CAS CS 585 Image and Video Computing - Spring 2024

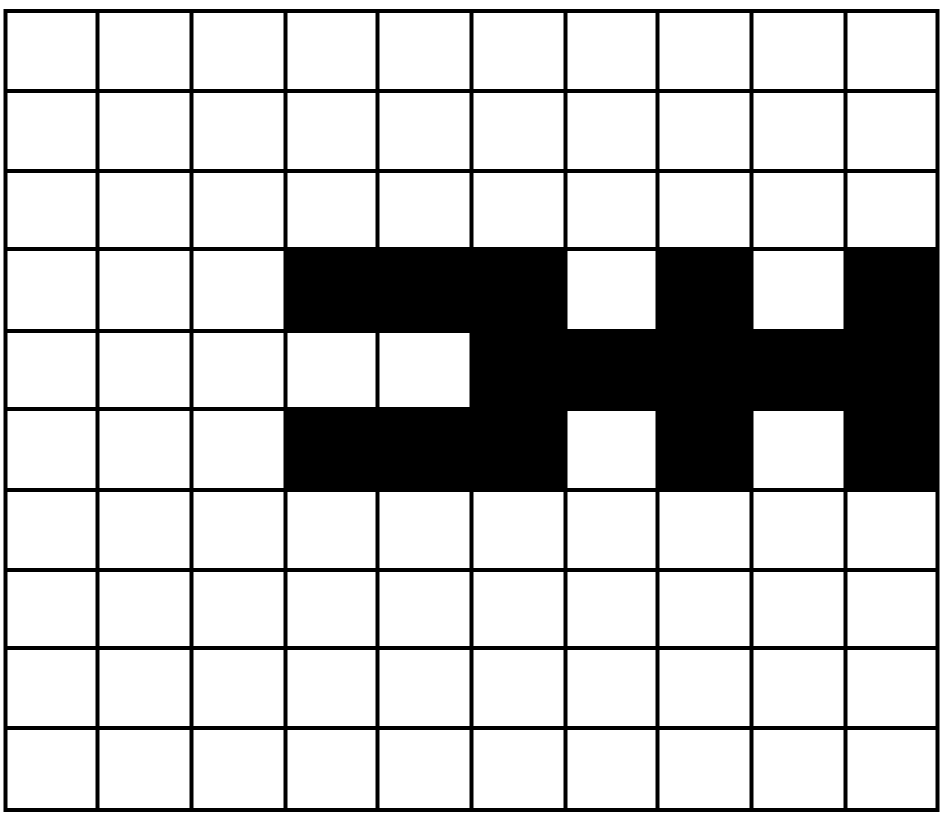
Assignment 1

Due on GradeScope, Wednesday, January 31, 2024, 11:59 pm

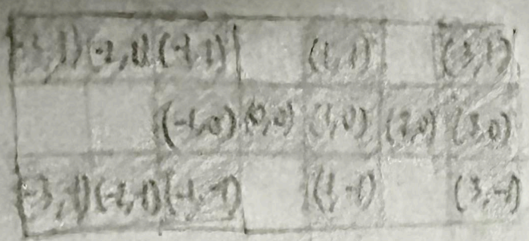
Include a "readme file" with acknowledgements of any help you may have received in solving this assignment**.**

Exercise 1

Consider the following binary image:



(a) Compute the coordinates of the centroid of the binary object using pencil and paper.
Hint: You can select a coordinate system origin that simplifies your computation.



