18.12.1 The gamma function

The *gamma function* (n) is defined by

$$\Gamma(n) = \int_0^\infty x^{x-1} e^{-1} dx, \tag{18.153}$$

which converges for $n > 0$, where in general n is a real number. Replacing n by $n + 1$ in (18.153) and integrating the RHS by parts, we find

$$\begin{aligned}\Gamma(n+1) &= \int_0^\infty x^n e^{-1} dx \\ &= [-x^n e^{-x}]_0^\infty + \int_0^\infty n x^{n-1} e^{-x} dx \\ &= n \int_0^\infty x^{n-1} e^{-x} dx\end{aligned}$$

from which we obtain the important result

$$\Gamma(n+1) = n\Gamma(n). \tag{18.154}$$

From (18.153), we see that (1) = 1, and so, if n is a positive integer,

$$\Gamma(n+1) = n!. \tag{18.155}$$

In fact, equation (18.155) serves as a definition of the factorial function even for non-integer n. For negative n the factorial function is defined by

$$n! = \frac{(n+m)!}{(n+m)(n+m-1)...(n+1)}. \tag{18.156}$$

where m is any positive integer that makes n + m > 0. Different choices of m *(> -n)* do not lead to different values for n!. A plot of the gamma function is given in figure 18.9, where it can be seen that the function is infinite for negative integer values of n, in accordance with (18.156). For an extension of the factorial function to complex arguments, see exercise 18.15.

By letting $x = y^2$ in (18.153), we immediately obtain another useful representation of the gamma function given by

$$\Gamma(n) = 2\int_0^\infty e^{-y^2} dy. \tag{18.157}$$

Setting $n = \frac{1}{2}$ we find the result

$$\Gamma(n) = 2\int_0^\infty e^{-y^2} dy = \int_{-\infty}^\infty e^{-y^2} dy = \sqrt{\pi},$$

where have used the standard integral discussed in section 6.4.2. From this result, $\Gamma(n)$ for half-integral n can be found using (18.154). Some immediately derivable factorial values of half integers are

$$\left(-\frac{3}{2}\right)! = -2\sqrt{\pi}, \qquad \left(-\frac{1}{2}\right) = \sqrt{\pi}, \qquad \left(\frac{1}{2}\right)! = \frac{1}{2}\sqrt{\pi}, \qquad \left(\frac{3}{2}\right)! = \frac{3}{4}\sqrt{\pi}.$$