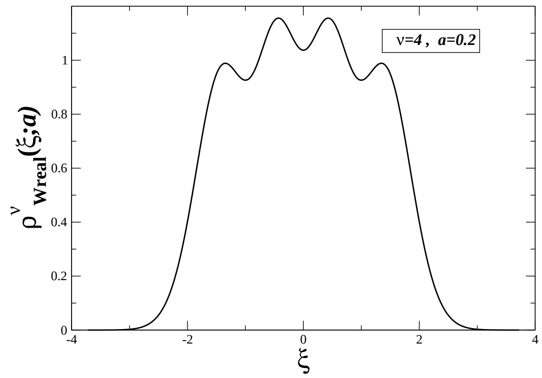

FIG. 3: The quenched density of the real eigenvalues of D_W.

Distribution of Tail States. For $|\hat{x} - \hat{m}|/\hat{a}^2 \gg 1$ and $8\hat{a}^2 \ll 1$ the tail of the spectral density inside the gap follows from a saddle point analysis. For $\hat{x} > 0$ we find

$$\rho_5(\hat{x}) \sim \exp[-(\hat{x} - \hat{m})^2/16\hat{a}^2], \tag{14}$$

and a similar result for $\hat{x} < 0$. This result applies to the tail of the blue and (marginally) of the red curve in Fig. 1. Reinstating physical parameters we find that the width parameter, σ, in (14) is given by $\sigma^2 = 8a^2W_8/(V\Sigma^2)$ so that, in the microscopic domain and sufficiently small $\hat{a}$, σ scales with $1/\sqrt{V}$. Such a scaling has been observed for $N_f = 2$ in [13, 22].

Tail states may also be studied in their own right and for applications in condensed matter physics. In particular, in the thermodynamic limit, $\hat{m}, \hat{x}, \hat{a}^2 \gg 1$, the average level density of D_5 can be obtained by a saddle point analysis. For $8\hat{a}^2/\hat{m} < 1$ it vanishes inside $[-\hat{x}_c, \hat{x}_c]$ with $\hat{x}_c$ given by $\hat{x}_c = 8\hat{a}^2\left[(\hat{m}/8\hat{a}^2)^{2/3} - 1\right]^{3/2}$. It has exactly the same form as for superconductors with magnetic impurities [2]. In the scaling limit where $V^{2/3}(x_c - x)$ is kept fixed, the spectral density can be computed inside the gap by a saddle point approximation of (5), and it agrees with universal Random Matrix Theory results for the so-called soft edge. Such universal behaviour has also been found in condensed matter systems [2, 3].

Conclusions. Using a graded chiral Lagrangian for Wilson fermions at finite lattice spacings, we have obtained an analytical form for the Wilson Dirac spectrum at fixed number, ν, of real eigenvalues. These results, and their extensions to dynamical fermions, should be useful for lattice simulations at finite volume. We have shown how the leading low-energy constant for Wilson fermions, W_8, can be extracted from lattice spectra of the Wilson Dirac operator in the ϵ-regime.

The problem can also be reformulated in terms of a new chiral Random Matrix Theory that describes spectral correlation functions of the Wilson Dirac operator in the appropriate scaling regime. These results open a new domain of Random Matrix Theory where chiral ensembles merge with Wigner-Dyson ensembles. Essential to this study is an analysis of the gap of the Wilson Dirac operator at finite mass. Lattice QCD simulations depend crucially on control of this gap and its variation as a function of the lattice spacing. As we have stressed, our results are not only important for understanding the finite lattice spacing effects of Wilson fermions near the chiral limit, but may have interesting applications to tail states in condensed matter systems.

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