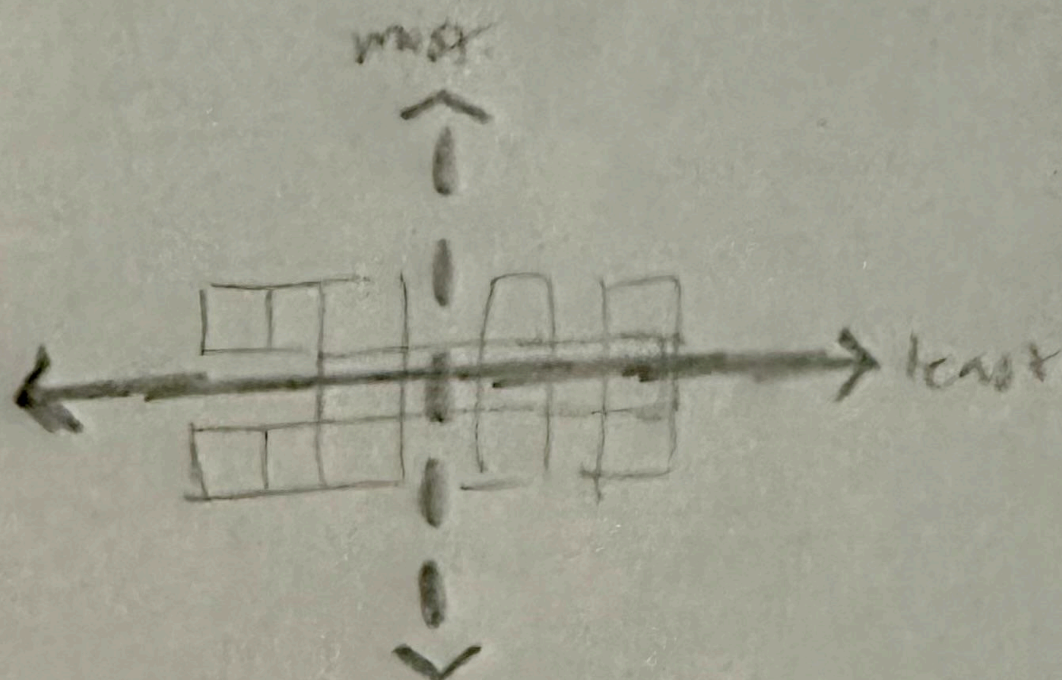
$\Sigma Y_i = 0$

$\Sigma X_i = -3\times2 - 2\times2 - 1\times2 - 1 + 1\times3 + 2 + 3\times3 = -6-4-2-1+3+2+9 = 1$

Centroid $\rightarrow (\frac{1}{15}, 0)$

Centroid = (1/15,0)

(b) Draw the axes of least and most inertia of the object. To distinguish them, use a dashed line for the axis of most inertia.



