When Does Distance Represent Robustness? A Dimensional Boundary from Neighbourhood Overlap

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Abstract

We study when distance provides a proportional measure of robustness under neighbourhood-based comparison. Robustness is formalized by coverage, defined as the fraction of a reference neighbourhood that remains overlapping after displacement. The analysis is purely geometric and applies to settings in which alternatives are evaluated relative to overlapping neighbourhoods rather than pointwise distinctions. We find that coverage, as a function of displacement, is affine if and only if variation is effectively one-dimensional, so that overlap is lost through uniform truncation along a single independent direction. In this case, distance admits a global interval-scale representation of robustness. When variation is not effectively one-dimensional, this relationship is strictly convex and proportionality fails. The result identifies a sharp geometric boundary for when distance can serve as a linear proxy for refinement, independent of behavioural, informational, or economic assumptions.

Keywords: Neighbourhood overlap; Robustness; Distance; Interval-scale representation; Effective Dimensionality; Geometric representation

1 Introduction

Neighbourhood-based reasoning arises naturally in environments where alternatives are described or compared through neighbourhoods, rather than precise pointwise distinctions. In such settings, the notion of similarity is captured by the overlap of neighbourhoods. Alternatives are treated as locally indistinguishable when their neighbourhoods intersect, and as decisively distinct once this overlap vanishes. Robustness to refinement or perturbation is therefore naturally tied to the extent of neighbourhood overlap.

A common simplification is to index such perturbations by a scalar notion of proximity, such as distance or neighbourhood radius, and to reason about robustness in terms of changes in this scalar. Proximity is then interpreted not merely as an ordering, but as a measure so that greater separation is taken to imply more stable conclusions and equal changes in distance are often treated as equally informative refinements. Whether such proportional interpretations are globally valid, however, is rarely examined.

This paper isolates the geometric conditions under which a measure of neighbourhood overlap supports proportional reasoning. Robustness is formalized through the notion of coverage, defined as a function measuring the fraction of a reference neighbourhood that remains overlapping after

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displacement. The analysis abstracts from behavioural or informational primitives and focuses entirely on the geometry of overlap. The central question is whether coverage varies affinely with distance, so that displacement provides a valid interval-scale representation of robustness. The question is not whether neighbourhood overlap decreases with distance, which is immediate, but whether coverage admits a global affine representation in distance, allowing distance to be interpreted as an interval-scale rather than merely an ordinal measure of robustness.

The main result identifies a sharp dimensional boundary. When variation is effectively one-dimensional, in the sense that overlap is lost through uniform truncation along a single independent direction, coverage is affine as a function of distance. When variation is not effectively one-dimensional, overlap is lost through curved or joint boundary regions, and coverage becomes strictly convex. As a result, distance remains ordinally meaningful but no longer supports proportional comparisons of robustness. Proportionality is therefore a knife-edge structural property of effectively one-dimensional variation. We show that this boundary reflects effective dimensionality rather than the choice of norm or neighbourhood shape, and persists under alternative neighbourhood structures such as product neighbourhoods. The results do not challenge neighbourhood-based reasoning but clarify when its scalar proxies are representationally valid.

Neighbourhood-based perturbations indexed by a scalar measure of proximity arise naturally in economic analysis, including local comparative statics, approximate rationality, and neighbourhood-based representations such as indistinguishability (Mas-Colell, Whinston, and Green, 1995; Varian, 1990; Luce, 1956; Fishburn, 1970). In these settings, distance is used to order refinements or perturbations, with smaller neighbourhoods corresponding to sharper comparisons or weaker deviations. While such arguments are formally ordinal, it is tempting to reason proportionally about changes in neighbourhood size. The analysis here characterizes the purely geometric conditions under which such proportional interpretations are valid, rather than asserting them through intuition.

The result is intended as a modular geometric component that can be embedded in richer models of coarse perception, robustness, or neighbourhood-based comparison. By separating monotonic refinement from proportional refinement, the analysis clarifies when distance can, and cannot, be interpreted as a linear measure of robustness.

The remainder of the paper proceeds as follows. Section 2 introduces the geometric framework and defines coverage as a measure of robustness under neighbourhood-based comparison. Section 3 characterizes the relationship between coverage and displacement and introduces effective dimensionality as the key determinant of proportionality. Section 4 analyzes coverage loss and interval-scale representation. Section 5 illustrates the implications for certification problems under bounded measurement error. Lastly, section 6 concludes.

2 Setup

Let $X \subseteq \mathbb{R}^n$ be a subset of the Euclidean space. Fix a neighbourhood system

$$x \longmapsto \mathcal{N}(x) \subseteq \mathbb{R}^n,$$

where $\mathcal{N}(x)$ is a measurable set representing the neighbourhood around x. Assume $\lambda(\mathcal{N}(x)) \in (0, \infty)$ for all $x \in X$, where λ denotes Lebesgue measure on $\mathbb{R}^n$.

