Homework 1 Solution

W203 Statistics for Data Science

Fall 2020

1. Gas Station Analytics (9 points)

At a certain gas station, 40% of the customers use regular gas (event R), 35% use mid-grade gas (event M), and 25% use premium (event P). Of the customers that use regular gas, 30% fill their tanks (event F). Of the customers that use mid-grade gas, 60% fill their tanks, while those that use premium, 50% fill their tanks. Assume that each customer is drawn independently from the entire pool of customers.

1. (3 points) What is the probability that the next customer will request regular gas and fill the tank?
2. (3 points) What is the probability that the next customer will fill the tank?
3. (3 points) Given that the next customer fills the tank, what is the conditional probabilty that they use regular gas?

Solution 1.1:

Given that a customer can only take regular gas, mid-graded gas or premium gs during a visit to the gas station, event R, event M and event P are mutually exclusive events. Also because each customer must chose one of the three types of gass to fill, event R, event M and event O are exhaustive events.

We have:

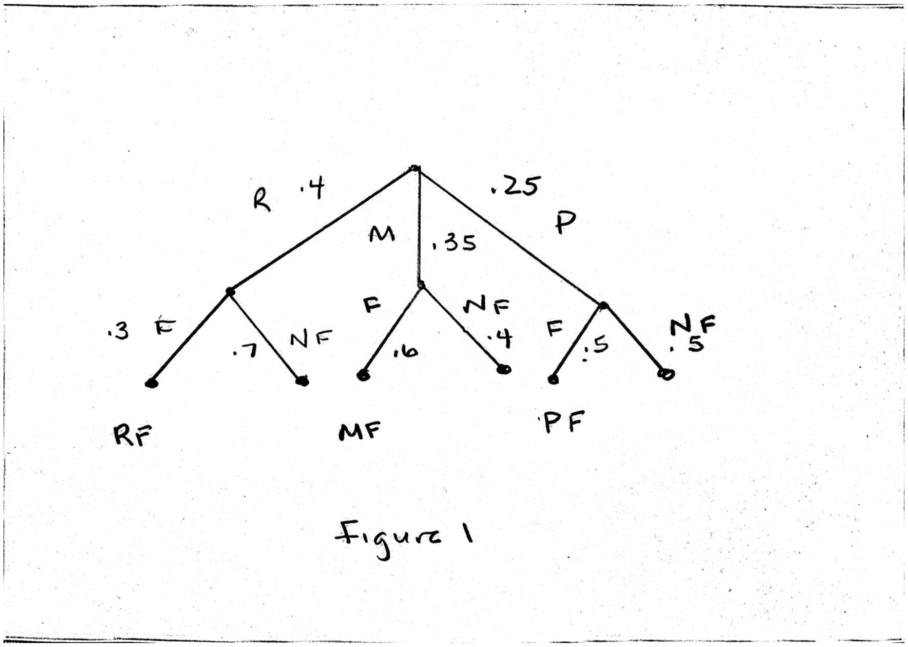
$$Pr(R) = .4,\ Pr(M) = .35,\ Pr(P) = .25,\ Pr(F|R) = .3,\ Pr(F|M) = .6,\ Pr(F|P) = .5$$

The probability that the next customer will request regular gas and fill the tank is $Pr(R \cap F)$. By the law of conditional probability $Pr(R \cap F) = Pr(R) \cdot Pr(F|R) = .3 \cdot .4 = .21$

One way to think about this problem graphically is the following.

In [41]:
```python
from IPython.display import Image
Image(filename="img/HW_S_Fig_1.jpg", width=450, height=450)
```

Out[41]:



Solution 1.2:

$Pr(F) = Pr(F|R) \cdot Pr(R) + Pr(F|M) \cdot P(M) + Pr(F|P) \cdot Pr(P) = .12 + .35 \cdot .6 + .25 \cdot .5 = .455$

Solution 1.3:

$P(R|F) = (Pr(F|R) \cdot Pr(R))/Pr(F)$ by Bayes Rule, $.12/.455 \approx .264$

2. The Toy Bin (12 points)

Solution 2.1 (3 points)

