The r_σ part can be absorbed into a redefinition of the σ coordinate and we essentially have $r = r(\lambda)$. These simplify the equations to give (up to $2nd$ order),

$$\dot{f}_\lambda = \frac{1}{r^2}\left[\left(f''_\sigma + f'^2_\sigma\right) + 3f'^2_\sigma\right] - \frac{\alpha'}{r^4}\left[\left(f''_\sigma + f'^2_\sigma\right)^2 + 3f'^4_\sigma\right] \tag{31}$$

$$\dot{r} = \frac{4}{r}\left[f''_\sigma + f'^2_\sigma\right] - \frac{4\alpha'}{r^3}\left[f''_\sigma + f'^2_\sigma\right]^2 \tag{32}$$

Since the equations are now in a variable separable form, for consistency at the leading order we must have $f''_\sigma + f'^2_\sigma = const_1$ and $(f''_\sigma + f'^2_\sigma) + 3f'^2_\sigma = const_2$; i.e. $f'_\sigma = k = const.$ and $f_\sigma = k\sigma$. Note that this condition comes only from the leading order terms. The equations in λ become $\frac{r^2}{e^{2f_\lambda}}(\dot{e^{2f_\lambda}}) = (\dot{r^2})$, which gives $e^{2f_\lambda} = r^2 = \Omega(\lambda) \geq 0$.

Thus we have a solution with the metric Eq.(26) which is conformal to the Anti-de Sitter metric, as stated in Prop.IV.1. The Ricci and Kretschmann scalars, $R = g^{ij}R_{ij}$ and $K = R_{ijkl}R^{ijkl}$ respectively, for such a metric are given by –

$$R = -\frac{4}{r^2}\left[2f'' + 5f'^2\right] = -\frac{20k^2}{\Omega} \tag{33}$$

$$K = \frac{8}{r^4}\left[2\left(f'' + f'^2\right)^2 + 3f'^4\right] = \frac{40k^4}{\Omega^2} \tag{34}$$

Thus the curvature diverges and the manifolds become singular at $\Omega = 0$.

Using the separable form for the functions in Eq.(5), we get (at third order),

$$\begin{aligned}
\beta^{(3)}_{\mu\mu} &= \frac{e^{2f}}{4r^6}\left(-32f'^6_\sigma - 16f'^4_\sigma f''_\sigma - 12f'^2_\sigma f''^2_\sigma - 14f''^3_\sigma + 4f'^3_\sigma f^{(3)}_\sigma + 8f'_\sigma f''_\sigma f^{(3)}_\sigma + {f^{(3)}_\sigma}^2\right)\\
\beta^{(3)}_{\sigma\sigma} &= \frac{1}{r^4}\left(-8f'^6_\sigma - 12f'^4_\sigma f''_\sigma + 3f'^2_\sigma f''^2_\sigma - 3f''^3_\sigma + 5f'^3_\sigma f^{(3)}_\sigma + 6f'_\sigma f''_\sigma f^{(3)}_\sigma + {f^{(3)}_\sigma}^2 + f'^2_\sigma f^{(4)}_\sigma + f''_\sigma f^{(4)}_\sigma\right)
\end{aligned} \tag{35}$$

where $F^{(n)} = \frac{d^nF}{d\sigma^n}$. But, from the separability at leading order, we already have $f' = k$ and thus both the terms reduce to $e^{-2f}r^6\beta^{(3)}_{\mu\mu} = r^4\beta^{(3)}_{\sigma\sigma} = -8k^6$. Similarly all higher derivative terms vanish at order 4 leaving $e^{-2f}r^8\beta^{(4)}_{\mu\mu} = r^6\beta^{(4)}_{\sigma\sigma} = 2\left(3 + 5\zeta(3)\right)k^8$.

This leads to the following ODE for Ω –

$$\frac{1}{8k^2}\frac{d\Omega}{d\lambda} = 1 - \frac{\alpha' k^2}{\Omega} + 2\left(\frac{\alpha' k^2}{\Omega}\right)^2 - \frac{3 + 5\zeta(3)}{2}\left(\frac{\alpha' k^2}{\Omega}\right)^3 \tag{36}$$

We can readily see that $k = 0$ is a fixed point of the flow. This corresponds to a 5