1 Linear Momentum

Linear Momentum

The linear momentum of a particle of mass m moving with velocity v is defined to be

$$\vec{p} = m\vec{v}$$

Units: kgm/s, since its dimensions are ML/T

Momentum is a vector, and hence can be split up into x, y and z axis.

$$p_x = mv_x, p_y = mv_y, p_z = mv_z$$

Newton Second Law

If m is constant,

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} = m\vec{a}$$

So, in terms of linear momentum, Newton's second law can be written as:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

The time rate of change of linear momentum of a particle is equal to the resultant force acting on the particle.

Impulse

The impulse of force **F**(t) acting on a particle from time t_i to t_f

$$\vec{J} = \vec{p}_f - \vec{p}_i = \int_{t_f}^{t_i} \vec{F} dt$$

Impulse of a force equals to the change in momentum of the particle caused by the force.

Impulse is also a vector, and has the same dimensions as momentum.

p_f and p_i represent the momentum immediately before and after the collision.

The impulsive force can be approximated as such:

$$\vec{J} = \Delta\vec{p} = (\vec{F}_a v)(t_f - t_i) =$$

In general, the impulse $(p_f - p_i)$ is the area under the F-t curve. The magnitude of impulse is affected by the magnitude of the force and the duration of which the force acts.

Momentum and KE

From $K = \frac{1}{2}mv^2$ and $p = mv$, we get

$$K = \frac{p^2}{2m}$$

$$p = \sqrt{2mK}$$

Conservation of Linear Momentum

If there are no external forces acting on an isolated system of particles, the total momentum of an isolated system remains constant.

$$\sum \vec{p}_i = \sum \vec{p}_f$$

The total momentum of an isolated system just before a collision equals the total momentum of the system just after the collision.

2 Collisions

In a collision, momentum and total energy is always conserved. KE is only conserved for elastic collision

Types of Collisions

The coefficient of restitution is defined as

$$e = \frac{\text{Relative speed of separation}}{\text{Relative speed of approach}} = \frac{v_{2f} - v_{1f}}{v_{1i} - v_{2i}}$$

1. **Perfectly Inelastic** $e = 0$
   The two (or more) bodies stick together after collision and move together.
   Kinetic energy is not conserved in an inelastic collision, but CoE may still be applicable.
2. **Inelastic** $e > 0$
   Kinetic energy is not conserved in an inelastic collision
3. **Elastic** $e = 1$
   The kinetic energy is conserved. $KE_i = KE_f$
4. **Explosive** $e > 1$

Elastic Collisions

The kinetic energy before and after collision is conserved.

Momentum Conservation

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

Energy Conservation

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

Coefficient of restitution

$$e = \frac{v_{2f} - v_{1f}}{v_{1i} - v_{2i}} = 1$$

And putting the equation for coefficient of restitution into the momentum conservation, we get

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right) v_{2i}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_{2i}$$

This gives rise to the following special cases

Case 1. $m_1 = m_2$
$v_{1f} = v_{2i}$ and $v_{2f} = v_{1i}$ i.e. the particles exchange velocities

Case 2. m_2 initially at rest ($v_{2i} = 0$), $m_1 >>> m_2$
$v_{1f} \approx v_{1i}$ and $v_{2f} \approx 2v_{2i}$

Case 3. m_2 initially at rest ($v_{2i} = 0$), $m_2 >>> m_1$
$v_{1f} \approx -v_{1i}$ and $v_{2f} \approx 0$

Note 2.1 The maximum transfer of KE occurs when $m_1 = m_2$. Compare the cases.

Note 2.2 For 2D/3D collisions, the momentum conservation can be split up into x, y and z axis.

Perfectly Inelastic Collisions

Two particles collide head on, and stick together and move with some common velocity v_f after collision.

Only momentum conservation applies here, unless the height is affected, then you can take into energy conservation.

Momentum Conservation

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m2) v_f$$
$$v_f = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2}$$

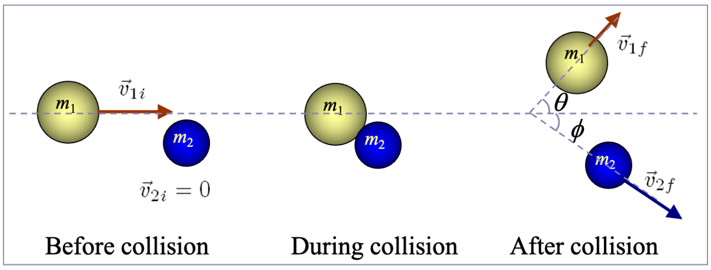
2D Collisions

For 2D collisions, the momentum in each direction is conserved.

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$

$$m_1 v_{1iy} + m_2 v_{2iy} = m_1 v_{1fy} + m_2 v_{2fy}$$

Glancing Collision



Momentum Conservation

x-axis only m_1 is moving initially

$$m_1 v_1 = m_1 v_{1f} \cos\theta + m_2 v_{2f} \cos\phi$$

y-axis No momentum in the y-axis initially.

$$0 = m_1 v_{1f} \sin\theta - m_2 v_{2f} \sin\phi$$

Energy Conservation Only if Elastic Collision

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

Using the equations from above, we get

$$v_{1i} = v_{1f} + v_{2f}$$
$$v_{1i}^2 = v_{1f}^2 + v_{2f}^2$$

By Pythagorean's Theorem, $\theta + \phi = 90°$

General Strategy

1. Choose an appropriate axis, usually the x-axis to coincide with the inital velocity
2. Sketch and label all vectors
3. Write down the momentum conservation in 2 directions
4. If elastic, write down the kinetic energy conservation

Center of Mass

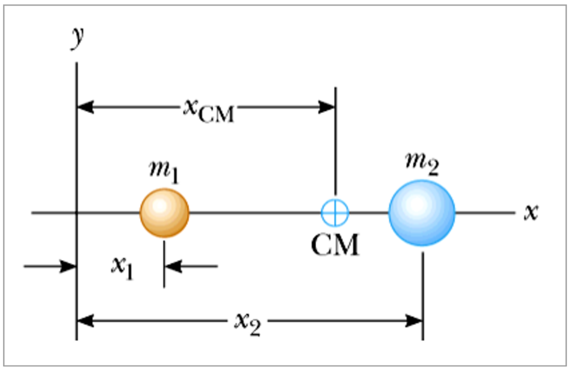
A mechanical system moves as if all its mass was concentrated at its center of mass.

