To prove the lemma, it suffices to show that the Hessian $\nabla^2 G$ is positive semidefinite. We need to show that all the principal minors of $\nabla^2 G$ in (76) are nonnegative.

$$\nabla^2 G = \begin{array}{c c} & \begin{array}{ccccc} \omega_1 & & \omega_N & \mu \end{array} \\ \begin{array}{c} \omega_1 \\ \ldots \\ \omega_N \\ \mu \end{array} & \begin{bmatrix} \frac{\partial^2 G}{\partial \omega_1^2} & \cdots & \frac{\partial^2 G}{\partial \omega_1 \partial \omega_N} & \frac{\partial^2 G}{\partial \omega_1 \partial \mu} \\ \ldots & & & \ldots \\ \frac{\partial^2 G}{\partial \omega_N \partial \omega_1} & \cdots & \frac{\partial^2 G}{\partial \omega_N^2} & \frac{\partial^2 G}{\partial \omega_N \partial \mu} \\ \frac{\partial^2 G}{\partial \mu \partial \omega_1} & \cdots & \frac{\partial^2 G}{\partial \mu \partial \omega_N} & \frac{\partial^2 G}{\partial \mu^2} \end{bmatrix} \end{array} \tag{76}$$

However, because we already know that all the principal minors of the matrix in (75) are nonnegative, it only remains to show that $\frac{\partial^2 G}{\partial \mu^2} \geq 0$ and $\det \nabla^2 G \geq 0$.

First, we show that $\frac{\partial^2 G}{\partial \mu^2} \geq 0$

Find the first derivative of G.

$$\begin{aligned} \frac{\partial G}{\partial \mu} &= \ln\left(\sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}}\right) + \mu \frac{\sum_{i=1}^{N} \theta_i \frac{-\omega_i}{\mu^2} e^{\frac{\omega_i}{\mu}}}{\sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}}} \\ &= \ln\left(\sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}}\right) - \frac{1}{\mu} \frac{\sum_{i=1}^{N} \theta_i \omega_i e^{\frac{\omega_i}{\mu}}}{\sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}}} \end{aligned} \tag{77}$$

Find the second derivative of G.

$$\begin{aligned} \frac{\partial^2 G}{\partial \mu^2} &= \frac{\sum_{i=1}^{N} \theta_i \frac{-\omega_i}{\mu^2} e^{\frac{\omega_i}{\mu}}}{\sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}}} + \frac{1}{\mu^2} \frac{\sum_{i=1}^{N} \theta_i \omega_i e^{\frac{\omega_i}{\mu}}}{\sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}}} \\ &\quad - \frac{1}{\mu} \frac{\sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}} \sum_{i=1}^{N} \theta_i \frac{-\omega_i^2}{\mu^2} e^{\frac{\omega_i}{\mu}} - \sum_{i=1}^{N} \theta_i \frac{-\omega_i}{\mu^2} e^{\frac{\omega_i}{\mu}} \sum_{i=1}^{N} \theta_i \omega_i e^{\frac{\omega_i}{\mu}}}{\left[\sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}}\right]^2} \end{aligned} \tag{78}$$

$$\begin{aligned} \frac{\partial^2 G}{\partial \mu^2} &= \frac{1}{\mu} \frac{\sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}} \sum_{i=1}^{N} \theta_i \frac{\omega_i^2}{\mu^2} e^{\frac{\omega_i}{\mu}} - \sum_{i=1}^{N} \theta_i \frac{\omega_i}{\mu^2} e^{\frac{\omega_i}{\mu}} \sum_{i=1}^{N} \theta_i \omega_i e^{\frac{\omega_i}{\mu}}}{\left[\sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}}\right]^2} \\ &= \frac{\sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}} \sum_{i=1}^{N} \theta_i \omega_i^2 e^{\frac{\omega_i}{\mu}} - \left[\sum_{i=1}^{N} \theta_i \omega_i e^{\frac{\omega_i}{\mu}}\right]^2}{\mu^3 \left[\sum_{i=1}^{N} \theta_i e^{\frac{\omega_i}{\mu}}\right]^2} \end{aligned} \tag{79}$$

The denominator of (79) is positive. Hence, it only remains to show that the numerator is nonneg-