

1 Linear Momentum and Collisions

Definitions

Linear Momentum: Particle of mass m moving at velocity v .

$$\vec{p} = m\vec{v}, \text{ in kgms}^{-1}$$

Newton 2nd Law in terms of p: The time rate of change of linear momentum is equal to the resultant force acting on the particle.

$$\vec{F} = \frac{d\vec{p}}{dt}$$

Impulse: Δ in momentum.

$$\vec{J} = \Delta\vec{p} = p_f - p_i = \int_{p_i}^{p_f} \vec{F} dt \text{ in kgms}^{-1}$$

Impulse Approximation: Impulse can be approximated with the area under $F - t$ graph.

$$\vec{J} = \Delta\vec{p} = \vec{F}_{av} \Delta t$$

Momentum and KE: Relationship between \vec{p} and E_k

$$K = \frac{p^2}{2m} \text{ or } p = \sqrt{2mK}$$

Conservation of Linear Momentum

If there are no external forces acting on a system, the total momentum of an isolated system at all times equals its initial momentum.
 $\sum p_f = \sum p_i$ and $\sum p = \text{constant}$

Collisions

The total momentum of an isolated system just before a collision equals the total momentum of the system just after the collision.

Coefficient of restitution

$$e = \frac{\text{Relative speed of separation}}{\text{Relative speed of approach}} = \frac{v_{2f} - v_{1f}}{v_{1i} - v_{2i}}$$

1. **Perfectly Inelastic** $e = 0$
 - The two bodies stick together after collision.
 - KE is not conserved.
 - Momentum is conserved.
2. **Inelastic** $e < 1$
 - KE is not conserved.
 - Momentum is conserved.
3. **Elastic** $e = 1$
 - KE conserved $K_f = K_i$
 - Momentum conserved
4. **Explosive** $e > 1$
 - KE is not conserved.
 - Momentum is conserved.

Elastic Collisions

Momentum Conservation $m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f}$

KE Conservation $\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$

Coefficient of Restitution Since $v_{1i} - v_{2i} = v_{2f} - v_{1f}$, thus $e = 1$

Manipulating the above equations,

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) v_{1i} + \left(\frac{2m_2}{m_1 + m_2} \right) v_{2i}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2} \right) v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2} \right) v_{2i}$$

Case 1: $m_1 = m_2$

$v_{1f} = v_{2i}$ and $v_{2f} = v_{1i}$ i.e. the particles exchange velocities

Case 2: m_2 initially at rest ($v_{2i} = 0$), $m_1 \gg m_2$

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) v_{1i} \text{ and } v_{2f} = \left(\frac{2m_1}{m_1 + m_2} \right) v_{1i}$$
$$v_{1f} \approx v_{1i} \text{ and } v_{2f} \approx 2v_{1i}$$

Case 3: m_2 initially at rest ($v_{2i} = 0$), $m_2 \gg m_1$

$$v_{1f} \approx -v_{1i} \text{ and } v_{2f} \approx 0$$

Note The maximum transfer of KE occurs when $m_1 = m_2$.

$$K_{2f} = \frac{1}{2} \left(\frac{2m_1}{m_1 + m_2} \right)^2 v_{1i}^2 = \frac{4m_2m_1}{(m_1 + m_2)^2} K_{1i}$$

Note For elastic collisions in 2D or 3D, we split the momentum conservation up into x , y and z axis as required.

Perfectly Inelastic Collisions

Momentum Conservation $m_1v_{1i} + m_2v_{2i} = (m_1 + m_2)v_f$

$$v_f = \frac{m_1v_{1i} + m_2v_{2i}}{m_1 + m_2}$$

Glancing Collision

2D Collision. Note the initial and final momentum in the x and y directions.

Momentum Conservation

x -axis: $m_1v_{1i} = m_1v_{1f} \cos \theta + m_2v_{2f} \cos \phi$

y -axis: $0 = m_1v_{1f} \sin \theta - m_2v_{2f} \sin \phi$

Conservation of KE If elastic glancing collision.

$$\frac{1}{2}m_1v_{1i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$$

If $m_1 = m_2$, then $v_{1i} = v_{1f} + v_{2f}$ and hence $v_{1i}^2 = v_{1f}^2 + v_{2f}^2$.

By Pythagoras Thm, $\phi + \theta = 90^\circ$.

