COMP 182: Homework 2

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Proofs

1. [5*pts*] Prove that given a nonnegative integer n, there is a unique nonnegative integer m such that $m^2 \leq n < (m+1)^2$

   Consider an arbitrary $n \in \mathbb{Z}, n \geq 0$. Since $n \in \mathbb{Z}$, we know that $\sqrt{n} \in \mathbb{R}$. We can then consider two cases, that $\sqrt{n} \in \mathbb{Z}$, or $\sqrt{n} \notin \mathbb{Z}$.

   Case 1: $\sqrt{n} \in \mathbb{Z}$. Then let $m = \sqrt{n} < m+1$. Therefore, $m^2 \leq n$ and $n < (m+1)^2$. Therefore, $m^2 \leq n < (m+1)^2$.

   Case 2: $\sqrt{n} \notin \mathbb{Z}$. Since we also know $\sqrt{n} \in \mathbb{R}$, consider the greatest integer less than n. In other words, consider $max(m)$ for all $m < \sqrt{n}, m \in \mathbb{Z}$. Since we have defined m to be the greatest integer less than $\sqrt{n}$, we can conclude that $\sqrt{n} < m+1$, where $m+1$ is the least integer greater than m. $\sqrt{n} < m+1 \implies n < (m+1)^2$. Therefore, $m^2 \leq n < (m+1)^2$.

   To prove uniqueness, we assume that there exist two distinct solutions, p, q, such that $p < q$ and satisfy $p^2 \leq n < (p+1)^2$ and $q^2 \leq n < (q+1)^2$. $p, q \in \mathbb{Z}^+ \implies p+1 \leq q \implies n < (p+1)^2 \leq q^2 \implies n < q^2$. This is in direct contradiction with our assumption that q is a solution and satisfies $q^2 \leq n < (q+1)^2$. Therefore, the solution to $m^2 \leq n < (m+1)^2$ is unique. $\square$

2. [5*pts*] Assuming that $\sqrt{n}$ is irrational for every positive integer n that is not a perfect square, prove that $\sqrt{2} + \sqrt{3}$ is irrational.

   We assume the negation, that $\sqrt{2}+\sqrt{3}$ is rational. By definition, $\exists p, q \in \mathbb{Z}(\sqrt{2}+\sqrt{3} = p/q)$. $\sqrt{2}+\sqrt{3} = p/q \implies (\sqrt{2}+\sqrt{3})^2 = p^2/q^2 \implies 5 + 2\sqrt{6} = p^2/q^2 \implies 2\sqrt{6} = (p^2 - 5q^2)/q^2 \implies \sqrt{6} = (p^2 - 5q^2)/(2q^2)$. $p, q \in \mathbb{Z}, q \neq 0 \implies [(p^2 - 5q^2) \in \mathbb{Z} \wedge (2q^2) \in \mathbb{Z}] \implies \sqrt{6}$ is rational. But 6 is not a perfect square. Thus, we have a contradiction, our assumption is false, and $\sqrt{2} + \sqrt{3}$ is irrational. $\square$

3. [5*pts*] Prove that $\sqrt[3]{2}$ is irrational.

   Let us assume $\sqrt[3]{2}$ is rational. Then $\sqrt[3]{2} = p/q$ where $p, q \in \mathbb{Z}$ and without loss of generality, p, q relatively prime. $\sqrt[3]{2} = p/q \implies 2 = p^3/q^3 \implies p^3 = 2q^3 \implies p^3$ is even $\implies p$ is even $\implies \exists m \in \mathbb{Z}(p = 2m) \implies p^3 = 8m \implies q^3 = 4m \implies q$ is even. (p is even) $\wedge$ (q is even) is a contradiction. Therefore, our assumption is incorrect and $\sqrt[3]{2}$ is irrational. $\square$

   Remark: Note that if m^3 is even, m is even. If m is odd, $m = 2n+1$ for some $n \in \mathbb{Z}$. Thus, $m^3 = 8n^3 + 12n^2 + 6n + 1 = 2(4n^3 + 6n^2 + 3n) + 1$, which is contradictory.

