Tensor Models for Velocity Field Extraction

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1 GADIM

$$
\begin{array}{ll}
\min & f(u) + g(v) + h(w) \\
\text{s.t.} & Au + Bv + Cw = 0
\end{array}
$$

$$
\left\{
\begin{array}{l}
u^k = \arg\min_u f(u) + \frac{1}{2\alpha}\|u - p^k\|_2^2 \\
v^k = \arg\min_v g(v) + \frac{1}{2\alpha}\|v - q^k\|_2^2 \\
w^k = \arg\min_w h(w) + \frac{1}{2\alpha}\|w - r^k\|_2^2 \\
\lambda^{k+1} = (I + \beta^2 AA^* + \beta^2 BB^* + \beta^2 CC^*)^{-1}\big[\lambda^k - \beta\big(A(2u^k - p^k) + B(2v^k - q^k) + C(2w^k - r^k)\big)\big] \\
p^{k+1} = p^k + \omega(u^k - p^k + \beta A^*\lambda^{k+1}) \\
q^{k+1} = q^k + \omega(v^k - q^k + \beta B^*\lambda^{k+1}) \\
r^{k+1} = r^k + \omega(w^k - r^k + \beta C^*\lambda^{k+1})
\end{array}
\right.
$$

2 Tensor Models

$$
\begin{array}{ll}
\min & f(u) + g(v) + h(w) \\
\text{s.t.} & \begin{bmatrix} \nabla_x \\ \nabla_y \\ \nabla_t \end{bmatrix} u - \begin{bmatrix} I_x & 0 \\ 0 & I_y \\ 0 & 0 \end{bmatrix} v - \begin{bmatrix} 0 \\ 0 \\ I_t \end{bmatrix} w = 0
\end{array}
$$

u-update:

$$
f(u) = \|d_x \odot u_x + d_y \odot u_y + d_t\|_1
$$

$$
\min_x \mu|a^*x - b| + \frac{1}{2}\|x - c\|_2^2
$$

$$
x_{\text{opt}} = \left\{
\begin{array}{ll}
c - a \cdot \text{sign}\big(\frac{a^*c-b}{a^*a}\big) \cdot \min\big\{\big|\frac{a^*c-b}{a^*a}\big|, \mu\big\}, & \text{if } a \neq 0 \\
c, & \text{if } a = 0
\end{array}
\right.
$$

$$
x_{\text{opt}} = c - \frac{a}{a^*a} \cdot \text{sign}(a^*c - b) \cdot \min\big\{|a^*c - b|, \mu a^*a\big\}
$$

v-update:

$$
g(v) = \rho\big\||[v_{xx}, v_{xy}, v_{yx}, v_{yy}]|\big\|_1
$$

$$
\min_x \mu\|x\|_2 + \frac{1}{2}\|x - c\|_2^2
$$

$$
x_{\text{opt}} = c \cdot \max\left\{1 - \frac{\mu}{\|c\|_2}, 0\right\}
$$

w-update:

$$
h(w) = \frac{\tau}{2}\|w\|_2^2
$$

$$\min_x \frac{\mu}{2}\|x\|_2^2 + \frac{1}{2}\|x - c\|_2^2$$

$$x_{\text{opt}} = \frac{c}{1+\mu}$$

λ-update:

$$
\begin{aligned}
&(I + \beta^2 AA^* + \beta^2 BB^* + \beta^2 CC^*)^{-1} \\
&= \left((1+\beta^2)I + \beta^2 \begin{bmatrix} \nabla_x \\ \nabla_y \\ \nabla_t \end{bmatrix} \begin{bmatrix} \nabla_x^* & \nabla_y^* & \nabla_t^* \end{bmatrix} \right)^{-1} \\
&= \frac{1}{1+\beta^2} I - \frac{\beta^2}{1+\beta^2} \begin{bmatrix} \nabla_x \\ \nabla_y \\ \nabla_t \end{bmatrix} \Big((1+\beta^2)I + \beta^2(\nabla_x^*\nabla_x + \nabla_y^*\nabla_y + \nabla_t^*\nabla_t) \Big)^{-1} \begin{bmatrix} \nabla_x^* & \nabla_y^* & \nabla_t^* \end{bmatrix}
\end{aligned}
$$

$$\begin{bmatrix} \xi_x^k \\ \xi_y^k \\ \xi_t^k \end{bmatrix} = \lambda^k - \beta\big(A(2u^k - p^k) + B(2v^k - q^k) + C(2w^k - r^k)\big)$$

$$\eta^{k+1} = \Big((1+\beta^2)I + \beta^2(\nabla_x^*\nabla_x + \nabla_y^*\nabla_y + \nabla_t^*\nabla_t) \Big)^{-1} (\nabla_x^*\xi_x + \nabla_y^*\xi_y + \nabla_t^*\xi_t)$$

$$\lambda^{k+1} = \frac{1}{1+\beta^2} \begin{bmatrix} \xi_x^k \\ \xi_y^k \\ \xi_t^k \end{bmatrix} - \frac{\beta^2}{1+\beta^2} \begin{bmatrix} \nabla_x \\ \nabla_y \\ \nabla_t \end{bmatrix} \eta^{k+1}$$

p-update:

$$A^*\lambda^{k+1} = \eta^{k+1}$$

$$p^{k+1} = p^k + \omega(u^k - p^k + \beta\eta^{k+1})$$

q-update:

$$q^{k+1} = q^k + \omega\left(v^k - q^k - \beta \begin{bmatrix} \lambda_x^{k+1} \\ \lambda_y^{k+1} \end{bmatrix} \right)$$

r-update:

$$r^{k+1} = r^k + \omega(w^k - r^k - \beta\lambda_t^{k+1})$$