Perceptual Topology and the Geometry of Preference

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Abstract

This paper studies classical consumer preference relations when the decision maker has bounded perceptual resolution. Bounded resolution induces neighbourhoods of indistinguishable bundles which generate a perceptual topology that blurs and filters the underlying complete and continuous classical preference relation. This interaction results in a grey zone of alternatives that are perceptually distinct from a reference bundle but not robustly rankable against it implying systematic incompleteness of weak preferences and failure of negative transitivity of strict preferences. These features arise solely from bounded perceptual resolution and are invariant across metrics and coordinate systems used to represent the consumption space. We show that heterogeneous and state-dependent resolution environments admit a canonical normalization under which perceptual neighbourhoods are Euclidean. As perceptual blur disappears, the perceptual topology converges to the classical topology making it invisible and the standard model of consumer choice is recovered.

1 Introduction

The standard model of consumer choice begins with a geometric commitment. The commodity space is identified with $\mathbb{R}^n_+$ and preferences are assumed to be continuous with respect to the default natural Euclidean topology on this space. This choice has an intuitive economic interpretation. Each coordinate x_i represents a physical quantity of good i, and differences in bundles are measured by the familiar Euclidean metric.

This approach then, necessarily encodes, implements and reinforces the idea that small physical changes, tiny or infinitesimal additions and subtractions of goods, result in nearby bundles. This representation and interpretation is convenient, simple and well understood, and to an extent, with seemingly little abstraction, behaviourally relatable and thus, lays the foundation for the theory of the consumer. Preferences then imposed on the geometry of bundles, modelled as elements of $\mathbb{R}^n_+$, naturally inherit the Euclidean topology where

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continuity becomes a resulting regularity assumption. It is then understood that minute changes in physical quantities should not lead to large and discontinuous jumps in ranking. Much of modern microeconomic rests on this structure, the status quo, as it should be.

Despite that, the Euclidean representation is not behaviourally neutral. It imports into consumer theory the full structure of a continuous metric space and bundles are infinitely divisible. Between any two bundles, a continuum of intermediate bundles await. And for every such bundle, arbitrarily small Euclidean neighbourhoods exist for continuity to be evaluated over. Smoothness of preferences, examined at the margin, is therefore not derived from behaviour alone, but enabled by the geometry of the commodity space. Continuity becomes meaningful only relative to this geometry, and the Euclidean topology embeds a form of "unbounded resolution" perspective on changes in consumption. This implicit relaxation of human capacity is an inherent cost of the Euclidean representation.

Nevertheless, this idealization is appropriate for many economic environments. For example, when goods are divisible at the scale at which decisions are made, no conflicts arise. Alternatively, when smooth approximations obtained reliably by empirical means are available, the approach seems quite fitting. At the same time, in other instances, it can be conceptually fragile. Nothing in the economic interpretation of bundles requires that our individual notion of Euclidean "closeness" align with the decision maker's capacity to perceive or evaluate differences. Physical divisibility does not imply perceptual discriminability, and the endowed capacity to consider arbitrarily small variations in each coordinate is a mathematical convenience rather than a behavioural necessity or fact.

For these reasons, a number of empirical regularities inconsistent with this theory arise. For example, perceptual intransitivity is used to contradict the transitivity of the classical indifference relation. Scenarios where a sequence of shades of white may be arranged so that each adjacent pair is indistinguishable to the decision maker implies an indifference. Yet, the extreme endpoints differ so starkly, from white to black, and so a strict preference may be expressed by a decision maker. As shades are assumed indistinguishable to the decision maker, transitivity of the classical indifference relation will force the implausible and contradictory conclusion that the first and the last shades must also be indifferent. This paradox arises not from irrationality, but from the assumption of arbitrarily fine perceptual resolution. Put differently, what does it really mean for shades to be indistinguishable? This concept is ill-defined if the underlying topology deployed is Euclidean.

Another example is to consider three bundles, $x = (2, 2)$, $y = (3, 3)$ and $z = (4, 1)$. If one assumes monotonicity of preferences, it follows that $(3, 3) \succ (2, 2)$ in that y is strictly preferred to x. However, from the decision maker's perspective, it may be very difficult to evaluate y with z and z with x in the strict preference sense. This then directly violates negative transitivity of the $\succ$ relation. In turn, from the completeness of the classical weak preference relation, $\succsim$, the indifference relation is induced by the symmetric pairs via $\succsim$. Thus, the lack of comparability suggests that y is indifferent to z, and z is indifferent to x. Hence, the failure of negative transitivity also points to the failure of transitivity of the indifference relation since, in this example, $y \sim z$ and $z \sim x$, yet $y \not\sim x$.

The above two examples highlight a common point of contention in the classical approach surrounding this concept of indistinguishability and incomparability which is ultimately masked by the notion of indifference. After all, preferences founded on the Euclidean topology leaves no room for concepts like “too close to call”. These core incompatibilities are not necessarily geometric in nature, rather, premised by some arbitrary notion of “closeness” which strongly points to topological considerations instead. The remainder of the paper investigates this very question by introducing a new perceptual topology layered on top of the classical model. Perception here does not rely on geometry but on the decision maker’s ability to discriminate among nearby bundles. At any given bounded resolution $\kappa > 0$, the decision maker perceives bundles only up to a coarse “blob” of indistinguishable alternatives. These blobs may be irregular, asymmetric and non-metric in shape but their economic meaning remains true in that the decision maker cannot reliably tell those bundles apart.

The key insight is that observed behaviour depends only on these coarse discrimination neighbourhoods and not on the geometric structure of the commodity space. As a result, perceptual choice is fundamentally topological and only the pattern in which bundles fall inside each neighbourhood matters, not their distances, shapes or directions. Classical preferences lives in a Euclidean world, but perceived choice is a projection of that world through a blurred perceptual lens.

This paper connects to three strands of work. First, it is related to the theory of semiorders and constant-threshold representations (Luce (1956), Fishburn (1970), and Cantone, Giarlotta and Watson (2019)). These models impose a fixed just-noticeable-difference (JND) either through a uniform utility gap or through coordinatewise thresholds. They maintain the standard Euclidean topology and modify the preference relation itself. In contrast, the present paper retains the classical preference as primitive and instead modifies the topology through which alternatives are perceived. Indifference thickening, incompleteness, and violations of negative transitivity arise not from an assumed gap in utility but from the geometry of perceptual neighbourhoods.

Second, the framework is linked to incomplete preference models (Aumann (1962), Bewley (1986), Dubra, Maccheroni and Ok (2004), and Gerasimou (2018)). In those approaches, incomparability is introduced as a behavioural postulate or motivated by ambiguity or Knightian uncertainty. Here, incompleteness is neither axiomatic nor psychological. Rather, it emerges endogenously from finite perceptual resolution. The grey zone, which we will later characterize, is not an independent primitive but the locus where strict preference is topologically infeasible given the fixed resolution scale.

Third, the paper relates to work on the topological foundations of preference, including Debreu’s classical analysis (Debreu (1954, 1959)) and metric invariant formulations (Mas-Colell (1974)). The contribution differs in that the topology is not exogenously fixed but is generated by perceptual limits. A canonical homeomorphism shows that heterogeneous or non-Euclidean perceptual structures admit a Euclidean representation once expressed in perceptual units. The behavioural implications, the grey zone, the geometry of indistinguishability, and the emergence of incompleteness, are therefore invariant across all resolution structures that share the same minimal perceptual scale.

Taken together, the paper provides a topological foundation for preference under finite resolution that is distinct from utility gap models, distinct from axiomatic incompleteness, and distinct from classical Euclidean representations. It identifies bounded perceptual precision as an independent and geometrically disciplined source of observable choice phenomena.

Three main results follow.

First, introducing a positive perceptual resolution immediately alters the local geometry of choice. Indistinguishability regions acquire nonzero thickness, and strict preference is feasible only outside these regions. The resulting perceptual grey zone bundles that are perceptually distinguishable from a reference point, but not cleanly or robustly comparable to it, forms a structurally determined region of incompleteness. Within this zone, weak preference must remain unresolved and negative transitivity necessarily fails. These properties arise directly from the perceptual topology itself rather than from auxiliary assumptions about hesitation, ambiguity, or partial preference.

