Appendix.

The expressions for collective mode frequencies contain series such as the dipole sums $\sigma(\vec{k})$ and $\sigma_c(\vec{k})$. Here and below in this Section we will use the dimensionless vector $\vec{l}$ and the condition $\vec{l} \neq 0$ in the sums is implied. The mathematical properties of such series are important not only for this problem, but also for any example of lattice systems with identical particles coupled by the dipole interaction.

Let us discuss properties of these series. As we will demonstrate these series have very singular behavior as a functions of quasimomentum $\vec{k}$, and the double sum converges rather slowly. This manifests in sum properties near symmetrical points of the reciprocal lattice $\vec{k}_0$, especially near the point of origin $\vec{k}_0 = 0$ or near the points of the type of M (1,1), for which $\vec{k}_0 = \pm(\vec{e}_x \pm \vec{e}_y)\pi/a$, and X (1,1), for which $\vec{k}_0 = \pm(\vec{e}_x)\pi/a$ or $\vec{k}_0 = (\pm\vec{e}_y)\pi/a$. Analyzing small deviations from these points, $\vec{k} = \vec{k}_0 + \vec{q}$, where $\vec{q}$ is small, one normally has to calculate derivatives such as $[\partial^2\sigma(\vec{k})/\partial q_i \partial q_j]$ at the point $\vec{k} = \vec{k}_0$. Term by term differentiation of the dipole sums gives series like $\sum (1/|\vec{l}|) \cdot \exp(i\vec{k}_0\vec{l})$, which are alternating and converge only conditionally. Moreover, for $\vec{k}_0 = 0$, i.e., for the physically most interesting case of long wave oscillations, the corresponding coefficient of $\vec{q}^2$ is described by the divergent series $\sum 1/|\vec{l}|$. The same property is present for the complex sum $\sigma_c(\vec{k})$, which is also important for the description of dipole-coupled modes. Therefore, the dipole sums at $\vec{k} \simeq 0$ can be non-analytical and it results in a non-standard dispersion relation for oscillations described above.

Let us start with study of the sum $\sigma(\vec{k})$ for small $|\vec{k}|/k_B$. To analyze the behavior of $\sigma(\vec{k})$ near the point $\vec{k} = 0$ one can write $\vec{k} = \vec{q}$, $|\vec{q}| \ll 1/a$ and present this series as

$$\sum \frac{e^{i\vec{q}\vec{l}}}{\left|\vec{l}\right|^3} = \sum \frac{1}{\left|\vec{l}\right|^3} \cdot \left[e^{i\vec{q}\vec{l}} \cdot e^{-\alpha \vec{l}^2} + e^{i\vec{q}\vec{l}} \cdot (1 - e^{-\alpha \vec{l}^2}) \right] , \tag{27}$$

where the multiplier $\exp(-\alpha|\vec{l}|^2)$ is chosen to provide a fast convergence of the corresponding series. The first term on the right hand side of Eq. (27) converges rapidly for $|\vec{l}| > 1/\sqrt{\alpha}$ and at $q = |\vec{q}| \to 0$ it contributes as $-q^2 D'$, where $D' = \sum (1/|\vec{l}|) \exp(-\alpha \vec{l}^2)$ is an absolutely converging series. Non-analyticity as $q \to 0$ is defined by the second term, calculation of which can be simplified. Indeed, at $|\vec{l}| \ll 1/\sqrt{\alpha}$ it comprises the small multiplier $\alpha \vec{l}^2$, and the contribution from the region $\left|\vec{l}\right| < 1/\sqrt{\alpha}$ is expected to be small. For the outer region $|\vec{l}| > 1/\sqrt{\alpha}$ one can expect that discreteness effects are small and the sum can be replaced