and with the chain rule respectively Faá di Bruno's formula

$$\frac{d\bar{\varepsilon}}{dT} = \frac{d\bar{\varepsilon}}{d\mu_{E_0}}\frac{d\mu_{E_0}}{dT}, \tag{A17}$$

$$\frac{d^n\bar{\varepsilon}}{dT^n} = \sum \frac{n!}{k_1!\dots k_n!}\frac{d^k\bar{\varepsilon}}{d\mu_{E_0}^k} \times \left(\frac{1}{1!}\frac{d\mu_{E_0}}{dT}\right)^{k_1}\cdots\left(\frac{1}{n!}\frac{d^n\mu_{E_0}}{dT^n}\right)^{k_n}, \tag{A18}$$

$$k = k_1 + \cdots + k_n,$$

where the sum runs over all integer numbers $k_1, \dots, k_n \geq 0$ which fulfill

$$k_1 + 2k_2 + \cdots + nk_n = n. \tag{A19}$$

We do not have to find this numbers, we only need to know that for $n > 1$ at least one $k_j > 0$, therefore each term in (A18) contains a factor $\frac{d^j\mu_{E_0}}{dT^j}$, so for $n_s =$ const., $T \to 0$ the sum vanishes and from Eq. (A16) we get

$$\bar{\varepsilon} \simeq \varepsilon_{F0} \quad \text{for } k_B T \ll \varepsilon_{F0}. \tag{A20}$$

## Appendix B: Ground state energy of the inversion layer

As mentioned before we calculate the ground state energy of the inversion layer with help of the Ritz variational principle using the Fang-Howard test envelope wave function[34]

$$\varphi(z,b) = \begin{cases} \sqrt{\frac{b^3}{2}}\, z\exp\left(-\frac{bz}{2}\right) & \text{for } z \geq 0 \\ 0 & \text{for } z < 0 \end{cases} \tag{B1}$$

and as an approximation for the potential

$$U(z) = \begin{cases} U_d(z) + U_s(z) + U_i(z) & \text{for } z \geq 0 \\ \infty & \text{for } z < 0, \end{cases} \tag{B2}$$

$$U_d(z) \simeq \begin{cases} \frac{e^2 n_{\text{depl}}}{\epsilon_{\text{sc}}\epsilon_0} z\left(1 - \frac{z}{2z_d}\right) & \text{for } z < z_d \\ \frac{e^2 n_{\text{depl}}}{2\epsilon_{\text{sc}}\epsilon_0} z_d & \text{for } z > z_d, \end{cases} \tag{B3}$$

$$U_s(z) \simeq \frac{e^2 n_s}{2b\epsilon_{\text{sc}}\epsilon_0}\left[6 - \left((bz)^2 + 4bz + 6\right)\exp(-bz)\right], \tag{B4}$$

$$U_i(z) \simeq \frac{be^2}{32\pi\epsilon_{\text{sc}}\epsilon_0}\frac{\epsilon_{\text{sc}} - \epsilon_{\text{ins}}}{\epsilon_{\text{sc}} + \epsilon_{\text{ins}}}\frac{1}{4z}. \tag{B5}$$

The term $U_d$ comes from the charged acceptors within the depletion layer with thickness $z_d$, $U_s$ describes the interaction with all other electrons in the inversion layer, and $U_i$ the interaction with image charges. To write $U_s(z)$ in this form we have to assume that only the first subband is occupied, this is the so called quantum limit. The conduction band edge is built by $U_d(z) + U_s(z)$, therefore the zero point of the energy scale was chosen to get $U_d(0) + U_s(0) = 0$ which means that the resulting ground state energy is measured against the conduction band edge at the interface $E_{Cs}$ as requested.

The Hamiltonian is given by $\hat{H} = \hat{T} + U$ with the operator for the kinetic energy $\hat{T} = -\frac{\hbar^2}{2m_z}\frac{\partial^2}{\partial z^2}$, where $m_z$ is the $z$-component of the effective mass of the semiconductor in the bulk. The ground state energy is the expectation value of the Hamiltonian,

$$\varepsilon_0 = \left\langle \hat{T} \right\rangle + \left\langle U_d \right\rangle + \left\langle U_s \right\rangle + \left\langle U_i \right\rangle, \tag{B6}$$

calculated with the value of $b$ which makes the total energy per electron

$$\tilde{\varepsilon} = \left\langle \hat{T} \right\rangle + \left\langle U_d \right\rangle + \frac{1}{2}\left\langle U_s \right\rangle + \left\langle U_i \right\rangle \tag{B7}$$

minimal. The factor 1/2 in the third term prevents from double counting the electron-electron interaction. With the density $n^* = n_{\text{depl}} + \frac{11}{32}n_s$ introduced by AFS[28] one gets

$$\tilde{\varepsilon} = \frac{\alpha}{2}b^2 + \beta b + \gamma b^{-1} - \frac{\delta(b)}{2}b^{-2}, \tag{B8}$$

$$\alpha = \frac{\hbar^2}{4m_z}, \tag{B9}$$

$$\beta = \frac{e^2}{32\pi\epsilon_{\text{sc}}\epsilon_0}\frac{\epsilon_{\text{sc}} - \epsilon_{\text{ins}}}{\epsilon_{\text{sc}} + \epsilon_{\text{ins}}}, \tag{B10}$$

$$\gamma = \frac{3e^2 n^*}{\epsilon_{\text{sc}}\epsilon_0}, \tag{B11}$$

$$\delta = \frac{12e^2(N_A - N_D)}{\epsilon_{\text{sc}}\epsilon_0} \tag{B12}$$

$$\times \left\{1 - \left[\frac{(bz_d)^2}{12} + \frac{bz_d}{2} + 1\right]\exp(-bz_d)\right\}. \tag{B13}$$

$N_A$ and $N_D$ are the densities of the acceptors and donors respectively. The coefficients have been chosen in order to get $\alpha$, $\beta$, $\gamma$, $\delta > 0$ and to get a most simple equation

$$\frac{d\tilde{\varepsilon}}{db} = \alpha b + \beta - \gamma b^{-2} + \delta(b)\, b^{-3} = 0. \tag{B14}$$

As AFS used $\frac{e^2 n_{\text{depl}}}{\epsilon_{\text{sc}}\epsilon_0} z\left(1 - \frac{z}{2z_d}\right)$ for $0 < z < \infty$ instead of Eq. (B3) they did not get the term proportional to $\exp(-bz_d)$, under normal circumstances it is very small but formally it is necessary to see that $b \to 0$, $\tilde{\varepsilon} \to -\infty$ is not the global minimum. Furthermore they neglected $\beta$ and $\delta(b)\, b^{-3}$ and got

$$b = \left(\frac{\gamma}{\alpha}\right)^{1/3} = \left(\frac{12e^2 n^* m_z}{\epsilon_{\text{sc}}\epsilon_0\hbar^2}\right)^{1/3}. \tag{B15}$$