This means that the model is not a gauge invariant theory.

The gauge symmetry inside the model will be disclosed via a new gauge-invariant formalism that does not require more than one WZ field. The fundamental concept behind the symplectic gauge-invariant formalism dwells in the extension of the original phase space with the introduction of two arbitrary function $\Psi(\phi,\pi_\phi,A_0,A_1,\pi_1,\theta)$ and $G(\phi,\pi_\phi,A_0,A_1,\pi_1,\theta)$, depending on both the original phase space variables and the WZ variable θ, into the first-order Lagrangian, right on the kinetic and symplectic potential sector, respectively. In this way, the first-order Lagrangian that governs the dynamics of the bosonized CSM, given in Eq. (5.2), is rewritten as

$$\tilde{L}^{(0)} = \pi_\phi\dot{\phi} + \pi_1\dot{A}_1 + \dot{\theta}\Psi - \tilde{U}^{(0)}, \tag{5.13}$$

where

$$\begin{aligned}\tilde{U}^{(0)} &= \frac{1}{2}(\pi_1^2+\pi_\phi^2+\phi'^2) - A_0\big[\pi_1' + \frac{1}{2}q^2(a-1)A_0 + q^2A_1 + q\pi_\phi + q\phi'\big] \\ &\quad - A_1\big[-q\pi_\phi - \frac{1}{2}q^2(a+1)A_1 - q\phi'\big] + G(\phi,\pi_\phi,A_0,A_1,\pi_1,\theta).\end{aligned} \tag{5.14}$$

The gauge-invariant formulation encompasses two steps: one is dedicated to the computation of Ψ while the other is addressed to the calculation of G.

The enlarged symplectic variables are now $\tilde{\xi}_\alpha^{(0)} = (\phi,\pi_\phi,A_0,A_1,\pi_1,\theta)$ with the following one-form canonical momenta

$$\begin{aligned}\tilde{A}_\phi^{(0)} &= \pi_\phi, \\ \tilde{A}_{A_1}^{(0)} &= \pi_1, \\ \tilde{A}_{A_0}^{(0)} &= \tilde{A}_{\pi_\phi}^{(0)} = \tilde{A}_{\pi_1}^{(0)} = 0, \\ \tilde{A}_\theta^{(0)} &= \Psi.\end{aligned} \tag{5.15}$$

The corresponding symplectic matrix $\tilde{f}^{(0)}$ reads

$$\tilde{f}^{(0)} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & \frac{\partial\Psi^y}{\partial\phi^x} \\ 1 & 0 & 0 & 0 & 0 & \frac{\partial\Psi^y}{\partial\pi_\phi^x} \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial\Psi^y}{\partial A_0^x} \\ 0 & 0 & 0 & 0 & -1 & \frac{\partial\Psi^y}{\partial A_1^x} \\ 0 & 0 & 0 & 1 & 0 & \frac{\partial\Psi^y}{\partial\pi_1^x} \\ -\frac{\partial\Psi^x}{\partial\phi^y} & -\frac{\partial\Psi^x}{\partial\pi_\phi^y} & -\frac{\partial\Psi^x}{\partial A_0^y} & -\frac{\partial\Psi^x}{\partial A_1^y} & -\frac{\partial\Psi^x}{\partial\pi_1^y} & f_{\theta_x\theta_y}\end{pmatrix}\delta(x-y), \tag{5.16}$$

where

$$f_{\theta_x\theta_y} = \frac{\partial\Psi_y}{\partial\theta_x} - \frac{\partial\Psi_x}{\partial\theta_y}, \tag{5.17}$$

with $\theta_x \equiv \theta(x)$, $\theta_y \equiv \theta(y)$, $\Psi_x \equiv \Psi(x)$ and $\Psi_y \equiv \Psi(y)$. Note that this matrix is singular since $\frac{\partial\Psi^x}{\partial A_0^y} = 0$. Due to this, we conclude that $\Psi \equiv \Psi(\phi,\pi_\phi,A_1,\pi_1,\theta)$.

To unveil the gauge symmetry hidden inside the model, we assume that this singular matrix has a zero-mode ($\nu^{(0)}$) that satisfies the following relation,

$$\int \nu_\alpha^{(0)}(x)\tilde{f}_{\alpha\beta}^{(0)}(x-y)\,d\,y = 0. \tag{5.18}$$

From this relation a set of equations will be obtained and consequently, the arbitrary function Ψ can be determined. We will now investigate the symmetry related to the following zero-mode,

$$\bar{\nu}^{(0)} = \begin{pmatrix} q & -q\partial_x & 1 & \partial_x & -q^2 & -1\end{pmatrix}, \tag{5.19}$$

with bar representing a transpose matrix.

To start, we multiply the zero-mode (5.19) by the symplectic matrix (5.16), as shown in equation (5.18). Due to this, some equations arise and after an integration Ψ is determined as

$$\Psi = \pi' + q\phi' + q\pi_\phi + q^2A_1. \tag{5.20}$$