4. [5*pts*] Prove that between every rational number and every irrational number there is an irrational number.

First, we will show that $p \times q, p \in \mathbb{Q}, q \in \mathbb{R}\backslash\mathbb{Q}$, is irrational. We assume $p \times q$ to be rational $\implies$ $p \times q = r/s$ such that $r, s \in \mathbb{Z}$. $p \times q = r/s \implies q = r/(sp)$. This shows q is rational, and thus we have a contradiction. Therefore $p \times q, p \in \mathbb{Q}, q \in \mathbb{R}\backslash\mathbb{Q}$, is irrational.

Now, we return to the original problem. Let $p \in \mathbb{Q}, q \in \mathbb{R}\backslash\mathbb{Q}$. We have two cases:

Case 1: $|p - q| > \sqrt{2} \implies \exists x = m\sqrt{2}, m \in \mathbb{Z}$ such that x is between p, q. Since $x = m\sqrt{2}, m \in \mathbb{Z}$, we have an irrational number between a rational and an irrational.

Case 2: $|p - q| \leq \sqrt{2}$ We consider the fact that $\exists M \in \mathbb{Z}$ such that $M|p - q| > \sqrt{2}$, in which case it can be shown as previously that $M|p - q| > \sqrt{2} \implies \exists x = m\sqrt{2}, m \in \mathbb{Z}$ such that x is between p, q. If we consider $y = x/M = m\sqrt{2}/M$, we have an irrational number y between p, q. $\square$

Sets and Functions

1. [10pts] The symmetric difference of sets A and B, denoted by $A\Delta B$, is the set containing those elements in A or B, but not in both A and B. For example, for $A = \{1, 2, 3\}$ and $B = \{2, 3, 4, 5\}$, $A\Delta B = \{1, 4, 5\}$

   (a) Prove that $(A\Delta B)\Delta B = A$.

   We first show that $A\Delta B = (A \cup B) \cap (\bar{A} \cup \bar{B})$.

   1. $A\Delta B$
   2. $(A \cup B)\backslash(A \cap B)$
   3. $(A \cup B) \cap \overline{(A \cap B)}$
   4. $(A \cup B) \cap (\bar{A} \cup \bar{B})$

   Now, we use the above to expand $(A\Delta B)\Delta B$

   1. $(A\Delta B)\Delta B$
   2. $C\Delta B$
   3. $(C \cup B) \cap (\bar{C} \cup \bar{B})$
   4. $\big[((A \cup B) \cap (\bar{A} \cup \bar{B})) \cup B\big] \cap \Big[\overline{((A \cup B) \cap (\bar{A} \cup \bar{B}))} \cup \bar{B}\Big]$

   Simplifying the left side of the intersection expression:

   1. $((A \cup B) \cap (\bar{A} \cup \bar{B})) \cup B$
   2. $((A \cup B) \cup B) \cap ((\bar{A} \cup \bar{B}) \cup B)$
   3. $(A \cup B) \cap ((\bar{A} \cup B) \cup (\bar{B} \cup B))$
   4. $(A \cup B) \cap ((\bar{A} \cup B) \cup U)$
   5. $(A \cup B) \cap U$
   6. $A \cup B$

   Simplifying the right side of the intersection expression:

   1. $\overline{((A \cup B) \cap (\bar{A} \cup \bar{B}))} \cup \bar{B}$
   2. $(\overline{(A \cup B)} \cup \overline{(\bar{A} \cup \bar{B})}) \cup \bar{B}$
   3. $((\bar{A} \cap \bar{B}) \cup (A \cap B)) \cup \bar{B}$
   4. $((\bar{A} \cap \bar{B}) \cup \bar{B}) \cup ((A \cap B)) \cup \bar{B})$
   5. $\bar{B} \cup ((A \cup \bar{B}) \cap (B \cup \bar{B}))$
   6. $\bar{B} \cup ((A \cup \bar{B}) \cap U)$
   7. $\bar{B} \cup (A \cup \bar{B})$
   8. $A \cup \bar{B}$

