COMP 182: Homework 6

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Recursion and Induction [100 pts]

— ALL PROOFS MUST USE INDUCTION —

1. [10pts] Prove that for every positive integer n, $\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}$.

   Hypothesis: $P(n) : \sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

   Base Case: $P(1)$. $1(1+1)(1+2) = 6$, $1(1+1)(1+2)(1+3)/4 = 6$.

   Inductive Case: Assume $P(k)$:

$$\sum_{i=1}^{k} i(i+1)(i+2) = \frac{k(k+1)(k+2)(k+3)}{4}$$
$$(k+1)(k+2)(k+3) + \sum_{i=1}^{k} i(i+1)(i+2) = \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)$$
$$\sum_{i=1}^{k+1} i(i+1)(i+2) = \frac{k(k+1)(k+2)(k+3)}{4} + \frac{4(k+1)(k+2)(k+3)}{4}$$
$$\sum_{i=1}^{k+1} i(i+1)(i+2) = \frac{(k+1)(k+2)(k+3)(k+4)}{4} \quad \blacksquare$$

2. [10pts] Prove that for all $n \geq 0$, $\sum_{i=0}^{n} i^3 = (\sum_{i=0}^{n} i)^2$.

   Hypothesis: $P(n) : \sum_{i=0}^{n} i^3 = (\sum_{i=0}^{n} i)^2$

   Base Case: $P(0) : \sum_{i=0}^{0} i^3 = 0, (\sum_{i=0}^{0} i)^2 = 0^2 = 0$

   Inductive Case: Assume $P(k)$:

$$\sum_{i=0}^{k} i^3 = (\sum_{i=0}^{k} i)^2$$
$$\sum_{i=0}^{k} i^3 = (\frac{k(k+1)}{2})^2$$
$$(k+1)^3 + \sum_{i=0}^{k} i^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$$
$$\sum_{i=0}^{k+1} i^3 = \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$
$$\sum_{i=0}^{k+1} i^3 = \frac{(k+1)^2(k^2+4k+4)}{4}$$
$$\sum_{i=0}^{k+1} i^3 = \frac{(k+1)^2(k+2)^2}{4}$$
$$\sum_{i=0}^{k+1} i^3 = (\sum_{i=0}^{k+1} i)^2 \quad \blacksquare$$

3. [10*pts*] Prove that n lines separate the plane into $(n^2+n+2)/2$ regions if no two of these lines are parallel and no three pass through a common point.

   Hypothesis: $P(n:)$ n lines separate the plane into $(n^2+n+2)/2$ regions if no two of these lines are parallel and no three pass through a common point.

   Base Case: $P(1)$ separates the plane into $(1^2+1+2)/2 = 2$ regions.

   Inductive Case: Assume $P(k)$: that k lines separate the plane into $(k^2+k+2)/2$ regions. Now, consider adding an additional line to the plane that is not parallel to any other line and does not intersect any existing intersections. Then, the new line must intersect each new line.

   Now consider the path the new line makes. It must intersect k lines, and $k+1$ regions (along the path, k partitions leads to $k+1$ sections). Then, with $k+1$ lines, there are an additional $k+1$ sections.

$$(k^2+k+2)/2 + k + 1 = (k^2+2k+1+k+1+1)/2 = ((k+1)^2+(k+1)+1)/2 \quad \blacksquare$$

4. [10*pts*] Prove that a postage of n cents, for $n \geq 18$, can be formed using just 4-cent stamps and 7-cent stamps.

   Hypothesis: $P(n) : n = 4x + 7y$, where $x, y \in \mathbb{N} \cup 0$

   Base Case: $P(18) = 4 * 1 + 7 * 2$

   Inductive Case: Since $k \geq 18$

   If $y = 0$ then $x > 4$. Then:

$$\begin{aligned} k &= 4x + 7y \\ k+1 &= 4x + 7y + 1 \\ k+1 &= 4x + 7y + 1 \\ k+1 &= 4(x) + 7y - 20 + 21 \\ k+1 &= 4(x-5) + 7(y+3) \end{aligned}$$

   If $y > 0$, then:

$$\begin{aligned} k &= 4x + 7y \\ k+1 &= 4x + 7y + 1 \\ k+1 &= 4x + 7y - 7 + 8 \\ k+1 &= 4(x+2) + 7(y-1) \end{aligned}$$

   In both cases, $k+1 = 4x' + 7y'$, x', y' are nonnegative integers $\blacksquare$.

5. [10*pts*] Let x be a real number such that $x + \frac{1}{x}$ is an integer. Prove that $x^n + \frac{1}{x^n}$ is an integer for all integers $n \geq 0$.

