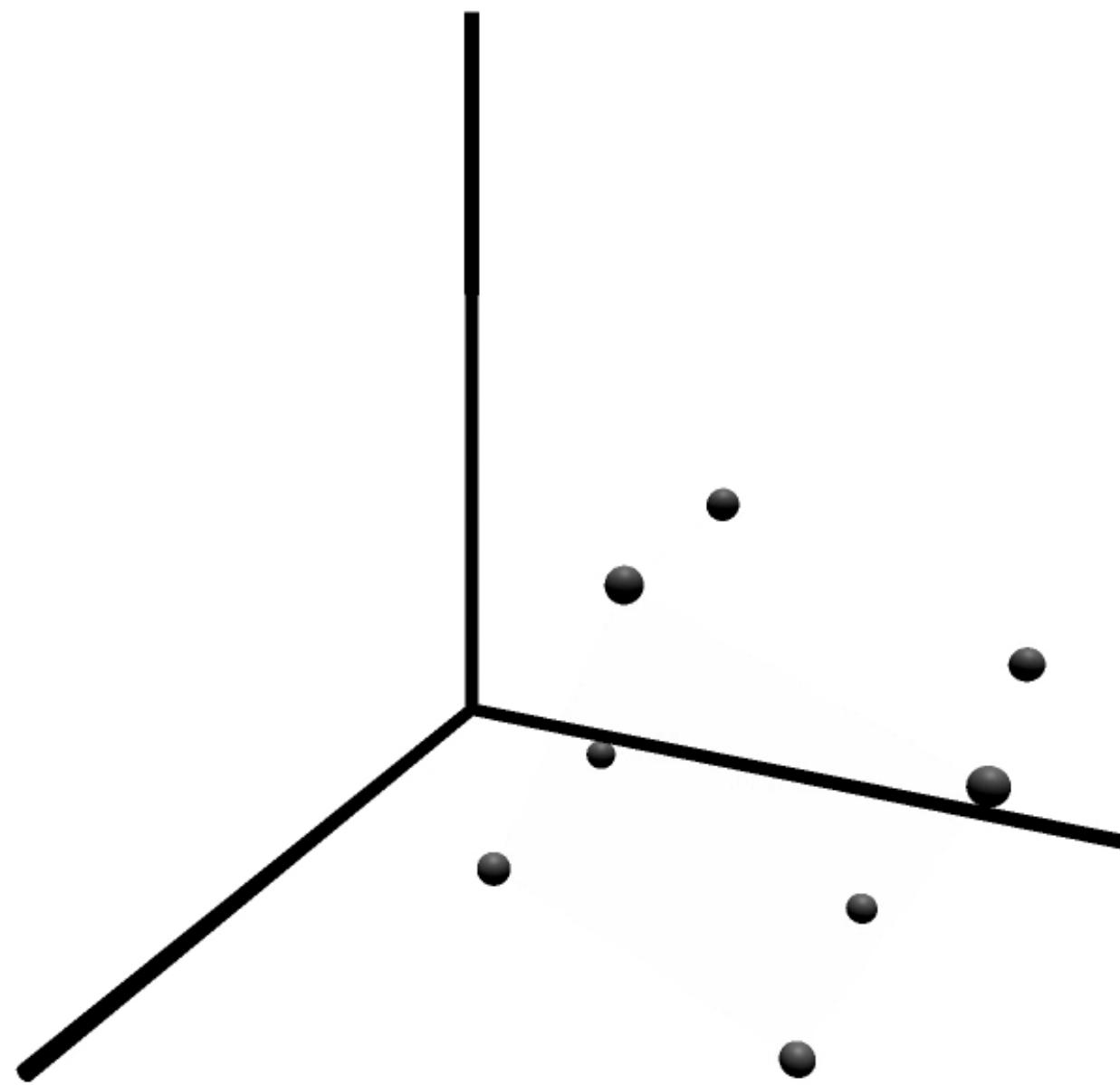


Spatial Transformations

Computer Graphics
CMU 15-462/15-662

Spatial Transformation

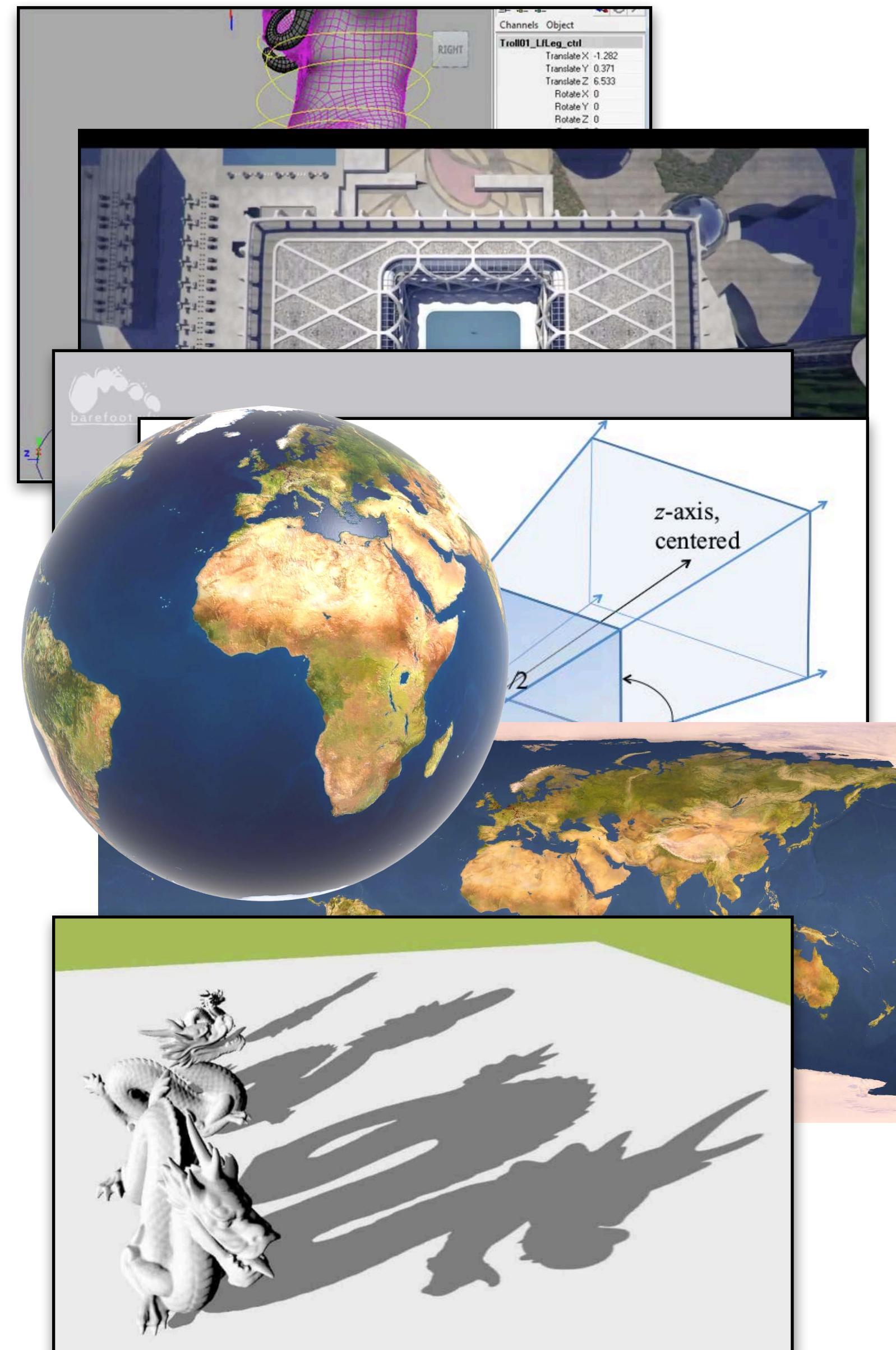
- Basically any function that assigns each point a new location
- Today we'll focus on common transformations of space (rotation, scaling, etc.) encoded by linear maps



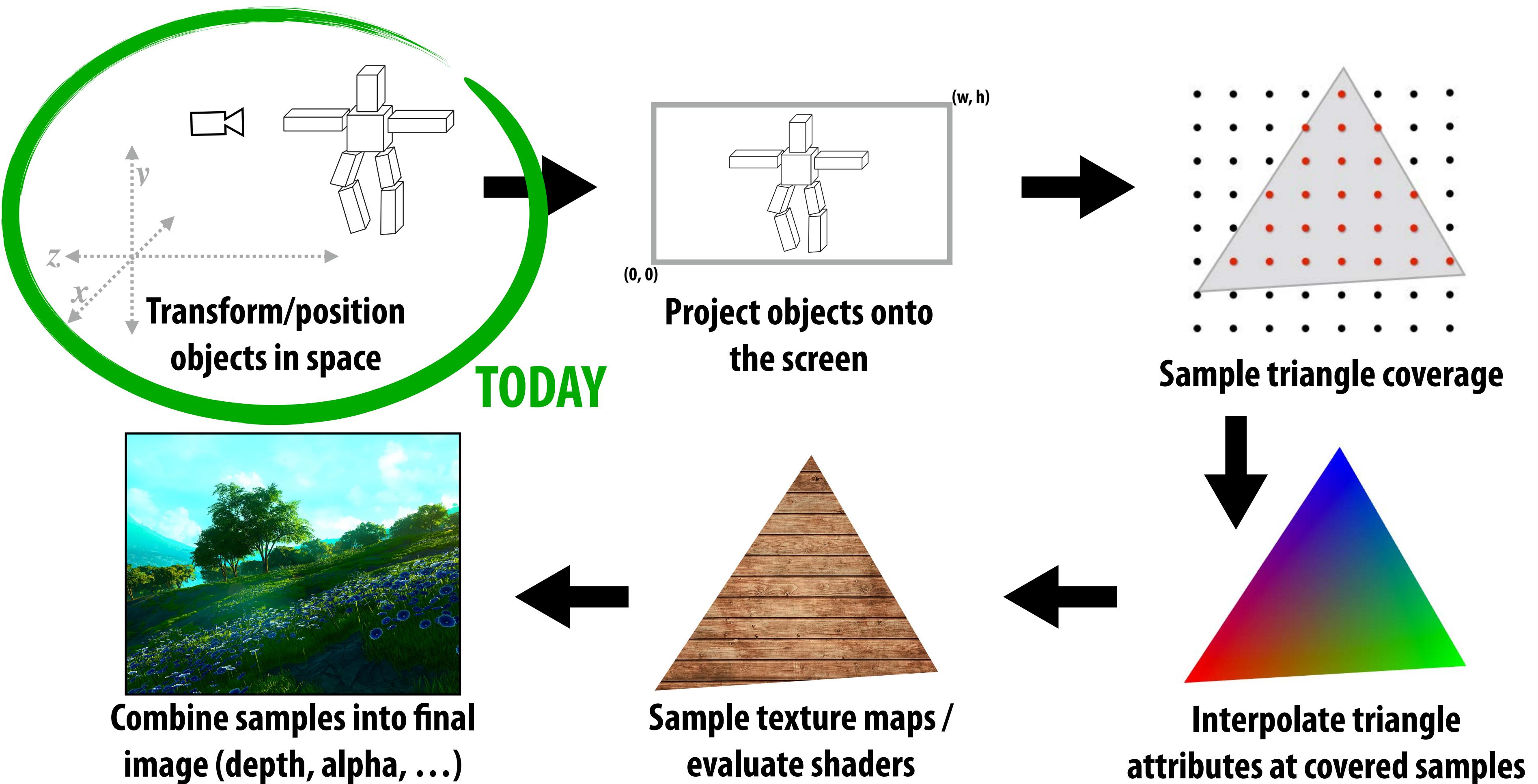
$$f: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

Transformations in Computer Graphics

- Where are linear transformations used in computer graphics?
- All over the place!
 - Position/deform objects in space
 - Move the camera
 - Animate objects over time
 - Project 3D objects onto 2D images
 - Map 2D textures onto 3D objects
 - Project shadows of 3D objects onto other 3D objects
 - ...



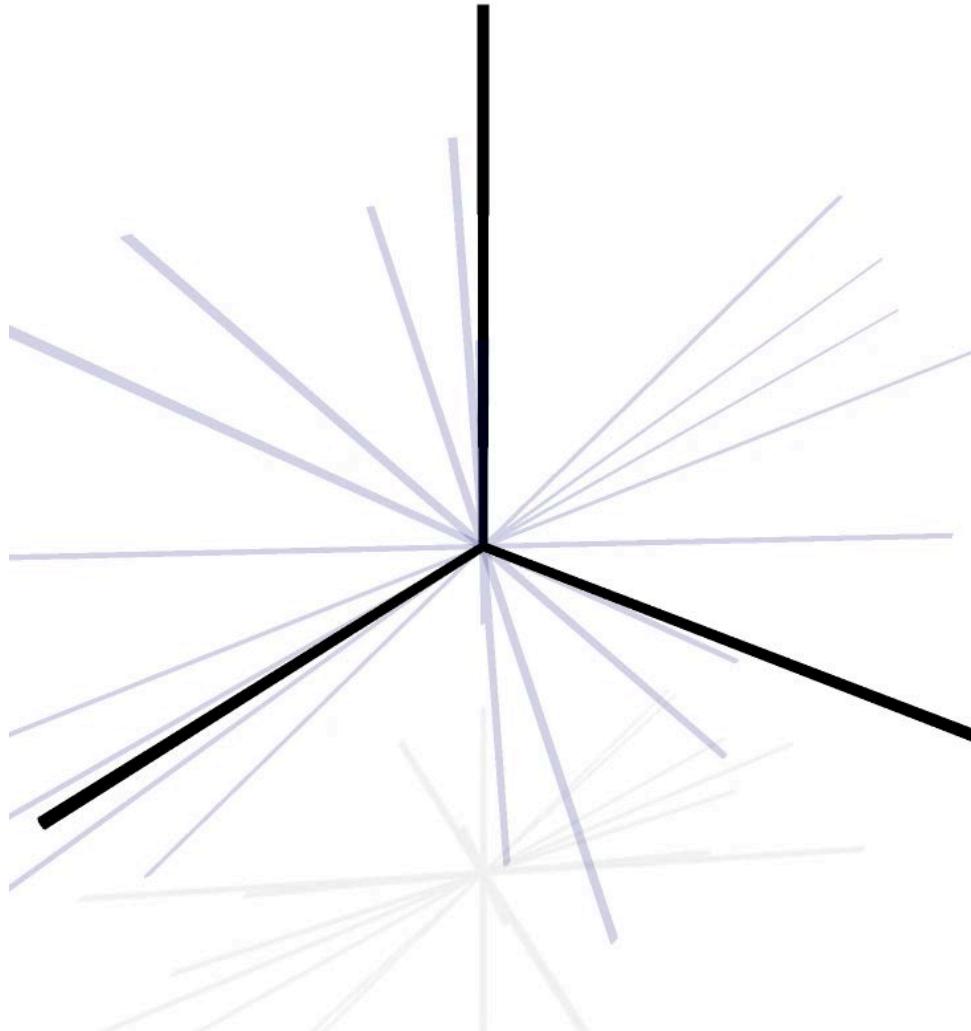
The Rasterization Pipeline



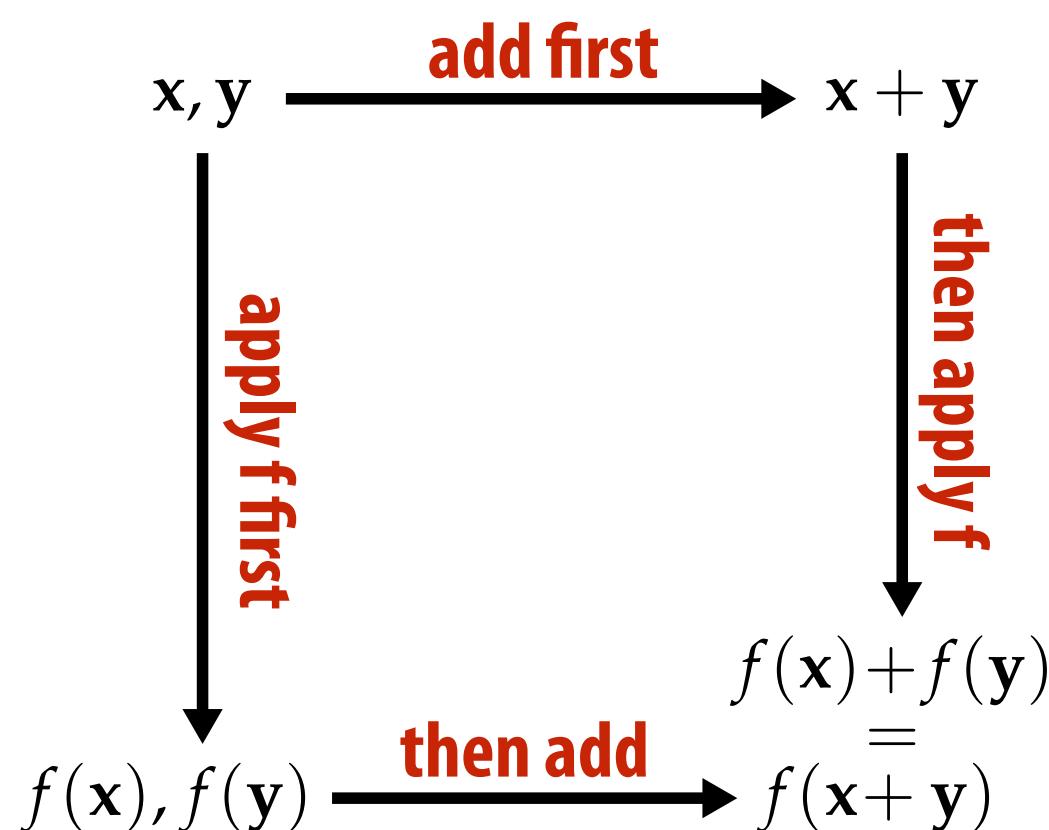
Review: Linear Maps

Q: What does it mean for a map $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ to be linear?

Geometrically: it maps lines to lines, and preserves the origin



Algebraically: preserves vector space operations (addition & scaling)



Why do we care about *linear* transformations?

- Cheap to apply
- Usually pretty easy to solve for (linear systems)
- **Composition of linear transformations is linear**
 - product of many matrices is a single matrix
 - gives uniform representation of transformations
 - simplifies graphics algorithms, systems (e.g., GPUs & APIs)

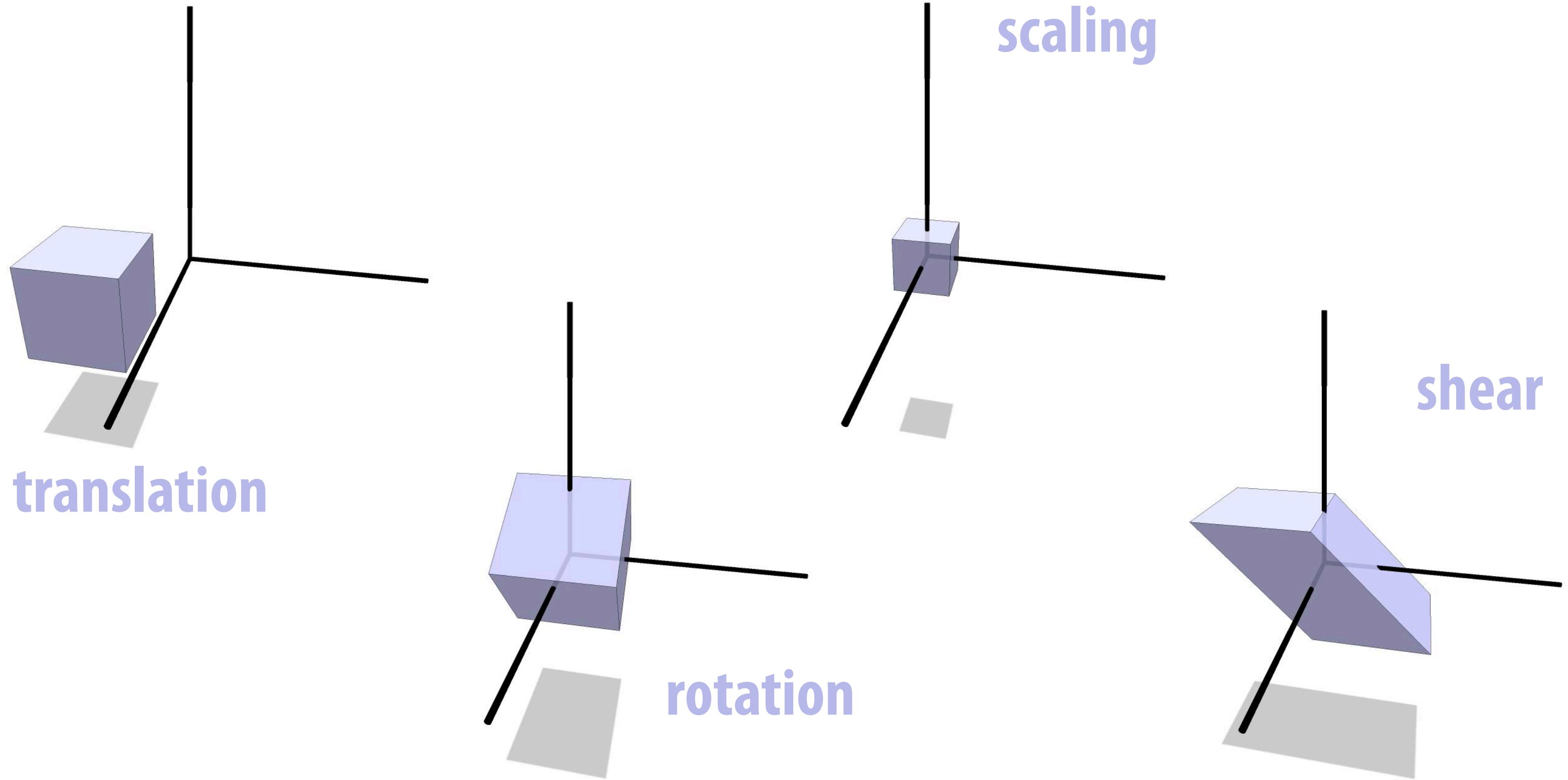
$$\begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \cdots = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

rotation *scale* *rotation* *composite transformation*

**What kinds of linear
transformations can we compose?**

Types of Transformations

What would you call each of these types of transformations?



Q: How did you know that? (Hint: you did not inspect a formula!)

Invariants of Transformation

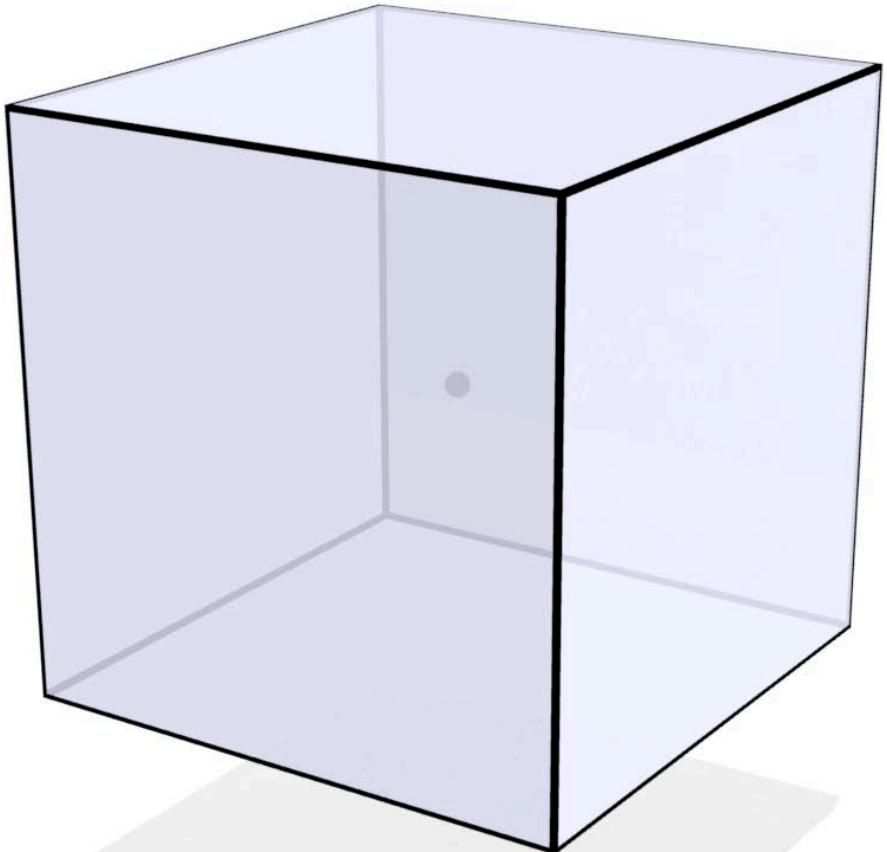
A transformation is determined by the invariants it preserves

transformation	invariants	algebraic description
linear	<i>straight lines / origin</i>	$f(ax+ay) = af(x) + f(y),$ $f(0) = 0$
translation	<i>differences between pairs of points</i>	$f(\mathbf{x}-\mathbf{y}) = \mathbf{x}-\mathbf{y}$
scaling	<i>lines through the origin / direction of vectors</i>	$f(\mathbf{x})/ f(\mathbf{x}) = \mathbf{x}/ \mathbf{x} $
rotation	<i>origin / distances between points / orientation</i>	$ f(\mathbf{x})-f(\mathbf{y}) = \mathbf{x}-\mathbf{y} ,$ $\det(f) > 0$
...

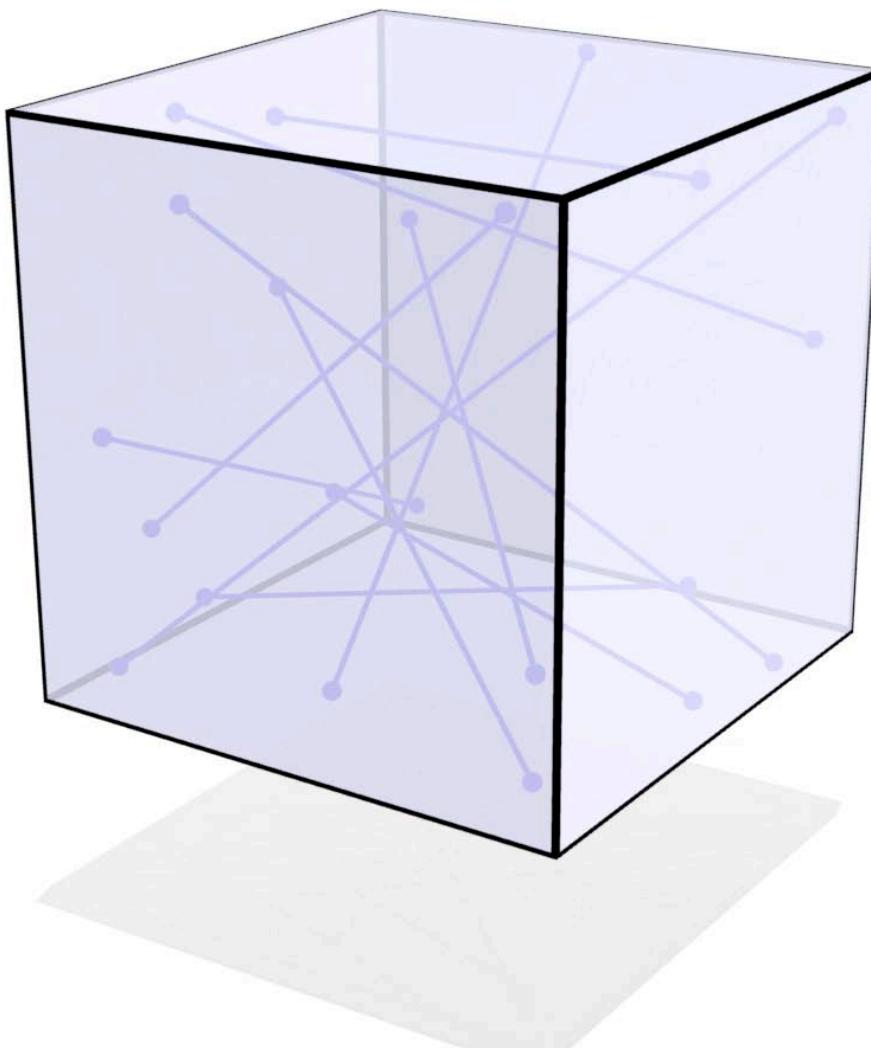
(Essentially how your brain “knows” what kind of transformation you’re looking at...)

Rotation

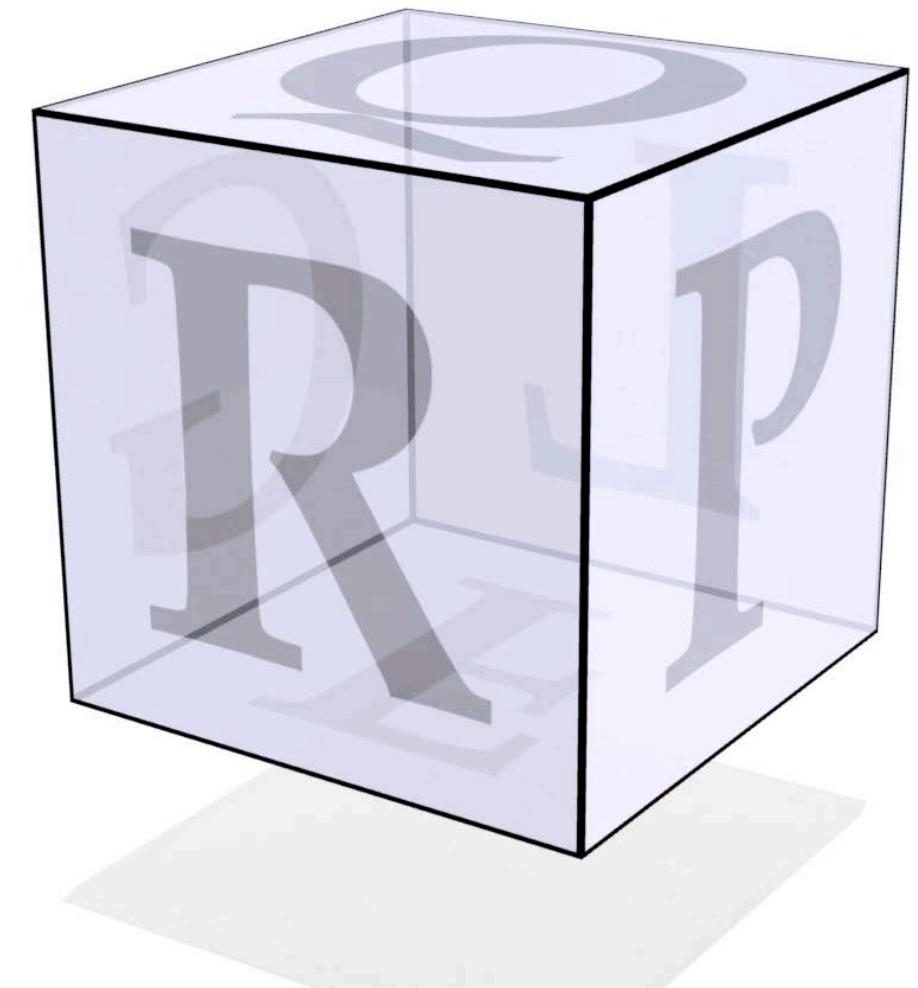
Rotations defined by three basic properties:



keeps origin fixed



preserves distances



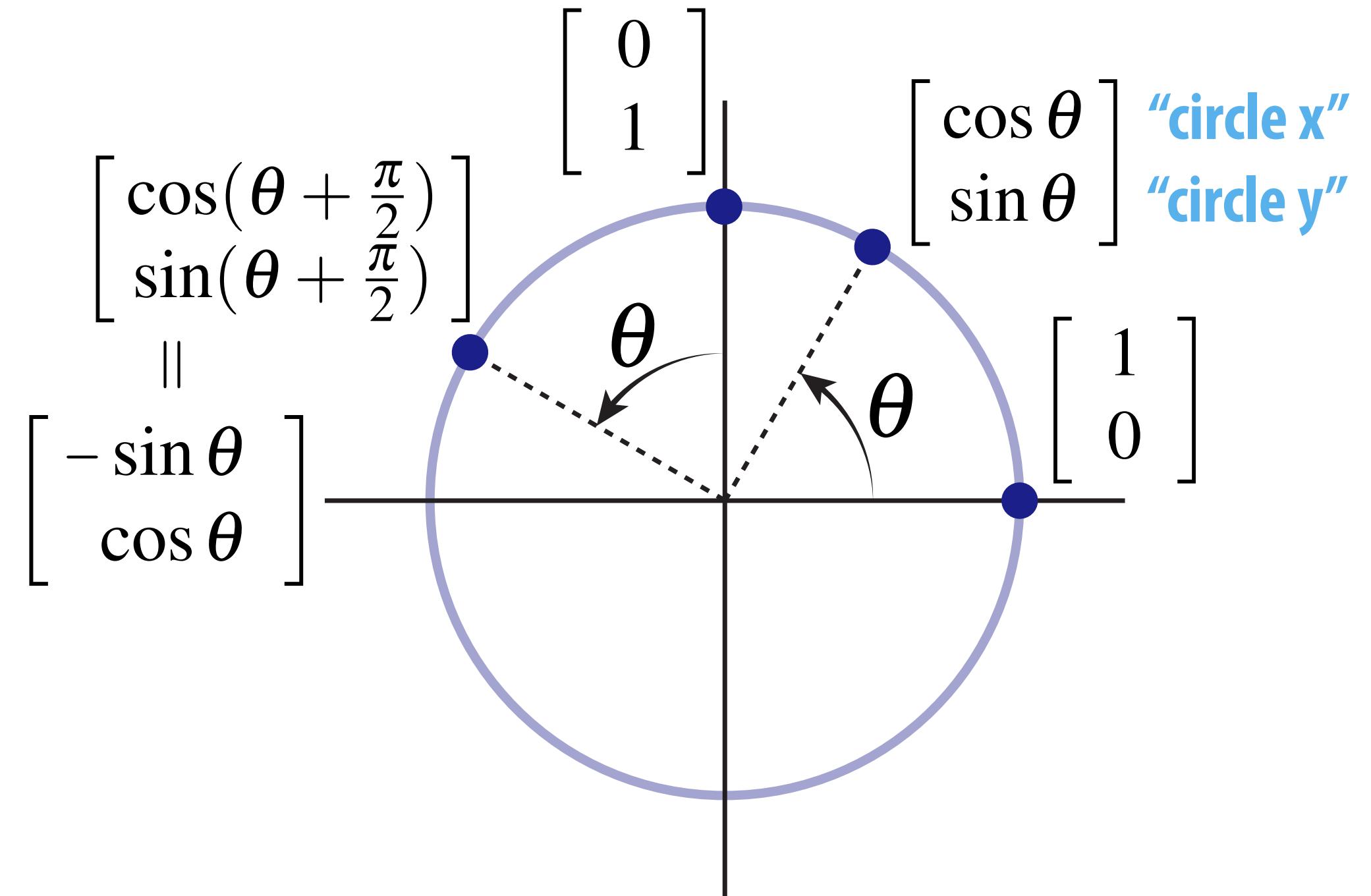
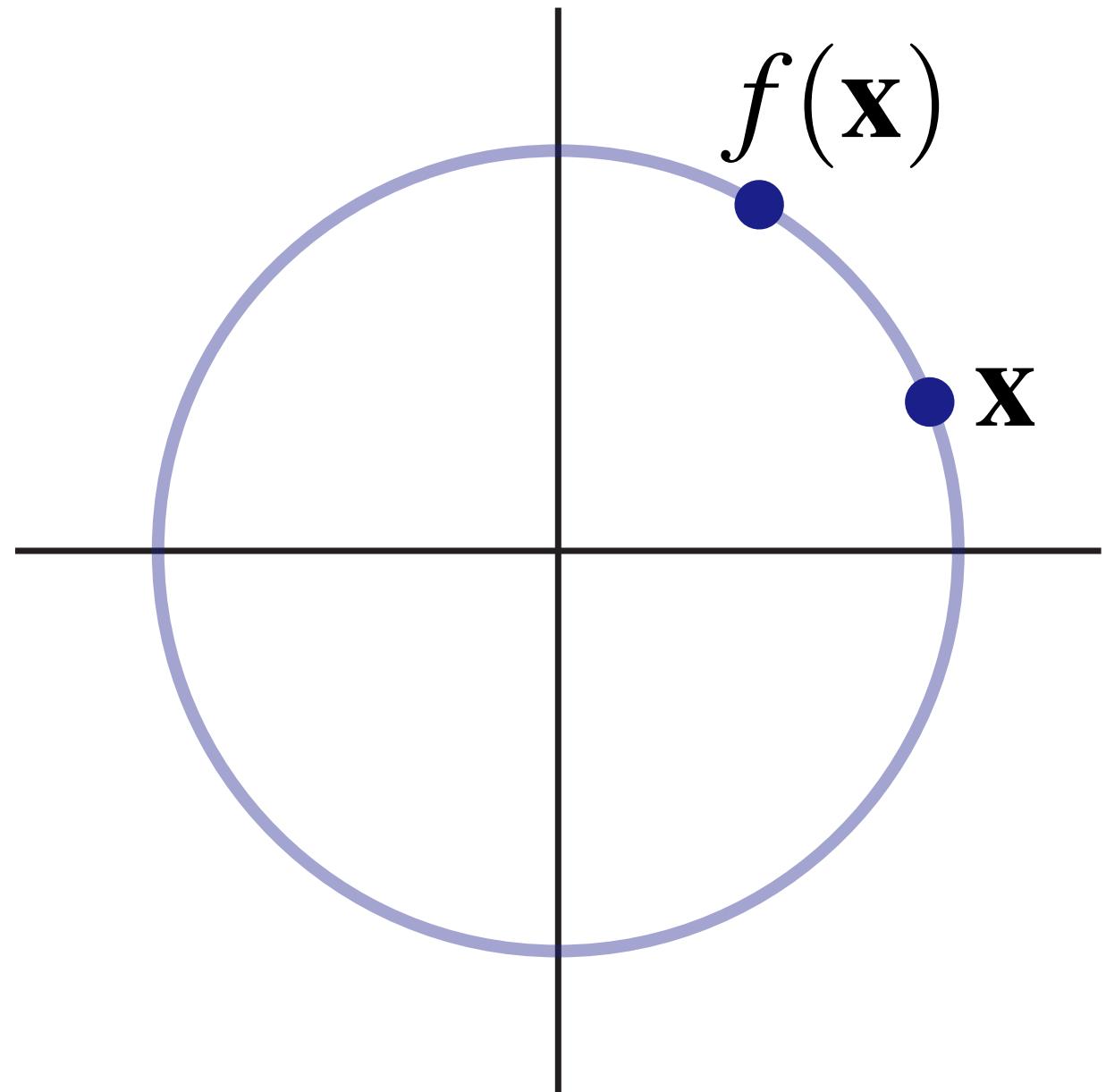
preserves orientation

First two properties together imply that rotations are linear.