Problem Solving Techniques

1. Sketch and label all velocity vectors
2. Momentum Conservation Equation $p_f = p_i$
3. Conservation of Energy $E_f = E_i$
4. Conservation of KE (only for elastic collisions) $K_f = K_i$

2 Center of Mass

Center of mass is positional information. If a force F acts on an object with a total mass of M , the CM accelerates at $a = \frac{F}{m}$

Two Distinct Points in 1D

$$x_{CM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

CM of a 2D Object

We can split up the object into different sections to calculate its CM.

$$\left. \begin{aligned} r_{CM1} &= 10\hat{i} + 25\hat{j} \\ r_{CM2} &= 5\hat{i} + 10\hat{j} \\ r_{CM3} &= 20\hat{i} + 5\hat{j} \end{aligned} \right\} \begin{aligned} x_{CM} &= \frac{m(10) + m(5) + m(20)}{3m} \\ y_{CM} &= \frac{m(25) + m(10) + m(5)}{3m} \end{aligned}$$

CM of a System of Particles

$$x_{CM} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \dots + m_nx_n}{m_1 + m_2 + m_3 + \dots + m_n} = \frac{\sum_i m_i x_i}{m_i}$$
$$y_{CM} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + \dots + m_ny_n}{m_1 + m_2 + m_3 + \dots + m_n} = \frac{\sum_i m_i y_i}{m_i}$$
$$z_{CM} = \frac{m_1z_1 + m_2z_2 + m_3z_3 + \dots + m_nz_n}{m_1 + m_2 + m_3 + \dots + m_n} = \frac{\sum_i m_i z_i}{m_i}$$
$$\vec{r}_{CM} = x_{CM}\hat{i} + y_{CM}\hat{j} + z_{CM}\hat{k}$$
$$\vec{r}_{CM} = \frac{\sum_i m_i \vec{r}_i}{M}$$

CM of Extended Object

For extended objects, there are no discrete points, so we have to take the integration.

$$x_{CM} = \frac{1}{M} \int x dm$$

$$y_{CM} = \frac{1}{M} \int y dm$$

$$z_{CM} = \frac{1}{M} \int z dm$$

$$\vec{r}_{CM} = \frac{1}{M} \int \vec{r} dm$$

Calculating the CM

Calculate the x_{CM} of a extended object

1. Using $x_{CM} = \frac{1}{M} \int x dm$, convert dm into mass per unit area $dm = m \times A$ (for 2D) or mass per unit volume $dm = m \times V$ (for 3D).
2. Then we express dm in terms of m and A .
3. Solve the integral.

Motion of a System of Particles

Velocity of CM

$$\vec{v}_{CM} = \frac{d\vec{r}_{CM}}{dt} = \frac{\sum_i m_i v_i}{M}$$

Momentum of System The total momentum of a system is equal to that of a single particle of mass M moving with a velocity v_{CM} .

$$M\vec{v}_{CM} = \sum_i m_i v_i = \sum \vec{p}_i = \sum \vec{p}_{total}$$

Acceleration of CM

$$\vec{a}_{CM} = \frac{d\vec{v}_{CM}}{dt} = \frac{1}{M} \sum_i m_i \frac{d\vec{v}_i}{dt} = \frac{1}{M} \sum_i m_i \vec{a}_i$$

External Force on System Internal forces in F_i cancel out. Left with F_{ext}

$$\sum_i \vec{F}_i = \sum_i \vec{F}_{ext} = M\vec{a}_{CM} = \sum_i m_i \vec{a}_i$$

Rocket Propulsion

$$mv = (m + dv)(v + dv) + (-dm)(v - v_{ex})$$

$$m \frac{dv}{dt} = -v_{ex} \frac{dm}{dt}$$

$$a = \frac{-v_{ex}}{m} \frac{dm}{dt}$$

$$v - v_0 = -v_{ex} \ln \left(\frac{m}{m_0} \right)$$

Thrust Force exerted on the rocket by the ejected exhaust gases

$$\text{Thrust} = m \frac{dv}{dt} = \left| v_{ex} \frac{dm}{dt} \right|$$

3 Rotation Motion

Definitions

Angle of Rotation: θ is the angle, r is the radius, and s is the arc length.

$$s = r\theta, \text{ in radians}$$

Average Angular Velocity

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t}$$

Instantaneous Angular Velocity

$$\omega = \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Average Angular Acceleration

$$\alpha = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

Instantaneous Angular Acceleration

$$\alpha = \lim_{\Delta t \rightarrow 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

Linear Velocity

$$v = r\omega$$

Tangential Acceleration

$$a_t = r\alpha$$

Radial Acceleration

$$a_{rad} = \frac{v^2}{r} = r\omega^2$$

Constant angular acceleration, α

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

The direction of angular velocity, ω is derived using the right-hand rule.

