Time Value of Money and the Risk Free Rate
Modern finance is predicated on the concept of "Time Value of Money" (TVM). TVM is simply the fact that $1 today is worth more than a guaranteed $1 at a future time.

Would you loan someone $100 dollars with a promise to pay $100 back in 1 year?

At a minimum, there is an economic opportunity cost to forgoing money until a later date. In the presence of inflation, where purchasing power drops over time, there is a real economic cost as well.

If money is worth more now than later, then we need a concept of a discount factor, or rate, that equates money now with money later.

The rate at which money can be borrowed or lent, without risk, is called the risk free rate:

$$P_{t+1} = P_t(1 + r_{rf})$$

Generally when discussing the risk free rate, this is the rate that the government or banks can borrow/lend short term (overnight). It represents the 1 day risk free rate. This is generally quoted as an annual rate. Compounding rules differ by market.

Arbitrage, Present and Future Value
We use interest rates as a discounting mechanism. If you know, without a doubt, that you will receive $\$X$ as some future time t, what is that worth now?

To understand what that value is, we need to understand arbitrage. Arbitrage is the ability to make a return, in excess of the risk free rate, without taking any risk. For example, if student A will buy 1 BTC from me for $50,000 and another student, B, will sell me 1 BTC for $30,000, I can buy from B, and sell immediately to A and make $20,000.

In this example, I made a 67% return $\left(\frac{20{,}000}{30{,}000}\right)$. I did this intantanesouly so the regardless of what the risk free rate was, the rate for 0 time is always 0% $67\% > 0\%$

Another example, Student A agrees to buy 1 BTC from me in 1 year for $50,000. Student B will sell me 1 BTC for $30,000. I can buy a 1 year government bond with a rate of 1%. I still make a 67% return, but now the risk free rate is 1%. If there was no chance of Student A not being able to do the deal (credit risk), then this is arbitrage.

(The example above is a forward agreement)

Arbitrage is free money.

When arbitrage exists, market actors are incentivized to take action and pick up the free money. This change in supply and demand causes pricing to shift toward a state where there is no arbitrage.

"All pricing problems in finance can be described as, 'find the no-arbitrage condition'" – some old finance professor

Present value is the current value of some asset. We find this by discounting the cash flows from that asset using an interest rate. For risky assets, we take the risk free rate and add a risk premium to it.

$$DiscountRate = RiskFreeRate + RiskPremium$$

How to come up with the Risk Premium is the subject of another class. Just understand that it exists. There can be multiple premiums for different risks added.

The question at the top of the section, " If you know, without a doubt, that you will receive $\$X$ as some future time t, what is that worth now?"

Here this is a risk free asset, we know with 100% certainty we will receive $X at time t. There is no risk premium. The discount rate is equal to the risk free rate.

$$DiscountRate = RiskFreeRate + 0$$

Let's assume the risk free rate is quoted as a continuously compounded annual return. t is in terms of years. Present Value is the discounted value of the cash flow:

$$PV = Xe^{-rt}$$

Why? If I invested PV into a government bond now, it's future value would be

$$FV = PVe^{rt}$$

I would receive FV at time t. If $FV \neq X$, then there is an arbitrage opportunity.

If our asset was a bond that paid cashflows X_i on times $i \in [1, 2, ..., t]$ then, and

$$PV = \sum_{i=1}^{t} X_i e^{-ri}$$

If the discounting rate is instead compounded discreetly, and a N times per year basis, then the formula would be

$$PV = \sum_{i=1}^{t} \frac{X_i}{\left(1+\frac{r}{N}\right)^{N*i}}$$

Recognize that

$$\lim_{N \to \infty} \frac{1}{\left(1+\frac{r}{N}\right)^{N}} = e^{-r}$$

Options

An option is the right, but not the obligation, to exchange at a later date at a determined price. Anytime you see a contract where someone has the ability to decide to transact at a later date, that is an option.

“All options have value.” – another old finance professor

Examples:

1. Stock option, ability to buy or sell a stock at a given price at some future date. The classic example

2. Loans with no prepayment penalty. US Mortgages, Auto Loans etc. A customer can hand the lender the remaining principal of the loan at any time. This has positive value when a borrower can refinance at a lower rate, lowering their payment but keeping the amount owed the same.

3. Callable Bonds. Corporate lenders sometimes issue callable bonds that allow them to pay the face value plus some premium to retire the debt. This can generally only happen on set dates. When rates fall, corporations will call their bonds, issuing new bonds at the lower rate. Same as #2, but for corporations with set dates.