If an external force F acts on this system of total mass M, the center of mass (CM) accelerates at $a = \frac{F}{M}$.

This is independent of other motion e.g. vibration and rotation. Note that CM $\neq$ Mass!!! CM is position info, where we assume the mass can be seen as acting through. However, when we look at mass in terms of Moment of Inertia, we see it as distributed.

Determining CM

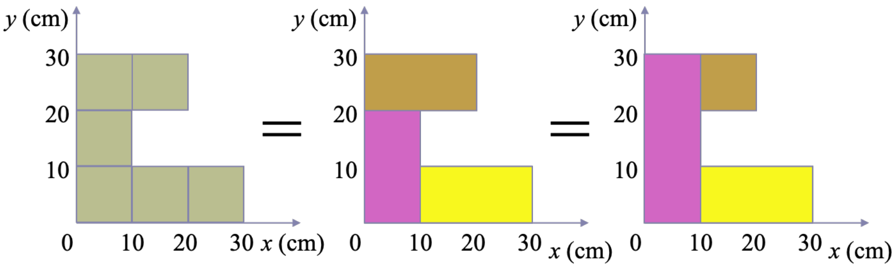
1D



In the above case, the center of mass of the two particles occur at CM, and is determined by

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

2D



We can split up the sections to make it easier for us to calculate the CM.

Example Using the middle combination

$$r_{CM1} = 10\hat{i} + 25\hat{j}$$
$$r_{CM2} = 5\hat{i} + 10\hat{j}$$
$$r_{CM3} = 20\hat{i} + 5\hat{j}$$

Then, assuming each section has a mass of m, we can determine the x_{CM} and y_{CM}.

$$x_{CM} = \frac{m(10) + m(5) + m(20)}{3m} = 11.7cm$$

$$y_{CM} = \frac{m(25) + m(10) + m(5)}{3m} = 13.3cm$$

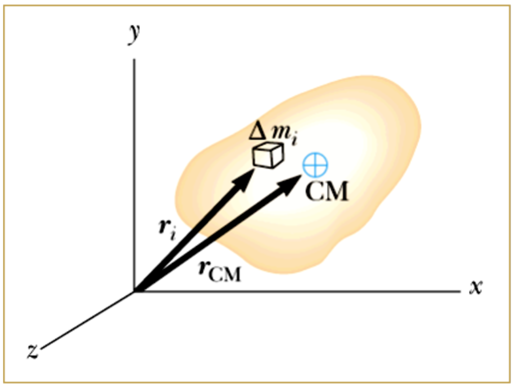
CM of system of particles

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + ... + m_n x_n}{m_1 + m_2 + m_3 + ... + m_n} = \frac{\sum_i m_i x_i}{m_i}$$
$$y_{CM} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + ... + m_n y_n}{m_1 + m_2 + m_3 + ... + m_n} = \frac{\sum_i m_i y_i}{m_i}$$
$$z_{CM} = \frac{m_1 z_1 + m_2 z_2 + m_3 z_3 + ... + m_n z_n}{m_1 + m_2 + m_3 + ... + m_n} = \frac{\sum_i m_i z_i}{m_i}$$

And therefore,

$$\vec{r}_{CM} = x_{CM}\hat{i} + y_{CM}\hat{j} + z_{CM}\hat{k}$$
$$\vec{r}_{CM} = \frac{\sum_i m_i \vec{r}_i}{M}$$

CM of an extended (continuous) object



The CM of a extended object is given by

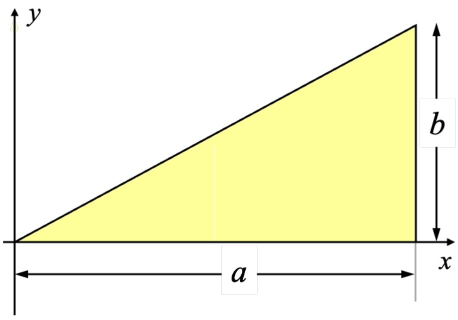
$$x_{CM} = \frac{1}{M}\int x dm$$

$$y_{CM} = \frac{1}{M}\int y dm$$

$$z_{CM} = \frac{1}{M}\int z dm$$

$$\vec{r}_{CM} = \frac{1}{M}\int \vec{r} dm$$

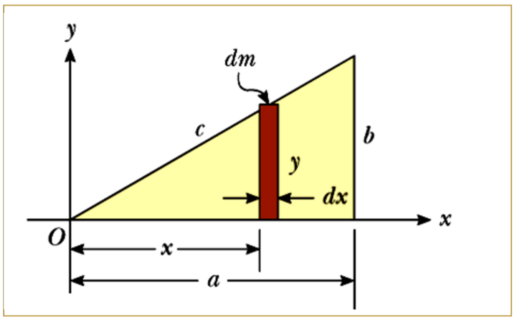
Example The CM of right angled triangle



$$x_{CM} = \frac{1}{M}\int x dm$$

$$y_{CM} = \frac{1}{M}\int y dm$$

Find x_{CM} by choosing a dm such that every point in dm has the same x value.



