Proportional Robustness and Dimensionality

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February 18, 2026

Abstract

Scalar perturbation parameters are widely used to index robustness in economic theory. An equilibrium may be ε-stable, a mechanism ε-incentive compatible, or a constraint satisfied up to a tolerance ε. These parameters clearly order environments, but they are often interpreted proportionally so that equal changes in ε are taken to represent equal changes in robustness. This paper characterizes when such proportional interpretation is justified. Robustness is formalized as neighbourhood coverage, the fraction of a reference neighbourhood that remains admissible after displacement of a boundary. We show that coverage is affine in the displacement parameter if and only if admissible variation is effectively one-dimensional. When variation is multidimensional, coverage is strictly convex in displacement magnitude, and no affine transformation of the scalar parameter yields a global interval-scale representation. The result identifies a sharp geometric boundary underlying scalar robustness indices and applies to local incentive compatibility, equilibrium refinements, and related perturbation arguments.

Keywords: Neighbourhood overlap; Robustness; Distance; Interval-scale representation; Effective Dimensionality; Geometric representation

1 Introduction

Scalar perturbation parameters are central to modern economic theory. Equilibrium refinements, robustness arguments, and incentive constraints are routinely indexed by a scalar tolerance parameter ε. An equilibrium is ε-stable; a mechanism is ε-incentive compatible; a constraint holds up to a perturbation of magnitude ε. These parameters clearly order environments. They are also frequently interpreted proportionally: moving from ε to 2ε is implicitly treated as a proportional weakening of the underlying requirement.

This paper asks when such proportional interpretation is justified. When do equal marginal changes in a scalar perturbation parameter correspond to equal marginal changes in robustness?

The answer depends not on the economic primitives but on geometric structure. Robustness can be viewed as neighbourhood coverage: the fraction of a reference neighbourhood that remains admissible after a boundary is displaced. We show that coverage is affine in the displacement parameter if and only if admissible variation is effectively one-dimensional. When variation is multidimensional, coverage is strictly convex in displacement magnitude. In that case, no affine transformation of the scalar parameter yields a global interval-scale representation of robustness.

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The result identifies a sharp dimensional boundary. In one-dimensional environments, relaxing a constraint shifts a cutoff point and admissible mass changes linearly in the perturbation parameter until saturation. In higher-dimensional environments, the same relaxation alters admissible regions along curved or jointly determined boundary segments, inducing strict convexity. Equal marginal increases in the scalar tolerance then generate unequal marginal changes in admissible mass. The perturbation parameter continues to rank environments, but it does not measure robustness proportionally.

The analysis applies to familiar settings. Incentive constraints play a central role in mechanism design (Myerson, 1981; Rochet, 1987), and robustness arguments frequently introduce tolerance parameters or perturbation magnitudes (Selten, 1975; Myerson, 1978; Van Damme, 1991; Bergemann and Morris, 2005). Section 2 illustrates the geometric mechanism in a local incentive compatibility environment. Section 5 presents a certification example under measurement error. In each case, relaxing a constraint displaces a boundary within a neighbourhood, and robustness is measured by the surviving fraction.

More broadly, scalar indices are pervasive tools for compressing multidimensional variation into a single number. A patient reports pain on a 1-10 scale; a wine expert rates a bottle from 1-20; film critics evaluate movies from 1-100. Such scales are used not only to rank outcomes but to support marginal reasoning. Equal numerical increments are implicitly treated as equal increments in the underlying phenomenon. Finer scales are viewed as more informative because they allow smaller step sizes, reinforcing this interval-scale interpretation.

Ultimately, numerical representations are central to economic analysis (Luce, 1956; Debreu, 1954). In measurement theory, interval-scale interpretations are characterized by invariance under positive affine transformations (Krantz et al., 1971). The contribution here is representational rather than computational. Rather than computing overlap volumes, the paper characterizes the structural condition under which scalar perturbation parameters inherit affine structure from the geometry of neighbourhood truncation.