Will have a *lot* more to say about rotations next lecture...

2D Rotations—Matrix Representation

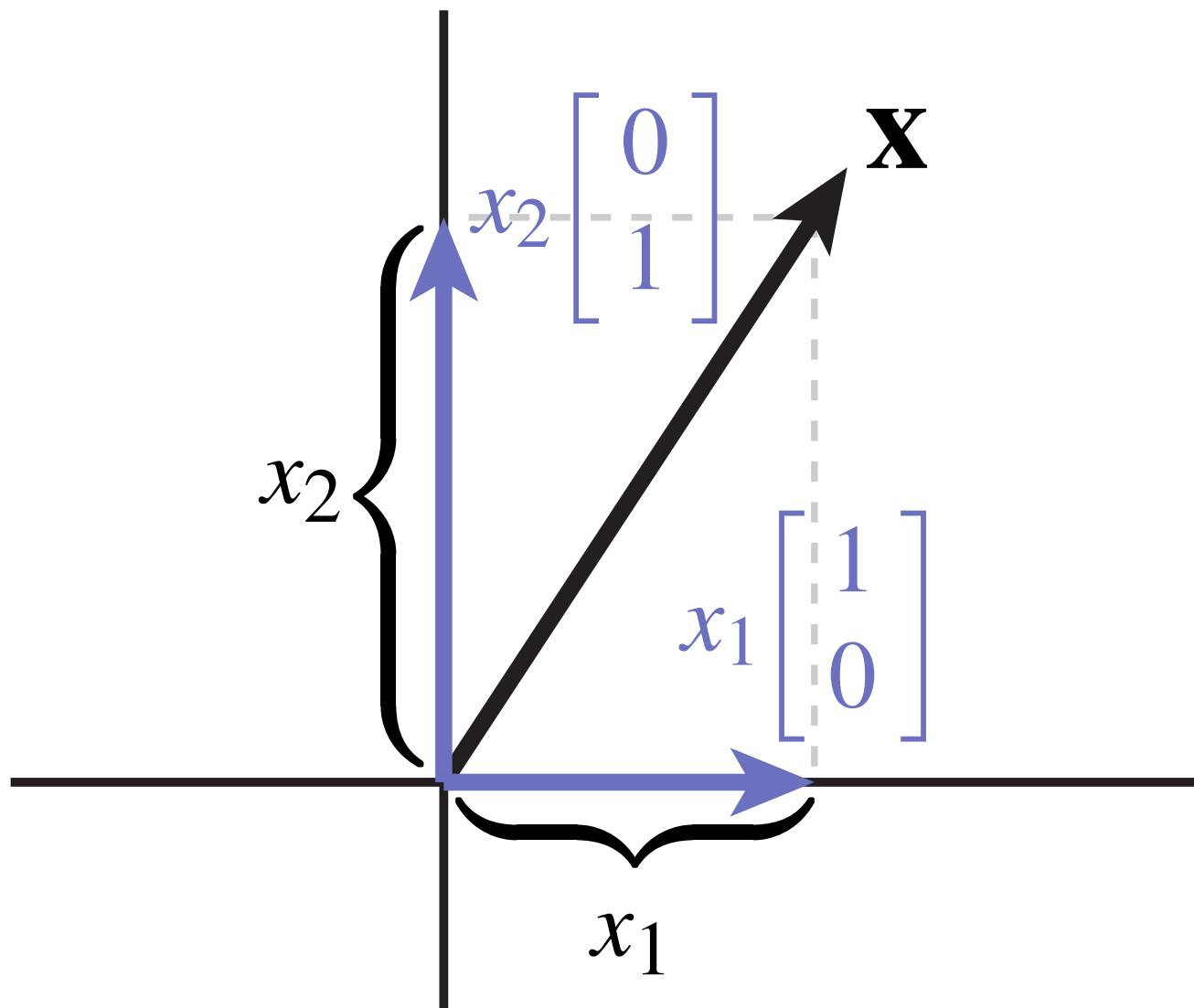
Rotations preserve distances and the origin—hence, a 2D rotation by an angle θ maps each point \mathbf{x} to a point $f_\theta(\mathbf{x})$ on the circle of radius $|\mathbf{x}|$:



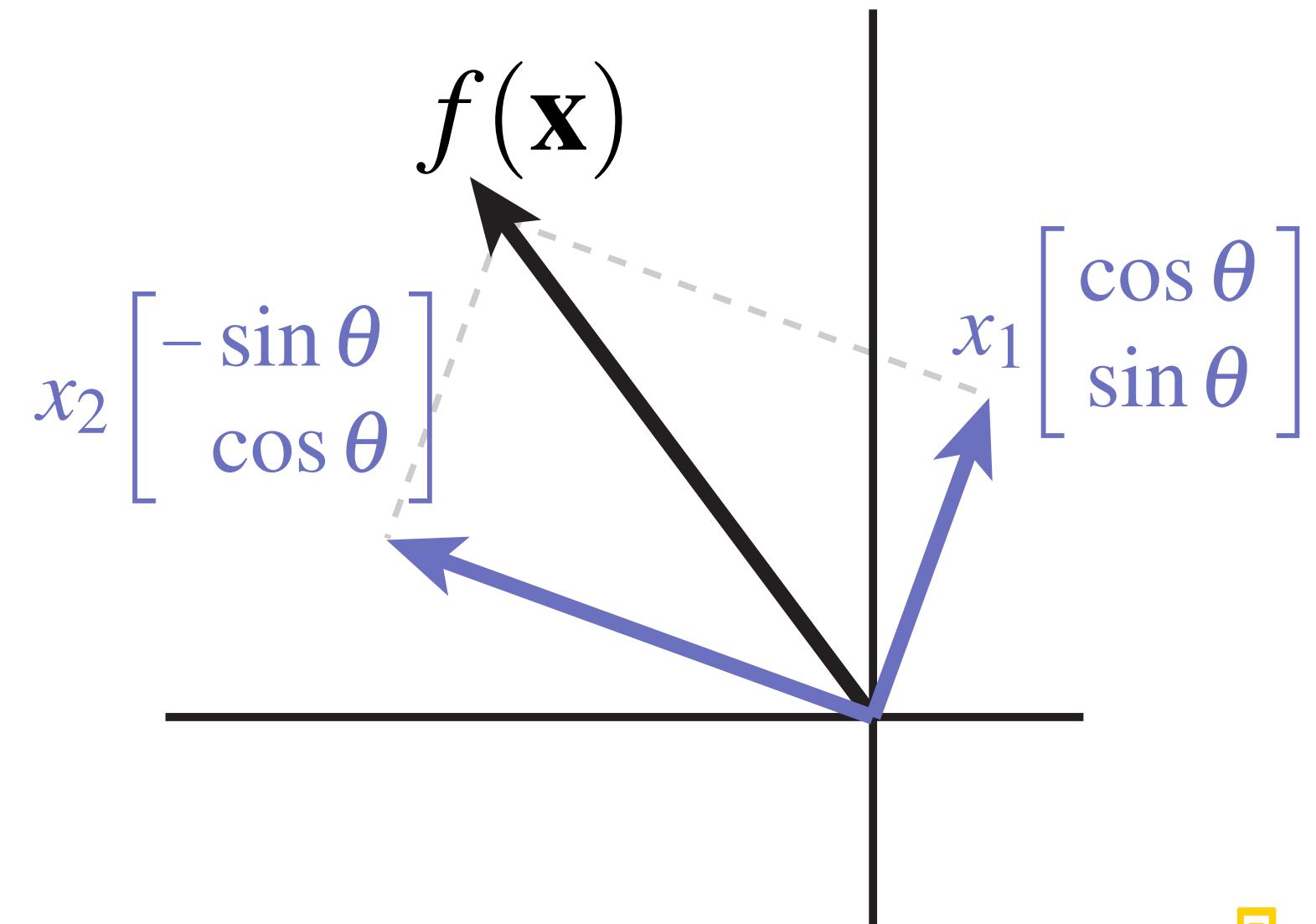
- Where does $\mathbf{x} = (1,0)$ go if we rotate by θ (counter-clockwise)?
- How about $\mathbf{x} = (0,1)$?

What about a general vector $\mathbf{x} = (x_1, x_2)$?

2D Rotations—Matrix Representation



$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



$$f(\mathbf{x}) = x_1 \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} + x_2 \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$$

So, How do we represent the 2D rotation function $f_\theta(\mathbf{x})$ using a matrix?

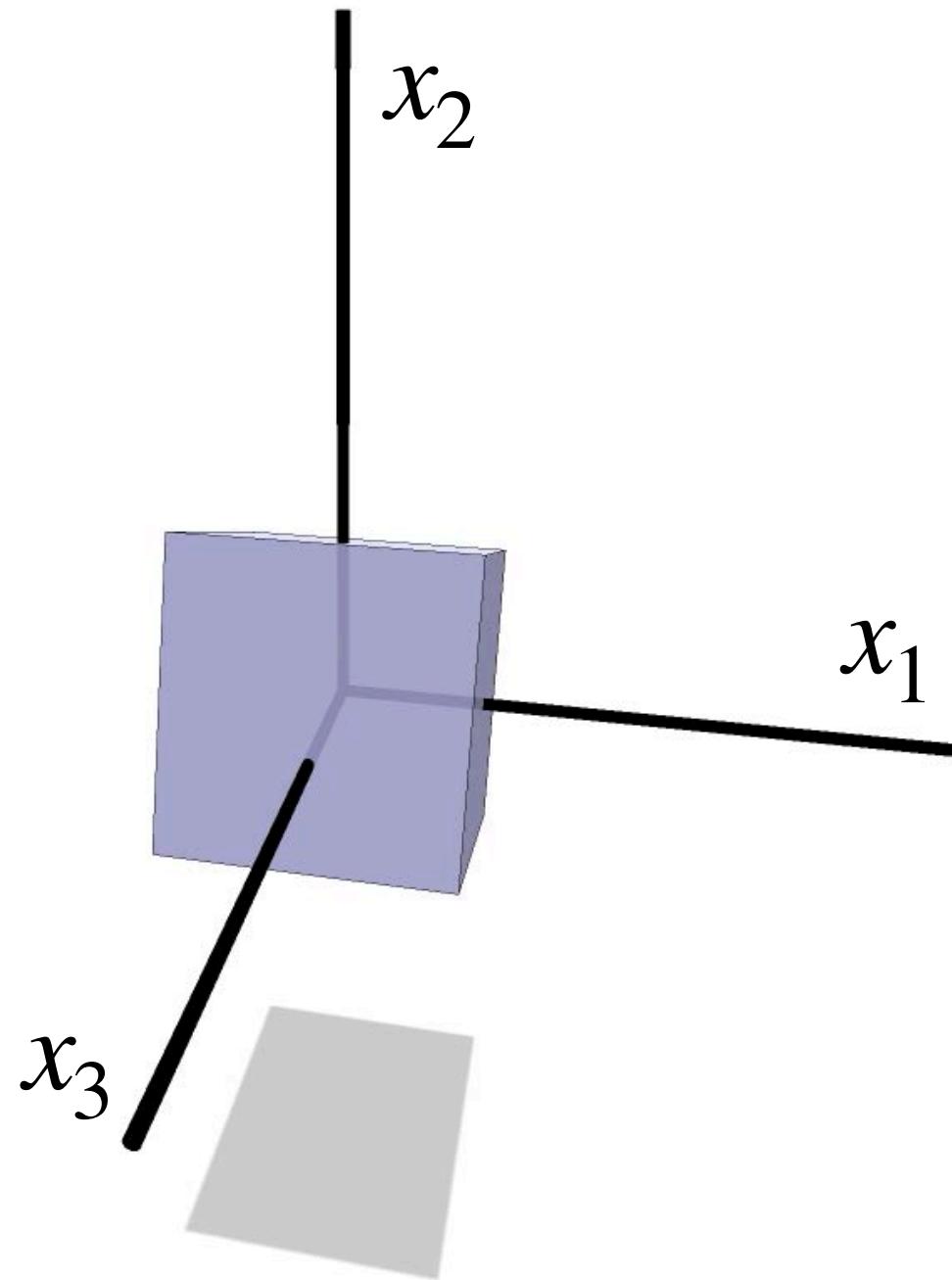
$$f_\theta(\mathbf{x}) = \begin{bmatrix} \cos \theta & -\sin(\theta) \\ \sin \theta & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

3D Rotations

- Q: In 3D, how do we rotate around the x_3 -axis?
- A: Just apply the same transformation of x_1, x_2 ; keep x_3 fixed

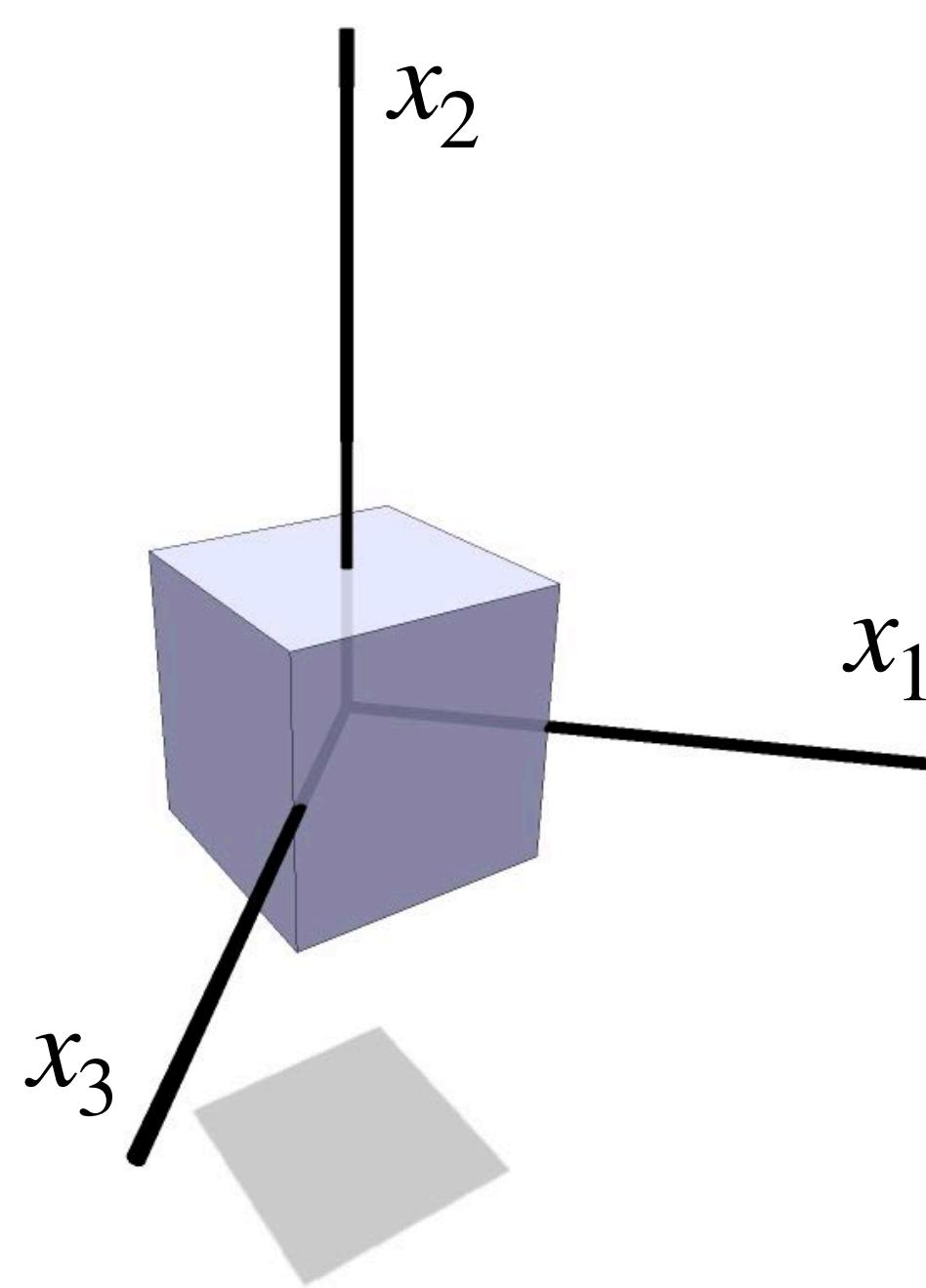
rotate around x_1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin(\theta) \\ 0 & \sin \theta & \cos(\theta) \end{bmatrix}$$



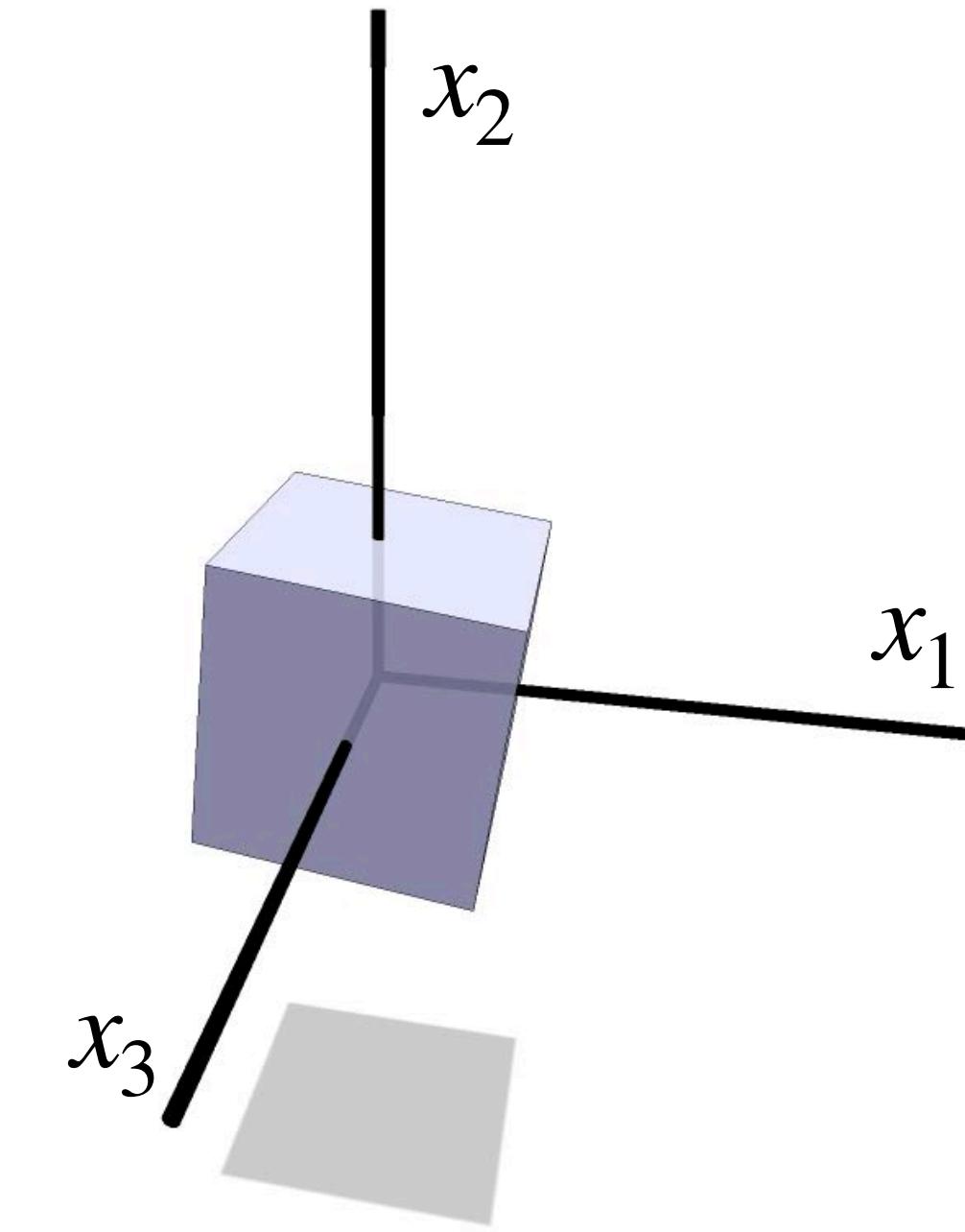
rotate around x_2

$$\begin{bmatrix} \cos \theta & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos(\theta) \end{bmatrix}$$



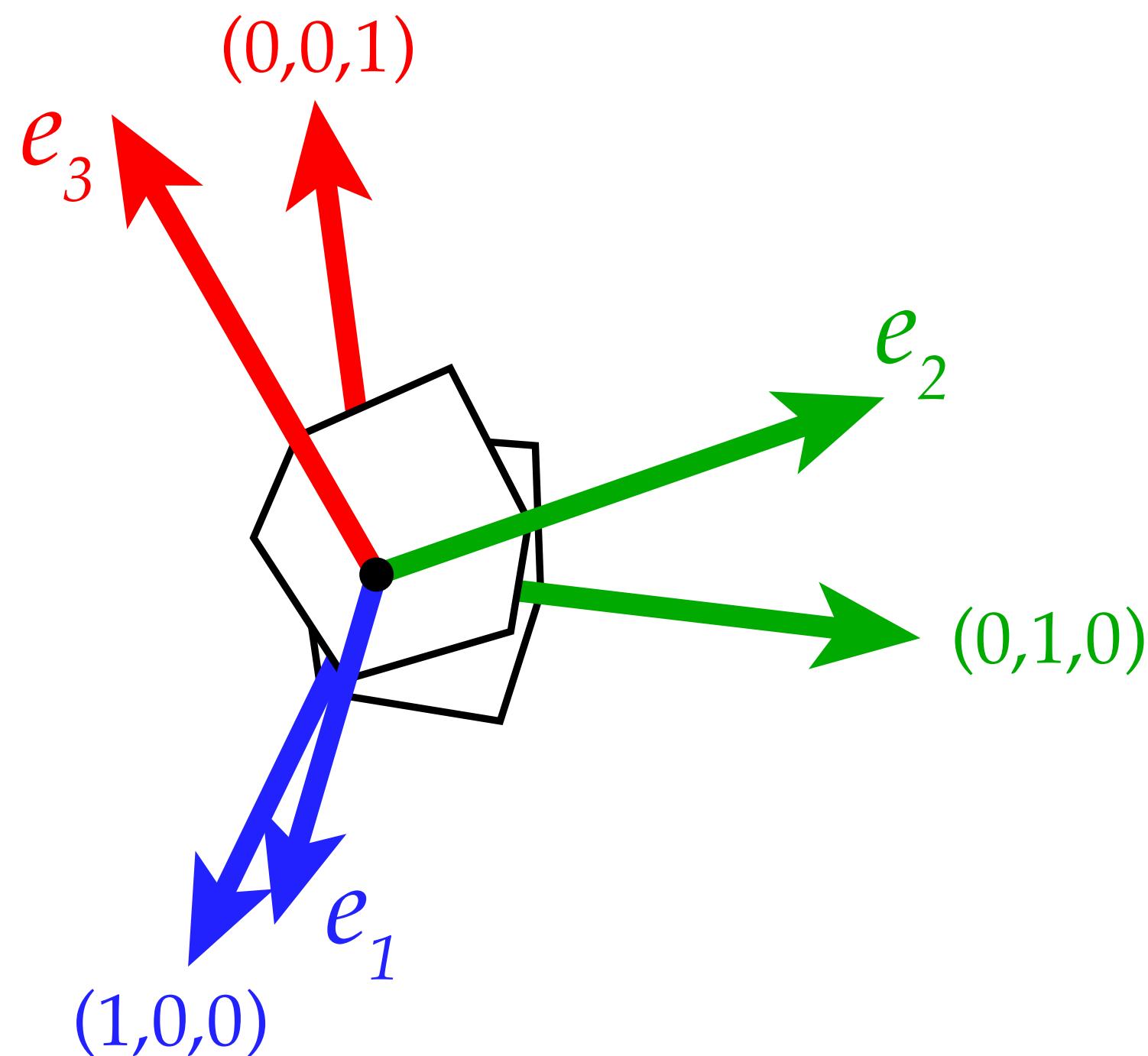
rotate around x_3

$$\begin{bmatrix} \cos \theta & -\sin(\theta) & 0 \\ \sin \theta & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Rotations—Transpose as Inverse

Rotation will map standard basis to orthonormal basis e_1, e_2, e_3 :



$$\begin{aligned} R^T &= \begin{bmatrix} e_1^T \\ e_2^T \\ e_3^T \end{bmatrix} \quad R = \begin{bmatrix} | & | & | \\ e_1 & e_2 & e_3 \\ | & | & | \end{bmatrix} \\ &= \begin{bmatrix} e_1^T e_1 & e_1^T e_2 & e_1^T e_3 \\ e_2^T e_1 & e_2^T e_2 & e_2^T e_3 \\ e_3^T e_1 & e_3^T e_2 & e_3^T e_3 \end{bmatrix} \\ &\quad I \end{aligned}$$

The diagram illustrates the transpose of a rotation matrix R^T as a mapping from the standard basis e_1, e_2, e_3 to an orthonormal basis. The columns of R^T represent the transformed basis vectors e_1^T, e_2^T, e_3^T . The matrix multiplication $R^T R = I$ is shown as a mapping from the orthonormal basis back to the standard basis, where the columns of R represent the standard basis vectors e_1, e_2, e_3 . The resulting identity matrix I indicates that the transformation is its own inverse.

Hence, $R^T R = I$, or equivalently, $R^T = R^{-1}$.

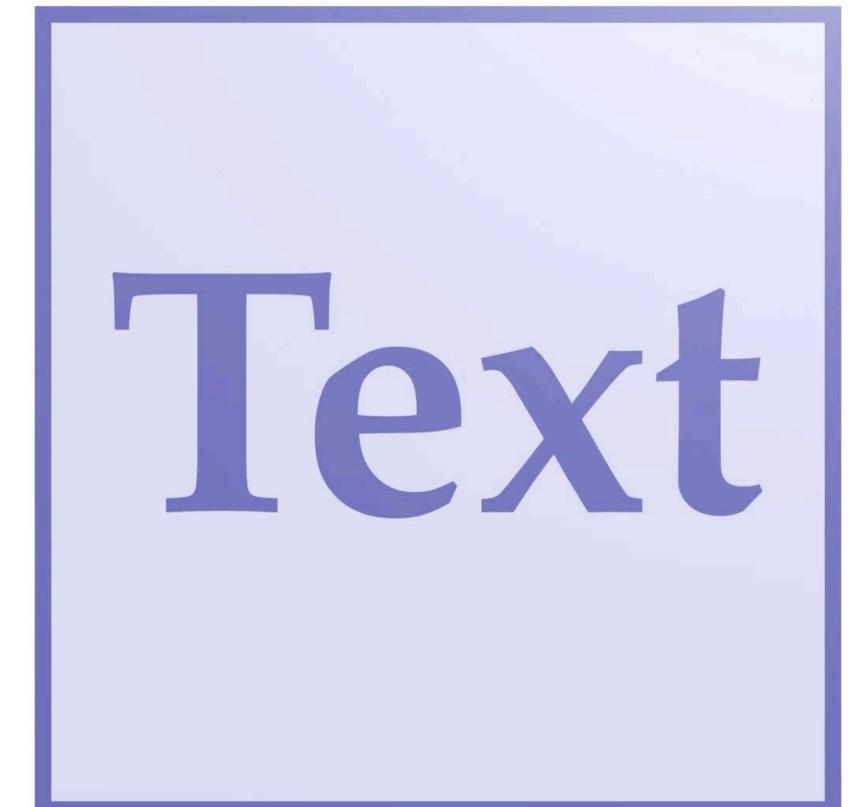
Reflections

- Q: Does every matrix $Q^T Q = I$ describe a rotation?
- Remember that rotations must preserve the origin, preserve distances, and preserve orientation
- Consider for instance this matrix:

$$Q = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad Q^T Q = \begin{bmatrix} (-1)^2 & 0 \\ 0 & 1 \end{bmatrix} = I$$

