Proof. Using the technique for quasi-linear equations [13], one can confirm under the conditions $(C1)-(C3)$ that system (4.2) possesses a unique bounded on $\mathbb{R}$ solution $\phi(t)$ which satisfies the relation

$$\phi(t) = \int_{-\infty}^{t} e^{A(t-u)}[f(\phi(u)) + g(u)]du. \tag{4.3}$$

Moreover, $\sup_{t\in\mathbb{R}} \|\phi(t)\| \leq M_\phi$, where $M_\phi = \dfrac{K(M_f + M_g)}{\omega}$ and $M_g = \sup_{t\in\mathbb{R}} \|g(t)\|$. The solution $\phi(t)$ is uniformly continuous on $\mathbb{R}$ since $\sup_{t\in\mathbb{R}} \|\phi'(t)\| \leq \|A\| M_\phi + M_f + M_g$.

Suppose that $x(t)$ is a solution of (4.2) such that $x(t_0) = x_0$ for some $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^m$. It can be verified that

$$\|x(t) - \phi(t)\| \leq K \|x_0 - \phi(t_0)\| e^{(KL_f - \omega)(t-t_0)}, \ t \geq t_0,$$

and therefore, $\phi(t)$ is uniformly exponentially stable.

Since the function $g(t)$ is unpredictable, there exist a positive number $\epsilon_0 \leq 1$ and sequences $\{t_n\}$, $\{\tau_n\}$, both of which diverge to infinity, such that $d(g_{t_n}, g) \to 0$ as $n \to \infty$ and $d(g_{t_n+\tau_n}, g_{\tau_n}) \geq \epsilon_0$ for all $n \in \mathbb{N}$, where the distance function d is given by (3.1).

First of all, we shall show that $d(\phi_{t_n}, \phi) \to 0$ as $n \to \infty$. Fix an arbitrary small positive number $\epsilon < 1$ and suppose that α is a positive number satisfying $\alpha \leq \dfrac{\omega - KL_f}{2\omega + K - 2KL_f}$. Let k_0 be a sufficiently large natural number such that

$$k_0 \geq \max\left\{ \frac{\ln(1/\alpha\epsilon)}{\ln 2}, \frac{1}{\omega - KL_f} \ln\left(\frac{2K(M_f + M_g)}{\omega\alpha\epsilon} \right) \right\}. \tag{4.4}$$

There exists a natural number n_0 such that if $n \geq n_0$ then $d(g_{t_n}, g) < 2^{-2k_0}\alpha\epsilon$. Therefore, for $n \geq n_0$, the inequality $\rho_{2k_0}(g_{t_n}, g) < \alpha\epsilon$ is valid. Since $\alpha\epsilon < 1$, we have that $\|g(t_n + s) - g(s)\| < \alpha\epsilon$ for $s \in [-2k_0, 2k_0]$.

Making use of the relation (4.3), one can obtain that

$$\phi(t_n + s) - \phi(s) = \int_{-\infty}^{s} e^{A(s-u)} \left[f(\phi(t_n + u)) - f(\phi(u)) + g(t_n + u) - g(u) \right] du.$$

Thus, if s belongs to the interval $[-2k_0, 2k_0]$, then it can be verified that

$$\begin{aligned} \|\phi(t_n + s) - \phi(s)\| \quad &\leq \frac{2K(M_f + M_g)}{\omega} e^{-\omega(s+2k_0)} + \frac{K\alpha\epsilon}{\omega} \left(1 - e^{-\omega(s+2k_0)} \right) \\ &+ KL_f \int_{-2k_0}^{s} e^{-\omega(s-u)} \|\phi(t_n + u) - \phi(u)\| du. \end{aligned} \tag{4.5}$$

Now, let us define the functions $\psi_n(s) = e^{\omega s} \|\phi(t_n + s) - \phi(s)\|$, $n \geq n_0$. Inequality (4.5) implies that

$$\psi_n(s) \leq \frac{K\alpha\epsilon}{\omega} e^{\omega s} + \left(\frac{2K(M_f + M_g) - K\alpha\epsilon}{\omega} \right) e^{-2\omega k_0} + KL_f \int_{-2k_0}^{s} \psi_n(u) du.$$