The remainder of the paper proceeds as follows. As a motivating example, Section 2 presents a local incentive compatibility illustration. Section 3 introduces the geometric framework and formalizes robustness as neighbourhood coverage. Section 4 characterizes the relationship between displacement and coverage and establishes the dimensional boundary across Euclidean and product neighbourhoods. Section 5 provides a contrasting certification example under measurement error. Lastly, a brief discussion and concluding remarks follow in Section 6.

2 A Local Incentive Compatibility Illustration

This section illustrates how the geometric properties of coverage translate into a familiar economic setting. We consider a standard local incentive compatibility environment and examine how robustness of the IC constraint varies with displacement in type space.

Consider a direct mechanism in which agents have types $\theta \in \Theta \subseteq \mathbb{R}^n$. Fix a reference type θ^* at which the mechanism satisfies the incentive compatibility (IC) constraint exactly.

To examine local robustness of incentive compatibility, consider nearby types within a small Euclidean neighbourhood

$$B_r(\theta^*) = \{\theta : \|\theta - \theta^*\|_2 \leq r\}.$$

Interpret this ball as small perturbations of the agent's type.

Let $U(\theta, \hat{\theta})$ denote the utility of a type-θ agent who reports $\hat{\theta}$. Exact incentive compatibility at θ^* requires

$$U(\theta^*, \theta^*) \geq U(\theta^*, \hat{\theta}) \quad \text{for all reports } \hat{\theta}.$$

We say the mechanism is ε*-incentive compatible* at θ^* if deviations are bounded by ε:

$$U(\theta, \theta) \geq U(\theta, \hat{\theta}) - \varepsilon$$

for types θ in a neighbourhood of θ^*.

Fix a particular deviation direction (for example, a locally most tempting misreport), and define

$$R(\varepsilon) = \frac{\lambda\big(\{\theta \in B_r(\theta^*) : U(\theta, \theta) \geq U(\theta, \hat{\theta}) - \varepsilon\}\big)}{\lambda\big(B_r(\theta^*)\big)},$$

where λ denotes Lebesgue measure. This ratio quantifies local robustness as the fraction of types in the neighbourhood for which the IC constraint is satisfied within tolerance ε.

A common interpretation treats ε proportionally: larger ε is viewed as proportionally weaker incentive compatibility. We ask whether this interpretation is geometrically justified.

Local linear structure. Assume U is differentiable and that the IC constraint binds smoothly at θ^*. For types close to θ^*, a first-order approximation of the deviation payoff difference yields

$$U(\theta, \hat{\theta}) - U(\theta, \theta) \approx v \cdot (\theta - \theta^*),$$

for some nonzero vector $v \in \mathbb{R}^n$.

Thus, locally, the set of types satisfying ε-IC is approximately a half-space:

$$\{\theta : v \cdot (\theta - \theta^*) \leq \varepsilon\}.$$

Increasing ε shifts this hyperplane outward, expanding the admissible region within the ball.

One-dimensional types. Let $n = 1$, so $B_r(\theta^*) = [\theta^* - r, \theta^* + r]$. Under the local linear approximation

$$U(\theta, \hat{\theta}) - U(\theta, \theta) \approx v(\theta - \theta^*), \qquad v \neq 0,$$

the ε-IC condition becomes

$$\theta \leq \delta, \qquad \delta := \varepsilon/|v|.$$

For $\delta \in [-r, r]$, the admissible set is the interval obtained by truncating $B_r(\theta^*)$ at the cutoff $\theta = \delta$. An increase in ε shifts this cutoff outward. The additional admissible mass equals a slab of constant width proportional to $d\varepsilon$.

Because this marginal width does not depend on the location of the cutoff, equal increments of ε generate equal increments in admissible mass until the entire interval is included. The scalar tolerance therefore admits a proportional interpretation in one dimension.

Two-dimensional types. Let $n = 2$, translate so $\theta^* = 0$, and rotate coordinates so the local linear approximation takes the form

$$U(\theta, \hat{\theta}) - U(\theta, \theta) \approx |v|\, \theta_1, \qquad |v| > 0.$$

Then the ε-IC condition is $\theta_1 \leq \delta$, where $\delta := \varepsilon/|v|$.

