denote the dynamical variables (fields) collectively by A and the θ-energy associated with a particular set of these variables as $H[A]$. Observed over long times θ, the system will visit all points on the hypersurface defined by $\delta(H[A]-H_0)$, where H_0 is the initial θ-energy of our system. The long θ-time average of an observable $\mathcal{O}[A]$ is then equal to the microcanonical average:

$$\begin{aligned} \langle\mathcal{O}\rangle &= \lim_{\theta\to\infty} \frac{1}{\theta}\int_0^\theta d\theta'\,\mathcal{O}[A(\theta')] \\ &= V(H_0)^{-1}\int \mathcal{D}A\,\delta(H[A]-H_0)\,\mathcal{O}[A], \end{aligned} \quad (22)$$

where $V(H_0)$ is the volume of the constant-H hypersurface. The last expression in (22) is the definition of the quantum mechanical average of the observable $\mathcal{O}$ in a formalism called *micro-canonical quantization* [52, 53]. Using the Fourier representation of the delta function, the micro-canonical average can be written in the form

$$\langle\mathcal{O}\rangle = V(H_0)^{-1}\int_{-\infty}^{\infty}\frac{d\tilde{\theta}}{2\pi}\int \mathcal{D}A\,e^{i(H[A]-H_0)\tilde{\theta}}\,\mathcal{O}[A]. \quad (23)$$

For systems with an infinite number of degrees of freedom, such as field theories, the microcanonical average is known to agree with the canonical average under very general conditions [52]. When this equivalence holds, we can express the expectation value of the observable $\mathcal{O}$ as

$$\langle\mathcal{O}\rangle = Z(\beta)^{-1}\int \mathcal{D}A\,e^{-\beta H[A]}\,\mathcal{O}[A], \quad (24)$$

where $Z(\beta)=\int\mathcal{D}A\,e^{-\beta H[A]}$ is a partition function and the parameter β is, as usual, defined by the condition

$$\beta\int\mathcal{D}A\,e^{-\beta H[A]}H[A] \equiv -\beta\frac{\partial Z}{\partial\beta} = \beta H_0. \quad (25)$$

Upon the identification $\beta=\hbar^{-1}$, the expression (24) becomes the usual definition of the (canonical) quantum mechanical expectation value of the observable $\mathcal{O}$ as a functional integral in euclidean space. $H[A]$ is recognized as the euclidean space action associated with the system. In our five-dimensional space-time H is simply the energy variable associated with the additional time dimension. The quantum of action (Planck's constant) $\hbar$ appears as the quasi-temperature associated with the ergodic finite-energy trajectory of the system in the five-dimensional space-time. Note that this definition of the expectation variable does not involves a thermal ensemble, only a (θ)-time average over an ergodic, classical trajectory in the five-dimensional space-time.

B. Stochastic and chaotic quantization

The micro-canonical ensemble has been widely used in lattice gauge theory to calculate vacuum expectation values of various operators [54–59]. In practice, the micro-canonical ensemble is generated by a stochastic trajectory obeying a Langevin equation in an auxiliary "time" variable. This is the essence of *stochastic quantization* [60–62], which is thus recognized as a schematic implementation of micro-canonical quantization [63].

Of course, the micro-canonical average can be generated by any other appropriate method, e. g. by a deterministic, but ergodic process in an additional *physical* (not "auxiliary") time dimension. Such a process is naturally realized when the evolution of the five-dimensional system in the additional time variable θ is strongly chaotic. The method of defining the vacuum expectation value of a system by means of the ergodic average over a chaotic trajectory has been called *chaotic quantization* [47, 64].

An example of a physical system that thus generates its own quantum analogue in a reduced dimension is the classical lattice gauge theory [48]. A numerical study comparing the "self-quantizing" five-dimensional classical gauge theory with the canonically quantized gauge theory in four dimensions, which was undertaken by Biró and Müller for the compact U(1) lattice gauge theory [65], showed excellent agreement for the expectation value of the Polyakov loop as a function of the gauge coupling if Planck's constant $\hbar$ is identified with the product $T_5 a$, where a is the lattice spacing.

C. Analytic continuation

In order to describe physics in real physical time, one must analytically continue the euclidean time coordinate τ to the real time coordinate $t=-i\tau$. For the canonical average (24), this is exactly what is usually done in quantum field theory, where the Minkowski space vacuum expectation value of an observable $\mathcal{O}$ is *defined* by the analytic continuation of the euclidean space functional integral. The critical question is whether the same analytic continuation also applies to, and makes sense for, the micro-canonical and ergodic time averages (22).

The analytic continuation involves a number of subtleties. First, the real-time analogue of the functional integral (24) does not involve a real integrand and is thus not mathematically well defined. For the micro-canonical average (22) the problem is that the analytically continued θ-energy (the Minkowski space action) is not positive definite, and thus the constant-H hypersurface does not have a finite volume, even for a field theory with infrared and ultraviolet cut-offs. In the stochastic average, the mathematical subtleties reside in the definition of a Langevin process for a system governed by a Minkowski space action with indefinite sign.

There has been a limited amount of work on stochastic quantization in Minkowski space [66, 67]. For certain systems, a complex Langevin process can be defined and simulated, which generates the appropriate micro-canonical ensemble [68, 69]. In other cases, the complex Langevin process can be stabilized by a suitable kernel [70]. Recently, complex Langevin evolution has been applied to generate the real-time quantum evolution of