tion, we will denote the limit $\lim_{L_x,\ L_y\to\infty} sc[(L_x)_{BCx} \times (L_y)_{BCy} \times (L_z)_{BCz}]$ simply as $S_{(L_z)_{BC_z}}$, where S stands for "slab". We will consider both free (F) and periodic (P) boundary conditions in the z direction, and thus slabs such as S_{3_F}, S_{3_P}, etc. For technical reasons (to get an expression involving a trace of a coloring matrix, as explained below) we will use periodic boundary conditions in the x direction. Note that the proper q-coloring constraint implies that FBC_z and PBC_z are equivalent if $L_z = 2$. The number of vertices for $G = sc[(L_x)_{BCx} \times (L_y)_{BCy} \times (L_z)_{BCz}]$ is $n = L_x L_y L_z$. The specific form of Eq. (1.1) for our calculation is

$$W(S_{(L_z)_{BCz}}, q) = \lim_{L_y\to\infty} \lim_{L_x\to\infty} [P(sc[(L_x)_P \times (L_y)_{BCy} \times (L_z)_{BCz}], q)]^{1/n} \ . \tag{2.1}$$

To derive a lower bound on $W(S_{(L_z)_{BC_z}}, q)$, we generalize the method of Refs. [11]-[14] from two to three dimensions. We consider two adjacent transverse slices of the slab orthogonal to the x direction, with x values x_0 and x_0+1. These are thus sections of the square lattice of dimension $L_y \times L_z$, which we denote $G_{x_0} = sq[(L_y)_{BC_y} \times (L_z)_{BC_z}]_{x_0}$ and $G_{x_0+1} = sq[(L_y)_{BC_y} \times (L_z)_{BC_z}]_{x_0+1}$. We label a particular color assignment to the vertices of G_{x_0} that is a proper q-coloring of these vertices as $C(G_{x_0})$ and similarly for G_{x_0+1}. The total number of proper q-colorings of G_{x_0} is

$$\mathcal{N} = P(G_{x_0}, q) = P(G_{x_0+1}, q) \ . \tag{2.2}$$

Now let us add the edges in the x direction that join these two adjacent transverse slices of the slab together. Among the $\mathcal{N}^2$ color configurations that yield proper q-colorings of these two separate yz transverse slices, some will continue to be proper q-colorings after we add these edges that join them in the x direction, while others will not. We define an $\mathcal{N} \times \mathcal{N}$-dimensional coloring compatibility matrix T with entries $T_{C(G_{x_0}),C(G_{x_0+1})}$ equal to (i) 1 if the color assignments $C(G_{x_0})$ and $C(G_{x_0+1})$ are proper q-colorings after the edges in the x direction have been added joining G_{x_0} and G_{x_0+1}, i.e., if the color assigned to each vertex $v(x_0, y, z)$ in G_{x_0} is different from the color assigned to the vertex $v(x_0+1, y, z)$ in G_{x_0+1}; and (ii) 0 if the color assignments $C(G_{x_0})$ and $C(G_{x_0+1})$ are not proper q-colorings after the edges in the x direction have been added, i.e., there exists some color assigned to a vertex $v(x_0, y, z)$ in G_{x_0} that is equal to a color assigned to the vertex $v(x_0+1, y, z)$ in G_{x_0+1}. Clearly, $T_{ij} = T_{ji}$. The chromatic polynomial for the slab is then given by the trace

$$P(sc[(L_x)_P \times (L_y)_{BCy} \times (L_z)_{BC_z}], q) = \mathrm{Tr}(T^{L_x}) \ . \tag{2.3}$$

Since T is a real symmetric matrix, there exists an orthogonal matrix A that diagonalizes T: $ATA^{-1} = T_{diag.}$. Let us denote the $\mathcal{N}$ eigenvalues of T as $\lambda_{T,j}$, $1 \le j \le \mathcal{N}$. Since T is a real non-negative matrix, we can apply the generalized Perron-Frobenius theorem [17, 18] to infer that T has a real maximal eigenvalue, which we denote $\lambda_{T,max}$. It follows that

$$\lim_{L_x\to\infty} [P(sc[(L_x)_P \times (L_y)_{BCy} \times (L_z)_{BC_z}], q)]^{1/L_x} = \lambda_{T,max} \ . \tag{2.4}$$

Now for the transverse slices G_{x_0} and G_{x_0+1}, denoted generically as $ts((L_z)_{BC_z})$, the chromatic polynomial has the form

$$P(G_{x_0}, q) = P(G_{x_0+1}, q) = \sum_j c_j \, (\lambda_{ts((L_z)_{BC_z}),j})^{L_y} \tag{2.5}$$

where the c_j are coefficients whose precise form is not needed here. The set of $\lambda_{ts((L_z)_{BC_z}),j}$'s is independent of the length L_y and although this set depends on BC_y, the maximal one (having the largest magnitude), $\lambda_{ts((L_z)_{BC_z}),max}$, is independent of BC_y (e.g., [16] and references therein). Hence,

$$\begin{aligned} \lim_{L_y\to\infty} [P(G_{x_0}, q)]^{1/L_y} &\equiv \lim_{L_y\to\infty} (\mathcal{N})^{1/L_y} \\ &= \lambda_{ts((L_z)_{BC_z}),max} \ . \end{aligned} \tag{2.6}$$

The two adjacent slices together with the edges in the x direction that join them constitute the graph $sc[2_F \times (L_y)_{BCy} \times (L_z)_{BC_z}]$. We denote the chromatic polynomial for this section (tube) of the sc lattice as $P(sc[2_F \times (L_y)_{BCy} \times (L_z)_{BC_z}], q)$ (which is equal to $P(sc[2_P \times (L_y)_{BCy} \times (L_z)_{BC_z}], q)$ because of the proper q-coloring condition). This has the form

$$\begin{aligned} & P(sc[2_F \times (L_y)_{BCy} \times (L_z)_{BC_z}], q) \\ &= \sum_j c'_j \, (\lambda_{tube((L_z)_{BC_z}),j})^{L_y} \end{aligned} \tag{2.7}$$

where c'_j are coefficients analogous to those in (2.5). Therefore,

$$\begin{aligned} \lim_{L_y\to\infty} \quad & [P(sc[2_F \times (L_y)_{BCy} \times (L_z)_{BC_z}], q)]^{1/L_y} = \\ &= \lambda_{tube((L_z)_{BC_z}),max} \ . \end{aligned} \tag{2.8}$$

Now let us denote the column sum (CS)

$$CS_j(T) = \sum_{i=1}^{\mathcal{N}} T_{ij} \ , \tag{2.9}$$