COMP 182: HW 1

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Problem 1

The goal of this problem is for you to read the paper and describe each of the five steps as they relate to this paper. In particular:

- Understanding the problem: What is the question that Zachary wanted to answer? What was the motivation for the question? What data did he collect? How did he think about solving the problem?

  Zachary wanted to discern the reason and the process of fisson in a small social group, in this case, a voluntary association. Such a problem was motivated by years of prior study, the centrality of the problem to social anthropologists, and the decision to apply a social network model to the small group. He collected historical information about the karate club in order to get a sense of the context of the fission, as well as statements from the club members with regard to their involvement in the club, their factional alignment, the club they joined after the fission, and the existence and strength of out-of-club connections with other members. He thought of using a novel method to formalize the problem into a mathematical model, to which he could apply his theories.

- Formulating the problem: What was the mathematical formulation of the problem that Zachary arrived at after understanding the problem?

  After Zachary understood the problem, he decided to craft a capacitative network, represented in both graph and matrix form, depicting nodes (people), the edges between nodes (affective friendships between friends), and weights (the capacity for information to flow through a given connection). Zachary then used the idea of a ”maximum flow-minimum cut labeling procedure” to test his theory of information flow in a voluntary associative group.

- Algorithm design: What algorithmic technique did Zachary use to solve the problem? You do not need to describe the algorithm itself, but rather the algorithmic technique he employed (1-2 sentences).

  Zachary cites a method that he wishes to use in order to execute the maximum flow-minimum cut procedure, by the name of NETFLOW. The maximum flow-minimum cut is a partition of the network along a *cut*, a collection of edges where the flow is equal to the capacity.

- Algorithm implementation: Comment on the algorithm implementation (1-2 sentences).

  Zachary used NETFLOW, an APL program to implement the algorithm on his model, developed by Smillie in his publication, *STATPACK 2*. To determine uniqueness, the program was run on the capacitative model with reversed source and sink.

- Solving the original problem: What were Zachary's findings in terms of the original problem he set out to solve?

  Zachary found that his model predicted the factional alignments and club membership with remarkable precision. With respect to the original problem, his results signify that fission in small social groups can be modeled and predicted using the concept of information flow in a capacitative network.

Problem 2

Write each of the following sentences using propositional logic. Clearly introduce propositions (using p, q, r, etc.) and give the compound proposition that correspond to the English sentence.

1. For you "to get an A in this course" (p), it is necessary and sufficient that "you learn how to solve discrete mathematics problems" (q).

   $p \leftrightarrow q$

2. "I go to the beach" (p) whenever "it is a sunny summer day" (q).

   $q \rightarrow p$

3. "The beach erodes" (p) whenever "there is a storm" (q).

   $q \rightarrow p$

4. "To be a citizen of this country" (p), it is sufficient that "you were born in the United States" (q).
   $q \rightarrow p$

5. "To take COMP 182" (p), you must "have taken COMP 140" (q).

   $p \rightarrow q$

6. If "you read the newspaper every day" (p), "you will be informed" (q), and conversely.

   $p \leftrightarrow q$

Recall that $\oplus$ denotes "exclusive or." For each of the following compound propositions, determine if it is satisfiable or not. If it is satisfiable, determine also if it is a tautology. Prove your answer.

1. $p \oplus p$

p	$p \oplus p$
T	F
F	F

   It is clear from the truth table that for all values of p, $p \oplus p$ is false. As such, $p \oplus p$ is a contradiction, and unsatisfiable.

2. $p \oplus \neg p$

p	$\neg p$	$p \oplus \neg p$
T	F	T
F	T	T

   As before, we use the truth table to show that $p \oplus \neg p$ is true for all values of p. Therefore, $p \oplus \neg p$ is satisfiable, and a tautology.

