Grothendieck-Teichmüller group and Absolute Galois Group

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Algebraic geometry, is the study of the connections between algebra and geometry. Historically algebraic geometers would study geometric surfaces that are cut out from solutions of polynomials (for example the solutions of the polyonmial $x^2 + y^2 - 1$ form a circle), but modern formulation of the field spearheaded by Alexandre Grothendieck in the mid-20th century redifines our geometrical objects using the modern tools of category theory. This translation between geometry and algebra lets us use tools from one subject to solve problems in the other. For example a problem in number theory could be finding integer solutions to

$$x^2 + y^2 = z^2$$

Where x, y, z are coprime. Dividing both sides by z^2, one finds that this problem is equivalent to solving, $(x/z)^2 + (y/z)^2 = 1$, i.e. finding the rational points on a circle which is a geometry problem. The study of algebraic geometry finds application in various fields, for example in cryptography, coding theory, number theory, computer-aided geometric design, and theoretical physics [1]

A seminal document relating to algebraic geometry is Grothendieck's research proposal, "Esquisse d'un programme", this document has gave research directions to the algebraic geometry community that are still being pursued to this day. Amongst them, he outlines a way to study the Absolute Galois Group of the rationals in a purely algebraic way, by exhibiting it as what we now call the Grothendieck-Teichmüller group, which is a subgroup of a group that we have a better understanding of.[2]

For reference the Absolute Galois Group is defined as the set of structure preserving maps from the algebraic closure of $\mathbb{Q}$ to itself fixing $\mathbb{Q}$. This is indeed a very important structure in number theory, since from Galois Theory we know that understanding Galois Groups is equivalent to understanding the intermidiate fields of a field extension. So in this specific case we can in some sense understand all finite field extensions of $\mathbb{Q}$. The Absolute Galois Group of $\mathbb{Q}$ is a complicated structure for which we don't currently have a clear description for, and Grothendieck's idea of identifying it with the Grothendieck-Teichmüller group will give us a description of the group that we can work with, and so will be a huge leap forward in Number Theory.

My research propsal would be to investigate some more properties of this Grothendieck-Teichmüller group, research in this area include studying: Braid groups, certain types of graphs called Dessins d'enfant and structures in category theory [3]

References

[1] Gowers, T., Barrow-Green, J., amp; Leader, I. (2010). In ˜The Princeton Companion to Mathematics (p. 372), Princeton University Press.

[2] Miller, D. (2014, September 2). A brief tour of Grothendieck-Teichmüller theory. https://pi.math.cornell.edu/ dkmiller/bin/grothendieck-teichmuller.pdf

[3] Lochak, P. (1994). The Grothendieck-Teichmüller group and automorphisms of braid groups. In L. Schneps (Author), The Grothendieck Theory of Dessins d'Enfants (London Mathematical Society Lecture Note Series, pp. 323-358). Cambridge: Cambridge University Press. doi:10.1017/CBO9780511569302.015