**Q: Does this matrix represent a rotation?
(If not, which invariant does it fail to preserve?)**

**A: No! It represents a reflection across the y-axis
(and hence fails to preserve orientation)**

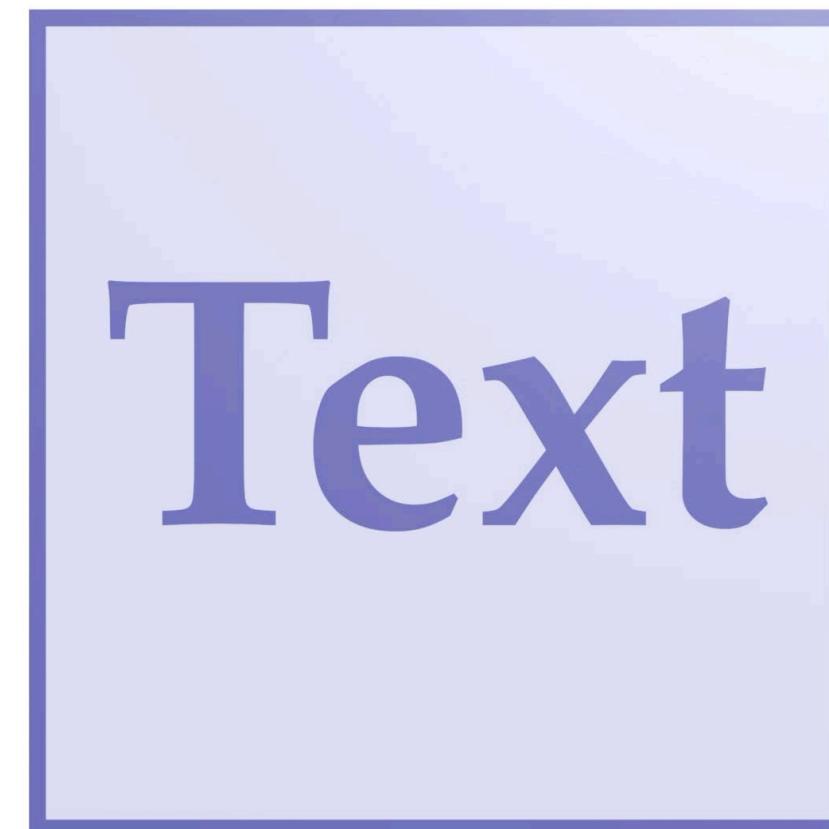


Orthogonal Transformations

- In general, transformations that preserve distances and the origin are called *orthogonal transformations*
- Represented by matrices $Q^T Q = I$
 - **Rotations** additionally preserve orientation: $\det(Q) > 0$
 - **Reflections** reverse orientation: $\det(Q) < 0$



rotation



reflection

Scaling

*assuming $a \neq 0, \mathbf{u} \neq 0$

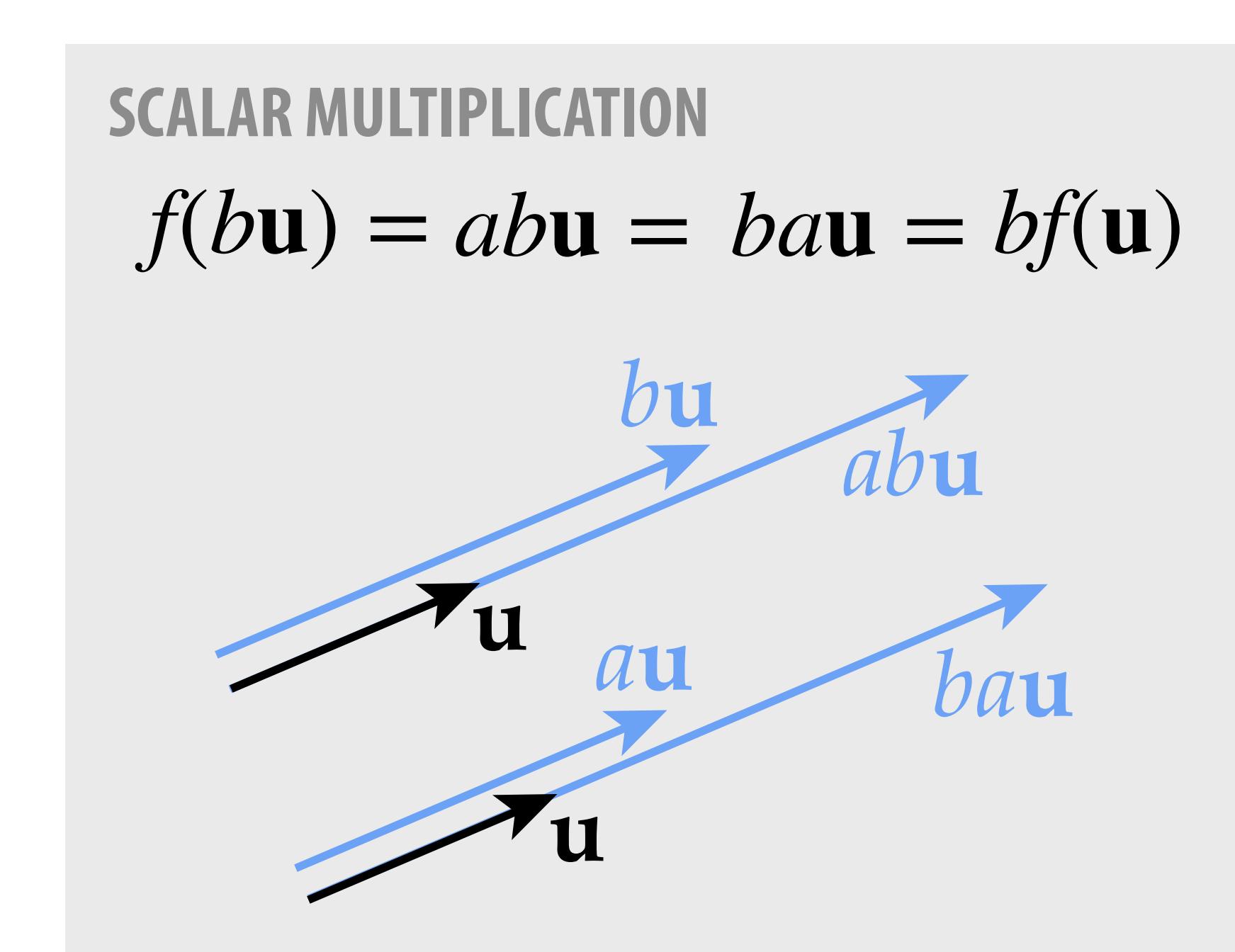
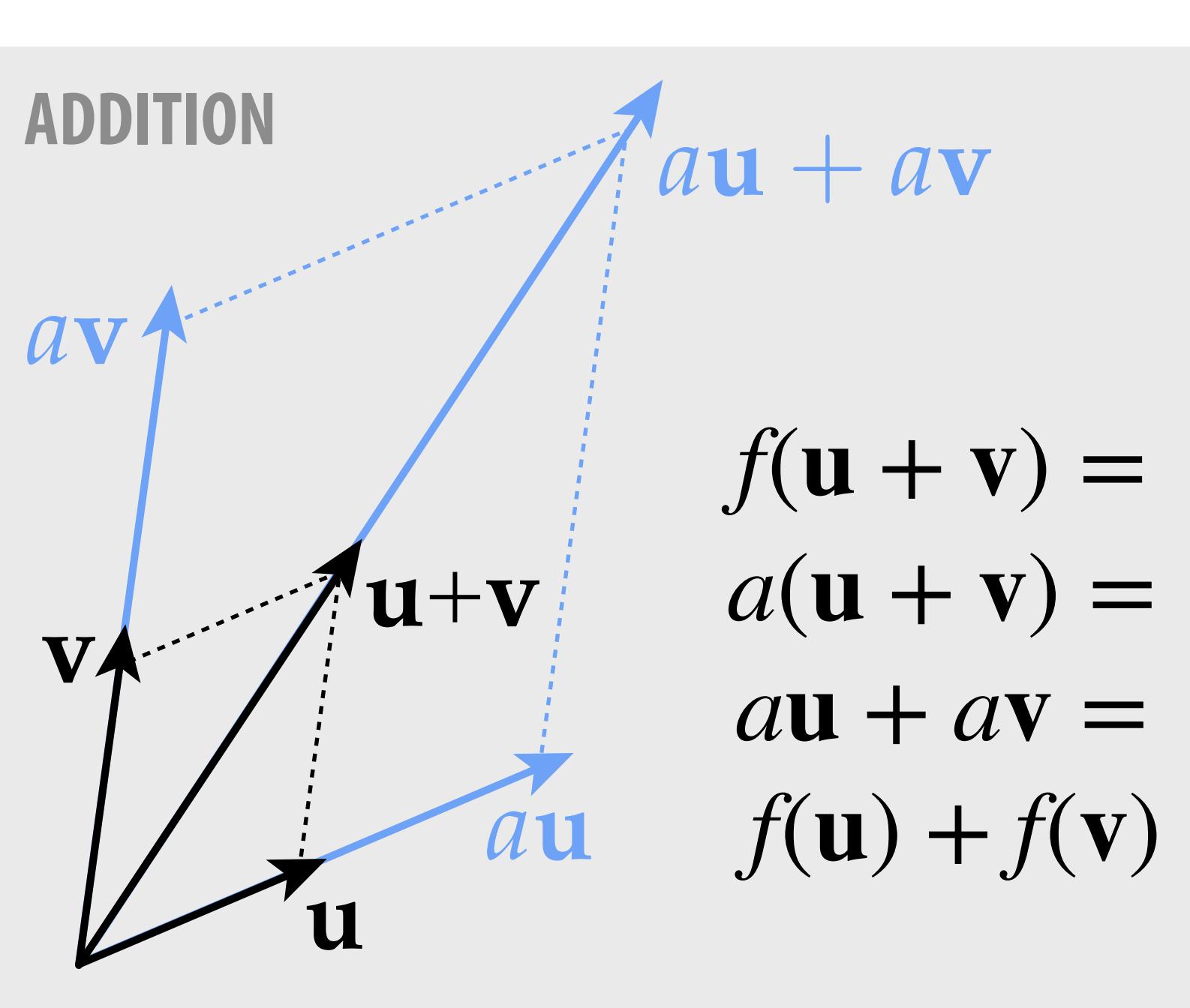
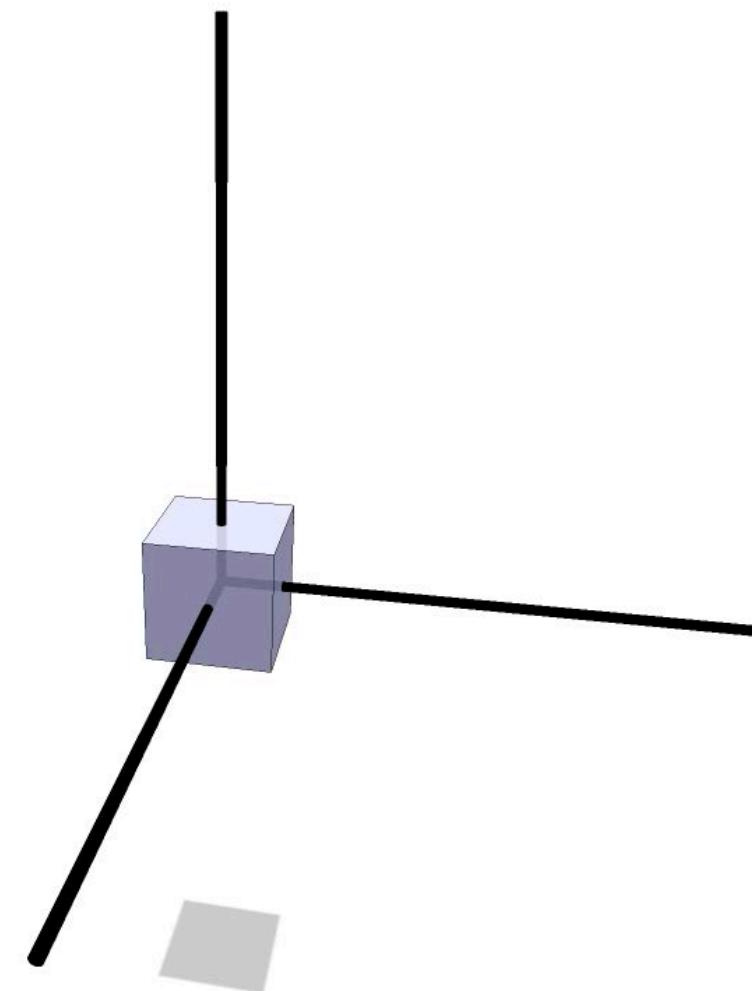
- Each vector \mathbf{u} gets mapped to a scalar multiple

- $f(\mathbf{u}) = a\mathbf{u}, \quad a \in \mathbb{R}$

- Preserves the direction of all vectors*

$$\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{a\mathbf{u}}{|a\mathbf{u}|}$$

- Q: Is scaling a linear transformation? A: Yes!



Scaling — Matrix Representation

Q: Suppose we want to scale a vector $\mathbf{u} = (u_1, u_2, u_3)$ by a .
How would we represent this operation via a matrix?

A: Just build a *diagonal* matrix D , with a along the diagonal:

$$\underbrace{\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}}_D \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} au_1 \\ au_2 \\ au_3 \end{bmatrix}}_{a\mathbf{u}}$$

Q: What happens if a is negative?

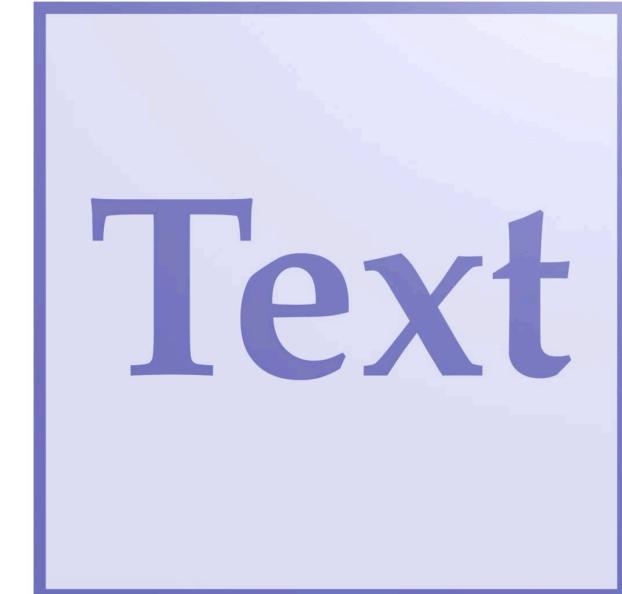
Negative Scaling



For $a = -1$, can think of scaling by a as sequence of reflections.

E.g., in 2D:

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



Since each reflection reverses orientation, orientation is preserved.

What about 3D?

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$



Now we have *three* reflections, and so orientation is reversed!

Nonuniform Scaling (Axis-Aligned)

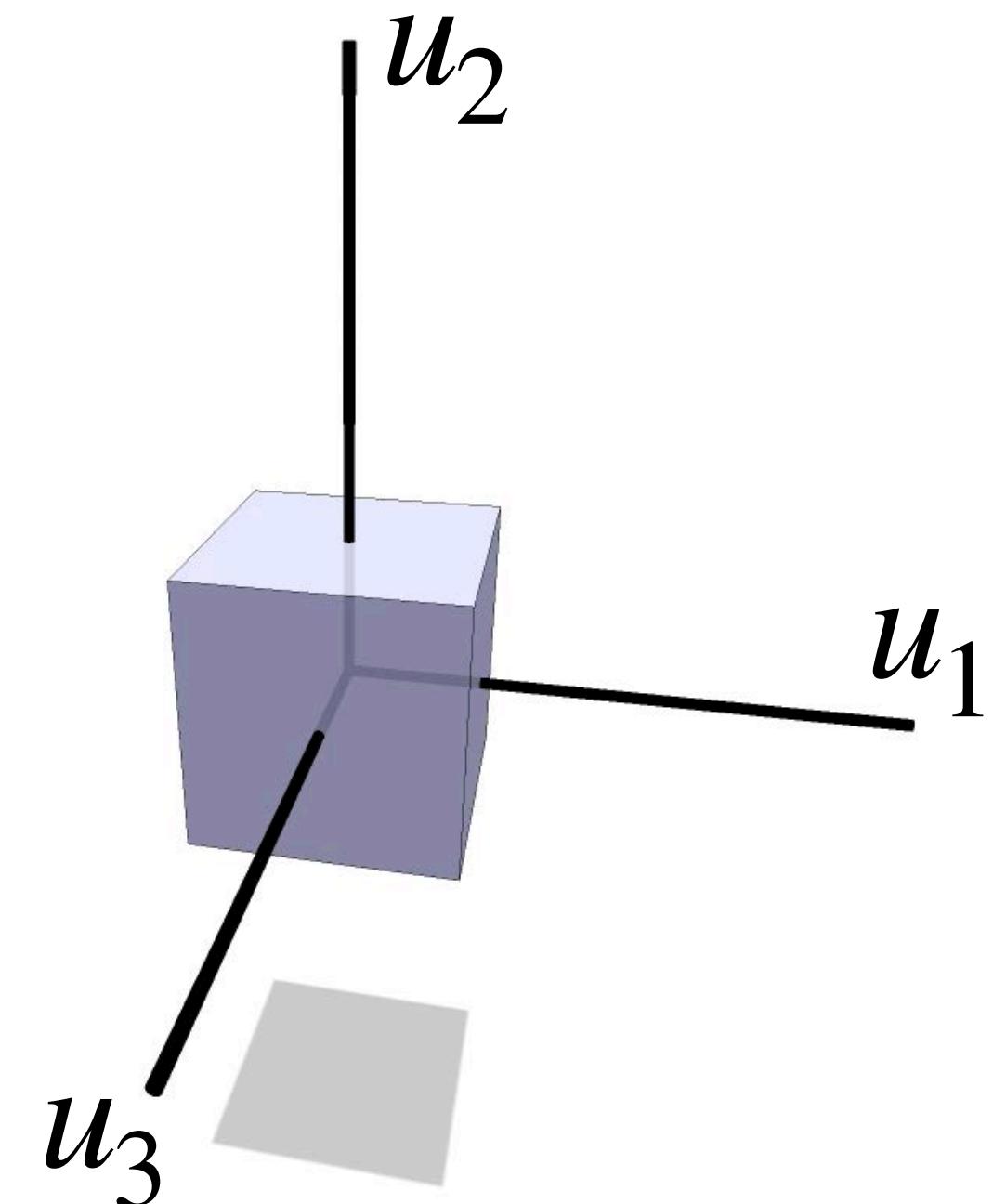
- We can also scale each axis by a different amount

- $f(u_1, u_2, u_3) = (au_1, bu_2, cu_3), \quad a, b, c \in \mathbb{R}$

- Q: What's the matrix representation?

- A: Just put a, b, c on the diagonal:

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} au_1 \\ bu_2 \\ cu_3 \end{bmatrix}$$



Ok, but what if we want to scale along some *other* axes?

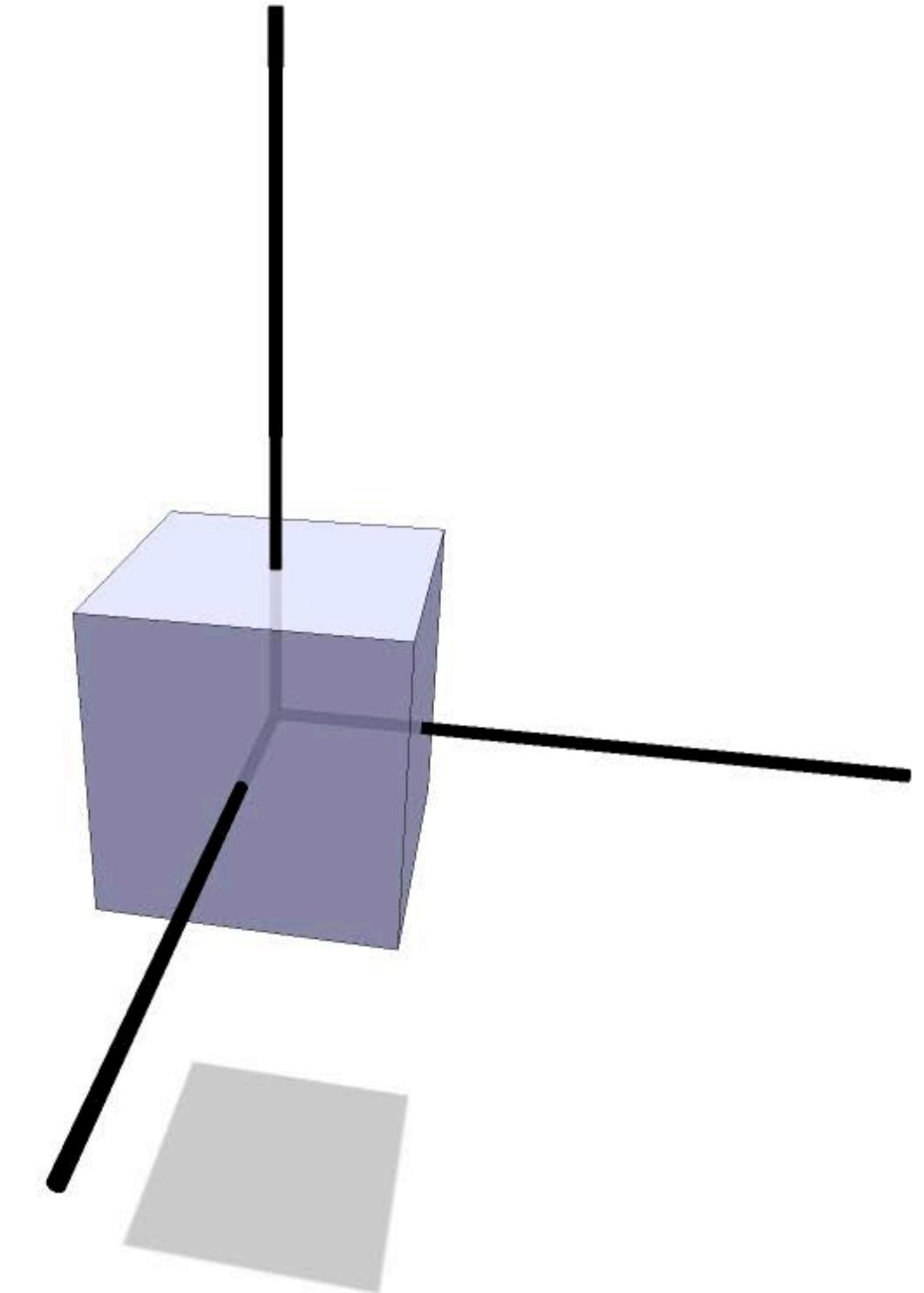
Nonuniform Scaling

- **Idea.** We could:

- rotate to the new axes (R)
- apply a diagonal scaling (D)
- rotate *back** to the original axes (R^T)

- Notice that the overall transformation is represented by a symmetric matrix

$$A := R^T D R$$



$$f(\mathbf{x}) = R^T D R \mathbf{x}$$

Q: Do all symmetric matrices represent nonuniform scaling (for some choice of axes)?

*Recall that for a rotation, the inverse equals the transpose: $R^{-1} = R^T$

Spectral Theorem

- A: Yes! **Spectral theorem** says a symmetric matrix $A = A^\top$ has
 - orthonormal eigenvectors $e_1, \dots, e_n \in \mathbb{R}^n$
 - real eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{R}$
- Can also write this relationship as $AR = RD$, where

$$R = \begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix} \quad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

- Equivalently, $A = RDR^\top$
- Hence, **every symmetric matrix performs a non-uniform scaling along some set of orthogonal axes.**
- If A is *positive definite* ($\lambda_i > 0$), this scaling is positive.

Shear

- A shear displaces each point \mathbf{x} in a direction \mathbf{u} according to its distance along a fixed vector \mathbf{v} :

$$f_{\mathbf{u}, \mathbf{v}}(\mathbf{x}) = \mathbf{x} + \langle \mathbf{v}, \mathbf{x} \rangle \mathbf{u}$$

- Q: Is this transformation *linear*?
- A: Yes—for instance, can represent it via a matrix

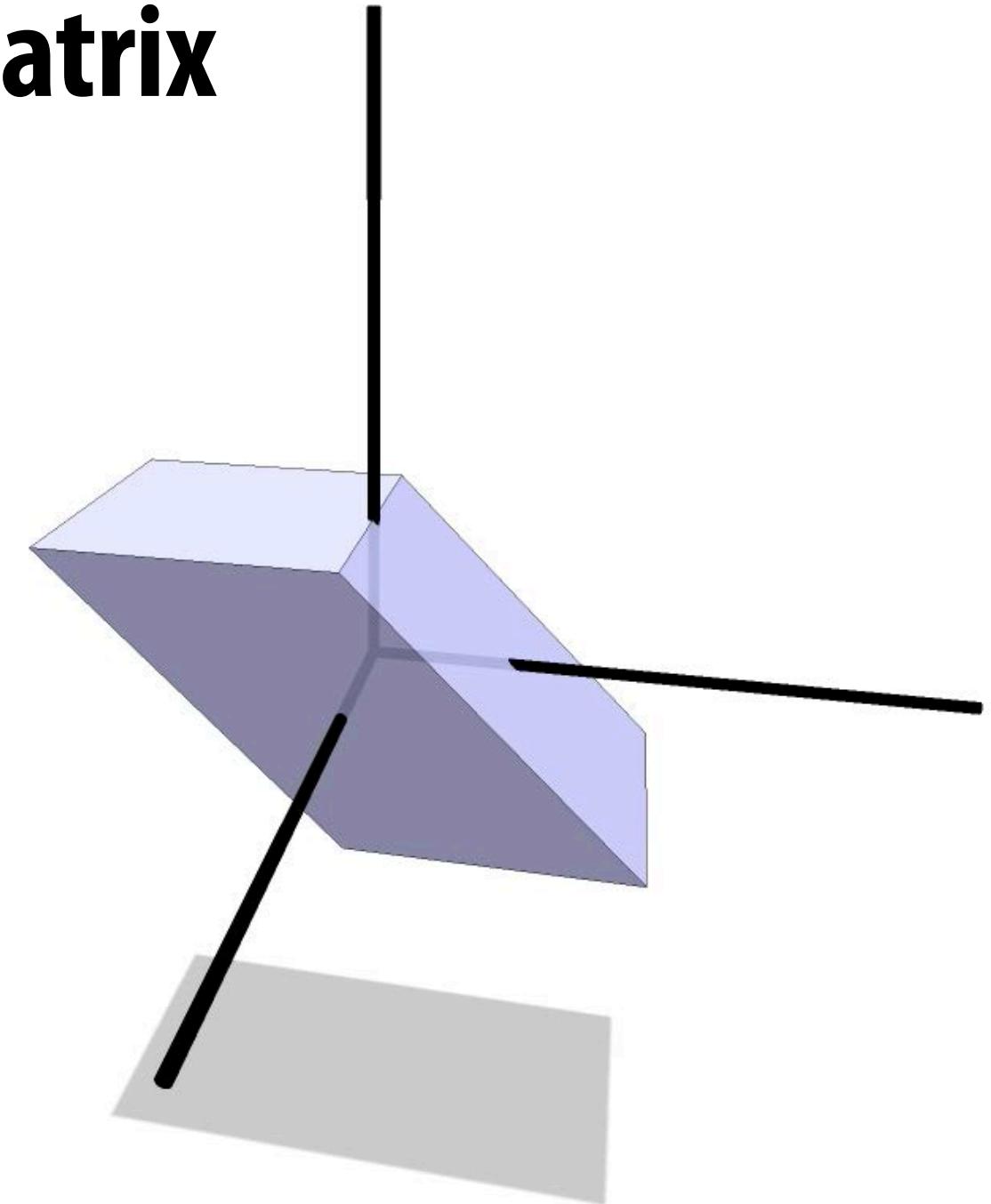
$$A_{\mathbf{u}, \mathbf{v}} = I + \mathbf{u}\mathbf{v}^T$$

Example.

$$\mathbf{u} = (\cos(t), 0, 0)$$

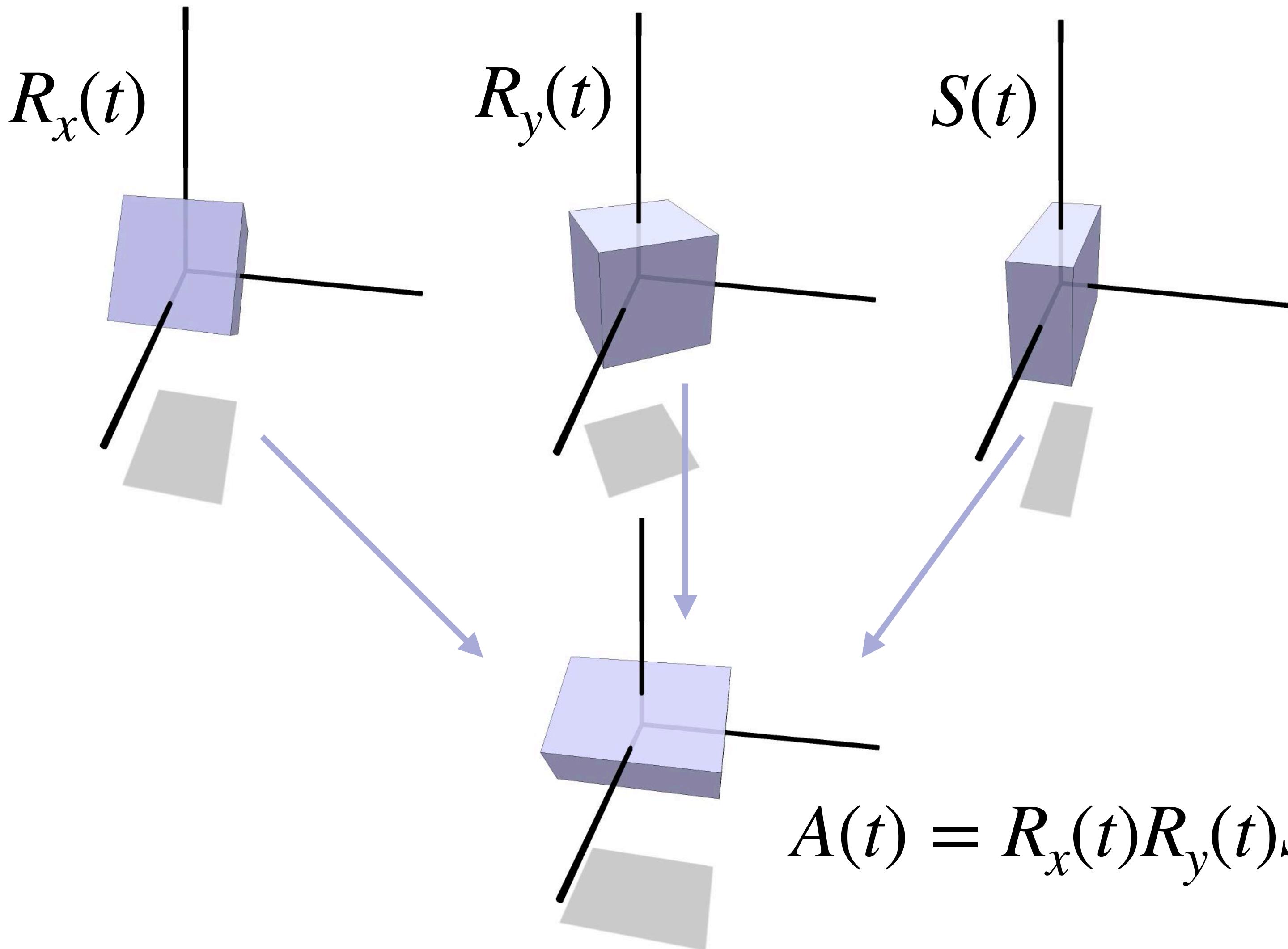
$$\mathbf{v} = (0, 1, 0)$$

$$A_{\mathbf{u}, \mathbf{v}} = \begin{bmatrix} 1 & \cos(t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Composite Transformations

From these basic transformations (rotation, reflection, scaling, shear...) we can now build up composite transformations via matrix multiplication:

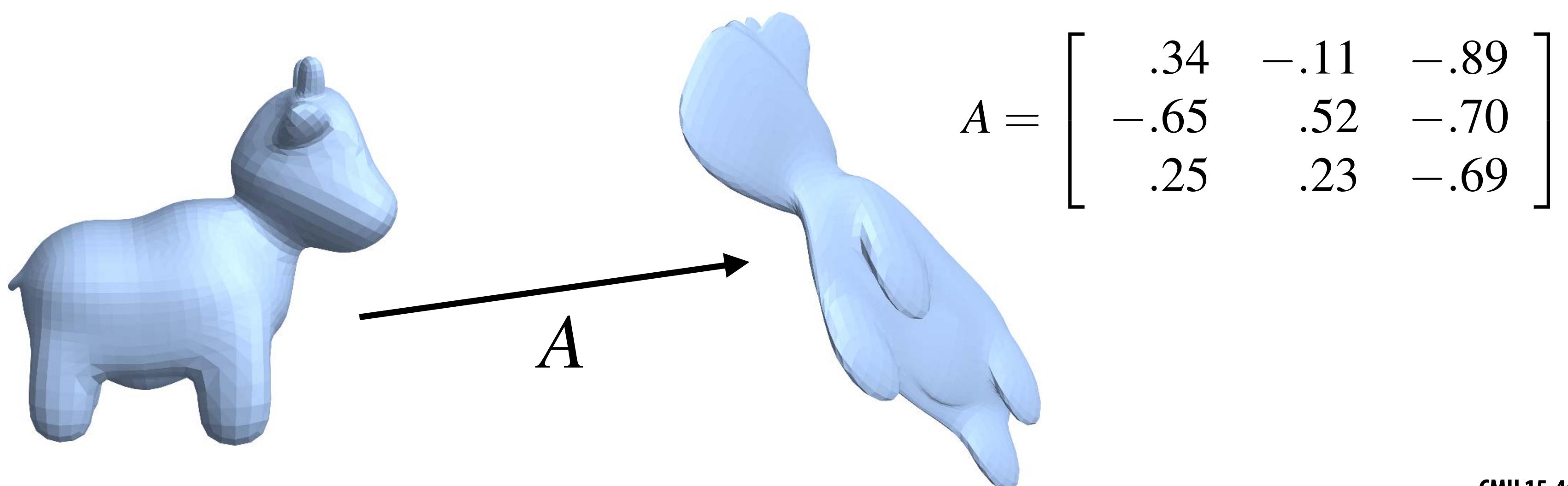


**How do we decompose a linear
transformation into pieces?**

(rotations, reflections, scaling, ...)

