where

$$\tilde{S}_{l_1 l_2 l_3} = 1, \quad \tilde{S}_{\mathrm{I}\, l_1 l_2 l_3} = -\frac{\{l_1\} + \{l_2\} + \{l_3\}}{6q^2}, \tag{47}$$

$$\tilde{S}_{\mathrm{II}\, l_1 l_2 l_3} = \frac{\{l_1\}^2 + \{l_2\}^2 + \{l_3\}^2 - 2\{l_1\}\{l_2\} - 2\{l_2\}\{l_3\} - 2\{l_3\}\{l_1\}}{12q^4}, \tag{48}$$

$$\tilde{K}^{l_1 l_2}_{l_3 l_4}(L) = 1, \quad \tilde{K}_{\mathrm{I}\,l_3 l_4}^{\ l_1 l_2}(L) = -\frac{\{l_1\} + \{l_2\} + \{l_3\} + \{l_4\}}{8q^2}, \tag{49}$$

$$\tilde{K}_{\mathrm{II}\,l_3 l_4}^{\ l_1 l_2}(L) = \frac{\{L\}^2 - (\{l_1\} + \{l_2\})(\{l_3\} + \{l_4\})}{16q^4}, \tag{50}$$

$$\tilde{K}_{\mathrm{III}\,l_3 l_4}^{\ l_1 l_2}(L) = \frac{(\{l_1\} + \{l_2\} - \{L\})(\{l_3\} + \{l_4\} - \{L\})}{32q^4}. \tag{51}$$

The above forms of skewness parameters [Eq. (47), (48)] were already appeared in [35].

Resemblances of the above results to those of the flat space are obvious if the integrands of Eqs. (30)–(32) are symmetrized with respect to l_1, l_2, l_3, and l_4 [conversely, one can desymmetrize the Eqs. (47)–(51) to have the similar form with Eqs. (30)–(32)]. Noting the all-sky and flat-sky correspondence [44, 46, 47], it is a straightforward exercise to show that the above all-sky equations reduce to those of flat-sky in the large-l limit. Following [44, 46, 47], but applying an improved approximation

$$Y_l^m(\theta, \phi) \approx (-1)^m \sqrt{\frac{2l+1}{4\pi}}\, J_m\left[\left(l + \frac{1}{2}\right)\theta\right] e^{im\phi}, \tag{52}$$

for $\theta \ll 1$, $l \gg 1$, the correspondences between all-sky and flat-sky spectra are derived as

$$C_l \approx C\left(l + \frac{1}{2}\right), \tag{53}$$

$$B_{l_1 l_2 l_3} \approx I_{l_1 l_2 l_3} B\left(l_1 + \frac{1}{2}, l_2 + \frac{1}{2}, l_3 + \frac{1}{2}\right), \tag{54}$$

$$P^{l_1 l_2}_{l_3 l_4}(L) \approx I_{l_1 l_2 L} I_{l_3 l_4 L} \times P\left(l_1 + \frac{1}{2}, l_2 + \frac{1}{2}; l_3 + \frac{1}{2}, l_4 + \frac{1}{2}; L + \frac{1}{2}\right), \tag{55}$$

and

$$T(l_1, l_2, l_3, l_4; l_{12}, l_{23}) = P(l_1, l_2; l_3, l_4; l_{12}) + P(l_1, l_3; l_2, l_4; l_{13}) + P(l_1, l_4; l_2, l_3; l_{23}), \tag{56}$$

where $l_{13} = [l_1^2 + l_2^2 + l_3^2 + l_4^2 - l_{12}^2 - l_{23}^2]^{1/2}$.

Since the all-sky multipole ℓ and the flat-sky wavenumber $|\boldsymbol{l}|$ are related by $|\boldsymbol{l}| = \ell + 1/2$, contributions of the multipole ℓ to the all-sky summation is approximately represented by the flat-sky integration over the range $\ell - 1/2 \le |\boldsymbol{l}| - 1/2 < \ell + 1/2$, i.e., $\ell \le |\boldsymbol{l}| < \ell + 1$. Thus, all-sky summations over $\ell = 2, 3, \ldots$ correspond to the flat-sky integrations with the limit $|\boldsymbol{l}| \ge 2$, as noted above. We confirm that the flat-sky approximations of Eqs. (24)–(26) with the above correspondences numerically reproduce the values calculated from all-sky formula of Eqs. (44)–(46) within several percent for $\theta_s < 100'$.