In [43]:
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Image(filename="img/HW_S_Fig_2.jpg", width=400, height=400)
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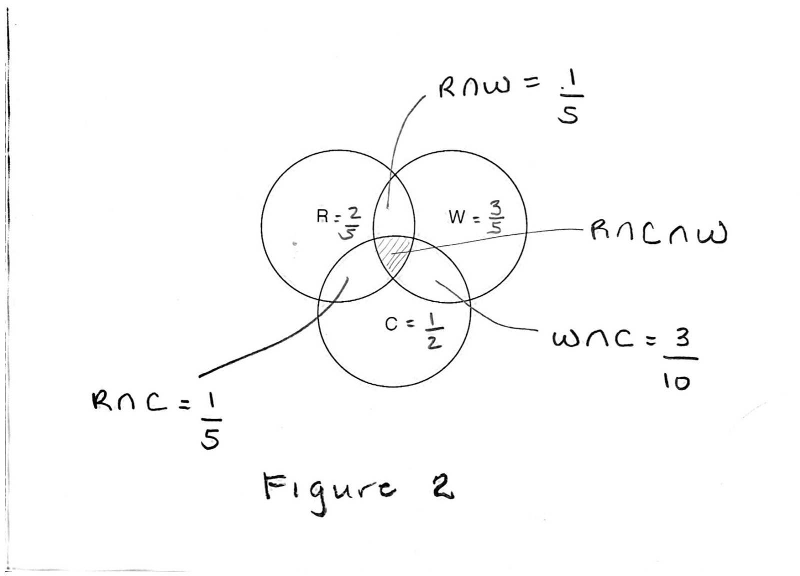
Out[43]:



Solution 2.2 (3 points)

Let $x = Pr(R \cap W \cap R)$

$1 - Pr(!R \cap !W \cap !C) = Pr(R) + Pr(W) + Pr(C) - Pr(R \cap W) - Pr(W \cap C) - Pr(R \cap C) + P(R \cap W$

$\frac{9}{10} = \frac{2}{5} + \frac{3}{5} + \frac{1}{2} - \frac{1}{5} - \frac{1}{5} - \frac{3}{10} + x$

$x = \frac{1}{10}$

Note there are a number of ways of combining the areas to solve for x.

Solution 2.3 (3 points)

We want the probability that a toy is not cool, given that it is red, $Pr(!C|R)$). This is the ratio $\frac{Pr(R) - Pr(R \cap C)}{Pr(R)} = \frac{2/5 - 1/5}{2/5} = \frac{1}{2}$

Solution 2.4 (3 points)

$Pr(C|R \cup W) = \frac{P(C \cap (R \cup W))}{Pr(R \cup W)}$ by the law of conditional probability

$= \frac{Pr((C \cap R) \cup (C \cap W))}{Pr(R \cup W)}$

$= \frac{Pr(C \cap R) + Pr(C \cap W) - Pr(R \cap W \cap C)}{Pr(R \cup W}$

We know $Pr(R \cup W) = Pr(R) + Pr(w) - Pr(R \cap W) = \frac{4}{5}$ (demoninator)

and $Pr(C \cap R) + Pr(C \cap W) - Pr(R \cap W \cap C) = \frac{1}{5} + \frac{3}{10} - \frac{1}{10} = \frac{2}{5}$ (numerator)

So $Pr(C|R \cup W) = \frac{1}{2}$

3. On the Overlap of Two Events (6 points)

Solution 3.1

Suppose for events A and B, $P(A) = \frac{1}{2}$ and $P(B) = \frac{3}{4}$ but we have no additional information about the events.

1. (3 points) What are the maximum and the minimum of $Pr(A \cap B)$?
2. (3 points) What are the maximum and minimum possible values for $P(A|B)$?