4. Puttable Bonds. Some corporate bonds are puttable, meaning the lender (owner of the bond) can force the corporation to buy back the bond at face value minus some premium. When rates rise, bond buyers will do this in order to invest their capital at higher rates.

5. Convertible Bonds. Convertible bonds allow the lender (owner of the bond) to convert the bond to common stock at a set price on certain dates.

6. Auto leases. In the US, you can lease (rent) a car for a set period. At the end of that period you can buy the car for a predetermined price or give it back to the car dealer.

Option Terms
The security whose price determines the payoff of the option is called the **underlying**. The option's value is derived from the value of the underlying (among other parameters). This is why we call options a derivative (Futures, Forwards, Swaps, etc. are other types of derivatives).

The price at which the underlying can be bought or sold is called the **strike**. If I buy an Apple option that allows me to buy 1 share of stock at $100 in 1 year, then the strike is $100.

The option to purchase something in the future is a **call**. If I buy an Apple option that allows me to buy 1 share of Apple stock in 1 year at a strike of $100, then I have bought a call.

The option to sell something in the future is a **put**. If I buy an Apple option that allows me to sell 1 share of Apple in 1 year at a strike of $100, then I have bought a put.

When an option is used, it is called the **exercise**. When you exercise a call option, you buy the contracted amount at the contracted price.

Maturity or Expiration Date is the date on which the option contract expires. The option cannot be exercised after this date and the contract is over.

Time to Maturity – the time, usually expressed as a fraction of a year, until the maturity date of the option.

Options that can only be exercised at the maturity date are called **European Options**. In the US, most options on Indexes (i.e. the S&P 500) are European options.

Options that can be exercised at any time are called **American Options**. Most options on single name equities are American options.

Options that can be exercised at certain dates between the start and end of the contract are called **Bermuda Options**. Bermuda is an island roughly halfway between the Americas and Europe – that's how these options got their names.

Asian Options set either the strike price or the underlying price by some averaging mechanism. This averaging lowers the volatility and ability to manipulate prices. Energy and some commodity options use this. Asian options tend to be cash settled (see below).

There are many other types of options that vary in how they are exercised.

Physical Settlement occurs when at option exercise, the underlying asset is exchanged. US stock options are physically settled. If you exercise the option, you either are given shares or have to deliver shares.

Financial (or Cash) Settlement occurs when the difference between the strike and underlying price is delivered instead of underlying asset. US Index options are usually cash settled.

Implied Volatility – the value of an option depends on the range of possible outcomes at exercise. This implies there is some probability distribution to prices. While we cannot directly observe the market's expectation of the forward price volatility, we can observe option prices. We can back out the market implied volatility from an option pricing function and the option price.

The **payoff** (or profit, π) of the option at the time of exercise is the amount of money made when the option is exercised.

The payoff for a call option is

$$\pi = max(0, S - X)$$

Where S is the underlying price and X is the strike price.

- If the value of the underlying is greater than the strike price, $S > X$, then I can buy the underlying, spending X and selling it for S. My profit is $S - X$.
- If the value of the underlying is less than the strike price, $S < X$, then I do nothing. It doesn't make sense to buy a stock for X when I could buy it for S where $S < X$

The payoff of a put option at the time of exercise is

$$\pi = max(0, X - S)$$

- If the value of the underlying is less than the strike price, $S < X$, then I can buy the underlying for S and sell it back for X.
- If the value of the underlying is greater than the strike price, $S > X$, then I do nothing.

If the strike price and underlying price are close together, the option is said to be **At the Money (ATM)**. If the payoff for the option (assuming immediate exercise) is positive, the option is said to be **In the Money**. If the payoff of the option (assuming immediate exercise) is 0, then the option is said to **Out of the Money.**

The degree to which an option is In or Out of the Money is called **Moneyness.**

Option Valuation
So how do we put a value on an option?

In general, we need the present value of the average of the payoff function at exercise.

Assume that $f(S)$ is the PDF of the distribution of the underlying price at the exercise time, t, then the value of the option would be

$$Value \ = \ e^{-rt}\int_{0}^{\infty} \pi \ * \ f(S) \, ds$$

Example: Value a European call option with:
- Underlying Price = 100
- Strike Price = 100
- 125 days until maturity
- 255 trading days in a year
- Risk free rate of 5%
- Implied Volatility of 20%

Further assume prices are lognormally distributed.