Decomposition of Linear Transformations

- In general, no **unique** way to write a given linear transformation as a composition of basic transformations!
- However, there are *many* useful decompositions:
 - singular value decomposition (good for signal processing)
 - LU factorization (good for solving linear systems)
 - polar decomposition (good for spatial transformations)
 - ...
- Consider for instance this linear transformation:



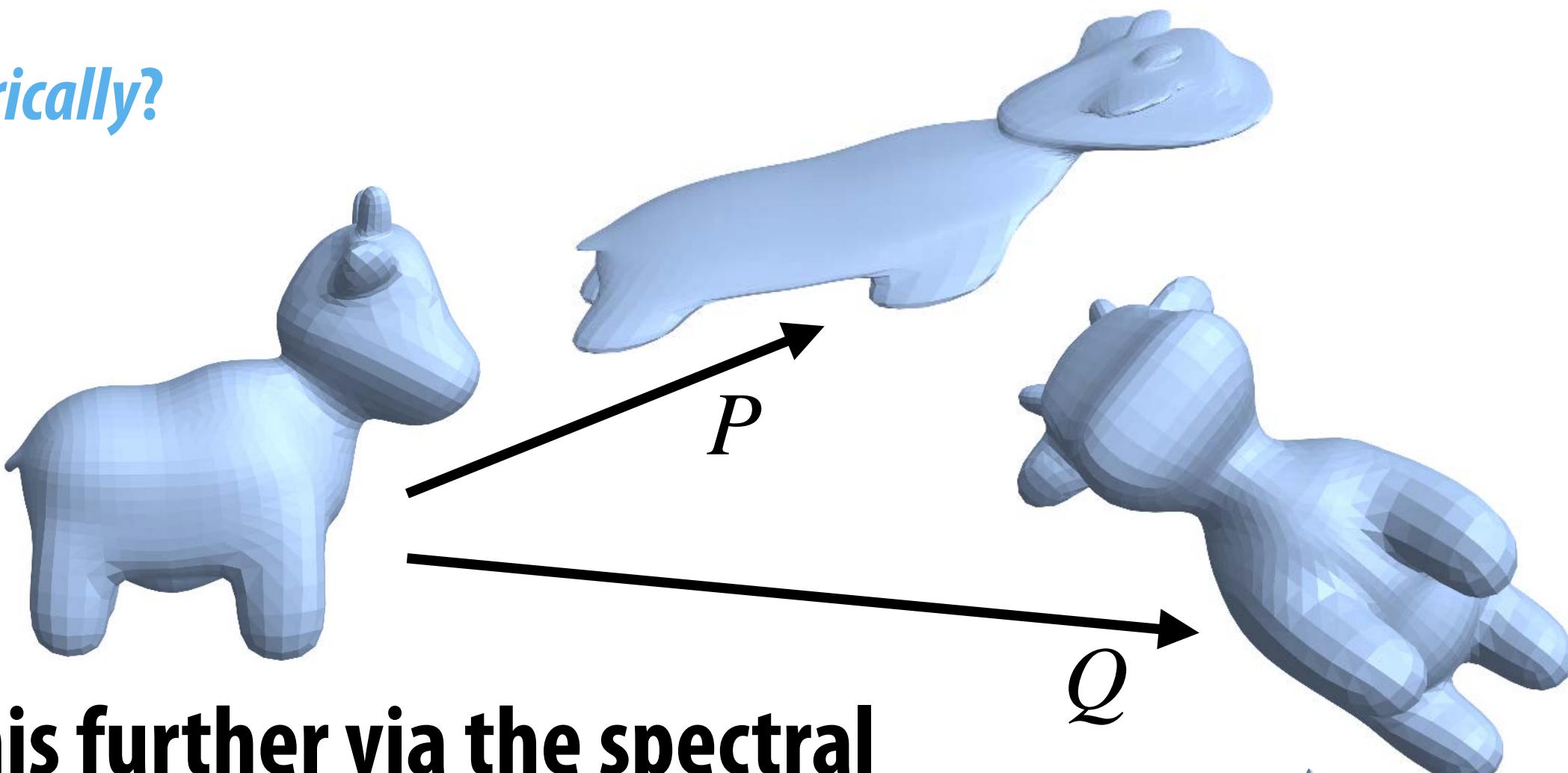
Polar & Singular Value Decomposition

For example, polar decomposition decomposes any matrix A into orthogonal matrix Q and symmetric positive-semidefinite matrix P :

Q: What do each of the parts mean *geometrically*?

rotation/reflection nonnegative,
nonuniform scaling

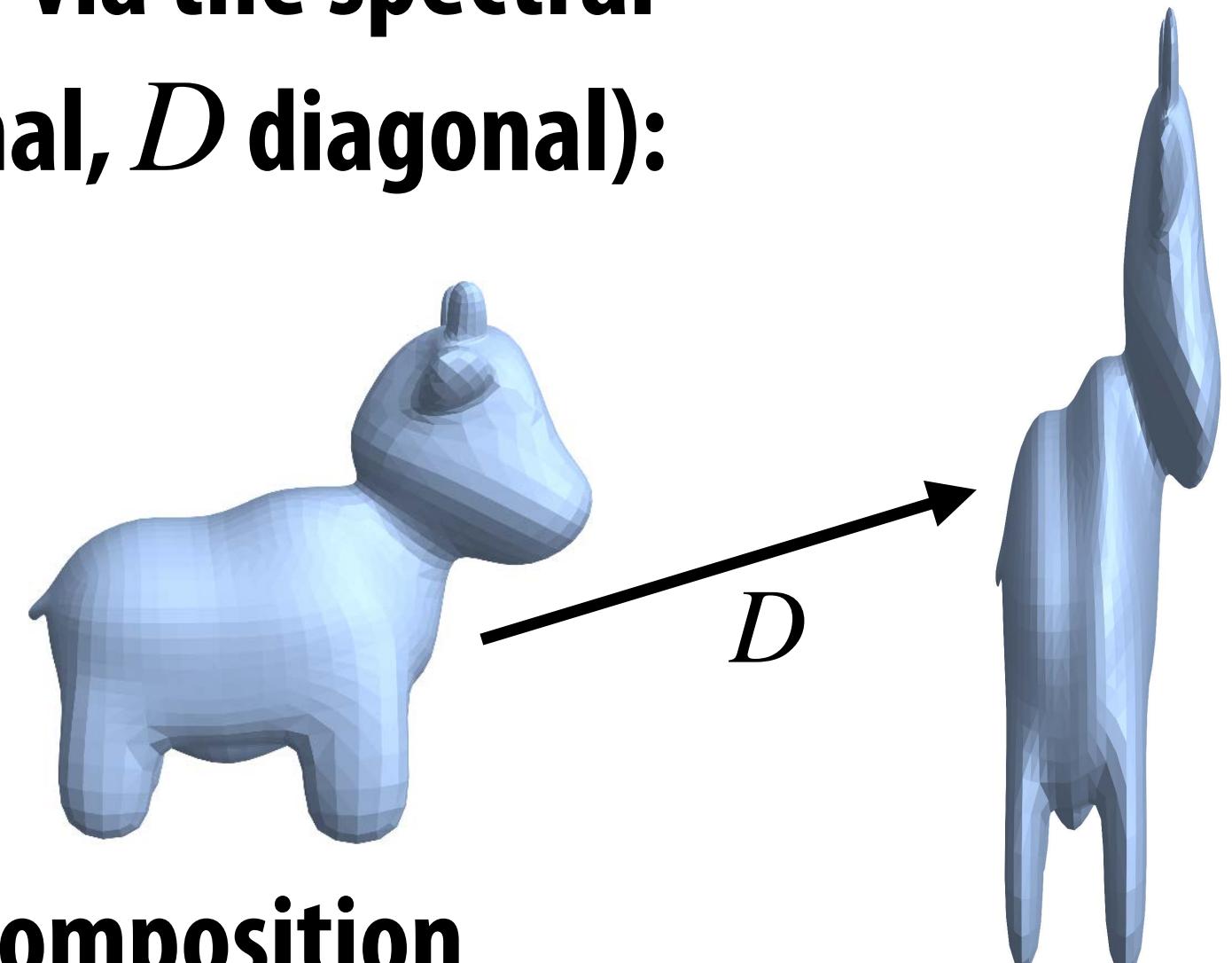
$$A = QP$$



Since P is symmetric, can take this further via the spectral decomposition $P = VDV^T$ (V orthogonal, D diagonal):

$$A = \underbrace{QP}_{U} V D V^T = U D V^T$$

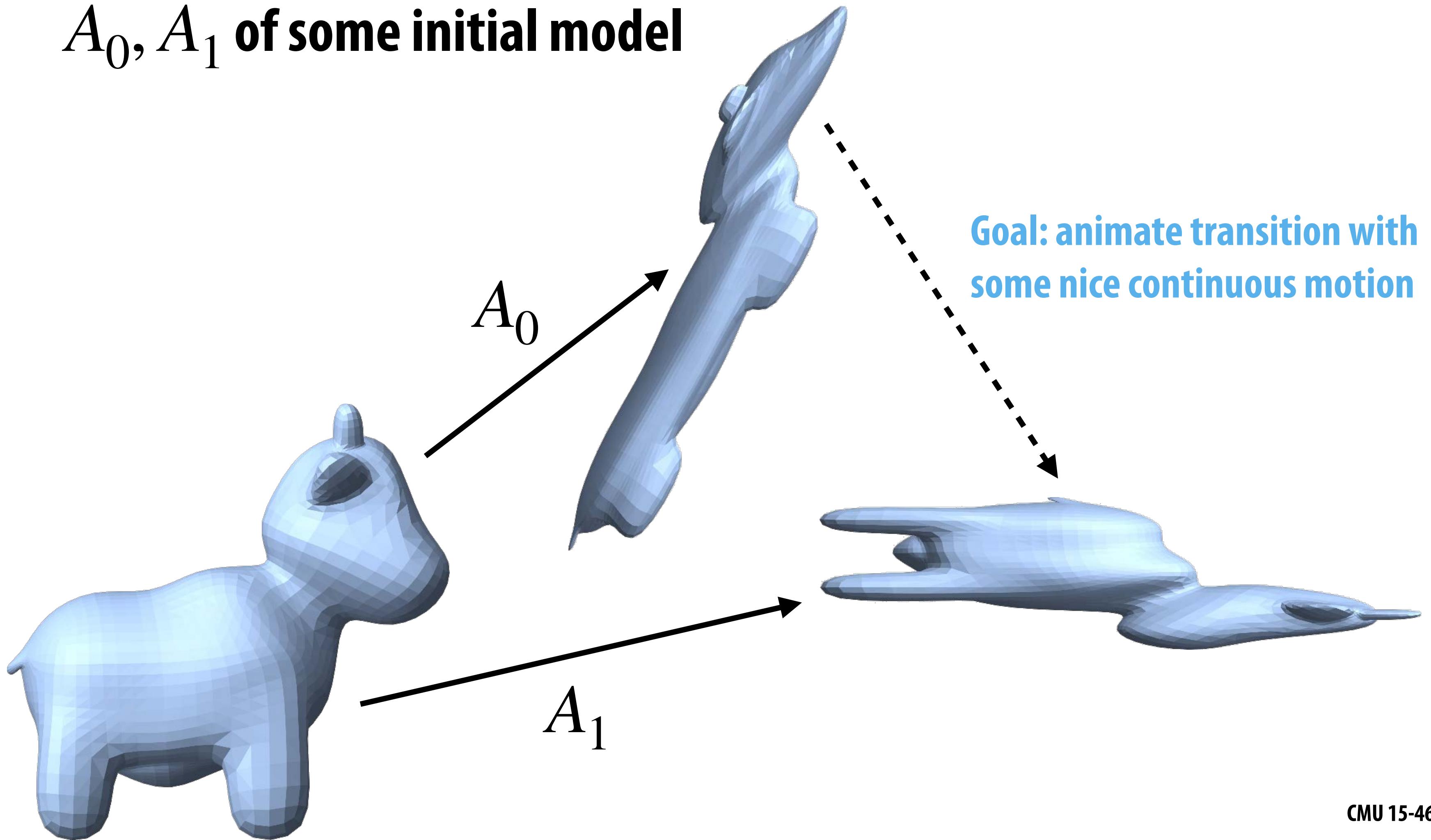
rotation rotation
axis-aligned scaling



Result UDV^T is called the singular value decomposition

Interpolating Transformations

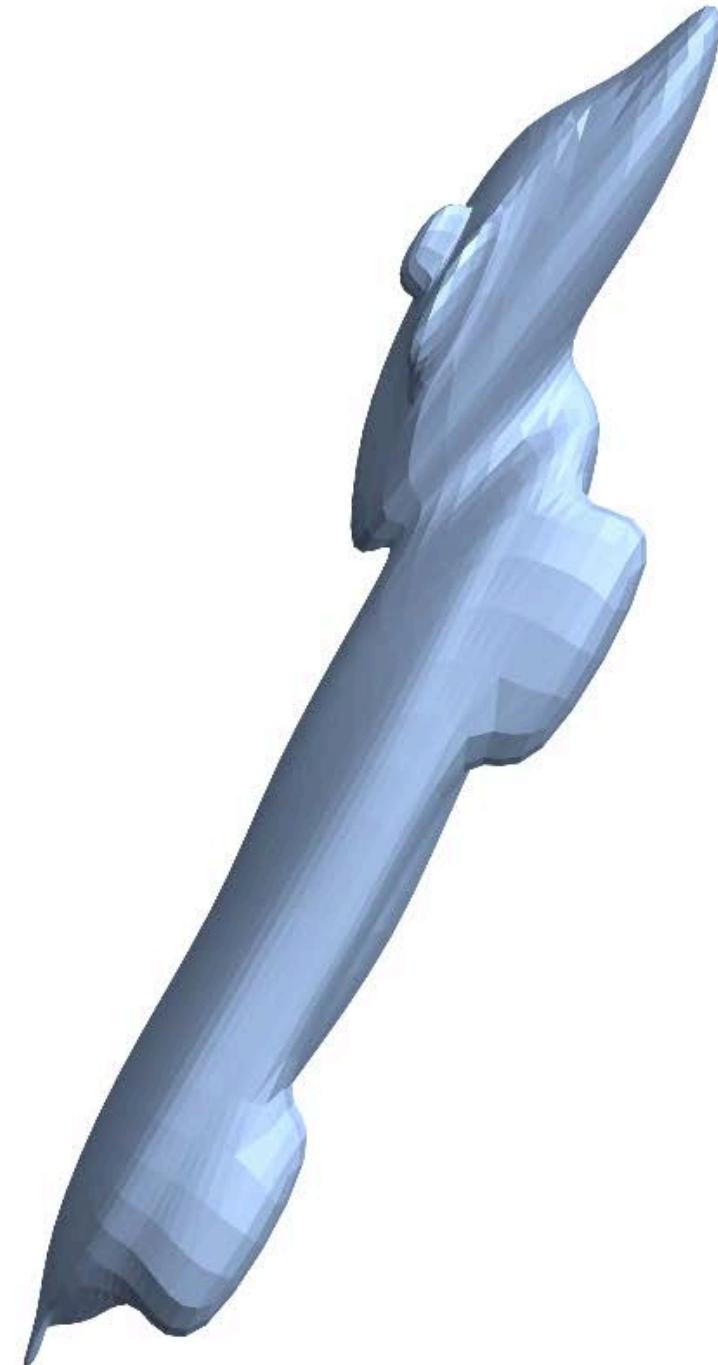
- How are these decompositions useful for graphics?
- Consider interpolating between two linear transformations A_0, A_1 of some initial model



Interpolating Transformations—Linear

One idea: just take a linear combination of the two matrices,
weighted by the current time $t \in [0, 1]$

$$A(t) = (1 - t)A_0 + tA_1$$



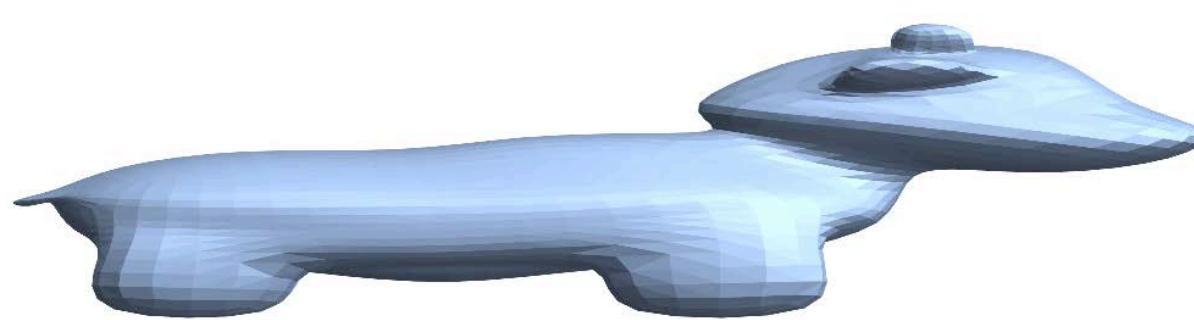
Hits the right start/endpoints... but looks awful in between!

Interpolating Transformations—Polar

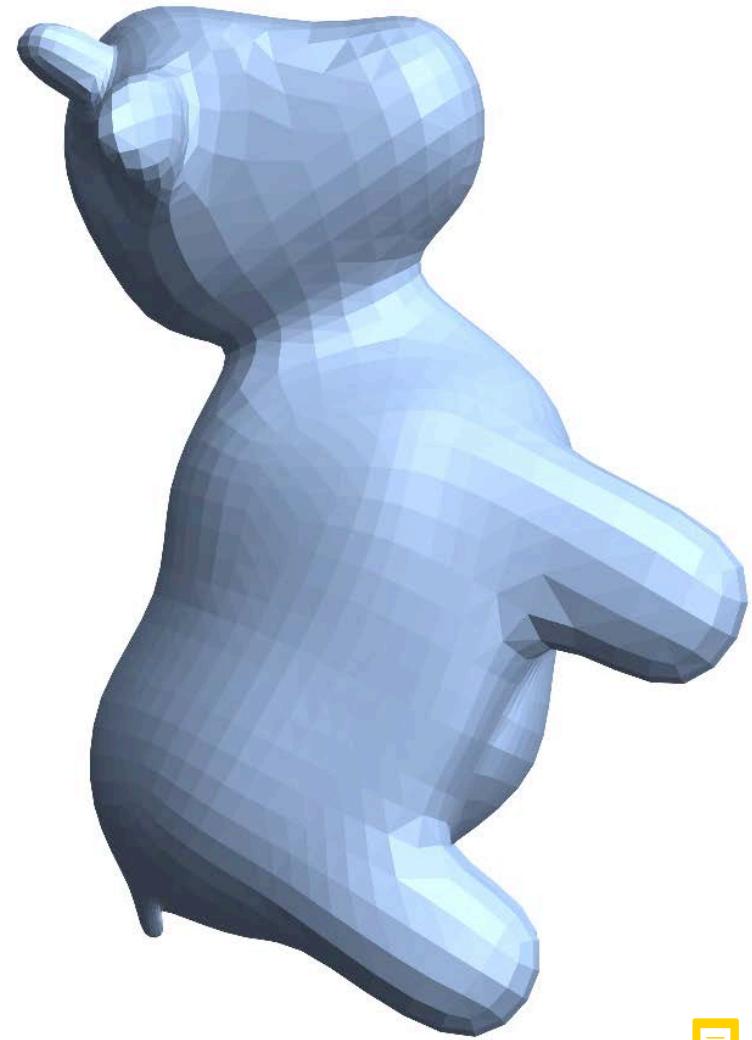
Better idea: *separately* interpolate components of polar decomposition.

$$A_0 = Q_0 P_0, \quad A_1 = Q_1 P_1$$

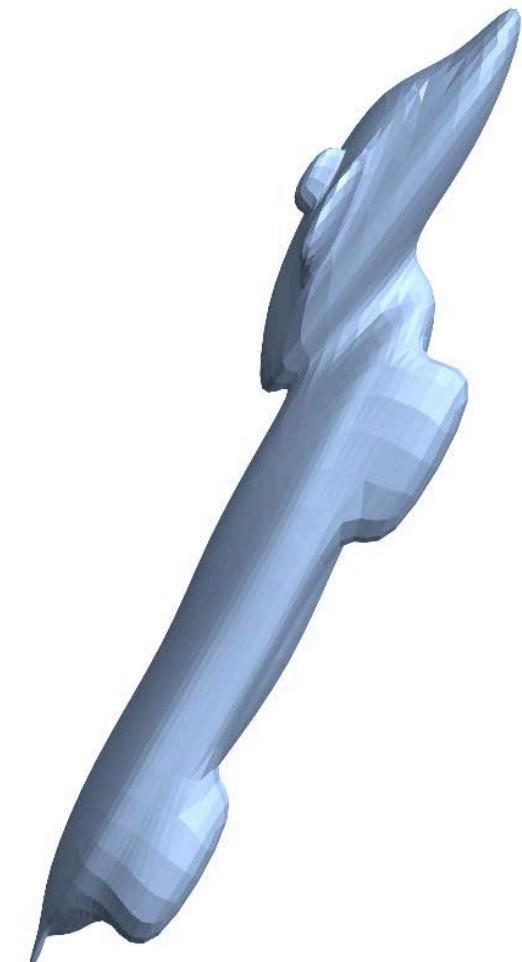
scaling



rotation



final interpolation



$$P(t) = (1 - t)P_0 + tP_1$$

$$\widetilde{Q}(t) = (1 - t)Q_0 + tQ_1$$

$$\widetilde{Q}(t) = Q(t)X(t)$$

$$A(t) = Q(t)P(t)$$

...looks better!

Example: Linear Blend Skinning

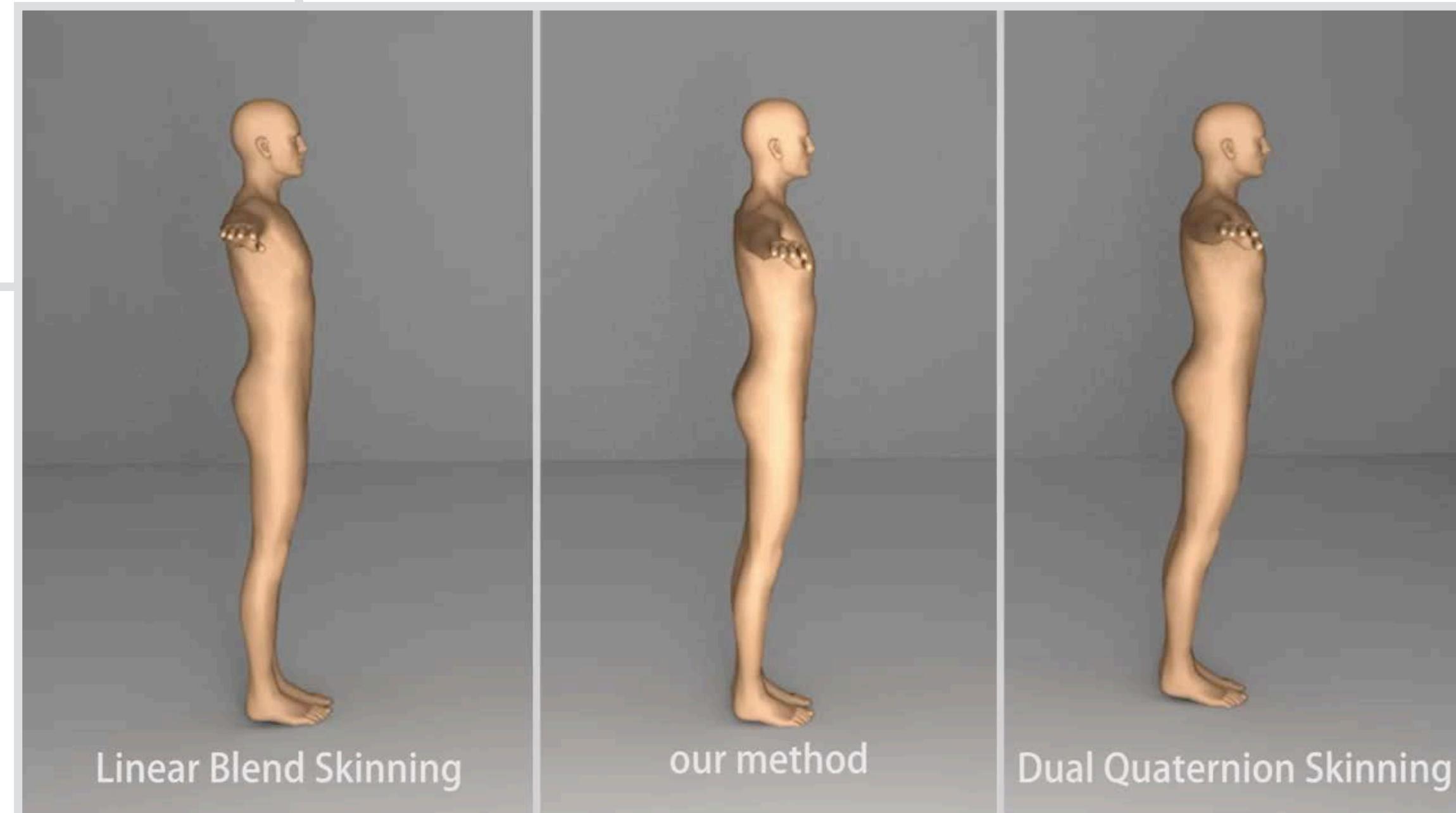
- Naïve linear interpolation also causes artifacts when blending between transformations on a character (“candy wrapper effect”)
- Lots of research on alternative ways to blend transformations...

LBS: candy-wrapper artifact



Rumman & Fratarcangeli (2015)
“Position-based Skinning for Soft Articulated Characters”

Jacobson, Deng, Kavan, & Lewis (2014)
“Skinning: Real-time Shape Deformation”



Translations



- So far we've ignored a basic transformation—translations
- A translation simply adds an offset \mathbf{u} to the given point \mathbf{x} :

$$f_{\mathbf{u}}(\mathbf{x}) = \mathbf{x} + \mathbf{u}$$

**Q: Is this transformation linear?
(Certainly seems to move us along a line...)**

Let's carefully check the definition...

additivity

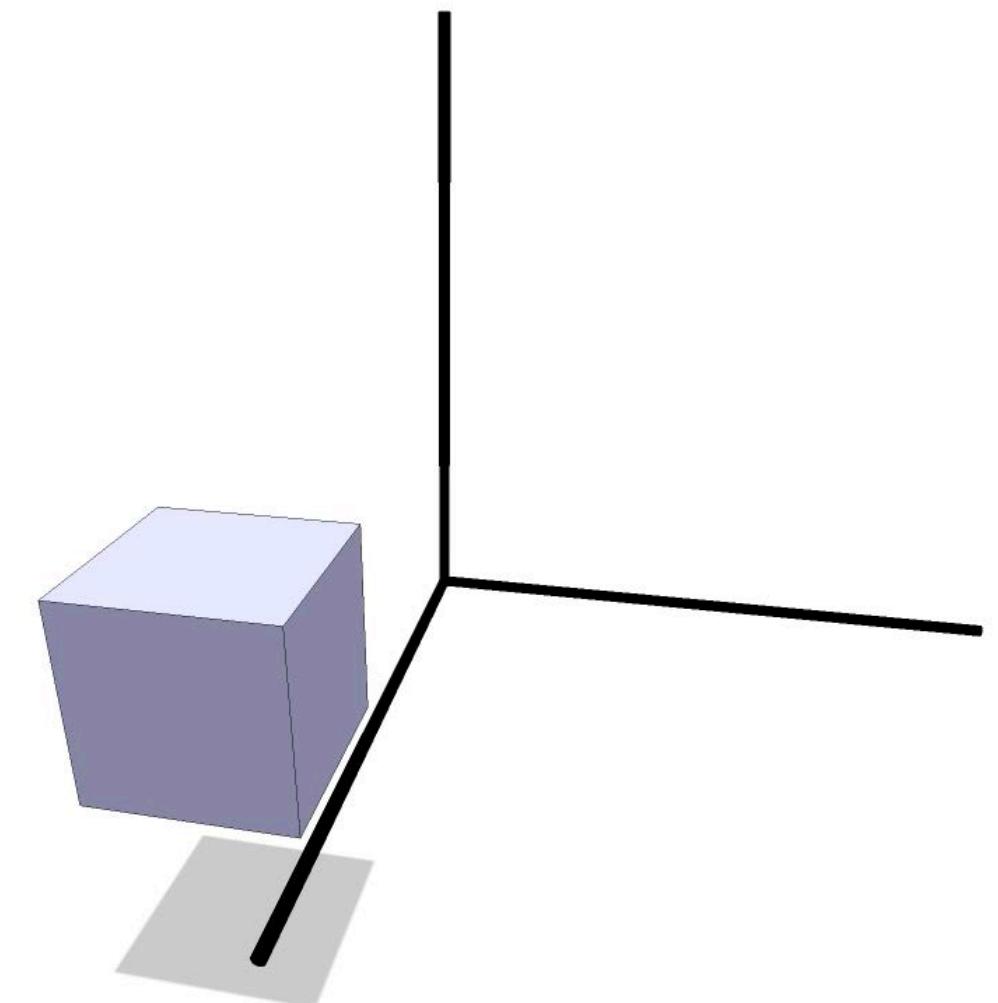
$$f_{\mathbf{u}}(\mathbf{x} + \mathbf{y}) = \mathbf{x} + \mathbf{y} + \mathbf{u}$$

$$f_{\mathbf{u}}(\mathbf{x}) + f_{\mathbf{u}}(\mathbf{y}) = \mathbf{x} + \mathbf{y} + 2\mathbf{u}$$

homogeneity

$$f_{\mathbf{u}}(a\mathbf{x}) = a\mathbf{x} + \mathbf{u}$$

$$af_{\mathbf{u}}(\mathbf{x}) = ax + a\mathbf{u}$$



A: No! Translation is affine, not linear!

Composition of Transformations

- Recall we can compose linear transformations via matrix multiplication:

$$A_3(A_2(A_1 \mathbf{x})) = (A_3 A_2 A_1) \mathbf{x}$$

- It's easy enough to compose translations—just add vectors:

$$f_{\mathbf{u}_3}(f_{\mathbf{u}_2}(f_{\mathbf{u}_1}(\mathbf{x}))) = f_{\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3}(\mathbf{x})$$

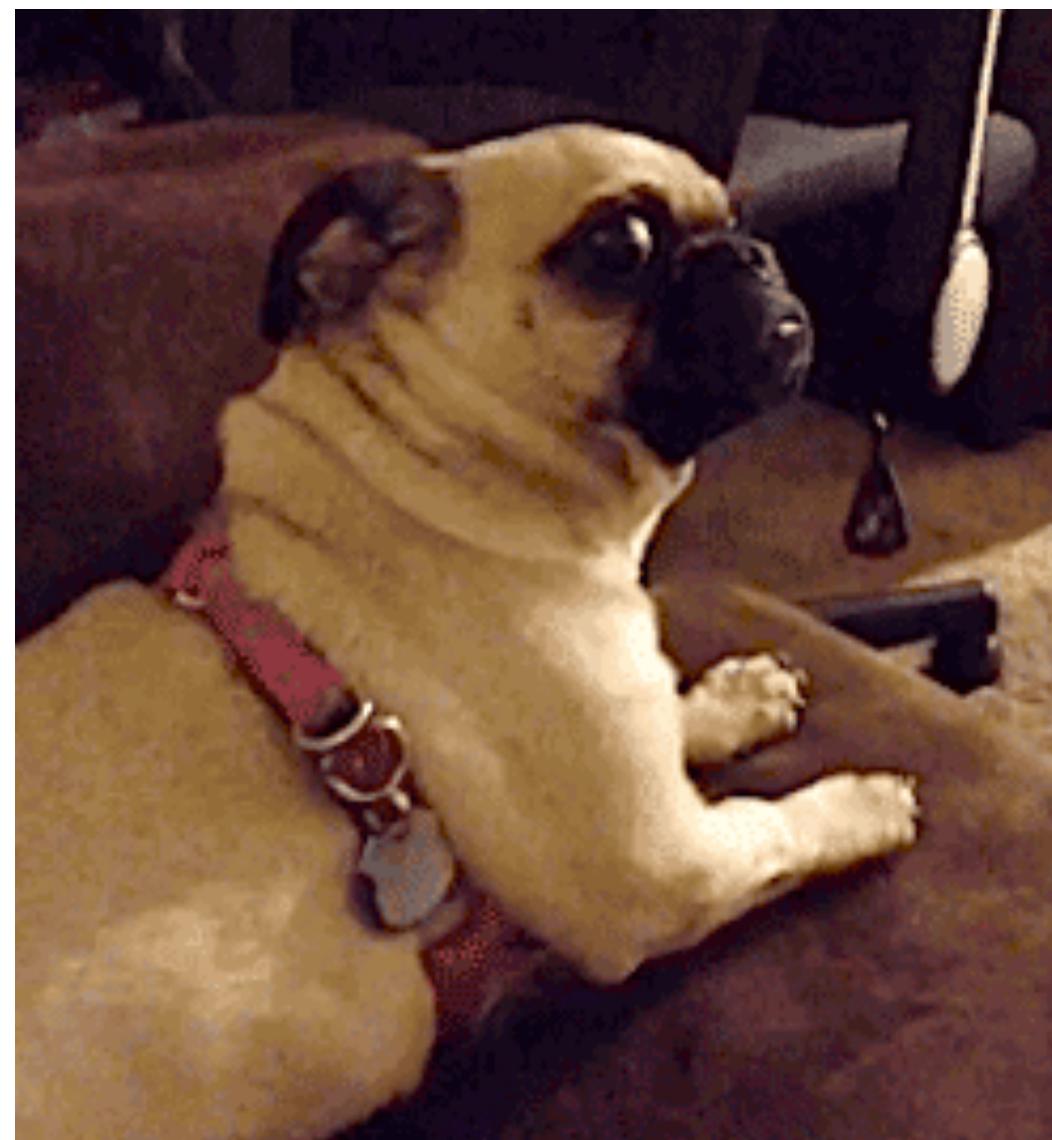
- What if we want to intermingle translations and linear transformations (rotation, scale, shear, etc.)?

$$A_2(A_1 \mathbf{x} + \mathbf{b}_1) + \mathbf{b}_2 = (A_2 A_1) \mathbf{x} + (A_2 \mathbf{b}_1 + \mathbf{b}_2)$$

- Now we have to keep track of a matrix *and* a vector
- Moreover, we'll see (later) that this encoding won't work for other important cases, such as perspective transformations

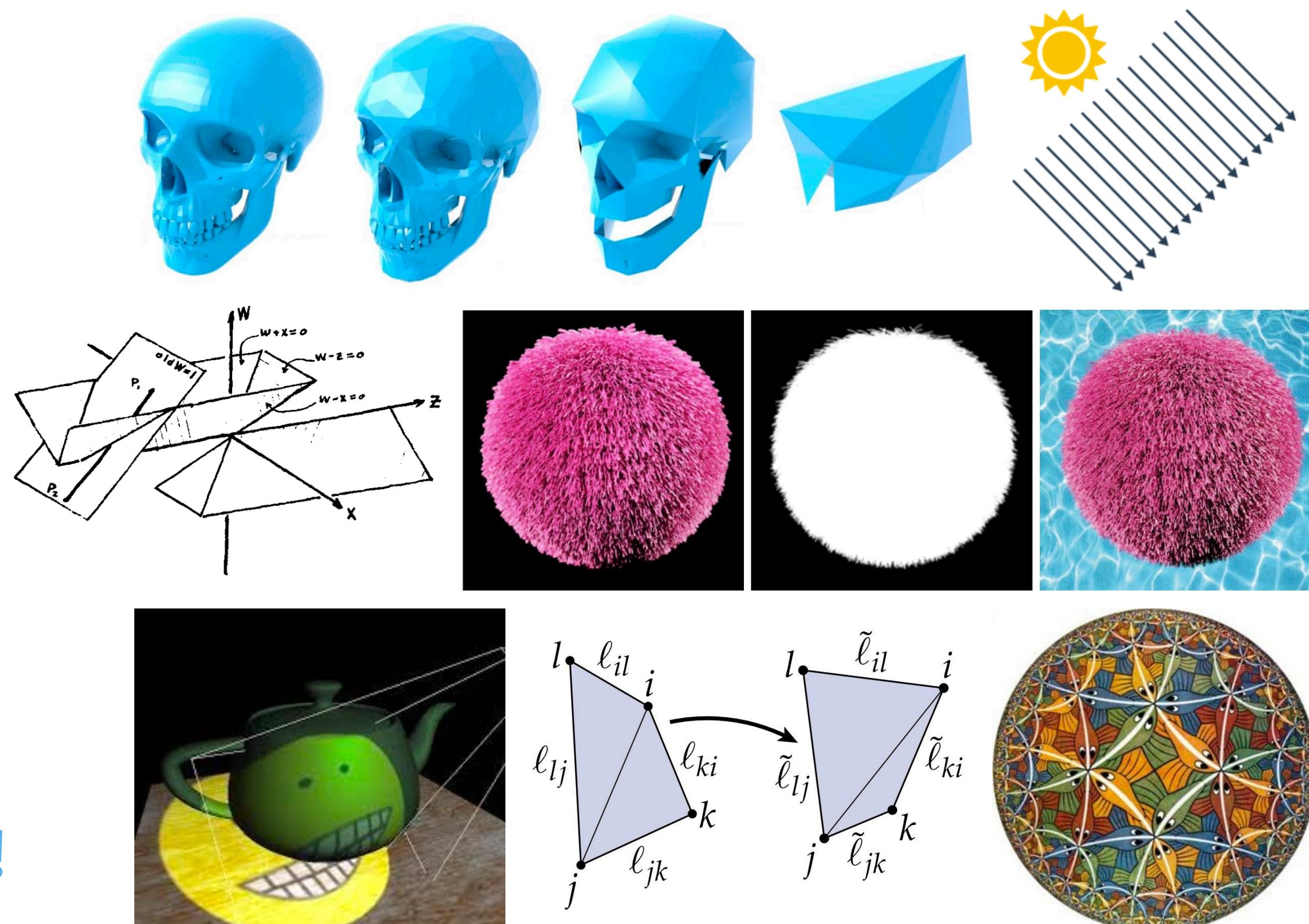
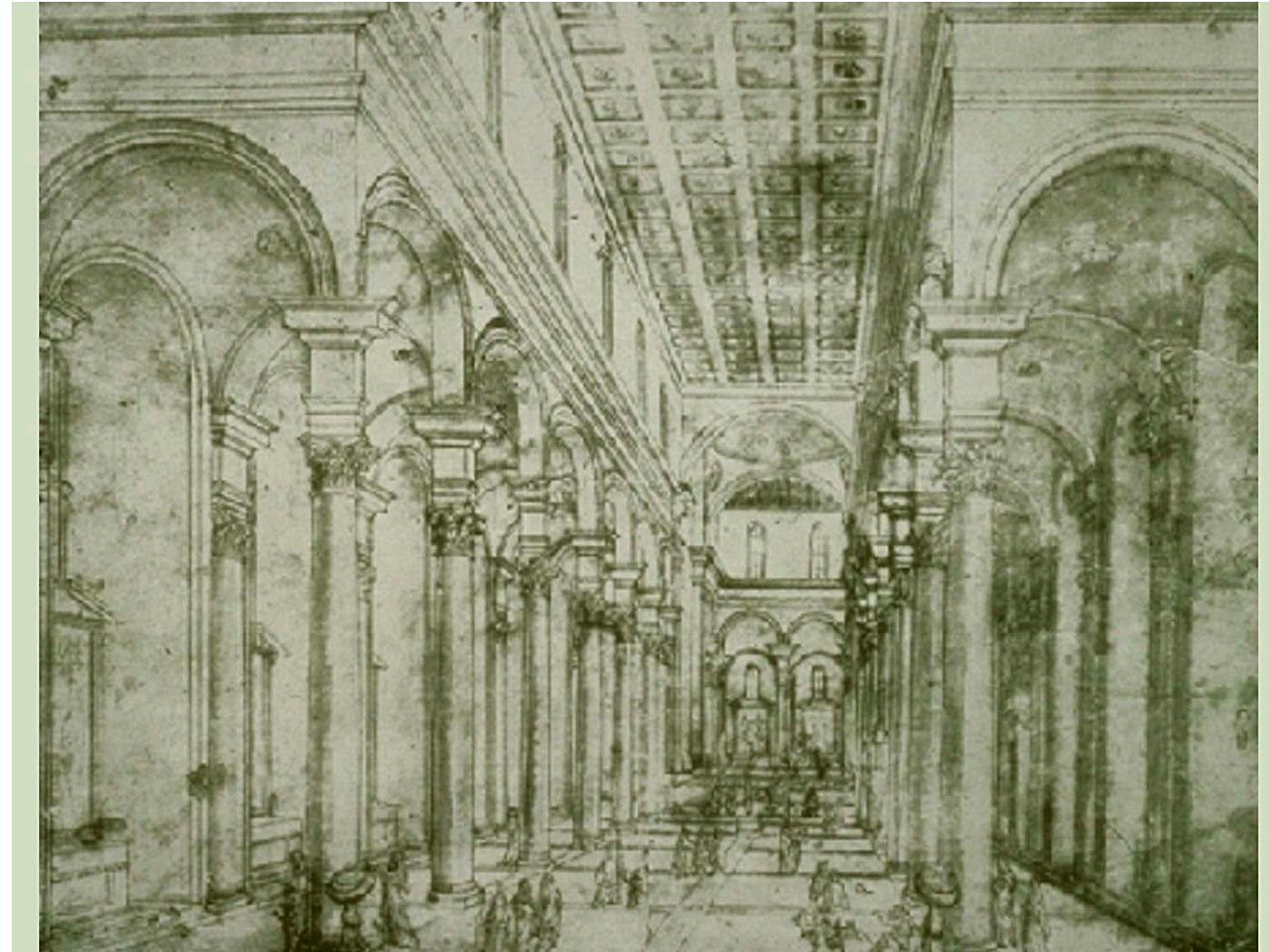
But there is a better way...

