Since $\Delta|M| = \langle |M|^2 \rangle - \langle |M| \rangle \langle |M| \rangle$ is a variance,

$$\langle M^2 \rangle \geq (\langle |M| \rangle)^2,$$

so stating $\langle |M| \rangle = \epsilon > 0$ implies $\langle M^2 \rangle \geq \epsilon^2 > 0$. On the other hand, since $M^2 \leq |M|$, the converse is true, so $\langle M^2 \rangle > 0$ and $\langle |M| \rangle > 0$ are equivalent.

As we have now defined the main quantities we'll be studying, our next step is to prove that we can substitute the sum over configurations of the graph with a sum over possible *border classes*, that we now define.

EQUIVALENCE BETWEEN SETS OF BORDERS AND SPIN CONFIGURATIONS

Definition. *A* **border class** *is a class $C = \{C^i\}$ of border sets $C^i = \left\{B^i_1, B^i_2, \ldots, B^i_{N_i}\right\}$, where $i = 1, \ldots, N_C$, such that*

- $B^i_u \cap B^i_v = \emptyset$ *for all* $i = 1, \ldots, N_C$ *and* $u, v = 1, \ldots, N_i$,
- $\cup^{l=1,\ldots,N_i} B^i_l = \cup^{m=1,\ldots,N_j} B^j_m$ *for all* $i, j = 1, \ldots, N_C$.

Theorem. *To any given border class corresponds one and only one configuration of spins on $\mathcal{P}$, once we set the value of a single spin.*

Proof. To prove that, for any border class and a given spin $p \in \mathcal{P}$, we can construct a single spin configuration, we first choose an arbitrary representative $C^i = \{B^i_1, \ldots, B^i_N\}$ of C and set all the spins to the value of p, then for each $B^i_k \in C^i$ we flip all the spins of the subgraph which doesn't contain p. The result is independent of the order in which we choose the B^i_k, since each spins changes sign once for every border that separates it from the fixed spin p, and is independent of the specific i.

To prove that for any given spin configuration we can create a single border class, we proceed as follows: let $R^{\pm}$ be the sets of all plus (minus) spins,

$$R^{\pm} \equiv \{i \in \mathcal{G} : \sigma_i = \pm 1\};$$

We now choose the subsets $R^{\pm}_i$ of $R^{\pm}$, so that each $R^{\pm}_i$ is connected, while for all $i \neq j$ $R^{\pm}_i$