it behaves the same as in the compactified theory. For subcritical boundary perturbations, $g < \frac{1}{2}$, propagating states and momentum dependence for the energy eigenvalues within a band arise from mixing between the states on neighboring unit cells. On the other hand, for $g \approx 0$, the NFS view of the band structure starts with the continuous relativistic dispersion relation of the free string at $R = \infty$. The boundary perturbation injects discrete units of momentum which are integer multiples of R_0^{-1}, causing left- and right-propagating string states near the edges of the Brillouin zones to mix, opening up band gaps at the zone boundaries.

To obtain the spectrum of the compactified $R = R_0$ theory, define the $SU(2)$ Kac-Moody currents constructed from the holomorphic (left-moving) part of the coordinate $X_L(z)$ (where $z = \tau + i\sigma$),

$$J_3 = \frac{i}{\sqrt{2}}\partial X_L, \quad J^{\pm} \equiv J_1 \pm iJ_2 = e^{\pm i\sqrt{2}X_L} \tag{17}$$

There is also a set of right-moving Kac-Moody currents, but by standard arguments, the open string Hamiltonian can be written entirely in terms of the left-moving fields defined on the doubled line segment $-l < \sigma < l$ with periodic boundary conditions. The free open string Hamiltonian density is given by a Sugawara construction which, by $SU(2)$ symmetry, can be written in several ways

$$\mathcal{H}_0(\sigma) =: J^1 J^1 :=: J^3 J^3 := \frac{1}{3} : \mathbf{J} \cdot \mathbf{J} : \tag{18}$$

In terms of the Kac-Moody currents, the marginal boundary perturbation (16) is simply

$$\mathcal{H}_{int} = \lambda(X(\sigma = 0)) = gJ^1(\sigma = 0) \tag{19}$$

For the free string in the $: J^3 J^3 :$ form, the partition function is easily calculated by observing that the states fall into $U(1)$ Kac-Moody modules, one for each allowed value of the $U(1)$ charge $Q = \int d\sigma/(2\pi)J^3(\sigma)$. The module with conformal weight Q^2 contributes $w^{-1/24}f(w)^{-1}w^{Q^2}$ to the partition function. The Kac-Moody charges are equal to the allowed translational zero mode momenta of the string, which for the $R = R_0$ compactified theory are integers. The sum over modules gives the free string partition function,

$$Z_{NN}^{R_0} = \frac{w^{-\frac{1}{24}}}{f(w)} \sum_{Q \epsilon \mathbf{Z}} w^{Q^2} \tag{20}$$

By $SU(2)$ symmetry, the same partition function is obtained from the $: J^1 J^1 :$ form of the Sugawara Hamiltonian. In this latter form, the marginal boundary perturbation can be included