where γ is the Euler constant. The Bethe logarithm is defined as

$$\ln(k_0) = \frac{\left\langle \sum_a \vec{p}_a \,(H_0 - \mathcal{E}_0)\, \ln\big[2\,(H_0 - \mathcal{E}_0)\big]\, \sum_b \vec{p}_b \right\rangle}{2\,\pi\, Z \left\langle \sum_c \delta^3(r_c) \right\rangle}. \tag{13}$$

The operator $H^{(5)}_{\rm fs}$ is the anomalous magnetic moment correction to the spin-dependent part of the Breit-Pauli Hamiltonian. $H^{(5)}_{\rm fs}$ does not contribute to the energies of the singlet states and to the spin-orbit averaged levels but it yields the $m\alpha^5$ contribution to the fine structure splitting. It is given by

$$\begin{aligned} H^{(5)}_{\rm fs} &= \frac{Z}{4\pi}\sum_a \vec{\sigma}_a \cdot \frac{\vec{r}_a}{r_a^3} \times \vec{p}_a \\ &+ \sum_{a<b}\biggl\{ \frac{1}{4\,\pi}\frac{\sigma_a^i\,\sigma_b^j}{r_{ab}^3}\left(\delta^{ij} - 3\,\frac{r_{ab}^i\, r_{ab}^j}{r_{ab}^2}\right) \\ &+ \frac{1}{4\,\pi\, r_{ab}^3}\big[2\left(\vec{\sigma}_a\cdot\vec{r}_{ab}\times\vec{p}_b - \vec{\sigma}_b\cdot\vec{r}_{ab}\times\vec{p}_a\right) \\ &+ \left(\vec{\sigma}_b\cdot\vec{r}_{ab}\times\vec{p}_b - \vec{\sigma}_a\cdot\vec{r}_{ab}\times\vec{p}_a\right)\big]\biggr\}. \end{aligned} \tag{14}$$

We note that despite the presence of terms with $\ln Z$ in Eq. (10), the correction $\mathcal{E}^{(5)}_\infty(\text{nlog})$ does not have logarithmic terms in its $1/Z$ expansion.

The recoil correction $\mathcal{E}^{(5)}_M$ consists of four parts [10],

$$\mathcal{E}^{(5)}_M = \frac{m}{M}\left(\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3\right) + \langle H^{(5)}_{\rm fs,rec}\rangle\,, \tag{15}$$

where

$$\mathcal{E}_1 = -3\,\mathcal{E}^{(5)}_\infty + \frac{4Z}{3}\sum_a \langle\delta^3(r_a)\rangle - \frac{14}{3}\sum_{a<b}\langle\delta^3(r_{ab})\rangle\,, \tag{16}$$

$$\begin{aligned} \mathcal{E}_2 &= Z^2\left[-\frac{2}{3}\ln(Z\alpha) + \frac{62}{9} - \frac{8}{3}\ln\left(\frac{k_0}{Z^2}\right)\right]\sum_a \langle\delta^3(r_a)\rangle \\ &\quad - \frac{14\,Z^2}{3}\sum_a \widetilde{Q}_a\,, \end{aligned} \tag{17}$$

with $\widetilde{Q}_a$ defined analogously to Eq. (11), and $(m/M)\,\mathcal{E}_3$ is the first-order perturbation of $\mathcal{E}^{(5)}_\infty$ due to the mass-polarization operator (6). The operator $H^{(5)}_{\rm fs,rec}$ yields a nonvanishing contribution to the fine-structure splitting only. It is given by

$$H^{(5)}_{\rm fs,rec} = \frac{m}{M}\,\frac{Z}{4\pi}\sum_{ab}\frac{\vec{r}_a}{r_a^3}\times\vec{p}_b\cdot\vec{\sigma}_a\,. \tag{18}$$

We note that the last term in Eq. (16) was omitted in the original derivation of Ref. [10].

The complete result for the $m\,\alpha^6$ correction $\mathcal{E}^{(6)}_\infty$ to the energy levels was derived by one of the authors (K.P.) in a series of papers [6, 7, 11, 12]

$$\begin{aligned} \mathcal{E}^{(6)}_\infty =& -\ln(Z\alpha)\,\pi\sum_{a<b}\langle\delta^3(r_{ab})\rangle + E_{\rm sec} \\ &+ \left\langle H^{(6)}_{\rm nrad} + H^{(6)}_{R1} + H^{(6)}_{R2} + H^{(6)}_{\rm fs} + H^{(6)}_{\rm fs,amm}\right\rangle\,. \end{aligned} \tag{19}$$

The first term in the above expression contains the complete logarithmic dependence of the $m\,\alpha^6$ correction. The part of it proportional to $\ln\alpha$ was first obtained in Ref. [13]. The remaining logarithmic part proportional to $\ln Z$ was implicitly present in formulas reported in Ref. [6, 7] (it originates from the expectation value of the operator $1/r_{ab}^3$). In Eq. (19), we group all logarithmic terms together so that the remaining part does not have any logarithms in its $1/Z$ expansion.

The term $E_{\rm sec}$ in Eq. (19) is the second-order perturbation correction induced by the Breit-Pauli Hamiltonian. (More specifically, it is the finite residual after separating divergent contributions that cancel out in the sum with the expectation value of the effective $m\alpha^6$ Hamiltonian.) The first part of the effective Hamiltonian, $H^{(6)}_{\rm nrad}$, originates from the non-radiative part of the electron-nucleus and the electron-electron interaction. The next two terms, $H^{(6)}_{R1}$ and $H^{(6)}_{R2}$, are due to the one-loop and two-loop radiative effects, respectively. The last two parts $H^{(6)}_{\rm fs}$ and $H^{(6)}_{\rm fs,amm}$ are the spin-dependent operators first derived by Douglas and Kroll [14]. They do not contribute to the energies of the singlet states and to the spin-orbit averaged levels. Expressions for these operators are well known and are given, e.g., by Eqs. (3) and (7) of Ref. [15]. The non-radiative part of the $m\,\alpha^6$ effective Hamiltonian is rather complicated. For simplicity, we present it specifically for a two-electron atom. The corresponding expression reads [6, 7]