Anticlockwise rotation: Upwards

Clockwise rotation: Downwards

The direction of angular acceleration, α is in the same direction as ω if the angular velocity is increasing, but it is in the opposite direction as ω if angular velocity is decreasing.

Rotational Energy

Kinetic energy of the i^{th} particle $K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i r_i^2 \omega^2$

Total Rotational KE

$$K_R = \sum_i K_i = \frac{1}{2} \sum_i m_i r_i^2 \omega^2$$

where ω , the angular velocity is the same for every particle.

4 Moment of Inertia

A measure of the resistance of an object to changes in its rotational motion.

Depends on the axis of rotational and the physical arrangement of the mass.

$$I = \sum_i m_i r_i^2 \text{ in kgm}^2$$

$$K_R = \frac{1}{2} I \omega^2$$

For extended objects,

$$I = \lim_{\Delta m_i \rightarrow 0} \sum_i \Delta m_i r_i^2 = \int r^2 dm = \int \rho r^2 dV$$

where ρ is the density of the object.

Discrete Points

$$I = \sum_i m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + \dots$$

Look at the axis and the distance of the point from the axis to determine what is r_i .

Extended Objects

$$I = \int_a^b r^2 dm$$

1. Express a small mass, dm in terms of:

- 2-D Objects

Find a small area perpendicular to the distance from the axis.

$$dm = \lambda dx = \frac{M}{L} dx$$

Where dm is a small mass, dx is a small length, λ is the mass per unit length.

The value of b = length on the right and a = length on the left

- 3-D Objects (that you can find a small volume)

Find a small volume perpendicular to the distance from the axis.

Then, express the small volume in terms of a small mass.

$$dm = \rho L(2\pi r) dr$$

Where dm is a small mass, $\rho = \frac{m}{v}$, density, $2\pi r L$ is the area, and dr is the small length.

Note that the area formula here only applies to the cylinder

The value of b = larger radius and a = smaller

- 3-D Objects (cannot find a small volume perpendicular)

Use the formula $I = \int dI$ instead, and find dI , see the sphere.

Parallel Axis Theorem

The moment of inertia about any axis parallel to and at distance D away from the axis that passes through the center of the mass is:

$$I = I_{CM} + MD^2$$

Moment of inertia is equals to the moment of inertia through the CM of the object + mass of the object \times distance from the CM.

Common Values of I

1. **Uniform Rigid Rod:** $I = \frac{1}{3}M(L^2 - 3Lh + 3h^2)$

Axis through one end of the rod, $h = 0$: $I = \frac{1}{3}ML^2$

Axis through the CM of the rod, $h = \frac{1}{2}$: $I = \frac{1}{12}ML^2$

I is the highest when $h = 0$ as the mass of the rod is far away from the axis of rotation

2. **Hollow/Solid Cylinder:** $I = \frac{1}{2}M(R_2^2 + R_1^2)$

Hoop or thin cylindrical shell: $R_1 = R_2 = R$ and hence $I = MR^2$

Disk or solid cylinder: $R_1 = 0$ and hence $I = \frac{1}{2}MR^2$

I is the highest for hoops as most of the mass is away from the axis of rotation!

3. **Rectangular Plate:** $I = \frac{1}{12}M(a^2 + b^2)$ where a is the breadth and b is the length.

4. **Solid Sphere:** $I = \frac{2}{5}MR^2$

Thin Spherical Shell (hollow inside): $I = \frac{2}{3}MR^2$

5 Rotational Dynamics

Torque

Torque, τ (moment) of a force F about a point O is defined as:

$$\tau = rF \sin \phi$$

Take the product of the force and the perpendicular distance from the pivot to the line of action of the force.

SI Unit: Newton Metre (Nm)

Clockwise: Negative torque

Anticlockwise: Positive torque

Use the right-hand rule to find the direction of torque.

Anticlockwise: Out of paper

Clockwise: Into the paper

Torque and Angular Acceleration

The torque acting on a particle is proportional to its angular acceleration.

For discrete objects,

$$\begin{aligned}\tau &= rF_t \\ &= r(ma_t) \\ &= r(mr\alpha) && \text{since } a_t = r\alpha \\ &= (mr^2)\alpha \\ &= I\alpha && \text{since } I = mr^2\end{aligned}$$

For extended objects,

$$\begin{aligned}\tau &= \int (r^2 dm) \alpha \\ &= \alpha \int r^2 dm \\ &= I\alpha\end{aligned}$$

Work, Power and Energy

Work Only the tangential component of F does work.

$$dW = \tau d\theta$$

Power

$$P = \frac{dW}{dt} = \tau\omega$$

Energy The net work done by external forces in rotating a rigid object about a fixed axis equal the change in the object's rotational energy.

$$W = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2$$

Solving Problems

Use conservation of energy to solve $K_f + U_f = K_i + U_i$

K is the linear kinetic energy $\frac{1}{2}mv^2$ and the rotational energy $\frac{1}{2}I\omega^2$

U is the potential energy e.g. GPE.