   Hypothesis: $P(n) : (x \in \mathbb{R} \wedge x + \frac{1}{x} \in \mathbb{Z}) \implies x^n + \frac{1}{x^n} \in \mathbb{Z}$

   Base Case: If $x + \frac{1}{x}$ is an integer, then $x \neq 0$. Then, $x^0 = 1$, and $x^n + \frac{1}{x^n} = 1 + 1/1 = 2$

   Inductive Case: Here, let us utilize strong induction. Thus, we assume $P(n)$ is true for all $n \in \{1, 2, ..., k\}$. Consider the product of two integers:

$$\left(x + \frac{1}{x}\right)\left(x^k + \frac{1}{x^k}\right) = \left(x^{k+1} + \frac{1}{x^{k+1}} + x^{k-1} + \frac{1}{x^{k-1}}\right) = \left(x^{k+1} + \frac{1}{x^{k+1}}\right) + \left(x^{k-1} + \frac{1}{x^{k-1}}\right)$$

   The product of two integers is an integer. So is the difference of two integers. Therefore,

$$\left(x^{k-1} + \frac{1}{x^{k-1}}\right) \in \mathbb{Z} \implies \left(x^{k+1} + \frac{1}{x^{k+1}}\right) \in \mathbb{Z} \quad \blacksquare$$

6. [10*pts*] Consider the sequences $a_1, a_2, a_3, ...$ where $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for all $n \geq 4$. Prove that $a_n < 2^n$ for all positive integers n.

   Hypothesis: $P(n) : a_n < 2^n$

   Base Case: $P(4) : a_4 = 1 + 2 + 3 = 6 < 2^4 = 16$

Inductive Case: Here, let us utilize strong induction. Thus, we assume $P(n)$ is true for all $n \in \{4, 5, ..., k\}$. Consider:

$$\begin{aligned} a_{k+1} &= a_k + a_{k-1} + a_{k-2} \\ a_{k+1} &< 2^k + 2^{k-1} + 2^{k-2} \\ a_{k+1} &< 7 \times 2^{k-2} < 8 \times 2^{k-2} = 2^{k+1} \quad \blacksquare \end{aligned}$$

7. [10pts] A subsequence of a string w is a sequence of letters from w in the same order in which they appear in w, but not necessarily contiguously. We denote by $Sub(w)$ the set of all subsequences of string w. For example, $Sub(aba) = \{\epsilon, a, b, ab, ba, aa, aba\}$. Give a recursive definition of $Sub(w)$.

   Basis Step: $\epsilon \in Sub(w)$

   Recursive Step: If $w = xa$, where x is a substring and $a \in \Sigma$ (set of the letters of w); then we define $Sub(w) = \{ya : y \in Sub(x)\} \cup Sub(x)$

8. [10pts] Give a recursive definition of the set of all binary strings (strings over the alphabet $\{0, 1\}$) that have more 0's than 1's.

   Basis Step: 0 has more 0s than ones. Thus $0 \in S$

   Recursive Step: Assume $w, z \in S$. Then, $wz, 1wz, w1z, wz1 \in S$

9. [20pts] Recall the recursive definitions $n(T)$ and $h(T)$ of the number of nodes and height, respectively, of a full binary tree. Further, consider the following recursive definitions of the set of leaves and the set of internal nodes of a full binary tree:

   - The root r is a leaf of the full binary tree with exactly one node r. This tree has no internal nodes.
   - The set of leaves of the tree $T = T_1 \cdot T_2$ is the union of the sets of leaves of T_1 and of T_2. The internal nodes of T are the root r of T and the union of the sets of internal nodes of T_1 and of T_2.

   Denote by $\ell(T)$ and $i(T)$ the numbers of leaves and of internal nodes, respectively, of a full binary tree T. Use structural induction to prove the following two results on full binary trees T.

   Note: The below relationships follow from the definitions of $\ell(T), i(T)$. Consider $T = T_1 \cdot T_2$, T_1, T_2 disjoint.

$$\ell(T) = \ell(T_1) + \ell(T_2) \qquad i(T) = i(T_1) + i(T_2) + 1$$

   (a) $n(T) \geq 2h(T) + 1$
   Hypothesis: $P(T) : n(T) \geq 2h(T) + 1$
   Base Case: $P(r) : n(r) = 1 \geq 2h(T) + 1 = 0 + 1$
   Inductive Case: We assume $P(T_1)$ and $P(T_2)$. For a tree $T = T_1 \cdot T_2$:,

$$\begin{aligned} n(T) &= n(T_1) + n(T_2) + 1 \\ n(T) &\geq 2h(T_1) + 1 + 2h(T_2) + 1 + 1 \\ &\geq 2 \max_{T_1, T_2} h(T_n) + 3 = 2 \max_{T_1, T_2} h(T_n) + 3 \\ &= 2h(T) + 1 \quad \blacksquare \end{aligned}$$

   (b) $\ell(T) = 1 + i(T)$
   Hypothesis: $P(T) : \ell(T) = 1 + i(T)$
   Base Case: $P(r) : \ell(r) = 1,\ 1 + i(r) = 1$

Inductive Case: We assume $P(T_1)$ and $P(T_2)$ for disjoint full binary trees T_1, T_2. For a tree $T = T_1 \cdot T_2$:

$$\ell(T) = \ell(T_1) + \ell(T_2)$$
$$\ell(T) = 2 + i(T_1) + i(T_2) = 1 + (1 + i(T_1) + i(T_2))$$
$$\ell(T) = 1 + i(T) \quad \blacksquare$$