$\Sigma (x-\mu)$.

vì Σ đối xứng $\Rightarrow \Sigma^{-1}$ đối xứng.

1 Prove that:

a, Prove Gaussian distribution is normalize

Normalization of Univariate Gaussian is given by

$$\int_{-\infty}^{+\infty} P(x \mid \mu, \sigma^2)\, dx = 1$$

$$\Leftrightarrow \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) dx = 1$$

Đầu tiên chúng ta sẽ c/minh base case khi mean = 0 ($\mu = 0$):

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2} x^2\right) dx = 1$$

$$\Leftrightarrow \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2\pi\sigma^2}\right) dx = \sqrt{2\pi\sigma^2}$$

$$I = \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2} x^2\right) dx$$

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Bình phương 2 vế, ta có:

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}x^2\right)\exp\left(-\frac{1}{2\sigma^2}y^2\right)$$

$$\Leftrightarrow I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)dx\,dy$$

Đổi biến hệ tọa độ (x, y) sang hệ tọa độ (r, θ)

$x = r\cos\theta$
$y = r\sin\theta$

Và sử dụng tính chất:
$\cos^2\theta + \sin^2\theta = 1$
Mà $x^2 + y^2 = r^2$

→ Dùng Jacobian sự thay đổi các biến được đưa ra:

$$dx\,dy = \begin{vmatrix} \frac{\partial(x)}{\partial(r)} & \frac{\partial(x)}{\partial(\theta)} \\ \frac{\partial(y)}{\partial(r)} & \frac{\partial(y)}{\partial(\theta)} \end{vmatrix}$$

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$$dx\,dy = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r$$

$$\Rightarrow dx\,dy = r\,dr\,d\theta$$

Thay vào I, ta có:

$$I^2 = \int_0^{2\pi}\int_0^{\infty} \exp\left(-\frac{r^2}{2\sigma^2}\right) r\,dr\,d\theta$$

$$= 2\pi \int_0^{\infty} \exp\left(-\frac{r^2}{2\sigma^2}\right) r\,dr$$

$$= 2\pi\pi \int_0^{\infty} \exp \exp\left(-\frac{r^2}{2\sigma^2}\right) \frac{d(r^2)}{2}$$

$$= \pi\left[\exp\left(-\frac{u}{2\sigma^2}\right)(-2\sigma^2)\right]_0^{\infty}$$

$$= 2\pi\sigma^2$$

$\Rightarrow I = \sqrt{2\pi\sigma^2}$, độc lập khi mean $\neq 0$, giả

sử $t = x - u$

$$\int_{-\infty}^{\infty} p(x|u,\sigma^2)\,dx = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} \exp\left(\frac{-t^2}{2\sigma^2}\right)dt$$

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$$= \frac{I}{\sqrt{2\pi\sigma^2}} = \frac{\sqrt{2\pi\sigma^2}}{\sqrt{2\pi\sigma^2}} = 1$$

b) Expectation of Gaussian distribution is : μ

Ta có PDF:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

From the definition of the expect value:

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

Sử Thay $t = \frac{x-\mu}{\sqrt{2}\sigma}$

$\rightarrow dt = \frac{1}{\sqrt{2}\sigma} dx$

$\rightarrow \sqrt{2}\sigma\, dt = dx$

$$E = \frac{1}{\sqrt{\pi}} \left(\sqrt{2}\sigma \int_{-\infty}^{\infty} t \exp(-t^2) dt + \mu \int_{-\infty}^{\infty} \exp(-t^2) dt \right)$$

$$= \frac{1}{\sqrt{\pi}} \left(\sqrt{2}\,\sigma \left[-\frac{1}{2} \exp(-t^2) \right]_{-\infty}^{\infty} \right.$$

$\left. + u\sqrt{\pi} \right)$

$$= \frac{u\sqrt{\pi}}{\sqrt{\pi}} = u$$

c) Variance of Gaussian distribution is σ^2

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-u)^2}{2\sigma^2}\right)$$

$$\mathrm{var}(X) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx - (E(X))^2$$

$$\Rightarrow \mathrm{var}(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{(x-u)^2}{2\sigma^2}\right) dx - u^2$$

