the effective quantum mechanical description of section IV. In section VII we discuss the phase diagram of the $O(2)$ model which emerges from our work.

II. MODEL AND OBSERVABLES

The action of the $O(2)$ non-linear sigma model on a lattice with a finite chemical potential that we study here is given by

$$S = -\beta \sum_{x,\alpha} \Big\{ \cos(\theta_x - \theta_{x+\alpha} - i\mu\delta_{\alpha,t}) \Big\}, \tag{1}$$

where x is the lattice site on a three dimensional cubic lattice, $\alpha = 1, 2$ represent the spatial directions and $\alpha = t$ represents the temporal direction. We will use L to represent the spatial size and L_t the temporal size and assume periodic boundary conditions. The constant β plays the role of the coupling. The chemical potential μ is introduced in the standard way and couples to the conserved charge of the global $O(2)$ symmetry [26]. When $\mu \neq 0$ the action becomes complex and Monte Carlo algorithms to generate configurations $[\theta]$ that contribute to the partition function

$$Z = \int [d\theta_x] \mathrm{e}^{-S}, \tag{2}$$

suffer from a sign problem. In particular the Wolff cluster algorithm [7] is no longer useful at non-zero chemical potential. Hence the phase diagram of the model in the (β, μ) plane remains unexplored.

It is possible to avoid the sign problem if we rewrite the partition function in the world-line representation [9]. Using the identity

$$\exp\{\cos\theta\} = \sum_{k=-\infty}^{\infty} I_k(\beta) e^{ik\theta}, \tag{3}$$

where I_k is the modified Bessel function of the first kind, on each bond (x, α), and performing the angular integration over θ_x the partition function can be rewritten as

$$Z = \sum_{[k]} \prod_{x} \Big\{ I_{k_{x,\alpha}}(\beta) e^{\mu\delta_{\alpha,t} k_{x,\alpha}} \Big\} \; \delta\Big(\sum_{\alpha} (k_{x,\alpha} - k_{x-\alpha,\alpha}) \Big), \tag{4}$$

where the bond variables $k_{x,\alpha}$ describe "world-lines" or "current" of particles moving from lattice site x to the site $x + \hat{\alpha}$ and take integer values. A configuration of these bond variables, denoted by $[k]$, is thus a world-line configuration. The global $U(1)$ symmetry of the model is manifest