**Strange idea:
Maybe translations turn into linear
transformations if we go into the
4th dimension...!**



Homogeneous Coordinates

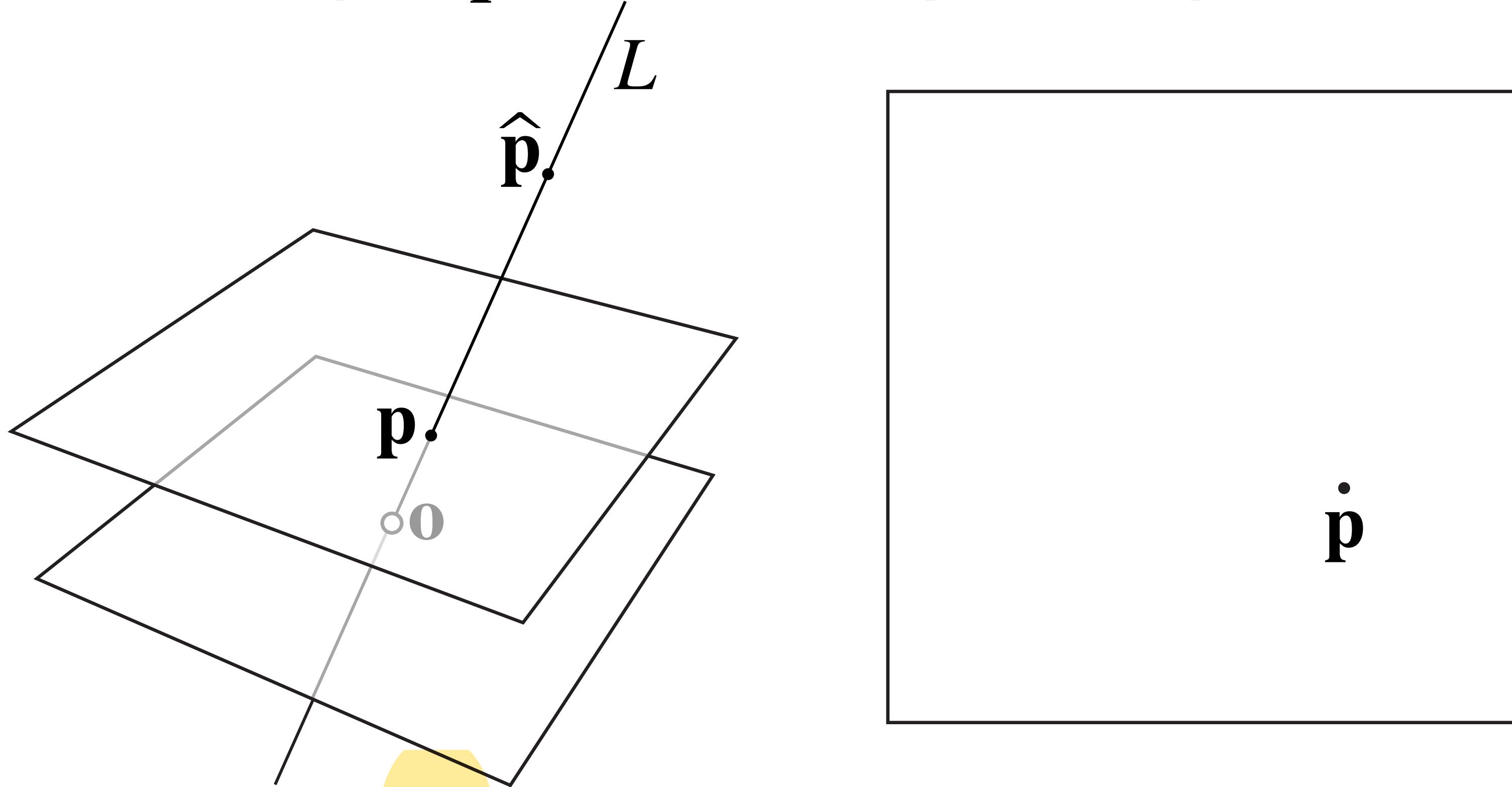
- Came from efforts to study perspective
- Introduced by Möbius as a natural way of assigning coordinates to lines
- Show up naturally in a surprising large number of places in computer graphics:
 - 3D transformations
 - perspective projection
 - quadric error simplification
 - premultiplied alpha
 - shadow mapping
 - projective texture mapping
 - discrete conformal geometry
 - hyperbolic geometry
 - clipping
 - directional lights
 - ...



Probably worth understanding!

Homogeneous Coordinates—Basic Idea

- Consider any 2D plane that does not pass through the origin o in 3D
- Every line through the origin in 3D corresponds to a point in the 2D plane
 - Just find the point p where the line L pierces the plane

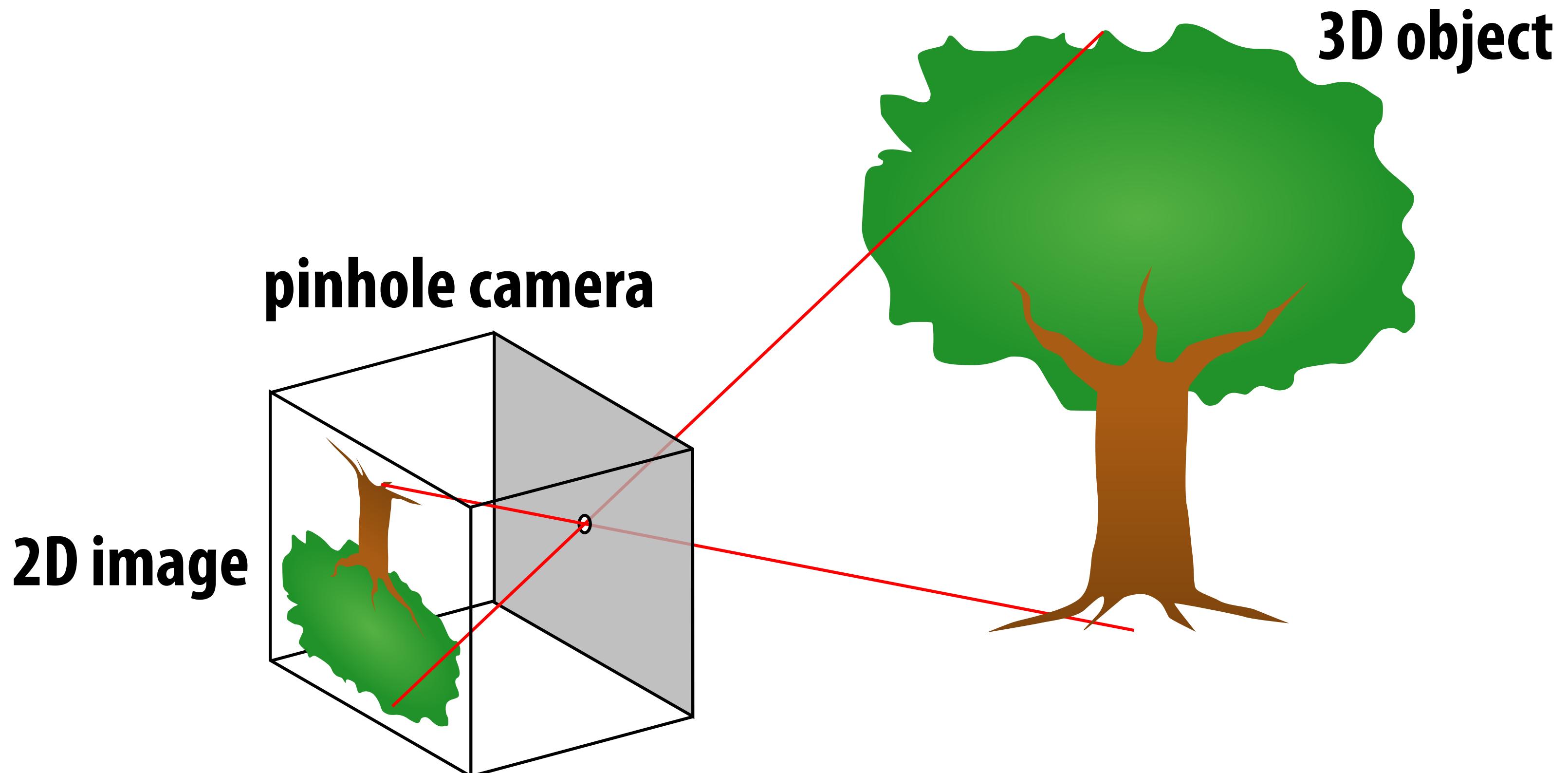


Hence, any point \hat{p} on the line L can be used to represent the point p .

Q: What does this story remind you of?

Review: Perspective projection

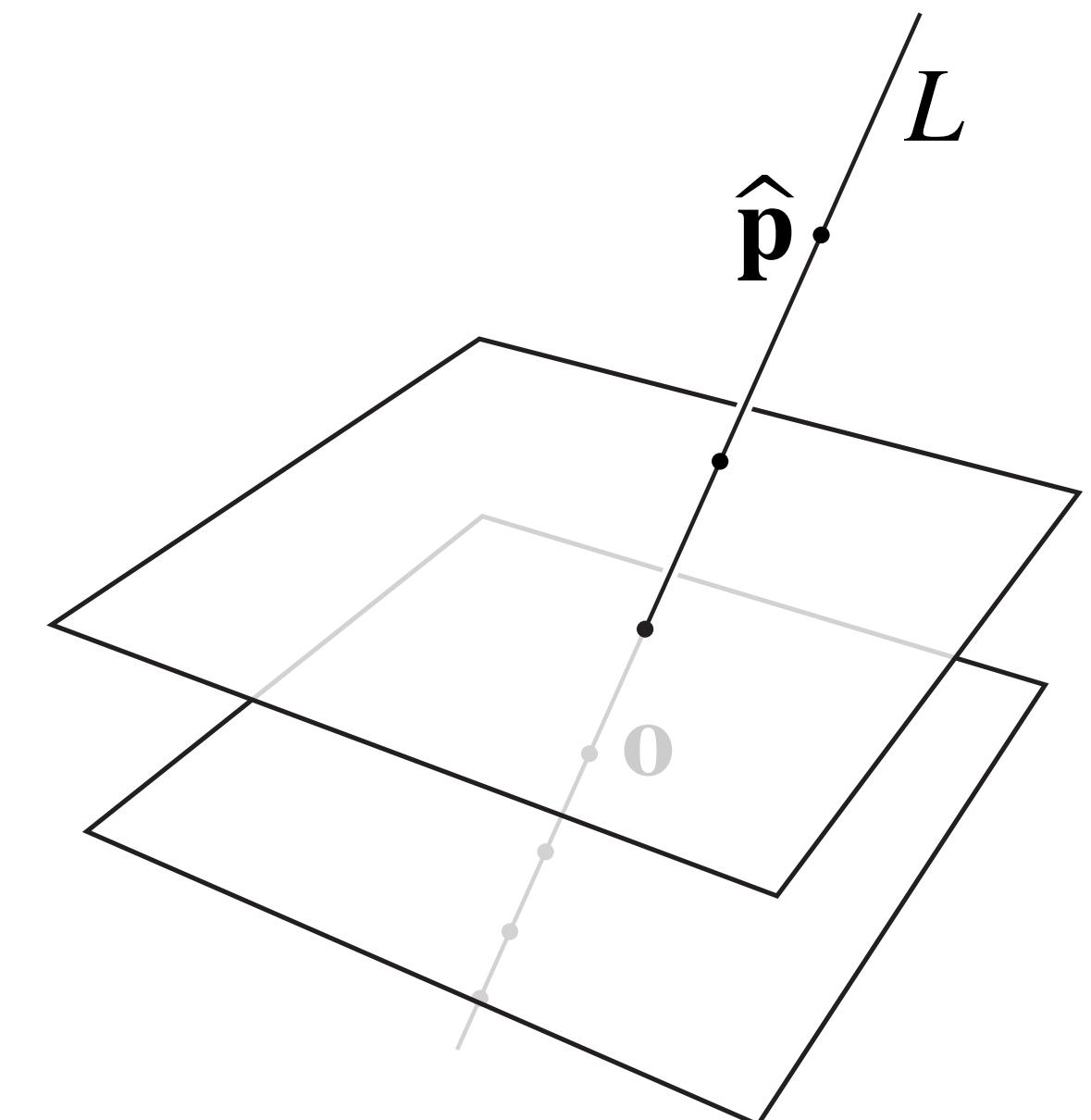
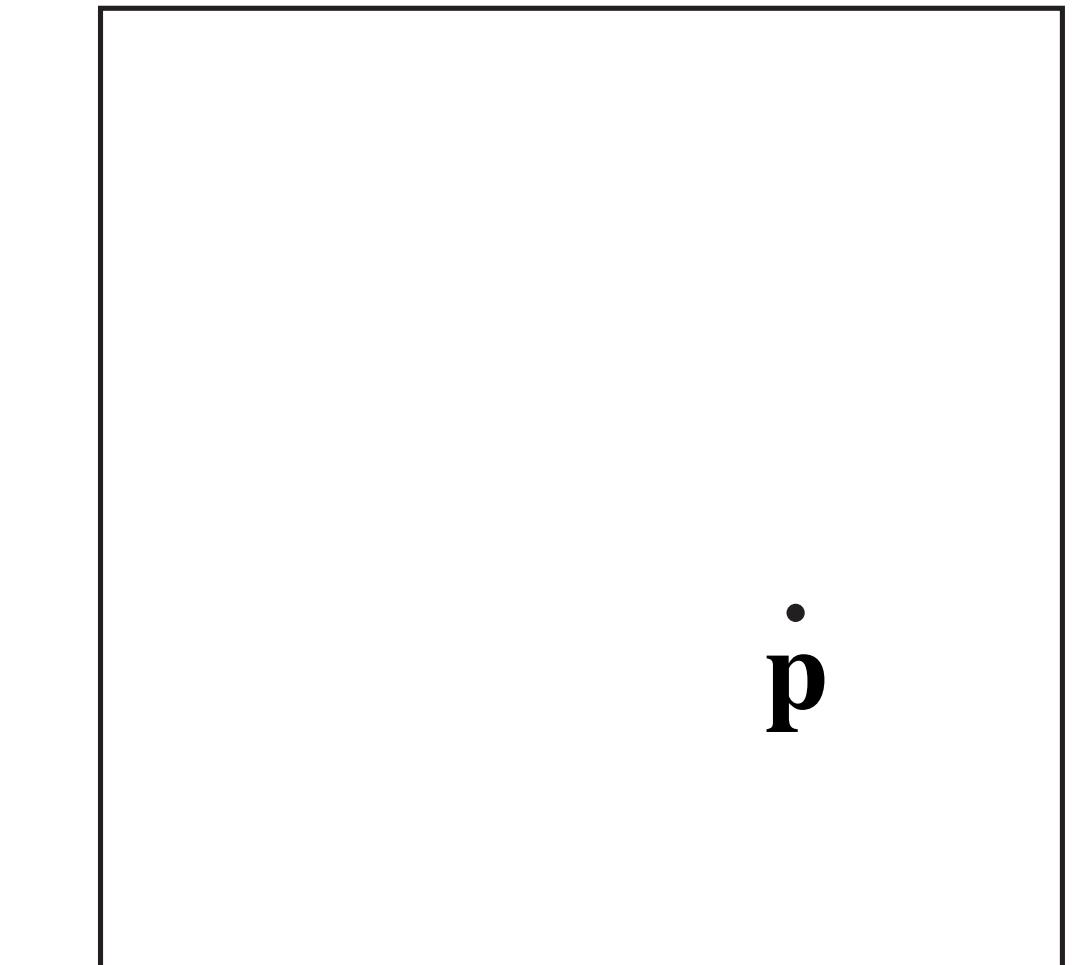
- Hopefully it reminds you of our “pinhole camera”
- Objects along the same line project to the same point



If you have an image of a single dot, can't know where it is!
Only which line it belongs to.

Homogeneous Coordinates (2D)

- More explicitly, consider a point $\mathbf{p} = (x, y)$, and the plane $z = 1$ in 3D
- Any three numbers $\hat{\mathbf{p}} = (a, b, c)$ such that $(a/c, b/c) = (x, y)$ are homogeneous coordinates for \mathbf{p}
 - E.g., $(x, y, 1)$
 - In general: (cx, cy, c) for $c \neq 0$
- Hence, two points $\hat{\mathbf{p}}, \hat{\mathbf{q}} \in \mathbb{R}^3 \setminus \{O\}$ describe the same point in 2D (and line in 3D) if $\hat{\mathbf{p}} = \lambda \hat{\mathbf{q}}$ for some $\lambda \neq 0$

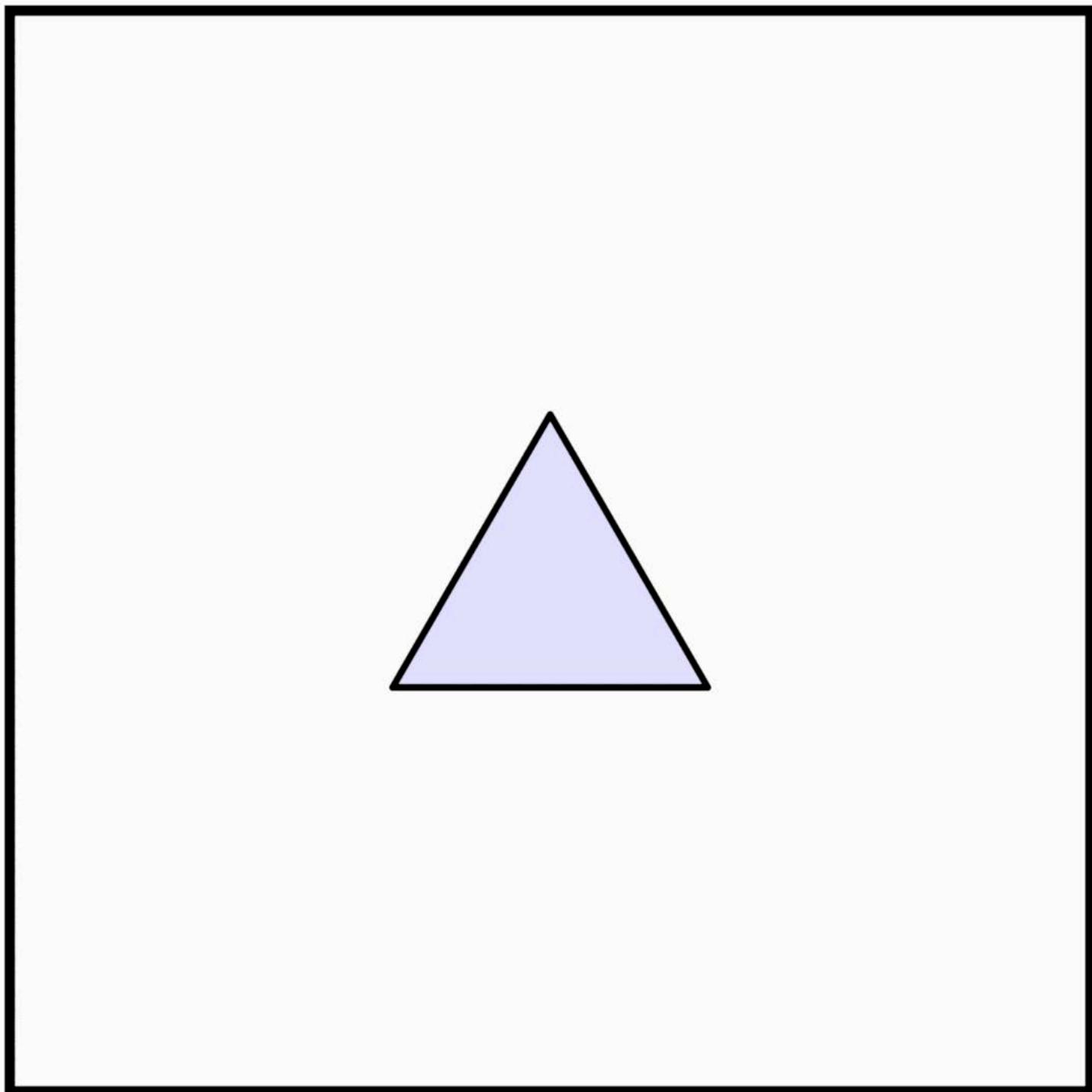


Great... but how does this help us with transformations?

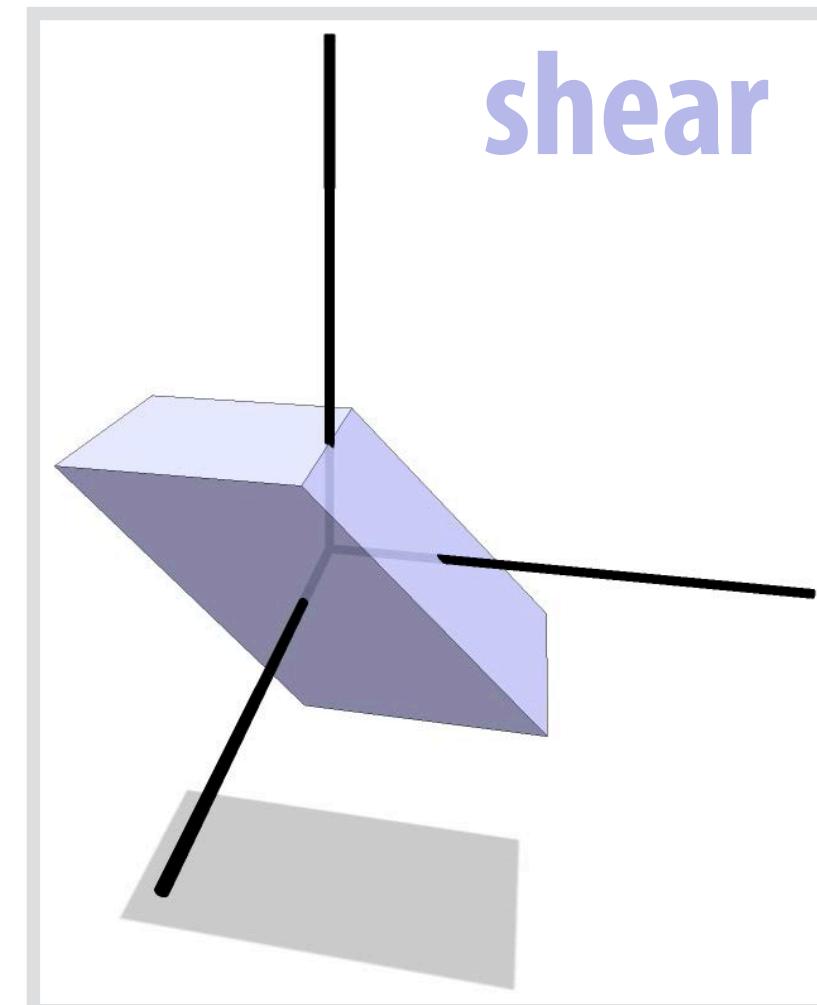
Translation in Homogeneous Coordinates

Let's think about what happens to our homogeneous coordinates \hat{p} if we apply a translation to our 2D coordinates p

2D coordinates



Q: What kind of transformation does this look like?



Translation in Homogeneous Coordinates

- But wait a minute—shear is a linear transformation!
- Can this be right? Let's check in coordinates...
- Suppose we translate a point $\mathbf{p} = (p_1, p_2)$ by a vector $\mathbf{u} = (u_1, u_2)$ to get $\mathbf{p}' = (p_1 + u_1, p_2 + u_2)$
- The homogeneous coordinates $\hat{\mathbf{p}} = (cp_1, cp_2, c)$ then become $\hat{\mathbf{p}}' = (cp_1 + cu_1, cp_2 + cu_2, c)$
- Notice that we're shifting $\hat{\mathbf{p}}$ by an amount $c\mathbf{u}$ that's proportional to the distance c along the third axis—a shear

Using homogeneous coordinates, we can represent an affine transformation in 2D as a linear transformation in 3D

Homogeneous Translation—Matrix Representation

- To write as a matrix, recall that a shear in the direction $\mathbf{u} = (u_1, u_2)$ according to the distance along a direction \mathbf{v} is

$$f_{\mathbf{u}, \mathbf{v}}(\mathbf{x}) = \mathbf{x} + \langle \mathbf{v}, \mathbf{x} \rangle \mathbf{u}$$

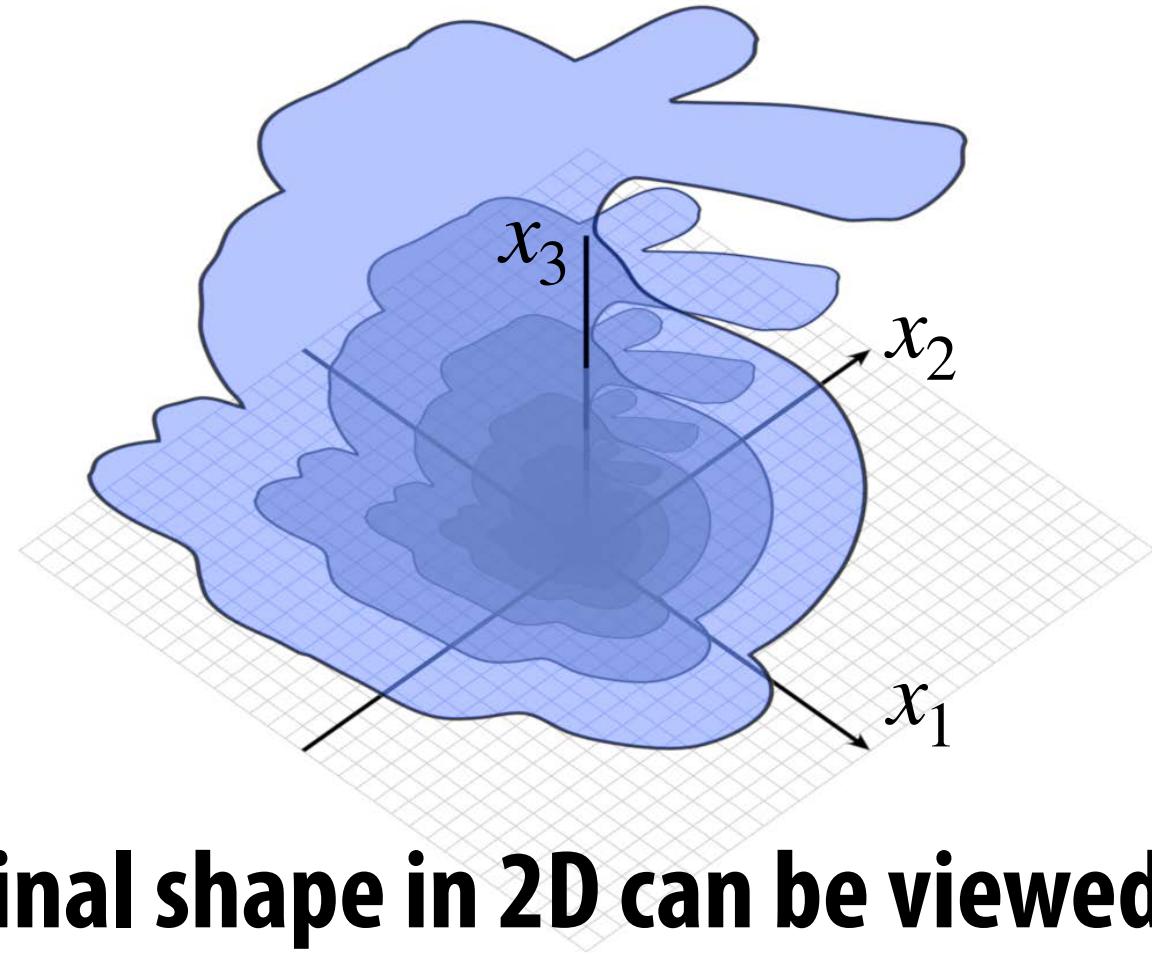
- In matrix form:

$$f_{\mathbf{u}, \mathbf{v}}(\mathbf{x}) = (I + \mathbf{u}\mathbf{v}^T) \mathbf{x}$$

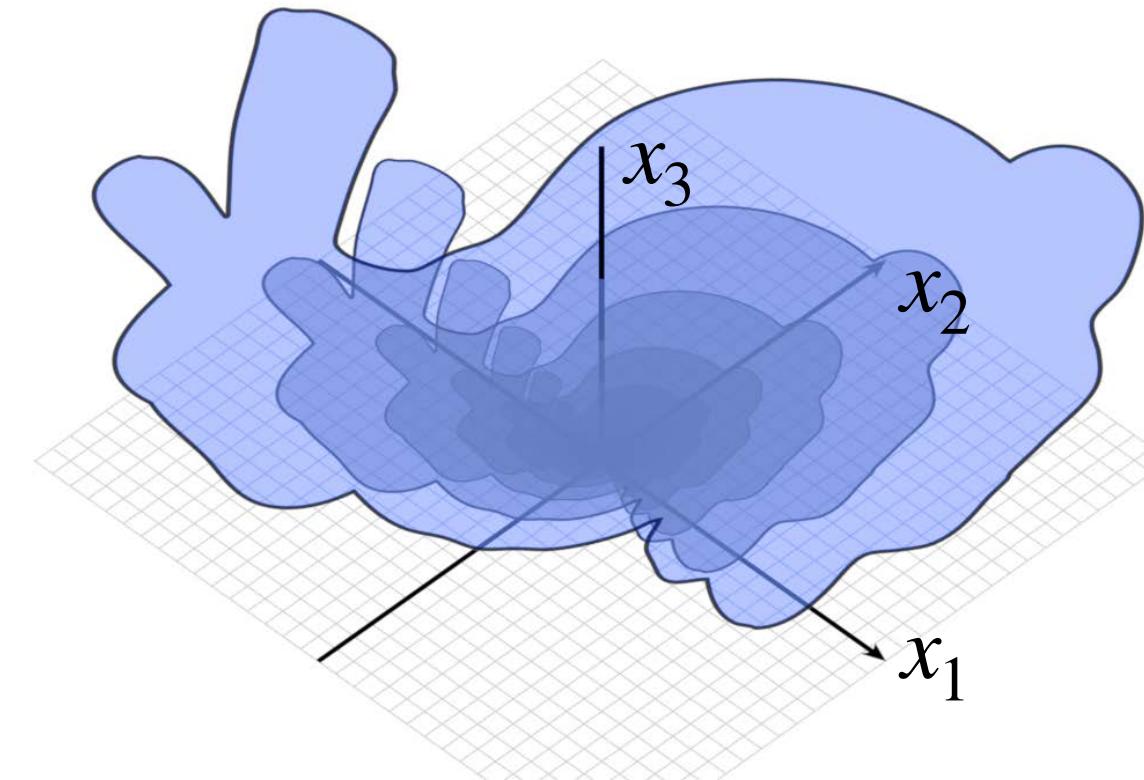
- In our case, $\mathbf{v} = (0, 0, 1)$ and so we get a matrix

$$\begin{bmatrix} 1 & 0 & u_1 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cp_1 \\ cp_2 \\ c \end{bmatrix} = \begin{bmatrix} c(p_1 + u_1) \\ c(p_2 + u_2) \\ c \end{bmatrix} \xrightarrow{1/c} \begin{bmatrix} p_1 + u_1 \\ p_2 + u_2 \\ 1 \end{bmatrix}$$

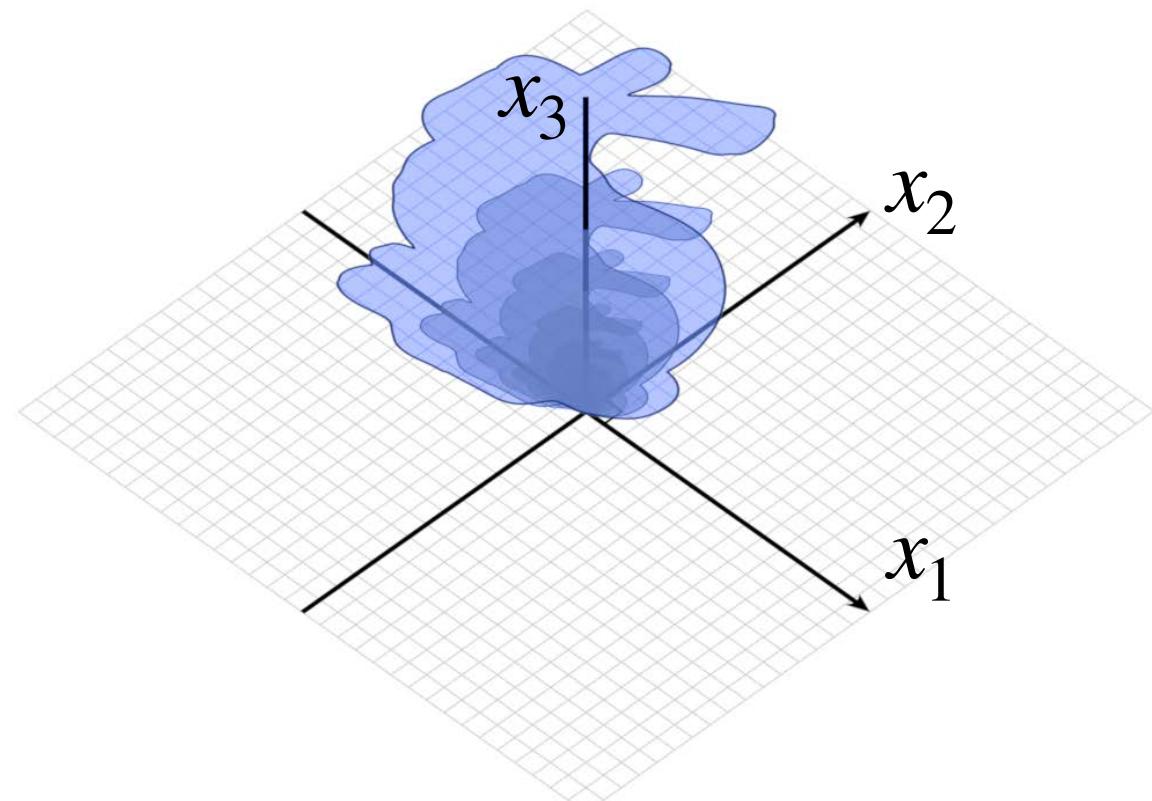
Other 2D Transformations in Homogeneous Coordinates



Original shape in 2D can be viewed as many copies, uniformly scaled by x_3

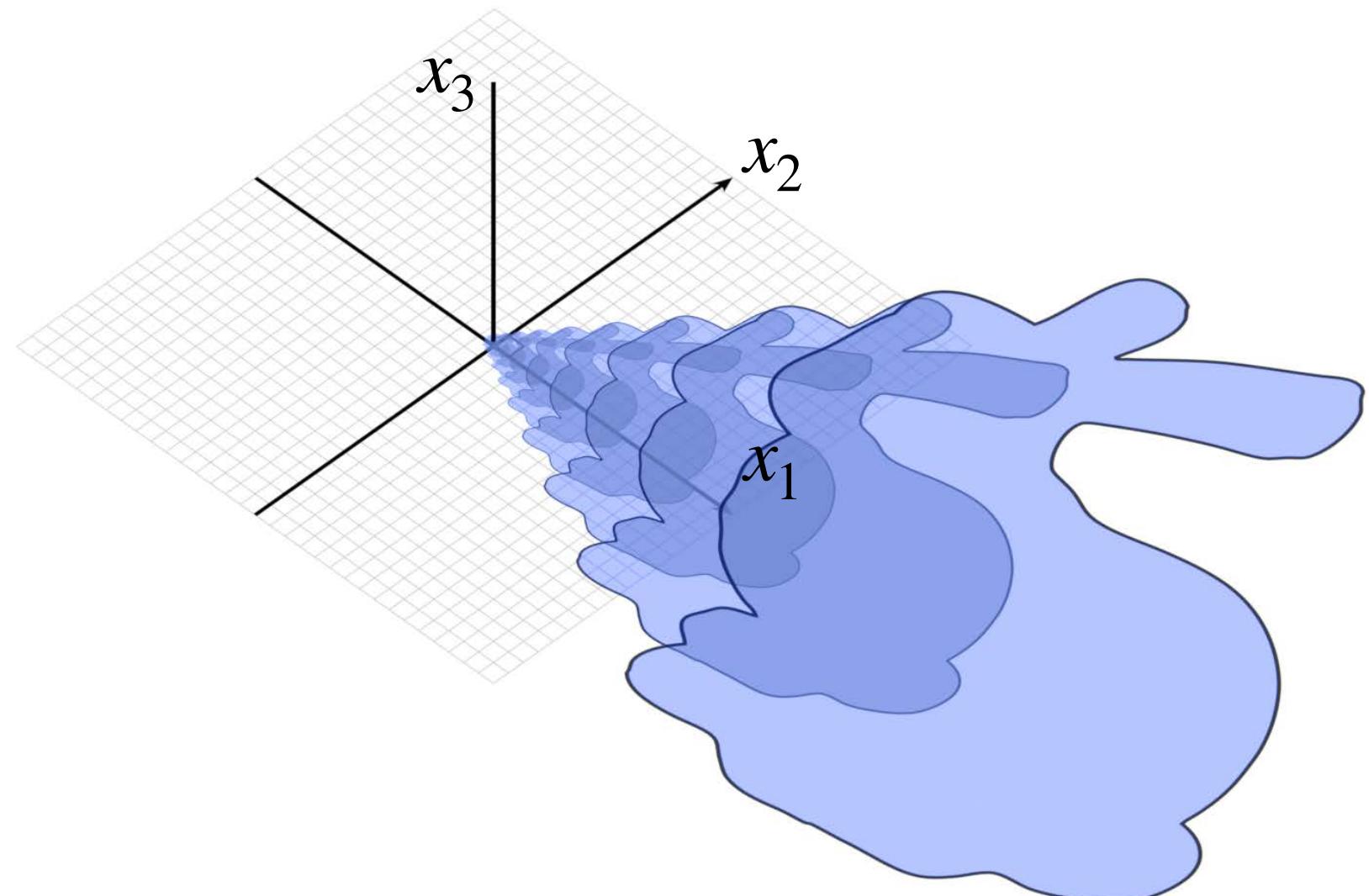


2D rotation \leftrightarrow rotate around x_3



2D scale \leftrightarrow scale x_1 and x_2 ; preserve x_3

(Q: what happens to 2D shape if you scale x_1 , x_2 , and x_3 uniformly?)