Second, the behavioural implications of finite resolution are strikingly robust. Perceptual neighbourhoods need not have any special geometric form. Nothing requires symmetry, convexity, homogeneity, or Euclidean structure in the original commodity coordinates. The invariance theorem in Section 5 shows that all qualitative features of the model, the decomposition into strictly comparable, indistinguishable, and grey zone regions, the topology of incomplete comparisons, and the failures of classical rationality properties, depend only on the underlying resolution structure. Any coordinate system or metric that generates the same minimal neighbourhoods yields identical observable behaviour.

Third, the perceptual environment admits a transparent canonical representation. Under mild regularity conditions, each coordinate can be reparametrized into perceptual units so that local discriminatory power is normalized and perceptual neighbourhoods become Euclidean balls of radius κ. This homeomorphic transformation preserves the ordinal content of the underlying classical preference while converting heterogeneous or state dependent resolution scales into a uniform geometric structure. In this representation, the model becomes particularly transparent. Perceptual blur corresponds exactly to finite Euclidean thickness, and the limit as $\kappa \to 0$ recovers the classical framework with zero thickness indifference curves, complete weak preference, negative transitivity, and ultimately, the standard utility representation.

Taken together, these results identify finite perceptual resolution as a fundamental geometric primitive for consumer theory. Properties often treated as primitive requirements of rationality, such as completeness of weak preference and negative transitivity of the strict preference, appear here as limiting features that reemerge only when discriminatory power becomes unbounded. Rather than fixing the Euclidean topology and imposing structure on preferences, this paper endogenizes the topology from perceptual constraints. Classical consumer theory thus arises as the unbounded perceptual resolution boundary case of a

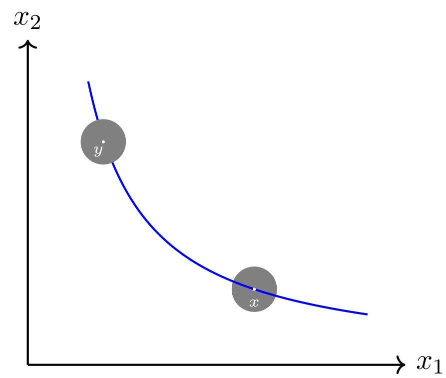

Figure 1: Despite the existence of a complete underlying classical ordering, the perceptual blur obscures the decision maker's ability to locate y relative to the smooth indifference curve induced by the classical ordering.

more general perceptual geometry.

As a final precursor to the analysis, Figure 1 illustrates a classical indifference curve together with perceptual neighbourhoods around two bundles x and y. For expositional clarity, the indifference curve in the figure is unmasked about the bundle x. Even with the added benefit of this visual aid, one that a decision maker with bounded perceptual resolution would not be privy to, a final question remains: Is y indifferent to, strictly better than, or strictly worse than x?

The remainder is as follows. Section 2 introduces the perceptual topology generated by a uniform minimal resolution scale $\kappa > 0$ and develops the induced perceptual relations. Section 3 analyzes the geometry of perceptual indistinguishability, establishes the decomposition into indistinguishable, strictly comparable, and grey-zone regions, and characterizes the resulting failures of weak completeness and negative transitivity. Section 4 establishes the impossibility of positively identifying the underlying classical preference structure with bounded perceptual resolution. Section 5 proves that all behavioural predictions of the model are invariant across metrics and coordinate systems that generate the same resolution structure. Additionally, it constructs the canonical homeomorphism that transforms a broad class of perceptual resolution structures including all separable environments into one with Euclidean perceptual neighbourhoods and shows that classical consumer theory is recovered in the limit as perceptual blur vanishes. Section 6 provides a brief discussion of the results and Section 7 concludes.

2 The Model

Let $X \subseteq \mathbb{R}^n$ denote the commodity space, equipped with its Euclidean metric. The natural topology induced on the commodity space is the Euclidean (classical) topology, $\mathcal{T}_c$, where open sets are defined by arbitrary unions of open balls:

$$B_r(x) = \{y \in \mathbb{R}^n : d(x, y) < r\},$$

for all $r \in \mathbb{R}_{++}$. On the classical topology, we build the primitives of the model by assuming the standard classical preference structure along with continuity defined in the usual way. We state them explicitly below:

Definition 1. *A* **Classical Weak Preference** *is a binary relation $\succsim_c$ on X satisfying:*

(C1) **Completeness.** *For all $x, y \in X$, $x \succsim_c y$ or $y \succsim_c x$.*

(C2) **Transitivity.** *For all $x, y, z \in X$, if $x \succsim_c y$ and $y \succsim_c z$, then $x \succsim_c z$.*

(C3) **Continuity.** *For each $x \in X$, the weak upper contour set:*

$$\succsim_c (x) := \{y \in X : y \succsim_c x\}$$

and weak lower contour set

$$\precsim_c (x) := \{y \in X : x \succsim_c y\}$$

are closed (or, equivalently, the associated strict contour sets are open) in $\mathcal{T}_c$.

The induced **Classical Strict Preference**, $\succ_c$, and corresponding **Classical Indifference**, $\sim_c$, relations are derived in their usual ways so that:

$$x \succ_c y \quad \Longleftrightarrow \quad (x \succsim_c y \quad \wedge \quad y \not\succsim_c x)$$

and

$$x \sim_c y \quad \Longleftrightarrow \quad (x \succsim_c y \quad \wedge \quad y \succsim_c x).$$

Similar to $\succsim_c (x)$ and $\precsim_c (x)$, the classical strict upper contour set is:

$$\succ_c (z) := \{y \in X : y \succ_c z\},$$

for $z \in X$, and the classical strict lower contour set is:

$$\prec_c (z) := \{y \in X : z \succ_c y\},$$

for $z \in X$, and are both open in $\mathcal{T}_c$. In the case of $Z \subseteq X$, we write the strict upper and lower contour sets as:

$$\succ_c (Z) := \{y \in X : y \succ_c z, \forall z \in Z\}$$

and

$$\prec_c (Z) := \{y \in X : z \succ_c y, \forall z \in Z\},$$

respectively.

On top of this structure, we now introduce and formalize the concept of bounded perceptual resolution.

Definition 2. *For each bundle $x \in X$, the* **Perceptual Indistinguishability Ball** *of radius κ is defined as:*

$$B_\kappa(x) = \{y \in X : d(x, y) < \kappa\}$$

These balls generate a topology, $\mathcal{T}_\kappa$, on X, given by:

$$\mathcal{T}_\kappa \equiv \left\{ \bigcup_{i \in I} B_\kappa(x_i) : x_i \in X \right\}.$$

Equivalently, $\mathcal{T}_\kappa$ is the perceptual topology whose basis consists of all sets $B_\kappa(x)$ for $x \in X$.[1]

It should be noted that if d is the Euclidean metric and $\kappa \to 0$, then the perceptual topology, $\mathcal{T}_\kappa$, approaches the classical topology, $\mathcal{T}_c$. For any fixed $\kappa > 0$, however, Euclidean open balls refine perceptual open balls strictly. Perceptual neighbourhoods are thick, typically containing regions that classical preferences dictate as indifference. The perceptual topology is the behavioural topology naturally suited for modelling choice under limited perceptual discriminability.

It should also be noted that the perceptual topology does not replace the Euclidean structure. Instead, it is a coarser topology superimposed on the same commodity space. Classical preferences are defined relative to the classical topology, $\mathcal{T}_c$, while perceptual preferences are generated by filtering the classical ordering through the perceptual topology, $\mathcal{T}_\kappa$.

Thus, two distinct topological structures operate throughout the analysis. The classical topology governs the underlying space of alternatives and encodes physical proximity, continuity of preferences, and the geometry of feasible paths. The perceptual topology, generated by the decision maker's bounded perceptual resolution governs which comparisons are perceptually admissible. These two structures serve different roles and are not interchangeable. In particular, true preferences are ontological and exist solely in the classical topology $\mathcal{T}_c$. Perception, on the other hand, is epistemological and exist solely in the perceptual layer governed by the perceptual topology $\mathcal{T}_\kappa$. The geometry of choice then, arises from the interaction of these two layers and is the basis of this analysis.

We now formally define the Perceptual Preference Relation.

[1]For any fixed $\kappa > 0$, the perceptual topology is not Hausdorff. Any two bundles within distance $< 2\kappa$ have intersecting κ-neighbourhoods and cannot be topologically separated.

