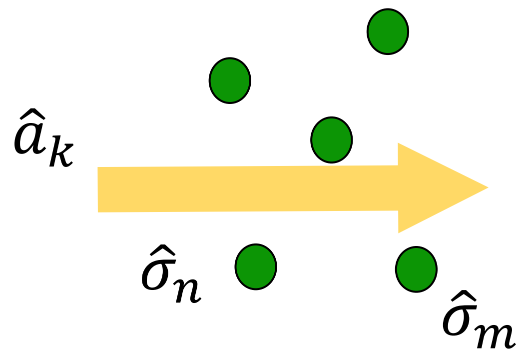
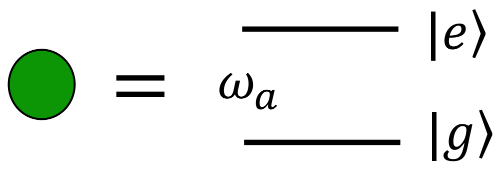
Model: atoms + photon modes



$$H = H_A + H_P + H_{AP}$$

"atoms": two-level emitters $n = 1, \dots, N$ at positions $\mathbf{r}_n$

$$H_A = \hbar\omega_a \sum_{n=1}^{N} \hat{\sigma}_n^\dagger \hat{\sigma}_n \qquad \hat{\sigma}_n = |g\rangle_n \langle e|$$



"photons" = continuum of photon modes $\hat{a}_k$ (modes defined by geometry) $\{k\}$ = mode index

$$H_P = \sum_k \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k$$

"atom-photon": dipole interaction (+rotating wave approx.)

mode profile

$$H_{AP} = -\sum_n \hat{E}(\mathbf{r}_n)\hat{\sigma}_n^\dagger d + h.c. \qquad \hat{E}(\mathbf{r}_n) = -i\sum_k \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0}}\, u_k(\mathbf{r}_n)\hat{a}_k$$

Heisenberg-Langevin approach

[see Lehmberg PRA 2, 883 (1970) for a full derivation]

$$H = \hbar\omega_a \sum_{n=1}^{N} \hat{\sigma}_n^\dagger \hat{\sigma}_n + \sum_k \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k - \sum_n \left[\hat{E}(\mathbf{r}_n)\hat{\sigma}_n^\dagger d + h.c.\right]$$

$$\hat{E}(\mathbf{r}_n) = -i\sum_k \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0}} u_k(\mathbf{r}_n)\hat{a}_k$$

$$\hat{\sigma}_n = |g\rangle_n\langle e|$$

Heisenberg Eq. for atom n:

$$\partial_t \hat{\sigma}_n = \frac{i}{\hbar}[H, \hat{\sigma}_n]$$

Use: $[\hat{\sigma}_n^\dagger, \hat{\sigma}_n] = \hat{\sigma}_n^z$ $\quad \hat{\sigma}_n^z = |e\rangle_n\langle e| - |g\rangle_n\langle g|$

$$\partial_t \hat{\sigma}_n = -i\omega_a \hat{\sigma}_n - i\frac{d}{\hbar}\hat{\sigma}_n^z \hat{E}(\mathbf{r}_n)$$

Now wish to find Eq. for field $\hat{E}(\mathbf{r}_n)$ and insert it into the above

Heisenberg Eq. for field = Maxwell's Eqs. (=Hamilton's Eq. for field...)

$$\nabla \times \hat{E}(r,t) = -\partial_t \hat{B}(r,t) \qquad \nabla \times \hat{B}(r,t) = \frac{1}{c^2}\partial_t\left[\hat{E}(r,t) + \frac{1}{\varepsilon_0}\hat{P}(r,t)\right]$$

Polarization density

= dipoles (of atoms) per volume

$$\hat{P}(\mathbf{r},t) = \sum_n \hat{d}_n(t)\delta(\mathbf{r}-\mathbf{r}_n)$$

$$\hat{d}_n = d\hat{\sigma}_n + \mathrm{h.c.}$$

Freq. domain $\quad \partial_t \to -i\,\omega$

$$\nabla \times \hat{E}(r,\omega) = i\,\omega \hat{B}(r,\omega) \qquad \nabla \times \hat{B}(r,\omega) = -i\frac{\omega}{c^2}\left[\hat{E}(r,\omega) + \frac{1}{\varepsilon_0}\hat{P}(r,\omega)\right]$$