For $\delta \in [0, r]$, the admissible set is the disk $B_r(0)$ truncated by the vertical line $\theta_1 = \delta$. An increase in ε shifts this line outward. The additional admissible mass equals the area of a vertical slice whose width is proportional to $d\varepsilon$ and whose height equals the chord length

$$2\sqrt{r^2 - \delta^2}.$$

Unlike the one-dimensional case, this chord length depends on δ and decreases as the boundary moves outward. Hence equal increments of ε generate unequal increments in admissible mass. The scalar tolerance therefore fails to measure robustness proportionally once the type space is multidimensional.

The incentive compatibility illustration isolates the core structure driving the result. Relaxing the IC constraint shifts a boundary within a small neighbourhood of types, and robustness is measured by the fraction of that neighbourhood that remains admissible. In one dimension, the boundary truncates the neighbourhood uniformly along a single axis. In higher dimensions, the same displacement removes mass along a curved or jointly determined boundary segment. The resulting nonlinearity is geometric rather than economic.

To separate geometry from economic primitives, the analysis now abstracts from the incentive compatibility example and studies neighbourhood overlap directly. This allows us to characterize the relationship between displacement and robustness independently of the institutional setting.

3 The Framework

Let $X \subseteq \mathbb{R}^n$ be a subset of the Euclidean space. Fix a neighbourhood system

$$x \longmapsto \mathcal{N}(x) \subseteq \mathbb{R}^n,$$

where $\mathcal{N}(x)$ is a measurable set representing the neighbourhood around x. Assume $\lambda(\mathcal{N}(x)) \in (0, \infty)$ for all $x \in X$, where λ denotes Lebesgue measure on $\mathbb{R}^n$.

Definition 1. *For $x, y \in X$, define* **Coverage** *by*

$$\mathsf{C}(x, y) \equiv \frac{\lambda\big(\mathcal{N}(x) \cap \mathcal{N}(y)\big)}{\lambda\big(\mathcal{N}(x)\big)}.$$

Coverage measures the fraction of the neighbourhood around x that remains overlapping when y is considered.[1] By construction, $\mathsf{C}(x, y) \in [0, 1]$ and $\mathsf{C}(x, x) = 1$. The geometry of overlap loss, and hence the functional form of coverage as a function of displacement, depends on the structure of the neighbourhood system $\mathcal{N}(\cdot)$.

To understand how geometry shapes this relationship, we examine two canonical neighbourhood structures. Euclidean neighbourhoods represent isotropic perturbations, in which proximity is measured radially. Product neighbourhoods, by contrast, capture coordinatewise perturbations, in which variation occurs independently along each dimension. Comparing these cases allows us to distinguish effects driven by curvature from those driven by dimensional structure.

We first examine Euclidean neighbourhoods and characterize how coverage varies with displacement. This leads to the notion of effective dimensionality, which governs when proportionality of robustness can arise.[2]

3.1 Euclidean Neighbourhoods

We begin with Euclidean neighbourhoods, which are rotationally symmetric balls. Fix a radius $r > 0$ and define

$$\mathcal{N}(x) = B_r(x) := \{y \in \mathbb{R}^n : \|y - x\|_2 \le r\}.$$

[1] All results would be unchanged under symmetric normalization (e.g., dividing by the average or union measure), but the asymmetric form aligns naturally with certification and robustness interpretations.

[2] The analysis draws on basic metric and volume properties of neighbourhoods. See (Burago et al., 2001).

Under this neighbourhood system, coverage specializes to

$$\mathsf{C}(x,y) = \frac{\lambda\big(B_r(x)\cap B_r(y)\big)}{\lambda\big(B_r(x)\big)}.$$

By translation and rotation invariance, coverage depends only on the distance $d = \|x-y\|_2$.

