Week 5 - Logistic Regression

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Exercise 1

Let C_1 be event that new data belongs to class 1 and C_2 be event that new data belongs to class 2

$\Rightarrow$ Posterior probability for C_1 is:

$$\begin{aligned} p(C_1|x) &= \frac{p(x|C_1)\,p(C_1)}{p(x|C_1)\,p(C_1) + p(x|C_2)\,p(C_2)} \\ &= \frac{1}{1 + \frac{p(x|C_2)\,p(C_2)}{p(x|C_1)\,p(C_1)}} \\ &= \frac{1}{1+e^{-a}} \end{aligned}$$

$$with\ a = log\frac{p(x|C_1)\,p(C_1)}{p(x|C_2)\,p(C_2)}$$

$\Rightarrow \sigma(a) = \frac{1}{1+e^{-a}}$ is the logistic sigmoid function

With $\hat{y} = w_0+w_1\phi(x_2)+...+w_n\phi(x_n) = w^T\phi$, the model of logistic regression is defined as

$p(C_1|\phi) = y(\phi) = \sigma(w^T\phi)$

$p(C_2|\phi) = 1 - p(C_1|\phi)$

For dataset ϕ_n and t_n where $t_n \in 0;1$ and $\phi_n = \phi(x_n)$ the likelihood function can be written:

$$p(t|w) = \prod_{n=1}^{N} y_n^{t_n}(1-y_n)^{1-t_n}$$

where $t = (t_1, ..., t_n)^T$ and $y_n = p(C_1|\phi)$

We difined the cross function as:

$$L = -log\,p(t|w) = -\Sigma\,t_n\,log(y_n) - \Sigma\,(1-t_n)\,log(1-y_n) = -t_n\,log(y_n) - (1-t)^T log(1-y)$$

Taking gradient of the function with respect to w
Using the Chain rule,

$$\frac{\partial L}{\partial w} = \frac{\partial L}{\partial y}\text{x}\frac{\partial y}{\partial z}\text{x}\frac{\partial z}{\partial w} \quad \text{where } z = w^T\phi$$

With:

- $\frac{\partial L}{\partial y} = \left[\frac{-t^T}{y} + \frac{(1-t)^T}{1-y}\right] = \frac{y-t}{y\,(1-y)}$
- $\frac{\partial y}{\partial z} = \frac{\partial \sigma(z)}{\partial z} = \left(\frac{1}{1+e^{-a}}\right)^{'} = \sigma(z)\,(1-\sigma(z)) = y\,(1-y)$
- $\frac{\partial z}{\partial w} = \phi$

$\Rightarrow \frac{\partial L}{\partial w} = \frac{y-t}{y\,(1-y)}\ y\,(1-y)\ \phi = (y-t)\phi$

Exercise 4

$$f^{'}(x) = f(x)\,(1-f(x))$$

$$\frac{f^{'}(x)}{f(x)\,(1-f(x))} = 1$$

$$\int \frac{f^{'}(x)}{f(x)\,(1-f(x))}dx = \int 1\,dx$$

$$\int \left[\frac{1}{f(x)} + \frac{1}{1-f(x)}\right] d(f(x)) = x + C$$

$$ln|f(x)| - ln|1-f(x)| = x + C$$

$$\frac{f(x)}{1-f(x)} = e^{x+C}$$

$$f(x) = e^{x+C} - f(x)\,e^{x+c}$$

$$f(x) = \frac{e^{x+C}}{1+e^{x+C}} = \frac{e^x}{C+e^x}$$