Derive wave equation: $\quad \nabla \times \nabla \times \hat{E} = i\,\omega \nabla \times \hat{B} = \frac{\omega^2}{c^2}\left[\hat{E} + \frac{1}{\varepsilon_0}\hat{P}\right]$

$$\nabla \times \nabla \times \hat{E}(\mathbf{r},\omega) - \frac{\omega^2}{c^2}\hat{E}(\mathbf{r},\omega) = \frac{\omega^2}{c^2}\frac{1}{\varepsilon_0}\hat{P}(\mathbf{r},\omega) = \frac{\omega^2}{c^2}\frac{1}{\varepsilon_0}\sum_n \delta(\mathbf{r}-\mathbf{r}_n)d\hat{\sigma}_n(\omega)$$

Heisenberg Eq. for field: formal solution (1)

$$\underbrace{\nabla \times \nabla \times \hat{E}(\mathbf{r},\omega) - \frac{\omega^2}{c^2}\hat{E}(\mathbf{r},\omega)}_{\text{Linear operator (~ "Helmholtz" Eq. for the field)}} = \underbrace{\frac{\omega^2}{c^2}\frac{1}{\varepsilon_0}\sum_n \delta(\mathbf{r}-\mathbf{r}_n)d\hat{\sigma}_n(\omega)}_{\text{point sources ("dipoles") @ atom positions}}$$

→ Solution to point source (=dipole) @ $\mathbf{r}_n$ – Green's function: $G(\mathbf{r}-\mathbf{r}_n)$

$$\hat{E}(\mathbf{r},\omega) = \hat{E}_0(\mathbf{r},\omega) + \underbrace{\frac{\omega^2}{c^2}\frac{d}{\varepsilon_0}\sum_n G(\mathbf{r}-\mathbf{r}_n,\omega)\hat{\sigma}_n(\omega)}_{\text{Field from all "dipole" sources (=radiating atoms)}}$$

Total field

"Source free" solution
= "free" field (in the absence of atoms)
= vacuum fluctuations + incident laser

field propagator = Green's function $G(\mathbf{r}_1,\mathbf{r}_2) = \mathbf{e}_d{}^\dagger \cdot \overline{\overline{\boldsymbol{G}}}(\mathbf{r}_1,\mathbf{r}_2)\cdot \mathbf{e}_d$

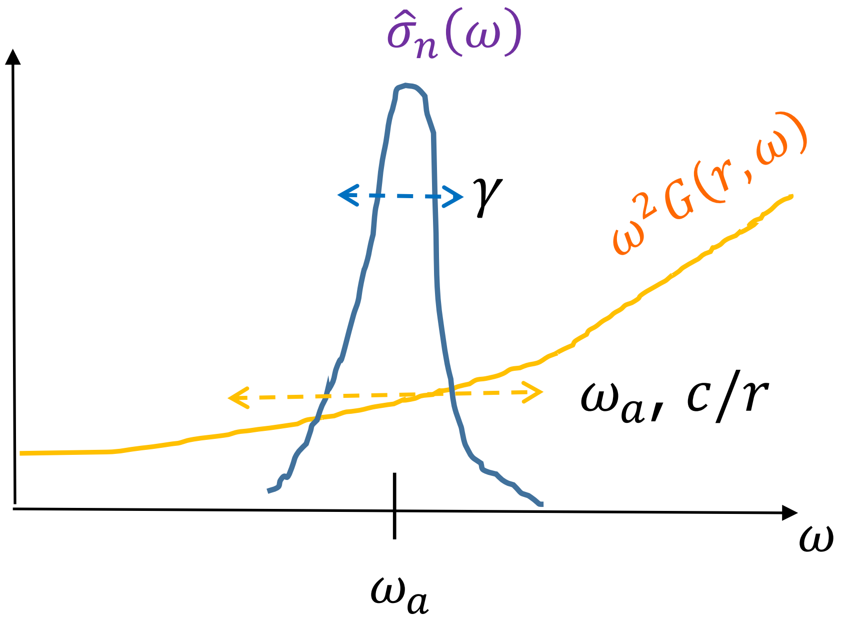
$$\mathbf{e}_i{}^\dagger \cdot \overline{\overline{\boldsymbol{G}}}(\mathbf{r}_1,\mathbf{r}_2)\cdot \mathbf{e}_j = G_{ij}(k,\mathbf{r}_1,\mathbf{r}_2) = \frac{e^{ikr}}{4\pi r}\left[\left(1+\frac{ikr-1}{k^2r^2}\right)\delta_{ij} + \left(-1+\frac{3-3ikr}{k^2r^2}\right)\frac{r^ir^j}{r^2}\right]$$

with $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, , $r = |\mathbf{r}|$ and $r^i = \mathbf{e}_i \cdot \mathbf{r}$.