Definition 1. *For $x, y \in X$, define* **Coverage** *by*

$$\mathsf{C}(x, y) \equiv \frac{\lambda\big(\mathcal{N}(x) \cap \mathcal{N}(y)\big)}{\lambda\big(\mathcal{N}(x)\big)}.$$

Coverage measures the fraction of the neighbourhood around x that remains overlapping when y is considered.[1] By construction, $\mathsf{C}(x, y) \in [0, 1]$ and $\mathsf{C}(x, x) = 1$. The geometry of overlap loss, and hence the functional form of coverage as a function of displacement, depends on the structure of the neighbourhood system $\mathcal{N}(\cdot)$.

The analysis below studies how coverage varies with displacement and, in particular, when distance can be treated as a proportional measure of robustness.

3 Geometric Structure of Coverage

This section characterizes how coverage varies with displacement and introduces the notion of effective dimensionality, which governs when proportionality of robustness holds.[2]

3.1 Euclidean Neighbourhoods

We begin with Euclidean neighbourhoods, which are rotationally symmetric balls. Fix a radius $r > 0$ and define

$$\mathcal{N}(x) = B_r(x) := \{y \in \mathbb{R}^n : \|y - x\|_2 \leq r\}.$$

Under this neighbourhood system, coverage specializes to

$$\mathsf{C}(x, y) = \frac{\lambda\big(B_r(x) \cap B_r(y)\big)}{\lambda\big(B_r(x)\big)}.$$

By translation and rotation invariance, coverage depends only on the distance $d = \|x - y\|_2$.

Define the Euclidean coverage function

$$C(d) := \mathsf{C}(x, y) \quad \text{for any } x, y \text{ with } \|x - y\|_2 = d.$$

Definition 2. *Consider Euclidean neighbourhoods $B_r(x) \subset \mathbb{R}^n$. A displacement from x to y has* **Effective Dimensionality** *k if overlap loss occurs along k independent Euclidean directions, where dimensionality refers to the geometry of truncation rather than the dimension of the path of displacement.*

Note that in Euclidean space, coverage depends only on the radial displacement $d = \|x - y\|_2$. However, radial movement corresponds to uniform truncation along a single direction only in $\mathbb{R}^1$, where neighbourhoods are intervals. In dimensions $n \geq 2$, even when displacement is along a single line, overlap is lost along a curved boundary involving multiple independent directions. Thus effective dimensionality equals 1 if and only if $n = 1$ for Euclidean neighbourhoods.

Lemma 1. *Let $B_r(x) \subset \mathbb{R}^n$ be Euclidean neighbourhoods and let $v = y - x$ with $d = \|v\|_2$.*

1. *If displacement is effectively one-dimensional, then for $0 \leq d \leq 2r$,*

$$\mathsf{C}(x, y) = 1 - \frac{d}{2r}.$$

[1] All results would be unchanged under symmetric normalization (e.g., dividing by the average or union measure), but the asymmetric form aligns naturally with certification and robustness interpretations.

[2] The analysis draws on basic metric and volume properties of neighbourhoods. See (Burago et al., 2001).

2. If displacement is effectively multidimensional, then the map $d \mapsto \mathsf{C}(x, y)$ *is strictly convex on* $(0, 2r)$.

Proof. For part 1, in one dimension, neighbourhoods are intervals of length $2r$:

$$B_r(x) = [x - r, x + r].$$

If $|x - y| = d \leq 2r$, then the intersection $B_r(x) \cap B_r(y)$ is an interval of length $2r - d$. Therefore

$$C_1(d) = \frac{2r - d}{2r} = 1 - \frac{d}{2r}.$$

For part 2, fix $n \geq 2$ and let $d = \|x - y\|_2$. By symmetry, the volume

$$V_n(d) := \lambda\big(B_r(x) \cap B_r(y)\big)$$

depends only on d. Differentiating the standard closed form expression (Li, 2011), for $d \in (0, 2r)$,

$$V_n'(d) = -\omega_{n-1}\big(r^2 - (d/2)^2\big)^{(n-1)/2},$$

where ω_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$. Differentiating again,

$$V_n''(d) = \omega_{n-1}\frac{n-1}{4}d\big(r^2 - (d/2)^2\big)^{(n-3)/2} > 0 \quad \text{for } d \in (0, 2r).$$

Since $C_n(d) = V_n(d)/V_n(0)$, it follows that $C_n''(d) > 0$ on $(0, 2r)$, proving strict convexity. □

Although coverage depends only on the scalar distance $d = \|x - y\|_2$, the geometry of overlap loss differs sharply across dimensions. In one dimension, truncation occurs uniformly at a single boundary, while in dimensions $n \geq 2$ overlap is lost along a curved boundary involving multiple directions. Therefore, radial symmetry does not imply effective one-dimensionality.