Solution 3.1

To conceptualize the maximum consider the following diagram.

```
In [36]: Image(filename="img/HW_S_Fig_3.1.jpg", width=400, height=400)
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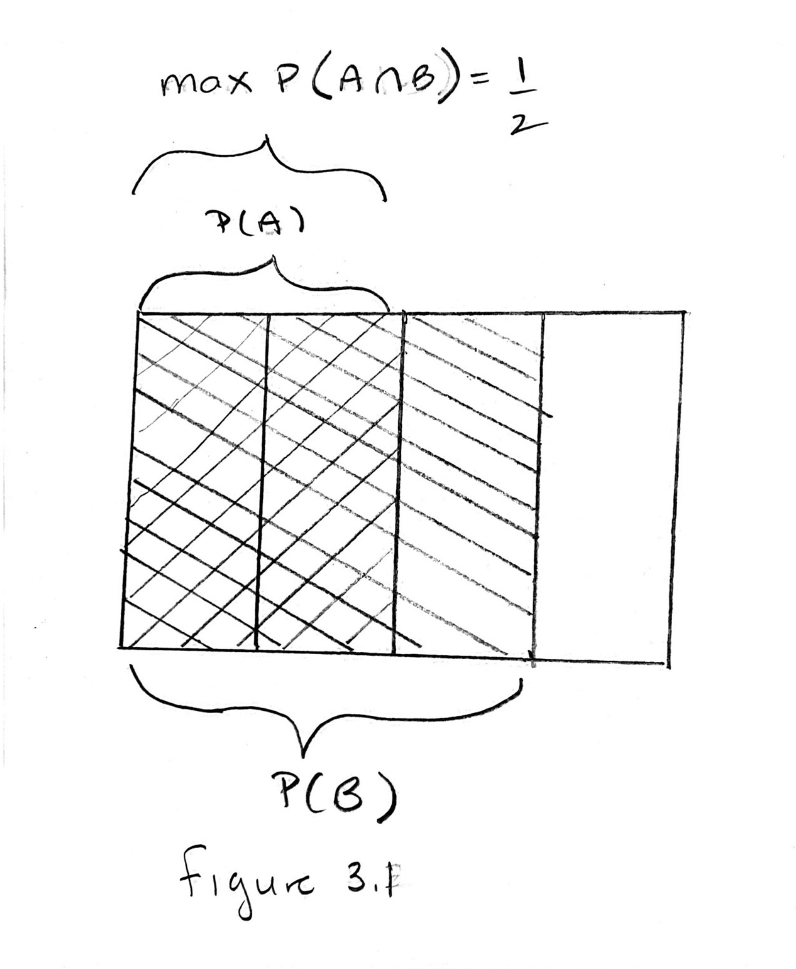
Out[36]:



Given that $Pr(A) + Pr(B) > 1$ A and B cannot be mutally exclusive events. So $Pr(A \cap B) > 0$. The intersection is maximized when all A events happen with B events. See diagram below. The maximum overlap, intersection, $Pr(A] \cap B)$, is $Pr(B) = \frac{1}{2}$.

To think about the minimum overlap consider the following diagram.

In [42]:
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Image(filename="img/HW_S_Fig_3.2.jpg", width=400, height=400)
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Out[42]:

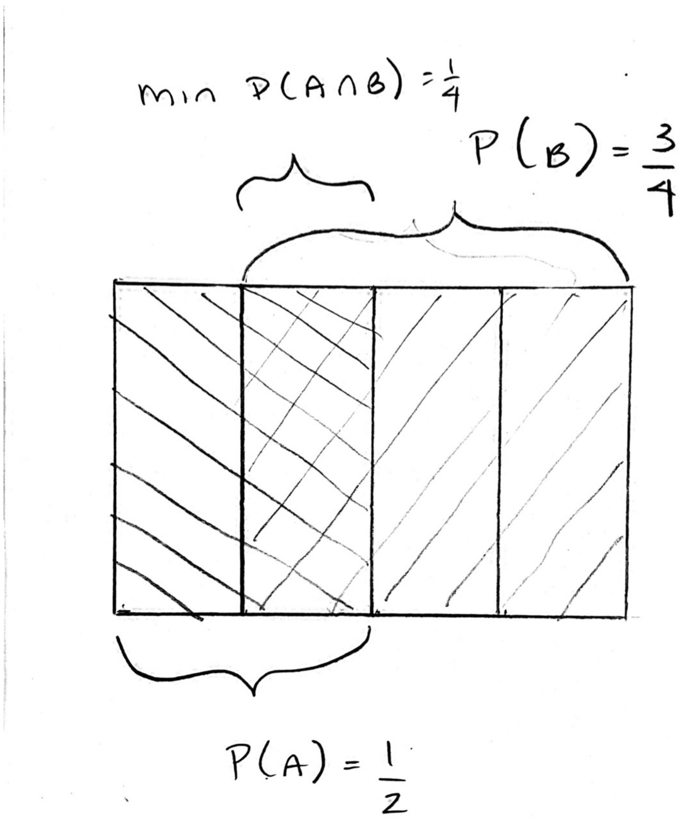

Figure 3.2

The minimum overlap, intersection, happens when Pr(A \cup B) fills the entire probability space, $Pr(A \cup B) = 1$. We know $Pr(A) + Pr(B) - Pr(A \cap B) = 1$.

$\frac{1}{2} + \frac{3}{4} - Pr(A \cap B) = 1$

Thus the minimum $Pr(A \cap B) = \frac{1}{4}$

So $\frac{1}{4} \leq Pr(A \cap B) \leq \frac{1}{2}$

Solution 3.2

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

We found in part 1 the numerator is maximized at $\frac{1}{2}$. Since $P(B) = \frac{3}{4}$, the maximum of the ratio is $\frac{2}{3}$.

We found in part 1 that the minimum is $\frac{1}{4}$. Thus the minimum is $\frac{1}{3}$.

So $\frac{1}{3} \leq Pr(A|B) \leq \frac{2}{3}$