$x > B_1 + B_2$, we have

$$0 \geq -\int_{-\infty}^{\infty} \phi(y-x) \log p_Y(y;\mu_0) \mathrm{d}y - \lambda x^2 - (c+\lambda) \tag{72}$$

$$\geq \int_{x-B_2}^{x+B_2} \phi(y-x) \frac{1}{2}(y-B_1)^2 \mathrm{d}y - \lambda x^2 - (c+\lambda) \tag{73}$$

$$= \int_{B_2}^{B_2} \phi(t) \frac{1}{2}(x-B_1+t)^2 \mathrm{d}t - \lambda x^2 - (c+\lambda) \tag{74}$$

$$\geq \frac{1}{2}(x-B_1)^2(1-\epsilon) - \lambda x^2 - (c+\lambda). \tag{75}$$

For (72) to hold for large x, λ must satisfy $\lambda \geq \frac{1}{2}$.

To finish the proof, it suffices to show that $\lambda < \frac{1}{2}$ for any $\gamma > 0$, so that contradiction arises, which implies that S_{μ_0} must be countably infinite. For fixed $q \in (0,1)$, denote the Lagrange multiplier in (58) as $\lambda(\gamma)$. Denote $C_G(\gamma) = \frac{1}{2}\log(1+\gamma)$, which is the channel capacity of a Gaussian channel with the average power constraint only. By the envelope theorem [19], $\lambda(\gamma)$ is the derivative of $C(\gamma, q)$ w.r.t. γ. Since $C(0,q) = C_G(0) = 0$ and the derivative of $C_G(\gamma)$ at $\gamma = 0$ is $\frac{1}{2}$, we have $\lambda(0) \leq \frac{1}{2}$, otherwise we could find a small enough γ such that $C(\gamma, q)$ would exceed $C_G(\gamma)$ which is obviously impossible. Next we show that $C(\gamma, q)$ is strictly concave for $\gamma \geq 0$. Suppose μ_1 and μ_2 are the capacity-achieving input distributions of (5) for different power constraints γ_1 and γ_2, respectively. Due to Property (b) in Theorem 1, μ_1 and μ_2 must be different. Define $\mu_\theta = \theta\mu_1 + (1-\theta)\mu_2$ for $\theta \in (0,1)$. It is easy to see that μ_θ satisfies that the duty cycle is no greater than $1-q$ and the average input power is no greater than $\theta\gamma_1 + (1-\theta)\gamma_2$. Now we have

$$C(\theta\gamma_1 + (1-\theta)\gamma_2, q) \geq I(\mu_\theta) \tag{76}$$

$$> \theta I(\mu_1) + (1-\theta) I(\mu_2) \tag{77}$$

$$= \theta C(\gamma_1, q) + (1-\theta) C(\gamma_2, q), \tag{78}$$

where (77) is due to the strict concavity of $I(\mu)$. Therefore, the strict concavity of $C(\gamma, q)$ for $\gamma \geq 0$ follows, which implies that $\lambda(\gamma) < \lambda(0) = \frac{1}{2}$ for all $\gamma > 0$.