Define the Euclidean coverage function

$$C(d) := \mathsf{C}(x,y) \quad \text{for any } x,y \text{ with } \|x-y\|_2 = d.$$

Definition 2. *Consider Euclidean neighbourhoods $B_r(x) \subset \mathbb{R}^n$. Let $v := y - x \neq 0$ denote a displacement vector and define the associated set of admissible directions*

$$\mathcal{U}(v) := \{u \in \mathbb{R}^n : \|u\|_2 = 1\}.$$

The **Effective Euclidean Dimensionality** *of the displacement v is*

$$k := \dim\big(\mathrm{span}(\mathcal{U}(v))\big).$$

Under the Euclidean norm in $\mathbb{R}^n$, $\mathcal{U}$ spans $\mathbb{R}^n$, so $k = n$. In particular, $k = 1$ if and only if $n = 1$.

Note that in Euclidean space, coverage depends only on the radial displacement $d = \|x-y\|_2$. However, radial movement corresponds to uniform truncation along a single direction only in $\mathbb{R}^1$, where neighbourhoods are intervals. In dimensions $n \geq 2$, even when displacement is along a single line, overlap is lost along a curved boundary involving multiple independent directions.

3.2 Product Neighbourhoods

The contrast observed under Euclidean neighbourhoods may reflect radial curvature rather than dimensional structure per se. To separate these effects, we next consider product neighbourhoods induced by the ℓ_∞ norm:

$$\mathcal{N}(x) = B_r^\infty(x) := \{y \in \mathbb{R}^n : \|y-x\|_\infty \leq r\} = \prod_{i=1}^{n} [x_i - r, x_i + r].$$

Coverage under product neighbourhoods is given by

$$\mathsf{C}^\infty(x,y) = \frac{\lambda(B_r^\infty(x)\cap B_r^\infty(y))}{\lambda(B_r^\infty(x))}.$$

Definition 3. *Let $v = y - x \in \mathbb{R}^n$. The* **Effective Product Dimensionality** *of the displacement v is*

$$k = \dim\big(\mathrm{span}\{\, e_i : v_i \neq 0 \,\}\big).$$

In product neighbourhoods, displacement operates along coordinates rather than radially. A Euclidean ball is isotropic in that moving a boundary outward in any direction removes mass along a curved surface that simultaneously spans multiple independent directions. Even if displacement occurs along a single vector, the boundary of a ball is curved, so the region lost or gained reflects joint variation across coordinates. By contrast, a product neighbourhood, such as a cube defined by coordinatewise bounds, separates variation across dimensions. Truncation occurs independently along each active coordinate. If displacement affects only one coordinate direction, the admissible region is reduced by shaving off a flat slab parallel to a face of the cube, leaving the remaining coordinates unchanged. Coverage therefore declines linearly in the magnitude of that single displacement. Strict convexity arises only when multiple coordinates are simultaneously active, so that overlap is reduced multiplicatively across independent directions. The difference is structural: radial neighbourhoods aggregate variation through curvature, whereas product neighbourhoods preserve coordinate separability. The dimensional boundary therefore reflects whether truncation occurs along one independent axis or through joint curvature across several.

4 Interval-Scale Representation

The preceding discussion suggests that proportional interpretation of scalar displacement depends on how coverage varies with the magnitude of that displacement. This section formalizes that relationship. We first characterize how coverage behaves under Euclidean and product neighbourhoods. We then use these results to define coverage lost and establish the dimensional boundary for interval-scale representation.

We begin with Euclidean neighbourhoods. Although coverage depends only on the scalar distance $d = \|x - y\|_2$, the geometry of overlap loss differs sharply across dimensions.

Lemma 1. *Fix $r > 0$ and Euclidean neighbourhoods $\mathcal{N}(x) = B_r(x) \subset \mathbb{R}^n$. Let $v := y - x$ and $d := \|v\|_2$. Let k denote the effective Euclidean dimensionality of the displacement.*[3] *Define $C(d) := \mathsf{C}(x, y)$ for any x, y with $\|x - y\|_2 = d$. Then, on $d \in [0, 2r]$:*

1. *If $k = 1$, the map $d \mapsto C(d)$ is affine.*
2. *If $k \geq 2$, the map $d \mapsto C(d)$ is strictly convex on $(0, 2r)$.*

Proof. Let $v := y - x$ and $d := \|v\|_2$. By translation and rotation invariance, coverage depends only on d, so we write $C(d) := \mathsf{C}(x, y)$.