Definition 3. *The* **Perceptual Indistinguishable Region** *for a bundle $x \in X$, B_x, is the closure of $B_\kappa(x)$ in $\mathcal{T}_c$. Formally:*

$$B_x := \overline{B_\kappa(x)} = \{y \in X : d(x, y) \leq \kappa\}.$$

The bundle y is said to be **Perceptually indistinguishable** *to x, $y \approx_\kappa x$, if $y \in B_x$.*

We say that y is **Perceptually Strictly Preferred** *to x, written $y \succ_\kappa x$, if:*

$$B_y \subset \succ_c (B_x).$$

Similarly, we say to y is **Perceptually Strictly Inferior** *to x, written $y \prec_\kappa x$, if:*

$$B_y \subset \prec_c (B_x).$$

The interpretation of the perceptual indistinguishability region is purely behavioural in that any bundle inside B_x cannot be perceptually distinguished from x by the consumer. Perception is uniform[2] in the sense that the JND is the same throughout the space. Perceptual strict preference here is a robustness condition in reflecting the confidence of the decision maker's ability to rank y against x. Any ambiguity that arises from the ranking is a result of the filtering of the latent classical preferences through the perceptual topology prohibiting the decision maker to confidently rank y against x.

We define the perceptual strict upper contour set as:

$$\succ_\kappa (x) := \{y \in X : B_y \subset \succ_c (B_x)\}.$$

Similarly, the perceptual strict lower contour set is:

$$\prec_\kappa (x) = \{y \in X : B_y \subset \prec_c (B_x)\}.$$

3 Properties of the Perceptual Relation

We begin our characterization of properties of the perceptual relation by first showing that for any $x \in X$, the perceptual strict upper contour set, the perceptual strict lower contour set, and the set of perceptually indistinguishable bundles do not partition the commodity space.

Lemma 1. *For any $x \in X$, there exists $y \in X$ such that:*

$$y \notin (\succ_\kappa (x) \cup \prec_\kappa (x) \cup B_x).$$

[2]The assumption of a uniform-κ is made here solely for expositional clarity. It does not restrict perceptual behaviour. The intended interpretation is behavioural in that any bundle lying in B_x is indistinguishable from x because the just-noticeable difference is taken to be the same at all locations. In later sections, we allow fully non-uniform or state-dependent resolution and show that the uniform case is merely a canonical representation under mild restrictions, and part of a hybrid canonical representation without restrictions.

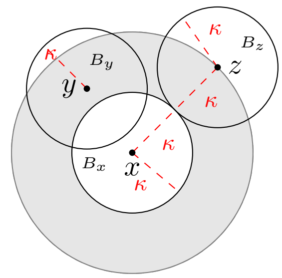

Figure 2: For $x \in X$ and given $\kappa > 0$, $B_y \cap B_x \neq \emptyset$ yet $y \notin B_x$, therefore, y falls into the grey zone. Geometrically, the grey zone arising from local blur, as stated in Lemma 3, is created by an annulus about B_x of width κ as shown by the point z.

Proof. Fix $x \in X$ and choose any $y \in X$ with $\kappa < d(x, y) < 2\kappa$. Then $y \notin B_x$ since $d(x, y) > \kappa$. Moreover, $B_x \cap B_y \neq \emptyset$, $B_y \not\subset B_x$ and $B_x \not\subset B_y$.

Because perceptual strict preference requires $B_y \subset \succ_c (B_x)$ some points of B_y lie in B_x, and no point in B_x can be strictly better than every points in B_x. Therefore, $y \notin \succ_\kappa (x)$. Similarly, since $B_y \cap B_x \neq \emptyset$, $B_y \not\subset \prec_c (B_x)$ so $y \notin \prec_\kappa (x)$. □

This finding represents the first major point of departure between the preference relation derived from the projection of the classical preference relation via the perceptual topology from the unfiltered classical preference relation itself. For any given $x \in X$, the strict upper and lower contour sets along with the associated indistinguishable set does not fully partition the underlying commodity space X. Immediately, a set of residual bundles remain. The collection of such bundles is what we will refer to as the grey zone which we now formally define and characterize its formulation after.

Definition 4. *For any $x \in X$, the* **Perceptual Grey Zone** *is the set:*

$$\parallel (x) := X \setminus (\succ_\kappa (x) \cup \prec_\kappa (x) \cup B_x)$$

The foundational structure of the perceptual grey zone is intuitive. Herein lies the bundles that are perceptually different from x but not sufficiently different enough for the decision maker to confidently rank such bundles due to their own perceptual blur. There are two distinct reasons why the grey zone arises. The first stems from the perceptual blur in the immediate neighbourhood, the local blur, about x (see Figure 2). By construction, a decision maker cannot distinguish bundles in B_x and thus, perceives them all equivalent from a ranking perspective. Now take a bundle $y \notin B_x$ so that $y \succ_c x$ but sufficiently close (i.e., $d(x, y) < 2\kappa$). In evaluating y under the perceptual lens, the decision maker understands that y itself, is indistinguishable from all bundles in B_y. And since $d(x, y) < 2\kappa$, $B_y \cap B_x \neq \emptyset$. Thus, if some bundles in B_y are clasically better than all points in

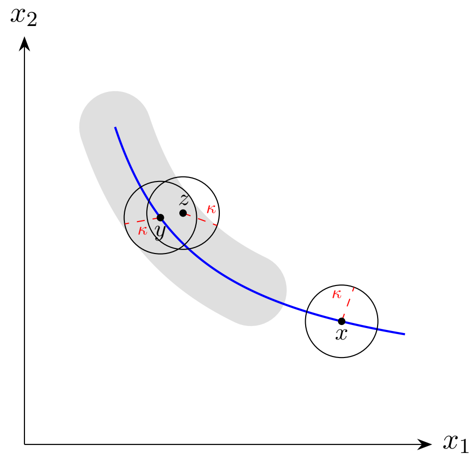

Figure 3: For $x \in X$ and given $\kappa > 0$, $B_z \not\subset \succ_c (B_x)$. Therefore, z falls into the grey zone. Geometrically, the grey zone arising from the classical indifference blur, as stated in Lemma 3, forms a 2κ-band around the classical indifference curve away from B_x.

B_x while other points in B_y are classically worse than some points in B_x, then y neither robustly dominates nor is robustly dominated by x. The decision maker, therefore, receives conflicting signals. While $y \succ_c x$, perceptual blur prevents a confident ranking resulting in such bundles falling into the grey zone.

The second reason for the existence of the grey zone is from conflicting signals from the classical structure away from the anchor bundle $x \in X$, the indifference blur (see Figure 3). Take any $z \notin B_x$ so that $z \sim_c x$. Then for any $y \in B_z$, it follows that $z \in B_y$. Combined with the disjoint nature of the sets $\sim_c (x)$, $\succ_c (x)$ and $\prec_c (x)$, $z \notin \succ_c (B_x)$. Naturally, $B_y \not\subset \succ_c (B_x)$ and $B_y \not\subset \prec_c (B_x)$ which results in y landing in the grey zone. Again, the decision maker is unable to assert dominance, in either direction, of y.

The grey zone is central to the geometry of choice under perceptual blur. It is worth noting that apparent counterexamples in which two "clearly different" bundles seem nonetheless incomparable under the perceptual topology typically arise from an implicit choice of the perceptual threshold that do not correspond to the assumed resolution of the decision maker. The grey zone is defined relative to the decision maker's perceptual ability, not the modeller's Euclidean intuition. Thus, if two distinctly different bundles is viewed as being easily ranked in any hypothetical example, then it simply implies that the perceptual threshold appropriate for that example is necessarily larger than the one under which the grey zone was defined. The magnitude of the grey zone should not be interpreted as large in an absolute sense. Its extent is entirely determined by the decision maker's perceptual capacity κ. We summarize our finding in the characterization of the grey zone in the following Lemma.

Lemma 2. *For any $x \in X$ and perceptual threshold $\kappa > 0$, the grey zone $\parallel(x)$ is contained within two geometric envelopes:*

1. **Local Blur Envelope.** *Any y that falls into the grey zone due to local blur must satisfy:*
$$d(x, y) \leq 2\kappa.$$
Equivalently:
$$B_x \cap B_y \neq \emptyset \quad \Longrightarrow \quad d(x, y) \leq 2\kappa.$$
Thus all local-blur points lie in the annulus:
$$\{y : \kappa < d(x, y) \leq 2\kappa\}.$$

2. **Indifference Blur Envelope.** *Any y that falls into the grey zone due to classical indifference blur lies within the tubular band:*
$$\bigcup_{z \sim_c x} B_z.$$
In particular, every point within perceptual distance κ of an indifference point $z \sim_c x$ is geometrically eligible to belong to the grey zone.

