bounded by

$$
\begin{aligned}
\frac{1}{n^2}\sum_{i=1}^n E[(g_{i,n}\epsilon_{1,i,n})^2|\mathcal{F}_{i-1}] &\lesssim n^{-1}\widetilde{b}_w^K(x_n)'\left(\widetilde{B}_w'\widetilde{B}_w/n\right)\widetilde{b}_w^K(x_n) && (66)\\
&\lesssim \zeta_{K,n}^2\lambda_{K,n}^2/n \quad \text{on } \mathcal{A}_n && (67)
\end{aligned}
$$

uniformly for $x_n \in \mathcal{S}_n$. Moreover,

$$
|n^{-1}g_{i,n}\epsilon_{1,i,n}| \lesssim \frac{\zeta_{K,n}^2\lambda_{K,n}^2 M_n}{n} \tag{68}
$$

uniformly for $x_n \in \mathcal{S}_n$. An tail bound for martingales (Freedman, 1975, Proposition 2.1) then provides that

$$
\begin{aligned}
&(\#\mathcal{S}_n)\max_{x_n\in\mathcal{S}_n}\mathbb{P}\left(\left\{\left|\frac{1}{n}\sum_{i=1}^n g_{i,n}\epsilon_{1,i,n}\right| > \frac{C}{2}\zeta_{K,n}\lambda_{K,n}\sqrt{(\log n)/n}\right\}\cap\mathcal{A}_n\right)\\
\lesssim\;& n^{\nu_1+\eta_2\nu_2}\exp\left\{-\frac{C^2\zeta_{K,n}^2\lambda_{K,n}^2(\log n)/n}{c_1\zeta_{K,n}^2\lambda_{K,n}^2/n + c_2\zeta_{K,n}^2\lambda_{K,n}^2 M_n/n\times C\zeta_{K,n}\lambda_{K,n}\sqrt{(\log n)/n}}\right\} && (69)\\
\lesssim\;& \exp\left\{\log n - \frac{C^2\zeta_{K,n}^2\lambda_{K,n}^2(\log n)/n}{c_3\zeta_{K,n}^2\lambda_{K,n}^2/n}\right\} + \exp\left\{\log n - \frac{C\sqrt{n\log n}}{c_4\zeta_{K,n}\lambda_{K,n}M_n}\right\} && (70)
\end{aligned}
$$

for finite positive constants $c_1,\ldots,c_4$. Thus (65a) vanishes asymptotically for all sufficiently large C provided $M_n = O(\zeta_{K,n}^{-1}\lambda_{K,n}^{-1}\sqrt{n/(\log n)})$. Choosing M_n as in the i.i.d. case completes the proof. ■

Proof of Remark 2.5. Take any $h \in L_{w,n}^\infty$ with $\|h\|_{\infty,w} \neq 0$. By the Cauchy-Schwarz inequality we have

$$
\begin{aligned}
|P_{K,w,n}(x)| &\leq \|\widetilde{b}_w^K(x)\|\|(\widetilde{B}_w'\widetilde{B}_w/n)^-\widetilde{B}_w'H/n\| && (71)\\
&\leq \zeta_{K,n}\lambda_{K,n}\|(\widetilde{B}_w'\widetilde{B}_w/n)^-\widetilde{B}_w'H/n\| && (72)
\end{aligned}
$$

uniformly over x, where $H = (h(X_1)w_n(X_1),\ldots,h(X_n)w_n(X_n))'$. When $\lambda_{\min}(\widetilde{B}_w'\widetilde{B}_w/n) \geq \frac{1}{2}$ (which it is wpa1 since $\|\widetilde{B}_w'\widetilde{B}_w/n - I_K\| = o_p(1)$), we have:

$$
\begin{aligned}
\|(\widetilde{B}_w'\widetilde{B}_w/n)^-\widetilde{B}_w'H/n\|^2 &= (H'\widetilde{B}_w/n)(\widetilde{B}_w'\widetilde{B}_w/n)^{-1}(\widetilde{B}_w'\widetilde{B}_w/n)^{-1}\widetilde{B}_w'H/n\\
&\leq 2(H'\widetilde{B}_w/n)(\widetilde{B}_w'\widetilde{B}_w/n)^{-1}\widetilde{B}_w'H/n\\
&\leq 2\|h\|_{w,n}^2 \leq 2\|h\|_{\infty,w}^2
\end{aligned}
$$