Exercise 2

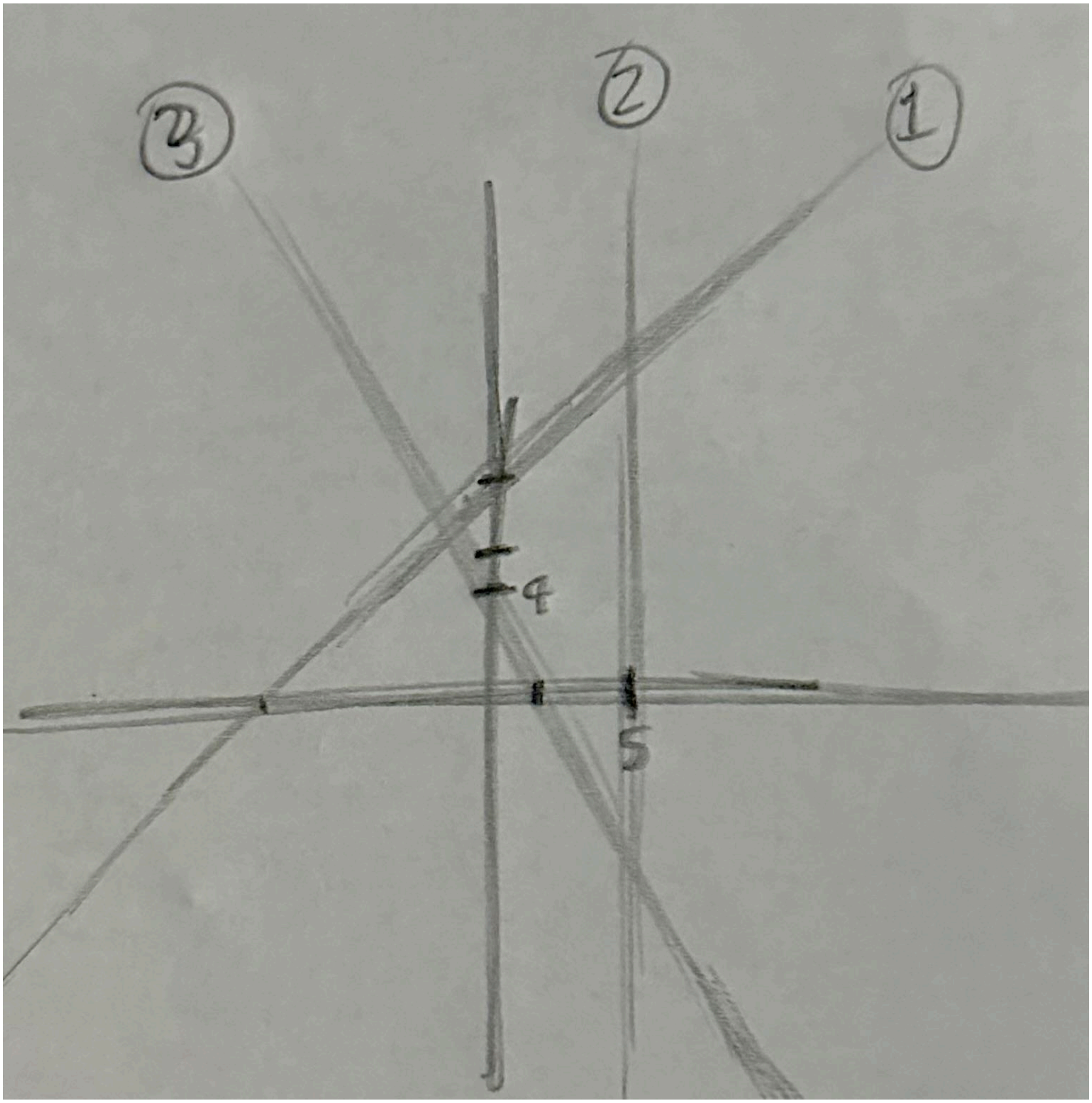
Consider the following three lines:

$(-1,1)^T \mathbf{x} - 7 = 0$

$(1,0)^T \mathbf{x} - 5 = 0$

$(2,1)^T \mathbf{x} - 4 = 0$

(a) Draw the lines in a 2D coordinate system using pencil and paper.