Lemma 3. *For any $x \in X$ and perceptual threshold $\kappa > 0$, a bundle $y \notin B_x$ lies in the grey zone if and only if at least one of the following holds:*

1. **Local Blur.**

   With $B_x \cap B_y \neq \emptyset$, $B_y \not\subset \succ_c (B_x)$ or $B_y \not\subset \prec_c (B_x)$.

2. **Classical Indifference Blur.**

   There exists $z \in B_y$ such that $z \sim_c x$.

Lemma 2 describes the geometric envelopes, a κ-annulus around B_x and a 2κ-thickening of the classical indifference set, within which grey-zone points must lie. These sets provide the correct metric intuition for the shape and extent of the grey zone, but they do not by themselves determine grey-zone membership. Lemma 3 gives the exact behavioural conditions. A point lies in the grey zone if and only if either local perceptual overlap prevents robust dominance, or a perceptually indistinguishable neighbour lies on the classical indifference curve. Thus the geometric envelopes delimit where grey-zone points can arise, while Lemma 3 specifies precisely which points within those envelopes actually do.

It should be pointed out that the local blur of the grey zone contains bundles where the decision maker knows that they are perceptually different, yet does not know how they are ordered. At its inner boundary is the indistinguishable region, thus, differences are perceptually annihilated and alternatives collapse into identity. At its outer boundary, differences are perceptible but not robust. Arbitrarily small perturbations in perception can reverse the ranking. It therefore, separates perceptual identity from ordinal confidence. It is neither ignorance nor indifference, but a structural inability to certify dominance.

We characterize the remaining geometric and topological structures.

Lemma 4. *Fix $x \in X$ and $\kappa > 0$. Then:*

1. *The sets $\succ_\kappa (x)$ and $\prec_\kappa (x)$ are open in $\mathcal{T}_c$.*

2. *The set B_x is closed in $\mathcal{T}_c$.*

3. *The sets $\succ_\kappa (x)$, $\prec_\kappa (x)$, B_x and $\|_\kappa (x)$ forms a partition for the commodity space X. That is:*

$$X = \succ_\kappa (x) \sqcup \prec_\kappa (x) \sqcup B_x \sqcup \|_\kappa (x).$$

4. *The boundaries of $\succ_\kappa (x)$ and $\prec_\kappa (x)$ are contained in $\|_\kappa (x)$:.*

$$\partial \succ_\kappa (x) \subset \|_\kappa (x) \quad \text{and} \quad \partial \prec_\kappa (x) \subset \|_\kappa (x)$$

The structure that the perceptual preference structure brings then immediately provides us with the following separation theorem.

Theorem 1. *Every continuous path connecting a point in $\succ_\kappa (x)$ to a point in $\prec_\kappa (x)$ must intersect the perceptual grey zone $\|_\kappa (x)$. Furthermore, every continuous path connecting a point in $\succ_\kappa (x)$, and $\prec_\kappa (x)$, to a point in B_x must also intersect the perceptual grey zone.*

Proof. As both sets are open, in $\mathcal{T}_c$, and disjoint with $\partial \succ_\kappa (x) \subset \|_\kappa (x)$ and $\partial \prec_\kappa (x) \subset \|_\kappa (x)$, any continuous path $p : [0,1] \to X$ with $p(0) \in \succ_\kappa (x)$ and $p(1) \in \prec_\kappa (x)$ must exit $\succ_\kappa (x)$ at some point t^*. It follows that $p(t^*) \in \partial \succ_\kappa (x) \subset \|_\kappa (x)$.

In the case of separation between $\succ_\kappa (x)$, and $\prec_\kappa (x)$, from B_x, it follows directly from the existence of the local blur shown in Lemma 2. $\square$

Thus, the geometric and topological structure given by the perceptual preference relation becomes clear. The classical preference structure has been coarsen in that for any given $x \in X$, the classical indifference set is enclosed by a 2κ-band away from the point x creating the classical indifference blur of the perceptual grey zone. At the same time, a κ width annulus surrounds the indistinguishability region B_x, anchored at x, creating the local blur completing the perceptual grey zone. The grey zone then serves as the separation between the perceptual strict upper contour set and the perceptual strict lower contour set as well as the perceptual indistinguishability region.

With this structure, we conclude this section by stating and proving the remaining properties of the induced perceptual preference structure.

Proposition 1. *The perceptual indistinguishable relation $\approx_\kappa$ is not transitive.*

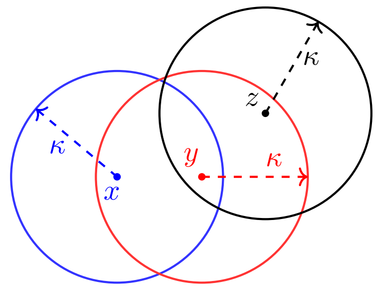

Figure 4: Illustration of the failure of transitivity of $\approx_\kappa$. Here, $y \in B_x$ and $z \in B_y$. Therefore, $x \approx_\kappa y$ and $z \approx_\kappa y$. However, $d(x,z) > \kappa$ and so $z \notin B_x$ and symmetrically, $x \notin B_z$ so $x \not\approx_\kappa z$. Together, $x \approx_\kappa y$ and $y \approx_\kappa z$ but $x \not\approx_\kappa z$ and hence, transitivity of $\approx_\kappa$ fails.

Proof. Take any $x, y \in X$ so that $\kappa < d(x,y) < 2\kappa$. It immediately follows that $x \not\approx_\kappa y$ by definition. Furthermore, there exists some $z \in X$ so that $d(x,z) < \kappa$ and $d(z,y) < \kappa$. Hence, $z \in B_x$ and so $x \approx_\kappa z$, $z \in B_y$ and so $z \approx_\kappa y$, along with our observation that $x \not\approx_\kappa y$ demonstrates that $\approx_\kappa$ is not transitive on X. □

The failure of transitivity of $\approx_\kappa$ (see Figure 4) is a geometric property of perceptual neighbourhoods and is thus a feature, not a defect. That is, fixed radius balls in a metric space need not overlap transitively. It reflects the plausible phenomenon that small, individually imperceptible differences can accumulate to produce a perceptible difference when taken together. Hence, the perceptual topology is capable of rationalizing the textbook example noted earlier regarding the sequence of indistinguishable darker shades of white yet still resulting in a strict preference between the extremes of white and black. Consequently, perceptual geometry, rather than the ambient Euclidean geometry in which classical preferences are predicated on, governs which distinctions the decision maker is able to make. This intransitivity is the source of several downstream phenomena in the model, including the incompleteness of weak preference and failure of negative transitivity of strict preference.

Proposition 2. *The perceptual strict preference relation $\succ_\kappa$ is:*

1. **Transitive:** *For all $x, y, z \in X$, if $x \succ_\kappa y$ and $y \succ_\kappa z$ then $x \succ_\kappa z$.*
2. **Fails Negative Transitivity:** *There exists $x, y, z \in X$ so that $x \not\succ_\kappa y$ and $y \not\succ_\kappa z$ but $x \succ_\kappa z$.*

Proof. For part 1, take any $x, y, z \in X$ so that $x \succ_\kappa y$ and $y \succ_\kappa z$. It follows that $B_x \subset \succ_c (B_y)$ and $B_y \subset \succ_c (B_z)$ which, by definition, implies that $x \succ_c y$ and $y \succ_c z$. If $x \not\succ_\kappa z$ then there must exist some $x' \in B_x$ so that $x' \notin \succ_c (B_z)$. But as $B_y \subset \succ_c (B_z)$, $x' \notin B_y$ which, in turn, contradicts the assumption that $B_x \subset \succ_c (B_y)$.