6 Angular Momentum

Background: Cross Products

$$\vec{C} = \vec{A} \times \vec{B}$$

The direction of \vec{C} is found using the right-hand rule. Line your hand against the first vector, and then rotate using the inside of your palms.

$$|\vec{C}| = |\vec{A}||\vec{B}| \sin \theta$$

Vector product is not commutative, but is distributive.

$$\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$$

Angular Momentum

The instantaneous angular momentum L of a particle relative to the origin O is defined by the cross product of the instantaneous vector position r and its instantaneous linear momentum p .

$$\vec{L} = \vec{r} \times \vec{p}$$

SI Unit: $\text{kgm}^2\text{s}^{-1}$

Torque and Angular Momentum

Torque Redefined: $\vec{\tau} = \vec{r} \times \vec{F}$

The torque acting on a particle is equal to the time rate of change of the particle's angular momentum.

$$\tau = \frac{d\vec{L}}{dt}$$

For a system of particles, the time rate of change of the total angular momentum of the system about some origin in an inertial frame equals the net external torque acting on the system about that origin.

$$\vec{L}_{tot} = \vec{L}_1 + \vec{L}_2 + \vec{L}_3 + \dots + \vec{L}_n = \sum_{i=1}^n \vec{L}_i$$

$$\sum \vec{\tau}_{ext} = \sum_{i=1}^n \frac{d\vec{L}_i}{dt} = \frac{d}{dt} \sum_{i=1}^n \vec{L}_i = \frac{d\vec{L}_{tot}}{dt}$$

Angular Momentum of an Object

For a single particle $L_i = m_i r_i^2 \omega$

For the whole object: $L_z = \sum m_i r_i^2 \omega$

$$L_z = I\omega$$

The net external torque acting on a rigid object rotating about a fixed symmetric axis equals the product of the moment of inertia and the angular acceleration about that axis.

$$\sum T_{ext} = \frac{dL_z}{dt} = I\alpha$$

Conservation of Angular Momentum

The total angular momentum of a system is constant if the resultant external torque acting on the system is zero.

$$\sum L_i = \text{constant and } L_i = L_f$$

Solving Problems

1. Conservation of linear momentum
2. Conservation of angular momentum: Includes linear motion, if it is not parallel to the axis. If clockwise w.r.t axis, negative.
3. Conservation of energy: KE, rotational energy

7 Temperature

Definitions

Thermal Contact: Energy can be exchanged between two bodies, but may not be in physical contact.

Thermal Equilibrium: No net exchange of energy by heat or EM radiation between the two objects. Two objects at the same temperature are in thermal equilibrium.

Temperature: The property that determines if objects are in thermal equilibrium and the direction of energy exchange.

Zeroth Law of Thermodynamics

If A and B are separately in thermal equilibrium with C, then A and B are in thermal contact with each other.

Temperature Scales

1. **Kelvin Scale** Used for scientific work. 0K is the lowest temperature possible.

$$T_K = 273.16 \left(\frac{P}{P_{\text{triple}}} \right)$$

P_{triple} is the triple point of water, where water in its three states coexists in equilibrium.

2. **Celsius Scale**

$$T_C = T_K - 273.15$$

3. **Fahrenheit Scale**

$$T_F = \frac{9}{5} T_C + 32$$

$$1^\circ\text{C} = 1\text{K} = \frac{9}{5}^\circ\text{F}$$

8 Heat and Ideal Gas

Mechanical Equivalent of Heat

Joule Experiment The loss in mechanical energy is proportional to ΔT

$$4.186 m_w \Delta T = 2mgh$$

m_w is the mass of water in grams, ΔT is the change of temperature, and mgh is the change in GPE of the masses.

1. **Calorie (cal)**
Amount of heat required to raise the temperature of 1g of water from 14.5°C to 15.5°C
2. **Joule (J)**
1/4.186 calorie
3. **Food value Calorie (Cal)**
Cal = 1 kcal = 1000 calorie
4. **British Thermal Unit (BTU)**
Amount of heat required to raise the temperature of 1 pound of water from 63 °F to 64 °F.
 $1\text{BTU} = 453.6 \times \frac{5}{9} = 252\text{cal} = 1055\text{J}$

Heat and Internal Energy

Internal Energy

All energy of a system (atoms and molecules) viewed from a reference frame at rest with respect to the CM of a system.