$k = \omega/c$

Heisenberg Eq. for field: formal solution (2)

$$\hat{E}(\mathbf{r},\omega) = \hat{E}_0(\mathbf{r},\omega) + \frac{\omega^2}{c^2}\frac{d}{\varepsilon_0}\sum_n \hat{\sigma}_n(\omega) G(\mathbf{r}-\mathbf{r}_n,\omega)$$



Back to time-domain – perform IFT:

$$\hat{E}(\mathbf{r},t) = \hat{E}_0(\mathbf{r},t) + \frac{1}{c^2}\frac{d}{\varepsilon_0}\sum_n \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} e^{-i\omega t}\hat{\sigma}_n(\omega)\ \omega^2 G(\mathbf{r}-\mathbf{r}_n,\omega)$$

- Dominant "free" dynamics: oscillations at ω_a
- Slow dynamics due to radiation ~ decay/shift γ

Wide function around ω_a

Width ~ $\omega_a,\ c/r$

Markov approximation: $\omega_a,\ c/r \gg \gamma$

$$\hat{E}(\mathbf{r},t) \approx \hat{E}_0(\mathbf{r},t) + \frac{1}{c^2}\frac{d}{\varepsilon_0}\sum_n \omega_a{}^2 G(\mathbf{r}-\mathbf{r}_n,\omega_a) \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} e^{-i\omega t}\hat{\sigma}_n(\omega)$$

$$= \hat{E}_0(\mathbf{r},t) + \frac{1}{c^2}\frac{d}{\varepsilon_0}\sum_n \omega_a{}^2 G(\mathbf{r}-\mathbf{r}_n,\omega_a)\hat{\sigma}_n(t)$$

Heisenberg-Langevin Eq. for atoms (1)

$$\partial_t \hat{\sigma}_n = -i\omega_a \hat{\sigma}_n - i\frac{d}{\hbar}\hat{\sigma}_n^z \hat{E}(\mathbf{r}_n)$$

We got for field: $\hat{E}(\mathbf{r}) = \hat{E}_0(\mathbf{r}) + \frac{1}{c^2}\frac{d}{\varepsilon_0}\sum_n \omega_a{}^2 G(\mathbf{r}-\mathbf{r}_n,\omega_a)\hat{\sigma}_n$

→ Field "felt" by **atom** n, $\mathbf{r} = \mathbf{r}_n$:

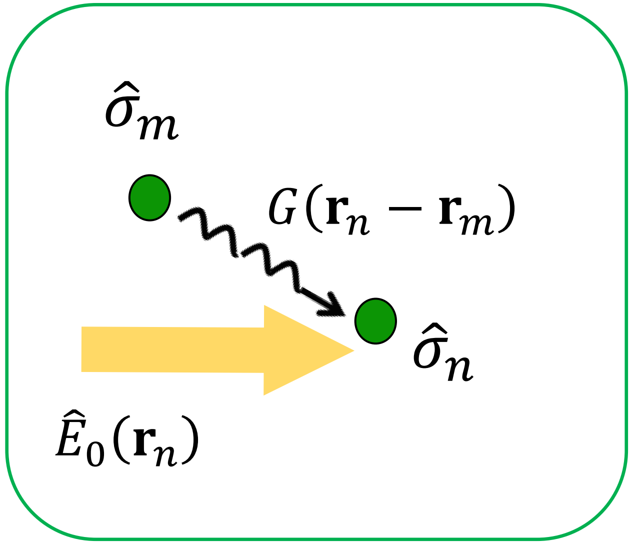
$$\hat{E}(\mathbf{r}_n) = \hat{E}_0(\mathbf{r}_n) + \frac{1}{c^2}\frac{d}{\varepsilon_0}\sum_m \omega_a{}^2 G(\mathbf{r}_n-\mathbf{r}_m)\hat{\sigma}_m$$

= "free" field (in the absence of atoms)
= vacuum fluctuations + incident laser

Field from all "dipole" sources
(=radiating atoms)

$$\hat{E}_0(\mathbf{r}_n) = -i\sum_k \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0}}\,u_k(\mathbf{r}_n)\hat{a}_k(0)e^{-i\omega_k t}$$

field propagator = $G(\mathbf{r}_n - \mathbf{r}_m)$



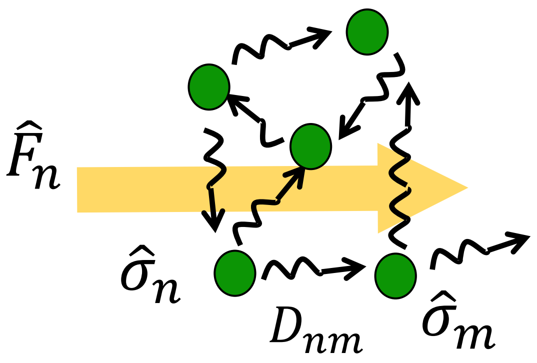
Heisenberg-Langevin Eq. for atoms (2)