If $k = 1$, Euclidean neighbourhoods are intervals (equivalently $n = 1$). For $d \in [0, 2r]$, the overlap $B_r(x) \cap B_r(y)$ has length $2r - d$, so

$$C(d) = \frac{2r - d}{2r} = 1 - \frac{d}{2r},$$

which is affine in d.

If $k \geq 2$ (equivalently $n \geq 2$), let

$$V_n(d) := \lambda\big(B_r(x) \cap B_r(y)\big).$$

By symmetry, $V_n(d)$ depends only on d. Differentiating the standard closed-form expression (Li, 2011) yields, for $d \in (0, 2r)$,

$$V_n'(d) = -\omega_{n-1}\big(r^2 - (d/2)^2\big)^{(n-1)/2}, \qquad V_n''(d) = \omega_{n-1}\frac{n-1}{4}d\big(r^2 - (d/2)^2\big)^{(n-3)/2} > 0.$$

Since $C(d) = V_n(d)/V_n(0)$, it follows that $C''(d) > 0$ on $(0, 2r)$. Hence $C(d)$ is strictly convex. $\square$

[3] For Euclidean balls, $k = 1$ if and only if $n = 1$; when $n \geq 2$ overlap loss occurs along a curved boundary spanning multiple independent directions.

Lemma 1 highlights the essential geometric distinction. In one dimension, neighbourhoods are intervals and overlap is lost through uniform truncation at a single boundary point. Coverage therefore declines linearly in distance. In higher dimensions, however, even displacement along a single direction removes overlap along a curved boundary that spans multiple independent directions. Radial symmetry ensures that coverage depends only on distance, but it does not imply linearity. Once $n \geq 2$, curvature induces strict convexity.

The convexity in Lemma 1 arises from curvature of radial neighbourhoods. To separate curvature from dimensionality, we next consider product neighbourhoods, where truncation occurs coordinatewise rather than radially.

Lemma 2. *Fix $r > 0$ and product neighbourhoods $\mathcal{N}(x) = B_r^\infty(x) \subset \mathbb{R}^n$. Let $v := y - x$ and $d := \|v\|_\infty$. Let k denote the effective product dimensionality of the displacement. Define $C(d) := \mathsf{C}(x, y)$ for any x, y with $\|x - y\|_\infty = d$. Then, on $d \in [0, 2r]$:*

1. *If $k = 1$, the map $d \mapsto C(d)$ is affine.*
2. *If $k \geq 2$, the map $d \mapsto C(d)$ is strictly convex on $(0, 2r)$.*

Proof. Let $v := y - x$ and $d := \|v\|_\infty$. For fixed displacement vector v, write $C(d) := \mathsf{C}(x, y)$ with $d = \|v\|_\infty$.

Let $S = \{i : v_i \neq 0\}$ denote the active coordinates and define $\alpha_i := |v_i|/(2r)$ for $i \in S$. Under product neighbourhoods, overlap occurs independently along active coordinates. Hence, for $d \in [0, 2r]$,

$$C(d) = \prod_{i \in S} \left(1 - \alpha_i \frac{d}{\|v\|_\infty}\right).$$

If $k = 1$, only one coordinate is active. The product reduces to a single linear term in d, so $C(d)$ is affine.

