Haar measure $dU_r = (8\pi^3)^{-1}\sin(2\phi)d\phi d\psi d\chi d\alpha$ [11, 15, 16]. In order to evaluate the entanglement set within the system, we use Wootters' concurrence [17]. For a general bipartite pure or mixed state described by the density matrix ϱ, concurrence is given by

$$\mathcal{C} = \max[0, \sqrt{\lambda_1} - \sum_{k=2}^{4}\sqrt{\lambda_k}] \tag{4}$$

where $\lambda_1 \geq \lambda_j$ $(j = 2,3,4)$ are the eigenvalues of $\rho\,(\sigma_2\otimes\sigma_2)\,\rho^*\,(\sigma_2\otimes\sigma_2)$. When A is prepared in its ground state and upon evolution of the $e-A$ system as $\varrho = \hat{U}_h\hat{U}_r\rho\hat{U}_r^\dagger\hat{U}_h^\dagger$, we get the density matrix

$$\varrho = \begin{pmatrix} C^2(\phi) & -\frac{i}{2}e^{-2iJt+i(\psi+\chi)}S(2Jt)S(2\phi) & -\frac{1}{2}e^{-2iJt+i(\psi+\chi)}C(2Jt)S(2\phi) & 0 \\ \frac{i}{2}e^{2iJt-i(\psi+\chi)}S(2Jt)S(2\phi) & S^2(\phi)S^2(2Jt) & -\frac{i}{2}S(4Jt)S^2(\phi) & 0 \\ -\frac{1}{2}e^{2iJt-i(\psi+\chi)}C(2Jt)C(2\phi) & \frac{i}{2}S(4Jt)S^2(\phi) & C^2(2Jt)S^2(\phi) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5}$$

with $C(x) = \cos(x)$ and $S(x) = \sin(x)$. It is then straightforward to see that the concurrence shared by e and A after the application of $\hat{U}_r$ and $\hat{U}_h$ results in the elegant expression

$$\mathcal{C}_{eA} = \sin^2(\phi)|\sin(4Jt)|. \tag{6}$$

This shows that, given a specific preparation of the environment, the only parameters governing the entanglement are ϕ and Jt. The typical value of $\mathcal{C}_{eA}$ is obtained by averaging the above expression over any possible unitary matrix U_r uniformly drawn according to the proper Haar measure. Explicitly, we have to calculate

$$\begin{aligned}\overline{\mathcal{C}}_{eA} &= \frac{1}{8\pi^3}\int_0^{\pi/2}\sin(2\phi)\,d\phi\int_0^{2\pi}d\psi\int_0^{2\pi}d\chi\int_0^{2\pi}d\alpha\,\mathcal{C}_{eA} \\ &= \frac{1}{2}\left|\sin(4J\tau)\right|,\end{aligned} \tag{7}$$

which may seem a special result arising from the pure-state preparation of the ancilla. However, this is definitely not the case, as we now demonstrate. By starting with the mixed ancilla state

$$\rho_A = \begin{pmatrix}\rho_0 & 0 \\ 0 & 1-\rho_0\end{pmatrix}, \quad \rho_0 \in [0,1] \tag{8}$$

and calculating the evolution of the $e-A$ system, one gets a density matrix that is an easy generalization of Eq. (5). By inspecting the eigenvalues of $\rho\,(\sigma_2\otimes\sigma_2)\,\rho^*\,(\sigma_2\otimes\sigma_2)$, whose explicit form is too lenghty to be reported here, one finds no dependence on χ, ψ or α, in full analogy with the case of Eq. (7). The explicit calculation of concurrence leads us to the expression

$$\mathcal{C}_{eA} = \frac{1}{2}|[-1+(2\rho_0-1)\cos(2\phi)]\sin(4Jt)|. \tag{9}$$

As $\int_0^{\pi/2}\cos(2\phi)\sin(2\phi)d\phi = 0$, the average concurrence (calculated using the appropriate Haar measure, as done before) turns out to be identical to Eq. (6). The study can be straightforwardly generalised to the case of qubit A being prepared in any coherent-superposition state, the only difficulty being a slightly more complicated expression for the concurrence corresponding to any set preparation of E. The message, however, is rather clear: regardless of the state into which the ancilla is prepared, the typical $e-A$ entanglement is simply set by the rescaled interaction time Jt. This is strictly valid only in the statistical sense: if specific instances of preparation of both A and e are taken, such an independence does not hold anymore. However, contrary to a naive expectation, the typical entanglement does not vanish. As we see later, this result is the key to understand what occurs in the two-environment case.

We now approach the second step of our proof by studying the invariance of the $e-A$ entanglement with respect to $\dim(E)$ when A is prepared in a pure state. For this task, we consider a simple extension of the previous case to a two-qubit environment $E = \{e_1, e_2\}$, initially prepared in a state described by

$$\begin{aligned}\rho_E = \frac{1}{4}(&\mathbb{1}_{e_1}\otimes\mathbb{1}_{e_2} + \sum_{k=1}^{3}\beta_{k,e_2}\mathbb{1}_{e_1}\otimes\hat{\sigma}_{k,e_2} \\ &+ \sum_{k=1}^{3}\beta_{k,e_1}\hat{\sigma}_{k,e_1}\otimes\mathbb{1}_{e_2} + \sum_{k,l=1}^{3}\chi_{kl}\hat{\sigma}_{k,e_1}\otimes\hat{\sigma}_{l,e_2})\end{aligned} \tag{10}$$

with $\boldsymbol{\chi}$ the elements of the tensor accounting for the correlations between e_1 and e_2 and $\boldsymbol{\beta}_{e_j}$ $(j=1,2)$ the Bloch vector of qubit e_j $(j = 1,2)$ [9, 18]. This form holds for both entangled and separable two-qubit states and is thus a formal description of an arbitrary preparation of E. By taking $e \equiv e_2$ (an arbitrary choice that does not affect the generality of our discussion) we follow the recipe for evolution described above. This time, before calculating the $e-A$ concurrence, we have to trace over e_1's degrees of freedom. Through a tedious but otherwise straightforward calculation, we see that although the tripartite $E-A$ density matrix depends on $\boldsymbol{\chi}$ and $\boldsymbol{\beta}_{e_1}$, the reduced density matrix of the $e-A$ system only depends