HW ML 3

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1 We will prove that

$$t_n = y(x, w) + \epsilon \Longleftrightarrow W = \left(X^t X\right)^{-1} X^T t$$

Suppose that the observations are drawn independently from a Gaussian distribution we have:

$$p\left(t_n\right) = \mathcal{N}\left(t_n \mid y(x, w), \epsilon^2\right) p(t \mid x, w, \beta) = \prod_{n=1}^{N} \mathcal{N}\left(t_n \mid y\left(x_n, w\right); \sigma^2\right)$$

$$\beta^{-}1 = \frac{1}{\sigma^2}$$

Maximize the logarithm of the likelihood function:

$$\log \prod_{i=1}^{N} \mathcal{N}\left(t_n \mid y\left(x_n, w\right), \beta^{-1}\right) = \sum_{i=1}^{N} \log\left(\frac{1}{\sqrt{2\pi\beta^{-1}}} e^{\left(t_n - y\left(x_n, w\right)\right)^2 \frac{\beta}{2}} = \sum_{i=1}^{N}\left(\frac{-1}{2}\log\left(2\pi\beta^{-1}\right) - \left(t_n - y\left(x_n, u\right.\right.\right.$$

Minimize: $\sum_{i=1}^{N}\left(t_n - y\left(x_n - w\right)\right)^2$

$$\mathrm{X} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \ldots & \ldots \\ 1 & x_n \end{bmatrix}, \mathrm{w} = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}, \mathrm{t} = \begin{bmatrix} t_1 \\ t_2 \\ \ldots \\ t_n \end{bmatrix} \mathrm{y} = \begin{bmatrix} y_1 \\ y_2 \\ \ldots \\ y_n \end{bmatrix} = \begin{bmatrix} w_1 x_1 + w_0 \\ w_1 x_2 + w_0 \\ \ldots \\ w_1 x_n + w_0 \end{bmatrix} = \mathrm{Xw}, \mathrm{t{-}y} = \begin{bmatrix} t_1 - y_1 \\ t_2 - y_2 \\ \ldots \\ t_n - y_n \end{bmatrix}$$

$$\Longrightarrow \|t - y\|_2^2 = \sum_{i=1}^{n}\left(t_i - y_i\right)^2 = \textit{Loss Function} \Longrightarrow L = \|t - y\|_2^2 = \|t - XW\|_2^2 \Longrightarrow \frac{\partial L}{\partial W} = 2X^T(t - XW) = 0 =$$

2 Prove that $X^t X$ is invertable when X full rank

Solution
If X is full rank, X is linear independent.

$$\Rightarrow \vec{v}^T X^T X \vec{v} = \vec{v}^T \overrightarrow{0} = 0 \Rightarrow (X\vec{v})^T X\vec{v} = 0 \Rightarrow (X\vec{v}) \cdot (X\vec{v}) = 0 \Rightarrow X\vec{v} = \overrightarrow{0}$$

So we have: if $\vec{v} \in N\left(X^T X\right) \Rightarrow \vec{v} \in N(X)$

$$\Rightarrow \vec{v} canonlybe \overrightarrow{0} \Rightarrow N\left(X^T X\right) = N(X) = \{\overrightarrow{0}\}$$

$\Rightarrow$ $X^T X$ is linearly independent; and $X^T X$ is a square matrix $\Rightarrow$ $X^T X$ is invertible