From the chosen strip,

$$\frac{y}{x} = \frac{b}{a}$$
$$y = \frac{b}{a}x$$

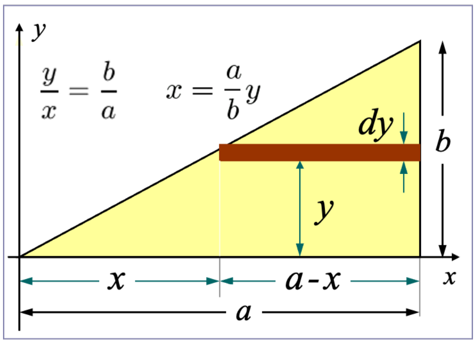
And then,

$$\rho = \text{mass per unit area}$$
$$M = \rho\left(\frac{1}{2}ab\right)$$
$$\rho = \frac{2M}{ab}$$
$$dm = \rho(ydx) = \rho\left(\frac{b}{a}x\right)dx$$

And thus,

$$\begin{aligned} x_{CM} &= \frac{1}{M}\int xdm \\ &= \frac{\rho}{M}\int_0^a x\left(\frac{b}{a}\right)dx \\ &= \frac{1}{M}\frac{2M}{ab}\left[\frac{b}{a} - \frac{x^3}{3}\right]_0^a \\ &= \frac{2}{ab}\left(\frac{b}{a}\frac{a^3}{3}\right) \\ &= \frac{2}{3}a \end{aligned}$$

Find y_{CM} by choosing a dm such that every point in dm has the same x value.



From the chosen strip,

$$\frac{y}{x} = \frac{b}{a}$$
$$x = \frac{a}{b}y$$

And then,

$$\rho = \text{mass per unit area}$$
$$M = \rho\left(\frac{1}{2}ab\right)$$
$$\rho = \frac{2M}{ab}$$
$$dm = \rho(a - xdy) = \rho\left(a - \frac{a}{b}y\right)dy$$

And thus,

$$\begin{aligned} y_{CM} &= \frac{1}{M}\int ydm \\ &= \frac{\rho}{M}\int_0^b y\left(a - \frac{a}{b}y\right)dy \\ &= \frac{1}{M}\frac{2M}{ab}\left[\frac{ay^2}{2} - \frac{ay^3}{3b}\right]_0^b \\ &= \frac{2}{ab}\left(\frac{ab^2}{2} - \frac{ab^3}{3b}\right) \\ &= \frac{1}{3}b \end{aligned}$$

Example The CM of a Cone

Refer to slides. For a uniform cone, you can assume the CM along the x and y axis to be 0 as it has a line of symmetry along the x and y axis.

CM Notes

For a homogeneous (constant density) body that has a geometric center, the CM is the geometric center (e.g. sphere, cube)

The CM of a symmetric object lies on the axis of symmetry and on any plan of symmetry

CM doesn't have to be within the body itself.

If **g** is constant over the mass distribution, the center of gravity coincides with the centre of mass. If an object is hung freely from any point, the vertical line through this point must pass through the centre of mass.

Motion of a system of a particles

The total linear momentum of the system is equal to that of a single particle of mass M moving with a velocity of v_{CM}.

$$\vec{r}_{CM} = \frac{\sum_i m_i \vec{r}_i}{M}$$

$$\vec{v}_{CM} = \frac{d\vec{r}_{CM}}{dt} = \frac{1}{M}\sum_i m_i \frac{\vec{r}_i}{dt} = \frac{\sum_i m_i v_i}{M}$$

Then, shifting the M to the front,

$$M\vec{v}_{CM} = \sum_i m_i v_i = \sum \vec{p}_i = \sum \vec{p}_{total}$$

From the formula for velocity, we get

$$\vec{a}_{CM} = \frac{d\vec{v}_{CM}}{dt} = \frac{1}{M}\sum_i m_i \frac{\vec{v}_i}{dt} = \frac{1}{M}\sum_i m_i \vec{a}_i$$

Moving the M to the front,

$$M\vec{a}_{CM} = \sum_i m_i \vec{a}_i = \sum_i \vec{F}_i$$

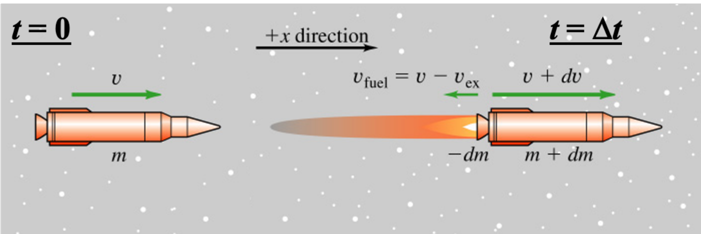
F_i includes both internal and external forces. Since internal forces cancel out,

$$\sum \vec{F}_{ext} = M\vec{a}_{CM} = \frac{d\vec{p}_{total}}{dt}$$

The CM moves like an imaginary particle of mass M under the influence of the external resultant force on the system.

Rocket Propulsion

A rocket moves upwards because the gases exhausted out moves downwards. So by conservation of momentum, the rocket has to move upwards.



$$mv = (m + dv)(v + dv) + (-dm)(v - v_{ex}$$

$$m\frac{dv}{dt} = -v_{ex}\frac{dm}{dt}$$

$$a = \frac{-v_{ex}}{m}\frac{dm}{dt}$$

Integration the above formula,

$$v - v_0 = -v_{ex}\ln\left(\frac{m}{m_0}\right)$$

Note

$v_e x$ refers to the exhaust speed of the gas. Since the gas is moving together with the rocket at initial speed v, the speed of $v_{fuel} = v - v_{ex}$.

dm refers to the mass of the fuel

m is the final mass, m_0 is the initial mass (rocket + fuel), so $\ln\frac{m}{m_0} < 0$ and this shows us $v > v_0$.

Thrust

Thrust is the force exerted on the rocket by the ejected exhaust gases

$$\text{Thrust} = m\frac{dv}{dt} = \left|v_{ex}\frac{dm}{dt}\right|$$

3 Rotation and Moment of Inertia

Rigid Body Rotation

Rotation of an extended object - different parts of an object have different linear velocities and accelerations.

Rigid means non-deformable and internal motion neglected.

When rotating about a fixed axis, every point on a rigid body has the same angular speed and the same angular velocity

Definitions

1. **Angle of Rotation** $\theta = \frac{s}{r}$
   θ is in radians
   s is the arc length, and r is the radius of arc
2. **Average Angular Velocity** $\bar{\omega} = \frac{\Delta\theta}{\Delta t}$
3. **Instantaneous Angular Velocity** $\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$
4. **Average Angular Acceleration** $\bar{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$
5. **Instantaneous Angular Acceleration** $\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$
6. **Linear velocity** $v = r\omega$
7. **Tangential Acceleration** $a_t = r\alpha$
8. **Radial Acceleration** $a_{rad} = \frac{v^2}{r} = r\omega^2$

Constant Angular Acceleration, α

$$\omega = \omega_0 + \alpha t$$
$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$
$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

Angular Velocity ω and Angular Acceleration, α

ω and α are vectors.

