Convex Optimization Homework 3

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1. This is the convex optimization problem we want to solve:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \sum_{c<50, r<250} [i_{rc} - t(x_1 r + x_2 c + x_3)]^2 \\ \text{subject to} \quad & Gx \leq h \end{aligned}$$

Where r is the *row* we are indexing, and c is the *column* we are indexing. i_{rc} is the pixel quantity at row r and column c, $t = 255$ is the actual pixel value we are given at the top left corner, and $x = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^T$ is our optimization variable. Here are the values of G and h:

$$G = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}$$

$$h = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Calling $v_{rc} = \begin{pmatrix} r & c & 1 \end{pmatrix}^T$, we derive our optimization function in standard quadratic programming form

$$\begin{aligned} \sum_{c<50, r<250} [i_{rc} - t(x_1 r + x_2 c + x_3)]^2 &= \sum_{c<50, r<250} [i_{rc} - t v_{rc}^T x]^2 \\ &= \sum_{c<50, r<250} [i_{rc}^2 - 2 i_{rc} + t v_{rc}^T x + t^2 x^T v_{rc} v_{rc}^T x]^2 \end{aligned}$$

Therefore, our optimization problem in QP standard form is:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \frac{1}{2} x^T P x + q^T x \\ \text{subject to} \quad & Gx \leq h \end{aligned}$$

where

$$P = 2t^2 \sum_{c<50, r<250} v_{rc} v_{rc}^T$$

and

$$q = -2t \sum_{c<50, r<250} i_{rc} v_{rc}$$

At this stage, we can now plug our model into a QP solver, such as that of `cvxopt` in Python. The code submitted does exactly this to obtain the observed parameters. It has been observed that

$$x = \begin{pmatrix} -1.00e-3 \\ -2.00e-3 \\ 10e-1 \end{pmatrix}$$

2. This is the convex optimization problem we want to solve:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \sum_{I \in S} \sum_{t \in I} [m(t) - s(t)]^2 \\ \text{subject to} \quad & Ax = b \end{aligned}$$

where I is an interval containing t's, S is the set of all intervals I. $m(t) = a_0 t + a_1 t + a_2 t^2 + a_3 t^3$ is the model, and $s(t)$ is the observed value at the point. Note that this can be either an x or a y value. A and b encode the continuity and derivative continuity constraints, and will be explained later. We parameterize the points simply by calling t at each point it's index in the list of all the points taken with `ginput`. Call

$$v(t) = \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

and

$$x_I = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

Then, we derive the standard QP form of the cost function as follows:

$$\begin{aligned}
\sum_{I \in S} \sum_{t \in I} [m(t) - s(t)]^2 &= \sum_{I \in S} \sum_{t \in I} [m^2(t) - 2m(t)s(t) + s^2(t)]^2 \\
&= \sum_{I \in S} \sum_{t \in I} [x_I^T v(t) v(t)^T x_I - 2s(t) v(t)^T x_I] + C \\
&= \sum_{I \in S} \frac{1}{2} x_I^T P_I x_I + q_I^T x_I + C
\end{aligned}$$

where

$$P_I = 2 \sum_{t \in I} v(t) v(t)^T$$

and

$$q_I = -2 \sum_{t \in I} s(t) v(t)$$

We can now state the minimization problem in standard form:

$$\begin{aligned}
\underset{x}{\text{minimize}} \quad & \frac{1}{2} x^T P x + q^T x \\
\text{subject to} \quad & Ax \leq b
\end{aligned}$$

where

$$x = \begin{pmatrix} x_{I1} \\ x_{I2} \\ \cdots \\ x_{IN} \end{pmatrix}$$

$$q = \begin{pmatrix} q_{I1} \\ q_{I2} \\ \cdots \\ q_{IN} \end{pmatrix}$$

and

$$P = \begin{pmatrix} P_{I1} & & 0 \\ & \ddots & \\ 0 & & P_{IN} \end{pmatrix}$$

where the subscript I_i indicates the ith interval, so the ith set of I, and $N = |I|$. Note that these matrices are given in block matrix form. All that is left is to take account the constraints. The constraints are that the endpoints should be equal, and the functions should be continuous in both the first and second derivative. Noting that

$$(a_0 + a_1 t + a_2 t^2 + a_3 t^3)' = a_1 + 2a_2 t + 3a_3 t^2$$

and

$$(a_0 + a_1 t + a_2 t^2 + a_3 t^3)'' = 2a_2 + 6a_3 t$$

we are now ready to encode these constraints in A. For the equality constraints, for endpoint i, we encode the constraints by putting the following vector in A:

$$\begin{pmatrix} 1 & t & t^2 & t^3 & -1 & -t & -t^2 & -t^3 & 0 & ... \end{pmatrix}$$

where everything else are zeros and the starting index is equal to four times the endpoint index. We do this for all endpoints, which will encode the equality constraints. Similarly, by looking at the formula for the first and second derivatives of the model, we put the vector

$$\begin{pmatrix} 0 & 1 & 2t & 3t^2 & 0 & -1 & -2t & -3t^2 & 0 & ... \end{pmatrix}$$

in for every first derivative constraint and the vector

$$\begin{pmatrix} 0 & 0 & 2 & 6t & 0 & 0 & -2 & -6t & 0 & ... \end{pmatrix}$$

in for every second derivative constraint. Then, all that is left is to make b a column vector of 0's, which will finish our definition of constraints in standard form.

At this point, the problem is ready to be given to a QP solver, which was done in the code submitted. The results are also available in the report.