3. $(p \oplus q) \vee (p \oplus \neg q)$

p	q	$p \oplus q$	$\neg q$	$p \oplus \neg q$	$(p \oplus q) \vee (p \oplus \neg q)$
T	T	F	F	T	T
T	F	T	T	F	T
F	T	T	F	F	T
F	F	F	T	T	T

   In this case, the truth table demonstrates that $(p \oplus q) \vee (p \oplus \neg q)$ is true for all possible combinations of p, q. Therefore, it can be concluded that $(p \oplus q) \vee (p \oplus \neg q)$ is both satisfiable, and a tautology.

Two compound propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology (see Section 1.3 in your textbook). Which of the following pairs of compound propositions are logically equivalent. Prove your answer.

1. $(p \wedge q) \to r$ and $(p \to r) \wedge (q \to r)$

   Consider $p = T, q = F, r = F$. Then $(p \wedge q) \to r$ is T, and $(p \to r) \wedge (q \to r)$ is F. Since there exist p, q, r, such that $[(p \wedge q) \to r] \leftrightarrow [(p \to r) \wedge (q \to r)]$ is false, the statement is not a tautology, and therefore $(p \wedge q) \to r$ and $(p \to r) \wedge (q \to r)$ are not logically equivalent.

2. $(p \vee q) \to r$ and $(p \to r) \vee (q \to r)$

   Consider $p = T, q = F, r = F$. Then $(p \vee q) \to r$ is F, and $(p \to r) \vee (q \to r)$ is T. Since there exist p, q, r, such that $[(p \vee q) \to r] \leftrightarrow [(p \to r) \vee (q \to r)]$ is false, the statement is not a tautology, and therefore, $(p \vee q) \to r$ and $(p \to r) \vee (q \to r)$ are not logically equivalent.

3. $p \to (q \leftrightarrow r)$ and $(p \to q) \leftrightarrow (p \to r)$

p	q	r	$p \to (q \leftrightarrow r)$	$(p \to q) \leftrightarrow (p \to r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

   As the truth table notes, $p \to (q \leftrightarrow r)$ and $(p \to q) \leftrightarrow (p \to r)$ have the same truth value for all possible combinations of p, q, and r. Therefore, the statements are logically equivalent.

Problem 3

Determine the truth value of each of these statements if the domain is $\mathbb{R}$. Explain your answer.

1. $\exists x(x^3 = -1)$ The statement is true, and the existential quantifier is satisfied with $x = -1$.
2. $\forall x((-x)^2 = x^2)$ The statement is true. Consider an arbitrary x. Then, $(-x)^2 = (-1 \cdot x)^2 = (-1)^2 x^2 = x^2$
3. $\forall x(2x > x)$ The statement is false, as the universal quantifier is negated by the existence of $x = -1$.
4. $\forall x \exists y(x \cdot y = y \cdot x)$ The statement is true, as multiplication on $\mathbb{R}$ is commutative.
5. $\forall x \forall y \exists z(z = (x + y)/2)$ The statement is true, as $z = (x + y)/2$ is a function defined on all points of the domain $\mathbb{R}^2$.
6. $\exists x \forall y(x \leq y^2)$ The statement is true, as $x = -1$ satisfies the statement for all $y \in \mathbb{R}$, by the property $\forall y \in \mathbb{R}(y^2 \geq 0)$.
7. $\exists x \forall y(x \leq y)$ The statement is false, as there does not exist an x such that $x \leq y$ for all y. We assume the statement to be true, and then consider an arbitrary y, and let there be some x such that $x \leq y$. Consider y' in the same domain as y, where $y' = x - 1$. As such, it is clear that we have reached a contradiction and the statement is false.

Which of the following pairs of expressions are logically equivalent? Prove your answer.

1. $\forall x(P(x) \lor Q(x))$ and $\forall x P(x) \lor \forall x Q(x)$

   Consider the domain $\mathbb{R}$, $P(x) = x > 0$, and $Q(x) = x \leq 0$. Given these $P(x), Q(x)$, $\forall x(P(x) \lor Q(x))$ is true, as any $x \in \mathbb{R}$ must either be > 0 or ≤ 0; while $\forall x P(x) \lor \forall x Q(x)$ is false, as it is clear that $\forall x \in \mathbb{R}$, $x \not> 0$.

