and $C_{s+1}^{(j)}$ are disjoint events, and the fact that $D_1^{(j)} = C_1^{(j)}$ we obtain:

$$\sum_{r=1}^{R_1(p_1)} \Pr\left(C_r^{(j)} \mid P_{2j} \leq \frac{rq_2}{R_1(p_1)}, P_1 = p_1\right) =$$

$$\Pr\left(D_1^{(j)} \mid P_{2j} \leq \frac{q_2}{R_1(p_1)}, P_1 = p_1\right) +$$

$$\sum_{r=2}^{R_1(p_1)} \left[\Pr\left(D_r^{(j)} \mid P_{2j} \leq \frac{rq_2}{R_1(p_1)}, P_1 = p_1\right) - \Pr\left(D_{r-1}^{(j)} \mid P_{2j} \leq \frac{rq_2}{R_1(p_1)}, P_1 = p_1\right)\right]$$

$$= \sum_{r=1}^{R_1(p_1)} \Pr\left(D_r^{(j)} \mid P_{2j} \leq \frac{rq_2}{R_1(p_1)}, P_1 = p_1\right) - \sum_{r=1}^{R_1(p_1)-1} \Pr\left(D_r^{(j)} \mid P_{2j} \leq \frac{(r+1)q_2}{R_1(p_1)}, P_1 = p_1\right)$$

$$\leq \sum_{r=1}^{R_1(p_1)} \Pr\left(D_r^{(j)} \mid P_{2j} \leq \frac{rq_2}{R_1(p_1)}, P_1 = p_1\right) - \sum_{r=1}^{R_1(p_1)-1} \Pr\left(D_r^{(j)} \mid P_{2j} \leq \frac{rq_2}{R_1(p_1)}, P_1 = p_1\right) \tag{F.20}$$

$$= \Pr\left(D_{R_1(p_1)}^{(j)} \mid P_{2j} \leq q_2, P_1 = p_1\right) = 1,$$

where the inequality in (F.20) follows from (F.19).

Proof of item 2. Let p_1 be arbitrary fixed. Then,

$$E\left(\sum_{j \in I_{10}} R_j / \max(R, 1) \mid P_1 = p_1\right) =$$

$$\sum_{j \in I_{10} \cap \mathcal{R}_1(p_1)} \sum_{r=1}^{R_1(p_1)} \frac{1}{r} \mathbf{I}\left[p_{1j} \leq \frac{rq_1}{m}\right] \Pr\left(P_{2j} \leq \frac{rq_2}{R_1(p_1)}, C_r^{(j)} \mid P_1 = p_1\right)$$

$$\leq \sum_{j \in I_{10} \cap \mathcal{R}_1(p_1)} \sum_{r=1}^{R_1(p_1)} \frac{1}{r} \Pr\left(P_{2j} \leq \frac{rq_2}{R_1(p_1)}, C_r^{(j)} \mid P_1 = p_1\right) \tag{F.21}$$

$$= \sum_{j \in I_{10} \cap \mathcal{R}_1(p_1)} \sum_{r=1}^{R_1(p_1)} \frac{1}{r} \Pr\left(P_{2j} \leq \frac{rq_2}{R_1(p_1)} \mid P_1 = p_1\right) \Pr\left(C_r^{(j)} \mid P_{2j} \leq \frac{rq_2}{R_1(p_1)}, P_1 = p_1\right)$$

$$\leq \frac{q_2}{R_1(p_1)} \sum_{j \in I_{10} \cap \mathcal{R}_1(p_1)} \sum_{r=1}^{R_1(p_1)} \Pr\left(C_r^{(j)} \mid P_{2j} \leq \frac{rq_2}{R_1(p_1)}, P_1 = p_1\right) \leq \frac{q_2}{R_1(p_1)} |I_{10} \cap \mathcal{R}_1(p_1)|. \tag{F.22}$$