For numerical evaluations of the kurtosis and its derivatives by the summation of Eq. (46), the number of terms to add is of order $O(l_{\max}^6)$, where $l_{\max}$ is the maximum multipole required for a given smoothing scale, e.g., $l_{\max} \sim$ several $\times\, \theta_s^{-1}$. The computational cost becomes progressively high for large $l_{\max}$, if the summation is naively performed. Efficient evaluations are necessary when the smoothing angle θ_s is small. In Appendix A, numerical schemes for the efficient evaluations are summarized.

IV. A SIMPLE EXAMPLE: THE LOCAL MODEL OF NON-GAUSSIANITY

The analytic MFs are evaluated once the power spectrum, bispectrum and trispectrum are given. These spectra depend on models of primordial density fluctuations. As a simple example, we consider below the local model of non-Gaussianity, although our formulas are not restricted to this particular model.

In the local model, the primordial curvature perturbations during the matter era is assumed to take the form [42, 43, 48–50]

$$\Phi(\boldsymbol{x}) = \phi(\boldsymbol{x}) + f_{\mathrm{NL}}\left(\phi^2(\boldsymbol{x}) - \langle\phi^2\rangle\right) + g_{\mathrm{NL}}\phi^3(\boldsymbol{x}), \tag{57}$$

in configuration space, where ϕ is an auxiliary random Gaussian field. The comoving curvature perturbation ζ is given by $\zeta = 3\Phi/5$. The CMB fluctuations generated by the curvature perturbations have the harmonic coefficients

$$a_{lm} = 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3} \tilde{\Phi}(\boldsymbol{k}) g_l^{(\mathrm{T})}(k) Y_l^{m*}(\hat{\boldsymbol{k}}), \tag{58}$$

where $\tilde{\Phi}(\boldsymbol{k})$ is the Fourier transform of the primordial curvature perturbation $\Phi(\boldsymbol{x})$, and $g_l^{(\mathrm{T})}(k)$ is the radiation transfer function.

The bispectrum and trispectrum of CMB in the local model of Eq. (57) are derived in literature [43, 49]:

$$B_{l_1 l_2 l_3} = 2 f_{\mathrm{NL}} I_{l_1 l_2 l_3}\left[\int r^2 dr \alpha_{l_1}(r)\beta_{l_2}(r)\beta_{l_3}(r) + \text{cyc.}\right], \tag{59}$$

$$\begin{aligned}\mathcal{T}^{l_1 l_2}_{l_3 l_4}(L) = I_{l_1 l_2 L} I_{l_3 l_4 L} &\times \Bigg\{4 f_{\mathrm{NL}}^2 \int r_1^2 dr_1 r_2^2 dr_2 F_L(r_1, r_2)\alpha_{l_1}(r_1)\beta_{l_2}(r_1)\alpha_{l_3}(r_2)\beta_{l_4}(r_2) \\ &+ g_{\mathrm{NL}} \int r^2 dr \beta_{l_2}(r)\beta_{l_4}(r)\left[\alpha_{l_1}(r)\beta_{l_3}(r) + \beta_{l_1}(r)\alpha_{l_3}(r)\right]\Bigg\}\end{aligned} \tag{60}$$

where

$$F_L(r_1, r_2) \equiv 4\pi \int \frac{k^2 dk}{2\pi^2} P_\phi(k) j_L(kr_1) j_L(kr_2), \tag{61}$$

$$\alpha_l(r) \equiv 4\pi \int \frac{k^2 dk}{2\pi^2} g_l^{(\mathrm{T})}(k) j_l(kr), \tag{62}$$

$$\beta_l(r) \equiv 4\pi \int \frac{k^2 dk}{2\pi^2} P_\phi(k) g_l^{(\mathrm{T})}(k) j_l(kr), \tag{63}$$