First, find the volatility to maturity.

$$dailyVol \ = \ \frac{0.2}{\sqrt{255}} \ = \ 1.252\%$$

$$\sigma \ = \ dailyVol \ * \ \sqrt{125} \ = \ 14\%$$

Next find the mean expected price

$$Y \ \sim \ LN(\mu, \sigma) \ \Rightarrow E(Y) \ = e^{\mu + \frac{\sigma^2}{2}}$$

$$E\left(P_t\right) = P_0 e^{ttm*r + \frac{\sigma^2}{2}}$$

Where ttm is the time to maturity or $\frac{125}{255}$.

$$\mu \ = \ ln\left(P_0\right) + \ ttm \ * \ r$$

We grow the asset by the risk free rate. This is part of the no arbitrage condition.

However, we do not want the volatility term to change this. So we set the μ for the Log Normal distribution to remove this.

$$\mu = ln\left(P_0\right) + ttm * r - \frac{\sigma^2}{2} = ln(100) + \frac{125}{255}(0.05) - 0.0098 = 4.6357$$

The price of the call option is then

$$Value = e^{\frac{125}{255}(.05)} \int_0^\infty max(0, S - 100)\left[\frac{1}{S\sqrt{2\pi\sigma^2}} e^{-\left(\frac{(ln(s)-\mu)^2}{2\sigma^2}\right)}\right] dS$$

$$Value \approx 6.8089$$

This integral has to be solved numerically. In the case of a European option, there is a closed form solution.

Fischer Black, and Myron Scholes published a paper in 1973 setting up a second order partial differential equation that had to hold for the no arbitrage condition (via a replicating portfolio of continuous delta hedging) that is consistent with the Capital Asset Pricing Model (a general equilibrium model for financial markets). Robert Merton published a paper shortly after with the same derivation.

Scholes and Merton were awarded the Nobel Prize in Economics in 1997 for this work. Fischer Black had passed away from cancer in 1995 and the Nobel Prize committee doesn't care about dead people.

There were a number of papers that take the same equation and modify it for different underlying markets.

1. Black Scholes 1973 Options on non-dividend paying equities. Solution equals the integral in the example above.
2. Merton 1973 Options on equities paying a continuously compounded dividend. μ is updated to include the dividend accrual.
3. Black 1976 Options on Futures. Futures prices are quoted in future value. μ is updated so that the price does not grow
4. Garman and Kohlhagen 1983 Options on FX. Here the risk free rate of the foreign currency needs to be taken into account.

All of these (plus some other cases) can be generalized into a single formula.

- S - Underlying Price
- X - Strike Price
- T - is the Time to Maturity
- σ - is the implied volatility
- r - is the risk free rate
- b - is the cost of carry
- Φ() - is the normal CDF function

$$d_1 = \frac{ln\left(\frac{S}{X}\right)+\left(b+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$Call = Se^{(b-r)T}\Phi\left(d_1\right) - Xe^{-rT}\Phi\left(d_2\right)$$

$$Put = Xe^{-rT}\Phi\left(-d_2\right) - Se^{(b-r)T}\Phi\left(-d_1\right)$$

- $r = b$
  - Black Scholes 1973 Option on Equities with no dividends
- $b = r - q$
  - Merton 1973 Options on Equities paying a continuous dividend rate q
- $b = 0$
  - Black 1976 Options on Futures
- $b = r - rff$
  - Garman and Kohlhagen 1983 FX option where rff is the foreign risk free rate

This is equivalent to updating the integral's formula for μ to:

$$\mu = ln\left(P_0\right) + ttm * b - \frac{\sigma^2}{2}$$

The underlying asset grows at rate b, the cost of carry.

When the limiting distribution of the final price is unknown, you can always use a Monte Carlo Simulation.

1. Simulate Prices through time for N iterations
2. Evaluate the payoff function at the final time.
3. Take the average of the payoff function values across the simulations
4. Discount that value back to the present value.

The value of a European option will always be greater than the payoff function prior to maturity. This premium to the payoff decays with time (called theta decay) until maturity when the option value equals the payoff function.

$$Value_t > \pi_t \quad \forall\, t > 0$$

Put Call Parity

First described at the turn of the 20th century by Higgans (1902) and Nelson (1904), put call parity describes a fundamental relationship between the price of an European put and a call on the same underlying, at the same strike price, and with the same maturity.

$$C + Xe^{-rT} = P + S$$

This holds as a no arbitrage condition assuming the underlying can be easily shorted, you can lend and borrow at the risk free rate, and there are no transaction costs.

