By Corollary 1, $\frac{x}{S_p} - 1 = \frac{x}{p} - \frac{S_p}{p} + o(p^{-1})$ a.s.. Using this fact, after some algebra, we get

$$\tilde{\theta}_{pj}\widetilde{v}_{pj} = \theta_{pj}v_{pj} + \theta_{pj}v_{pj}^2\left(\frac{x}{p} - \frac{S_p}{p}\right) + O\left(\left(\frac{x}{p}-1\right)^2\right),$$

$$\ln\left(2\tilde{\theta}_{pj}\right) = \ln\left(2\theta_{pj}\right) + \left(\frac{x}{p} - \frac{S_p}{p}\right) + O\left(\left(\frac{x}{p}-1\right)^2\right),$$

and

$$\begin{aligned}\sum_{i=1}^{p}\ln\left(K_p^{MP}\left(2\tilde{\theta}_{pj}\right) - \lambda_{pi}\right) &= \sum_{i=1}^{p}\ln\left(K_p^{MP}\left(2\theta_{pj}\right) - \lambda_{pi}\right) - p\left(1 - 4c_p\theta_{pj}^2 v_{pj}^2\right)\left(\frac{x}{p} - \frac{S_p}{p}\right) \\ &\quad + O\left(\left(\frac{x}{p}-1\right)^2\right).\end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{I}(0,\infty) &= \int_{p-\alpha\sqrt{p}}^{p+\alpha\sqrt{p}} x^{\frac{np}{2}-1}e^{-\frac{n}{2}x}e^{p\sum_{j=1}^{r}\left[\theta_{pj}v_{pj} - \frac{1}{2p}\sum_{i=1}^{p}\ln(1+2\theta_{pj}v_{pj}-2\theta_{pj}\lambda_{pi})\right]} \\ &\quad \times e^{\sum_{j=1}^{r}\theta_{pj}v_{pj}(x-S_p)}\left(\prod_{j=1}^{r}\prod_{s=1}^{j}\sqrt{1-4\left(\theta_{pj}v_{pj}\right)\left(\theta_{ps}v_{ps}\right)c_p} + o(1)\right)\mathrm{d}x \\ &= (1+o(1))\prod_{j=1}^{r}\left(1+h_j\right)^{\frac{np}{2}} L_p(h;\lambda_p)\int_{p-\alpha\sqrt{p}}^{p+\alpha\sqrt{p}} x^{\frac{np}{2}-1}e^{-\frac{n}{2}x}e^{\sum_{j=1}^{r}\theta_{pj}v_{pj}(x-S_p)}\mathrm{d}x,\end{aligned} \tag{55}$$

where the last equality in (55) follows from (3) and Proposition 2.

The last equality in (55), (4) and the fact that

$$\int_{p-\alpha\sqrt{p}}^{p+\alpha\sqrt{p}} x^{\frac{np}{2}-1}e^{-\frac{n}{2}x}e^{\sum_{j=1}^{r}\theta_{pj}v_{pj}(x-S_p)}\mathrm{d}x = e^{\sum_{j=1}^{r}-\frac{h_j}{2c_p}S_p}\left(\frac{n}{2} - \sum_{j=1}^{r}\frac{h_j}{2c_p}\right)^{-\frac{np}{2}}\Gamma\left(\frac{np}{2}\right)(1+o(1))$$

imply that

$$\begin{aligned}L_p\left(h;\mu_p\right) &= (1+o(1))\,L_p(h;\lambda_p)e^{\sum_{j=1}^{r}-\frac{h_j}{2c_p}S_p}\left(1 - \sum_{j=1}^{r}\frac{h_j}{nc_p}\right)^{-\frac{np}{2}} \\ &= (1+o(1))\,L_p(h;\lambda_p)e^{-\frac{S_p-p}{2c_p}\sum_{j=1}^{r}h_j + \frac{1}{4c_p}\left(\sum_{j=1}^{r}h_j\right)^2},\end{aligned}$$

which establishes (11). The rest of the statements of Theorem 1 follow from (10), (11), and Lemmas 12 and A2 of OMH.□