a bipartite stabilizer state is generated by all minimal stabilizer generators that have support on both parts of the system. The minimal stabilizer generators are those stabilizer operators which generate the stabilizer group (through multiplication) while having the smallest support of all such sets of operators. There are in general a lot of such sets, so the choice is not unique. For example, the stabilizer generators $Z \otimes \mathbb{1}$ and $\mathbb{1} \otimes Z$ would stabilize the product state $\psi = |\uparrow\rangle \otimes |\uparrow\rangle$. Obviously the combination $Z \otimes Z$ and $\mathbb{1} \otimes Z$ would do the same, but with one generator with larger support. In this case the first set would be minimal, while the second wouldn't. In the case of translation-invariant pure stabilizer states we don't have this issue. The set of generators is always translation invariant and fulfills the conditions introduced in Section III A. Assume that a given generator of a translation-invariant pure stabilizer state is not minimal. Then it is composed of at least two stabilizer generators which also have to fulfill the conditions from Section III A. In particular, all of the generators have to be the same. This implies that the polynomials of the original non-minimal generator have common divisors. But this is not possible, because it is required that the polynomials are coprime. Thus the generators of translation-invariant pure stabilizer states are always minimal.

Unfortunately the proof in [10] relies heavily on the fact that only finite systems are considered. We use a different approach and come to essentially the same result that holds for translation-invariant stabilizer states on infinite chains.

1. The Bipartite Case

First we will investigate the case of a bipartite splitting of the chain. We have two parties, say Alice and Bob, where Alice controls the part $\mathscr{A}$ and Bob the part $\mathscr{B}$ of the chain. This is shown in Figure 4. Let us first define

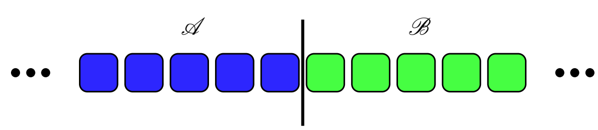

Figure 4: The spin chain is cut into two halfchains $\mathscr{A}$ and $\mathscr{B}$. We study the entanglement between these halfchains.

bipartite entanglement for stabilizer states.

Definition III.2. *The entanglement $E(\omega_\xi)$ of a translation-invariant stabilizer state ω_ξ in a bipartite setting is the number of maximally entangled qubit pairs with respect to any bipartite cut. These qubit pairs are logical ones, that is they are each localized on several physical qubits. By local operations on each part of the chain one could localize the logical qubits onto one physical qubit each.*

In this case we have the following theorem:

Theorem III.3. *A pure translation-invariant stabilizer state of stabilizer generator length $2n+1$ entangles n qubit pairs maximally with respect to any bipartite cut.*

Proof. Here we will only present the idea of the proof. The technical parts can be found in the appendix of [2]

Obviously, all bipartite cuts are equivalent as the state is translation-invariant. Thus, we can look at any particular cut to prove the general result. We have a stabilizer generator centered around each site. Unless our stabilizer generators are single site operators ($n = 0$), the cut will always leave several stabilizer generators cut into parts on both systems $\mathscr{A}$ and $\mathscr{B}$. As one can see in Figure 5 there will be $2n$ stabilizer generators affected. These operators generate the correlation subgroup $\mathcal{S}_{AB}$. Thus, the correlation subgroup has $2n$ generators and therefore dimension $|\mathcal{S}_{AB}| = 2n$. The interesting fact about these stabilizer generators is that, despite commuting as a whole, their restrictions to $\mathscr{A}$ or $\mathscr{B}$ don't necessarily commute. Now we try to find commuting pairs of anti-commuting Pauli products in the restriction of $\mathcal{S}_{AB}$ to $\mathscr{A}$ or $\mathscr{B}$ by multiplying stabilizer generators from the correlation subgroup (We check for the anti-commuting parts only on one side, but carry out the multiplication on both sides to preserve the stabilized state). We know from the theory of quantum error correction codes that each such pair on $\mathscr{A}$ encodes one qubit [9]. Because we carried out the multiplication on both sides, the corresponding parts of the operators on $\mathscr{B}$ fulfill the same commutation relations and therefore also encode qubits. The pairs of operators on each halfchain behave like pairs of X and Z; thus, we call them $\bar{X}$ and $\bar{Z}$. We only required them to anti-commute, so we did not fix which of them is the X and which the Z. Thus, we can choose this and we choose it in such a way that the corresponding operators on each side are either bot $\bar{Z}$ or both $\bar{X}$. Thus, we have a pair of stabilizer generators that reads $\bar{X}_{\mathscr{A}} \otimes \bar{X}_{\mathscr{B}}$ and $\bar{Z}_{\mathscr{A}} \otimes \bar{Z}_{\mathscr{B}}$. But, as seen in Example III.1, $\mathbb{S} = \{X \otimes X, Z \otimes Z\}$ encodes a Bell state, which is maximally entangled. Thus, each pair of anticommuting pairs stabilizes a maximally entangled (logical) qubit pair.

What is left to show is that we can always find n such pairs. The proof is rather lengthy and technical. It is carried out in [2] and based on methods from quantum error correction codes to directly construct the pairs. □

C. The Tripartite Case

In this setting we cut the chain into three parts, one middle part of length L and two infinite ends, as shown in Figure 6. We now want to calculate the entanglement between the finite part and the two infinite parts. To do this calculation, we use the same method as above, and arrive at the following theorem.