$$c_2^2 = \frac{\lambda}{2\rho_{n0}} + \frac{1}{2\rho_0\chi} - \gamma - \frac{1}{2}\sqrt{\left(\frac{\lambda}{\rho_{n0}} + \frac{1}{\rho_0\chi} - 2\gamma\right)^2 - \frac{4\rho_{s0}}{\rho_{n0}}\left[\frac{\lambda}{\chi\rho_0^2} - \gamma^2\right]} . \tag{49}$$

In particular, when $\rho_{s0} = 0$ (normal solids), c_2 vanishes, and we only have

$$c_{\rm NS}^2 = (\tilde{\lambda} + 2\tilde{\mu} + 1/\chi)/\rho_0 - 2\gamma, \tag{50}$$

which agrees with the longitudinal sound speed obtained by Zippelius *et al.*[44] once we identify $\tilde{\lambda} = \lambda^{\rm ZHN} + 2\gamma^{\rm ZHN} + 1/\chi$ and $\gamma = (\gamma^{\rm ZHN} + 1/\chi)/\rho_0$. Moreover, when the Lamé coefficients and the coupling constant γ vanish we recover the sound speed of a normal fluid. As discussed earlier, there is one pair of transverse sound modes with speed

$$c_T = \sqrt{\frac{\tilde{\mu}}{\rho_{n0}}}. \tag{51}$$

Finally, we can calculate the correlation functions from Eq. (45):

$$\langle \delta\rho_\Delta(\mathbf{Q})\delta\rho_\Delta(-\mathbf{Q})\rangle = \rho_{s0}q^2\frac{\rho_0\omega_n^2 + (\lambda - 2\rho_0\gamma + 1/\chi)q^2}{\Delta_\mathbf{A}}, \tag{52}$$

$$\langle \delta\rho_\Delta(\mathbf{Q})u_L(-\mathbf{Q})\rangle = iq\frac{\rho_{s0}}{\rho_0}\frac{\rho_0\omega_n^2 - (\rho_0\gamma - 1/\chi)q^2}{\Delta_\mathbf{A}}, \tag{53}$$

and

$$\langle u_L(\mathbf{Q})u_L(-\mathbf{Q})\rangle = \frac{1}{\rho_0^2\chi}\frac{\rho_0^2\chi\omega_n^2 + \rho_{s0}q^2}{\Delta_\mathbf{A}}. \tag{54}$$

Since the density fluctuation is related to the defect density fluctuation and the strain tensor by Eq. (34), the density-density correlation function becomes

$$\begin{aligned}\langle \delta\rho(\mathbf{Q})\delta\rho(-\mathbf{Q})\rangle &= A\left(\frac{1}{i\omega - c_L q} - \frac{1}{i\omega + c_L q}\right) \\ &+ B\left(\frac{1}{i\omega - c_2 q} - \frac{1}{i\omega + c_2 q}\right),\end{aligned} \tag{55}$$

where

$$A = -q\frac{\rho_0\rho_{n0}c_L^2 - \rho_{s0}\lambda}{2c_L\rho_{n0}(c_L^2 - c_2^2)}, \tag{56}$$

$$B = -q\frac{\rho_0\rho_{n0}c_2^2 - \rho_{s0}\lambda}{2c_2\rho_{n0}(c_2^2 - c_L^2)}. \tag{57}$$

Then, by performing the analytic continuation $i\omega_n = \omega + i\delta$, the density-density response function can be obtained from the imaginary part of the density-density correlation function:

$$\begin{aligned}\chi''_{\rho\rho}(\mathbf{q},\omega) &= -\pi A\Big[\delta(\omega - c_L q) - \delta(\omega + c_L q)\Big] \\ &\quad -\pi B\Big[\delta(\omega - c_2 q) - \delta(\omega + c_2 q)\Big],\end{aligned} \tag{58}$$

where we have used the identity

$$\frac{1}{\omega' - \omega - i\epsilon} = P\frac{1}{\omega' - \omega} + i\pi\delta\left(\omega - \omega'\right). \tag{59}$$

It is easy to show that the response function satisfies the thermodynamic sum rule (for the derivation of the static correlation function see Appendix B)

$$\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\frac{\chi''_{\rho\rho}(\mathbf{q},\omega)}{\omega} = \frac{\rho_0^2\chi\lambda}{\lambda - \rho_0^2\gamma^2\chi}, \tag{60}$$

and the f-sum rule

$$\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\omega\chi''_{\rho\rho}(\mathbf{q},\omega) = \rho_0 q^2. \tag{61}$$

B. Dynamics of supersolid with dissipation

We continue our discussion of the density correlation and response functions by including dissipative terms in the equations of motion. As mentioned above, the dissipative terms will broaden the δ-function peaks in the density response function. In addition, as noted by Martin et al.[38], the dissipation is necessary to identify the "missing" defect diffusion mode in normal solids. The dissipative hydrodynamic equations of motion for a supersolid were first obtained by Andreev and Lifshitz,[1] who used standard entropy-production arguments to generate the dissipative terms. For an isotropic supersolid (including the nonlinear term neglected by Andreev and Lifshitz) we have (with the new dissipative terms on the right hand side)

$$\partial_t\rho + \partial_i j_i = 0, \tag{62}$$

$$\begin{aligned}\partial_t j_i + \partial_j \Pi_{ij} &= \zeta\partial_i\partial_k v_{nk} + \eta\partial^2 v_{ni} \\ &\quad -\Sigma\partial_i\partial_k\Big[\rho_s(v_{nk} - v_{sk})\Big],\end{aligned} \tag{63}$$

$$\partial_t u_i - v_{ni} + v_{nk}\partial_k u_i + u_i\partial_k v_{nk} = \Gamma\partial_k\lambda_{ki}, \tag{64}$$

$$\begin{aligned}\partial_t v_{si} + \partial_i\left(\mu + \frac{1}{2}{v_s}^2\right) &= -\Lambda\partial_i\partial_k\Big[\rho_s(v_{nk} - v_{sk})\Big] \\ &\quad +\Sigma\partial_i\partial_k v_{nk},\end{aligned} \tag{65}$$

where $\mathbf{j} = \rho_n\mathbf{v}_n + \rho_s\mathbf{v}_s$ is the total mass current, Σ and Λ coefficients of viscosity, ζ the bulk viscosity coefficient, η the shear viscosity coefficient, Γ the diffusion coefficient for defects.

We next linearize the dissipative hydrodynamic equations by considering small fluctuations from the equilibrium values. Writing $\delta\mu$ and λ_{ij} in terms of $\delta\rho$ and δw_{ij},

$$\delta\mu = \frac{1}{\rho_0^2\chi}\delta\rho + \gamma w_{ii}, \tag{66}$$