the functional integrals can be achieved introducing the superfields $\Psi_{r,r'}(R)$, $R=\{z,\theta,\theta^*\}$, as follows

$$\Psi_{r,r'}(R)=a_{r,r'}(z)\,\theta+f^T_{r,r'}(z)\,\theta^*+\eta_{r,r'}(z)+\eta^+_{r,r'}(z)\,\theta^*\theta \tag{3.5}$$

where $a_{r,r'}(z)$ and $f_{r,r'}(z)$ are real fields of the coordinates r,r' and the variables $z=(\tau,\sigma,u)$. The hermitian conjugation means complex conjugation " $*$ " supplemented by the transposition, such that $\eta^+_{r,r'}=\eta^*_{r,'r}$. The variables θ, θ^* are artificially introduced Grassmann variables that help us to write the exponent in the compact form.

As a result, we write the functional $Z[\phi]$, Eq. (2.31), in the form

$$Z[\phi]=Z_0\int\exp\left(-\mathcal{S}_0[\Psi]-\mathcal{S}^{(u\phi)}[\Psi]\right)D\Psi \tag{3.6}$$

where $S_0[\Psi]$ is the bare part of the action

$$\mathcal{S}_0=\frac{i}{2}\sum_{r,r'}\int\left[\Psi_{r',r}(R)\left(\frac{\partial}{\partial\tau}+(\hat{\varepsilon}_r-\hat{\varepsilon}_{r'})\right)\Psi_{r,r'}(R)\right]dR \tag{3.7}$$

The second term $\mathcal{S}^{(u\phi)}[\Psi]$ in the exponent in Eq. (3.6) is linear with respect to the HS field $\phi_{r\sigma}(\tau)$. Its explicit form reads

$$\begin{aligned}\mathcal{S}^{(u\phi)}[\Psi]=-i\sum_{r,r'}\int\phi_{r\sigma}(\tau)\Big[u\left(\Psi_{r',r}(R)-n_{r',r}\theta\right)\\ \times\left(\Psi_{r,r'}(R)-n_{r,r'}\theta\right)-i\sigma\delta_{r,r'}\Psi_{r,r}(R)\,\theta^*\Big]dR\end{aligned} \tag{3.8}$$

It is interesting to remark here that the superfield $\Psi(R)$ is anticommuting, which is a rather unusual feature of the field theory considered here. However, it is very important that this field describes bosons and not fermions. This follows from the periodic boundary condition $\Psi(\tau)=\Psi(\tau+\beta)$. It is this condition that determines unambiguously the statistics of the particles. The fact that $\Psi(R)$ is anticommuting is only a formal property.

The form of Eqs. (3.6-3.8) allows us to average immediately over the HS field $\phi_{r\sigma}(\tau)$. This integration is Gaussian and is specified by Eq. (2.18). The analytic supersymmetric field theory written here relies crucially on the presence of a bath, whose function is to break the translational symmetry of the system. In order to take the bath into account, we model the interaction in the same way as in Eq. (2.37). The pair correlation of the fields with the distribution W can be written as

$$\begin{aligned}&\left\langle\sigma\sigma'\phi_{r\sigma}(\tau)\,\phi'_{r'\sigma'}(\tau')\right\rangle_W=U_{r,r'}(R,R') \qquad (3.9)\\ &U_{r,r'}(R,R')=\delta(\tau-\tau')\left(\sigma\sigma' V_0^{total}\delta_{r,r'}+V^{(1),total}_{r,r'}\right),\end{aligned}$$

where V_0^{total} and $V^{(1),total}_{r,r'}$ are defined from V_0 and $V^{(1)}$ according to Eqn. (2.37). The coupling constants V_0 and $V^{(1)}$ contain the constant $\bar{V}$ that has been introduced in a rather arbitrary manner (see, below Eq. (2.5). However, the same constant enters the renormalized chemical potential μ', Eq. (2.14), and the renormalized thermodynamical potential Ω written below Eq. (2.14). Of course, the constant $\bar{V}$ should disappear from the final result for the thermodynamical potential, which can serve as a check of any computation. Actually, considering only perturbation theory in the interaction there is no necessity to introduce this constant. In such calculations the sign of $\tilde{V}_{r,r'}$ in Eq. (2.4) does not play an important role because one could decouple the interaction term with $\tilde{V}$ integrating over purely imaginary HS. We have added $\bar{V}$ because we want to keep the HS fields real, keeping in mind a possibility of applying the method to numerical investigations.