3.2 Product Neighbourhoods

The dimensional boundary identified above is not an artifact of spherical geometry. Consider product neighbourhoods induced by the ℓ_∞ norm:

$$\mathcal{N}(x) = B_r^\infty(x) := \{y \in \mathbb{R}^n : \|y - x\|_\infty \leq r\} = \prod_{i=1}^{n}[x_i - r, x_i + r].$$

Coverage under product neighbourhoods is given by

$$\mathsf{C}^\infty(x, y) = \frac{\lambda(B_r^\infty(x) \cap B_r^\infty(y))}{\lambda(B_r^\infty(x))}.$$

Definition 3. *Let* $v = y - x \in \mathbb{R}^n$*. The displacement* v *has* **Effective Dimensionality** k *if*

$$k = \dim\big(\mathrm{span}\{\, e_i : v_i \neq 0 \,\}\big).$$

Under product neighbourhoods, truncation occurs independently along coordinate directions, so effective dimensionality is determined by the number of active coordinates.

Lemma 2. *Fix $r > 0$ and $x, y \in \mathbb{R}^n$, let $v := y - x$, and suppose the effective dimensionality of v is k. Define*

$$C(t) := \prod_{i:v_i \neq 0} \left(1 - \frac{t|v_i|}{2r}\right), \qquad t \in \left[0, \frac{2r}{\|v\|_\infty}\right].$$

Then:

1. *If $k = 1$, $C(t)$ is affine on its domain.*
2. *If $k \geq 2$, $C(t)$ is strictly convex on $\left(0, \frac{2r}{\|v\|_\infty}\right)$.*

Proof. For i with $v_i \neq 0$ define $\alpha_i := |v_i|/(2r)$. Then

$$C(t) = \prod_{i \in S} (1 - \alpha_i t),$$

so

$$C'(t) = -\sum_{i \in S} \alpha_i \prod_{j \in S \setminus \{i\}} (1 - \alpha_j t),$$

and

$$C''(t) = \sum_{\substack{i,j \in S \\ i \neq j}} \alpha_i \alpha_j \prod_{\ell \in S \setminus \{i,j\}} (1 - \alpha_\ell t).$$

For Part 1, let $k = 1$. Then, for all $i \neq j$ we have $\alpha_i \alpha_j = 0$, so every term in the above sum vanishes and $C''(t) = 0$ on the domain. Hence $C(t)$ is affine.

For Part 2, let $k \geq 2$. Then, there exist $i \neq j$ with $\alpha_i, \alpha_j > 0$. For $t \in (0, 2r/\|v\|_\infty)$, each factor $(1 - \alpha_\ell t)$ is strictly positive. Therefore every summand in $C''(t)$ is nonnegative and at least one summand is strictly positive, implying $C''(t) > 0$ on the interior of the domain. Hence $C(t)$ is strictly convex. □

4 Coverage Misalignment and Interval-Scale Representation

Fix a reference point $x \in X$.

Definition 4. *The* coverage misalignment *between x and y is*

$$M(x, y) := 1 - \mathsf{C}(x, y),$$

that is, the fraction of the reference neighbourhood around x that does not overlap with the neighbourhood around y.

Definition 5. *Distance provides an **interval-scale representation of coverage** if there exist constants $a \in \mathbb{R}$ and $b \neq 0$ such that*

$$\mathsf{C}(x, y) = a + b\|x - y\| \quad \text{for all } y \text{ with } \|x - y\| \leq 2r,$$

where $\|\cdot\|$ denotes the norm associated with the neighbourhood geometry.

It is important to distinguish exact proportionality from local approximation. When coverage is strictly convex in displacement, distance cannot provide a global interval-scale representation. That is, no affine transformation of distance can match coverage over the full range of admissible displacements. This does not preclude local linearization. As with any smooth function, coverage is locally well approximated by its first-order expansion around a reference point, so sufficiently small perturbations can always be treated as approximately proportional. The results below concern exact proportionality rather than first-order approximation. They identify when distance provides a globally valid interval-scale measure of robustness, independent of local linearizations that hold generically but only infinitesimally.

Proposition 1. *Distance provides an interval-scale representation of coverage if and only if displacement is effectively one-dimensional.*

Proof. If displacement is effectively one-dimensional, coverage decays affinely by Lemmas 1 and 2, yielding an interval-scale representation.