- C - the price of the Call
- Xe^{-rT} - the present value of the strike price
- P - the price of the put
- S - the current price of the underlying

Each term is a transaction we can make and putting them together gives us replicating portfolios for both a Call and a Put.

A replicating portfolio is a series of transactions we can do now that mirror the payoff of another portfolio. If the two portfolios have the same payoffs, then their value today should be equal. Otherwise I can buy one and sell the other and lock in an arbitrage profit.

Example

$C = P + S - Xe^{-rT}$

Show this must hold.

Cash Flow table for buying a call:

Transaction	t=0	t=T: $X < S_T$	t=T: $X \geq S_T$
Buy C	-C	S_T-X	0

Cash Flow table for replicating portfolio

Transaction	t=0	t=T: $X < S_T$	t=T: $X \geq S_T$
Buy P	-P	0	X-S_T
Buy Stock	-S_0	S_T	S_T
Borrow PV X	Xe^{-rT}	-X	-X
Total	$Xe^{-rT} - P - S_0$	S_T-X	0

In both future states of the world, buying the call or buying a put, buying the stock, and taking out a loan, gives you the same payoff.

Binomial Trees

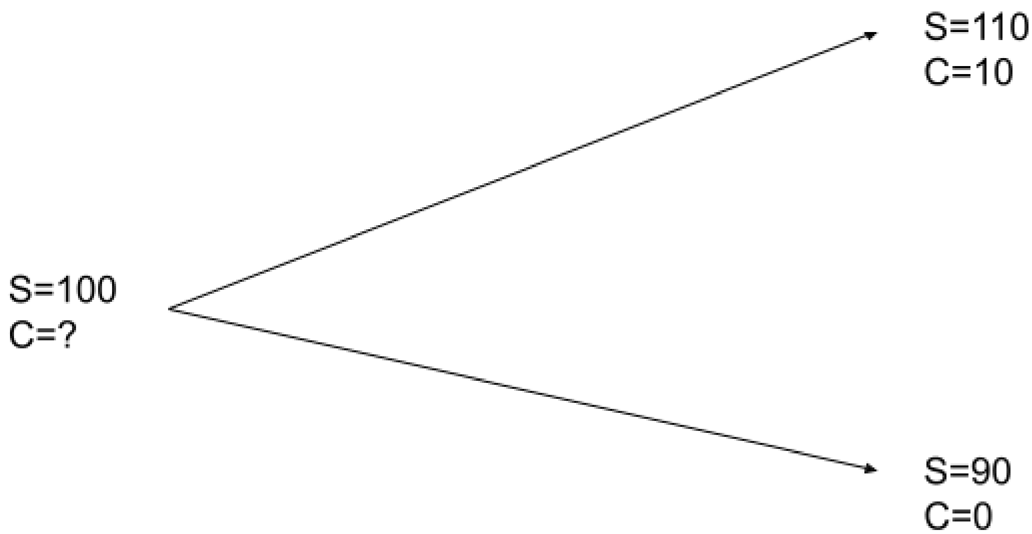
Binomial trees provide a way to value options with path dependencies. They were first published in 1979 by Cox, Ross, and Rubinstein.

A binomial tree is a recombining tree structure that discretely approximates geometric brownian motion. If taken to its limit with an infinite number of small steps, it converges to brownian motion.

Recommended reading: Chapter 13 of Hull, “Options Futures and Other Derivatives,” 11th edition (starting page 266).

1 step example.

You own a European call option with a strike of $100. The current share price of the stock is $100. The risk free rate is 5%. The time to maturity is 1 year. Using a 1 step binomial tree, approximate the value of the option.



In this tree we say the binomial states of the world are either the stock gains 10% or loses 10%. 10% are arbitrarily chosen for now. If this is the case, the value of the call at expiration is either \$10 or \$0.

What is the value of the call today? The intuitive answer is a probability weighted average of the final states discounted to present value.

$$C_0 = e^{-0.05}(p * 10 + (1 - p) * 0)$$

Finding the value of P requires us to go back to the risk neutral framework used in the option valuations above. This is the probability, in the risk neutral world not the real world, of the up move in price.

$$u, d = price\ multiplier\ in\ the\ (positive, negative)\ case$$

$$p = \frac{e^{bt} - d}{u - d},\ (1 - p) = \frac{u - e^{bt}}{u - d}$$

The book walks you through the proof.

Here we use the cost of carry, b, instead of the risk free rate to be consistent with our generalized Black Scholes model.