2. $\forall x(P(x) \land Q(x))$ and $\forall x P(x) \land \forall x Q(x)$

   Let us first assume $\forall x(P(x) \land Q(x))$ is true.

   $\forall x P(x) \land \forall x Q(x)$ will be shown to be true.

1.	$\forall x(P(x) \land Q(x))$	Assumption
2.	$P(a) \land Q(a)$ for an arbitrary a	Universal instantiation on (1)
3.	$P(a)$ for an arbitrary a	Simplification on (2)
4.	$Q(a)$ for an arbitrary a	Simplification on (2)
5.	$\forall x P(x)$	Universal generalization on (3)
6.	$\forall x Q(x)$	Universal generalization on (4)
7.	$\forall x P(x) \land \forall x Q(x)$	Conjunction on (5) and (6)

$\square$

Let us first assume $\forall x P(x) \wedge \forall x Q(x)$ is true.
$\forall x (P(x) \wedge Q(x))$ will be shown to be true.

1.	$\forall x P(x) \wedge \forall x Q(x)$	Assumption
2.	$\forall x P(x)$	Simplification on (1)
3.	$\forall x Q(x)$	Simplification on (1)
4.	$P(a)$ for an arbitrary a	Universal instantiation on (2)
5.	$Q(b)$ for an arbitrary b	Universal instantiation on (3)
6.	$P(a) \wedge Q(b)$ for an arbitrary a, b	Conjunction on (4) and (5)
7.	$P(a) \wedge Q(a)$ for an arbitrary a	Let b=a on (6)
8.	$\forall x (P(x) \wedge Q(x))$	Universal generalization on (7)

□

3. $\forall x P(x) \vee \forall x Q(x)$ and $\forall x \forall y (P(x) \vee Q(y))$ (here all quantifiers have the same nonempty domain).

Let us demonstrate $\forall x P(x) \vee \forall x Q(x) \rightarrow \forall x \forall y (P(x) \vee Q(y))$
by contraposition

1.	$\neg[\forall x \forall y (P(x) \vee Q(y))]$	Assumption
2.	$\exists x \exists y (\neg P(x) \wedge \neg Q(y))$	Apply De Morgan's to (1)
3.	$\exists y (\neg P(a) \wedge \neg Q(y))$ for some a	Existential instantiation on (2)
4.	$\neg P(a) \wedge \neg Q(b)$ for some a, b	Existential instantiation on (3)
5.	$\neg P(a)$ for some a	Simplification on (3)
6.	$\neg Q(b)$ for some b	Simplification on (3)
7.	$\exists x \neg P(x)$	Existential generalization on (4)
8.	$\exists x \neg Q(x)$	Existential generalization on (5)
9.	$\exists x \neg P(x) \wedge \exists x \neg Q(x)$	Conjunction on (7)
10.	$\neg[\forall x P(x) \vee \forall x Q(x)]$	Apply De Morgan's to (8)

□

Let us demonstrate $\forall x \forall y (P(x) \vee Q(y)) \rightarrow \forall x P(x) \vee \forall x Q(x)$
by contraposition

1.	$\neg[\forall x P(x) \vee \forall x Q(x)]$	Assumption
2.	$\exists x \neg P(x) \wedge \exists x \neg Q(x)$	Apply De Morgan's to (1)
3.	$\exists x \neg P(x)$	Simplification on (2)
4.	$\exists x \neg Q(x)$	Simplification on (2)
5.	$\neg P(a)$ for some a	Existential instantiation on (3)
6.	$\neg Q(b)$ for some b	Existential instantiation on (4)
7.	$\neg P(a) \wedge \neg Q(b)$ for some a, b	Conjunction on (6)
8.	$\exists y (\neg P(a) \wedge \neg Q(y))$ for some a	Existential generalization on (7)
9.	$\exists x \exists y (\neg P(x) \wedge \neg Q(y))$	Existential generalization on (8)
10.	$\neg[\forall x \forall y (P(x) \vee Q(y))]$	Apply De Morgan's to (9)

□