B. 2d case

The $d = 2$ case is somewhat more subtle due to the marginality of the coupling g. The fixed point $\widehat{g}_* = 0$ and the RG flows are depicted in Figure 1. The fixed point at $\widehat{g}_* = 0$ is just the free field theory, thus there is no analog of the BEC/BCS cross-over. One can nevertheless formally define the unitary limit as $S = -1$, as in $1d$ and $3d$. In this subsection we explore this possibility and interpret it using the renormalization group. As we'll see, this limit occurs at $g = \pm\infty$, and the scattering length diverges.

First begin with the beta-function in 2d, eq. (110):

$$\frac{dg}{d \log \Lambda} = \frac{mg^2}{4\pi} \tag{11}$$

Let $g = g_0$ at some arbitrary scale Λ_0. Integrating the beta function one finds:

$$g(\Lambda) = \frac{g_0}{1 - \frac{mg_0}{4\pi} \log(\Lambda/\Lambda_0)} \tag{12}$$

Note that g diverges at the scale:

$$\Lambda_* = \Lambda_0 e^{4\pi/mg_0} \tag{13}$$

This is the familiar Landau pole. Whereas in for example the relativistic ϕ^4 theory in $d = 3$ where the Landau pole is unphysical due to higher order corrections, here the beta function (11) is exact[15], thus this divergence is physical. There are two cases to consider:

Attractive case: If g_0 is negative, $\Lambda_* < \Lambda_0$, and thus $g = -\infty$ occurs in the infra-red.

Repulsive case: If g_0 is positive, $\Lambda_* > \Lambda_0$, and $g = +\infty$ occurs in the ultra-violet.

Both cases are consistent with the flows depicted in Figure 1.