where $\gamma(k)$ is also non-negative for real argument. The angle $\delta'(k)$ is defined by

$$\begin{aligned}\cosh(2K_2) &= \cosh(2K_1^*)\cosh(\gamma(k)) \\ &\quad - \sinh(2K_1^*)\sinh(\gamma(k))\cos(\delta'(k))\end{aligned} \tag{4.45}$$

and

$$\sin(\delta'(k))\sinh(\gamma(k)) = \sin k \sinh(2K_2) \tag{4.46}$$

Notice that the term multiplying $\sinh(\gamma(k))$ in (4.43) is itself a spin operator in the paired subspace. Raising (4.43) to the N^{th} power is, then, straightforward:

$$\begin{aligned}&(\mathbf{V}'(k))^N \\ &= \cosh(N\gamma(k)) \\ &\quad - \sinh(N\gamma(k))(\cos(\delta'(k))\tau^z(k) - \sin(\delta'(k))\tau^y(k))\end{aligned} \tag{4.47}$$

Since $\tau^z(k)|0\rangle = -|0\rangle$ and for M even, $\exp(i\pi M) = 1$, evaluating the matrix element in (4.8) is an easy matter:

$$\begin{aligned}&Z_f \\ &= \exp[N(K_1^* + K_2)] \\ &\quad \times \prod_{0<k<\pi} \{\cosh(N\gamma(k)) + \sinh(N\gamma(k))\cos(\delta'(k))\}\end{aligned} \tag{4.48}$$

Checking the behavior of $\delta'(\pi)$, we see that this equation can be written as

$$Z_f = \prod_{k\in\Omega_M(+)} \{\cos(N\gamma(k)) + \sinh(N\gamma(k))\cos(\delta'(k))\}^{1/2} \tag{4.49}$$

Thus

$$\begin{aligned}&\ln Z_f \\ &= \frac{1}{2}\sum_{k\in\Omega_M(+)} \ln\left(\cosh(N\gamma(k)) + \sinh(N\gamma(k))\cos(\delta'(k))\right)\end{aligned} \tag{4.50}$$

This is in fact correct whether M is odd or even. The result for Z_f^a, the case in which antiperiodicity is enforced in each M-spin row, is a simple variation on (4.50). One simply replaces the requirement $k \in \Omega_M(+)$ in the sum by $k \in \Omega_M(-)$.

B. Periodic and antiperiodic boundary conditions

The limit as $N \to \infty$ is easily taken, since $M < \infty$ implies that the sum is finite. Another way of examining this is to go back to (4.12). As $N \to \infty$, we expect only the contribution of the maximum eigenvector to survive, this provided its scalar product with $|0\rangle$ does not vanish, which turns out to be the case on detailed calculation. The summand in (1.45) becomes $N\gamma$, and the sum may be implemented as a contour integral, following the methods of [23, 24]:

$$\lim_{N\to\infty} N^{-1}\ln Z_f = -\frac{M}{4\pi}\oint_C \frac{d\omega}{1+e^{iM\omega}}\gamma(\omega) \tag{4.51}$$

where C is the rectangle in the complex ω plane with sides which are segments of the lines $\mathrm{Re}(\omega) = \pm\pi, \mathrm{Im}(\omega) = \pm\epsilon$. The number ϵ is chosen small enough so that the only singularities of the integrand inside the contour are the zeros of the denominator. Equation (1.46) can then be written as:

$$\begin{aligned}\lim_{N\to\infty} N^{-1}\ln Z_f &= \frac{M}{4\pi}\int_{-\pi}^{\pi} d\omega\ \gamma(\omega) \\ &\quad - \frac{M}{2\pi}\int_{-\pi+i\epsilon}^{\pi+i\epsilon} d\omega \frac{e^{iM\omega}}{1+e^{iM\omega}}\gamma(\omega)\end{aligned} \tag{4.52}$$

The first term on the right hand side is the bulk term. The Casimir contribution is contained in the second term on the right hand side of (4.52). The function γ has branch cuts on the imaginary axis symmetrically positioned about the real axis. Using the 2π periodicity of the integrand in (4.52), the contour in (4.52) will be deformed into one surrounding the branch cut in the upper half plane. It is convenient to make the transformation $\omega = i\hat{\gamma}(u)$ and interchange $K1$ and $K2$ in Eq. (4.44). Expanding the denominator, integrating by parts and resumming gives the Casimir free energy from (4.52) as

$$F_{cas} = -\frac{k_B T}{\pi}\int_0^{\pi} du \ln\left[1 + \exp\left(-M\hat{\gamma(u)}\right)\right] \tag{4.53}$$

The resulting Casimir force is denoted $f_{cas}(cyl.)$ and is just:

$$f_{cas}(cyl.) = -\frac{k_B T}{4\pi}\int_{-\pi}^{\pi} du\ \hat{\gamma}(u)\left[1 - \tanh\left(M\hat{\gamma}(u)/2\right)\right] \tag{4.54}$$

Note that this expression is invariant under interchange of K_1^* and K_2; that is, there is a dual symmetry; further, this force is *attractive*. The scaling form (with $K_1 = K_2$) is accessed by noting that $\hat{\gamma}(u) \sim [\hat{\gamma}(0)^2 + u^2]^{1/2}$ for small u and $\hat{\gamma}(0)$. We then introduce scaling variables $y = Mu/2$ and $x = M\hat{\gamma}(0)/2$ and take the scaling limit, giving:

$$\begin{aligned}M^2\bar{f}_{cas}(cyl.) &\to -\frac{k_B T}{\pi}\int_{-\infty}^{\infty} dy(x^2+y^2)^{1/2} \\ &\quad \left[1 - \tanh\left((x^2+y^2)^{1/2}\right)\right]\end{aligned} \tag{4.55}$$

This implies the same scaling function as in (3.15). The integral in (4.55) has the value $\pi^2/12$ at $x = 0$.