If displacement is effectively multidimensional, coverage is strictly convex in displacement magnitude along the relevant path, ruling out affinity and hence interval-scale representation. □

The equivalence is geometric rather than metric.

Across both Euclidean and product geometries, proportional decay of coverage is therefore not tied to the choice of norm, but to the effective dimensionality of variation instead. Proportional robustness is a knife-edge property of effectively one-dimensional truncation.

5 GPS Certification and the Dimensional Boundary

This section illustrates the dimensional boundary with a simple certification problem under bounded measurement error. The example uses a one-sided variant of neighbourhood overlap, in which robustness is measured relative to a fixed constraint set rather than a second neighbourhood.

While the analysis above is formulated in terms of overlap between two neighbourhoods, the same geometry governs robustness of classification against a boundary. In this one-sided setting, coverage is defined as the fraction of a neighbourhood that lies on the admissible side of a constraint. As a result, the functional relationship between coverage and displacement is identical to that arising from two-sided neighbourhood overlap.

5.1 Worked Example

An agent stands at a true location x but is observed through a GPS device that returns a reported location $\hat{x}$ with bounded error:

$$\hat{x} \in B_r(x),$$

where $B_r(x)$ is the Euclidean ball of radius $r > 0$ centered at x. Assume for simplicity that the GPS error is uniform on $B_r(x)$. Let the agent's property be given by the set P. Then the probability that the GPS report certifies the agent is on their property equals

$$\Pr(\hat{x} \in P \mid x) = \frac{\lambda\big(B_r(x) \cap P\big)}{\lambda\big(B_r(x)\big)}.$$

This is exactly a coverage ratio: the fraction of the error neighbourhood consistent with the property claim.

5.1.1 One Dimension: Linear Coverage

Let $n = 1$ and suppose the property is the half-line

$$P = \{s \in \mathbb{R} : s \geq 0\}.$$

If the agent stands at $x = t \geq 0$, then $B_r(t) = [t - r, t + r]$ and for $0 \leq t \leq r$,

$$\Pr(\hat{x} \in P \mid x = t) = \frac{1}{2} + \frac{t}{2r},$$

while for $t \geq r$ the probability equals 1. Thus certifiability increases linearly with distance from the boundary until it saturates.

5.1.2 Two Dimensions: Nonlinear (Convex) Coverage

Let $n = 2$ and suppose the property is the right half-plane

$$P = \{(s_1, s_2) \in \mathbb{R}^2 : s_1 \geq 0\}.$$

Place the agent at $x = (t, 0)$ with $t \geq 0$. For $t \geq r$, the disk $B_r(x)$ lies entirely in P and the certification probability is 1. For $0 \leq t \leq r$, the disk crosses the boundary and the certification probability equals the fraction of the disk that lies in the half-plane. This fraction admits a standard closed form and is strictly convex in t on $[0, r]$.[3] Thus equal increases in distance from the boundary correspond to equal gains in certifiability only in one dimension.

6 Conclusion

This paper isolates a simple geometric boundary for when proximity can be interpreted as a proportional measure of robustness under neighbourhood-based comparison. Using neighbourhood overlap as a canonical notion of robustness, we show that coverage is affine in distance if and only if variation is effectively one-dimensional. In this case, equal changes in distance correspond to equal losses of coverage, and proximity admits an interval-scale interpretation. When variation is not effectively one-dimensional, coverage is strictly convex and proportional comparability is lost.

Although overlap formulas in specific geometries are well known, the result here is not a calculation but a representation result: it identifies when coverage can be expressed as an affine function of distance, and shows that this property fails generically even in symmetric and smooth environments.

The result is not a critique of particular models, but a clarification of the geometric conditions under which common forms of local reasoning admit proportional interpretations. Many arguments rely on shrinking neighbourhoods, small perturbations, or marginal changes to extract sharper ordinal conclusions. The analysis shows that treating such changes as proportionate refinements requires an implicit one-dimensional structure.

Scalar representations impose this structure by construction. By collapsing multidimensional environments to a single index, they restore linearity of coverage loss and make robustness interval-comparable by construction. The usefulness of such representations therefore reflects a geometric commitment rather than a purely ordinal one.

[3] One convenient closed form is obtained from the area of a circular segment:

$$\Pr(\hat{x} \in P \mid x = (t, 0)) = 1 - \frac{1}{\pi r^2}\Big[r^2 \arccos(t/r) - t\sqrt{r^2 - t^2}\Big], \qquad 0 \leq t \leq r,$$

which is strictly convex on $[0, r]$.

By separating monotonicity from proportionality, the analysis clarifies when distance can, and cannot, be interpreted as a linear measure of refinement. The dimensional boundary identified here is geometric rather than behavioural, and applies independently of the economic or decision-theoretic context in which neighbourhood-based comparison is employed.

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