For part 2, pick any $x \in X$ and consider some $z \in X$ so that $x \succ_\kappa z$ and $d(x, z) < 3\kappa$. It follows that there exists some $y \in X$ so that $d(y, x) < \kappa$, $\kappa < d(y, z) < 2\kappa$. Then, $x \approx_\kappa y$ and so $x \not\succ_\kappa y$. Further, $\kappa < d(y, z) < 2\kappa$ implies that $B_y \cap B_z \neq \emptyset$ and $y \notin B_z$ and so $y \not\approx_\kappa z$ and $y \not\succ_\kappa z$. Therefore, $x \not\succ_\kappa y$, $y \not\succ_\kappa z$ and $x \succ_\kappa z$ demonstrating the failure of negative transitivity of $\succ_\kappa$. $\square$

It is transparent in the classical framework that when strict preference is taken as primitive, the induced weak preference is complete solely because it is defined by nondominance. If we defined a "naive" perceptual weak preference relation by analogy with the classical construction, then all grey zone points are identified as mutually weakly preferred and therefore collapses the grey zone into spurious perceptual indifference. This is not behaviourally meaningful under bounded perceptual resolution. Applied to $\succ_\kappa$, it forces every grey zone pair (x, y) into reciprocal weak preference and hence into perceptual indifference, eliminating the very incomparabilities generated by perceptual blur to begin with. Any weak perceptual preference relation that preserves the grey zone as a distinct behavioural category must therefore be incomplete. Completeness of weak preference is not a primitive property but a limiting one. It reemerges only when the grey zone vanishes in the limit of unbounded perceptual resolution.

4 Non-Identifiability of Classical Preferences

From the previous section, it was shown how the decision maker's preference relation is coarsened when the classical preference structure is projected through the perceptual lens. However, a critical question remains. Will a rational decision maker be able to identify the underlying classical preference structure that generated such a coarsened result? After all, if a decision maker is able to deduce, for example, $\sim_c (x)$ just by observing a symmetric κ-band, would it then not be possible to simply affirm that the latent indifference set is the one that runs through the middle?

Below, we formalize the identification problem and then conclude this section with the non-identifiability theorem along with a brief discussion.

Definition 5. *For any $x \in X$, given $\kappa > 0$, let:*

$$\mathcal{O} := \{\mathbf{B}, \mathbf{U}, \mathbf{L}, \mathbf{G}\}$$

*denote the four perceptual categories (indistinguishable, strictly preferred, strictly inferior, and grey zone). The **Perceptual Classification** around x is the map:*

$$C_x : X \to \mathcal{O}$$

defined by:

$$C_x(y) := \begin{cases} \mathbf{B}, & y \in B_x \\ \mathbf{U}, & y \in \succ_\kappa (x) \\ \mathbf{L}, & y \in \prec_\kappa (x) \\ \mathbf{G}, & y \in \parallel_\kappa (x) \end{cases}.$$

Definition 6. *Given $\kappa > 0$, a classical preference relation $\succ_c$ is said to be* **Identifiable** *if it is the unique classical preference relation that induces the family $\{C_x\}_{x\in X}$. Formally, for any classical preference $\succ_c'$, if:*

$$C_x^{\succ_c}(y) = C_x^{\succ_c'}(y),$$

for all $x, y \in X$, then $\succ_c' = \succ_c$.

Intuitively, arbitrarily small perturbations to any given classical preference relation $\succ_c$, by coarsening a neighbourhood to extend the indifference set to generate an alternative classical preference relation $\succ_c'$, will ultimately go undetected by the perceptual filter. Thus, unique identification of an underlying classical preference relation, given an observed classification, is not possible. Since perceptual strict preference depends only on how entire κ-neighbourhoods compare to one another, the fine structure of the classical ranking inside these regions is observationally irrelevant. Consequently, many distinct continuous classical preferences generate identical perceptual classifications. Even a fully rational decision maker who understands the perceptual mechanism cannot reverse engineer a unique classical preference from the perceptual one.

It should be noted that even by strengthening the structure of classical preference to include properties such as monotonicity and convexity does not overturn the non-identifiability result. The coarseness of the perceptual topology collapses infinitely many locally distinct classical preferences into the same perceptual classification. In the absence of finer perceptual resolution, the underlying preference ordering remains underidentified. Identification, therefore, fails in general not because preferences are insufficiently structured, but because perception cannot resolve the distinctions those structure imply.

Theorem 2. *Fix $\kappa > 0$. Let $\succ_c$ be any complete, transitive, and continuous classical preference relation on X, and let $\{C_x\}_{x\in X}$ be the associated family of perceptual classifications. Then $\succ_c$ is not identifiable from $\{C_x\}_{x\in X}$. There exist infinitely many distinct classical preferences $\succ_c' \neq \succ_c$, each complete, transitive, and continuous, that induce the exact same perceptual classifications.*

Proof. Fix $x \in X$ and choose any $\lambda \in (0, \kappa)$. Construct a new classical preference relation $\succ_c'$ so that $y \sim_c' x$, for all $y \in \overline{B_\lambda(x)}$ and $\succ_c' = \succ_c$, otherwise. Since $\overline{B_\lambda(x)} \subset B_x$, this modification is a local coarsening of $\succ_c$ within a region strictly smaller than the perceptual indistinguishability ball. It preserves completeness and transitivity, and continuity is maintained because upper contour sets are altered only by collapsing a subset of an already connected indifference region.

For any anchor x, the perceptual categories $\succ_\kappa (x)$, $\prec_\kappa (x)$, B_x, and $\|_\kappa (x)$ depend only on classical comparisons between the entire perceptual ball B_x and other perceptual balls B_y. The modification to $\succ_c$ occurs strictly inside $B_\lambda(x) \subset B_x$ and does not affect any classical comparisons on B_x that determine robust dominance. Hence $C_x^{\succ_c'}(y) = C_x^{\succ_c}(y)$, for all $x, y \in X$.

Since λ may be chosen arbitrarily in $(0, \kappa)$, $\succ_c$ is not identifiable. □

5 Invariance and Canonical Representation

A central feature of the framework is that its geometric and behavioural implications do not depend on the particular coordinate system or metric used to represent perception. What matters is only the resolution structure generated by projecting the classical topology through the perceptual filter and that the decision maker can only discriminate across bundles down to the scale of κ. The analysis is therefore fundamentally coordinate-free. All that is required is a neighbourhood system, around each point, reflecting the decision maker's fixed perceptual resolution.

Recall that the perceptual indistinguishability ball at x is represented by the open ball:

$$B_\kappa(x) = \{y \in X : d(x, y) < \kappa\}$$

The particular metric d here plays no essential role. The key structural property is that $\{B_\kappa(x)\}_{x \in X}$ forms the minimal neighbourhood system through which the decision maker perceives X. In particular, the agent cannot distinguish any two bundles whose separation lies strictly within B_x, regardless of the coordinate system used to describe X.

Definition 7. *A topology $\mathcal{T}$ on X satisfies the* **Fixed Resolution Property** *with scale κ if, for every $x \in X$:*

1. *$B_\kappa(x) \in \mathcal{T}$*
2. *No open neighbourhood of x is strictly contained in $B_\kappa(x)$*

That is, $B_\kappa(x)$ is the smallest perceptually admissible neighbourhood of x. Any further refinement is beyond the decision maker's discriminatory capacity.

The perceptual preference relation induced from a classical preference $\succ_c$, depends only on this fixed-resolution neighbourhood system. In particular, all results derived above, the existence and geometry of the grey zone, the openness of perceptual strict upper and lower contour sets, the separation theorem, the intransitivity of $\approx_\kappa$, the failure of negative transitivity of $\succ_\kappa$, and the non-identifiability of classical preferences, are invariant across all topologies or metrics that share the same minimal resolution structure. Thus, the behaviour implied by perceptual blur is independent of coordinates, norms, or even the choice of metric. Rather, it is determined entirely by the perceptual indistinguishable neighbourhoods $\{B_\kappa(x)\}_{x\in X}$.