Includes the kinetic energy of random translational, rotational and vibrational motion of the molecules.

Includes the potential energy within molecules (intramolecular) and between molecules (intermolecular).

Excludes the kinetic energy due to bulk movement (e.g. external forces)

Heat Transfer of energy across the boundary of a system due to temperature difference between system and surroundings.

Changing the T of a substance

1. **Heat** Energy transfer between the substance at different T
2. **Work** Energy transfer that doesn't require a T difference e.g. mechanical energy (joule experiment), electrical energy

Heat Capacity

Specific Heat c : Amount of heat needed to raise the temperature of unit mass of substance by 1 °C

$$Q = mc\Delta T$$

where Q is the heat energy, m is the mass of the substance, c is the specific heat, and ΔT is the change in temperature.

If the specific heat varies with temperature,

$$Q = m \int_{T_1}^{T_2} c \, dT$$

Molar Specific Heat C : Amount of heat needed to raise the temperature of one mole of substance by 1 °C.

$$Q = nC\Delta T$$

where Q is the heat energy, n is the number of moles, C is the molar specific heat, and ΔT is the change in temperature.

Molar Specific Heat at Constant Volume C_v : $Q = nC_v\Delta T$.

Molar Specific Heat at Constant Pressure C_p : $Q = nC_p\Delta T$

Note $m = n \times M$, where m is the mass of the substance, n is the number of moles, and M is the Molar mass.

Note $C = Mc$

Latent Heat

Latent heat refers to the amount of heat needed to change the phase of the substance per unit mass.

$$Q = mL$$

where Q is the heat energy, m is the mass of substance, and L is the latent heat.

L_v : Latent heat of vaporisation - from liquid to gas or vice versa

L_f : Latent heat of fusion - from solid to liquid or vice versa

Note Temperature is constant during state change

Ideal Gas Laws

1. **Charles & Gay-Lussac Law** $V \propto T$ for fixed P

2. **Boyle's Law** $P \propto \frac{1}{V}$ for fixed T

3. **Pressure Law** $P \propto T$ for fixed V

4. **Ideal Gas Law** $PV = nRT$

Ideal Gas Equation

$$PV = nRT$$

where n is the number of moles, R is the universal gas constant (same for all gases), and T is the temperature.

$R = 8.315 \text{ J/mol K}$

Avogadro's Hypothesis: Equal volume of gas at the same pressure and temperature contains equal numbers of molecules.

$$PV = NkT$$

where k is the Boltzmann's constant $= 1.381 \times 10^{-23} \text{ J/K}$ and N is the number of molecules.

$$N = n \cdot N_A$$

where N is the number of molecules, N_A is Avogadro's Constant $= 6.022 \times 10^{23} \text{ molecules/mole}$, and n is the number of moles.

$$k = \frac{R}{N_A}$$

where k is the Boltzmann's constant, and R is the universal gas constant.

9 Kinetic Theory of Gas

Assumptions

- Large number of molecules; and the separation is much larger than the dimensions of the molecules. The molecules occupy negligible volume, and are treated as point-like
- Obey Newton's laws of motion, but move randomly, different molecules with different speeds and directions
- Elastic collisions with each other and the wall i.e. KE and momentum are conserved
- Forces between molecules are negligible except during collision (only short range forces)
- All molecules are identical - consider only pure substances.

Molecular Model of Gases

The force exerted by walls on the molecules is

$$F = \frac{N_{\text{collide}}(dp_x)}{dt} = \frac{NAmv_x^2}{V}$$

Hence, by Newton's Third Law, the force exerted by the molecules on the walls is equal to the force exerted by the walls on the molecules, and hence pressure is:

$$p = \frac{F}{A} = \frac{Nmv_x^2}{V}$$

However, since the v_x is not the same for all molecules, we use the mean squared speed. Since the $v^2 = v_x^2 + v_y^2 + v_z^2$, for random motion, we conclude

$$\bar{v}_x^2 = \bar{v}_y^2 = \bar{v}_z^2 = \frac{1}{3}\bar{v}^2$$

Hence,

$$\begin{aligned} p &= \frac{Nmv_x^2}{V} \\ &= \frac{1}{3} \frac{N}{V} m\bar{v}^2 \\ &= \frac{1}{3} \rho \bar{v}^2 \end{aligned} \quad \text{since } \rho = \frac{Nm}{V}$$

