with

$$
\begin{aligned}
\left|\Psi_1\right| &< C_1 e^{-n f_{i0}} h_i^{-1} n^{-3/2} \sup_{z\in\bar{B}} \left|g\left(z\right)\right|, && (44)\\
\left|\Psi_2\right| &< C_1 e^{-n f_{i0}} e^{-nC_2} h_i^{-1} \sup_{z\in\mathcal{K}_{i1}\cup\bar{\mathcal{K}}_{i1}} \left|g\left(z\right)\right|, \text{ and} && (45)\\
\left|\Psi_3\right| &< C_1 \left|\oint_{\mathcal{K}_{i2}\cup\bar{\mathcal{K}}_{i2}} e^{-n f_i(z)} g\left(z\right) \mathrm{d}z\right|, && (46)
\end{aligned}
$$

where C_1 and C_2 are some positive constants, and $\bar{B}$ is a closed ball with center at z_{i0} and radius $r_i/2$.

Now, let $g\left(z\right) = g_j(z) = z^{j-1}\exp\left\{-\Delta_p\left(z\right)\right\}$. Lemma A2 in OMH implies that $\sup_{z\in\bar{B}\cup\mathcal{K}_{i1}\cup\bar{\mathcal{K}}_{i1}} \left|g\left(z\right)\right| = h_i^{1-j} O_p\left(1\right)$ uniformly in $h_i \in \left(0, \bar{h}\right]$. Therefore, by (44) and (45),

$$
\Psi_1 + \Psi_2 = e^{-n f_{i0}} h_i^{-j} n^{-3/2} O_p\left(1\right). \tag{47}
$$

Turning to the analysis of Ψ_3, note that by definition of $f_i\left(z\right)$ and $g\left(z\right)$,

$$
e^{-n f_i(z)} g\left(z\right) = e^{n\frac{h_i}{1+h_i}z} z^{j-1} \prod_{j=1}^{p} \left(z-\lambda_j\right)^{-1}. \tag{48}
$$

For $z \in \mathcal{K}_{i2}\cup\bar{\mathcal{K}}_{i2}$, we have $\left|\left(z-\lambda_j\right)^{-1}\right| < \left(3z_{i0}\right)^{-1}$, and $\left|z\left(z-\lambda_j\right)^{-1}\right| < 2$, for any $j = 1, ..., p$. Therefore, using (48), we get

$$
\begin{aligned}
\left|\oint_{\mathcal{K}_{i2}\cup\bar{\mathcal{K}}_{i2}} e^{-n f_i(z)} g\left(z\right) \mathrm{d}z\right| &< 2^{j-1}\left(3z_{i0}\right)^{-p+j-1} \oint_{\mathcal{K}_{i2}\cup\bar{\mathcal{K}}_{i2}} \left|e^{n\frac{h_i}{1+h_i}z}\mathrm{d}z\right| \\
&= 2^{j}\left(3z_{i0}\right)^{-p+j-1}\left(n\frac{h_i}{1+h_i}\right)^{-1} e^{n\frac{h_i}{1+h_i}z_{i0}} \\
&= 2^{j}\left(3z_{i0}\right)^{j-1}\left(n\frac{h_i}{1+h_i}\right)^{-1} e^{-n\left(c_p\ln(3z_{i0})-\frac{h_i}{1+h_i}z_{i0}\right)} \\
&= 2^{j}\left(3z_{i0}\right)^{j-1}\left(n\frac{h_i}{1+h_i}\right)^{-1} 3^{-p} e^{-n(c_p\ln(z_{i0})-h_i-c_p)}.
\end{aligned}
$$

On the other hand, for any $h_i \in \left[0, \bar{h}\right]$, $h_i < \sqrt{c_p}$ for sufficiently large n and p, and