COMP 182: Homework 5

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Problem 1: Relations [25 points]

1. Let R_1 be the relation defined on the set of ordered pairs of positive integers such that $(a, b)R_1(c, d)$ if and only if $ad = bc$. Is R_1 an equivalence relation? Prove your answer.

   - For every ordered pair of positive integers (a, b), $ab = ab \implies ((a, b), (a, b)) \in R_1$. Therefore, R_1 is reflexive.
   - For every $((a, b), (c, d)) \in R_1$, $ad = bc \implies ((c, d), (a, b)) \in R_1$. Therefore, R_1 is symmetric.
   - For every $((a, b), (c, d))$ and $((c, d), (f, g)) \in R_1$, $(ad = bc \wedge cg = fd) \implies a/b = c/d = f/g \implies ((a, b), (f, g)) \in R_1$. Therefore, R_1 is transitive.

   Thus, R_1 is an equivalence relation. $\blacksquare$

2. Give a partially ordered set, or poset, that has

   (a) a minimal element but no maximal element.
   Consider $S = \mathbb{N}, R = \{(a, b)|a \leq b\}$

   (b) neither a maximal nor a minimal element.
   Consider $S = \mathbb{Z}, R = \{(a, b)|a \leq b\}$

3. Let $(A, \preceq)$ be a partially ordered set that has no minimal element and A is not empty. Can A be finite?

   No. Consider an arbitrary poset $(A, \preceq)$ that has no minimal element and A is not empty, and an arbitrary element a_0. Since there is no minimal element, there must exist a_1 such that $a_1 \prec a_0$. The same can be said for a_1, that is, that there must exist a_2 such that $a_2 \prec a_1$. This process can be applied to any a_n indefinitely, and therefore, A cannot be finite. $\blacksquare$

Problem 2: Computing Rooted Directed MSTs (RDMSTs) [51 pts]

Lemma 1: Let $g = (V, E, w)$ be a weighted, directed graph with designated root $r \in V$. Let

$$E' = \{me(u) : u \in (V \setminus r)\}$$

. Then, either $T = (V, E')$ is an RDMST of g rooted at r or T contains a cycle.

Proof of Lemma 1:

It is evident that if T is a RDMST, then T does not contain a cycle.

To demonstrate the converse, assume T does not contain a cycle.

Ignoring the directions of the edges of E', and noticing are exactly $|V| - 1$ edges and no cycles, it is evident that T is a spanning tree. Then, if the directions of E' are taken into account, noting that T is a spanning tree and every node in $V \setminus r$ is the head of an edge, there is a path from r to every node in $V \setminus r$ and T is a RDST.

Now, let us also assume that there is a distinct RDST, $R \neq T$, that is the RDMST. Then, the weight of R must be less than that of T. However, since T is defined to have the edges of least weight for each node, and R is distinct, we have a contradiction: R cannot be the RDMST.

Therefore, either $T = (V, E')$ is an RDMST of g rooted at r or T contains a cycle. ■

Lemma 2: Let $g = (V, E, w)$ be a weighted, directed graph with designated root $r \in V$. Consider the weight function $w' : E \to R+$ defined as follows for each edge $e = (u, v)$:

$$w'(e) = w(e) - m(v).$$

Then, $T = (V, E')$ is an RDMST of $g = (V, E, w)$ rooted at r if and only if T is an RDMST of $g = (V, E, w')$ rooted at r.

Proof of Lemma 2:

To prove **Lemma 2**, let it first be shown that for all RDSTs $T = (V, E')$ of $g = (V, E, w)$ rooted at r, the weight of T differs by the same constant to the weight of T of $g = (V, E, w')$.

Consider an arbitrary RDST $T = (V, E')$ of $g = (V, E, w)$, and let the weight of T be W. Then, applying $w'(e) = w(e) - m(v)$ ensures that the weight of each edge in T decreases by the minimum weight of the edges heading into the same node. Since each RDST of g must have one and only one edge heading into each node in $V \setminus r$; the total weight of T, whether with regards to w or w', differs by the same constant as all other RDST.

The weight of T of $g = (V, E, w')$ becomes $W - \sum_{v \in V \setminus r} m(v)$.

