the anti-Hermitian strength of $\mathcal{H}_0$. The spectrum of $\mathcal{H}_0$ turns out to be real-valued for $\alpha < \alpha_c$, where $\alpha_c \geq 0$ corresponds to the transition from unbroken to broken $\mathcal{PT}$ symmetry. Numerical studies generally show that for $\alpha < \alpha_c$ the spectrum is composed by bands separated by gaps like in an ordinary crystal, whereas for $\alpha \geq \alpha_c$ band merging is observed with the appearance of pairs of complex-conjugate eigenvalues [13, 16]. For instance, for the potential defined by

$$V_R(x) = V_0 \cos(2\pi x/a) \ , \ \ V_I(x) = V_0 \sin(2\pi x/a), \quad (2)$$

one has $\alpha_c = 1$ [13].

In this section we consider the unbroken symmetry phase, i.e. the case $\alpha < \alpha_c$. As recently shown in Ref.[17], for $\alpha < \alpha_c$ the motion of a Bloch particle in presence of an external dc force F can be described following the same lines as in ordinary crystals by expanding the field $\psi(x,z)$ as a superposition of Bloch-Floquet eigenfunctions $\phi_n(x,\kappa) = u_n(x,\kappa)\exp(i\kappa x)$ of $\mathcal{H}_0$, where the wave number κ varies in the first Brillouin zone, i.e. $-k_B/2 \leq \kappa < k_B/2$, $k_B = 2\pi/a$ is the Bragg wave number, n is the band index, and $u_n(x,\kappa)$ is the periodic part of the Bloch-Floquet eigenfunction. After setting $\psi(x,z) = \sum_n \int d\kappa c_n(\kappa,z)\phi_n(\kappa,z)$ and assuming normalized eigenfunctions such that $\int dx \phi^*_{n'}(-x,-\kappa')\phi_n(x,\kappa) = \mathcal{D}_n \delta_{n,n'}\delta(\kappa-\kappa')$ with $\mathcal{D}_n = \pm 1$, the evolution equations for the spectral coefficients $c_n(\kappa,z)$ read [17]

$$i\lambda\left(\partial_z + \frac{F}{\lambda}\partial_\kappa\right) c_n = E_n(\kappa)c_n - F\mathcal{D}_n \sum_l X_{n,l}(\kappa)c_l \quad (3)$$

where $E_n(\kappa)$ is the energy of $\phi_n(x,\kappa)$ [with $E_n(-\kappa) = E_n(\kappa)$] and $X_{n,l}(\kappa) \equiv (2\pi i/a)\int_0^a dx u^*_n(-x,-\kappa)\partial_\kappa u_l(x,\kappa)$. The off-diagonal elements $X_{n,l}$ $(n \neq l)$ in Eq.(3) are responsible for interband transitions, i.e. Zener tunneling (ZT). If bands n and l are separated by a large gap and the ac force $F(z)$ is small enough such that $|FX_{n,l}(\kappa)| \ll |E_n(\kappa) - E_l(\kappa)|$ in the entire Brillouin zone, ZT is negligible as in ordinary lattices and one can make the single-band approximation by setting $X_{n,l} \simeq 0$ for $n \neq l$ in Eq.(3). In the single-band approximation one thus obtains

$$i\lambda\left(\partial_z + \frac{F}{\lambda}\partial_\kappa\right) c(z,\kappa) = \left[E(\kappa) - iF(z)\Phi(\kappa)\right] c(\kappa,z) \quad (4)$$

