and isotropic about every point and, therefore, (anti-)de Sitter.

THE JEBSEN-BIRKHOFF THEOREM IN SCALAR-TENSOR GRAVITY

With the advent of the Jordan [31] and Brans-Dicke [32] theories first and of scalar-tensor theories later [33], the validity of the Jebsen-Birkhoff theorem was investigated in alternative gravity [34–39, 41–46].

In general, the theorem does not hold in scalar-tensor gravity: one needs to impose that the effective stress-energy tensor $T_{ab}^{(\phi)}$ of the Brans-Dicke-like scalar field of the theory is time-independent in order for the metric to be static (this requirement is usually achieved by imposing that ϕ is time-independent, but is also obtained by a stealth field ϕ [29, 30]). The failure of the theorem in the presence of time-dependent scalars opens the door for new phenomenology in scalar-tensor gravity which is unknown in General Relativity. The failure of the Jebsen-Birkhoff theorem is to be expected: since scalar-tensor gravity has a new spin zero degree of freedom in comparison with General Relativity, scalar monopole radiation can occur. In Einstein's theory monopole radiation is forbidden by the Jebsen-Birkhoff theorem, which is a consequence of the fact that the gravitational field is represented only by a spin two field. Since in Einstein's theory gravitational radiation is necessarily quadrupole to lowest order, spherically symmetric pulsating sources cannot generate gravitational radiation and the metric must be static. This is no longer true in scalar-tensor gravity, in which the time-varying monopole moment of a radially pulsating spherical source generates propagating scalar radiation which makes also the metric non-static.

Let us consider a scalar-tensor theory of gravity described by the action

$$\begin{aligned} S_{ST} &= \frac{1}{16\pi}\int d^4x\,\sqrt{-g}\left[\phi R-\frac{\omega(\phi)}{\phi}\,g^{ab}\nabla_a\phi\,\nabla_b\phi-V(\phi)\right] \\ &+ S^{(m)}\,. \end{aligned} \tag{16}$$

The field equations can be written in the form of effective Einstein equations as

$$\begin{aligned} R_{ab}-\frac{1}{2}\,g_{ab}R &= \frac{8\pi}{\phi}\,T_{ab}^{(m)} \\ &+ \frac{\omega(\phi)}{\phi^2}\left(\nabla_a\phi\nabla_b\phi-\frac{1}{2}\,g_{ab}\nabla^c\phi\nabla_c\phi\right) \\ &+ \frac{1}{\phi}\left(\nabla_a\nabla_b\phi-g_{ab}\Box\phi\right)-\frac{V(\phi)}{2\phi}\,g_{ab} \\ &\equiv \frac{8\pi}{\phi}\left(T_{ab}^{(m)}+T_{ab}^{(\phi)}\right)\,, \end{aligned} \tag{17}$$

$$(2\omega+3)\,\Box\phi = 8\pi T^{(m)}-\frac{d\omega}{d\phi}\,\nabla^c\phi\nabla_c\phi+\phi\,\frac{dV}{d\phi}-2V\,, \tag{18}$$

where $T_{ab}^{(m)}$ is the matter energy-momentum tensor and we assume that $\phi>0$ in conjunction with $\omega>-3/2$ to guarantee the positivity of the effective gravitational coupling [47]

$$G_{eff}=\frac{2\,(\omega+2)}{2\omega+3}\,\frac{1}{\phi}\,. \tag{19}$$

In this form, it is easy to see that the scalar field ϕ acts as an effective form of matter in the field equations (17) and therefore, by imposing that the matter stress-energy tensor $T_{ab}^{(m)}$ vanishes, one is left with an effective stress-energy tensor $T_{ab}^{(\phi)}$ such that $T_{00}^{(\phi)}$ could be time-dependent and $T_{0i}^{(\phi)}\neq 0$ if ϕ depends on time. In other words, a time-dependent Brans-Dicke-like field ϕ spoils the validity of the Jebsen-Birkhoff theorem and only the assumption that ϕ is time-independent (or that $T_{ab}^{(\phi)}$ is static, as for a time-dependent stealth field ϕ) restores the staticity of a spherically symmetric solution. This can be checked explicitly using the field equations, which we do in the following.

The trivial case ϕ =constant

The case ϕ =const.$\equiv\phi_0>0$ is trivial and eq. (17) reduces to

$$R_{ab}-\frac{1}{2}\,g_{ab}R=\frac{8\pi}{\phi_0}\,T_{ab}^{(m)}-\frac{V_0}{2\phi_0}\,g_{ab} \tag{20}$$

where $V_0\equiv V(\phi_0)$, so that the theory degenerates to General Relativity with the cosmological constant $\Lambda\equiv V_0/(2\phi_0)$. If $T_{ab}^{(m)}$ is such that the energy distribution is static (including the case $T_{ab}^{(m)}=0$), version 1 of the Jebsen-Birkhoff theorem holds and the metric is static in the region in which the coordinate gradients preserve their causal character. The same happens if $T_{ab}^{(\phi)}$ vanishes with $\phi\neq$const.