Assume $T = (V, E')$ is an RDMST of $g = (V, E, w)$ rooted at r. Then, as shown above, the weights of all RDSTs will decrease by the same amount, and thus, T will have the least weight of all RDSTs. ■

Problem 3: Elucidating the Transmission of a Bacterial Infection [24 pts]

Analyze the provided data and report:

- Use the provided function infer_transmap to build a transmission map rooted at Patient 1. For this provide a drawing (upload a file). You can use graphviz or any drawing software of your choice.

  [The graph is provided below. Note that the weight values on the graph have been rounded to the nearest thousandth, and the original values are displayed in **Table 1**]

- The weight of the RDMST that you computed.

  $\boxed{43.05940784503632}$

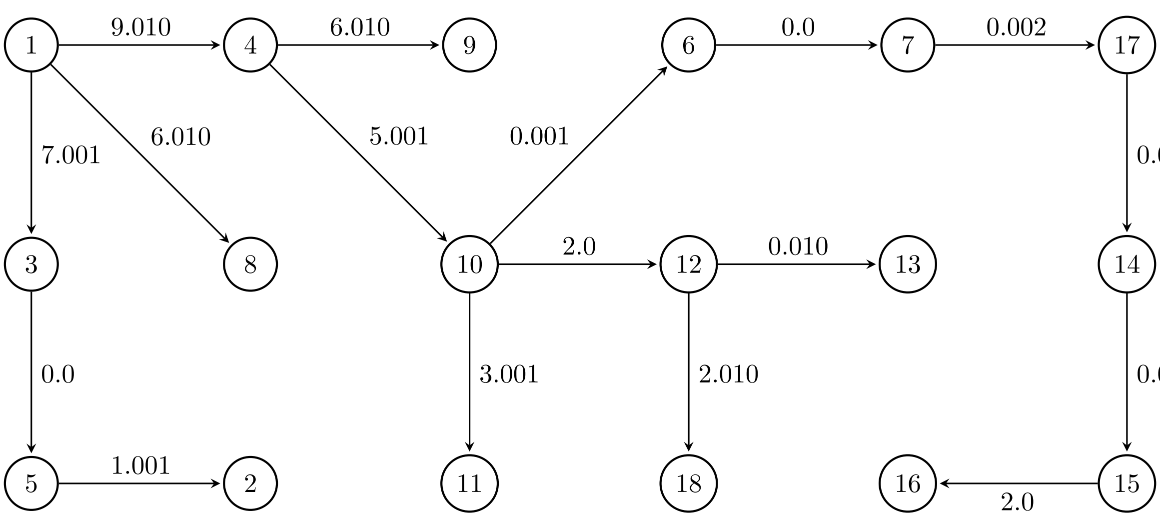
- A brief description of the transmission map (that is, what does it tell you about how the infection spread).

  The transmission map demonstrates that the infection only spreads to a few people at a time. The process creates a tree with a high degree of depth but relatively low width.

  This could signify that the patients are unlikely to spread it to many people at a time, but those who are infectious easily infect 1-3 people. The fact that this outbreak occurred at a hospital definitely affected the infection rate- another factor consider is the fact that this is a bacterial rather than viral infection.

- A brief description of whether this transmission map is unique (you need to check the complete weighted digraph that you built in order to answer this question).

  No, the transmission map is not unique. For example, the edge (12, 18) can be replaced by (13, 18); and (1, 4) Both edges have the same weight, 2.00999. Upon further inspection, this seems to be the only viable single substitution. The other single substitution option is (1, 4) by (18, 4) [w(18, 4) = w(1, 4) = 9.00999], but creates a cycle and is thus non-viable.



Edge	Weight
(1, 8)	6.00999
(1, 4)	9.00999
(1, 3)	7.001112687651331
(4, 9)	6.00999
4, 10)	5.000822421307506
(15, 16)	2.0
(14, 15)	0.0024914527845036317
(17, 14)	0.00123363196125908
(12, 18)	2.00999
(12, 13)	0.00999
(10, 11)	3.0007498547215494,
(10, 12)	2.0
(10, 6)	0.0005079661016949153
(6, 7)	0.0
(7, 17)	0.0020076755447941885
(3, 5)	0.0
(5, 2)	1.0005321549636803

Table 1: Weights of each edge of the transmission tree