Molecular Interpretation of T

$$\begin{aligned} T &= \frac{2}{3k} \left(\frac{1}{2} m\bar{v}^2 \right) \\ \frac{3}{2} kT &= \frac{1}{2} m\bar{v}^2 \\ \frac{1}{2} m\bar{v}_x^2 &= \frac{1}{2} m\bar{v}_y^2 = \frac{1}{2} m\bar{v}_z^2 = \frac{1}{2} kT \end{aligned}$$

Root-mean-square Speed

The total translational energy of all molecules is given by

$$K_{tr} = N \left(\frac{1}{2} m\bar{v}^2 \right) = \frac{3}{2} NkT = \frac{3}{2} nRT$$

$$v_{rms} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3RT}{M}}$$

where m is the mass of the molecule, and M is the molar mass. At the same temperature, the lighter the molecule, the faster it will move.

Heat Capacity

Inter-molecular potential energy = 0 for ideal gas. Thus,

$$E_{int} = K_{tr} = N \left(\frac{1}{2} m\bar{v}^2 \right) = \frac{3}{2} NkT = \frac{3}{2} nRT$$

Equating total rotational energy to heat input at constant volume,

$$dK_{tr} = \frac{3}{2} nRdT$$

$$dQ = nC_v dT$$

$$nC_v dT = \frac{3}{2} nRdT$$

$$C_v = \frac{3}{2} R$$

However, this value of C_v only holds for monoatomic.

Equipartition of Energy

Each degree of freedom has associated with it, on average, an energy of $\frac{1}{2}kT$ per molecule (or $\frac{1}{2}RT$ per mole).

Degree of Freedom Number of independent ways molecules can possess energy.

The following ways hold at room temperature. At higher temperatures, there will also be vibrational degrees of freedom.

1. **Monoatomic** e.g. He
3 degrees of freedom: 3 translational motion
 $C_v = \frac{3}{2}R$ per mole
2. **Diatomic** e.g. H_2
5 degrees of freedom at room temperature: 3 translational + 2 rotational
 $C_v = \frac{5}{2}R$ per mole

At high temperatures, it may have vibrational as well, so at high temperatures
 $C_v = \frac{7}{2}R$ per mole
3. **Polyatomic** More than 3 molecules, e.g. CO_2
6 degrees of freedom: 3 translational + 3 rotational
 $C_v = \frac{6}{2}R$ per mole

Molar Specific Heat of Solids

For solids, they cannot translate or rotate due to their structures, so they can only vibrate in 3 directions.

For each direction of vibration, there are 2 degrees of freedom.

$$E = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2$$

where $\frac{1}{2}mv_x^2$ is the vibrational kinetic energy of the atoms and the $\frac{1}{2}kx^2$ is the potential energy of the vibration between the atoms.

Hence,

$$C_v = 6 \times \frac{1}{2}R = 3R$$

10 First Law of Thermodynamics

Definitions

Quasi-static Process Processes that are carried out slowly enough so that the system passes through a sequence of thermal equilibrium states. (A non quasi-static process will be the adiabatic free expansion)

State Variables Temperature, volume, pressure and internal energy

Transfer Variables Heat and Work

Work Done by Gas

Work done by the gas is positive when the gas expands.

Work done by the gas is negative when the gas contracts (or when work is done on the gas).

$$W = \int_{V_1}^{V_2} p dV$$

Work done by the gas is path dependent. You see the area under the pV graph to determine the work done.

Kinetic energy of molecules decrease when positive work is done by the gas. And KE increases when negative work is done by the gas.

First Law of Thermodynamics

$$\Delta U = U_2 - U_1 = Q - W$$

where Q is the heat entering the system and W is the work done by the system (gas).

Q , heat transfer and W are path dependent.

$Q - W$ is not path dependent, it only depends on the initial and final state.

Note For an ideal gas, the ΔU depends on ΔT

Thermodynamic Processes

1. Isochoric (Isovolumetric) Process

The volume of the system remains unchanged. Heat flows into the system changes its internal energy (heat in, increase U , heat out, decrease U)

$$\begin{aligned} W &= 0 & \text{since } \Delta V &= 0 \\ \Delta U &= Q - W \\ &= nC_v\Delta T - 0 \\ &= nC_v(T_2 - T_1) \end{aligned}$$

2. Isobaric Process

The pressure of the system remains constant.

$$\begin{aligned} W &= \int_{V_a}^{V_b} p dV = p(V_b - V_a) \\ Q &= nC_p\Delta T \\ \Delta U &= nC_p\Delta T - p(V_b - V_a) \end{aligned}$$