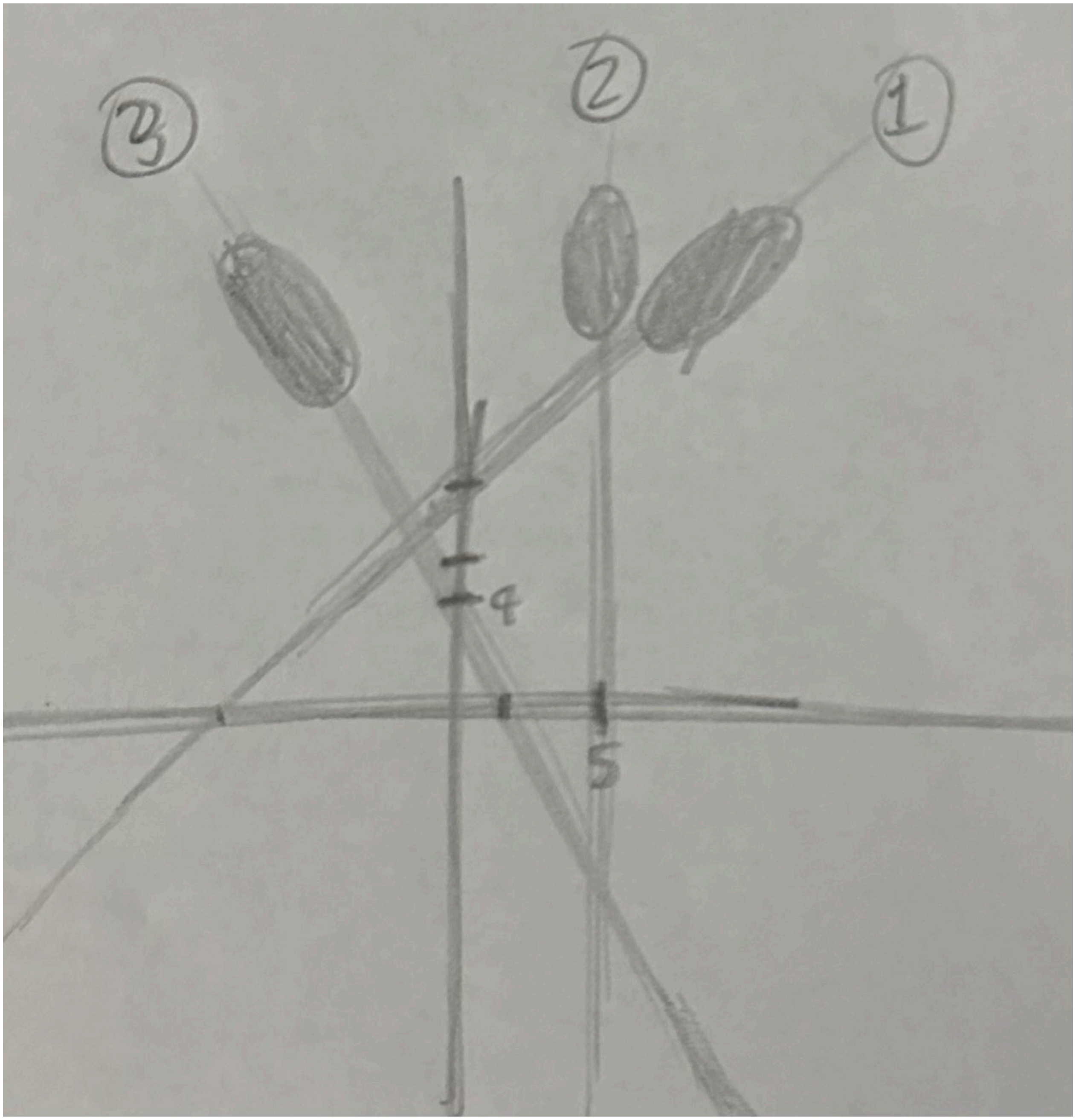
(b) Rewrite the line equations using the x sin α – y cos α + g = 0 notation of a line.

x*sin(-3*pi/4)-y*cos(-3*pi/4)+7*cos(-3*pi/4)=0

x*sin(pi/2)-y*cos(pi/2)-5=0

x*sin(pi-atan(2))-y*cos(pi-atan(2))+4*cos(pi-atan(2))=0

(c) For each line, sketch a binary object for which the line is its axis of least inertia.



Exercise 3

Second moments can be used to evaluate how longish or how circular the shape of a binary object is. This has been used, for example, to evaluate blood samples in microscopy images for sickle cell disease.

(a) Write down the relevant mathematical expression as a function of the second moments of the binary image $B(x,y)$.

$(0.5 (a+c) - 0.5 \text{ sqrt}(b^2 + (a-c)^2))/(0.5 (a+c) + 0.5 \text{ sqrt}(b^2 + (a-c)^2))$

(b) Is the expression larger or smaller for sickle cells compared to healthy blood cells?

Smaller, because the sickle cells are shriveled up— less round.

(c) Propose another computer vision problem where object circularity is a property that could aid in image interpretation.

One interpretation problem where measuring the circulatory of an object could be important is with differentiating between a collection of pencils or erasers in an image.