The direction of angular velocity, ω, is determined by the right-hand rule. ACW - Upwards and CW - Downwards.

The direction of angular acceleration, α is in the same direction as ω if the rotation is speeding up, and is in the opposite direction of ω if it is slowing down.

Rotational Energy

The kinetic energy of the ith particle is given by

$$K_i = \frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i r_i^2 \omega^2$$

So the total kinetic energy is given by

$$K_R = \sum_i K_i = \frac{1}{2}\sum_i m_i r_i^2 \omega^2$$

Note The ω is the same for every particle.

Moment of Inertia

Rewriting kinetic energy,

$$K_R = \frac{1}{2}I\omega^2$$

And the moment of inertia, I, is written as

$$I = \sum_i m_i r_i^2$$

For a extended rigid object,

$$I = \lim_{\Delta m_i \to 0} \sum_i \Delta m_i r_i^2 = \int r^2 dm = \int \rho r^2 dV$$

ρ is the density of the object.

The moment of inertia, I is a measure of the resistance of an object to changes in its rotational motion.

I depends on the physical arrangement of that mass (e.g. weight distribution) and also the axis of rotation (which affects r_i).

Units are kgm^2

Discrete Objects / Points

For discrete point or disjoint objects, you use the following formula

$$I = \sum_i m_i r_i^2$$

Note The axis of rotation will affect the r_i^2 and hence the value of I.

Continuous / Extended Object

For continuous / extended objects, use the following formula.

$$I = \int_a^b r^2 \; dm$$

Then, usually, you will need to convert the dm into dx, so the formula can be integrated.

The limits b and a depends on how far away it is from the axis.

Steps to find dm in terms of dx

1. 2-D Objects
   Find a small area perpendicular to the distance from the axis.
   Then express a small mass, dm, in terms of a small length dx.

   $dm = \lambda dx = \frac{M}{L}dx$

   Where dm is a small mass, dx is a small length, λ is the mass per unit length
   See uniform rod example

2. 3-D Objects (that you can find an small volume)
   Find a small volume perpendicular to the distance from the axis.
   Then, express the small volume in terms of a small mass.

   $dm = \rho L(2\pi r)dr$

   Where dm is a small mass, $rho = \frac{m}{v}$, density, $2\pi rL$ is the area, and dr is the small length.
   Note that the area formula here only applies to the cylinder

3. 3-D Objects (cannot find a small volume perpendicular)
   Use the formula $I = \int dI$ instead, and find dI

Parallel Axis Theorem

The moment of inertia about any axis parallel to and at distance D away from the axis that passes through the center of mass is

$$I = I_{CM} + MD^2$$

Where I_{CM} is the moment of inertia if the axis was at the CM, and MD^2 is the mass × distance^2 away from the CM.

4 Rotational Dynamics

Torque

The torque τ of a force F about O is defined as

$$\tau = rF \sin\phi$$

Also known as moment of a force. Take the product of the force and the perpendicular distance from the pivot to the line of action of the force.

The SI Unit is Newton metre (Nm).

The direction of torque τ is out of the paper for clockwise, and into the paper for anticlockwise, same direction as angular velocity, ω *Follow right hand rule*

When summing torques we only care about those that cause a rotation. The force that cause a clockwise rotation is taken to be negative.

Torque and Angular Acceleration

$$\begin{aligned} \tau &= rF_t \\ &= r(ma_t) \\ &= r(mr\alpha) && \text{since } a_t = r\alpha \\ &= (mr^2)\alpha \\ &= I\alpha && \text{since } I = mr^2 \end{aligned}$$

The torque acting on a particle is proportional to its angular acceleration.

The same formula applies for extended objects.

$$d\tau = rdF_t = r(dm)a_t = (r^2 dm)\alpha$$
$$\tau = \int (r^2 dm)\alpha = a \int r^2 dm$$

Work, Power, and Energy

Work done by rotation

$$\begin{aligned} dW &= \vec{F} \cdot d\vec{s} \\ &= F \sin\phi \ r \ d\theta \\ &= \tau d\theta \end{aligned}$$

Power of rotation is equals to torque times angular velocity

$$\begin{aligned} P &= \frac{dW}{dt} \\ &= \tau \frac{d\theta}{dt} \\ &= \tau\omega \end{aligned}$$

The net work done by external forces in rotating a rigid object about a fixed axis equals the change in the object's rotational energy.

$$W = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2$$

5 Angular Momentum

Vector Product

$$\vec{C} = \vec{A} \times \vec{B}$$

Find the direction of the vector $\vec{C}$ by using the right hand rule.

The magnitude of vector $\vec{C}$ can be obtained by using

$$|\vec{C}| = |\vec{A}||\vec{B}| \sin\theta$$

Note Vector product is not commutative. $\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$, but it is distributive.

Torque as a vector product

Torque can also be expressed as a vector product

$$\tau = \vec{r} \times \vec{F}$$

Angular Momentum

Angular momentum of a particle is defined as the cross product of instantaneous vector position $\vec{r}$ and the linear momentum $\vec{p}$.

$$\vec{L} = \vec{r} \times \vec{p}$$

SI Unit: kgm^2/s

Note A particle moving along a straight line may have angular momentum as well, it depends on where is the origin defined. If there is a vector position $\vec{r}$ of the particle that is not parallel to the $\vec{p}$ (or $\vec{v}$ – same direction since $p = mv$).

