# Convex Optimization Homework 3

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1. This is the convex optimization problem we want to solve:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \sum_{c<50, r<250} [i_{rc} - t(x_1 r + x_2 c + x_3)]^2 \\ \text{subject to} \quad & Gx \leq h \end{aligned}$$

Where $r$ is the *row* we are indexing, and $c$ is the *column* we are indexing. $i_{rc}$ is the pixel quantity at row $r$ and column $c$, $t = 255$ is the actual pixel value we are given at the top left corner, and $x = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^T$ is our optimization variable. Here are the values of $G$ and $h$:

$$G = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}$$

$$h = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Calling $v_{rc} = \begin{pmatrix} r & c & 1 \end{pmatrix}^T$, we derive our optimization function in standard quadratic programming form

$$\begin{aligned} \sum_{c<50, r<250} [i_{rc} - t(x_1 r + x_2 c + x_3)]^2 &= \sum_{c<50, r<250} [i_{rc} - t v_{rc}^T x]^2 \\ &= \sum_{c<50, r<250} [i_{rc}^2 - 2 i_{rc} + t v_{rc}^T x + t^2 x^T v_{rc} v_{rc}^T x]^2 \end{aligned}$$

Therefore, our optimization problem in QP standard form is:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \frac{1}{2} x^T P x + q^T x \\ \text{subject to} \quad & Gx \leq h \end{aligned}$$

where

$$P = 2t^2 \sum_{c<50, r<250} v_{rc} v_{rc}^T$$

and

$$q = -2t \sum_{c<50, r<250} i_{rc} v_{rc}$$

At this stage, we can now plug our model into a QP solver, such as that of `cvxopt` in Python. The code submitted does exactly this to obtain the observed parameters. It has been observed that

$$x = \begin{pmatrix} -1.00e-3 \\ -2.00e-3 \\ 10e-1 \end{pmatrix}$$

2. This is the convex optimization problem we want to solve:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \sum_{I \in S} \sum_{t \in I} [m(t) - s(t)]^2 \\ \text{subject to} \quad & Gx \leq h \end{aligned}$$

where $I$ is an interval containing $t$'s, and $S$ is the set of all intervals $I$. $m(t) = a_0 t + a_1 t + a_2 t^2 + a_3 t^3$ is the model, and $s(t)$ is the observed value at the point. Note that this can be either an $x$ or a $y$ value. We parameterize the points simply by calling $t$ at each point it's index in the list of all the points taken with `ginput`. Call

$$v(t) = \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

and

$$x_I = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

Then, we derive the standard QP form of the cost function as follows:

$$\begin{aligned}
\sum_{I \in S} \sum_{t \in I} [m(t) - s(t)]^2 &= \sum_{I \in S} \sum_{t \in I} [m^2(t) - 2m(t)s(t) + s^2(t)]^2 \\
&= \sum_{I \in S} \sum_{t \in I} [x_I^T v(t) v(t)^T x_I - 2s(t) v(t)^T x_I] + C \\
&= \sum_{I \in S} \frac{1}{2} x_I^T P_I x_I + q_I^T x_I + C
\end{aligned}$$

where

$$P_I = 2 \sum_{t \in I} v(t) v(t)^T$$

and

$$q_I = -2 \sum_{t \in I} s(t) v(t)$$

We can now state the minimization problem in standard form:

$$\begin{aligned}
&\underset{x}{\text{minimize}} && \frac{1}{2} x^T P x + q^T x \\
&\text{subject to} && Gx \leq h
\end{aligned}$$

where

$$x = \begin{pmatrix} x_{I1} \\ x_{I2} \\ \cdots \\ x_{IN} \end{pmatrix}$$

$$q = \begin{pmatrix} q_{I1} \\ q_{I2} \\ \cdots \\ q_{IN} \end{pmatrix}$$

and

$$P = \begin{pmatrix} P_{I1} & & 0 \\ & \ddots & \\ 0 & & P_{IN} \end{pmatrix}$$

where the subscript $I_i$ indicates the $i$th interval, so the $i$th set of $I$, and $N = |I|$. Note that these matrices are given in block matrix form. At this point, the problem is ready to be given to a QCQP solver, which was done in the program submitted. The results are also available in the report.