If $k \geq 2$, at least two coordinates are active. Differentiating twice yields

$$C''(d) = \sum_{\substack{i,j \in S \\ i \neq j}} \alpha_i \alpha_j \prod_{\ell \in S \setminus \{i,j\}} \left(1 - \alpha_\ell \frac{d}{\|v\|_\infty}\right),$$

which is strictly positive on $(0, 2r)$ because each factor in the product is positive on this interval and at least one term in the sum is strictly positive. Hence $C(d)$ is strictly convex. □

Lemma 2 shows that convexity is not a peculiarity of spherical geometry. Under product neighbourhoods, displacement affects coverage independently along active coordinates. When variation is confined to a single coordinate direction, truncation removes a flat slab and coverage declines linearly. Strict convexity arises precisely when multiple coordinates are jointly active, so that overlap is reduced multiplicatively across independent directions. In both Euclidean and product geometries, affinity holds if and only if effective dimensionality equals one.

The preceding lemmas describe how coverage varies with displacement. The representational question is therefore immediate: when does distance provide a proportional measure of coverage?

If coverage is affine in displacement, equal increments of distance generate equal increments in the fraction of the neighbourhood that remains admissible. If coverage is strictly convex, this proportionality fails: equal increments of distance correspond to unequal changes in coverage. The question of interval-scale representation therefore reduces to whether coverage is affine or strictly convex in displacement.

Definition 4. *Distance provides an* **Interval-Scale Representation of Coverage** *if there exist constants $a \in \mathbb{R}$ and $b \neq 0$ such that*

$$\mathsf{C}(x, y) = a + b\|x - y\| \quad \textit{for all } y \textit{ with } \|x - y\| \leq 2r,$$

where $\|\cdot\|$ denotes the norm associated with the neighbourhood geometry.

It is important to distinguish exact proportionality from local approximation. When coverage is strictly convex in displacement, distance cannot provide a global interval-scale representation. That is, no affine transformation of distance can match coverage over the full range of admissible displacements. This does not preclude local linearization. As with any smooth function, coverage is locally well approximated by its first-order expansion around a reference point, so sufficiently small perturbations can always be treated as approximately proportional. The results below concern exact proportionality rather than first-order approximation. They identify when distance provides a globally valid interval-scale measure of robustness, independent of local linearizations that hold generically but only infinitesimally.

Proposition 1. *Fix a reference point x and a neighbourhood system. Let k denote the effective dimensionality of displacement. Then distance provides a global interval-scale representation of coverage if and only if $k = 1$. Equivalently, $d \mapsto C(d)$ is affine on its domain if and only if $k = 1$, and is strictly convex on the interior whenever $k \geq 2$.*

Proof. If $k = 1$, Lemmas 1 and 2 imply that $d \mapsto C(d)$ is affine on its domain. Hence there exist constants $a \in \mathbb{R}$ and $b \neq 0$ such that $\mathsf{C}(x, y) = a + b\|x - y\|$ for all y in the relevant neighbourhood, so distance provides an interval-scale representation.

If $k \geq 2$, the same lemmas imply that $d \mapsto C(d)$ is strictly convex on the interior of its domain. Strict convexity rules out any global affine representation of $C(d)$ in terms of distance. □

Thus proportional interpretation of the scalar displacement parameter is possible globally if and only if overlap loss is effectively one-dimensional. The equivalence is geometric rather than metric.

Across both Euclidean and product geometries, proportional decay of coverage is therefore not tied to the choice of norm, but to the effective dimensionality of variation instead. Proportional robustness is a knife-edge property of effectively one-dimensional truncation.

5 GPS Certification and the Dimensional Boundary

This section provides a second illustration of the dimensional boundary in a certification problem with bounded measurement error. The setting differs from the local incentive compatibility illustration in Section 2, but the geometric structure is identical: a boundary is displaced within a neighbourhood, and robustness is the fraction of that neighbourhood that remains admissible. The example therefore serves to demonstrate that the dimensional boundary is not specific to incentive constraints, but intrinsic to the geometry of neighbourhood-based robustness.

In the analysis above, robustness was formulated as overlap between two neighbourhoods. The certification setting instead involves a fixed constraint set and a perturbed observation. Here robustness is measured as the fraction of an error neighbourhood that lies on the admissible side of a boundary. Although the formulation is one-sided rather than two-sided, the underlying geometry is identical: displacement of a boundary within a neighbourhood and measurement of the surviving fraction. Consequently, the functional relationship between coverage and displacement coincides with that obtained for neighbourhood overlap. The example therefore isolates the same dimensional boundary in a different economic context.

