In the incoherent regime, the adiabatic elimination leads to the set of coupled dynamical equations:

$$\begin{aligned} \frac{d\langle \hat{a}^\dagger \hat{a}\rangle}{dt} &= -(\kappa+R)\langle \hat{a}^\dagger \hat{a}\rangle + R\langle \hat{\sigma}_+\hat{\sigma}_-\rangle \\ \frac{d\langle \hat{\sigma}_+\hat{\sigma}_-\rangle}{dt} &= -(\gamma+R)\langle \hat{\sigma}_+\hat{\sigma}_-\rangle + R\langle \hat{a}^\dagger \hat{a}\rangle \,, \end{aligned} \tag{6}$$

where we have introduced the quantity

$$R = \frac{4g^2}{\kappa+\gamma+\gamma^*}\frac{1}{1+\left(\frac{2\delta}{\kappa+\gamma+\gamma^*}\right)^2} \,. \tag{7}$$

As it was underlined in Ref. [11], the quantity R can be seen as an effective coupling rate between the atom and the cavity mode, i.e. the system is formally equivalent to two coupled boxes (as represented in Fig. 1b). The "atomic" box is initially charged with a quantum of energy that can escape in the environment at rate γ, or in the "cavity" box at rate R. In the same way, the cavity box can lose its excitation with a rate κ, or give it back to the atomic box with a probability per unit time R. The parameter R is also involved in the efficiency of the corresponding single photon source, which reads

$$\beta = \frac{R\kappa/(R+\kappa)}{\gamma + R\kappa/(R+\kappa)} \,. \tag{8}$$

Seen from the atom point of view, the cavity mode appears as a further loss channel, whose effective rate is $R\kappa/(R+\kappa)$ (see Fig.1c). This result could have been straightforwardly derived from the classical picture. Such an expression for β is valid in any regime, even out of the incoherent regime, the only one in which the adiabatic elimination is supposed to be valid. Thus, the effective coupling rate R appears as a key parameter, allowing us to revisit the notions of good and bad cavity regimes, respectively. The bad cavity regime is achieved when $\kappa > R$, namely when the cavity damping time is shorter than the typical atom-cavity coupling time. In this case, the quantum exits the cavity mode as soon as it is released from the emitter: the cavity behaves as a supplementary loss channel. This is the usual regime for single photon sources and it is studied in the next two sections. In the good cavity regime, which is achieved when $R > \kappa$, the quantum of energy is emitted by the atom and can stay in the cavity mode before being lost in the environment. We stress here that the good cavity regime, which at resonance is achieved when $2g > \sqrt{\kappa(\kappa+\gamma+\gamma^*)}$, is more demanding than the strong coupling regime if γ^* becomes non negligible (whereas again,