Exercise 4

The minimum and maximum values of the moment of inertia can be

written as $E = 0.5 (a+c) \pm 0.5 \text{ sqrt}(b^2 + (a-c)^2)$,

where a, b, and c are defined as in class.

(a) Prove that $E \geq 0$.

To prove E >= 0, we can simply take the equation for the minimum value of E, since the maximum value must be larger than the minimum value, and therefore if the minimum value >= 0 then the maximum value >= 0. We start with:

$0.5 (a+c) \pm 0.5 \text{ sqrt}(b^2 + (a-c)^2)$? 0 (? because inequality is unknown)

We multiply by 2 to get:

$(a+c) - \text{sqrt}(b^2 + (a-c)^2)$? 0

We rearrange to get:

$(a+c)$? $\text{sqrt}(b^2 + (a-c)^2)$ (because regardless of inequality/equality rearrangement is valid)

We square (and expand) both sides to get:

a^2 + 2ac + c^2 ? b^2 + a^2 - 2ac + c^2

We rearrange again:

4ac ? b^2

Substituting we get (excluding dxdy for brevity):

4*(∬ x^2*b(x,y))(∬ y^2*b(x,y)) ? 4*(∬ x*y*b(x,y))

We divide both sides by 4 to get:

(∬ x^2*b(x,y))(∬ y^2*b(x,y)) ? (∬ x*y*b(x,y)) (because dividing by a positive number is valid for any inequality/equality)

From this it is clear that the LHS is greater than the RHS, as when we do the integration, ⅓

$*(x^3*y)*⅓*(x*y^3) > ¼*x^2*y^2$.

(b) When is E = 0?

We start with the equation:

$0.5\ (a+c) \pm 0.5\ \mathrm{sqrt}(b^2 + (a-c)^2) = 0$

We multiply by 2 to get:

$(a+c) \pm \mathrm{sqrt}(b^2 + (a-c)^2) = 0$

We rearrange to get:

$(a+c) = \pm\ \mathrm{sqrt}(b^2 + (a-c)^2)$

We square both sides to get:

$a^2 + 2ac + c^2 = b^2 + a^2 - 2ac + c^2$

We rearrange to get:

$4ac = b^2$

Thus we know that when $4ac = b^2$, E = 0. (Eerily similar to the determinant of the quadratic formula).

Exercise 5

Prove that the axis of most inertia of a binary object goes through the object centroid.

Axis of most inertia:

We prove this in the same way we proved that the axis of least inertia goes through the object centroid. We start with the equation for determining the inertia of an object to a axis defined by p and t (theta) from the reading (https://homepages.inf.ed.ac.uk/rbf/CVonline/LOCAL_COPIES/OWENS/LECT2/node3.html):

$\iint (x*\sin(t)-y*\cos(t)+p)^2*b(x,y)\ dxdy = I$

We can determine the value of p that maximizes this equation by taking the derivative by p and setting it equal to zero. By taking the derivative of this equation by p, expanding, and setting equal to zero, we get (assuming the dxdy component):

$\iint x*\sin(t)*b(x,y) - \iint y*\cos(t)*b(x,y) + \iint p*b(x,y) = 0$

We divide by $\iint b(x,y)$to get:

$(x_c\sin(t) - y_c\cos(t)+p)$

Where x_c and y_c are the coordinates of the center of mass (centroid). Because we know from the reading that the distance 'r' from point (x,y) to the line defined by p and t is given by $r = x*\sin(t)-y*\cos(t)+p$, $(x_c\sin(t)-y_c\cos(t)+p) = 0$ tells us that the object centroid (x_c,y_c) lies on both the axis that maximizes and the axis that minimizes inertia. NOTE: upon further thought it appears that this calculation does not actually optimize the function for the axis of most inertia, because the axis of most inertia would be indefinitely far from the center of the object.

Exercise 6

When we want to represent a binary object in an image with an object that has a simpler shape, we can use a region that has the shape of an ellipse with the same zeroth, first, and second moments. An ellipse can be defined by the equation

$$(x/\alpha)^2 + (y/\beta)^2 = 1,$$

where α is the semi-major axis along the x-axis and β is the semi-minor axis along the y-axis.

