevertheless, for clarity we restrict ourself to a linear frequency dispersion. We would like to point out that the imaginary part of $\tilde{\varepsilon}$ is related to the frequency dispersion and not to optical absorption in the material, which is neglected here. Nevertheless, a realistic $\Omega \neq \omega_\mathrm{r}$ with negative imaginary part introduces a kind of loss in the originally lossless medium which counteracts the nonphysical exponential increase of the intensity, thereby turning the negative quality factor into a positive one.

In the following it will be convenient to express the values of the derivatives of ε and μ at the fixed frequency ω_r by their dimensionless linear dispersions

$$\alpha_\varepsilon = -\frac{\partial \varepsilon}{\partial \omega}\Big|_{\omega_\mathrm{r}} \frac{\omega_\mathrm{r}}{\varepsilon} \,, \quad \alpha_\mu = -\frac{\partial \mu}{\partial \omega}\Big|_{\omega_\mathrm{r}} \frac{\omega_\mathrm{r}}{\mu} \,. \tag{28}$$

For NIMs ($\varepsilon < 0$, $\mu < 0$) these quantities α_ε, α_μ have to be chosen larger than 1 to satisfy the inequalities (17) (for conventional dielectrics α_ε, α_μ must be smaller than 1.). To satisfy the inequalities (18) for NIMs we must have

$$\alpha_\varepsilon > 2 - \frac{2}{\varepsilon} \,, \quad \alpha_\mu > 2 - \frac{2}{\mu} \,. \tag{29}$$

With the quantities in Eq. (28) we can rewrite Eq. (27) as

$$\tilde{\varepsilon}(\Omega) = \varepsilon(\omega_\mathrm{r}) \left(1 + \alpha_\varepsilon \frac{\omega_\mathrm{r} - \Omega}{\omega_\mathrm{r}} \right) \,. \tag{30}$$

We can derive a modified permeability in an analogue way

$$\tilde{\mu}(\Omega) = \mu(\omega_\mathrm{r}) \left(1 + \alpha_\mu \frac{\omega_\mathrm{r} - \Omega}{\omega_\mathrm{r}} \right) \,. \tag{31}$$

As a result of our considerations we can use Maxwell's equations (1)-(2) and the constitutive relations (3)-(4) for monochromatic waves with modified permittivity and permeability given by Eqs. (30) and (31). A direct consequence is the modified mode equation

$$-\nabla^2 \psi = \tilde{n}^2(\Omega) \frac{\Omega^2}{c^2} \psi \,, \tag{32}$$

with the modified refractive index

$$\tilde{n}(\Omega) \;=\; \sqrt{\tilde{\varepsilon}(\Omega) \tilde{\mu}(\Omega)} \approx n(\omega_\mathrm{r}) \left(1 + \alpha_n \frac{\omega_\mathrm{r} - \Omega}{\omega_\mathrm{r}} \right) \tag{33}$$

and the dimensionless linear dispersion

$$\alpha_n = -\frac{\partial n}{\partial \omega}\Big|_{\omega_\mathrm{r}} \frac{\omega_\mathrm{r}}{n} = \frac{\alpha_\varepsilon + \alpha_\mu}{2} \,. \tag{34}$$

In the derivation we have ignored terms of order $(\Omega - \omega_\mathrm{r})^2$, which is consistent with Eq. (24). For the square root in Eq. (33) we choose the positive branch. Note that the (modified) refractive index can be defined negative or positive. This does not matter for our purpose, as the sign of the refractive index neither enters the mode equation (32) nor the modified boundary conditions

$$\psi_1 \;=\; \psi_2 \,, \; \frac{1}{\tilde{\mu}_1} \partial_\nu \psi_1 = \frac{1}{\tilde{\mu}_2} \partial_\nu \psi_2 \quad \text{(TM)} \tag{35}$$

$$\psi_1 \;=\; \psi_2 \,, \; \frac{1}{\tilde{\varepsilon}_1} \partial_\nu \psi_1 = \frac{1}{\tilde{\varepsilon}_2} \partial_\nu \psi_2 \quad \text{(TE)} \,. \tag{36}$$