TABLE I: Transition temperatures of the pure and random-bond 2d BC model obtained in this paper. Second column from reference [48], third and last entries of third and fifth columns from Ref. [37].

Δ/J	k_BT/J				
	Pure		Random		
	Ref. [48]		$r = 0.9/1.1$	$r = 0.75/1.25$	$r = 0.6/1.4$
0	1.695	1.693(3)		1.674(2)	
0.5	1.567	1.564(3)		1.547(2)	
1	1.398	1.398(2)		1.381(1)	
1.2					1.277(3)
1.4					1.184(3)
1.5	1.150	1.151(1)	1.149(1)	1.144(2)	1.131(2)
1.6			1.084(1)		1.071(3)
1.7			1.005(1)		
1.75		0.958(1)		0.960(2)	
1.8			0.908(1)		0.917(3)
1.9		0.769(1)	0.774(2)	0.786(4)	
1.95	0.650	0.659(2)		0.702(3)	
1.975		0.574(2)		0.626(2)	

the pure model and for the random version the disorder strength $r = 0.75/1.25$ was chosen in both cases. The strong violation of universality observed appeared to be the result of the softening of the first-order transition due to bond-randomness. Specifically, it was concluded that the new strong-disorder universality class is well described by a correlation length exponent in the range $\nu = 1.30(6) - 1.35(5)$, and exponent ratios γ/ν and β/ν very close to the Ising values 1.75 and 0.125, respectively [37]. The above weak universality [16, 53, 54] seems to be valid between the Ising-like continuous transitions of the random-bond 2d BC model for small values of Δ ($\Delta = 1.0$ in Ref. [37]) and continuous transitions belonging to the strong-disorder universality class.

Therefore, the strong-disorder universality class may be characterized by the above distinct value of the correlation length exponent and a strong saturation of the specific heat. Qualitatively this saturating behavior is quite instructive and for illustrative reasons is reproduced here in Fig. 3. This figure contrasts, at $\Delta = 1.975$, the specific heat's finite-size behavior of the pure 2d BC model (first-order regime) and two disordered cases corresponding to disorder strengths $r = 0.85/1.15$ and $r = 0.75/1.25$. The saturation of the specific heat is very clear in both cases of the disorder strength.

It is of interest to point out here that these findings for the strong-disorder universality class appear to be fully compatible with the classification of phase transitions in disordered systems proposed recently by Wu [55]. According to this classification the strong-disorder transition is expected to be inhomogeneous and percolative with an expected exponent of the order $\nu = 1.34$ [56]. Furthermore, it has been suggested to us by Wu [57], that the strong lack of self-averaging of this transition stems from the above properties. This violation of self-averaging, together with the strong finite-size effects, make the systematic MC approach of the strong-disorder regime very demanding, if not impractical. On the other hand, the weak regime (or Ising universality regime) suffers a much weaker lack of self-averaging, by at least a factor of ~ 12 [37], and a smooth behavior is observed at moderate lattice sizes. Thus, aiming here to observe, even approximately, the extent of the involved universality classes we carried out our study at moderate values of the crystal field and disorder, and found a behavior quite convincing from which the frontier of the strong universality class can be estimated by observing the disappearance of the expected 2d random Ising universality class.

B. Pure and random-bond 2d BC model: Range of universality with the 2d Ising model

Let us now proceed with the analysis of our numerical data for the disorder strengths and crystal fields given in Table I and observe and contrast their FSS behavior with that of the pure model. Starting this comparative study with the FSS of the specific heat maxima (using for the random-bond version at the strength $r = 0.75/1.25$ the corresponding quantity averaged over disorder, i.e. $[C]^*_{av}$), we present in Fig. 4 fitting attempts for the same range of Δ for the pure model [Fig. 4(a)] and the random-bond version [Fig. 4(b)]. As indicated by the scales in the x-axis and the functions in the corresponding panels, the expected Ising logarithmic divergence has been assumed for the pure model $C^* = C_1 + C_2 \ln L$, whereas the double-logarithmic divergence $[C]^*_{av} = C_1 + C_2 \ln(\ln L)$ has been assumed for the random version. Although, it is very difficult to ir-