COMP 182: Homework 7

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Friday, April 3, 2020

Problem 1: Counting [70 pts]

1. [6pts] How many bit strings of length 10 either start with 000 or end with 111? Show your work.

   There are 2^7 bit strings that start with 000 and are of length 10. Similarly, there are 2^7 bit strings that end with 111. There are 2^4 bit strings that begin with 000 and end with 111. Therefore, there are $2^7 + 2^7 - 2^4 = 256 - 16 = \boxed{240}$ bit strings that start with 000 or end with 111.

2. [6pts] How many positive integers less than 1,000,000 have exactly one digit equal to 9 and have a sum of digits equal to 13? Show your work.

   If one digit is 9, the other digits must sum to 4. We can allow leading zeroes in our number to correspond to those number with less than 6 digits. Thus, we have 4 stars and 4 bars. Therefore, there are $8!/(4!4!) = 70$ possible choices of the 5 remaining integers. As such, there are $70 * 6 = \boxed{420}$ integers whose digits sum to 13, are less than 1,000,000, and have the digit 9.

3. [6pts] How many solutions are there to the inequality $x_1 + x_2 + x_3 \leq 11$, where x_1, x_2, and x_3 are nonnegative integers? Show your work.

   Again, we have the n stars and 2 bars problem, for $x_1 + x_2 + x_3 = n$ for $n = 0, ..., 11$.

   $$\sum_{n=0}^{11} \frac{(n+2)!}{2!n!} = \sum_{n=0}^{11} \frac{(n+1)(n+2)}{2} = \sum_{n=0}^{11} \frac{n^2+3n+2}{2} =$$

   $$\frac{1}{2}\left(\frac{11(2*11+1)(11+1)}{6} + 3\frac{11(11+1)}{2} + 22\right) + 1 = 11(23) + 99 + 11 + 1 = 364$$

4. [7pts] Prove, without using induction, that in any set of $n+1$ positive integers not exceeding $2n$ there must be two that are relatively prime.

   First, we establish that two consecutive integers must be relatively prime. Consider the contradiction, that n and $n+1$ are both divisible by q, a positive integer greater than one. Then, n/q and $n/q + 1/q$ are integers. However, this implies $1/q$ is an integer, and we thus have a contradiction, as $q > 1$.

   Since we choose $n+1$ integers from a 1 to $2n$, we can say we wish to choose an integer in the n pairs of consecutive integers. By the pidgeonhole principle, there must be two consecutive integers, who then must be coprime. ■

5. [7pts] Suppose that p and q are distinct prime numbers and that $n = pq$. What is the number of positive integers not exceeding n that are relatively prime to n? Show your work.

   Consider the number of positive integers that are not relatively prime. Thus, they must either be a multiple of q (of which there are p) or p (of which there are q). Thus, since there is only one integer that is both a multiple of p and of q, there are $\boxed{n - p - q + 1}$ positive integers not exceeding and relatively prime to n.

6. [18pts] In celebration of the end of the semester, all the students and instructors (including TA's; not just Luay) in COMP 182 were invited to take a group picture. Assume that are n students and m instructors. Show your work in each part.

(a) How many ways are there to arrange all the students and instructors in a line such that all students are standing next to each other (no instructors between them)?

All the students must stand together ($n!$), and all the instructors must stand together ($m!$), thus there are $\boxed{2 * n! * m!}$ ways.

(b) How many ways are there to arrange all the students and instructors in a line such that no two instructors are standing next to each other (the number of students in the course is so large that it allows us to consider this!)?

Consider the $n+1$ spaces that an instructor can fill between the students. Then, there are $\binom{n+1}{m}$ combinations of students and instructors. Thus, there are $\boxed{\binom{n+1}{m} m!n!}$ total permutations.

(c) How many ways are there to arrange all the students and instructors in a line such that no student is standing next to Luay?

Let us bind an instructor to either side of Luay (of which there are $(m-1)(m-2)$ ways to do so). Then there are $(m-1)(m-2)(n+m-2)!$ ways of organizing the n students, $m-3$ instructors and the bound Luay.

Now we also must consider the cases Luay is on either side of the photo. There are $m-1$ instructors to be by his side, so there are a total of $2(m-1)(m+n-2)!$ permutations with a Luay cap.

Thus, there are a total of $(m-1)(m-2)(n+m-2)!+2(m-1)(m+n-2)! = \boxed{m(m-1)(n+m-2)!}$ permutations such that no student touches Luay.

7. [4*pts*] What are the coefficients of x^4 and x^6 in $(x^2 - 2x^{-1})^8$? Show your work.

The binomial theorem then gives:

$$\sum_{j=0}^{8} \binom{8}{j} x^{2j}(-2x^{-1})^{8-j} = \sum_{j=0}^{8} \binom{8}{j} x^{3j-8}(-2)^{8-j}$$

Let $j = 4$:

$$\binom{8}{4} x^{12-8}(-2)^{8-4} = \frac{8!}{4!4!}16x^4 = \boxed{1120}x^4$$

For integer values of j, $3j - 8 = 6$ has no solution. Thus, the coefficient of x^6 is $\boxed{0}$

8. [10*pts*] Give combinatorial proofs for each of the following equalities.

(a) $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$. Let us consider the number of ways there are to choose n distinct integers from 1 to $2n$. One way to do this is $\binom{2n}{n}$.

Another way is to choose k integers from 1 to n, and $n-k$ integers from $n+1$ to $2n$, for $k = 0, ..., n$. There are $\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2$ ways to do so.

(b) $3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k$.

Consider the number of sequences of length n, made from the numbers $\{1, 2, 3\}$. Clearly, one way to count them is 3^n.

Now consider the same problem, and fix the number of 3's to be $n - k$. Then, there are $\binom{n}{k}$ ways to choose the non-3 elements, and 2^k different permutations of those elements. Now, simply vary $k = 0, ..., n$ the obtain the result: $\sum_{k=0}^{n} \binom{n}{k} 2^k$

9. [6*pts*] How many solutions does $x_1 + x_2 + x_3 = 50$ have, where x_1, x_2, and x_3 are nonnegative integers each of which is smaller than 20.

Let us fix $x_1 = 12, ..., 19$:

- $x_1 = 12$, 1 solution [19, 19]
- $x_1 = 13$, 2 solutions [18, 19], [19, 18]
- $x_1 = 14$, 3 solutions
- $x_1 = 15$, 4 solutions
- $x_1 = 16$, 5 solutions
- $x_1 = 17$, 6 solutions
- $x_1 = 18$, 7 solutions
- $x_1 = 19$, 8 solutions

Therefore, there are $8 * 9/2 = 36$ possible solutions