   Simplifying the original expression:

   1. $(A \cup B) \cap (A \cup \bar{B})$
   2. $((A \cup B) \cap A) \cup ((A \cup B) \cap \bar{B})$
   3. $A \cup ((A \cup B) \cap \bar{B})$
   4. $A \cup ((A \cap \bar{B}) \cup (B \cap \bar{B})$
   5. $A \cup ((A \cap \bar{B}) \cup \varnothing)$
   6. $A \cup (A \cap \bar{B})$
   7. A

   (b) Is the following true: $A\Delta(B\Delta C) = (A\Delta B)\Delta C$? Prove your answer.

First, let us prove that "exclusive or," $\oplus$, is commutative.

$$\begin{aligned} p \oplus q \\ (p \wedge \neg q) \vee (\neg p \wedge q) \\ (\neg p \wedge q) \vee (p \wedge \neg q) \\ (q \wedge \neg p) \vee (\neg q \wedge p) \\ q \oplus p \end{aligned}$$

Likewise, let use show that "exclusive or," $\oplus$, is associative.

$p \oplus q = (p \wedge \neg q) \vee (\neg p \wedge q)$
$(p \oplus q) \oplus r$
$= [((p \wedge \neg q) \vee (\neg p \wedge q)) \wedge \neg r] \vee [\neg((p \wedge \neg q) \vee (\neg p \wedge q)) \wedge r]$
$= [(p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r)] \vee [(\neg(p \wedge \neg q) \wedge \neg(\neg p \wedge q)) \wedge r]$
$= [(p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r)] \vee [((\neg p \vee q) \wedge (p \vee \neg q)) \wedge r]$
$= [(p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r)] \vee [((\neg p \wedge (p \vee \neg q)) \vee (q \wedge (p \vee \neg q))) \wedge r]$
$= [(p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r)] \vee [((\neg p \wedge \neg q) \vee (p \wedge q)) \wedge r]$
$= (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (p \wedge q \wedge r)$
$= p \wedge ((\neg q \wedge \neg r) \vee (q \wedge r)) \vee \neg p \wedge ((q \wedge \neg r) \vee (\neg q \wedge r))$
$= p \wedge ((\neg q \wedge \neg r) \vee (q \wedge r)) \vee \neg p \wedge (q \oplus r)$
Now consider $\neg(q \oplus r)$
$= \neg[(q \wedge \neg r) \vee (\neg q \wedge r)]$
$= \neg(q \wedge \neg r) \wedge \neg(\neg q \wedge r)]$
$(\neg q \vee r) \wedge (q \vee \neg r)]$
$= (\neg q \wedge \neg r) \vee (q \wedge r)]$
Finally, we have:
$= p \wedge ((\neg q \wedge \neg r) \vee (q \wedge r)) \vee \neg p \wedge (q \oplus r)$
$= p \wedge \neg(q \oplus r) \vee \neg p \wedge (q \oplus r)$
$= p \oplus (q \oplus r)$

We seek to prove $A\Delta(B\Delta C) = (A\Delta B)\Delta C$:

$$\begin{aligned} x \in A\Delta(B\Delta C) \\ x \in A \oplus (x \in B \oplus x \in C) \\ (x \in A \oplus x \in B) \oplus x \in C \\ x \in (A\Delta B)\Delta C \end{aligned}$$

$$\begin{aligned} x \in (A\Delta B)\Delta C \\ (x \in A \oplus x \in B) \oplus x \in C \\ x \in A \oplus (x \in B \oplus x \in C) \\ x \in A\Delta(B\Delta C) \end{aligned}$$

2. [$5pts$] Prove that if A and B are two finite sets, then $|A \cap B| \leq |A \cup B|$. Determine when this relationship is an equality.