(a) Prove that the minimum and maximum second moments of the region about an axis through the origin are $\pi/4\ \alpha\ \beta^3$ and $\pi/4\ \beta\ \alpha^3$, respectively.

We start with the equation of the ellipse in polar coordinates, substituting x = r*a*cos(t) and y = r*b*sin(t). With the Jacobian:

J = |(a*cos(t),b*sin(t)), (-r*a*sin(t),r*b*cos(t))| = r*a*b

We know that the maximum and minimum second moments of the ellipse will be from the major and minor axes. We observe that an ellipse can be rotated 90 degrees to effectively swap its semi-major and semi-minor axes values, so we first find the MOI across the x axis. We can integrate on the y position at each point squared and multiplied by the jacobian determinant to get the moment of inertia. We integrate across a full revolution of t and the full radius of the ellipse:

$\int_0^1\int_0^{2*pi}$ (r^2*b^2*sin(t)^2)*(r*a*b) dtdr

→ $\int_0^1\int_0^{2*pi}$ (r^3*a*b^3*sin(t)^2) dtdr → a*b^3* $\int_0^1\int_0^{2*pi}$ (r^3*sin(t)^2) dtdr

→ a*b^3*pi* $\int_0^1$ (r^3) dr → a*b^3*pi*(¼)

And by rotating the ellipse 90 degrees, we get a^3*b*pi*(¼). Thus the minimum and maximum second moments of the ellipse through the origin are π/4 α β^3 and π/4 β α^3.

~~We observe the integral of the area of the ellipse in the second quadrant:~~

~~$\int_0^\beta$(β/α)*(α^2-x^2)^0.5 dx~~

~~This derives the area of the ellipse as a sum of the areas of consecutive vertical slices of the ellipse. This means to get “a”, we need to get the sum of the y positions squared in each slice. The integral of y positions squared is ⅓*y^3, and we already have the expression for the vertical height of the ellipse at each x position. Thus, a = ⅓ * $\int_0^\beta$((β/α)*(α^2-x^2)^0.5)^3 dx. For c, we can simply reverse β and α to get the equivalent value, as this is the same as rotating the~~

~~ellipse 90 degrees or flipping x and y. So c = ⅓ * $\int_\theta^\beta$((α/β)*(β^2-x^2)^0.5)^3 dx. For b, we can think of each slice as summing the areas of overlapping rectangles with gradually increasing heights but the same width. Thus we can use 0.5*y^2*x, and so b = $\int_\theta^\beta$0.5*((β/α)*(α^2-x^2)^0.5)^2*x dx.~~

The second moment of any region about an axis inclined at an angle γ can be written in the form $E = a \sin^2 \gamma - b \sin \gamma \cos \gamma + c \cos^2 \gamma$.

(b) Compute the major and minor axes of an equivalent ellipse, which means an ellipse that has the same second moment about any axis through the origin.

We start with the minimum moment of inertia. We know that the minimum/maximum MOI axes of an ellipse will be its major and minor axes. The major axis of the ellipse, of length α, is zero degrees offset from the x axis, giving us:

E_{min} = c (because sin(0) is 0, so all terms with sin(y) are zero)

Thus we get:

c = π/4 α β^3

We rearrange to get:

β = (c/(π/4*α))^⅓

For the maximum MOI we get:

E_{max} = a (because cos(0) is 0, so all terms with cos(y) are zero)

Giving us:

α = (a/(π/4*β))^⅓

We plug this into our β equation to get:

β = (c^3/((π/4)^2*a))^⅛

We do the same thing for α to get:

α = (a^3/(π/4)^2*c)^⅛

Thus our equations for the major and minor axes of an ellipse given the maximum and minimum MOI are

α = (a^3/(π/4)^2*c)^⅛ and β = (c^3/((π/4)^2*a))^⅛