Proposition 3. *Let $(X, \mathcal{T})$ be a topological space and let $\succ_c$ be a complete, transitive, and continuous classical preference relation on X. Suppose that $\mathcal{T}$ satisfies the fixed-resolution property with scale $\kappa > 0$, so that for each $x \in X$, the perceptual indistinguishability ball $B_\kappa(x)$ is an open minimal neighbourhood of x, and let B_x be the corresponding perceptual indistinguishability region. Define the perceptual relations $\approx_\kappa$, $\succ_\kappa$, $\prec_\kappa$ and the grey zone $\|_\kappa (x)$ as above.*

Then all geometric and behavioural properties of the perceptual structure derived in the previous sections hold on $(X, \succ_c, \mathcal{T})$. In particular, for every $x \in X$:

1. **Grey Zone Decomposition and Blur:** *The space decomposes as:*

$$X \;=\; B_x \sqcup \succ_\kappa (x) \sqcup \prec_\kappa (x) \sqcup \|_\kappa (x),$$

   where B_x is closed in $\mathcal{T}_c$, $\succ_\kappa (x)$ and $\prec_\kappa (x)$ are open in $\mathcal{T}_c$, $\partial \succ_\kappa (x) \subset \|_\kappa (x)$ and $\partial \prec_\kappa (x) \subset \|_\kappa (x)$. The grey zone $\|_\kappa (x)$ captures both local blur (for bundles at distances in $(\kappa, 2\kappa)$) and classical indifference blur (a 2κ-band around classical indifference sets away from B_x).

2. **Path Separation:** *Every continuous path from a point in $\succ_\kappa (x)$ to a point in $\prec_\kappa (x)$ intersects the grey zone $\|_\kappa (x)$. Furthermore, every continuous path connecting a point in $\succ_\kappa (x)$, and $\prec_\kappa (x)$, to a point in B_x, must also intersect the perceptual grey zone.*

3. **Intransitivity of Perceptual Indistinguishability:** *The perceptual indistinguishability relation $\approx_\kappa$ is not transitive.*

4. **Transitivity and Failure of Negative Transitivity:** *The perceptual strict preference relation $\succ_\kappa$ is transitive but fails negative transitivity.*

5. **Non-identifiability:** *The classical preference $\succ_c$ is not identifiable from the induced perceptual classifications $\{C_x\}_{x\in X}$.*

All these properties depend only on the neighbourhood structure $\{B_\kappa(x)\}_{x\in X}$ and not on the particular choice of coordinates, metric, or norm used to represent perception.

The proof follows directly from the observation that each of the results above uses only finite perceptual resolution and does not require the ability to refine open neighbourhoods below scale κ. Therefore, any topology consistent with the constraint yields the same behavioural structure.

5.1 Nonuniform-κ

Recall when first introduced in Section 2, the perceptual indistinguishability region at $x \in X$, B_x, explicitly assumed the symmetry of the perceptual bound of the decision maker. In a way, this was relaxed in the topological invariance result in Proposition 3 insofar as the chosen metric allowed it to. We now extend and generalize perceptual bounds further.

Let $\kappa_i : \mathbb{R}_+ \to \mathbb{R}_+$ be a state-dependent locally bounded perceptual resolution in the i-th coordinate. Perceptual distance is here cumulative and the local resolution κ_i determines marginal discriminability.

Definition 8. *Let $\kappa_i : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous and strictly positive for each i. The* **Canonical Perceptual Transformation** *is the mapping $T = (T_1, \ldots, T_n)$ with:*

$$T_i(x_i) = \int_0^{x_i} \frac{1}{\kappa_i(t)} dt,$$

for all i. Each T_i is strictly increasing and T is a homeomorphism onto its image.

The commodity space X encodes physical quantities, whereas perceptual comparisons operate in units of discriminability rather than physical scale. Classical preferences are defined on coordinates of X, but their perceptual projection depends on the decision maker's resolution. The mapping T converts physical units into perceptual units. In the transformed space, distances reflect cumulative discriminability. In particular, T equalizes the local resolution across coordinates, so that perceptual neighbourhoods become spherical in T-space. Hence, the uniform bounded-resolution model used as our benchmark is precisely the canonical representation of any separable state-dependent resolution structure $\{\kappa_i\}$.

Define the perceptual metric d_k on X by:

$$d_k(x, y) := \left(\sum_{i=1}^{n} \left(\int_{y_i}^{x_i} \frac{1}{\kappa_i(t)} dt \right)^2 \right)^{1/2}.$$

Theorem 3. *Let T be a canonical perceptual transformation. Then:*

1. *Each T_i is a strictly increasing homeomorphism onto its image and hence T is a homeomorphism from X onto $T(X)$.*
2. *T is an isometry between (X, d_κ) and $(T(X), \| \cdot \|_2)$, so that for all $x, y \in X$:*

$$d_\kappa(x, y) = \|T(x) - T(y)\|_2.$$

3. *For every $x \in X$ and $r > 0$:*

$$T\big(B_r^{d_\kappa}(x)\big) = B_r^{\|\cdot\|_2}(T(x)).$$

Thus, a fixed perceptual resolution κ corresponds to Euclidean balls of radius κ in T-space.

Proof. For (1): Since κ_i is continuous and strictly positive, T_i is continuous and strictly increasing on $\mathbb{R}_+$. Hence T_i is a continuous bijection from $\mathbb{R}_+$ onto the interval $T_i(\mathbb{R}_+)$. Its inverse is continuous because every strictly monotone continuous bijection between intervals has a continuous inverse. Thus, each T_i is a homeomorphism onto its image, and the product map $T = (T_1, \ldots, T_n)$ is a homeomorphism from X onto $T(X)$.

For (2): For any $x, y \in X$ and each i:

$$T_i(x_i) - T_i(y_i) = \int_{y_i}^{x_i} \frac{1}{\kappa_i(t)}\, dt.$$

By the definition of d_κ:

$$d_\kappa(x,y) = \left(\sum_{i=1}^{n} \big(T_i(x_i) - T_i(y_i)\big)^2 \right)^{1/2} = \|T(x) - T(y)\|_2,$$

which shows that T is an isometry.

For (3): For any $x \in X$ and $r > 0$:

$$T(B_r^{d_\kappa}(x)) = \{T(y) : d_\kappa(x,y) < r\} = \{z \in T(X) : \|z - T(x)\|_2 < r\} = B_r^{\|\cdot\|_2}(T(x)),$$

where the last equality is simply the definition of the Euclidean ball. Hence, this yields the desired correspondence between perceptual balls in (X, d_κ) and Euclidean balls in T-space. $\square$

The above theorem demonstrates that the transformation T provides a canonical representation of perceptual discrimination that respects non-uniform JND structure. Heterogeneous or state-dependent κ_i functions merely distort the commodity space before perception is evaluated. In perceptual coordinates, the model exhibits the same spherical geometry as in the uniform-κ case.

It is worth emphasizing that the symmetry of perceptual neighbourhoods in the canonical representation does not impose symmetry in the original commodity coordinates. Because T rescales each coordinate by its local perceptual resolution $1/\kappa_i(\cdot)$, a unit of perceptual distance may correspond to commodity increments of very different magnitudes at different locations. In commodity space, the κ-neighbourhoods may therefore be highly irregular, asymmetric and state dependent. Their symmetry emerges only after normalization. This idea is shown in Figure 5. T converts heterogeneous local resolutions into uniform perceptual units under which, perceptual neighbourhoods become Euclidean balls. The two representations are topologically equivalent and T preserves exactly which points lie in which neighbourhoods even though the geometry differs sharply across the various coordinate systems.

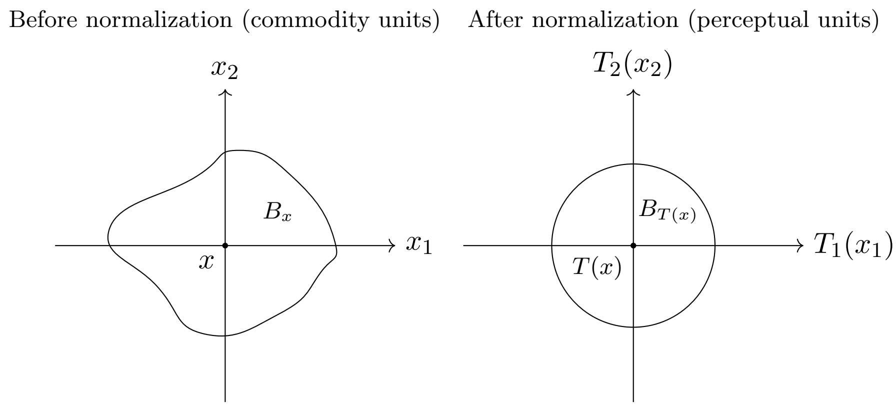

Figure 5: Illustration of perceptual geometry before and after normalization. Left: Perceptual indistinguishability region B_x in commodity units, where physical coordinates distort perceptual distances and κ-balls need not be symmetric, convex, or elliptical. Their shapes reflect arbitrary, potentially state-dependent variation in perceptual resolution across attributes. Right: Perceptual indistinguishability region $B_{T(x)}$ under the canonical normalization T, where perceptual units equalize local resolution and $B_{T(x)}$ become true Euclidean balls.