2D translate \leftrightarrow shear

Now easy to compose all these transformations

3D Transformations in Homogeneous Coordinates

- Not much changes in three (or more) dimensions: just append one “homogeneous coordinate” to the first three
- Matrix representations of 3D linear transformations just get an additional identity row/column; translation is again a shear

point in 3D

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

rotate (x, y, z) around y by θ

$$\begin{bmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

shear (x, y) by z in (s, t) direction

$$\begin{bmatrix} 1 & 0 & s & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

scale x, y, z by a, b, c

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

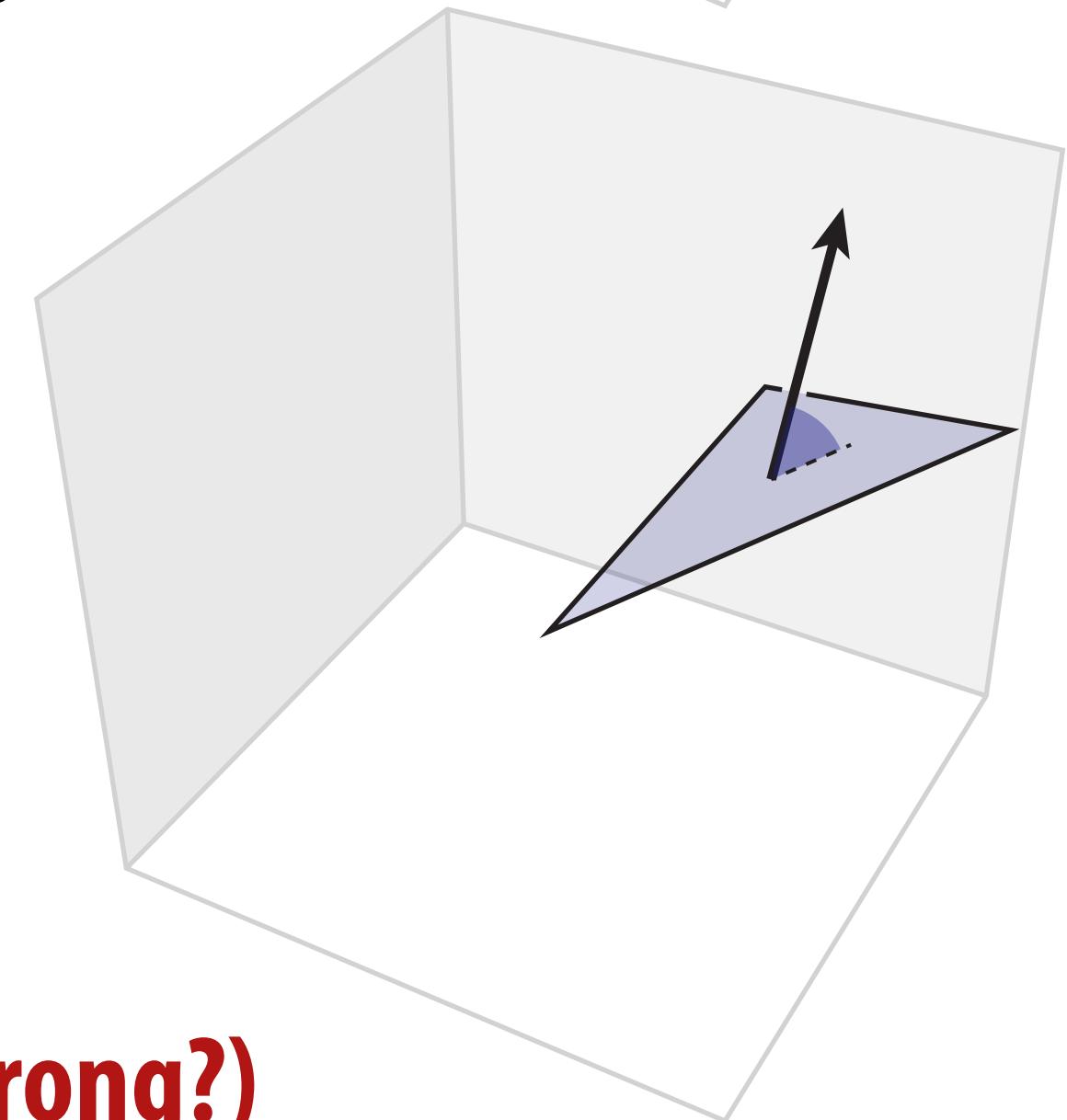
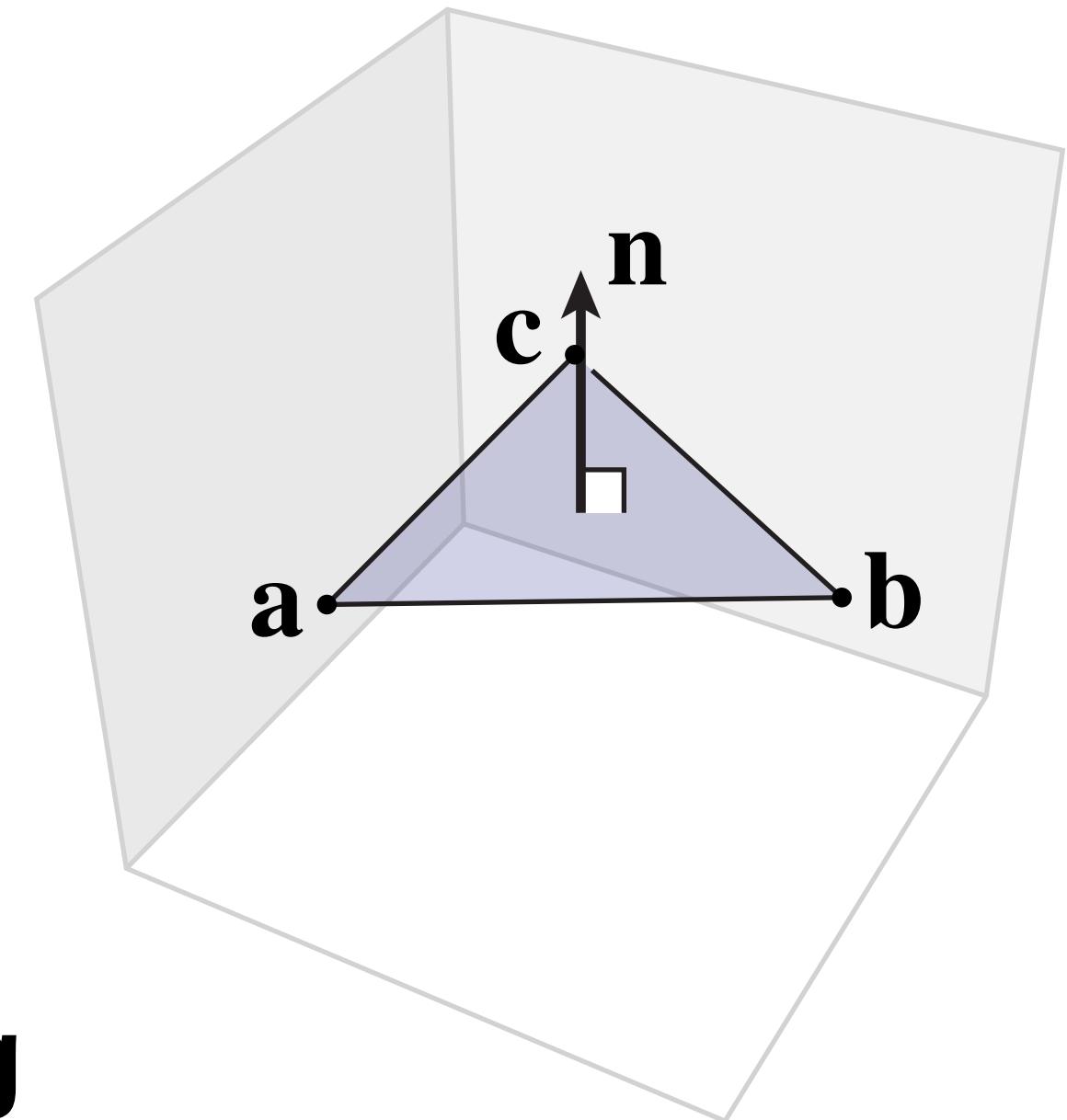
translate (x, y, z) by (u, v, w)

$$\begin{bmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Points vs. Vectors

- Homogeneous coordinates have another useful feature: distinguish between points and vectors
- Consider for instance a triangle with:
 - vertices $a, b, c \in \mathbb{R}^3$
 - normal vector $n \in \mathbb{R}^3$
- Suppose we transform the triangle by appending "1" to a, b, c, n and multiplying by this matrix:

$$\begin{bmatrix} \cos \theta & 0 & \sin \theta & u \\ 0 & 1 & 0 & v \\ -\sin \theta & 0 & \cos \theta & w \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Normal is not orthogonal to triangle! (What went wrong?)

Points vs. Vectors (continued)

- Let's think about what happens when we multiply the normal vector \mathbf{n} by our matrix:

rotate normal around y by θ

$$\begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ 1 \end{bmatrix} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ 1 \end{bmatrix}$$

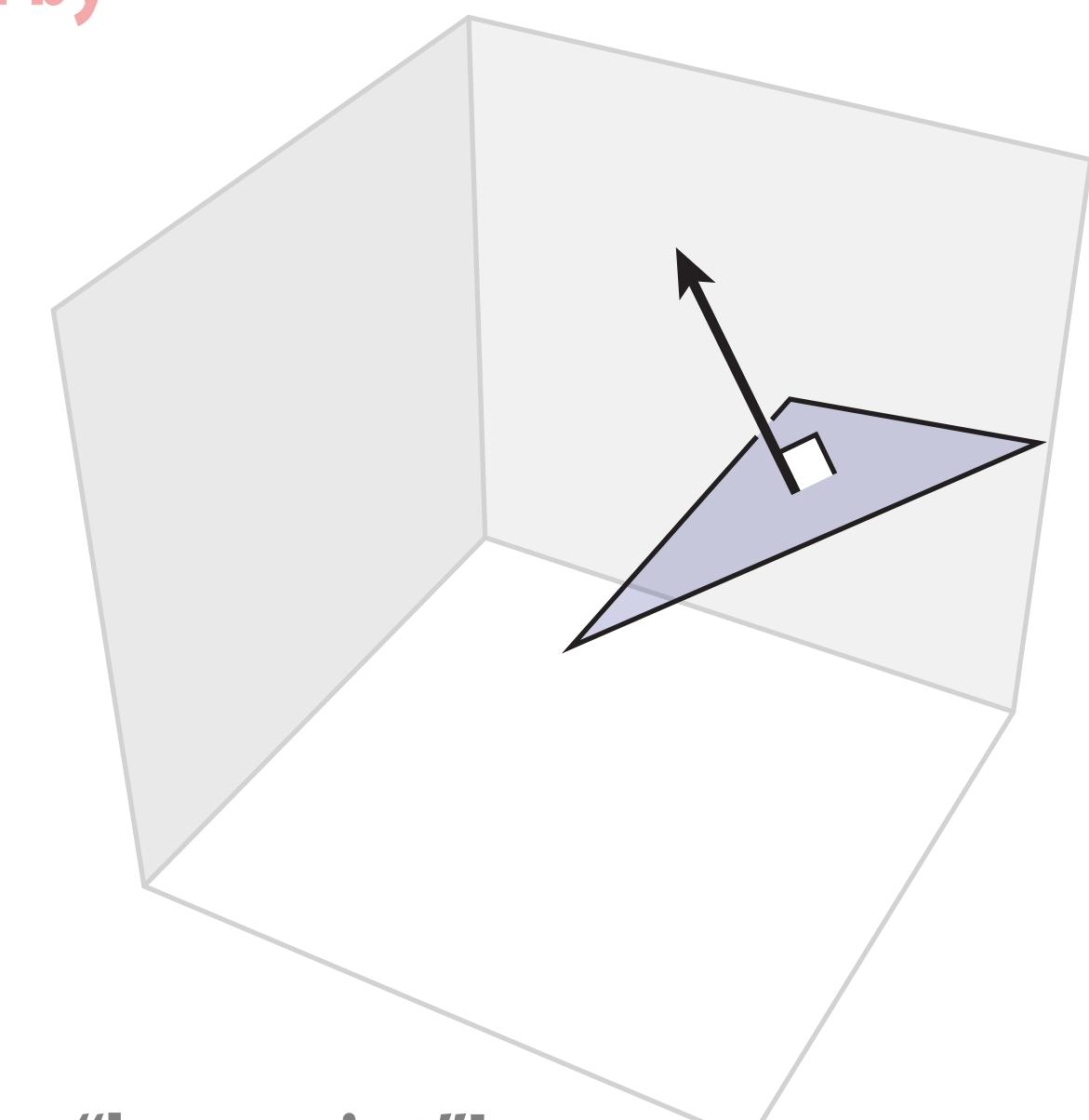
- But when we rotate/translate a triangle, its normal should just rotate!*

translate normal by (u, v, w)

- Solution? Just set homogeneous coordinate to zero!

$$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ 0 \end{bmatrix}$$

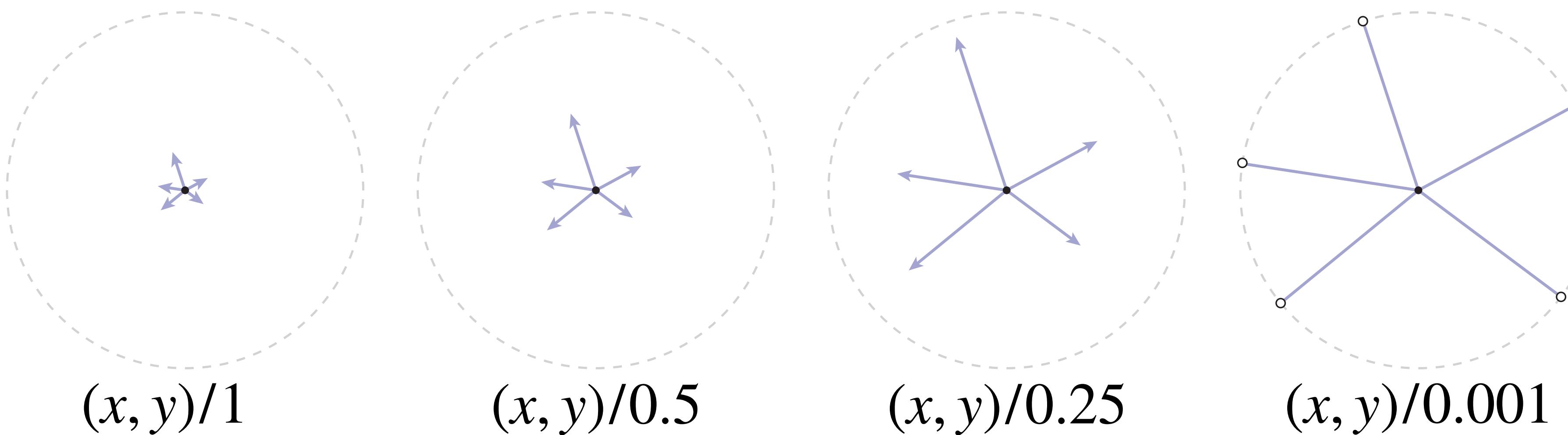
- Translation now gets ignored; normal is orthogonal to triangle



*Recall that vectors just have direction and magnitude—they don't have a “basepoint”!

Points vs. Vectors in Homogeneous Coordinates

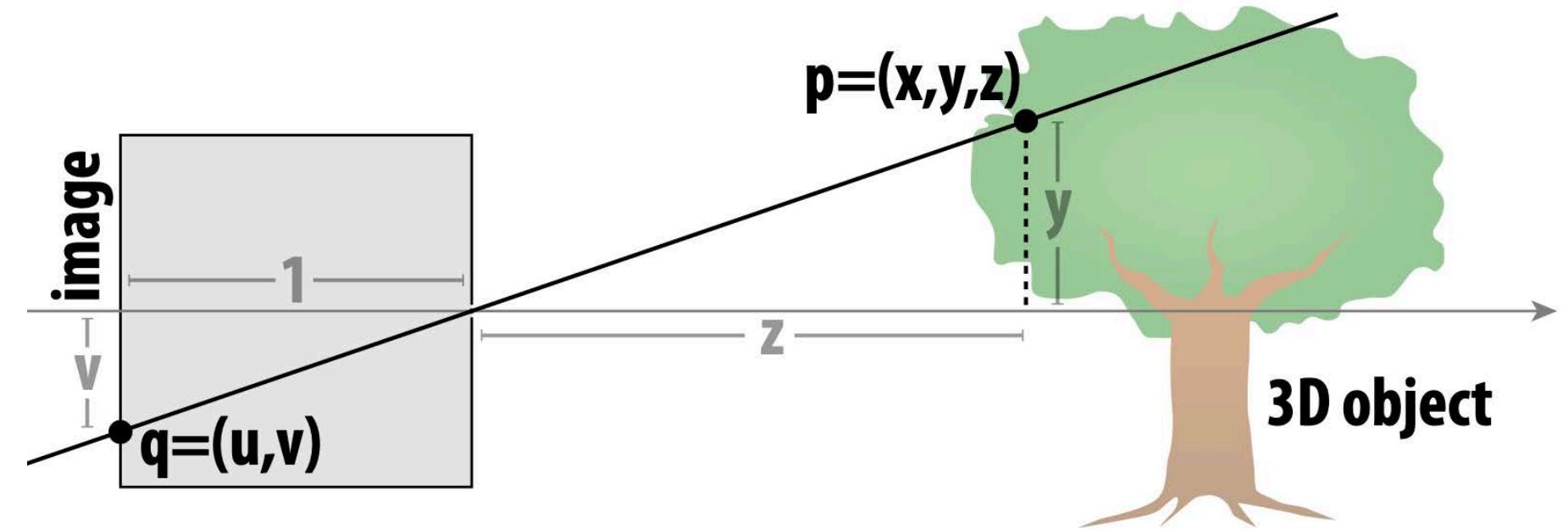
- In general:
 - A *point* has a nonzero homogeneous coordinate ($c = 1$)
 - A *vector* has a zero homogeneous coordinate ($c = 0$)
- But wait... what division by c mean when it's equal to zero?
- Well consider what happens as $c \rightarrow 0...$



Can think of vectors as “points at infinity” (sometimes called “ideal points”)
(In practice: still need to check for divide by zero!)

Perspective Projection in Homogeneous Coordinates

- Q: How can we perform perspective projection* using homogeneous coordinates?
- Remember from our pinhole camera model that the basic idea was to “divide by z ”
- So, we can build a matrix that “copies” the z coordinate into the homogeneous coordinate
- Division by the homogeneous coordinate now gives us perspective projection onto the plane $z = 1$



$$(x, y, z) \mapsto (x/z, y/z)$$

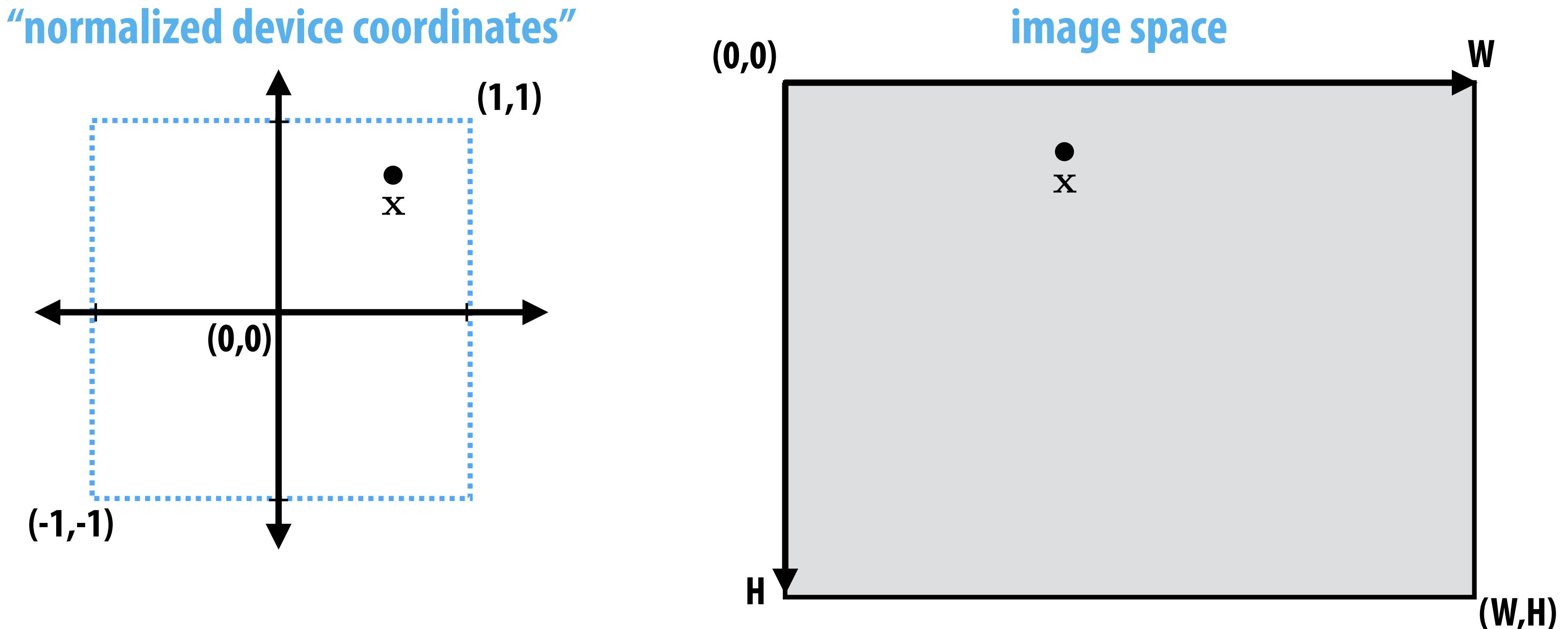
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ z \end{bmatrix}$$

$$\implies \begin{bmatrix} x/z \\ y/z \\ 1 \end{bmatrix}$$

*Assuming a pinhole camera at $(0,0,0)$ looking down the z -axis

Screen Transformation

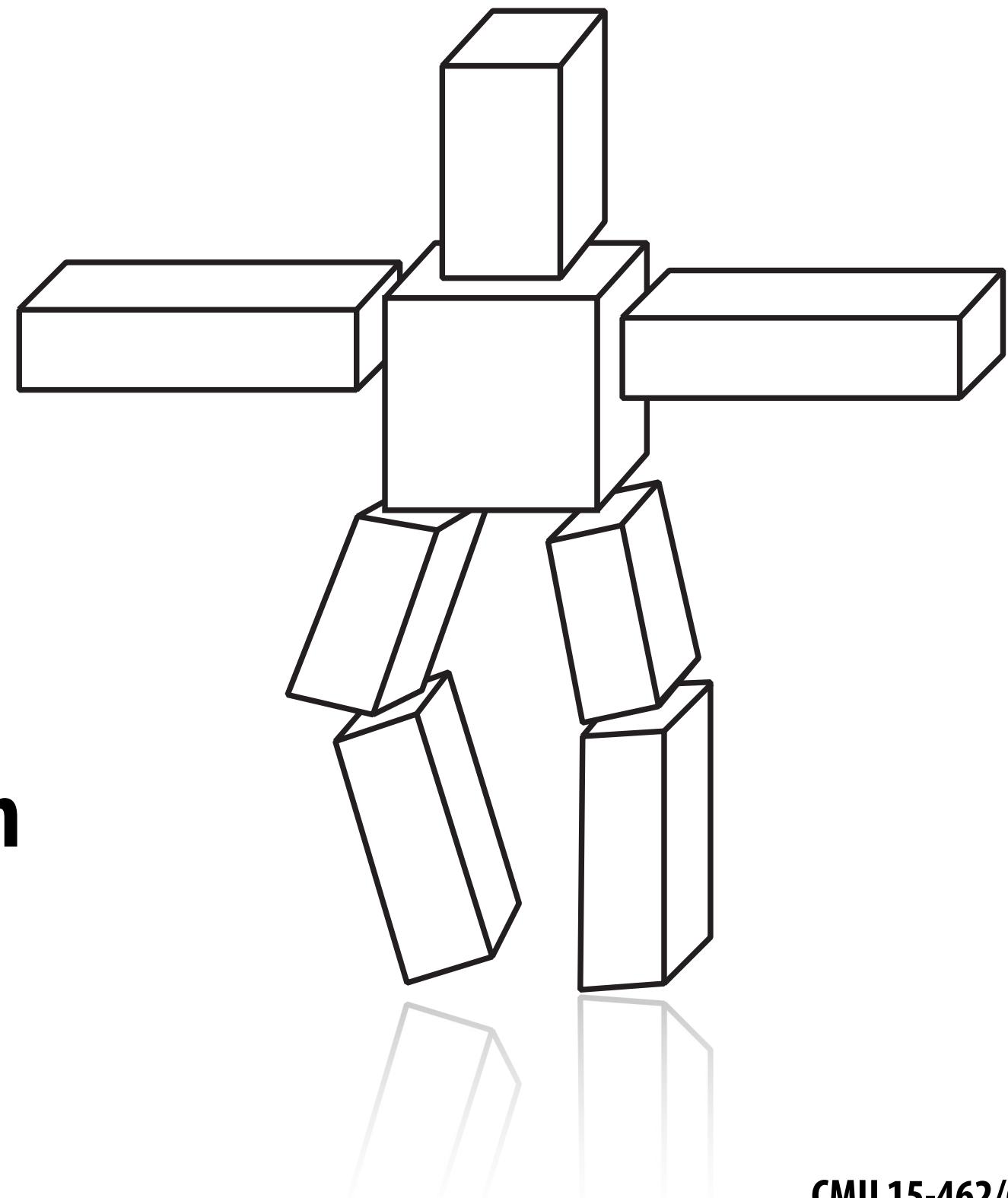
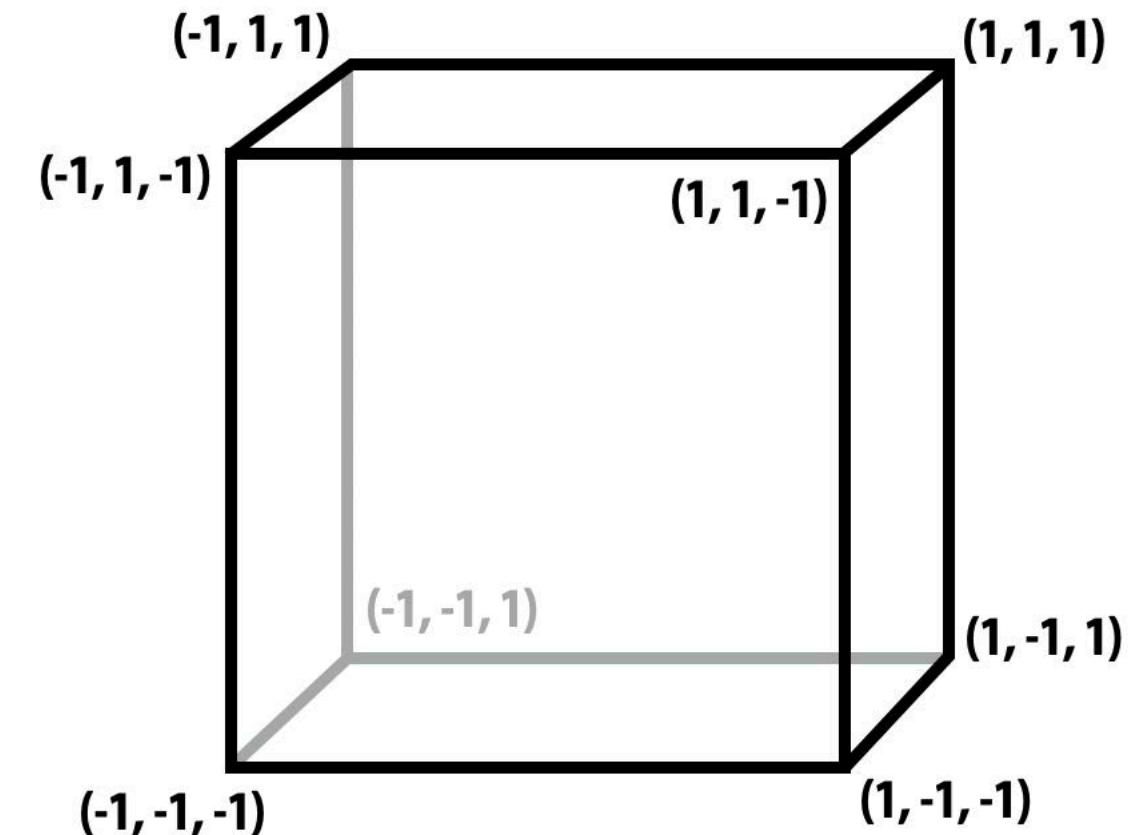
- One last transformation is needed in the rasterization pipeline: transform from viewing plane to pixel coordinates
- E.g., suppose we want to draw all points that fall inside the square $[0,1] \times [0,1]$ on the $z = 1$ plane, into a $W \times H$ pixel image



Q: What transformation(s) would you apply? (Careful: y is now down!)