   $|A \cap B| \leq |A \cup B|$ if there exists a one-to-one function from A to B. Such a function is easily found in this example: $f : A \to B, f(x) = x$. The one-to-one property is also easily verified, consider $a \neq b \implies f(a) = a \neq b = f(b) \implies f(a) \neq f(b)$.

   This relationship is an equality when $A = B$, as then $\forall x \in A \cup B \exists x \in A \cap B$ and $A \cup B = A \cap B$

3. [5*pts*] Find $f \circ g$ and $g \circ f$, where $f(x) = x^2 + 1$ and $g(x) = x + 2$ are functions from $\mathbb{R}$ to $\mathbb{R}$. (Definition of $f \circ g$ is in Section 2.3 in the textbook.)

   $f \circ g = f(g(x)) = f(x+2) = (x+2)^2 + 1 = x^2 + 4x + 4 + 1 = x^2 + 4x + 5$
   $g \circ f = g(f(x)) = g(x^2+1) = x^2 + 1 + 2 = x^2 + 3$

4. [5*pts*] If f and $f \circ g$ are one-to-one, does it follow that g is one-to-one? Prove your answer.

   Consider g to not be one to one. Then $\exists a \neq b(g(a) = g(b))$. $g(a) = g(b) \implies f(g(a)) = f(g(b))$. However, we now have $a \neq b \implies f(g(a)) = f(g(b))$, which contradicts our knowledge that $f \circ g$ is one-to-one. Therefore, our assumption is false, and g must be one-to-one.

5. [10*pts*] Let f be a function from A to B and let S and T be subsets of A (i.e., $S, T \subseteq A$). Prove that

   (a) $f(S \cup T) = f(S) \cup f(T)$

   To show $f(S \cup T) \subseteq f(S) \cup f(T)$, suppose $y \in f(S \cup T)$. Then there exists $x \in S \cup T$ st $f(x) = y$. If $x \in S$ then $y \in f(S)$. If $x \in T$ then $y \in f(T)$. $x \in S \cup T$ st $f(x) = y \implies x \in S \vee x \in T \implies y \in f(S) \vee y \in f(t)$. So $y \in f(S) \cup f(T)$.
   Conversely, $y \in f(S) \cup f(T) \implies y \in f(S)$ or $y \in f(T) \implies \exists x \in S(f(x) = y) \vee \exists x \in T(f(x) = y) \implies y \in f(S \cup T)$

   (b) $f(S \cap T) \subseteq f(S) \cap f(T)$

   Consider $y \in f(S \cap T)$. Then $\exists x \in S \cap T(f(x) = y) \implies x \in S \wedge x \in T \implies f(x) = y \in f(S) \wedge f(x) = y \in f(T) \implies y \in f(S) \cap f(T)$

Graphs

1. [15*pts*] Define the propositional function Q over the domain that consists of all undirected graphs, such that for a graph $g = (V, E), Q(g)$ is given by:
   $(|V| \geq 2) \wedge (\exists V' \subset V(V' \neq \varnothing \wedge \forall \{u, v\} \in E(\{u, v\} \cap V' \neq \varnothing \wedge \{u, v\} \cap (V \backslash V') \neq \varnothing)))$

   (a) Give a graph g_1 that has at least four nodes and at least two edges such that $Q(g_1)$ is True. Explain your answer.

   Consider $g_1 = (V, E), V = \{1, 2, 3, 4\}, E = \{\{1, 2\}, \{1, 3\}\}, V' = \{1\}$. This g_1 satisfies all conditions of $Q(g_1)$. $|V| = 4$, $V' \subset V$, $V' \neq \varnothing$, and for each edge in E, there is a node in both V and V'.

   (b) Give a graph g_2 that has at least four nodes and at least two edges such that $Q(g_2)$ is False. Explain your answer.

   Consider $V = \{1, 2, 3, 4\}, E = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$. The graph is constructed that for all $V' \subset V$, either V' is empty, or that there is an edge in E such that both nodes of the edge are not in V or are not in $V \backslash V'$.