3. Isothermal Process

The temperature of the system remains constant.

$$\begin{aligned} \Delta U &= 0 & \text{since } U \text{ depends on } T \text{ for ideal gas} \\ \Delta U &= Q - W \\ Q &= W \\ Q &= W = nRT \ln \left(\frac{V_f}{V_i} \right) \end{aligned}$$

4. Adiabatic Process

There is no heat transfer between the system and the surroundings. The internal energy decreases as work is done by the system, which usually means there is a fall in temperature.

$$\begin{aligned} Q &= 0 \\ \Delta U &= Q - W \\ &= -W \end{aligned}$$

5. Adiabatic Process Free Expansion Process

There is no heat entering the system, and there is no work done or by the system.

$$\Delta U = Q = W = 0$$

This cannot be represented by a line, since it is not a quasi-static process. It doesn't have points of thermal equilibrium during the process.

6. Cyclic Process

For a cyclic process, since the initial state and the final states are the same, there is no change in internal energy.

However, there is work done by the gas (see the area under the graph)

$$\begin{aligned} Q - W &= 0 \\ Q &= W \end{aligned}$$

If the cycle is clockwise, the work done by gas is positive. If the cycle is anticlockwise, the work done by gas is negative.

Heat Capacity of Ideal Gas

For an ideal gas, the ΔU only depends on ΔT . Hence, for all processes,

$$\Delta U = nC_v \Delta T$$

$$C_p - C_v = R$$

Ratio of Heat Capacities, γ

$$\gamma = \frac{C_p}{C_v} > 1 \text{ and } \gamma - 1 = \frac{R}{C_v}$$

1. Monoatomic

$$C_V = \frac{3}{2}R$$

$$C_P = C_V + R = \frac{5}{2}R$$

$$\gamma = \frac{C_P}{C_V} = \frac{5}{3} = 1.67$$

2. Diatomic

$$C_V = \frac{5}{2}R$$

$$C_P = C_V + R = \frac{7}{2}R$$

$$\gamma = \frac{C_P}{C_V} = \frac{7}{5} = 1.40$$

3. Polyatomic

$$C_V = \frac{6}{2}R$$

$$C_P = C_V + R = \frac{8}{2}R$$

$$\gamma = \frac{C_P}{C_V} = \frac{8}{6} = 1.33$$

Adiabatic Process for Ideal Gases

$$T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1}$$

$$P_1 V_1^\gamma = P_2 V_2^\gamma$$

This tells us that the gradient of the $P - V$ graph for an adiabatic process is steeper than for isothermal processes since the gradient is $1/V^\gamma$ where $\gamma > 1$, as compared to $1/V$.

Since $W = \Delta U = -nC_v \Delta T$, $W = nC_v \Delta T$, and hence

$$W = \frac{1}{\gamma - 1} (p_a v_a - p_b v_b)$$

11 Heat Engines and Heat Pumps

Definitions

Heat Engine

A device that converts heat into mechanical energy. Carries some working substance through a cyclic process.

Heat engines do work. The cycle is clockwise in the $P - V$ graph.

Heat Pump E.g. Refrigerator

A device that converts mechanical energy into heat.

Heat pump requires work to be done. The cycle is anti-clockwise in the $P - V$ graph.

Efficiency of Heat Engines

For a cyclic process, $\Delta U = 0$, so $Q = -W$.

$$W = |Q_H| - |Q_C|$$

where Q_H is the heat energy of the hot reservoir and Q_C is the heat energy of the cooler reservoir, and W is the effective work done.

Thermal Efficiency

$$e = \frac{W}{Q_H} = 1 - \left| \frac{Q_C}{Q_H} \right|$$

It is impossible for thermal efficiency to be 100% i.e. heat cannot be converted entirely to mechanical work.

Kevin Planck Statement "It is impossible to construct a heat engine that, operating in a cycle, produces no effect other than the input of energy by heat from a reservoir and the performance of an equal amount of work."

Coefficient of Performance: Heat Pumps

1. Cooling Mode e.g. Fridge or AC

$$K_c = \frac{|Q_C|}{W} = \frac{|Q_C|}{|Q_H| - |Q_C|}$$

2. Heating Mode e.g. Heat Pump

$$K_h = \frac{|Q_H|}{W} = \frac{|Q_H|}{|Q_H| - |Q_C|}$$

Clausius Statement "It is impossible to construct a cyclical machine whose sole effect is the continuous transfer of heat energy from one object to another object at a higher temperature without the input of energy by work."

Reversible Processes

Reversibility of Heat Transfer

Heat transfer between two objects of different finite temperatures is irreversible.