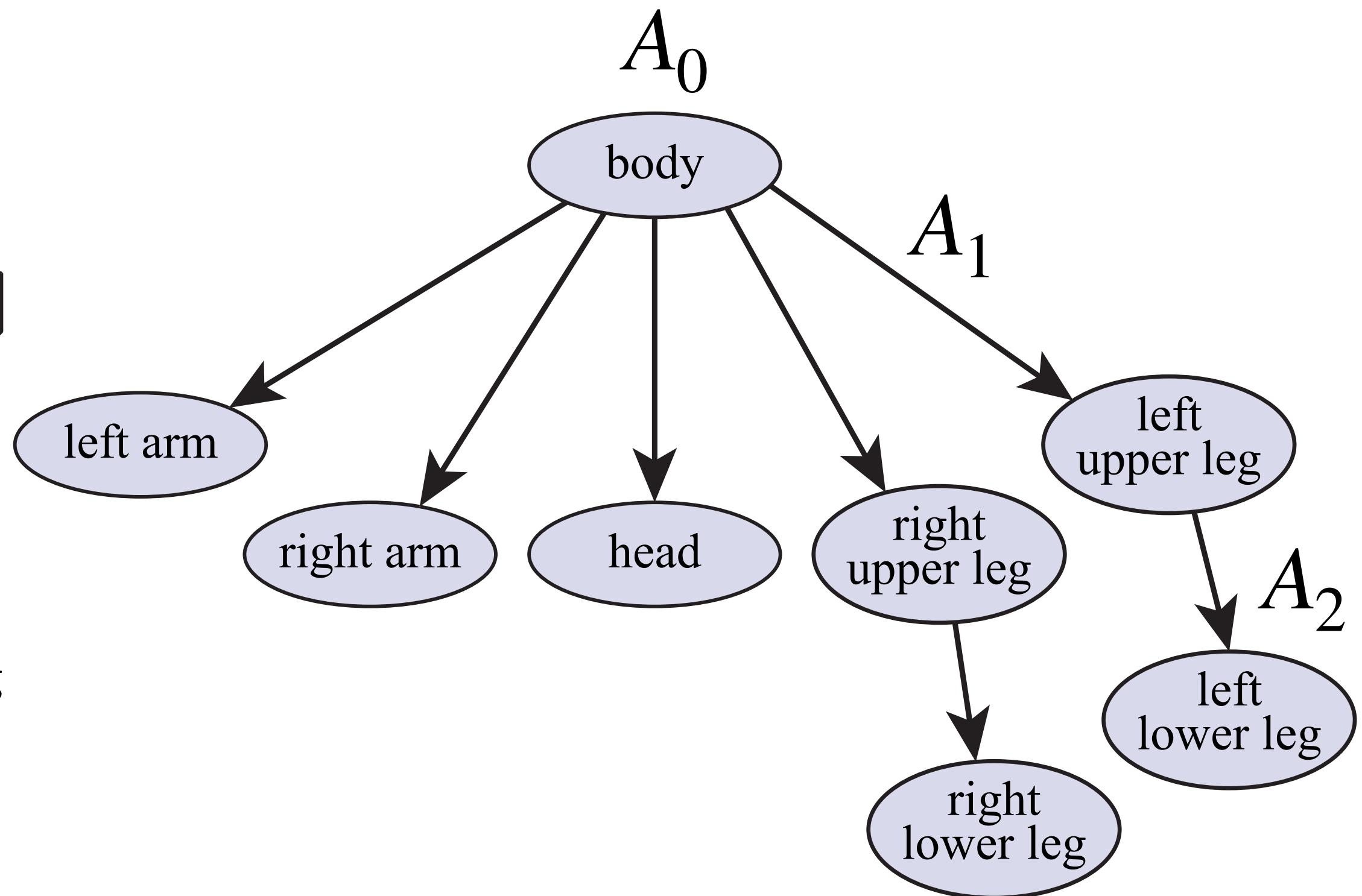
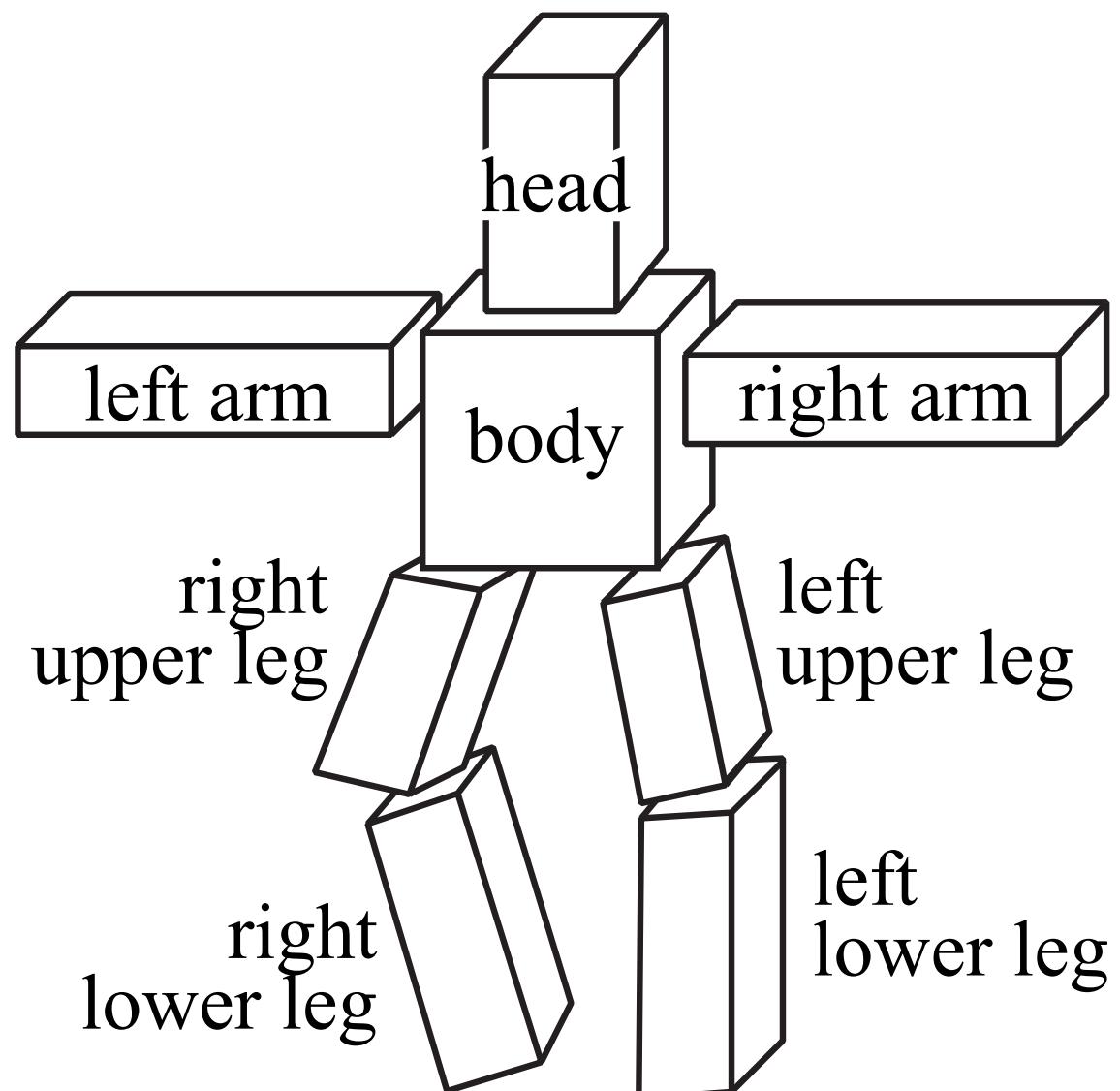
Scene Graph

- For complex scenes (e.g., more than just a cube!) scene graph can help organize transformations
- Motivation: suppose we want to build a “cube creature” by transforming copies of the unit cube
- Difficult to specify each transformation directly
- Instead, build up transformations of “lower” parts from transformations of “upper” parts
 - E.g., first position the body
 - Then transform upper arm relative to the body
 - Then transform lower arm relative to upper arm
 - ...



Scene Graph (continued)

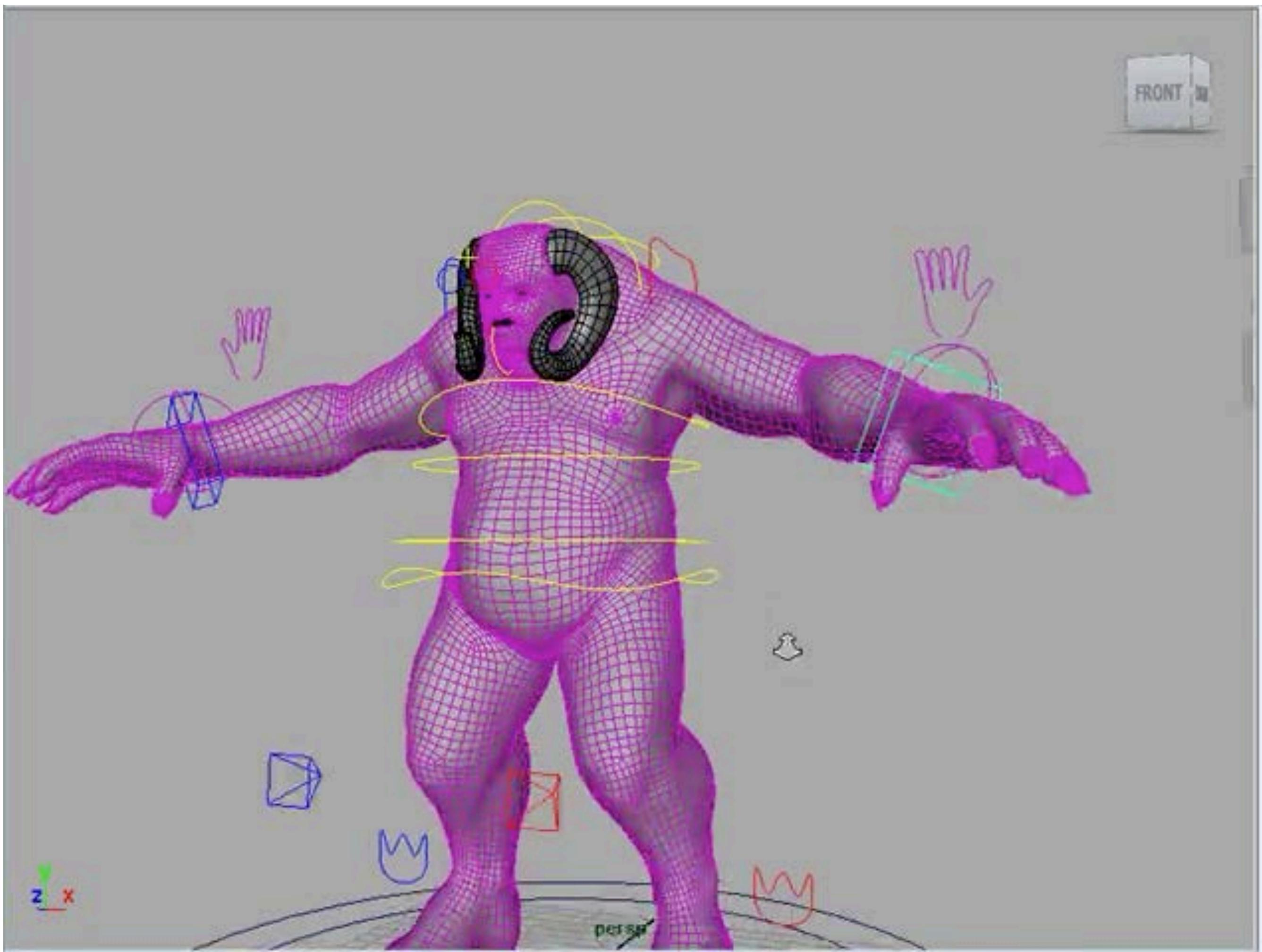
- Scene graph stores relative transformations in directed graph
- Each edge (+root) stores a linear transformation (e.g., a 4x4 matrix)
- Composition of transformations gets applied to nodes



- E.g., A_1A_0 gets applied to left upper leg; $A_2A_1A_0$ to left lower leg
- Keep transformations on a stack to reduce redundant multiplication

Scene Graph—Example

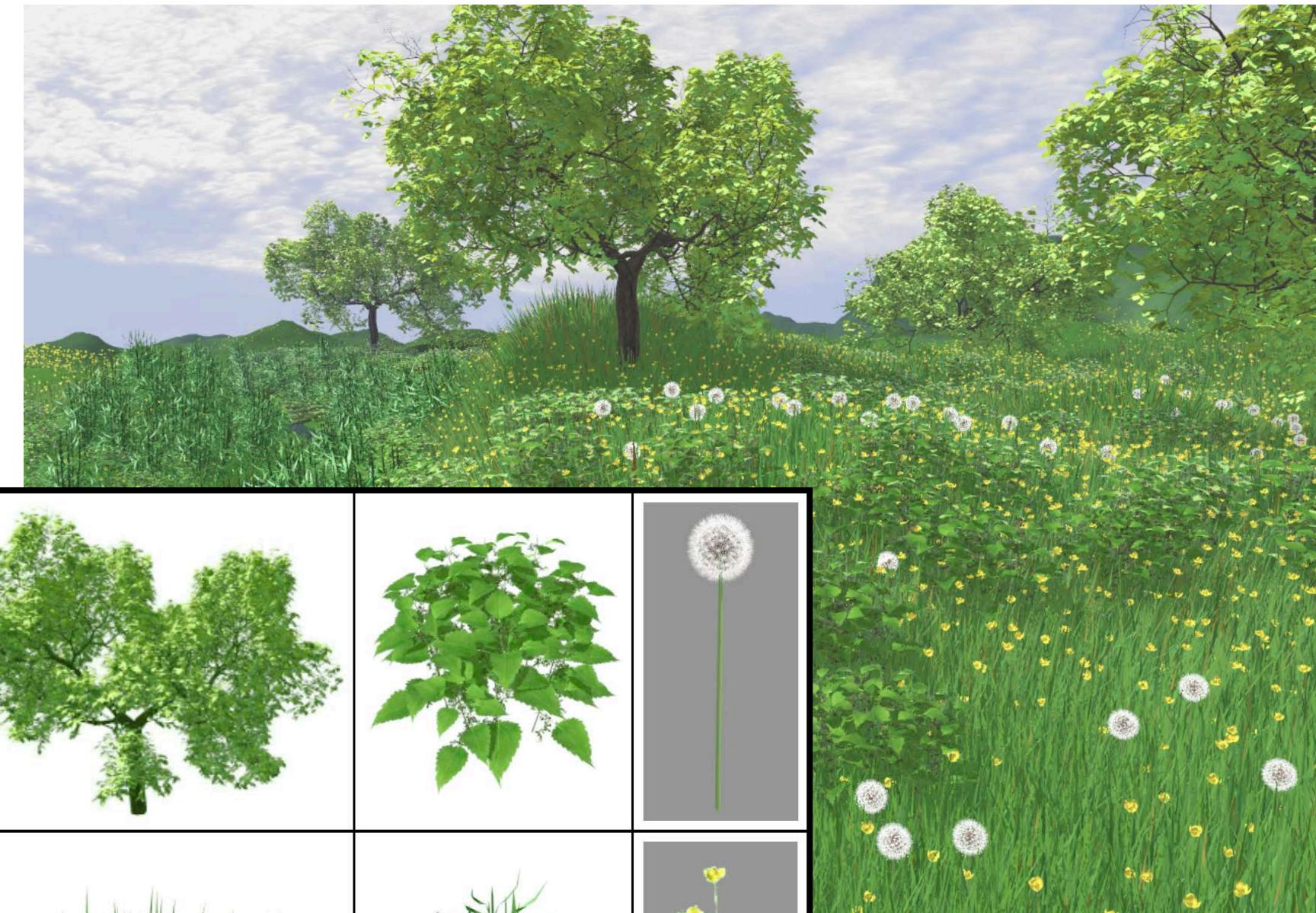
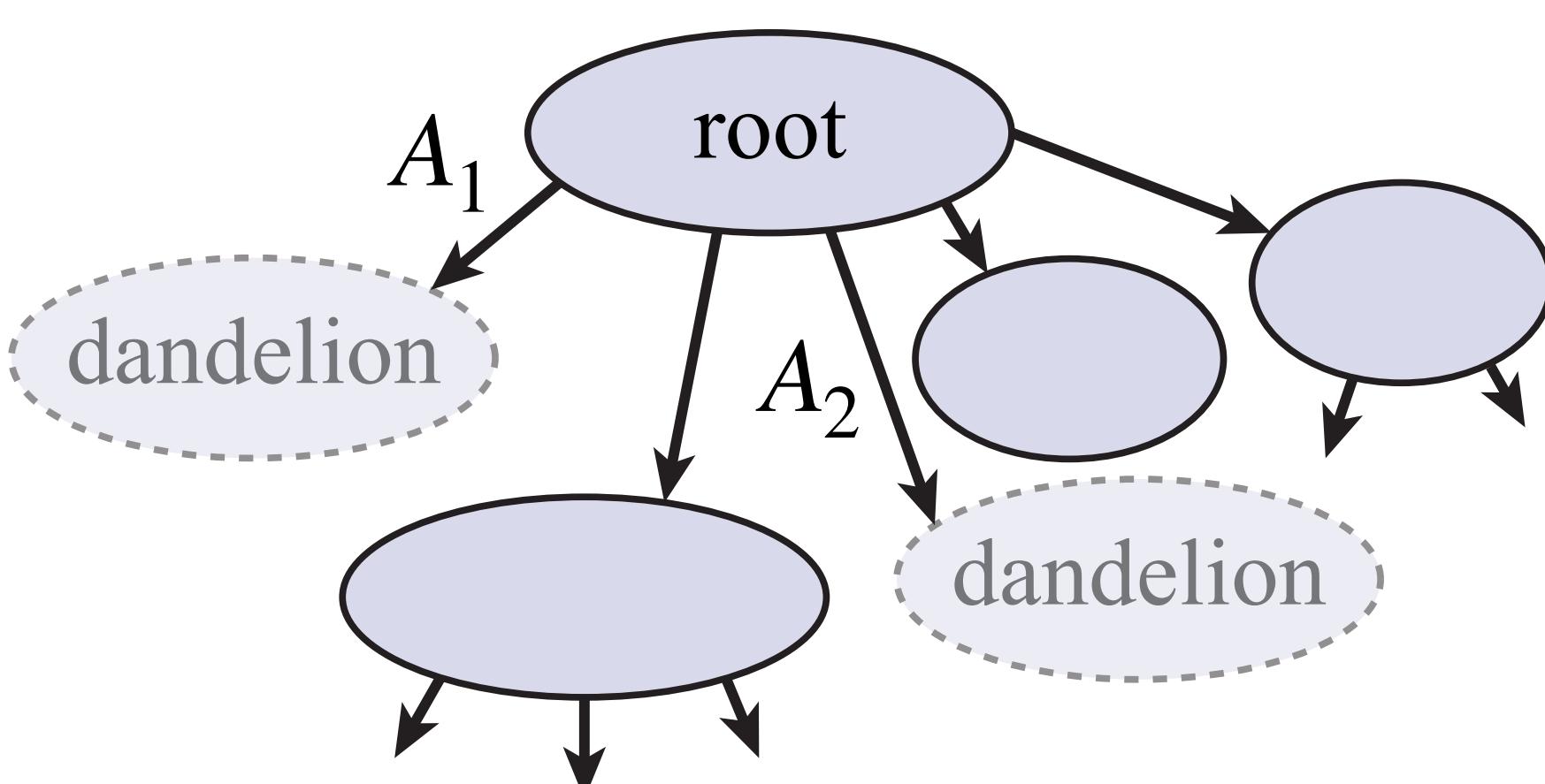
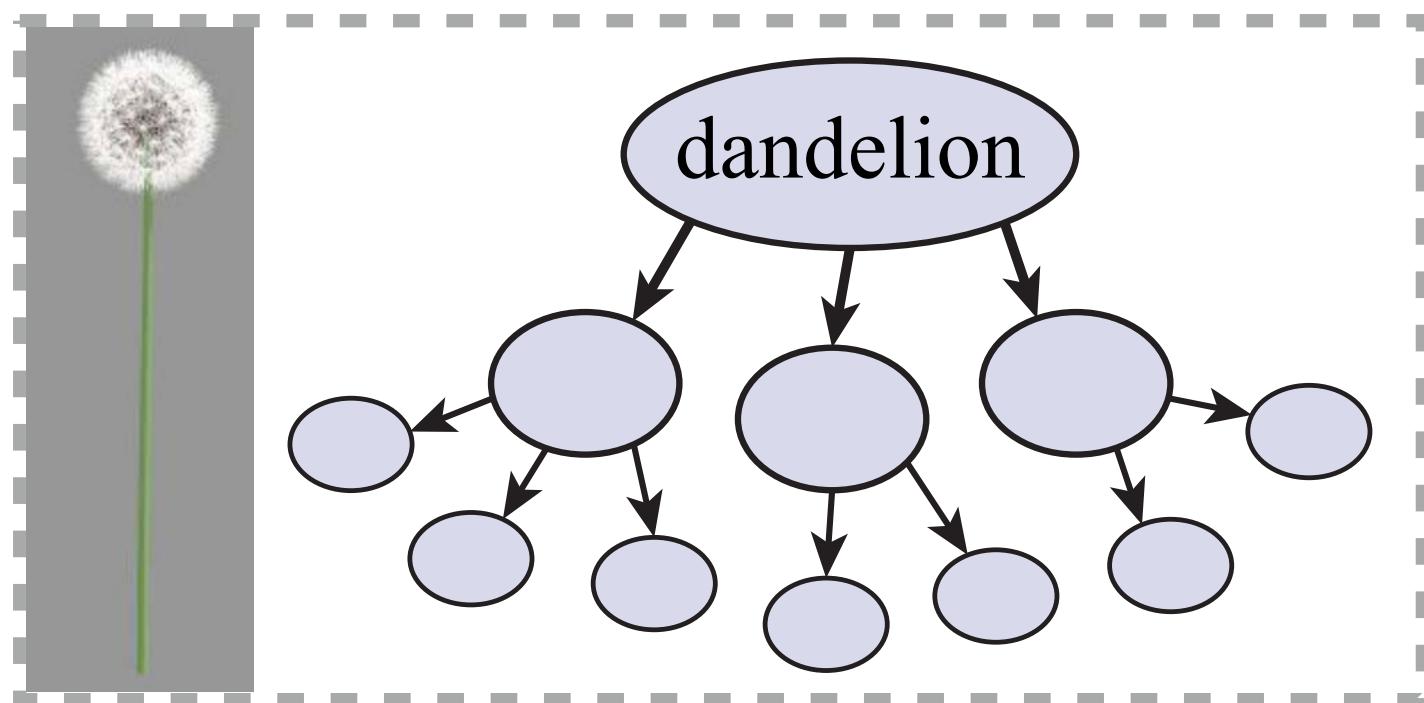
Often used to build up complex “rig”:



In general, scene graph also includes other models, lights, cameras, ...

Instancing

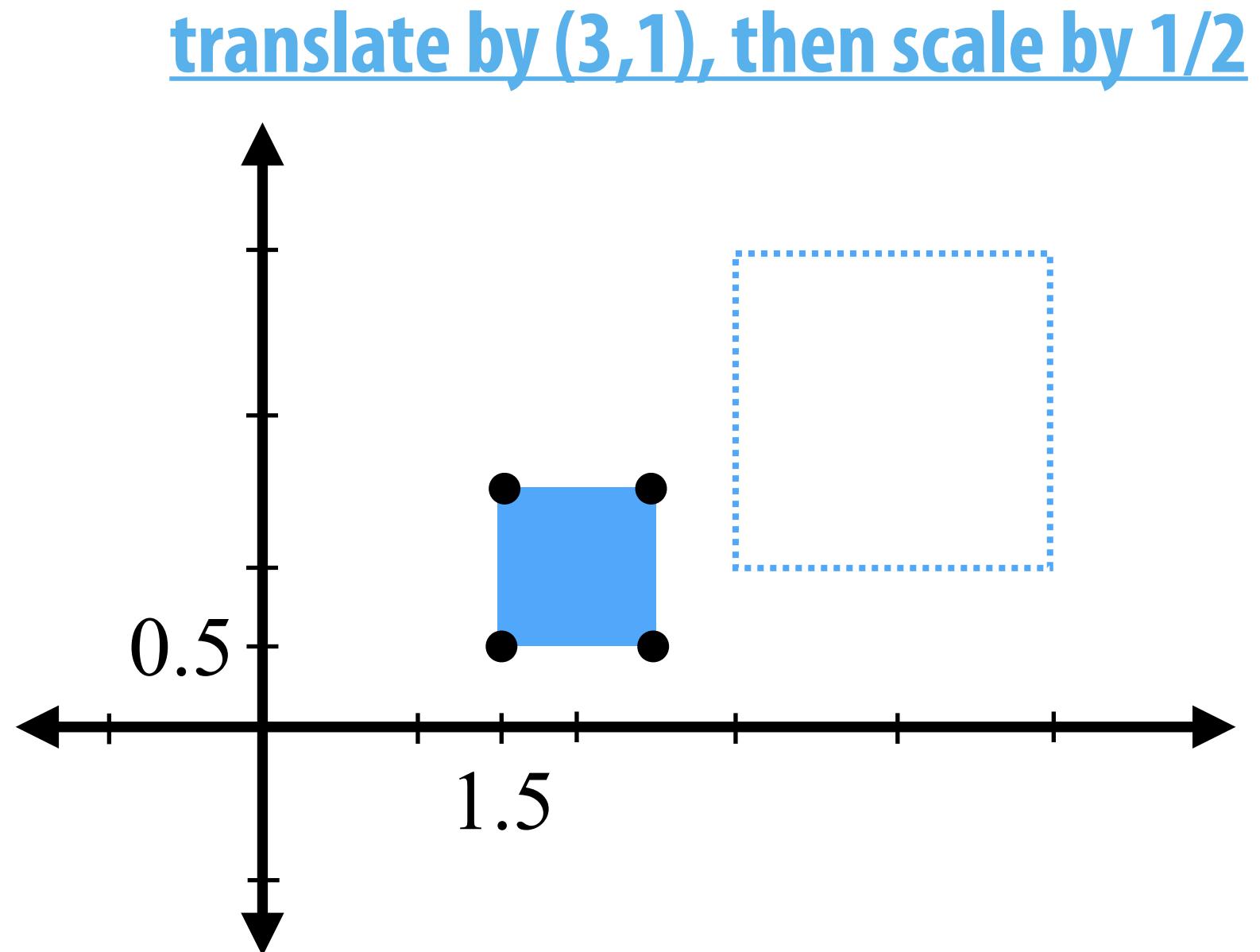
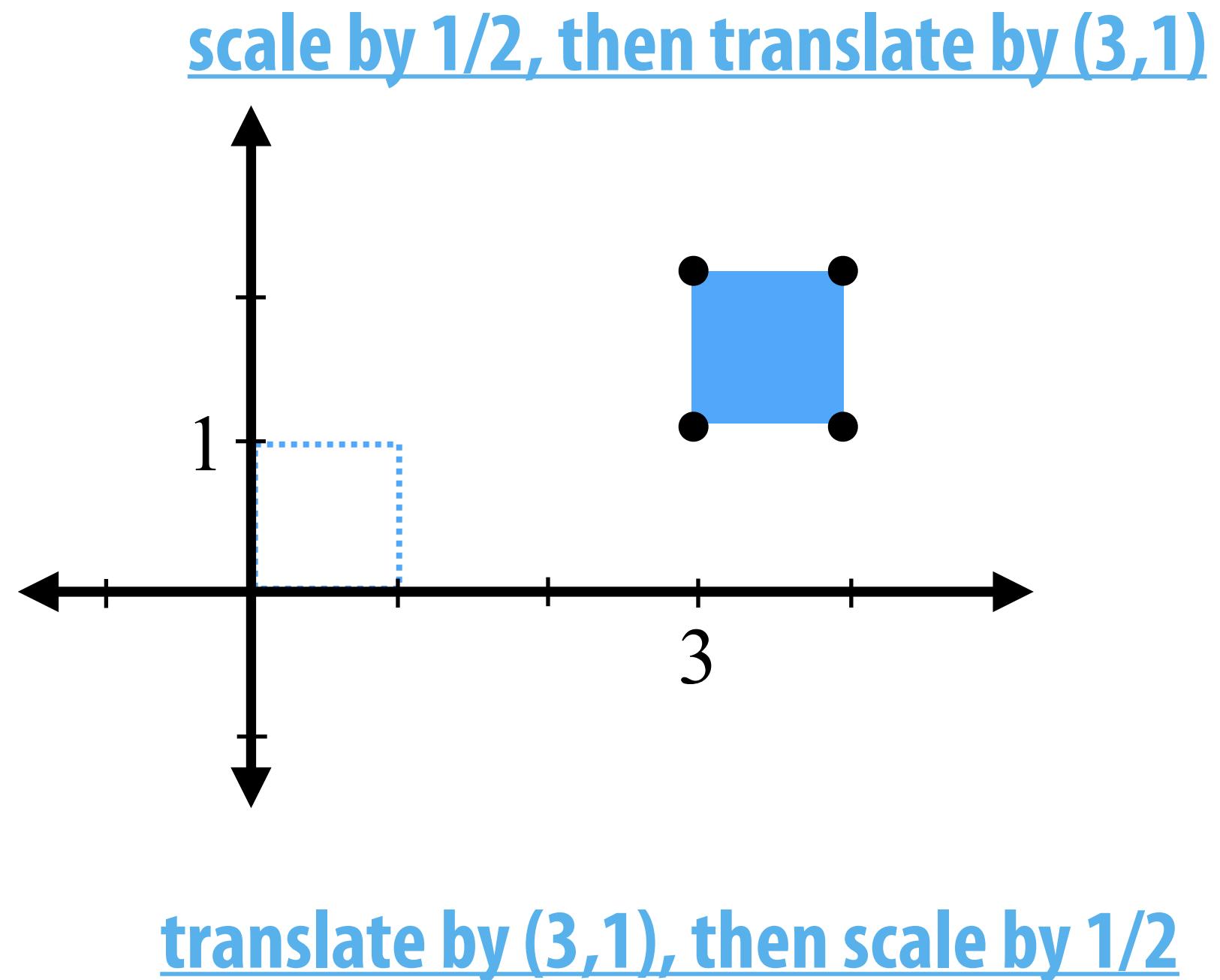
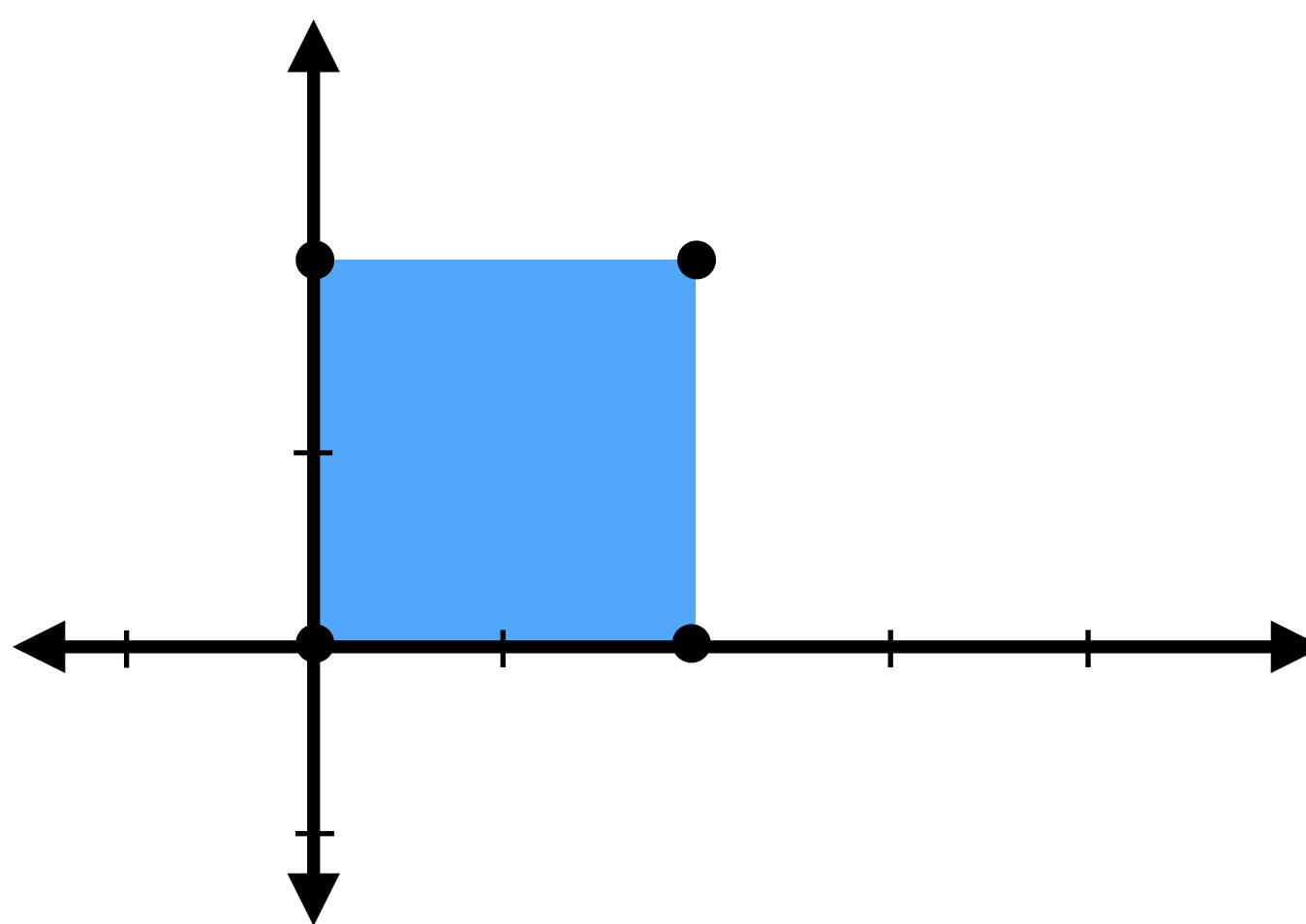
- What if we want many copies of the same object in a scene?
- Rather than have many copies of the geometry, scene graph, etc., can just put a “pointer” node in our scene graph
- Like any other node, can specify a different transformation on each incoming edge