Torque as Angular Momentum

The torque acting on a particle is equal to the time rate of change of the particle's angular momentum.
Valid for any origin in an inertia frame

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

For a system of particles, the same formula can be applied.

$$\vec{L}_{tot} = \vec{L}_1 + \vec{L}_2 + \vec{L}_3 + ... + \vec{L}_n = \sum_{i=1}^{n} \vec{L}_i$$

The total rate of change of angular momentum of the system about some origin in an inertia frame equals the net external torque acting on the system about that origin.

$$\sum \vec{\tau}_{ext} = \sum_{i=1}^{n} \frac{d\vec{L}_i}{dt} = \frac{d}{dt}\sum_{i=1}^{n} \vec{L}_i = \frac{d\vec{L}_{tot}}{dt}$$

Angular Momentum

For a particle with mass m_i, angular momentum L_i

$$L_i = m_i r_i^2 \omega$$

For a whole object,

$$\begin{aligned} L_z &= \sum_i m_i r_i^2 \omega \\ &= I\omega \end{aligned}$$

The net external torque acting on a rigid object rotating about a fixed symmetric axis equals the product of the moment of inertia and the angular acceleration about that axis.

$$\begin{aligned}\sum \tau_{\text{ext}} &= \frac{dL_z}{dt} \\ &= I\frac{d\omega}{dt} = I\alpha\end{aligned}$$

Conservation of Angular Momentum

The total angular momentum of a system is zero if the resultant external torque acting on the system is zero.

$$\sum \tau_{\text{ext}} = \frac{dL_z}{dt} = 0$$

$$\therefore \vec{L} = \text{const}$$

The conservation of angular momentum is

$$\begin{aligned}\vec{L}_i &= \vec{L}_f \\ I_i\omega_i &= I_f\omega_f\end{aligned}$$

Note Valid for rotation about a fixed axis or axis through the CM of a moving system as long as the axis remains fixed in direction.

Questions usually require a combination of conservation of linear momentum, conservation of angular momentum and conservation of energy as well.

6 Summary of Formulas

1. Angle of Rotation, θ in radians
   s is arc length and r is the radius of the arc
$$\theta = \frac{s}{r}$$

2. (Instantaneous) Angular Velocity, ω is the rate of change of angle of rotation w.r.t. time.
   θ is the angle of rotation and t is time
$$\omega = \frac{d\theta}{dt}$$

3. Linear Velocity, v
   r is the radius and ω is angular velocity
$$v = r\omega$$

4. (Instantaneous) Angular Acceleration, α
   ω is the angular velocity and t is time
$$\alpha = \frac{d\omega}{dt}$$

5. Tangential Acceleration, a_t
   r is the radius and α is angular acceleration
$$a_t = r\alpha$$

6. Radial Acceleration, a_{rad}
   v is the linear velocity, and r is the radius
$$a_{rad} = \frac{v^2}{r}$$
$$a_{rad} = r\omega$$

7. Rotational Kinetic Energy of an object
   m_i is the mass of each particle of the object, and m_i is the mass of each particle, r_i is the radius of each particle from the center of mass, and ω is the angular velocity of each particle (which is the same for every particle).
$$K_{tot} = \sum_i K_i = \frac{1}{2}\sum_i m_i r_i^2 \omega^2$$

8. Moment of Inertia for discrete objects or discrete points
   m_i is the mass of the point, r_i is the distance from the axis
$$I = m_i r_i^2$$

9. Moment of Inertia for continuous or extended objects
   r is the distance from the axis of rotation, dm is a small mass. Convert the dm and r in terms of x.
$$I = \int r^2 \, dm$$

10. Torque, also known as the moment of a force is the product of the force and the perpendicular distance to the line of action of the force.
    r is the distance from the axis, and $F \sin\phi$ is the force perpendicular to the distance from the axis.
$$\tau = rF\sin\phi$$
    I is the moment of inertia, and α is the angular acceleration
$$\tau = I\alpha$$

7 Temperature

Laws of Thermodynamics

0th Law: Temperature and Thermal Equilibrium

1st Law: Quantity of Energy (Energy Conservation law)

2nd Law: Quality of Energy (Process direction and extent)

3rd Law: 0 K is unattainable.

Definitions

1. **Thermal Contact** Energy can be exchanged between objects, need not be in physical contact
2. **Thermal Equilibrium** No net exchange of energy by heat or EM radiation between two objects i.e. the two objects have the same temperature

   Note If A and B are separately in thermal equilibrium with C, then A and B are in thermal equilibrium with each other.
3. **Temperature** Property that determines if objects are in thermal equilibrium, and the direction of energy exchange.

Temperature Measurement

1. Length of a solid (e.g. bimetallic strip)
2. The volume of liquid (e.g. mercury / alcohol thermometer)
3. Pressure of gas at constant volume

   Pressure increases linearly with temperature in the form: $P = aT_c + b$.
   T_c is the temperature in °C, and b is the pressure at 0°C, and a is the gradient of the line.

   The same formula can be written in the form for Kelvin. $P = cT_{\text{Kelvin}}$.

   Mainly used for calibrating temperature.
4. Volume of gas at constant pressure
5. The resistance of a conductor or semiconductor (e.g. Digital Thermometers)
6. The induced emf of dissimilar metals (e.g. thermocouple)
7. The color of an object
8. The color transmitted/reflected by a liquid crystal (e.g. Strip thermometer)
9. The intensity of thermal radiation

Temperature Scales

1. **Kelvin Scale** Mostly used for scientific work. Nothing can go below 0K.

   $$T_K = 273.16\left(\frac{P}{P_{\text{triple}}}\right)$$

   P_{triple} is the triple point of water, where water in its three states coexists in equilibirum
2. **Celcius Scale** Used in everyday life

   $$T_C = T_K - 273.15$$
3. **Fahrenheit Scale** Used in some countries

   $$T_F = \frac{9}{5}T_C + 32$$

   1°C = 1K = $\frac{9}{5}$°F

8 Heat and Ideal Gas Equation

Mechanical Equivalent of Heat

Joule's experiment found that the loss in mechanical energy is proportional to ΔT in the form:

$$4.186 m_w \Delta T = 2mgh$$

m_w is the mass of water, and mgh is the change in gravitational potential energy of the masses

Units of Heat

1. **calorie** (cal)
   Amount of heat required to raise the temperature of 1g of water from 14.5°C to 15.5°C
2. **joule** (J)
   1/4.186 calorie
3. **Food value Calorie** (Cal)
   Cal = 1 kcal = 1000 calorie
4. **British Thermal Unit** (BTU)
   Amount of heat required to raise the temperature of 1 pound of water from 63 °F to 64 °F.

$$1\text{BTU} = 453.6 \times \frac{5}{9} = 252\text{cal} = 1055J$$

Heat and Internal Energy

Internal Energy: All energy of a system (atoms and molecules) viewed from a reference frame at rest with respect to the CM of a system.