Using Eqs. (3.8,3.9) we easily integrate in Eq. (3.6) over the field $\phi_{r\sigma}(\tau)$ reducing the partition function Z to the form

$$Z=Z_0\int\exp\left(-\mathcal{S}_0[\Psi]-\mathcal{S}_{int}[\Psi]\right)D\Psi \tag{3.10}$$

with $\mathcal{S}_0[\Psi]$ given by Eq. (3.7) and

$$\mathcal{S}_{int}[\Psi]=\mathcal{S}_2[\Psi]+\mathcal{S}_3[\Psi]+\mathcal{S}_4[\Psi]. \tag{3.11}$$

The term $\mathcal{S}_{int}[\Psi]$ in the action describes the interaction between the fields Ψ. The terms $\mathcal{S}_2[\Psi]$, $\mathcal{S}_3[\Psi]$ and $\mathcal{S}_4[\Psi]$ contain quadratic, cubic and quartic in Ψ terms, respectively. They can be written in the form,

$$\begin{aligned}\mathcal{S}_2=\frac{1}{2}\int\sum_{r,r_1}\left\{i\Psi_{r,r}(R)\theta^*-[\Psi(R),\hat{n}]_{rr}u\theta\right\}U_{r,r_1}(R,R_1)\\ \times\left\{i\Psi_{r_1,r_1}(R_1)\theta_1^*-[\Psi(R_1),\hat{n}]_{r_1,r_1}u_1\theta_1\right\}\ dRdR_1,\end{aligned} \tag{3.12a}$$

$$\begin{aligned}\mathcal{S}_3=\int\sum_{r,r',r_1}\Psi_{r',r}(R)\Psi_{r,r'}(R)U_{r,r_1}(R,R_1)\\ \times\left\{i\Psi_{r_1,r_1}(R_1)\theta_1^*-[\Psi(R_1),\hat{n}]_{r_1,r_1}u_1\theta_1\right\}\ udRdR_1,\end{aligned} \tag{3.12b}$$

$$\begin{aligned}\mathcal{S}_4=\frac{1}{2}\int\sum_{r,r',r_1,r_{1'}}\Psi_{r',r}(R)\Psi_{r,r'}(R)U_{r,r_1}(R,R_1)\\ \times\Psi_{r_1',r_1}(R_1)\Psi_{r_1,r_1'}(R_1)u_1udRdR_1\end{aligned} \tag{3.12c}$$

where $[,]$ stands for the commutator. Integration over R in Eq. (3.12a-3.12c) implies summation over σ and integration over u,τ,θ,θ^*. The bare action $\mathcal{S}_0$ and the interaction term $\mathcal{S}_4$, being invariant under the transformation of the fields Ψ

$$\Psi_{r,r'}(\theta,\theta^*)\to\Psi_{r,r'}(\theta+\kappa,\theta^*+\kappa^*) \tag{3.13}$$

(κ and κ^* being anticommuting variables) are fully supersymmetric in the sense of Ref. 31, whereas the terms $\mathcal{S}_2$ and $\mathcal{S}_3$ break this invariance. The invariance under the transformation (3.13) is stronger than the standard BRST symmetry for stochastic field equations [invariance under the transformation $\Psi(\theta^*)\to\Psi(\theta^*+\kappa^*)$][31], and reflects additional symmetries of Eqs. (3.7,3.12c).