(c) Let g be a graph with five nodes. What is the smallest value k such that if g has k or more edges, then $Q(g)$ is False, regardless of how the nodes are connected by the edges. Explain your answer.

It can be shown that there is a graph with 5 nodes and 6 edges such that $Q(g)$ is true: $g = (V, E), V = \{1, 2, 3, 4, 5\}, E = \{\{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \}, V' = \{1, 2\}$.
The same cannot be said of a graph with 7 edges and 5 nodes. Such a graph is 3 edges away from being complete. $Q(g)$ can be shown to be false for any nonempty subset V'.
Consider V' of size one. Then for $Q(g)$ to be true, all the edges of g must include the node of V', which is impossible as there are only 4 other nodes.
Consider V' of size 2. Then for $Q(g)$ to be true, all the edges of g must include either node of V', but not both. One can consider the case where this is true to be when for each pair of nodes in an edge, one node is in V', and the other in $V \backslash V'$. To maximize the number of edges, the two nodes in V' connect to each of the three nodes in $V \backslash V'$, resulting in only 6 total edges.
Since arbitrary V's of size 3 and 4 are symmetrical to arbitrary V's of size 1 and 2, it can be concluded that $k = 7$.

2. [15*pts*] Let $g_1 = (V_1, E_1)$ and $g_2 = (V_2, E_2)$ be two graphs. We say that g_1 and g_2 are isomorphic if there exists a one-to-one and onto function $f : V_1 \to V_2$ such that $\forall u, v, \{u, v\} \in E_1 \leftrightarrow \{f(u), f(v)\} \in E_2$. We call such a function f an *isomorphism.*

(a) Given an example of two graphs (their node names are different), both on four nodes, that are isomorphic. Explain why they are isomorphic.

Consider $g_1 = (V_1, E_1), V_1 = \{1, 2, 3, 4\}, E_1 = \{\{1, 2\}, \{1, 3\}\}$.
Consider $g_2 = (V_2, E_2), V_2 = \{a, b, c, d\}, E_2 = \{\{a, b\}, \{a, c\}\}$.
Then there exists $f : V_1 \to V_2$ such that

$$f(x) = \begin{cases} a & x = 1 \\ b & x = 2 \\ c & x = 3 \\ d & x = 4 \end{cases}$$

These graphs are isomorphic, as there is a one-to-one and onto function such that for each edge in g_1, there is the image of the edge in g_2, and vice versa.

(b) Given an example of two graphs (their node names are different), both on four nodes, that are not isomorphic. Explain why they are not isomorphic.

Consider $g_1 = (V_1, E_1), V_1 = \{1, 2, 3, 4\}, E_1 = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$.
Consider $g_2 = (V_2, E_2), V_2 = \{a, b, c, d\}, E_2 = \{\{a, b\}, \{a, c\}, \}$.
These two graphs are not isomorphic, because there is no one-to-one and onto function $f : V_1 \to V_2$ that can ensure that $\forall u, v, \{u, v\} \in E_1 \to \{f(u), f(v)\} \in E_2$

(c) Let g_1, g_2, g_3 be three graphs such that g_1 and g_2 are isomorphic and g_2 and g_3 are isomorphic. Show that g_1 and g_3 are isomorphic by formally describing the isomorphism function and proving that it is indeed an isomorphism.

Let us define $g_1 = (V_1, E_1), g_2 = (V_2, E_2), g_3 = (V_3, E_3)$.
We assume that there exist bijections $f_{12} : V_1 \to V_2$ such that
$\forall u, v, \{u, v\} \in E_1 \leftrightarrow \{f_{12}(u), f_{12}(v)\} \in E_2$, and $f_{23} : V_2 \to V_3$ such that $\forall u, v, \{u, v\} \in E_2 \leftrightarrow \{f_{23}(u), f_{23}(v)\} \in E_3$.