where we omitted, for the sake of simplicity, the band index n and set $i\Phi(\kappa) \equiv \mathcal{D}_n X_{n,n}(\kappa)$. Because of the symmetry of $V(x)$, $\mathrm{Re}(u_n(k,x))$ and $\mathrm{Im}(u_n(k,x))$ have well defined and opposite parity under the inversion $x \to -x$; this implies that $\Phi(\kappa)$ is a real-valued function of κ, vanishing for a real potential (i.e. for $\alpha = 0$). As previously shown in [17], when a dc force F is applied to the crystal, from Eq.(4) it follows that the energy spectrum is described by a complex-valued Wannier-Stark ladder. The non-reality of the energy spectrum comes from the extra-term $\Phi(\kappa)$ in Eq.(4) and is physically due to the fact that the external dc force F breaks the $\mathcal{PT}$ symmetry of the full Hamiltonian $\mathcal{H} = \mathcal{H}_0 - Fx$. For an ac-like force with period $\Lambda = 2\pi/\omega$, because of the z-periodicity of the Hamiltonian its energy spectrum is replaced by a quasi-energy spectrum. Moreover, in the single-band approximation DL corresponds to a complete collapse of the quasi-energy band like in an ordinary crystal [2]. According to Floquet's theorem of periodic systems, the quasi-energy $\mathcal{E}(\kappa)$ for the n-th lattice band can be readily calculated by looking for a solution to Eq.(4) of the form $c(z,\kappa) = a(z,\kappa)\exp[-i\mathcal{E}(\kappa)z/\lambda]$ with $a(z+\Lambda,\kappa) = a(z,\kappa)$. One obtains

$$\mathcal{E}(\kappa) = \frac{1}{\Lambda}\int_0^\Lambda dz \left[E(\kappa') - iF(z)\Phi(\kappa')\right] \quad (5)$$

where we have set $\kappa' \equiv \kappa - k(\Lambda) + k(z)$ and $k(z) = (1/\lambda)\int_0^z d\xi F(\xi)$. Let us assume that the ac forcing $F(z)$ is an odd function with respect to some point z_0, i.e. that $F(z-z_0) = -F(z_0-z)$ for some z_0 in the oscillation cycle. This condition is satisfied, for instance, for the important case of an harmonic (e.g. sinusoidal or cosinusoidal) ac driving force, originally considered by Dunlap and Kenkre [1] and that will be assumed in the following. Owing to this additional temporal symmetry on the driving force, which is absent for the BO problem [17], a real-valued quasi-energy spectrum for the non-Hermitian time-periodic Hamiltonian $\mathcal{H}_0 - F(z)x$ is obtained. In fact, in this case the imaginary term on the right hand side of Eq.(5) vanishes after integration because $F(z)$ and $\Phi(\kappa')$ have opposite parity for the inversion $(z-z_0) \to -(z-z_0)$. The quasi-energy spectrum is thus real-valued and its expression takes the usual form as in a conventional crystal. DL corresponds to a collapse of the quasienergy band $\mathcal{E}(\kappa)$, i.e. to $d\mathcal{E}(\kappa)/d\kappa = 0$. For most driving fields like a sinusoidal field, DL can be attained exactly solely in the NNTB approximation, i.e. when the band shape $E(\kappa)$ is sinusoidal [3], $E(\kappa) = E_0 - \Delta\cos(\kappa a)$. In this case, assuming for the sake of definiteness a sinusoidal ac-like force $F(z) = F_0\cos(\omega z)$, the explicit form of the quasi-energy reads [2]

$$\mathcal{E}(\kappa) = E_0 - \Delta J_0\left(\frac{F_0 a}{\lambda\omega}\right)\cos(\kappa a). \quad (6)$$

Band collapse, leading to DL, is thus attained when $J_0(F_0 a/\lambda\omega) = 0$ [1, 2, 7].

To check the correctness of the analysis, we investigated DL for the complex crystal $V(x) = V_0[\cos(2\pi x/a) + i\alpha\sin(2\pi x/a)]$ in the unbroken $\mathcal{PT}$ symmetry phase ($\alpha < 1$) by a direct numerical analysis of Eq.(1) using a pseudo-spectral split-step method with absorbing boundary conditions. Figure 1 shows a typical example of band diagram below the phase transition point ($\alpha = 0.3$), numerically computed by a spectral analysis of the Hamiltonian $\mathcal{H}_0$. Note that the lowest band of the array is with excellent accuracy approximated by a sinusoidal curve and turns out to be separated by a large gap from the second