5.2 Vanishing Perceptual Blur and Classical Preferences

The canonical representation constructed above assumes strictly positive perceptual resolution $\kappa_i > 0$. This assumption is technical in nature as it forces each of the coordinatewise integrand of the canonical perceptual transformation to be well defined. We relax this technical requirement slightly and extend this framework to allow perceptual resolution to decay to zero at finite discriminability distances. This yields a natural boundary case of the model in which perceptual blur vanishes and classical preference is recovered.

Definition 9. *The* **Perceptual Resolution Function** *in coordinate i is a mapping $\kappa_i : [0, \infty) \to (0, \infty)$ that depends on discriminability distance d such that there exists a (possibly finite) critical value $\overline{d}_i \in (0, \infty]$ so that:*

$$\lim_{d \uparrow \overline{d}_i} \kappa_i(d) = 0,$$

with $\kappa_i(d) > 0$ for all $d < \overline{d}_i$.

We refer to $\overline{d}_i$ as the **Resolution Horizon** *for coordinate i.*

As an extreme and stylized example, if the perceptual resolution function is identical across all coordinates and strictly decreasing and reaches zero at a finite resolution horizon $\overline{d}$, the geometry of the grey zone undergoes a systematic transformation. Close to the anchor bundle x, where perceptual resolution is coarser, classical indifference curves

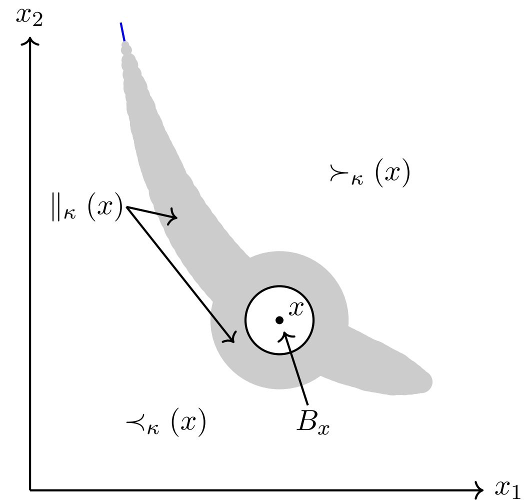

Figure 6: Perceptual indifference blur around a classical indifference curve under decaying resolution. Near the reference point, perceptual limitations create a thick band of indistinguishable bundles around the underlying indifference curve. As perceptual resolution improves with distance, this band tapers and ultimately collapses at the resolution horizon $\overline{d}$, beyond which the agent recovers perfect discrimination. The classical indifference curve is thus the limiting boundary of the perceptual blur under vanishing resolution.

thicken by 2κ as before, generating the largest cross-section of the perceptual blur. As one moves away, and so the local blur fades and classical indifference blur begins to take over, perceptual resolution improves. Given the strictly decreasing perceptual resolution function, the indistinguishable region surrounding each point $z \sim_c x$ shrink accordingly. The resulting perceptual blur contracts monotonically forming a tapered tube around the indifference curve whose width at any point reflects the local blur respecting the decay governed by the perceptual resolution function. The tapering progressively narrow and ultimately collapses at the resolution horizon $\overline{d}$ where perceptual blur vanishes and perfect discrimination is restored. This is illustrated in Figure 6.

Note that when $\overline{d}_i < \infty$, the decision maker's ability to discriminate in coordinate i increases without bound as distances approach $\overline{d}_i$. The canonical transformation extends naturally via an improper integral.

Given perceptual resolution functions $\{\kappa_i\}$, define:

$$T_i(x_i) \;=\; \int_0^{x_i} \frac{1}{\kappa_i(d(t))}\, dt,$$

where $d(t)$ denotes discriminability distance along coordinate i. If $\kappa_i(d) \to 0$ as $d \to \overline{d}_i$, the integral is interpreted in the improper sense and diverges as $x_i \to \overline{d}_i$. Define $T =$

$(T_1, \ldots, T_n)$. The key structural point is that divergence of T_i does not break the canonical representation. Instead, it enlarges the perceptual coordinate range and preserves the homeomorphism.

The perceptual coordinates induced by T are always interpreted relative to the anchor x. Consequently, a decaying resolution bound implying improved discriminability yields a stretching geometric effect on the underlying commodity coordinate projected. The region in which T_i becomes steep or divergent is local and depends on the location of the anchor point x along the i-th coordinate. As x varies, the point at which perceptual resolution becomes arbitrarily fine shifts accordingly, ensuring no global inconsistency despite the unbounded scaling of T_i near its resolution horizon.

Proposition 4. *Given perceptual resolution functions* $\{\kappa_i\}$*:*

1. *Each* T_i *is continuous and strictly increasing on* $[0, \overline{d}_i)$ *and map onto* $[0, \infty)$ *if* $\overline{d}_i < \infty$ *and onto* $[0, T_i(\infty))$ *otherwise.*

2. $T = (T_1, \ldots, T_n)$ *is a homeomorphism from* X *onto its image* $T(X) \subseteq \mathbb{R}^n_+$*, even when some coordinates satisfy* $\overline{d}_i < \infty$*.*

3. *Under the perceptual metric:*

$$d_\kappa(x, y) = \left(\sum_{i=1}^{n} (T_i(y_i) - T_i(x_i))^2 \right)^{\frac{1}{2}}$$

and the mapping T *is an isometry between* (X, d_κ) *and* $(T(X), \|\cdot\|_2)$*.*

Proof. For (1): Since $\kappa_i(t) > 0$ for all $t < \overline{d}_i$ and is continuous, $1/\kappa_i(t)$ is continuous and positive. Thus T_i is strictly increasing and continuous. If $\kappa_i(t) \to 0$, then:

$$\int_0^{\overline{d}_i} \frac{1}{\kappa_i(t)}\, dt = \infty,$$

hence T_i maps $[0, \overline{d}_i)$ onto $[0, \infty)$.

For (2): Each T_i is a continuous, strictly monotone bijection between intervals, so it is a homeomorphism onto its image. The product of coordinatewise homeomorphisms remains a homeomorphism onto the product image.

For (3): The distance identity:

$$d_\kappa(x, y) = \|T(y) - T(x)\|_2$$

follows from the definition of d_κ and the componentwise representation of T, exactly as in Theorem 3. □

From the proofs of both Theorem 3 and Proposition 4, it is clear that the existence of a canonical representation hinges solely on the existence of an underlying homeomorphism from commodity space into perceptual space. Although technically straightforward, this observation is conceptually fundamental for any generalized perceptual resolution function $\kappa : X \to \mathbb{R}^n_+$ for which a continuous coordinatewise strictly increasing transformation T exists with continuous inverse, the mapping T is a homeomorphism onto its image. Hence every such perceptual environment admits an equivalent canonical representation in perceptual units. All structural results derived above depend only on the resolution topology and therefore apply to any such representation. In other words, our findings here are metric invariant, coordinate invariant, and norm invariant and so the perceptual geometry determines the coordinate system rather than the other way around.

The characterization immediately produces the classical preference structure recovery in the limiting case when the perceptual blur vanishes. In such a limit, the perceptual topology, $\mathcal{T}_\kappa$, collapses to the classical topology, $\mathcal{T}_c$, rendering it invisible. Hence, any added structure arising from the perceptual topology vanishes and the classical preference relation is recovered.

