satisfied

$$y(t) = \int_{-\infty}^{\infty} G_\beta(t,s)(H_1(y(s),\nu_\beta(s),s) + h(x(s)))ds, \tag{2.10}$$

where

$$G_\beta(t,s) = \begin{cases} V_\beta(t)P_\beta V_\beta(s)^{-1} & t \geq s, \\ V_\beta(t)(I-P_\beta)V_\beta(s)^{-1} & t \leq s, \end{cases}$$

and

$$H_1(y_1,y_2,s) = f(y_1+y_2,s) - f(y_2,s) - D_u f(y_2,s)y_1.$$

Now, we are ready to prove the following result.

Lemma 2.1 *If conditions $(C1)-(C6)$ hold, then for each $x(t) \in \mathscr{A}_x$ there exists a unique solution $\phi^\beta_{x(t)}(t)$ of (2.9) such that $\sup\limits_{t\in\mathbb{R}} \left\|\phi^\beta_{x(t)}(t)\right\| \leq r_1$ for a constant $r_1 = \dfrac{\frac{4KP_2}{\alpha}M_h}{1+\sqrt{1-\frac{16K^2P_2^2}{\alpha^2}M_h}}$.*

Proof. Consider the set $C_0(\mathbb{R})$ of continuous functions $y(t)$ satisfying $\|y\|_0 \leq r_1$, where $\|y\|_0 = \sup\limits_{t\in\mathbb{R}} \|y(t)\|$. Define the operator Π on C_0 as

$$\Pi y(t) = \int_{-\infty}^{\infty} G_\beta(t,s)(H_1(y(s),\nu_\beta(s),s) + h(x(s)))ds.$$

Using the mean value theorem we obtain that

$$\begin{aligned} \|H_1(y(s),\nu_\beta(s),s)\| &\leq \int_0^1 \|D_u f(\theta y_1 + y_2,s) - D_u f(y_2,s)\| \, d\theta \|y_1\| \\ &\leq \int_0^1\int_0^1 \|D_{uu} f(\tau\theta y_1 + y_2,s)\| \, d\tau d\theta \|y_1\|^2 \\ &\leq P_2\|y_1\|^2. \end{aligned}$$

Hence, the inequality

$$\|\Pi y(t)\| \leq \frac{2K(P_2\|y\|_0^2 + M_h)}{\alpha}$$

is valid so that

$$\|\Pi y\|_0 \leq \frac{2K(P_2\|y\|_0^2 + M_h)}{\alpha}.$$

Let

$$H_2(y_1,y_2,y_3,s) = f(y_1+y_2,s) - f(y_3+y_2,s) - D_u f(y_2,s)(y_1-y_3).$$