What is interesting, and what was proved long ago, is that we don't need real probabilities of the moves. The risk neutral assumption gives the same option value as if we used appropriate probabilities and non-risk free discount rates.

Back to the example

$$p = \frac{e^{0.05} - 0.9}{1.1 - 0.9} = \frac{0.1513}{0.2} = 0.7564$$

$$C_0 = e^{-0.05}(0.7564 * 10 + (0.2436) * 0) \approx 7.20$$

The missing piece is how to incorporate volatility? Simply, the u & d values are chosen based on the following

$$u = e^{\sigma\sqrt{\Delta t}}, \ d = e^{-\sigma\sqrt{\Delta t}}$$

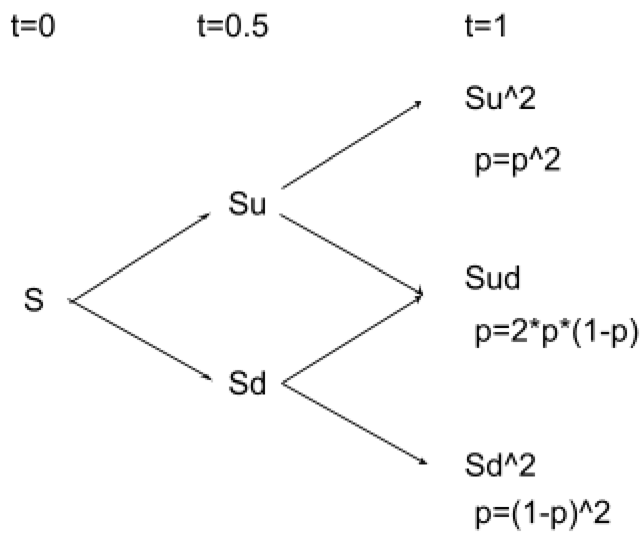
Where

$$\Delta t = \frac{T}{N}$$

N, being the number of steps in the tree and T being the time to maturity, expressed in years.

2 step example with volatility.
$T = 1$
$N = 2$
$S = 100$
$r = b = 0.05$
$\sigma = 0.1$ – notice this is roughly the same assumption from our first example…
Then
$u = e^{0.1*\sqrt{0.5}} = 1.073271$
$d = e^{0.1*\sqrt{0.5}} = 0.931731$
$p = \frac{e^{0.5*0.5} - 0.931731}{1.073271 - 0.931731} = .6612$

The general form of the recombining 2 step tree is.



There are 3 states of the world to consider.

1. The Up/Up case has a final stock value, probability and call value
    a. $S_T = Su^2 = 115.91$
    b. $p_T = p^2 = 0.4372$
    c. $C_T = max(0, S - X) = 15.91$
2. The Up/Down and Down/Up case can be reached through 2 paths.
    a. $S_T = Sud = 100$
    b. $p_T = 2 * p * (1 - p) = 0.4480$
    c. $C_T = 0$
3. The Down/Down case has values
    a. $S_T = Sd^2 = 86.81$
    b. $p_T = (1 - p)^2 = 0.1148$
    c. $C_T = 0$

The approximate value of the call option is then

$$C \approx e^{-0.05}(15.91 * 0.4372 + 0 + 0) = 6.32$$

Using the BSM model, the exact value of the call option is: $6.80

If we generalize the binomial tree out to N steps, then we have $N + 1$ terminal nodes.

For each node $i \in 0 \ldots N$

$$S_{T,i} = S * u^i * d^{N-i}$$

$$p_{T,i} = p^i * (1 - p)^{N-i}$$

$$nPaths = \left(\frac{N!}{i!(N-i)!}\right)$$

Where $nPaths$ are the number of pathways that arrive at node i

Therefore the final value of the option is

$$Value = e^{-rt} \sum_{i=0}^{N} \left(\frac{N!}{i!(N-i)!}\right) * p^i * (1 - p)^{N-i} * \pi\left(S * u^i * d^{N-i}\right)$$

Implied Volatility

So far, we have used the implied volatility as an input. However, the market priced volatility is not directly observable. Instead we have to solve for it using our option valuation routine and the observed market price.

Example:

The observed option price is $6. There are 125 days until maturity (255 days in a year). The risk free rate is 5%. The strike on the option is $100 and the current stock price is $100. What is the implied volatility?

Solution: Using the root finder in Julia and the Generalized Black Scholes Merton formula, we find the volatility implied by the market price is approximately 17%.