for the first particle [10], we obtain [11] the equation for $G_{(t,t+s)}(s)$

$$G_{(t,t+s)}(s) = \int_0^t G_O(G_{(t-u,t-u+s)}(s))\psi(u)du$$
$$+s\int_t^{t+s} G_O(G_{(0,t-u+s)}(s))\psi(u)du + \int_{t+s}^{\infty}\psi(u)du. \tag{2}$$

In the limit of $s \to 0$ the generating function goes to $P(Z(t,t+s)=0) := P_E(t,s)$, i.e. the probability that not a single particle appears on the boundary at the mentioned time interval, taking this limit in Eq. (2) we obtain

$$P_E(t,s) = \int_0^t \frac{1-p}{1-pP_E(t-u,s)}\psi(u)du + \int_{t+s}^{\infty}\psi(u)du, \tag{3}$$

where we have now used the explicit from of $G_O(s)$. Eq (3), a non-linear Volterra equation of the second kind, is our main equation for the unrestricted case of the model and it describes the time evolution of a two-time quantity. We haven't assumed anything as to the form of $\psi(t)$ and so Eq. (3) is quite general. The long-time behavior of P_E is, as we shall see, governed entirely by the long time behavior of $\psi(t)$. For normal diffusion in the bulk, this long-time behavior is $\psi_t \sim \psi_\infty t^{-(1+\beta)}$, with $\beta = 1/2$, $\psi_\infty = 1/\sqrt{4\pi D}$, where D is the diffusion constant (defined as usual). We also consider the more general case of $0 < \beta < 1$, and in such case the constant ψ_∞ is used for the normalization of ψ_t. The case of $0 < \beta < 1$ can be achieved if the diffusion in the bulk is anomalous [12]; e.g., subdiffusion for $0 < \beta < 1/2$ described by models such as the continuous time random walk (CTRW) [12, 13]. Herein we study P_E exclusively; other quantities such as the mean number of particles can also be obtained from Eq. (2) [11].

We will now explore the behavior of $P_E(t,s)$ in the asymptotic limit of large t for different values of the parameters p and β. Inspired by numerical solutions of Eq. (3), we adopt the ansatz

$$P_E(t,s) = P_E^{\infty} - A(s)t^{-\alpha} \qquad t \to \infty \tag{4}$$

In order to obtain a solution of Eq. (3) we substitute the asymptotic form in Eq. (4) into both sides of Eq. (3), perform an expansion in inverse powers of t and compare the coefficients in front of the appropriate leading terms. We leave the exact technical details for a longer publication [11] and now provide only the final results.

So doing, in the limit $t \to \infty$, s fixed, we obtain

$$P_E^{\infty} = \begin{cases} \frac{1-p}{p} & p \geq 1/2 \\ 1 & p \leq 1/2 \end{cases} \tag{5}$$

which clearly points out the existence of a critical $p = p_c = 1/2$. The location of the critical point at $p = 1/2$ is due to the fact that at this value, each particle on the boundary produces exactly one offspring. Below this point, the number of particles decreases exponentially with the number of past boundary particles, and above it it grows exponentially. The behavior of $A(s)$ and α for the off-critical state, $p \neq p_c$, is given by

$$A(s) = \begin{cases} \frac{1-p}{1-2p}\frac{\psi_0}{\beta}s & p < 1/2 \\ -1 & p > 1/2 \end{cases}, \tag{6}$$

$$\alpha = \begin{cases} 1+\beta & p < 1/2 \\ \beta & p > 1/2 \end{cases}. \tag{7}$$

The presence of the phase transition as we approach $p = 1/2$ from below is clearly manifest in the divergence of the coefficient A. The approach from above is not obvious from the large t behavior. What happens is that for $p \gtrsim 1/2$, P_E first rises toward unity, as happens below the transition. However, at very large t, the behavior crosses over toward the power-law decay toward P_E^{∞}. The details of this crossover will be presented in our longer presentation [11].

In the critical state, $p = p_c$ Eq. (5) remains valid and so $P_E^{\infty} = 1$ while the behavior of $A(s)$ and α shows a transition as a function of β. For α, we obtain

$$\alpha = \begin{cases} 1-\beta & \beta \leq 1/2 \\ \beta & \beta \geq 1/2 \end{cases}, \tag{8}$$

We can exhibit an analytic expression for $A(s)$ at p_c only for $\beta > 1/2$:

$$A(s) = -\frac{\pi\csc(\beta\pi)\Gamma(1-b)}{\Gamma(1-2\beta)\Gamma(1+\beta)} \qquad (\beta > 1/2) \tag{9}$$

For $\beta < 1/2$, $A(s)$ has to be calculated numerically in general. However, in the limit of large s, we have

$$A(s) \propto s^{1-\beta} \qquad (\beta < 1/2). \tag{10}$$

The existence of a special β for the behavior at criticality is very non-trivial and it is especially interesting that the critical β_c is equal to 1/2, i.e the normal diffusion case. We can treat this critical β_c as a critical fractal dimension since 2β is just the fractal dimension of the diffusion. For the special case of $\beta = 1/2$ our ansatz, i.e. Eq. (4), does not work and one needs to treat this case specially [11]; the result is

$$1 - P_E(t,s) \propto \frac{1}{\log(s/t)}\left(s/t\right)^{1/2} \qquad (\beta = 1/2). \tag{11}$$

The logarithmic corrections in the behavior support our claims as to the critical nature of $\beta = 1/2$. For a two time quantity like the survival probability $P_S(t,s) = 1 - P_E(t,s)$, one usually expects for a stationary process a dependence only on the time difference s; when the process is non-stationary this is generally not true. When the time dependence is that of a ratio of the two