bath. Let us consider a thermal nonequilibrium process in a finite-time schedule $t_0 = 0 \le t \le t_n = n\delta t$. Thermal fluctuations can be simulated by the master equation. The probability that the system is in a state σ_k at time t_k is denoted as $P(\sigma_k; t_k)$. The transition probability per unit time δt is defined as $M(\sigma_{k+1}|\sigma_k; t_k)$. In the original formulation of JE, the work is defined as the energy difference merely attributed to the change of the Hamiltonian, but we can construct JE also in the case of changing the inverse temperature by defining the work as $-\beta W(t_k) = -(\beta(t_{k+1}) - \beta(t_k))H_0$ for the state σ. The left-hand side of JE can then be expressed as

$$\begin{aligned}&\left\langle \mathrm{e}^{-\beta W} \right\rangle \\ &= \sum_{\{\sigma_k\}} \prod_{k=0}^{n-1} \left\{ \mathrm{e}^{-\beta W(t_k)} \mathrm{e}^{\delta t M(\sigma_{k+1}|\sigma_k; t_k)} \right\} \\ &\quad \times \tilde{P}(\sigma_0; t_0), \end{aligned} \tag{3}$$

where $\tilde{P}(\sigma_0; t_0)$ denotes the initial equilibrium distribution. Even if the transition term $\exp(\delta t M(\sigma_{k+1}|\sigma_k; t_k))$ is removed in this equation, JE is trivially satisfied as one can simply confirm. A non-trivial feature of JE is in the insertion of the transition term. From Eq. (3), it is straightforward to prove JE by use of the detailed-balance condition. An observant reader may think the above formulation without any consideration of quantum nature is not available for the application to QA. Nevertheless we can apply the classical JE to QA by aid of the classical-quantum mapping [10].

3. Classical-quantum mapping

The classical-quantum mapping leads us to a special quantum system, in which the (instantaneous) equilibrium state of the above stochastic dynamics can be expressed as the ground state. A general form of such a special quantum Hamiltonian is given as $H_q(\sigma'|\sigma; t) = I - \mathrm{e}^{\frac{\beta(t)}{2} H_0} M(\sigma'|\sigma; t) \mathrm{e}^{-\frac{\beta(t)}{2} H_0}$. This Hamiltonian has the ground state as $|\Psi_{\mathrm{eq}}(t)\rangle = \sum_\sigma \mathrm{e}^{-\beta(t) H_0/2} |\sigma\rangle / \sqrt{Z(t)}$. The ground state energy is 0, which can be explicitly shown by the detailed-balance condition. On the other hand, the excited states have positive-definite eigenvalues, which can be confirmed by the application of the Perron-Frobenius theorem.

In the above special quantum system, we can deal with a quasi-equilibrium stochastic process as an adiabatic quantum-mechanical dynamics in QA. Let us consider QA for the above special quantum system by setting the parameter corresponding to the temperature $T \to \infty$ ($\beta \to 0$). This condition gives the trivial ground state for H_1 in the preceding section with an uniform linear combination, similarly to the ordinary QA. If we change $T \to 0$ very slowly, one can obtain the ground state of H_q, which expresses the very low-temperature equilibrium state for H_0, the cost function of the optimization problem that we wish solve. We however consider to construct a protocol with the same spirit as JE by using the special quantum system to overcome the bottleneck of the ordinary QA as proposed in the following section.

4. Quantum Jarzynski annealing and its application

We prepare a trivial ground state with a uniform linear combination as the initial condition in the ordinary QA. This initial state corresponds to the high-temperature equilibrium state $|\Psi_{\mathrm{eq}}(t_0)\rangle \propto \exp(-\beta(t_0) H_0/2)|\sigma\rangle$ with $\beta(t_0) \ll 1$. We introduce the exponentiated work operator $W_{\mathrm{exp}}(t_k) = \exp(-(\beta(t_{k+1}) - \beta(t_k))H_0/2)$. It looks like a non-unitary operator, but we can construct this operation by considering an extended quantum system [11, 12]. If we apply $W_{\mathrm{exp}}(t_k)$ to the preceding quantum state $|\Psi_{\mathrm{eq}}(t_k)\rangle$, it is changed into a state corresponding to the equilibrium distribution with the inverse temperature $\beta(t_{k+1})$. After then, the time-evolution operator $U(\sigma'|\sigma; t_{k+1}) = \exp(-\mathrm{i}\delta t H_q(\sigma'|\sigma; t_{k+1})/\hbar)$ also does not alter this state, since it is the ground state of $H_q(\sigma'|\sigma; t_{k+1})$. The resulting state after the repetition of the above procedure is

$$\begin{aligned}|\Psi(t_n)\rangle \propto \prod_{k=0}^{n-1} \left\{ W_{\mathrm{exp}}(t_k) U_{k+1}(\sigma_{k+1}|\sigma_k; t_k) \right\} . \\ \times |\Psi_{\mathrm{eq}}(t_0)\rangle \end{aligned} \tag{4}$$

This is essentially of the same form as Eq. (3). We measure the obtained state by the projection onto a specified state σ'. The probability is then given by $|\langle\sigma'|\Psi(t_n)\rangle|^2$, which means that the ground state we wish to find is obtained with the probability proportional to $\exp(-\beta(t_n) H_0)$. If we continue the above procedure up to $\beta(t_n) \gg 1$, we can efficiently obtain the ground state of H_0. This is called the quantum Jarzynski annealing (QJA) in the present study. Most of the readers, who are familiar with the ordinary computation, have considered that it may seem unnecessary to apply the time-evolution operator $U(\sigma_{k+1}|\sigma_k; t_k)$, which expresses change between states by quantum fluctuations, at the middle step between the operations of the exponentiated work operators $W_{\mathrm{exp}}(t_k)$. The time-evolution operator does not mean an artificial control but describes the change by quantum nature during quantum computation, which is inherent property in quantum computation. However we here remember the nontrivial