Since $T_{l',l}$ depends only on the difference between l and l', we can write

$$T_{l',l} = T_{l'-l} \equiv T_j = \frac{1}{N} e^{i\frac{2\pi}{N}\frac{\phi}{\phi_0}j} \sum_{n=0}^{N-1} e^{i\frac{2\pi}{N}nj} \varepsilon_n(\phi) \; . \quad (59)$$

Inverting Eq. (59) we get

$$\varepsilon_n(\phi) = \sum_{j=0}^{N-1} \; e^{-ik_n(\phi)ja} T_j \; . \quad (60)$$

By means of Eq. (60), the ground state energy of the fully filled valence band is readily obtained in the form

$$E(\phi) = \sum_{n=0}^{N-1} \varepsilon_n(\phi) = N \cdot T_0(\phi) \; . \quad (61)$$

Finally, the persistent current at full-filling reads

$$I \; = \; -\frac{dE}{d\phi} \; = \; -N\frac{dT_0}{d\phi} \; . \quad (62)$$

One can see from Eq. (57) that the matrix element T_0 represents the on-site energy of the Wannier state $w_l(x)$. Since $w_l(x)$ depends on magnetic flux, so does T_0. This however means that the current (62) is in general not zero. We have thus arrived at the same conclusion as in the preceding section, where the persistent current in the insulating ring (Eq. 35) was derived from the Bloch states and general recipe.

C. Ring Wannier functions in the basis of crystal Wannier functions

The result (47), or equivalently the result (35), can be obtained from the result (62), if the ring Wannier functions are expanded over the basis of the crystal Wannier functions. To show this we proceed as follows.

The Bloch functions of the ring are given by Eq. (38) and normalized as $\int_0^{Na} \; \varphi_k^*(x)\varphi_k(x) \; dx = 1$, where $k = k_n(\phi)$ [see Eq. (48)]. The Bloch functions $\phi_k(x)$ coincide with $\varphi_k(x)$ except for the normalization constant. As they are normalized by means of Eq. (40), in the ring we have the relation

$$\varphi_k(x) = \frac{1}{\sqrt{N}}\phi_k(x) \; , \qquad k = k_n(\phi) \; , \quad (63)$$

We set for $\phi_k(x)$ in Eq. (63) the definition (43). Then we set Eq. (63) into the right hand side of Eq. (49). We obtain

$$w_l(x) = \frac{1}{N}\sum_k \sum_{l'=-\infty}^{\infty} e^{-ik(l-l')a} W_{l'}(x) \; , \quad (64)$$

where $k = k_n(\phi)$ and summation over k means the summation over $n = 0, 1, \ldots, N-1$. Using substitution $l' = Nr+s$, where $s = 0, 1, \ldots (N-1)$ and $r = 0, \pm1, \pm2, \cdots \pm \infty$, we further obtain

$$w_l(x) = \sum_{r=-\infty}^{\infty} \sum_{s=0}^{N-1} \; W_{Nr+s}(x) e^{i\frac{2\pi}{N}\frac{\phi}{\phi_0}(rN+s-l)} \times \frac{1}{N}\sum_{n=0}^{N-1} e^{i\frac{2\pi}{N}n(s-l)} \; . \quad (65)$$

Eventually

$$w_l(x) = \sum_{r=-\infty}^{\infty} e^{i2\pi\frac{\phi}{\phi_0}r} \; W_{Nr+l}(x) \; . \quad (66)$$

Equation (66) expresses the ring Wannier function at the ring site l, $w_l(x)$, through the crystal Wannier functions $W_{l'}(x)$. For $\phi = 0$ equation (66) coincides with equation (5.9) in Kohn's paper [17].

Setting Eq. (66) into Eq. (57) we obtain

$$\begin{aligned} T_j &= \int_0^{Na} \; w_j^*(x) \; \hat{H} \; w_0(x) \; dx \\ &= \sum_{r=-\infty}^{\infty} \sum_{r'=-\infty}^{\infty} e^{i2\pi\frac{\phi}{\phi_0}(r-r')} \int_0^{Na} W_{Nr'+j}(x) \hat{H} W_{Nr}(x) dx \\ &= \sum_{s=-\infty}^{\infty} e^{-i2\pi\frac{\phi}{\phi_0}s} \sum_{r=-\infty}^{\infty} \int_{0-rNa}^{Na-rNa} W_{Ns+j}(x)\hat{H}W_0(x)dx \\ &= \sum_{s=-\infty}^{\infty} \; e^{-i2\pi\frac{\phi}{\phi_0}s} \int_{-\infty}^{\infty} \; W_{Ns+j}(x)\hat{H}W_0(x)dx \; . \end{aligned} \quad (67)$$

By means of the matrix elements Γ_j, equation (67) can be written as

$$T_j = \sum_{s=-\infty}^{\infty} \; e^{-2\pi i\frac{\phi}{\phi_0}s} \Gamma_{sN+j} \; . \quad (68)$$

For $j = 0$ equation (68) gives

$$T_0 = \Gamma_0 + 2\sum_{s=1}^{\infty} \; \Gamma_{sN} \cos(2\pi\frac{\phi}{\phi_0}s) \; . \quad (69)$$

We substitute Eq. (69) into the right hand side of Eq. (62). We obtain the persistent current at full filling in the form

$$I = -\frac{4\pi N}{\phi_0}\sum_{s=1}^{\infty} \; s \; \Gamma_{sN} \; \sin(2\pi\frac{\phi}{\phi_0}s) \; , \quad (70)$$

or equivalently in the form (47) which coincides with the result (35) for $a_j = 2\Gamma_j$.

In the rest of this section we discuss the physical meaning of the coefficients Γ_N, Γ_{2N}, etc., appearing in expression (47). Since $a_j = 2\Gamma_j$, the meaning of the coefficients a_N, a_{2N}, etc., is the same and does not need an extra discussion.