Đặt $t = \frac{x-u}{\sqrt{2}\sigma}$

$$\Rightarrow \mathrm{var}(x) = \frac{\sqrt{2}\sigma}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sqrt{2}\sigma t + u)^2 \exp(-t^2)\,dt - u^2$$

$$= \frac{1}{\sqrt{\pi}} \left(2\sigma^2 \int_{-\infty}^{\infty} t^2 \exp(-t^2)\,dt + 2\sqrt{2}\sigma u \right.$$

$$\int_{-\infty}^{\infty} t \exp(-t^2)\,dt + u^2 \int_{-\infty}^{\infty} t \exp(-t^2)\,dt + u^2$$

$$\int_{-\infty}^{\infty} \exp(-t^2)\,dt) - u^2.$$

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$$= \frac{1}{\sqrt{\pi}} \left(2\sigma^2 \int_{-\infty}^{\infty} t^2 \exp(-t^2)\,dt + 2\sqrt{2}\,\sigma u \right.$$

$$\left[-\frac{1}{2} \exp(-t^2) \right]_{-\infty}^{\infty} + u^2 \sqrt{\pi} \Big) - u^2$$

$$= \frac{1}{\sqrt{\pi}} \left(2\sigma^2 \int_{-\infty}^{\infty} t^2 \exp(-t^2)\,dt + 2\sqrt{2}\,\sigma u \right.$$

$$+ u^2 - u^2$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 \exp(-t^2)\,dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \cdot \left(\left[-\frac{t}{2} \exp(-t^2) \right]_{-\infty}^{\infty} + 1/2 \int_{-\infty}^{\infty} \right.$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2} \int_{-\infty}^{\infty} \exp(-t^2)\,dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2} \int_{-\infty}^{\infty} \exp(-t^2)\,dt$$

$$= \frac{2\sigma^2 \sqrt{\pi}}{2\sqrt{\pi}}$$

$$= \sigma^2$$

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Σ_{11} $(y_1 - u_1) + (y_2 - u_2)^T \Sigma_{22} (y_2 - u_2)$

$(y_1 - u_1)$

$= (y_1 - u_*)^T \Sigma_*^{-1} (y_1 - u_*) + (y_2 - u_2)^T \Sigma_{22}^{-1} (y_2 - u_2)$

Conditional part : $(y_1 - u_*)^T \Sigma_*^{-1} (y_1 - u_*)$

Marginal part : $(y_2 - u_2)^T \Sigma_{22}^{-1} (y_2 - u_2)$

d

d) Multivariate Gaussian is normalized.

D dimension vector x, the multivariate gaussian distribution takes the form

$$p(x \mid \mu, \sigma^2) = \frac{1}{(2\pi)^{D/2} |\Sigma|^{1/2}} \exp\left\{ -\frac{1}{2} (x-\mu)^T \Sigma^{-1} (x-\mu) \right\}.$$

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The functional dependence of the gaussian on x is through the quadratic form:

$$\Delta^2 = (x-\mu)^T \Sigma^{-1} (x-\mu) \quad (1),$$

We consider eigenvector equation for covariance matrix:

$$\Sigma u_i = \lambda_i u_i \quad (i = 1, 2, \dots, D)$$

Because Σ is a real symmetric matrix $\Rightarrow$ its eigenvalues will be real, and its eigenvectors can be chosen to form an orthonormal set.

$$\Rightarrow u_i^T u_j = I_{ij} \quad (2) \quad \left(I_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0, & \text{oth} \end{cases} \right.$$

$$\Rightarrow \Sigma = \sum_{i=1}^{D} \lambda_i u_i u_i^T$$

$$\Sigma^{-1} = \sum_{i=1}^{D} \frac{1}{\lambda_i} u_i u_i^T$$

From (1) $\Rightarrow$

$$\Delta^2 = \sum_{i=1}^{D} \frac{1}{\lambda_i} (x-\mu)^T u_i u_i^T (x-\mu)$$

$$= \sum_{i=1}^{D} \frac{y_i^2}{\lambda_i}$$

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=> We form a vector y

From (2) => U is orthorgonal

=> Change from x to the y coordinate system, we have a Jacobian matrix J.