Excludes the kinetic energy due to bulk movement (e.g. external forces)
Includes the KE of random translational, rotational and vibrational motion of the molecules. *Includes* the PE within molecules (intramolecular) and PE between molecules (intermolecular)

Heat: Transfer of energy across the boundary of a system due to temperature difference between system and surroundings.

Heat is to internal energy like work done is to mechanical energy.

Heat, Internal Energy and Work

Increase the internal energy T of a substance by:

1. **Heat** Energy transfer between substances at different T e.g. Using a bursen burner to heat up water
2. **Work** Energy transfer that doesn't require a T difference e.g. Joule Experiment, using electric energy, or friction

Heat Capacity and Specific Heat

Specific Heat c
Amount of heat needed to raise the temperature of unit mass of substance by 1 °C

$$Q = mc\Delta T$$

where Q is the heat energy, m is the mass of the substance, c is the specific heat, and ΔT is the change in temperature.

If the specific heat varies with temperature,

$$Q = m\int_{T_1}^{T_2} c\ dT$$

Molar Specific Heat C
Amount of heat needed to raise the temperature of one mole of substance by 1 °C.

$$Q = nC\Delta T$$

where Q is the heat energy, n is the number of moles, C is the molar specific heat, and ΔT is the change in temperature.

Note $m = nM$, where m is the mass of the substance, n is the number of moles, and M Molar mass (mass of one mole).

Note This gives rise to $C = Mc$

Sometimes, the molar heat capacity depends on conditions such as constant pressure (C_P)

$$Q = nC_P\Delta T$$

Or constant volume (C_V) - usually for gases

$$Q = nC_V\Delta T$$

Latent Heat (Hidden Heat)

Transfer of heat does not result in a change of temperature during a phase change but there is change in internal energy e.g. during melting (solid $\rightarrow$ liquid), during boiling (liquid $\rightarrow$ gas) or a change in crysalline structure

Latent Heat L
Amount of heat needed to change the phase of the substance per unit mass

$$Q = mL$$

Q is the heat energy, m is the mass of substance, and L is the latent heat.
Note Use L_f for latent heat of fusion (solid to liquid or vice versa) and L_v for latent heat of vaporisation (liquid to gas or vice versa).

Idea Gas State Equation

Charles & Gay-Lussac Law $V \propto T$ for fixed P

Boyle's Law $P \propto \frac{1}{V}$ for fixed T

Pressure Law $P \propto T$ for fixed V

$\therefore \frac{PV}{T} =$ constant

With that, we arrive at:

$$PV = nRT$$

where n is the number of moles, R is the universal gas constant (same for all gases), and T is the temperature.

$$PV = NkT$$

where N is the number of molecules, k is the Boltzmann's constant ($1.381 \times 10^{-23} J/K$), T is the temperature.

$$n = \frac{N}{N_A}$$

where the number of moles is equals to the number of molecules divided by the Avogadro's number N_A ($6.022 \times 10^2 3$ molecules/mole).

$$k = \frac{R}{N_A}$$

Avogardo's Hypothesis Equal volume of gas at the same pressure and temperature contains equal number of molecules.

9 Kinetic Theory of Gases

Assumptions of the molecular model of an ideal gas

1. Large number of molecules; and the separation is much larger than the dimensions of the molecules. The molecules occupy negligible volume, and are treated as point-like
2. Obey Newton's laws of motion, but move randomly, different molecules with different speeds and directions
3. Elastic collisions with each other and the wall i.e. KE and momentum are conserved
4. Forces between molecules are negligible except during collision (only short range forces)
5. All molecules are identical - consider only pure substances.

Molecular Model

As molecules of a gas collide with the walls of the container, it changes direction, so there is a change in momentum in the x-direction.

$$dp_x = 2m|v_x|$$

In a time dt, the number of molecules that collide with the wall is

$$N_{collide} = \frac{1}{2}\left(\frac{N}{V}\right)A(|v_x|dt)$$

where $N_{collide}$ is the number of molecules colliding with the wall, N is the total number of molecules, V is the volume, A is the cross sectional area of the wall, $|v_x|$ is the velocity in the x-direction and dt is a small change in time.

Thus, the total change in momentum of all the molecules are

$$N_{collide}(dp_x) = \frac{1}{2}\left(\frac{N}{V}\right)A(|v_x|dt)(2m|v_x|)$$

And since $F = \frac{d\vec{p}}{dt}$, the force exerted by the wall on the molecules is

$$F = \frac{N_{collide}(dp_x)}{dt} = \frac{NAmv_x^2}{V}$$

By Newton's Third Law, the molecules exert the same magnitude of force on the wall. Since Pressure = force / area,

$$p = \frac{F}{A} = \frac{Nmv_x^2}{V}$$

v_x is not the same for all molecules, so we use the mean square speed instead. $\bar{v_x^2} = \frac{\sum_{i=1}^{N} v_x^2}{N}$.
And since the $v^2 = v_x^2 + v_y^2 + v_z^2$, for random motion, we can conclude

$$\bar{v_x^2} = \bar{v_y^2} = \bar{v_z^2} = \frac{1}{3}\bar{v^2}$$

And hence, we write

$$\begin{aligned} p &= \frac{Nmv_x^2}{V} \\ &= \frac{1}{3}\frac{N}{V}mv^2 \\ &= \frac{1}{3}\rho v^2 && \text{since } \rho = \frac{Nm}{V} \end{aligned}$$

Molecular Interpretation of T

Using the above equation,

$$
\begin{aligned}
p &= \frac{1}{3}\frac{N}{V}m\bar{v^2} \\
pV &= \frac{2}{3}N(\frac{1}{2}m\bar{v^2}) \\
NkT &= \frac{2}{3}N(\frac{1}{2}m\bar{v^2}) \\
T &= \frac{2}{3k}(\frac{1}{2}m\bar{v^2})
\end{aligned}
$$

Temperature is a direct measure of the average molecular kinetic energy

$$
\begin{aligned}
\frac{1}{2}m\bar{v^2} &= \frac{3}{2}kT \\
\frac{1}{2}m\bar{v_x^2} = \frac{1}{2}m\bar{v_y^2} = \frac{1}{2}m\bar{v_z^2} &= \frac{1}{2}kT
\end{aligned}
$$

Root-mean-Square (rms) Speed

The total translational energy of all molecules is given by

$$K_{tr} = N(\frac{1}{2}mv^2 = \frac{3}{2}NkT = \frac{3}{2}nRT$$

$$v_{rms} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3RT}{M}}$$

where m is the mass of the molecule, and M is the molar mass. At the same temperature, the lighter the molecule, the faster it will move.