Corollary 1. *Fix $x \in X$ and consider a sequence of resolution profiles $\{\kappa^{(m)}\}$ with corresponding resolution horizons $\overline{d}_i^{(m)} \to 0$ for each i. Then:*

1. *The perceptual neighbourhoods collapse:*
$$B_x^{(\kappa^{(m)})} \downarrow \{x\}.$$
2. *Perceptual equality converges to classical equality:*
$$x \approx_{\kappa^{(m)}} y \quad \Longrightarrow \quad x = y,$$
for large m.
3. *Perceptual strict preference converges pointwise to the classical strict preference:*
$$y \succ_c x \quad \Longrightarrow \quad y \succ_{\kappa^{(m)}} x,$$
for large m.
4. *The grey zone vanishes:*
$$\|_{\kappa^{(m)}} (x) \downarrow \emptyset.$$

Proof. Because $T^{(m)}$ is an isometry onto its image:

$$B_x^{(\kappa^{(m)})} = \left\{ y : \|T^{(m)}(y) - T^{(m)}(x)\|_2 < \kappa^{(m)} \right\}.$$

As $\overline{d}_i^{(m)} \to 0$, each coordinate $T_i^{(m)}$ collapses to a point in a neighbourhood of x_i, implying the Euclidean ball in T-space collapses to $\{T^{(m)}(x)\}$. Pulling back through a homeomorphism gives $B_x^{(\kappa^{(m)})} \downarrow \{x\}$, and the remaining statements follow exactly as in the uniform case. $\square$

When perceptual resolution decays sufficiently quick so that the resolution horizon $\overline{d}_i$ approaches zero, the decision maker acquires nearly perfect discrimination in coordinate i. Under perceptual coordinates, this corresponds to the collapse of that coordinate range. In commodity space, it corresponds to the disappearance of the perceptual blur. Classical preference is therefore the boundary case of the canonical representation and arises when perceptual resolution becomes bounded.

6 Discussion

The analysis above places the perceptual topology rather than any particular metric or coordinate system at the centre of choice under bounded perceptual resolution. Although the Euclidean metric provides a convenient representation of perceptual indistinguishable regions as balls of radius κ, none of the structural results depend on this choice. What matters is the resolution structure $\{B_\kappa(x)\}_{x\in X}$, the collection of minimal indistinguishability regions through which the agent perceives the commodity space. Any metric, norm, or system of coordinates that induces the same neighbourhood system yields identical geometric and behavioural properties. In this sense, perceptual geometry is fundamentally coordinate free.

This perspective clarifies how perceptual blur interacts with classical completeness. Classical weak preference is complete because, when strict preference is negatively transitive, nondominance uniquely determines the weak order. Under bounded perceptual resolution, this construction is no longer behaviourally meaningful. Strict perceptual preference remains transitive but fails negative transitivity. The grey zone consists of precisely those pairs that cannot be robustly ranked. A naive nondominance based weak preference would collapse all such pairs into spurious indifference, erasing the very incomparabilities generated by the perceptual structure. Any weak perceptual preference that respects the grey zone as a distinct behavioural category must therefore be incomplete. Completeness reemerges only in the vanishing perceptual blur limit, when the grey zone collapses and perceptual resolution becomes unbounded.

The canonical representation developed in Section 5 illustrates the geometric foundations of these results. State dependent or heterogeneous discriminability across coordinates poses no conceptual difficulty. After normalizing each coordinate into perceptual units, indistinguishability regions become Euclidean balls and the perceptual topology becomes standard. The uniform-κ environment is therefore, not a special case but a canonical representation of a broad class of perceptual systems. Classical convexity, smoothness, and other regularity properties have no inherent meaning in commodity space. They arise only when viewed through the perceptual coordinates in which local resolution is normalized.

The geometric structure of the grey zone is particularly informative. Local blur generates an annulus of width κ surrounding the indistinguishable region about x, B_x, while classical indifference blur produces a 2κ-thickened tube around the classical indifference set through x. These regions reflect genuine perceptual limitations. Bundles may be objectively ordered by the classical preference but remain incapable of robust comparison once the agent projects them through the perceptual filter resulting in blur. Such configurations, including cases in which two bundles each robustly dominate a third while remaining mutually incomparable, arise naturally in multidimensional perceptual geometry but are impossible under any unidimensional or scalar JND representation. This highlights the essential multidimensionality of perceptual discrimination.

Finally, the framework offers a geometric interpretation of several classical regularity assumptions. Local nonsatiation fails generically because every sufficiently small Euclidean neighbourhood lies within a perceptual indistinguishability region. Convexity becomes a coordinate artifact unless local resolution is constant across attributes. Continuity and completeness of weak preference, often treated as fundamental primitives, appear instead as idealized limits attained only when perceptual resolution becomes arbitrarily fine. Classical consumer theory therefore emerges as the degenerate high resolution limit of a richer perceptual geometry.

At first glance, the uniform-κ specialization of the model may appear closely related to classical JND or semiorder representations which also generate thickened indifference regions. This resemblance, however, is only partial. In standard JND models, indifference arises from a global scalar utility threshold and produces a uniform band around indifference curves that is independent of the reference bundle. By contrast, the present framework generates an additional and conceptually distinct source of incomparability, a local perceptual blur centered at the anchor bundle x.

This local blur arises from the collapse of discrimination within the perceptual neighbourhood B_x, where all bundles are perceptually indistinguishable from x. As a result, strict preference is topologically infeasible in a neighbourhood of x, and comparison failures persist even when classical utility differences exceed κ. The interaction between this local blur and the thickening of classical indifference sets produces the grey annulus characterized earlier. No analogue of this phenomenon exists in JND or constant threshold models, where indifference is imposed *ex ante* and does not interact with the local geometry of perception.

Thus, even in its uniform-κ representation, the model cannot be reduced to a scalar threshold on utility differences. The geometry of incomparability is anchored, multidimensional, and topologically induced, rather than globally uniform. This distinction explains why failures of negative transitivity and local monotonicity arise here as geometric consequences of finite resolution rather than as artifacts of utility perturbations.

A further distinction between the present framework and classical JND or semiorder models concerns the optics of perceptual failure. In standard threshold models, discrimination failure is global. Comparisons become indeterminate whenever utility differences fall below a fixed scalar threshold, independent of the location of the reference bundle. Such models therefore do not distinguish between perceptual difficulty arising near a reference point and difficulty arising farther away along an indifference surface.

By contrast, bounded perceptual resolution generates two qualitatively distinct sources of incomparability. The first is a near field blur: within the perceptual neighbourhood B_x, all bundles are indistinguishable from x, and strict preference is topologically infeasible regardless of classical utility differences. The second is a far-field blur: a thickening of classical indifference sets induced by perceptual uncertainty away from the anchor.

This distinction yields a discrepancy between near and far perceptual judgments that cannot be captured by any scalar JND representation. Agents may confidently rank distant improvements while remaining unable to discriminate between nearby alternatives, even when the underlying utility differences are comparable. This asymmetry between near and far discriminability is well documented empirically and reflects the intuitive fact that small differences near a reference point are harder to resolve than larger differences farther away.

In the present model, this asymmetry is not imposed behaviourally but follows directly from the geometry of perception. JND models capture indifference thickening but cannot account for local collapse of discrimination around a reference point, which is the primary driver of incompleteness and failure of negative transitivity in the perceptual topology.

Additionally, large utility differences can be concentrated in perceptually negligible physical neighbourhoods and bounded resolution breaks the identification of preference orderings independent of marginal utility. For example, a decision maker may have similar perceptual limitations when comparing between 100g to 101g of gold dust or flour despite significant differences in utility between the two scenarios resulting from an additional gram of the commodity concerned.

7 Conclusion

This paper develops a geometric and topological foundation for choice under bounded perceptual resolution. By treating the classical preference as the underlying evaluative relation and imposing a perceptual topology that limits the granularity of discrimination, the analysis identifies the conditions under which strict comparisons are perceptually robust. These conditions generate a distinctive structure. Indistinguishability is intransitive, strict perceptual preference is transitive but fails negative transitivity, and a well defined grey zone separates robustly better and worse alternatives from those that cannot be classified with confidence.

The grey zone is the central object of the theory. It reflects the bundles that are perceptibly different from an anchor but not different enough to yield robust dominance in either direction. Its geometry follows directly from finite perceptual resolution. Local blur creates an annulus around the indistinguishability region, and classical indifference blur creates a thickened tube around the indifference set. These regions encode the structural limits of perceptual judgment. They also imply that any weak perceptual preference that preserves the distinction between comparable and incomparable bundles must be incomplete. Completeness is recovered only in the limit of perfect perceptual clarity, when the grey zone collapses and perceptual neighbourhoods shrink to points.

The analysis further shows that all results depend solely on the perceptual neighbourhood system $\{B_\kappa(x)\}_{x\in X}$, not on the particular coordinates or metrics used to represent the commodity space. The canonical representation normalizes heterogeneous local resolutions into perceptual units, restoring spherical geometry and making explicit the invariances of the model. Classical regularity assumptions emerge as special cases of infinite perceptual resolution, illuminating their status as idealizations rather than primitive axioms.

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