ergetically higher branch has purely real stability eigenvalues $\pm\lambda^R$, corresponding to an unstable excited state $\mathrm{e}^{\mathrm{N}=1}$, the lower branch possesses purely imaginary eigenvalues $\pm\lambda^I$ and corresponds to the stable ground state $\mathrm{g}^{\mathrm{N}=1}$. At the bifurcation point the branches of the stability eigenvalues merge and vanish.

The situation is different if the condensate wave function is described by more than one Gaussian. As the scattering length is decreased from positive values towards the tangent bifurcation, the branch corresponding to the ground state $\mathrm{g}^{\mathrm{coupled}}$ turns into an unstable state $\mathrm{u}^{\mathrm{coupled}}$ at a scattering length of $a_{\mathrm{cr}}^{\mathrm{p}} = -0.00359$. This is evident from the stability analysis shown in Fig. 3 (b) where the stability eigenvalues for the ground state, calculated using 6 Gaussians, are plotted in a small interval of the scattering length around $a_{\mathrm{cr}}^{\mathrm{p}}$. Above $a_{\mathrm{cr}}^{\mathrm{p}}$ the eigenvalues are purely imaginary, below they are purely real. [Note that in a Bogoliubov analysis this instability should appear as a dynamical instability.] The ground state remains unstable down to the tangent bifurcation point at $a_{\mathrm{cr}}^{\mathrm{t}} = -0.01224$, where it joins the branch of the unstable excited state.

The quality of the calculation using 5 Gaussian wave packets is also demonstrated in Fig. 3 (a) where the results of a numerically grid calculation by imaginary time evolution are shown by crosses. Evidently the numerical results and the results obtained using 5 coupled Gaussians excellently agree. The imaginary time calculation, however, can only trace the stable branch of the solution and fails for the unstable branch. Thus it is demonstrated that the Gaussian wave packet method is not only numerically accurate but also capable of giving access to regions of the space of solutions of the Gross-Pitaevskii equation with dipolar interaction that are difficult to investigate by conventional numerical full quantum calculations.

The phenomenon of one smooth branch of solutions becoming unstable as a function of a control parameter is reminiscent of a *pitchfork* bifurcation. The two stable solutions on the fork arms which should also be born in a pitchfork bifurcation, and exist in a tiny neighborhood $(a_{\mathrm{cr}}^{\mathrm{p}} - \epsilon) < a < a_{\mathrm{cr}}^{\mathrm{p}}$, are numerically hard to trace and therefore not plotted in the figure. Their existence, and the pitchfork type of the bifurcation, however, can be proven by looking at the "phase portrait" plotted in Fig. 4 at a value of the scattering length $a = -0.036$ slightly below $a_{\mathrm{cr}}^{\mathrm{p}}$. Figure 4 shows contours of equal deviation of the mean field energy from that of the ground state in the plane spanned by the two eigenvectors whose eigenvalues are involved in the stability change in Fig 3 (b). The coordinate axes δ_1, δ_2 correspond to small variations of the Gaussian parameters in the eigenvector directions around the hyperbolic fixed point solution located at the origin. The portrait clearly reveals the existence of two nearby elliptic fixed points corresponding to two additional stable solutions. Therefore, in a small interval ϵ of the scattering length below $a_{\mathrm{cr}}^{\mathrm{p}}$, $(a_{\mathrm{cr}}^{\mathrm{p}} - \epsilon) < a < a_{\mathrm{cr}}^{\mathrm{p}}$, there exist two additional branches, besides the unstable solution. This proves that the bifurcation is of pitchfork type. Note that the classification of the condensate as unstable for $a < a_{\mathrm{cr}}^{\mathrm{p}}$ nevertheless remains true in physical terms due to the numerically small value of ϵ. We also note that for $a > a_{\mathrm{cr}}^{\mathrm{p}}$ the phase portrait possesses only one elliptic fixed point cooresponding to the stable stationary ground state.

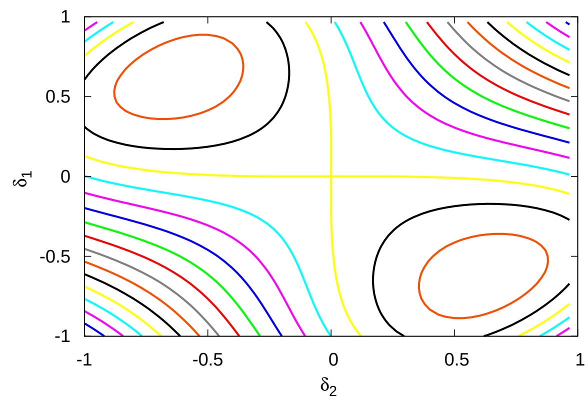

FIG. 4: Contour plot of the mean field energy with the eigenvectors corresponding to the eigenvalues of Fig. 3 (b) linearizing the vicinity of the fixed point (δ_1, δ_2 in arbitrary units). The figure shows $a = -0.0036$ close below the pitchfork bifurcation point, showing three fixed points: Two stable and one hyperbolic.

Is there a chance of observing dipolar BECs on the stable fork arms? The answer probably is no, in the same way as it is in the case of the question of observing the transition to structured ground states, possibly associated with a roton instability, shortly before collapse. The reason is the difficulty of adjusting trap frequencies and the scattering length to the necessary precision in a real experiment. Nevertheless theoretical investigations of this type close to the threshold of instability of dipolar condensates are valuable in their own right since they help to understand the nature of the collapse, and thus of "what's really going on".

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