operator with only two indices. With this in mind, we define the *reduced density matrix*:

$$\rho_{red}(x_i, x_f, \beta) \equiv \int dr \rho(x_i, r; x_f, r; \beta). \tag{6}$$

In the next section it will be the only operator practical for calculating thermal averages.

For the system defined by Eq. 1, we can now write a path-integral description of the reduced density matrix,

$$\begin{aligned}\rho_{red}(x_i, x_f, \beta) &= \int dr \int \mathcal{D}x\, \mathcal{D}r' \ \exp\left(-S_S^E[x] - S_I^E[x, r']\right) \\ &= \int \mathcal{D}x \exp\left(-S_x^E[x]\right) \int dr \int \mathcal{D}r' \exp\left(-S_I'^E[x, r']\right),\end{aligned} \tag{7}$$

where we integrate over paths with endpoints $x(0) = x_i$, $x(\beta) = x_f$, $r'(0) = r$, and $r'(\beta) = r - A[x, \beta]$. The final step in 7 allows the integral over the paths $r'(\tau)$ to be performed independently of the integral over $x(\tau)$. Using Eq. 4, a simple Gaussian integral yields

$$\begin{aligned}\rho_{red}(x_i, x_f, \beta) &= \int_{x(0)=x_i}^{x(\beta)=x_f} \mathcal{D}x \\ \times \ \exp &\left(-S_S^E[x] + \sum_i \frac{C_i^2}{2m\omega_i \sinh(\frac{\omega_i \beta}{2})} \int_0^\beta d\tau \int_0^\tau ds\, x(\tau)x(s)\cosh\left[\omega_i\left(\tau - s - \beta/2\right)\right]\right)\end{aligned} \tag{8}$$

where the summation is introduced because we have generalized this to a system where a heavy particle interacts with a bath of simple harmonic oscillators. This is now the path integral form for the reduced density matrix. This is analogous to the idea of influence functionals, worked out for real-time propagators many years ago [21].

B. Making a dissipative system

Any finite quantum-mechanical system is reversible and therefore inappropriate for describing Brownian motion. One might find it intuitive that if the bath of harmonic oscillators were taken to an infinite limit, it would be "large enough" so that energy from the heavy particle could dissipate into the system and never return. This intuition was proven to be true by the authors of [22], who considered the real-time evolution of the density matrix for our system and showed that when the bath of harmonic oscillators is determined by the continuous density of states

$$C^2(\omega)\rho_D(\omega) = \begin{cases} \frac{2m\eta\omega^2}{\pi} & \text{if } \omega < \Omega \\ 0 & \text{if } \omega > \Omega \end{cases} \tag{9}$$