## Theorem 1

**Theorem 1** *Under M.1 − M.4, sharp bounds on marginal distributions of $Y_0$ and $Y_1$, their joint distribution and the DTE are obtained as follows: for $d \in \{0,1\}$, $y \in \mathbb{R}$, $\delta \in \mathbb{R}$, and $(y_0, y_1) \in \mathbb{R} \times \mathbb{R}$,*

$$
\begin{aligned}
F_d(y) &\in \left[F_d^L(y), F_d^U(y)\right], \\
F(y_0, y_1) &\in \left[F^L(y_0, y_1), F^U(y_0, y_1)\right], \\
F_\Delta(\delta) &\in \left[F_\Delta^L(\delta), F_\Delta^U(\delta)\right],
\end{aligned}
$$

*where*

$$
\begin{aligned}
F_0^L(y) &= \sup_{z \in \Xi} \left[P\{y|0,z\}(1-p(z)) + L_{01}^{wst}(y,z)\right], \qquad (11)\\
F_0^U(y) &= \inf_{z \in \Xi} \left[P\{y|0,z\}(1-p(z)) + U_{01}^{wst}(y,z)\right], \\
F_1^L(y) &= \sup_{z \in \Xi} \left[P\{y|1,z\}p(z) + L_{10}^{wst}(y,z)\right], \\
F_1^U(y) &= \inf_{z \in \Xi} \left[P\{y|1,z\}p(z) + U_{10}^{wst}(y,z)\right], \\
F^L(y_0,y_1) &= \sup_{z \in \Xi} \left[\begin{array}{c} \max\left\{(P(y_0|0,z)-1)(1-p(z)) + L_{10}^{wst}(y_1,z), 0\right\} \\ + \max\left\{L_{01}^{wst}(y_0,z) + (P(y_1|1,z)-1)p(z), 0\right\} \end{array}\right], \\
F^U(y_0,y_1) &= \inf_{z \in \Xi} \left[\begin{array}{c} \min\left\{P(y_0|0,z)(1-p(z)), U_{10}^{wst}(y_1,z)\right\} \\ + \min\left\{U_{01}^{wst}(y_0,z), P(y_1|1,z)p(z)\right\} \end{array}\right], \\
F_\Delta^L(\delta) &= \sup_{z \in \Xi} \left[\begin{array}{c} \sup_{y \in \mathbb{R}} \max\left\{P(y|1,z)p(z) - U_{01}^{wst}(y-\delta,z), 0\right\} \\ + \sup_{y \in \mathbb{R}} \max\left\{L_{10}^{wst}(y,z) - P(y-\delta|0,z)(1-p(z)), 0\right\} \end{array}\right], \\
F_\Delta^U(\delta) &= 1 + \inf_{z \in \Xi} \left[\begin{array}{c} \inf_{y \in \mathbb{R}} \min\left\{P(y|1,z)p(z) - L_{01}^{wst}(y-\delta,z), 0\right\} \\ + \inf_{y \in \mathbb{R}} \min\left\{U_{10}^{wst}(y,z) - P(y-\delta|0,z)(1-p(z)), 0\right\} \end{array}\right].
\end{aligned}
$$

**Proof.** The proof consists of three parts: sharp bounds on (i) marginal distributions, (ii) the joint distribution, and (iii) the DTE.

**Part 1. Sharp bounds on marginal distributions $F_0(\cdot)$ and $F_1(\cdot)$**