to Remark 6, we can consider in $\mathbb{R}^{p\times q}\times\mathbb{R}^{q\times q}$ the topology induced by the norm

$$|||(\mathbf{A},\mathbf{V})||| = \sup\{\|\mathbf{A}\|_2, \|\mathbf{V}\|\}. \tag{A.5}$$

Given $(\mathbf{B}_0,\boldsymbol{\Sigma}_0)$ in $\mathbb{R}^{p\times q}\times\mathbb{R}^{q\times q}$, the proof consists in find an upper bound of $|d^2(\mathbf{B},\boldsymbol{\Sigma}) - d^2(\mathbf{B}_0,\boldsymbol{\Sigma}_0)|$ that tends to 0 when $|||(\mathbf{B},\boldsymbol{\Sigma})-(\mathbf{B}_0,\boldsymbol{\Sigma}_0)||| \to 0$.

Adding and subtracting $d^2(\mathbf{B}_0,\boldsymbol{\Sigma})$ we have that

$$\left|d^2(\mathbf{B},\boldsymbol{\Sigma}) - d^2(\mathbf{B}_0,\boldsymbol{\Sigma}_0)\right| \le \left|d^2(\mathbf{B},\boldsymbol{\Sigma}) - d^2(\mathbf{B}_0,\boldsymbol{\Sigma})\right| + \left|d^2(\mathbf{B}_0,\boldsymbol{\Sigma}) - d^2(\mathbf{B}_0,\boldsymbol{\Sigma}_0)\right|.$$

Let $r(\mathbf{B}_0) = (\mathbf{y} - \mathbf{B}_0'\mathbf{x})$. Using basic tools from linear algebra we obtain

$$\left|d^2(\mathbf{B},\boldsymbol{\Sigma}) - d^2(\mathbf{B}_0,\boldsymbol{\Sigma})\right| \le q|||(\mathbf{B},\boldsymbol{\Sigma})-(\mathbf{B}_0,\boldsymbol{\Sigma}_0)|||\left(\|\mathbf{x}\| + 2\left\|r(\mathbf{B}_0)\right\|\right)\|\mathbf{x}\|\lambda_1(\boldsymbol{\Sigma}^{-1}),$$

and by Weyl's Perturbation Theorem (see [4], pg. 63), we have that

$$\lambda_1(\boldsymbol{\Sigma}^{-1}) = \frac{1}{\lambda_q(\boldsymbol{\Sigma})} < \frac{1}{\lambda_q(\boldsymbol{\Sigma}_0) - |||(\mathbf{B},\boldsymbol{\Sigma})-(\mathbf{B}_0,\boldsymbol{\Sigma}_0)|||}, \tag{A.6}$$

combining these inequalities we obtain a bound of $|d^2(\mathbf{B},\boldsymbol{\Sigma}) - d^2(\mathbf{B}_0,\boldsymbol{\Sigma})|$ that tends to 0 when $|||(\mathbf{B},\boldsymbol{\Sigma})-(\mathbf{B}_0,\boldsymbol{\Sigma}_0)||| \to 0$.

If $r(\mathbf{B}_0) = \mathbf{0}$ the lemma is proved, otherwise using the Cauchy-Schwarz inequality and (A.6) we have

$$\left|d^2(\mathbf{B}_0,\boldsymbol{\Sigma}) - d^2(\mathbf{B}_0,\boldsymbol{\Sigma}_0)\right| \le \frac{\|\boldsymbol{\Sigma}_0^{-1}\|\|r(\mathbf{B}_0)\|^2|||(\mathbf{B},\boldsymbol{\Sigma})-(\mathbf{B}_0,\boldsymbol{\Sigma}_0)|||}{\lambda_q(\boldsymbol{\Sigma}_0) - |||(\mathbf{B},\boldsymbol{\Sigma})-(\mathbf{B}_0,\boldsymbol{\Sigma}_0)|||},$$

which completes the proof. ∎

Proof of Theorem 1: By Lemma 9, it sufices to show that there exist t_1 and t_2