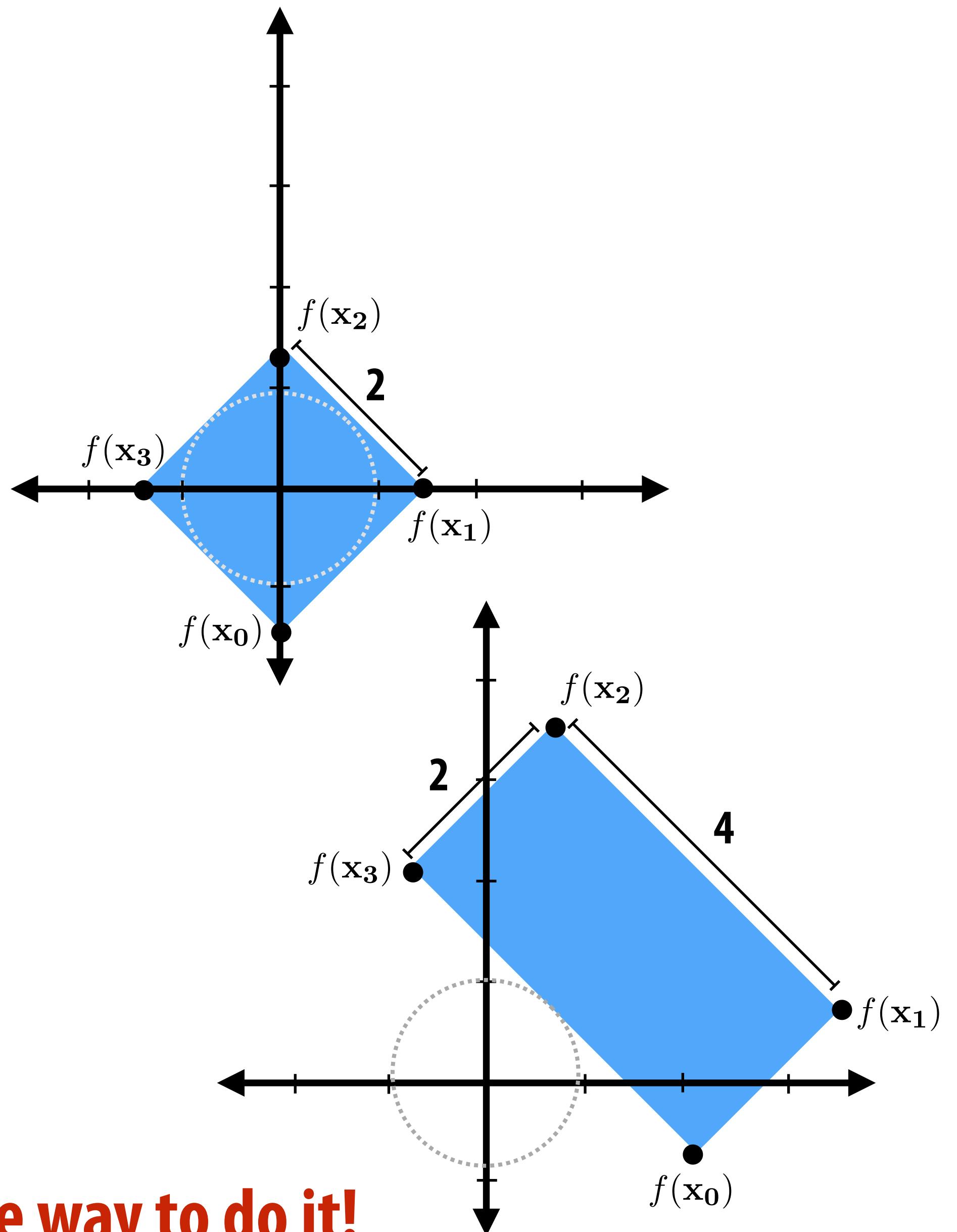
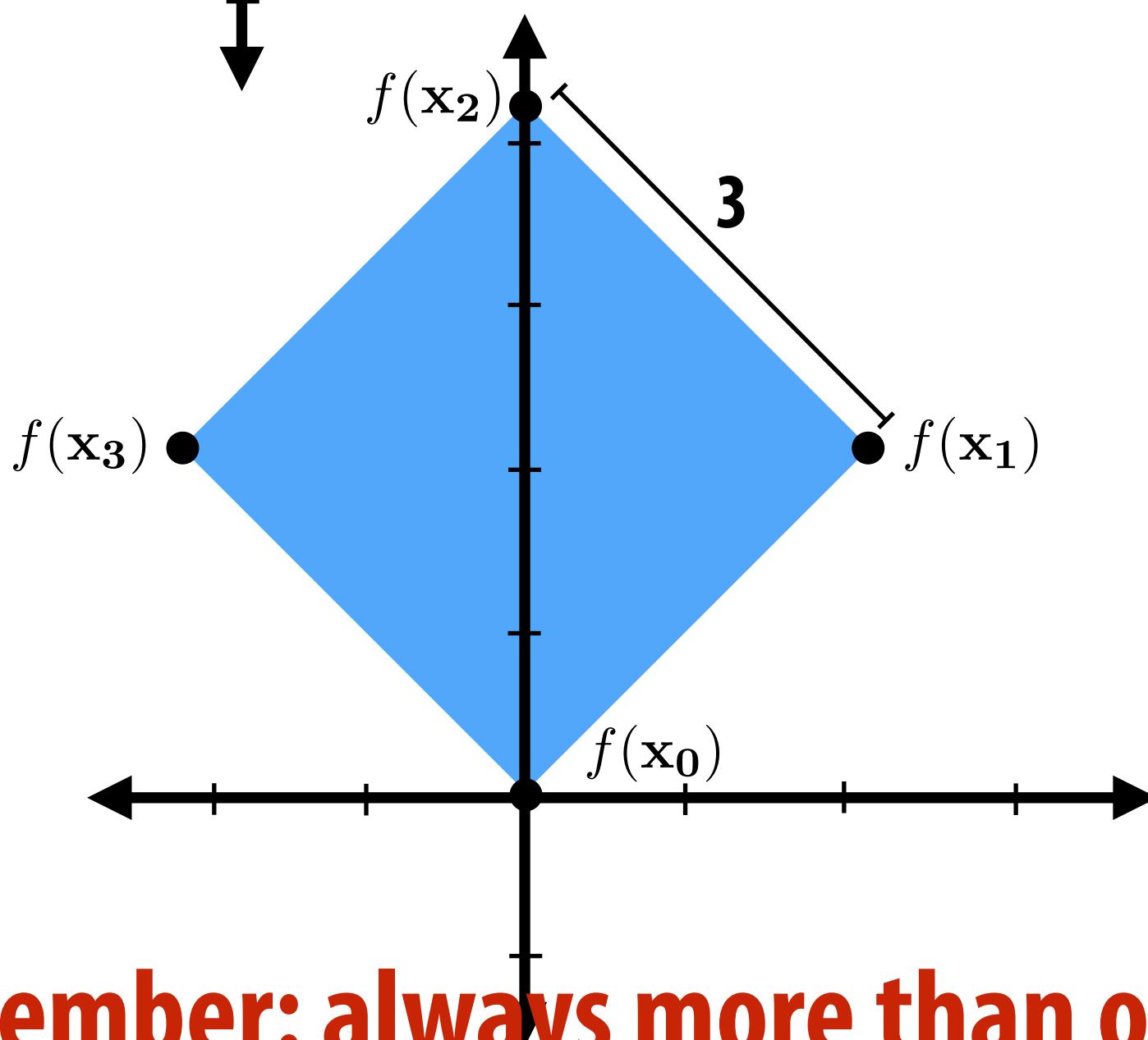
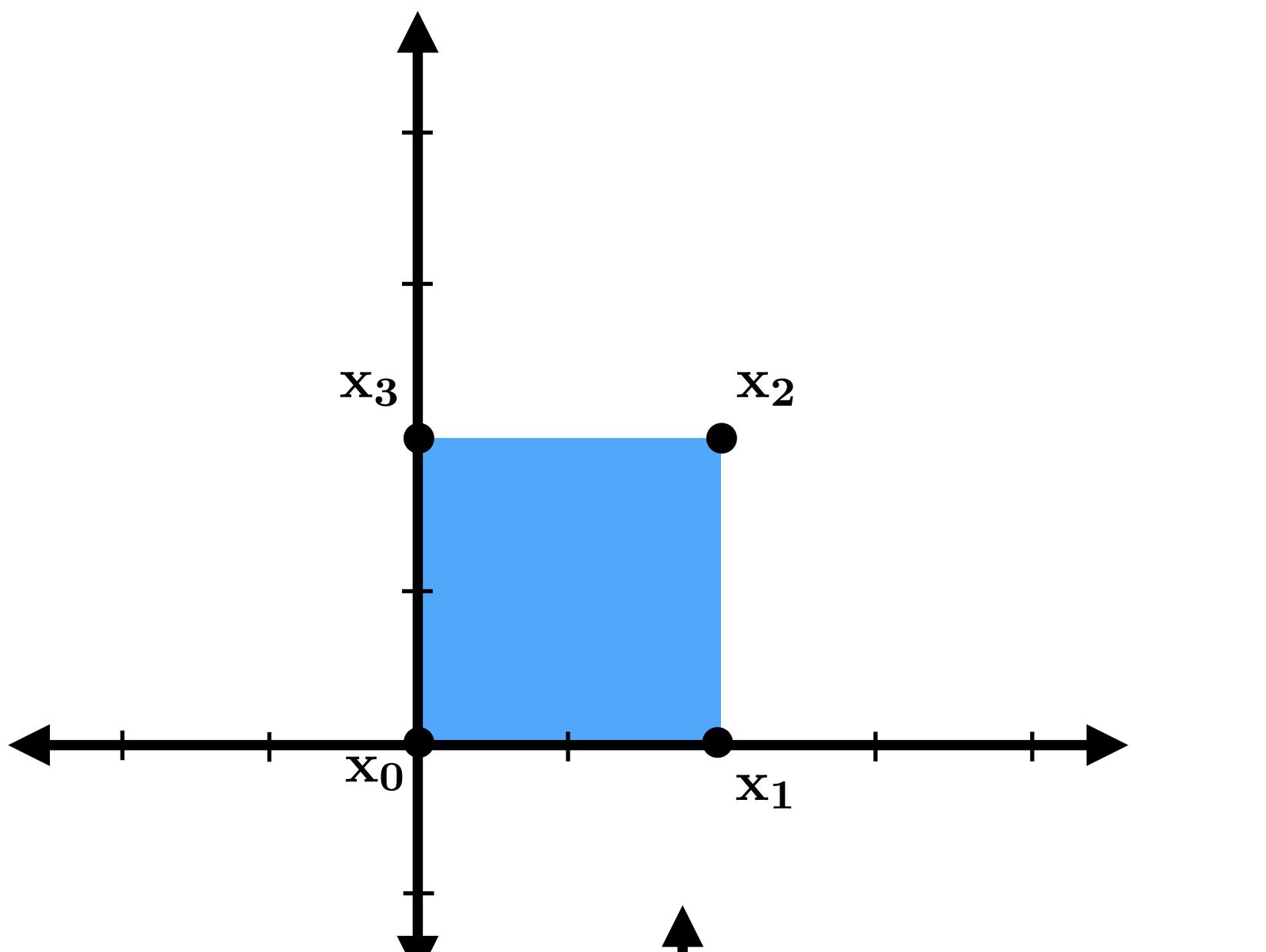
Instancing—Example



Order matters when composing transformations!

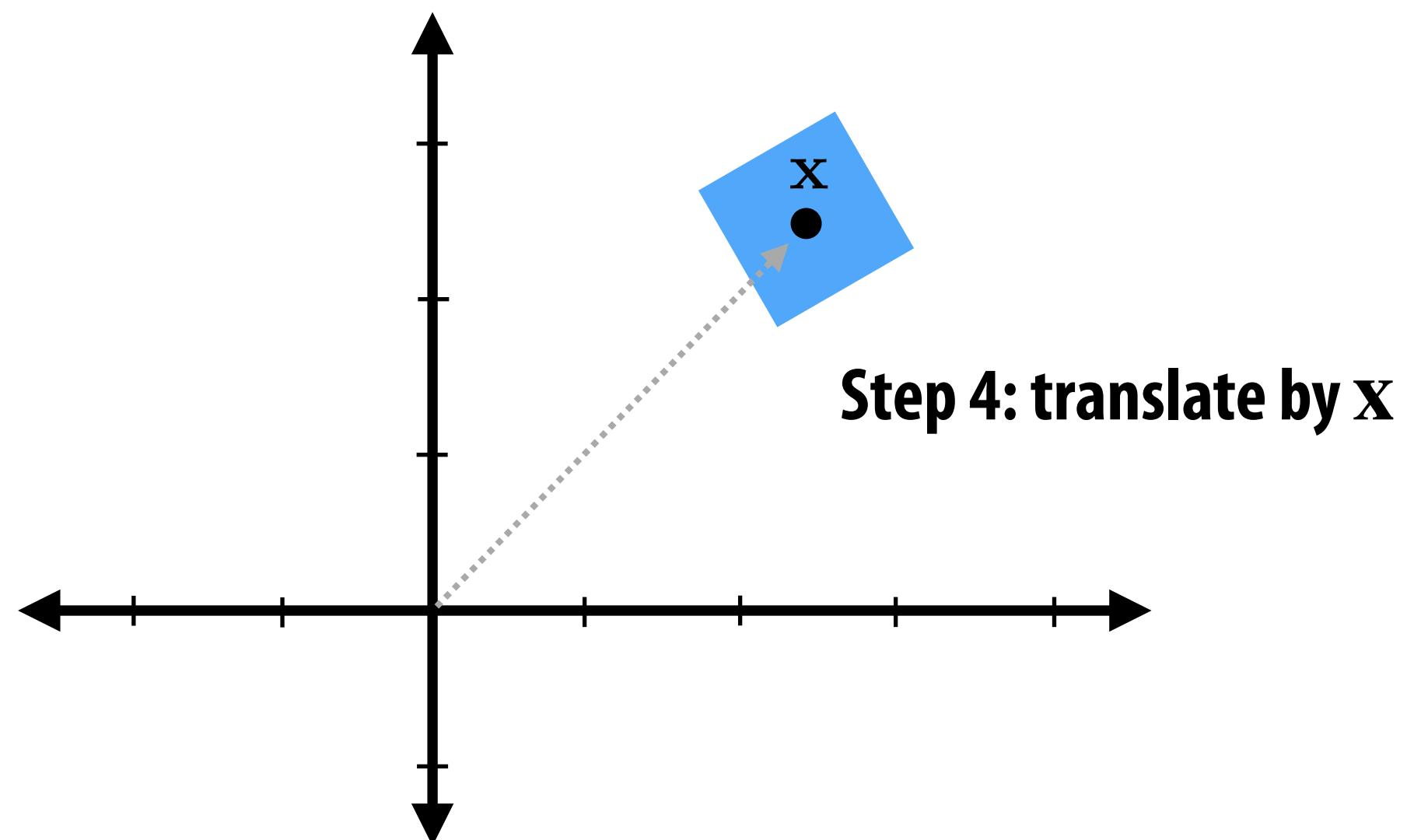
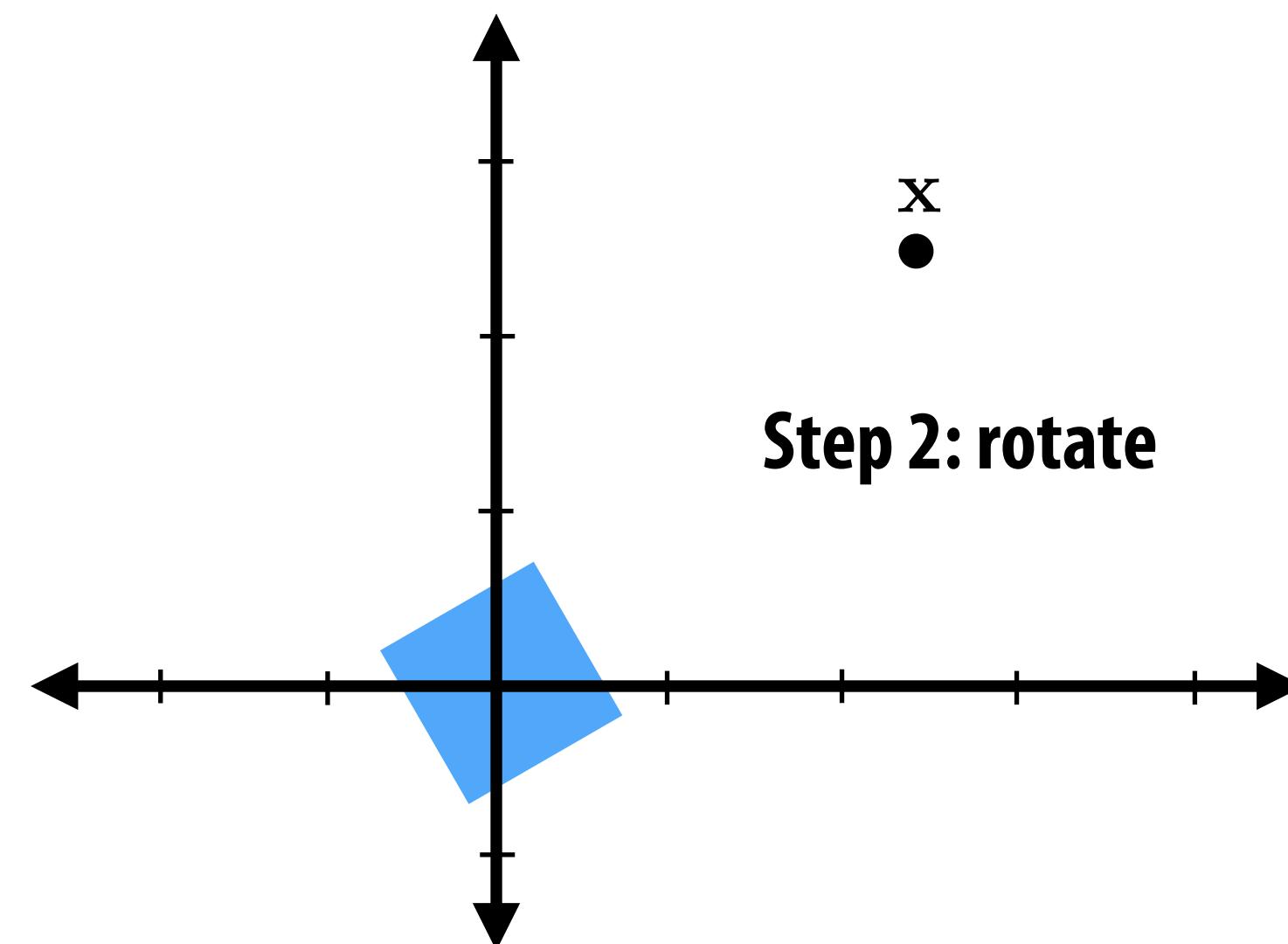
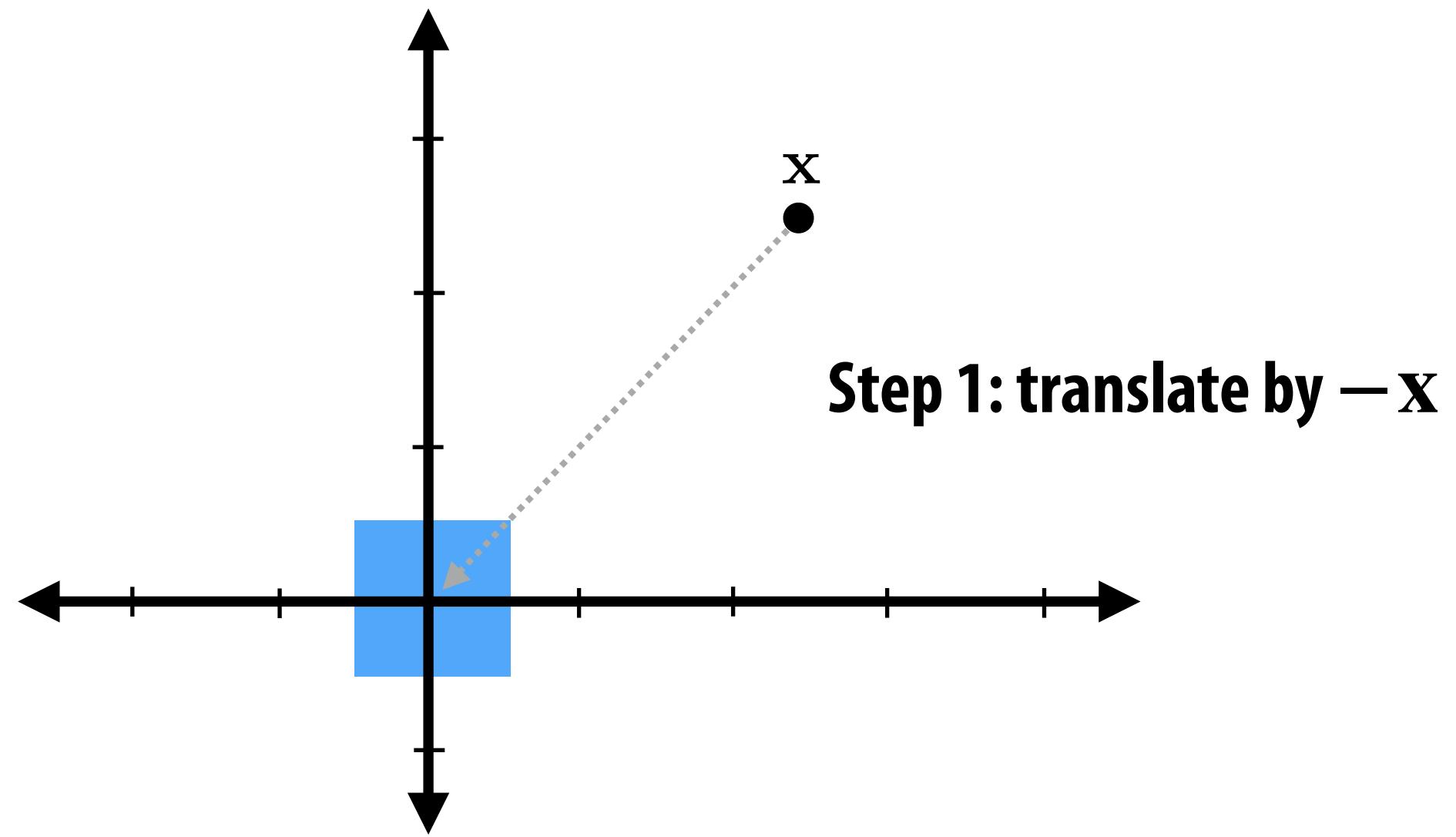
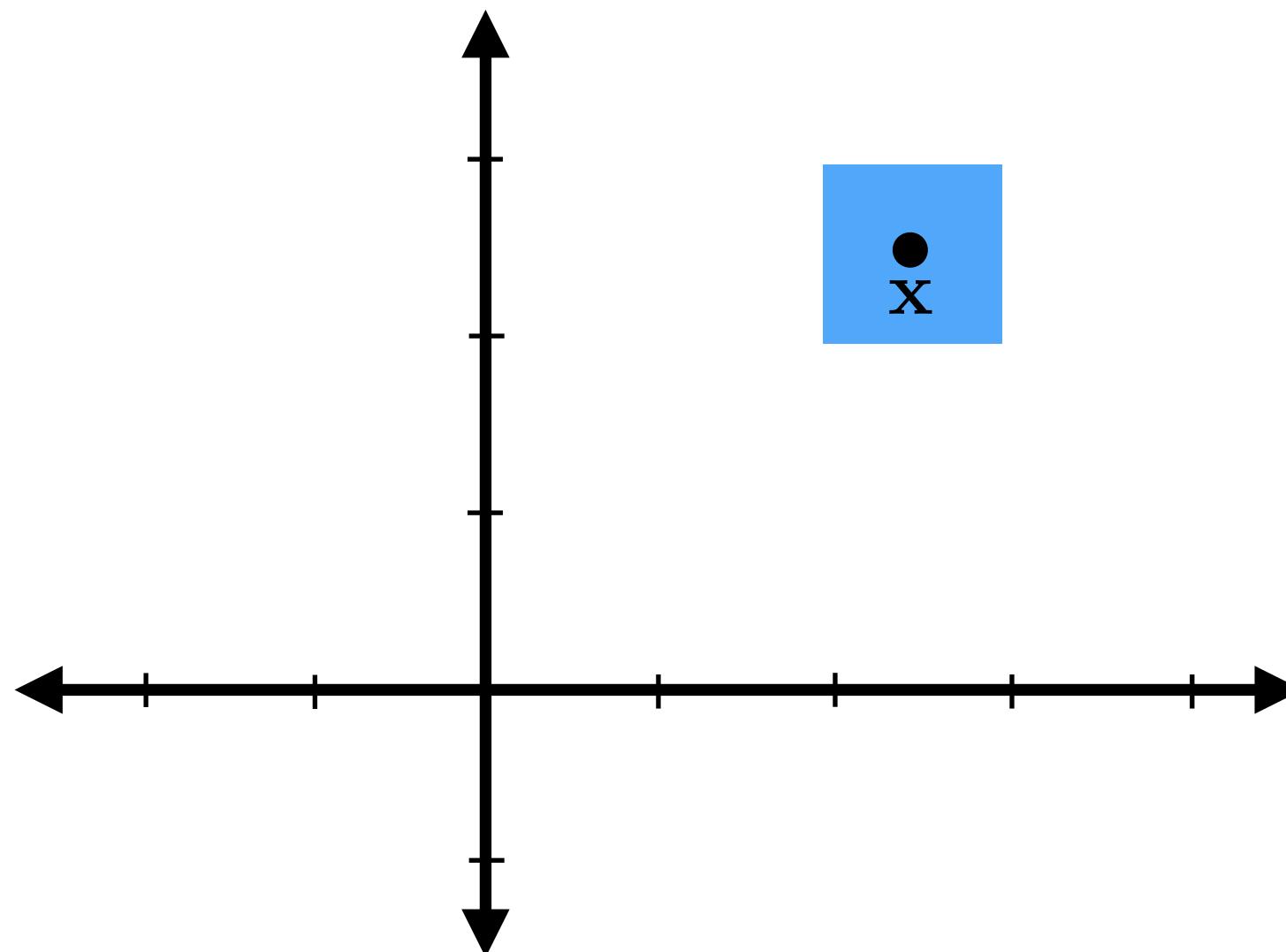


How would you perform these transformations?



Remember: always more than one way to do it!

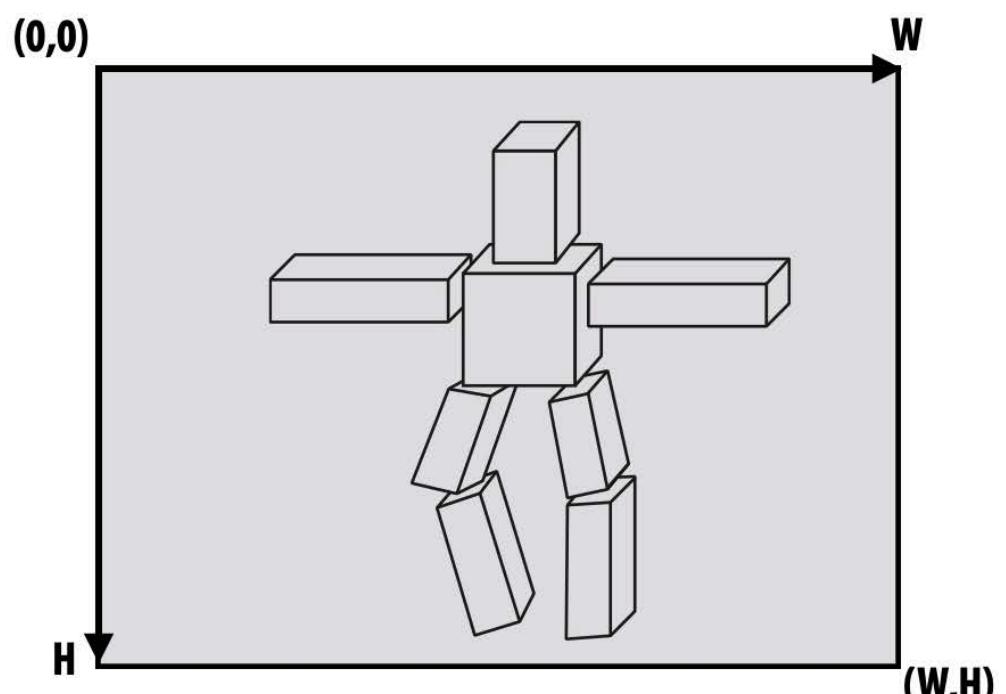
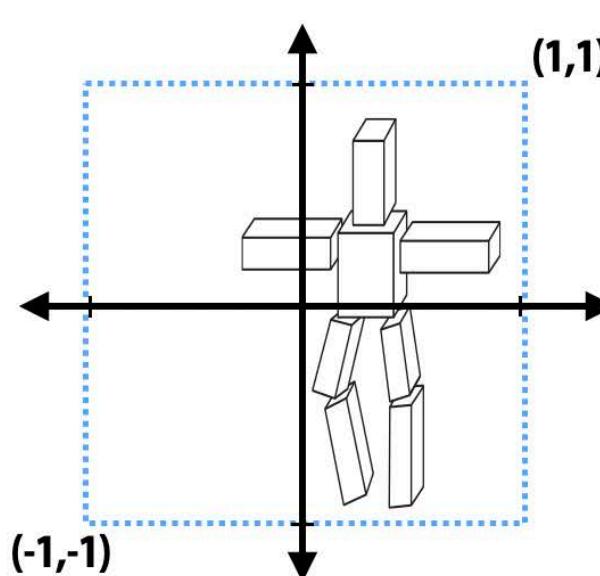
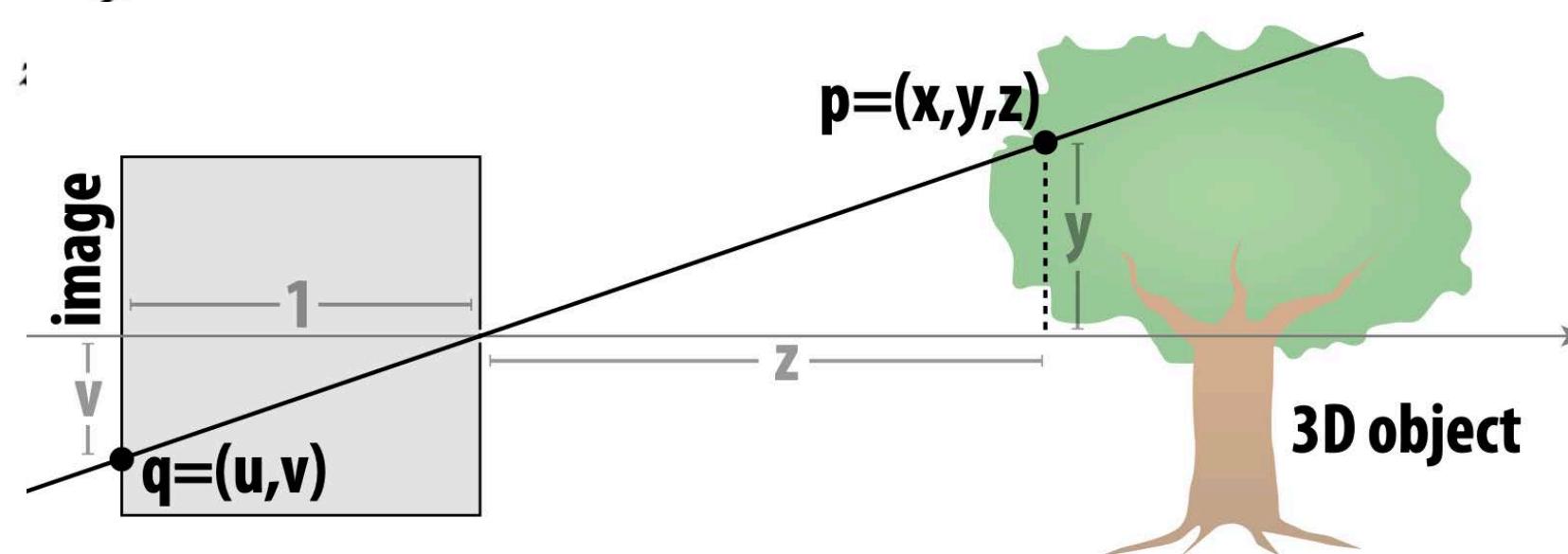
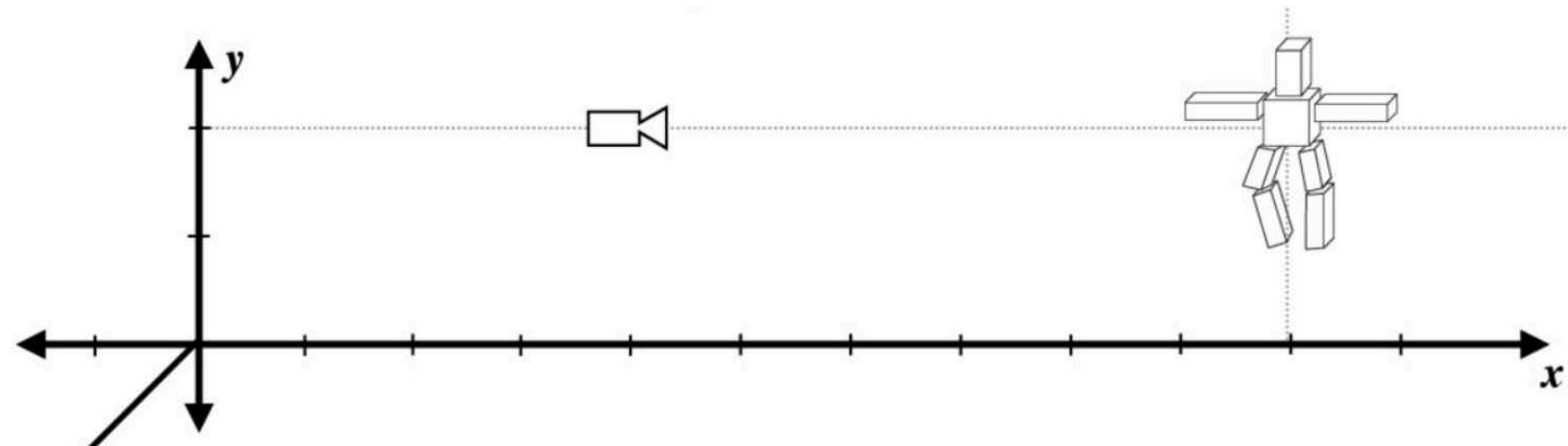
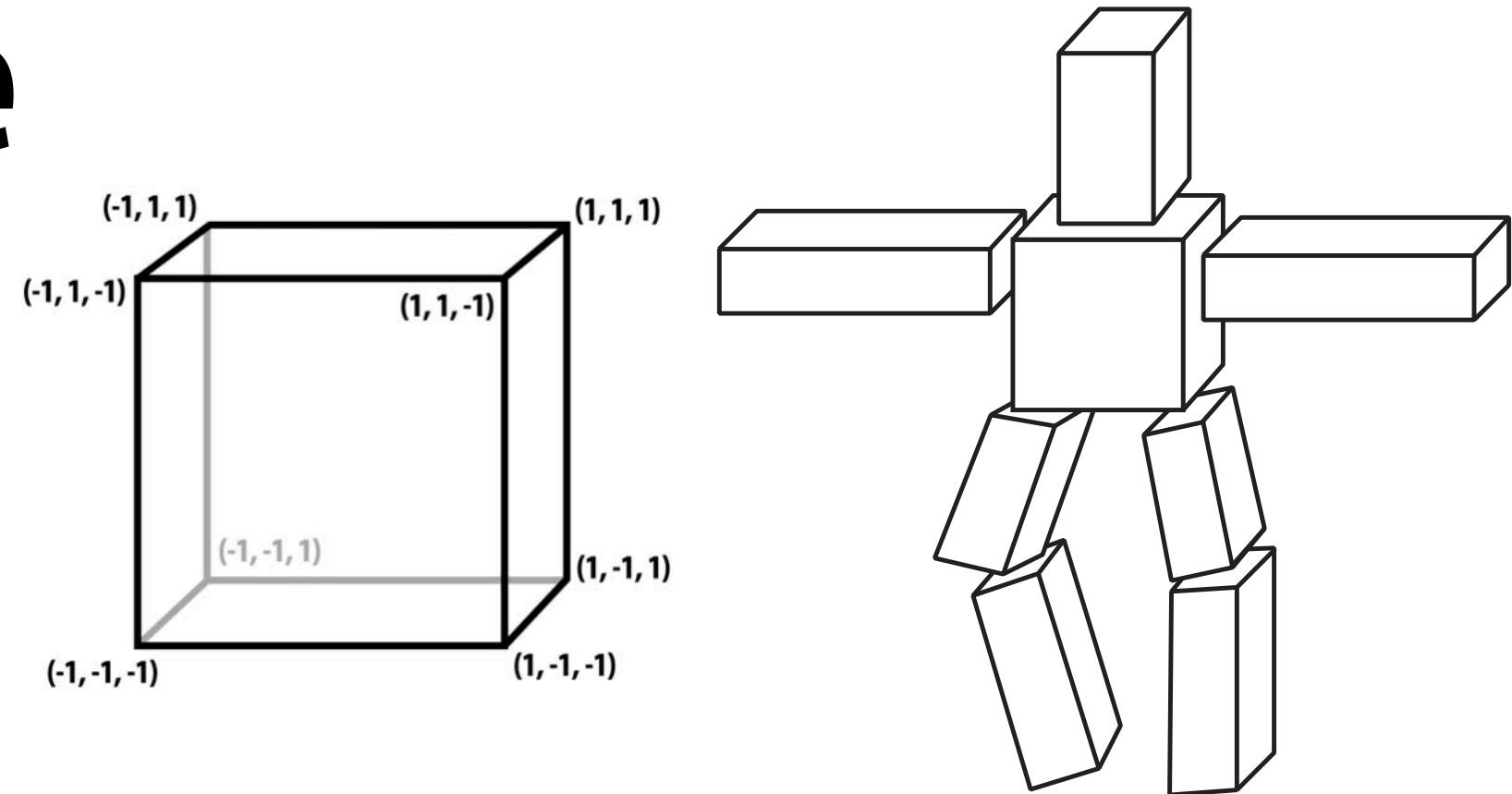
Common task: rotate about a point x



Q: What happens if we just rotate without translating first?

Drawing a Cube Creature

- Let's put this all together: starting with our 3D cube, we want to make a 2D, perspective-correct image of a "cube creature"
- First we use our scene graph to apply 3D transformations to several copies of our cube
- Then we apply a 3D transformation to position our camera
- Then a perspective projection
- Finally we convert to image coordinates (and rasterize)
- ...Easy, right? :-)



Spatial Transformations—Summary

transformation defined by its invariants

basic linear transformations

scaling
rotation
reflection
shear

composite transformations

- compose basic transformations to get more interesting ones
- always reduces to a single 4x4 matrix (in homogeneous coordinates)
 - simple, unified representation, efficient implementation
- order of composition matters!
- many ways to decompose a given transformation (polar, SVD, ...)
- use *scene graph* to organize transformations
- use *instancing* to eliminate redundancy

basic nonlinear transformations

translation
perspective projection

linear when represented via homogeneous coords
homogeneous coords also distinguish points & vectors

Next time: 3D Rotations