$$J_{ij} = \frac{\partial x_i}{\partial y_i} = U_{ji}$$

We have: $|J|^2 = |U^T|^2 = |U^T||U|$

$$= |U^T \cdot U| = |I| = 1$$

$$\Rightarrow |J| = 1.$$

$$\Rightarrow |\Sigma|^{1/2} = \prod_{j=1}^{D} \lambda_j^{1/2}$$

With y coordinate system, Gauss dist. Take the form.

$$P(y) = P(x) \cdot |J| = \prod_{j=1}^{D} \frac{1}{(2\pi \lambda_j)^{1/2}} \exp\left(-\frac{y_j^2}{2\lambda_j}\right)$$

$$\Rightarrow \int_{-\infty}^{+\infty} P(y)\,dy = \prod_{j=1}^{D} \int_{-\infty}^{+\infty} \frac{1}{(2\pi \lambda_j)^{1/2}} \exp\left(-\frac{y_i^2}{2\lambda_i}\right) dy_i$$

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Problem 2.

Let $Y \; N(\mu, \Sigma)$ Consider the partioning μ and Y into

$$\mu = \begin{vmatrix} u_1 \\ u_2 \end{vmatrix}$$

$$\Sigma = \begin{vmatrix} y_1 \\ y_2 \end{vmatrix}$$

With a simila partition of Σ into:

$$\begin{vmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{vmatrix}$$

Then $(y_1 / y_2 = a)$, the conditional distribution of first partition given the second is $N(\bar{u}, \Sigma)$ with mean:

$$\bar{u} = u_1 + \Sigma_{12} \Sigma_{22}^{-1} (a - u_2)$$

and covariance matrix:

$$\bar{\Sigma} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$$

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Use the blockwise inversion formula to rewrite the inverse-variance matrix as:

$$\Sigma^{-1} = \begin{vmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{vmatrix}^{-1} = \begin{vmatrix} \Sigma^*_{11} & \Sigma^*_{12} \\ \Sigma^*_{21} & \Sigma^*_{22} \end{vmatrix}$$

where:

$$\Sigma^*_{11} = \Sigma_*^{-1}$$

$$\Sigma^*_{12} = -\Sigma_*^{-1} \Sigma_{12} \Sigma_{22}^{-1}$$

$$\Sigma^*_{21} = -\Sigma_{22}^{-1} \Sigma_{21} \Sigma_*^{-1}$$

$$\Sigma^*_{22} = \Sigma_{22}^{-1} + \Sigma_{22}^{-1} \Sigma_{21} \Sigma_* \Sigma_{12} \Sigma_{22}^{-1}$$

$$(y-\mu)^T \Sigma^{-1} (y-\mu) = \begin{bmatrix} y_1 - \mu_1 \\ y_2 - \mu_2 \end{bmatrix}^T \begin{bmatrix} y_1 - \mu_1 \\ y_2 - \mu_2 \end{bmatrix} \begin{bmatrix} \Sigma^*_{11} & \Sigma^*_{12} \\ \Sigma^*_{21} & \Sigma^*_{22} \end{bmatrix}$$

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$$= (y_1 - u_1)^T \Sigma_{11}^{*} (y_1 - u_1) + (y_1 - u_1)^T \Sigma_{12}^{*} (y_2 - u_2)$$

$$+ (y_2 - u_2)^T \Sigma_{21}^{*} (y_1 - u_1) + (y_2 - u_2)^T \Sigma_{22}^{*} (y_2 - u_2)$$

$$+ (y_1 - u_1)$$

$$= (y_1 - u_*)^T \Sigma_{*}^{-1} (y_1 - u_*) + (y_2 - u_2)^T \Sigma_{22}^{-1} (y_2 - u_2)$$

Conditional part : $(y_1 - u_*)^T \Sigma_{*}^{-1} (y_1 - u_*)$

Marginal part : $(y_2 - u_2)^T \Sigma_{22}^{-1} (y_2 - u_2)$

d

d) Multivariate Gaussian is normalized.

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