In other words, in terms of

$$h(\nu) = \sum_{i=0}^{d-1} \epsilon_i(\nu) \left|i\right\rangle \left\langle i\right| \quad , \tag{A1}$$

the Hamiltonians of individual subsystems can be expressed as $H_{X_k} = h(\nu_k)$. The bath Hamiltonian H_B is the sum of the subsystem Hamiltonians and therefore the energy eigenvalue of the bath state $\left|n\right\rangle_B = \left|i_1, i_2, i_3, \cdots\right\rangle_B$ is

$$E_n = \sum_{k=1}^{\infty} \epsilon_{i_k}(\nu_k) \quad . \tag{A2}$$

Note that if $\{i_k\}$ had infinitely many nonzero entries, then the corresponding energy would be infinite. This justifies the definition of the Hilbert space $\mathcal{H}_B$ in the way described above.

Let $\sigma(\nu) = \exp(-\beta h(\nu))/\zeta(\nu)$ be the parameter-dependent density matrix and $\zeta(\nu)$ be the corresponding partition function. The shorthand notation $\sigma_k = \sigma(\nu_k)$ will be used for the thermal equilibrium states of X_k. The partition function of the bath is then given by

$$Z_B = \prod_{k=1}^{\infty} \zeta(\nu_k) \quad . \tag{A3}$$

Note that the convergence of this product depends only on the large k behavior of the sequence $\{\nu_k\}$. By adjusting the way ν_k converges to 1, it is possible obtain a finite Z_B.

Let us now compute the HTO. In the following expressions, the subscripts are used for indicating the subsystem a given density matrix applies to. Suppose that the initial state of the device A is ρ. The initial and final states of AB are given by the following self-explanatory expressions.

$$\rho_{AB} = (\ \rho\ )_A \otimes (\sigma_1)_{X_1} \otimes (\sigma_2)_{X_2} \otimes \cdots \ , \tag{A4}$$

$$\rho'_{AB} = (\left|\psi_0\right\rangle \left\langle\psi_0\right|)_A \otimes (\ \rho\ )_{X_1} \otimes (\sigma_1)_{X_2} \otimes \cdots \ . \tag{A5}$$

The change in the average energy of the bath is given by

$$\Delta E_B = \mathrm{tr}(\rho Q) - k_B T J + \Delta \tag{A6}$$

where $J = J(\beta Q)$ and

$$\Delta = k_B T J - q_0 - \mathrm{tr}\, \sigma_1 H_{X_1} + \sum_{k=2}^{\infty} \mathrm{tr}(\sigma_{k-1} - \sigma_k) H_{X_k} \quad , \tag{A7}$$