5.1 Worked Example

An agent stands at a true location x but is observed through a GPS device that returns a reported location $\hat{x}$ with bounded error:

$$\hat{x} \in B_r(x),$$

where $B_r(x)$ is the Euclidean ball of radius $r > 0$ centered at x. Assume for simplicity that the GPS error is uniform on $B_r(x)$. Let the agent's property be given by the set P. Then the probability that the GPS report certifies the agent is on their property equals

$$\Pr(\hat{x} \in P \mid x) = \frac{\lambda\big(B_r(x) \cap P\big)}{\lambda\big(B_r(x)\big)}.$$

This is exactly a coverage ratio: the fraction of the error neighbourhood consistent with the property claim.

One dimension: linear coverage. Let $n = 1$ and suppose the property is the half-line

$$P = \{s \in \mathbb{R} : s \geq 0\}.$$

If the agent stands at $x = t \geq 0$, then $B_r(t) = [t - r, t + r]$ and for $0 \leq t \leq r$,

$$\Pr(\hat{x} \in P \mid x = t) = \frac{1}{2} + \frac{t}{2r},$$

while for $t \geq r$ the probability equals 1. Thus certifiability increases linearly with distance from the boundary until it saturates.

Two dimensions: nonlinear (convex) coverage. Let $n = 2$ and suppose the property is the right half-plane

$$P = \{(s_1, s_2) \in \mathbb{R}^2 : s_1 \geq 0\}.$$

Place the agent at $x = (t, 0)$ with $t \geq 0$. For $t \geq r$, the disk $B_r(x)$ lies entirely in P and the certification probability is 1. For $0 \leq t \leq r$, the disk crosses the boundary and the certification probability equals the fraction of the disk that lies in the half-plane. This fraction admits a standard closed form and is strictly convex in t on $[0, r]$.[4] Thus equal increases in distance from the boundary correspond to equal gains in certifiability only in one dimension.

6 Discussion and Conclusion

The examples in the paper share a common structure. Robustness is measured as the fraction of a small neighbourhood that remains admissible after displacement of a boundary. In the incentive compatibility illustration, the boundary is an indifference condition separating admissible and inadmissible types. In the certification example, it is a property constraint. In the abstract framework, it is the boundary of a second neighbourhood. The unifying object is geometric: displacement of a boundary within a neighbourhood and measurement of the surviving fraction.

[4] One convenient closed form is obtained from the area of a circular segment:

$$\Pr(\hat{x} \in P \mid x = (t, 0)) = 1 - \frac{1}{\pi r^2}\Big[r^2 \arccos(t/r) - t\sqrt{r^2 - t^2}\Big], \qquad 0 \leq t \leq r,$$

which is strictly convex on $[0, r]$.

The analysis identifies a sharp dimensional boundary governing proportional interpretation of scalar perturbation parameters. When admissible variation is effectively one-dimensional, truncation occurs uniformly along a single independent direction and coverage declines linearly in displacement. Equal marginal changes in a scalar tolerance parameter then correspond to equal marginal changes in robustness. In higher-dimensional environments, truncation occurs along curved or jointly determined boundary segments. Coverage is strictly convex in displacement magnitude, and no affine transformation of the scalar parameter yields a global interval-scale representation of robustness.

The result is structural rather than application-specific. It applies to local incentive compatibility, certification under measurement error, and other settings in which robustness is measured by the surviving fraction of a neighbourhood. Scalar indices continue to order environments, and local linearization ensures approximate proportionality for sufficiently small perturbations. The contribution is, therefore, to identify the geometric condition under which proportional interpretation holds exactly rather than locally.

More broadly, the analysis separates monotonic refinement from proportional refinement. The former requires only that coverage decline with displacement and is ubiquitous in robustness arguments. The latter is more restrictive and depends on effective dimensionality. Proportional robustness is therefore not a general property of perturbation parameters, but a feature of effectively one-dimensional truncation.

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