However, heat transfer between two objects with an infinitesimal temperature difference is reversible.

Reversibility of Isothermal Processes

If heat flow is caused by a difference in temperature in a process such as the quasi-static isothermal expansion/contraction, the process is reversible i.e. the process can go both directions in the $P - V$ graph.

Reversibility of Adiabatic Processes

A process that doesn't convert heat into mechanical work or vice versa such as the quasi-static adiabatic expansion/compression can also be reversible. This is not referring to the free expansion adiabatic process.

The Carnot Engine

An idealised heat engine with maximum possible efficiency that is consistent with the 2nd Law of Thermodynamics.

Uses only reversible processes (adiabatic and isothermal).

The Carnot Cycle (Heat engine: Clockwise)

1. Isothermal Expansion
2. Adiabatic Expansion
3. Isothermal Compression
4. Adiabatic Compression

Efficiency of Carnot Engine

$$e = 1 - \frac{T_C}{T_H} = \frac{T_H - T_C}{T_H}$$

All Carnot engines operating between the same two temperatures have the same efficiency, regardless of the working substance.

No engine is more efficient than the Carnot engine. Any engine that is more efficient violates the 2nd Law.

Every step in the Carnot cycle is reversible. Change the direction and it works as a heat pump.

Thermodynamic Temperature

Since the Carnot cycle is independent of the working substance, the thermodynamic temperature can be defined in terms of the Carnot cycle.

Thermodynamic temperature is expressed as the ratio of heat flow. One reservoir is kept at triple point of water T_H or T_C , and the other is kept at the temperature we want to measure.

$$\frac{T_C}{T_H} = \frac{|Q_C|}{Q_H}$$

$$\frac{Q_H}{T_H} = \frac{|Q_C|}{T_C}$$

4-Stroke Gasoline Engine

Not reversible, only goes clockwise in the Otto Cycle.

The Otto Cycle

- (1) **Fuel Intake** Isobaric Process
Volume increases from V to rV at constant pressure as the intake valve is opened, and the piston moves down.
- (2) **Adiabatic Compression**
Volume decreases from rV to V , but pressure increases as the intake valve is closed, and the piston moves up.
- (3) **Isochoric Heating**
Pressure increases while volume remains constant at V , due to the spark plug.
- (4) **Adiabatic Expansion**
Piston is pushed down, and volume increases from V to rV , and pressure decreases.
- (5) **Isochoric Cooling**
Pressure decreases while volume remains constant at rV as exhaust valve is opened.
- (6) **Residual Gas Exhaustation** Isobaric Process
Piston moves up while the exhaust valve is open and the residual gas is removed at constant temperature. Volume drops from rV to V , at constant pressure.

4-Stroke: Piston moves in (1), (2), (4) and (6).

r : Compression ratio, larger more efficient.

Efficiency of Otto Cycle

$$\begin{aligned} e &= \frac{W}{Q_H} \\ &= 1 - \frac{|Q_C|}{Q_H} \\ &= 1 - \frac{T_d - T_a}{T_c - T_b} \\ &= 1 - \frac{1}{r^{\gamma-1}} \end{aligned}$$

Laws of Thermodynamics

Law 0. Temperature and Thermal Equilibrium

Law 1. Quantity of Energy $\Delta U = Q - W$

Law 2. Quality of Energy ΔS

Law 3. 0K is unattainable.

12 Entropy

Entropy is a measure of the degree of disorder of a closed thermodynamic system.

The larger the entropy, the greater the disorder.

In a reversible process, the entropy dS is given by an infinitesimal amount of heat transfer dQ at absolute temperature T

$$dS = \frac{dQ}{T}$$

Change of Entropy

The change of entropy, ΔS is given by

$$\Delta S = \int_a^b dS = \int_a^b \frac{dQ}{T}$$

Change of Entropy for Reversible Processes

The change of entropy in a Carnot cycle is $\Delta S = 0$.

Since any reversible cycle can be approximated to a series of Carnot cycles, for any reversible process, the change of entropy is zero. $\Delta S = 0$.

Change of Entropy for Irreversible Processes

For irreversible processes i.e. processes that are not adiabatic (free expansion not counted) and isothermal, we need to apply the integration.

Use 1st Law to find out the expression for dQ .

Entropy and the 2nd Law

When a system undergoes a process starting from one equilibrium state and ending in another, the total entropy of the system plus environment can never decrease.

Entropy can either remain the same or increase.

Natural processes tend to move towards a state of greater disorder, which means the final entropy increases.

This is because ordered arrangements require more information to classify than a disordered arrangement.