from the existence of nonlinearity in the system. The dispersion relation of effective phonons corresponding to the effective Hamiltonian (5) can be written in the form

$$\omega_k = 2\sqrt{f}\sin\frac{k}{2}. \tag{6}$$

It is then straightforward to calculate the dimensionless nonlinearity ϵ and the specific heat C from gaussian averages of SCPT, which yields

$$\epsilon = \frac{f-K}{f+K}, \tag{7}$$

$$C = \frac{3}{4} + \frac{K}{4\sqrt{K^2+12\lambda T}}. \tag{8}$$

The derivation of the effective phonon spectrum is based on the generalized equipartition theorem,

$$k_B T = \left\langle q_k \frac{\partial H}{\partial q_k} \right\rangle_c, \tag{9}$$

where the bracket $\langle \cdot \rangle_c$ stands for a thermal average in the canonical ensemble and q_k denotes the Fourier transform of the coordinate x_i. The next step consists of an approximative transformation of the right hand side of Eq. (9) into a more compact form (see [3, 12] for details),

$$k_B T = \tilde{\omega}_k^2 \left\langle q_k^2 \right\rangle_c. \tag{10}$$

Specifically, the effective phonon spectrum $\tilde{\omega}_k$ for the FPU-β model reads as

$$\tilde{\omega}_k = 2\sqrt{\alpha}\sin\frac{k}{2}. \tag{11}$$

Here α is given by

$$\alpha = K + \lambda\frac{\langle \sum_i (x_{i+1}-x_i)^4\rangle_c}{\langle \sum_i (x_{i+1}-x_i)^2\rangle_c} = \frac{2\lambda T Y_{1/4}(x)}{K[Y_{3/4}(x) - Y_{1/4}(x)]}, \tag{12}$$

where $Y_{1/4}(x)$ and $Y_{3/4}(x)$ are the modified Bessel functions of the second kind and $x \equiv K^2/(8\lambda T)$. The parameters for the nonlinearity $\tilde{\epsilon}$ and the specific heat $\tilde{C}$ that follow from EPT can be expressed in the form

$$\tilde{\epsilon} = \frac{K^2[Y_{1/4}(x) - Y_{3/4}(x)] + 2\lambda T Y_{1/4}(x)}{K^2[Y_{3/4}(x) - Y_{1/4}(x)] + 2\lambda T Y_{1/4}(x)}, \tag{13}$$

$$\tilde{C} = \frac{3}{4} + \frac{K^2}{16\lambda T}\frac{Y_{3/4}(x)}{Y_{1/4}(x)} + \frac{K^4}{64\lambda^2 T^2}\left(1 - \frac{Y_{3/4}(x)^2}{Y_{1/4}(x)^2}\right). \tag{14}$$