Heat Capacity

The inter-molecular potential energy $= 0$ for ideal gas

$$E_{int} = K_{tr}$$

If heat dQ enters the system at constant volume, and changes the temperature by dT, then

$$
\begin{aligned}
dK_{tr} &= \frac{3}{2}nR\ dt \\
dQ &= nC_v\ dt \\
nC_v\ dt &= \frac{3}{2}nR\ dt \\
C_v &= \frac{3}{2}R = 12.47\text{J/mol K}
\end{aligned}
$$

However, the above value only works for monoatomic molecules. For monoatomic molecules, there is 3 degrees of freedom at room temperature 20°C. The 3 modes of translation.

For diatomic, the values don't work because they have 2 more modes of motion. There are 2 independent modes of rotation. So there is a total of 5 degrees of freedom.

For polyatomic, there is translation in 3 directions, rotation in 3 directions, and hence it has 6 degrees of freedom.

Equipartition of Energy

Each degree of freedom has associated with it, on average, an energy of $\frac{1}{2}kT$ per molecule (or $\frac{1}{2}RT$ per mole). We also need to consider vibration.

Gas molecules can also vibrate, and each vibration mode is equivalent to two degrees of freedom (vibration KE $\frac{1}{2}mv^2$ and potential energy $\frac{1}{2}kx^2$), but usually only significant at high temperatures and in polyatomic gases.

In summary,

1. **Monoatomic** e.g. He
   3 degrees of freedom: 3 translational motion
   $C_v = \frac{3}{2}R$ per mole
2. **Diatomic** e.g. H_2
   5 degrees of freedom at room temperature: 3 translational + 2 rotational
   $C_v = \frac{5}{2}R$ per mole

   At high temperatures, it may have vibrational as well, so at high temperatures
   $C_v = \frac{7}{2}R$ per mole for diatomic
3. **Polyatomic** More than 3 molecules, e.g. CO_2
   6 degrees of freedom: 3 translational + 3 rotational
   $C_v = \frac{6}{2}R$ per mole

Molar Specific Heat of Solids

For solids, they cannot translate or rotate due to their structures, so they can only vibrate in 3 directions. There are two degrees of freedom for each vibrational motion.

$$E = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2$$

where $\frac{1}{2}mv_x^2$ is the vibrational kinetic energy of the atoms and the $\frac{1}{2}kx^2$ is the potential energy of the vibration between the atoms.

Hence,

$$C_v = 6 \times \frac{1}{2}R = 3R$$

10 First Law of Thermodynamics

Processes

1. **Quasi-static Process** Processes that are carried out slowly enough so that the system passes through a sequence of thermal equilibrium states
2. **State Variables** Temperature, volume, pressure and internal energy
3. **Transfer Variables** Heat and Work

Work Done by Gas

Work is done on/by the gas when there is a change in pressure or volume of the gas.

W is the work done by the thermodynamic system (the gas) and is positive when the volume of gas increases. W is negative when the volume of the gas decreases.

$$W = \int_{V_1}^{V_2} p dV$$

Work done is path-dependent. You see area under the pV graph to determine the work done.

Heat Transfer

Heat transfer Q is process dependent / path dependent.

First Law of Thermodynamics: Heat Transfer, Work Done and Internal Energy

The first law of thermodynamics state that when a system undergoes a change from one state another, the change in its internal energy U is given by

$$\Delta U = U_2 - U_1 = Q - W$$

where Q is the energy transferred into the system and W is work done by the system

Heat transfer, Q and work done by gas, W are related to the internal energy U in the form:

$$\Delta U = Q - W$$

$Q - W$ is not path dependent, it only depends on the inital and final state of the system.

ΔU depends on the ΔT for an ideal gas!

Thermodynamic Processes

1. **Isochoric (Isovolumetric) Process**
   The volume of the system remains unchanged. Heat flows into the system changes its internal energy (heat in, increase U, heat out, decrease U)

$$\begin{aligned} W &= 0 && \text{since } \Delta V = 0 \\ \Delta U &= Q - W \\ &= nC_v \Delta T - 0 \\ &= nC_v (T_2 - T_1) \end{aligned}$$

2. **Isobaric Process**
   The pressure of the system remains constant

$$\begin{aligned} W &= \int_{V_a}^{V_b} p dV = p(V_b - V_a) \\ Q &= nC_p \Delta T \\ \Delta U &= Q - p(V_b - V_a) \end{aligned}$$

3. **Isothermal Process**
   The temperature of the system remains constant.

$$\begin{aligned} \Delta U &= 0 && \text{since U is a function of T for ideal gas} \\ \Delta U &= Q - W \\ Q &= W \\ Q &= W = nRT \ln\left(\frac{V_f}{V_i}\right) \end{aligned}$$

4. **Adiabatic Process**
   There is no heat transfer between the system and the surroundings. The internal energy decreases as work is done by the system, which usually means there is a fall in temperature.

$$\begin{aligned} Q &= 0 \\ \Delta U &= Q - W \\ &= -W \end{aligned}$$

5. **Adiabatic Process** Free Expansion Process
   There is no heat entering the system, and there is no work done or by the system.

$$\Delta U = Q = W = 0$$

   This cannot be represented by a line, since it is not a quasi-static process. It doesn't have points of thermal equilibrium during the process.