Then, let us prove that $f_{23}(f_{12}) = f_{13} : V_1 \rightarrow V_3$ such that $\forall u, v, \{u, v\} \in E_1 \leftrightarrow \{f_{13}(u), f_{13}(v)\} \in E_3$

First, we must show that f_{13} is one-to-one. Consider two distinct points in V_1, a and b. Since f_{12} is one-to-one, $a \neq b \implies f_{12}(a) \neq f_{12}(b)$. Since f_{23} is one-to-one, $f_{12}(a) \neq f_{12}(b) \implies f_{23}(f_{12}(a)) \neq f_{23}(f_{12}(b))$. Therefore, $a \neq b \implies f_{23}(f_{12}(a)) \neq f_{23}(f_{12}(b))$ shows that $f_{23}(f_{12}) = f_{13}$ is indeed one-to-one.

Now, we must show f_{13} is onto. We know $\forall v_3 \in V_3, \exists v_2 \in V_2(f_{23}(v_2) = v_3)$ and $\forall v_2 \in V_2, \exists v_1 \in V_1(f_{12}(v_1) = v_2)$. If we consider an arbitrary $v_3 \in V_3, \exists v_2 \in V_2(f_{23}(v_2) = v_3)$, and for that $v_2, \exists v_1 \in V_1(f_{12}(v_1) = v_2)$, we have shown that for an arbitrary $v_3 \in V_3, \exists v_1 \in V_1(f_{23}(f_{12}(v_1))) = v_3)$. Therefore, f_{13} is onto.

Finally, we must show that the function is an isomorphism. We are given:

(1) $\forall u, v, \{u, v\} \in E_1 \leftrightarrow \{f_{12}(u), f_{12}(v)\} \in E_2$, and

(2) $\forall u, v, \{f_{12}(u), f_{12}(v)\} \in E_2 \leftrightarrow \{f_{23}(f_{12}(u)), f_{23}(f_{12}(v))\} \in E_3$.

If we take the hypothetical syllogism of the forward directions of the biconditional on (1) and (2), and conjoin it with the hypothetical syllogism of the backwards directions of the biconditional on (1) and (2), we have:

$\forall u, v, \{u, v\} \in E_1 \leftrightarrow \{f_{23}(f_{12}(u)), f_{23}(f_{12}(v))\} \in E_3$ $\square$

3. [15*pts*] The *chromatic number* of a graph $g = (V, E)$ is the smallest integer $k \in \mathbb{Z}^+$ that satisfies $\exists f : V \rightarrow [k], (\{u, v\} \in E \rightarrow f(u) \neq f(v))$, where $[k]$ denotes the set $\{1, 2, ..., k\}$. For each of the following graphs, what is its chromatic number? Justify your answer.

(a) $g_1 = (V_1, E_1)$ where $V_1 = \{1, 2, ..., n\}$ and $E_1 = \varnothing$

$k = 0$, since $E_1 = \varnothing$, and $(\{u, v\} \in E \rightarrow f(u) \neq f(v))$ is vacuously true.

(b) $g_2 = (V_2, E_2)$ where $V_2 = \{1, 2, ..., n\}$ and $E_2 = \{\{i, j\} | i, j \in V_2, i \neq j\}$

$k = n$, since g_2 is a complete graph, and each node is connected to every other node.

(c) Consider the map of the 50 States of the United States, plus Canada and Mexico, and think of it as a map of 52 "countries". We would like to color the map such that no two counties that share a border have the same color (for ease of visualization of the map). We are interested in finding the smallest number of colors needed to color the map. How would you solve this problem by formulating it as a graph-theoretic problem and an algorithm that computes the chromatic number of a graph? Explain your answer.

We formulate the set of the 52 "countries" as a graph $g = (V, E)$, where V is the set of all the states, Canada, and Mexico, and E is the set of edges $\{u, v\}$ such that u, v are neighboring countries. Then, the problem reduces to finding a "chromatic number" in order to color the map such that no two neighboring "countries" (nodes connected by an edge) have the same color (are mapped to the same number in [k]).

Then, our algorithm is one that takdes g as input and outputs k, the chromatic number.