$$\partial_t \hat{\sigma}_n = -i\omega_a \hat{\sigma}_n - i\hat{\sigma}_n^z \frac{d}{\hbar}\left[\hat{E}_0(\mathbf{r}_n) + \frac{\omega_a{}^2}{c^2}\frac{d}{\varepsilon_0}\sum_m G(\mathbf{r}_n - \mathbf{r}_m)\hat{\sigma}_m\right]$$

$$D_{nm} = -i\frac{3}{2}\gamma\lambda G(\mathbf{r}_n - \mathbf{r}_m) \qquad \gamma = \frac{d^2\omega_a{}^3}{3\pi\varepsilon_0\hbar c^3}$$

$$\lambda = 2\pi c/\omega_a$$

$$\boxed{\partial_t \hat{\sigma}_n = -i\omega_a \hat{\sigma}_n + \hat{\sigma}_n^z\left[\hat{F}_n + \sum_m D_{nm}\hat{\sigma}_m\right]}$$



→ Collective response

(= multiple scattering
= dipole-dipole)

quantum noise (vacuum) + incident laser

$$\hat{F}_n = -i\frac{d}{\hbar}\hat{E}_0(\mathbf{r}_n) = -d\sum_k \sqrt{\frac{\omega_k}{2\varepsilon_0\hbar}}u_k(\mathbf{r}_n)\hat{a}_k(0)e^{-i\omega_k t}$$

Dipole-dipole coupling (with all atomic dipoles)

$$D_{nm} = \gamma_{nm}/2 + i\Delta_{nm}$$

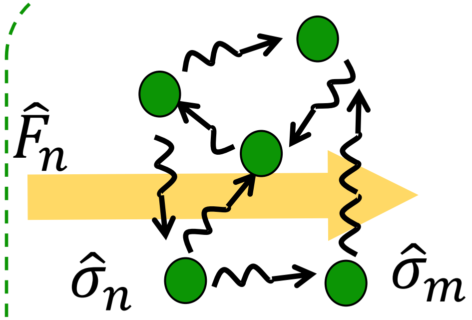
dipole-dipole kernel
(photon Green's function)

$$\partial_t \hat{\sigma}_n = -\left(i\omega_a + \frac{\gamma}{2}\right)\hat{\sigma}_n + \hat{\sigma}_n^z\left[\hat{F}_n + \sum_{m\neq n} D_{nm}\hat{\sigma}_m\right]$$

$$\mathrm{Re}\, D_{nn} = \gamma/2$$

$$\mathrm{Im}\, D_{nn} + \omega_a \to \omega_a$$

Many-body physics of quantum emitters (atoms)



$D_{nm} = \gamma_{nm}/2 + i\Delta_{nm}$

Heisenberg-Langevin Eq. for atoms:

$$\partial_t \hat{\sigma}_n = \frac{i}{\hbar}[H_{\text{eff}}, \hat{\sigma}_n] + \hat{\sigma}_n^z \hat{F}_n$$

Effective Hamiltonian (non-Hermitian): **collectivity**

$$H_{\text{eff}} = \hbar\left(\omega_a - i\frac{\gamma}{2}\right)\sum_{n=1}^{N}\hat{\sigma}_n^\dagger \hat{\sigma}_n + \hbar\sum_{n=1}^{N}\sum_{m\neq n}\left(\Delta_{nm} - i\frac{\gamma_{nm}}{2}\right)\hat{\sigma}_n^\dagger \hat{\sigma}_m$$

Dipole-dipole interaction
(reversible excitation exchange)

Collective radiation
(dissipation)

Quantum noise
(has to exist due to dissipation)

$$\hat{F}_n = -d\sum_k \sqrt{\frac{\omega_k}{2\varepsilon_0\hbar}}\, u_k(\mathbf{r}_n)\hat{a}_k(0)e^{-i\omega_k t}$$

Equivalent description: quantum master equation

$$\partial_t \hat{\rho} = -\frac{i}{\hbar}\left(H_{\text{eff}}\hat{\rho} - \hat{\rho}H_{\text{eff}}^\dagger\right) + \sum_{n,m}\gamma_{nm}\hat{\sigma}_n\hat{\rho}\hat{\sigma}_m^\dagger$$

Quantum noise: “jump term”

Output light

$$\hat{E}(\mathbf{r}) = \hat{E}_0(\mathbf{r}) + \frac{1}{c^2}\frac{d}{\varepsilon_0}\sum_n \omega_a{}^2 G(\mathbf{r}-\mathbf{r}_n, \omega_a)\hat{\sigma}_n$$