Proof. Suppose $\partial\Omega$ has a cusp at $\gamma(s)$. Then, using the terminology of definition 4.2.4 and the fact that γ is arc length parameterized, we have $\Gamma^+ = -\Gamma^-$. We let $w = 0$ and note that $w = \Gamma^+ + \Gamma^- \in T_\gamma(s)$. Letting $u, v \in \partial\Omega \cap D(\gamma(s), r)$ with $u \neq v$, we have $\langle w, u - v \rangle = 0$, contradicting the fact that $\partial\Omega$ is tangent-cone graph-like. Therefore, $\partial\Omega$ has no cusps. □

4.2.5 TCGL Boundary Properties

The following technical lemmas allow us to bound various distances and areas encountered in tangent-cone graph-like boundaries.

Lemma 4.2.16. *Suppose that $\partial\Omega$ is tangent-cone graph-like with radius r and points $p_1, p_2 \in \partial\Omega$ with $d(p_1, p_2) < r$. Then one of the arcs (call it P) along $\partial\Omega$ between p_1 and p_2 is such that, for any two points $q_1, q_2 \in P$, we have $d(q_1, q_2) < r$.*

Proof. Note that $p_2 \in D(p_1, r)$ so that there is an arc along $\partial\Omega$ from p_1 to p_2 which is fully contained in the interior of $D(p_1, r)$ by theorem 4.2.1. We will call this arc P.

For all x on P, let P_x denote the subpath of P from p_1 to x (so $P = P_{p_2}$). We claim that P_x is contained in $D(x, r)$ for all x on P (thus, P is contained in $D(p_2, r)$). Indeed, if this were not the case, then there must be some $\hat{x}$ on P such that $P_{\hat{x}}$ is contained in $D(\hat{x}, r)$ but $C(\hat{x}, r) \cap P_{\hat{x}}$ is nonempty (i.e., we can move the disk along P until some part of the subpath hits the boundary). That is, the subpath $P_{\hat{x}}$ has a tangency with the disk $D(\hat{x}, r)$ which is impossible because of theorem 4.2.1.

Let $q_1 \in P$ and note that since P_x is contained in $D(x, r)$ for all x on P, we have that P is contained in $D(q_1, r)$. Therefore, $d(q_1, q_2) < r$ for all $q_1, q_2 \in P$ as desired. □

Lemma 4.2.17. *If $q_1 = \gamma(s_1), q_2 = \gamma(s_2) \in P$ where P is as in the previous lemma, then the arc length between q_1 and q_2 along P is at most $\sqrt{2}d(q_1, q_2)$.*