6. **Cyclic Process**
   For a cyclic process, since the initial state and the final states are the same, there is no change in internal energy.

   However, there is work done by the gas (see the area under the graph)

$$\begin{aligned} Q - W &= 0 \\ Q &= W \end{aligned}$$

Heat Capacities of Ideal Gas

For constant volume (isovolumetric) process, where gas is heated from T_1 to T_2.

$$\Delta U = Q = nC_v \Delta T$$

For isobaric process,

$$\begin{aligned} W &= p\Delta T = nR\Delta T \\ \Delta U &= nC_p \Delta T - nR\Delta T \end{aligned}$$

However, since internal energy of an ideal gas is a function of temperature only, the change in internal energy when a gas goes to T_1 to T_2 is as such:

$$\Delta U = nC_v \Delta T$$

Using the above relation, for an isobaric process,

$$\begin{aligned} \Delta U &= nC_p \Delta T - nR\Delta T \\ nC_v \Delta T &= nC_p \Delta T - nR\Delta T \\ C_p - C_v &= R \end{aligned}$$

Ratio of Heat Capacities γ

$$\gamma = \frac{C_P}{C_V} > 1$$

1. **Monoatomic**

$$\begin{aligned} C_V &= \frac{3}{2}R \\ C_P &= C_V + R = \frac{5}{2}R \\ \gamma &= \frac{C_P}{C_V} = \frac{5}{3} = 1.67 \end{aligned}$$

2. **Diatomic**

$$\begin{aligned} C_V &= \frac{5}{2}R \\ C_P &= C_V + R = \frac{7}{2}R \\ \gamma &= \frac{C_P}{C_V} = \frac{7}{5} = 1.40 \end{aligned}$$

3. **Polyatomic**

$$\begin{aligned} C_V &= \frac{6}{2}R \\ C_P &= C_V + R = \frac{8}{2}R \\ \gamma &= \frac{C_P}{C_V} = \frac{8}{6} = 1.33 \end{aligned}$$

Adiabatic Process for Ideal Gas

$$\begin{aligned} T_1 V_1^{\gamma-1} &= T_2 V_2^{\gamma-1} \\ P_1 V_1^{\gamma} &= P_2 P_2^{\gamma} \end{aligned}$$

The pv graph of an adiabatic process is always steeper than the isothermal curve because for adiabtic, the curve is $p = \frac{1}{V^\gamma}$ while for isothermal is $p = \frac{1}{V}$.

And since $\gamma > 1$, the gradient of the curve is steeper.

Work done in Adiabatic Process

$$\begin{aligned} W &= -\Delta U = -nC_v \Delta T \\ &= nC_v(T_a - T_b) \\ &= \frac{1}{\gamma - 1}(p_a V_a - p_b V_b) \end{aligned}$$

11 Heat Engines

A heat engine is a mechanical device that converts heat into mechanical energy.

A heat engine carries some working substance through a cyclic process.

Heat engine does work. The cycle is clockwise in the pv graph.

Heat pump (e.g. refrigerator) work has to be done on it. The cycle is anticlockwise in the PV graph.

Efficiency of Heat Engines

Because work done is in a cycle, $\Delta U = 0$, so $Q = -W$.

$$W = |Q_H| - |Q_C|$$

where Q_H is the heat energy of the hot reservoir and Q_C is the heat energy of the cooler reservoir, and W is the effective work done.

Thermal Efficiency is defined as:

$$\begin{aligned} e &= \frac{W}{Q_H} \\ &= 1 - \left|\frac{Q_C}{Q_H}\right| \end{aligned}$$

It is impossible for thermal efficiency to be 100% i.e. heat cannot be converted entirely to mechanical work.

Heat Pumps

Refrigerators and heat pumps are heat engines running in reverse.

A refrigerator or heat pump carries some working substance through a cyclic process, during which the working substance absorbs heat from a low-temperature reservoir and delivers it to a high temperature reservoir by means of work supplied externally.

Coefficient of Performance of Heat Pumps

1. **Cooling Mode** e.g. Fridge or AC
$$K_c = \frac{|Q_C|}{W} = \frac{|Q_C|}{|Q_H| - |Q_C|}$$
2. **Heating Mode** e.g. Heat Pump
$$K_h = \frac{|Q_H|}{|W|} = \frac{|Q_H|}{|Q_H| - |Q_C|}$$

Clausius Statement

Heat doesn't flow spontaneously from a cold object to a hot object. Work has to be done on the object for that to happen.

Reversible Processes

Heat transfer between two objects of different finite temperatures is an irreversible process.

Heat transfer between two objects with a infinitesimal temperature difference is reversible.

Reversible Isothermal Process

If heat flow is caused by a difference in temperature in a process such as the quasi-static isothermal expansion / contraction, the process is reversible i.e. the process can go both directions in the pv graph.

Reversible Adiabatic Process

A process that doesn't convert heat into mechanical work or vice versa such as the quasi-static adiabatic expansion / compression can also be reversible

The Carnot Engine

An ideal heat engine with maximum possible efficiency that is consistent with the 2nd law of thermodynamics.

$$e = 1 - \frac{T_C}{T_H} = \frac{T_H - T_C}{T_H}$$
$$\frac{T_C}{T_H} = \frac{|Q_C|}{Q_H}$$
$$\frac{Q_H}{T_H} = \frac{|Q_C|}{T_C}$$

All Carnot engines operating between the two same temperatures have the same efficiency, regardless of the working substance

No engine is more efficient than the Carnot engine operating between two same reservoirs. Any engine that is more efficient will violate the 2nd law of thermodynamics

Each step in the Carnot engine is reversible.

Thermodynamic Temperatures

Since the Carnot cycle is universal and independent of the working fluid, the thermodynamic temperature can be defined in terms of the Carnot cycle.

$$\frac{T_C}{T_H} = \frac{|Q_C|}{Q_H}$$
$$\frac{Q_H}{T_H} = \frac{|Q_C|}{T_C}$$

One reservoir is kept at the triple point of water, and the other at the arbitrary temperature we want to measure.