on the CE trade-off curve, contradicting the optimality of this trade-off curve.

The next surface to consider is that formed by combining the CQ trade-off curve with the "inverse" of the entanglement distribution protocol. Recall that the entanglement distribution protocol exploits a noiseless qubit channel to establish a shared noiseless ebit. Let $(C_{\text{CQ}}(s_2), Q_{\text{CQ}}(s_2), 0)$ denote a parametrization of all points on the CQ trade-off curve with respect to some parameter $s_2 \in [0, 1/2]$, and recall that each point on the trade-off curve has corresponding entropic quantities of the form $(I(X;B), I(A\rangle BX), 0)$. Then the surface formed by combining the CQ trade-off curve with the inverse of entanglement distribution is

$$\{(C_{\text{CQ}}(s_2), Q_{\text{CQ}}(s_2) + E, E) : s_2 \in [0, 1/2], \quad E \geq 0\}.$$

This surface also forms an outer bound for the capacity region. Were it not so, then one could combine points outside it with entanglement distribution to outperform points on the CQ trade-off curve, contradicting the optimality of this trade-off curve.

The final surface to consider is the following regularization of the plane that (17) specifies:

$$C + 2Q \leq \frac{1}{n} h\left(\mathcal{N}^{\otimes n}\right), \tag{20}$$

for all $n \geq 1$, where

$$h(\mathcal{N}) \equiv \max_{\rho} I(AX;B),$$

and ρ is a state of the form in (15). Lemma 1 below states that $h(\mathcal{N}^{\otimes n})$ actually single-letterizes for any quantum channel $\mathcal{N}$:

$$h\left(\mathcal{N}^{\otimes n}\right) = nh(\mathcal{N}),$$

so that the computation of the boundary $h(\mathcal{N}^{\otimes n})/n$ is tractable. Its proof is a consequence of the single-letterization of the entanglement-assisted classical capacity [37], but we provide it in Appendix A for completeness.

Lemma 1 *The plane in (20) admits a single-letter characterization for any noisy quantum channel $\mathcal{N}$:*

$$h\left(\mathcal{N}^{\otimes n}\right) = nh(\mathcal{N}).$$

The above three surfaces all form outer bounds on the CQE capacity region, but is it clear that we can achieve points along the boundaries? To answer this question, we should consider the intersection of the first and second surfaces, found by solving the following equation for Q and E:

$$\begin{aligned}(C_{\text{CE}}(s_1) - 2Q, Q, E_{\text{CE}}(s_1) - Q) \\ = (C_{\text{CQ}}(s_2), Q_{\text{CQ}}(s_2) + E, E).\end{aligned}$$

Using the entropic expressions for the trade-off curves and solving the above equation gives that all points along the intersection have entropic quantities of the following form:

$$\left(I(X;B), \frac{1}{2} I(A;B|X), \frac{1}{2} I(A;E|X)\right).$$

Ref. [40] constructed a protocol, dubbed the "classically-enhanced father protocol," that can achieve the above rates for CQE communication. Thus, by combining this protocol with super-dense coding and entanglement distribution, we can achieve all points inside the first and second surfaces with entanglement consumption below a certain rate. Finally, by combining this protocol with super-dense coding, entanglement distribution, and the wasting of entanglement, we can achieve all points that lie inside all three surfaces, and thus we can achieve the full CQE capacity region. We summarize these results as the following theorem.

Theorem 1 *Suppose the CQ and CE trade-off curves of a quantum channel $\mathcal{N}$ single-letterize. Then the full CQE capacity region of $\mathcal{N}$ single-letterizes:*

$$\mathcal{C}_{\text{CQE}}(\mathcal{N}) = \mathcal{C}^{(1)}_{\text{CQE}}(\mathcal{N}).$$

We apply the above theorem in the next section. We first show that both the CQ and CE trade-off curves single-letterize for all Hadamard channels, and it then follows that the full CQE capacity region single-letterizes for these channels.

IV. SINGLE-LETTERIZATION OF THE CQ AND CE TRADE-OFF CURVES FOR HADAMARD CHANNELS

A. CQ Trade-off Curve

For the CQ region, we would like to maximize both the classical and quantum communication rates, but we cannot have both be simultaneously optimal. Thus, we must trade between these resources. If we are willing to reduce the quantum communication rate by a little, then we can communicate more classical information and vice versa.

Our main theorem below states and proves that the following function generates points on the CQ trade-off curve for Hadamard channels:

$$f_{\lambda}(\mathcal{N}) \equiv \max_{\rho} I(X;B)_{\rho} + \lambda I(A\rangle BX)_{\rho}, \tag{21}$$

where the state ρ is of the form in (15) and $\lambda \geq 1$.

Theorem 2 *For any fixed $\lambda \geq 1$, the function in (21) leads to a point*

$$(I(X;B)_{\rho}, I(A\rangle BX)_{\rho})$$

on the CQ trade-off curve, provided ρ maximizes (21).