Mathew Alex

3. Consider the wave equation with a "mass" term $u_{tt} = c^2\Delta u - m^2 u = 0$ in $\mathbb{R}^n \times \mathbb{R}$, where c and m are positive constants.

   (i) In dimension $n = 1$, show that the light cones $ct = \pm x$ are characteristic lines for the equation.

The highest order derivative is of second order. We can write the equation as,

$$a_{(2)}\partial^2 u - m^2 u = 0,$$

where,

$$a_{(2)} = \begin{bmatrix} 1 & 0 \\ 0 & -c^2 \end{bmatrix}.$$

Consider an arbitrary hyper-plane through the origin given by $\alpha t + \beta x = 0$, where $\hat{n} = (\alpha, \beta)$ is its normal vector and $\alpha, \beta \in \mathbb{R}$ are constants.

$$\begin{aligned} a_{(2)}.\hat{n}^2 &= a_{00}\hat{n_0}^2 + a_{11}\hat{n_1}^2 \\ &= \alpha^2 - c^2\beta^2 \qquad (*) \\ &= 0 \text{ for characteristic lines.} \end{aligned}$$

For light cones, $x \pm ct = 0 \implies \alpha = \pm c, \beta = 1$.
$\alpha^2 - c^2\beta^2 = c^2 - c^2 = 0 \implies$ characteristic lines of the PDE are tangent to the light cone through $(0, 0)$.
The non-characteristic lines consists of

$$\alpha t + \beta x = 0 \text{ such that } \alpha \neq \pm c\beta$$

These lines are nowhere tangent to the light cone and $t = 0$ is a straightforward choice for a non-characteristic hyperplane to specifify the initial data.

   (ii) Show that the energy $E = \int_{-\infty}^{+\infty}(u_t^2 + c^2u_x^2 + m^2u^2)dx$ in dimension $n = 1$ is conserved under a suitable decay condition on u as $|x| \to \infty$.

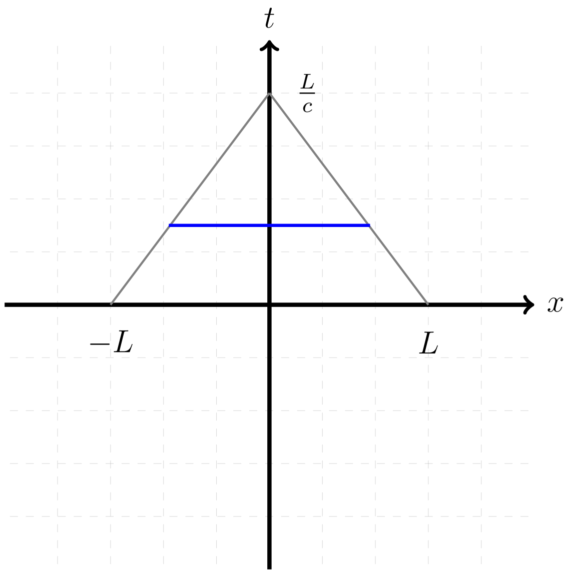
Route 1:

$$\begin{aligned} \frac{dE}{dt} &= 2\int_{\mathbb{R}}(u_t \underbrace{u_{tt}}_{c^2u_{xx} - m^2u} + c^2u_xu_{xt} + m^2uu_t)dx \\ &= 2\int_{\mathbb{R}}(c^2\underbrace{(u_tu_{xx} + u_xu_{xt})}_{(u_tu_x)_x} \underline{-m^2uu_t} + \underline{m^2uu_t})dx \\ &= 2c^2u_tu_x|_{x=-\infty}^{\infty} \text{ (Gauss's theorem)} \\ &= 0 \text{ if } u_tu_x \text{ is even.} \end{aligned}$$

Route 2:

$$\begin{aligned} \frac{dE}{dt} &= 2\int_{\mathbb{R}}(u_t \underbrace{u_{tt}}_{0} + c^2\underbrace{u_xu_{xt}}_{2\frac{\partial}{\partial t}u_x^2} + m^2\underbrace{uu_t}_{2\frac{\partial}{\partial t}u^2})dx \\ &= \frac{d}{dt}\int_{\mathbb{R}}(c^2u_x^2 + m^2u^2)dx \end{aligned}$$

(iii) Suppose the initial data $u|_{t=0} = u_0(x)$ and $u_t|_{t=0} = u_1(x)$ vanish inside an interval $-L \le x \le L, L > 0$. Show that the energy contained inside the forward light cone $e(t) = \int_{-L+ct}^{L-ct} (u_t^2 + c^2u_x^2 + m^2u^2)dx \ge 0$ obeys the inequality $e'(t) \le (2c\,|u_tu_x| - u_t^2 - c^2u_x^2 - m^2u^2)|_{-L+ct}^{-L+ct} \le 0$ for all $0 \le t \le L/c$. Explain why this implies that disturbances (caused by a change in initial data) cannot propagate faster than speed c.



$$
\begin{aligned}
e'(t) &= \frac{d}{dt}\int_{-L+ct}^{L-ct}(u_t^2 + c^2u_x^2 + m^2u^2)dx \\
&= \int_{-L+ct}^{L-ct}\underbrace{\frac{\partial}{\partial t}(u_t^2 + c^2u_x^2 + m^2u^2)}_{2c^2(u_tu_x)_x}dx + \overbrace{\frac{d}{dt}(L-ct)}^{-c}\left(u_t^2 + c^2u_x^2 + m^2u^2\right)|_{x=L-ct} \\
&- \underbrace{\frac{d}{dt}(-L+ct)}_{c}\left(u_t^2 + c^2u_x^2 + m^2u^2\right)|_{x=-L+ct}(\text{Leibniz integral rule}) \\
&= 2c^2\underbrace{(u_tu_x)|_{x=-L+ct}^{L-ct}}_{\text{1D Gauss's theorem}} - c\left\{\left(u_t^2 + c^2u_x^2 + m^2u^2\right)|_{x=-L+ct} + \left(u_t^2 + c^2u_x^2 + m^2u^2\right)|_{x=L-ct}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&- c\left\{\left(u_t^2 + c^2u_x^2 + m^2u^2\right)|_{x=-L+ct} + \left(u_t^2 + c^2u_x^2 + m^2u^2\right)|_{x=L-ct}\right\} \\
&\le -c\left\{\left(u_t^2 + c^2u_x^2 + m^2u^2\right)|_{x=-L+ct} - \left(u_t^2 + c^2u_x^2 + m^2u^2\right)|_{x=L-ct}\right\} \\
&= c\left(u_t^2 + c^2u_x^2 + m^2u^2\right)|_{x=-L+ct}^{L-ct}
\end{aligned}
$$

So

$$
\begin{aligned}
e'(t) &\le c(\overbrace{2cu_tu_x - u_t^2 - c^2u_x^2}^{-(u_t-cu_x)^2\le 0} - m^2u^2)|_{x=-L+ct}^{L-ct} \\
&= -m^2cu^2|_{x=-L+ct}^{L-ct}
\end{aligned}
$$

If we look for seprable solutions of the form $u(t,x) = A(x)B(t)$, B will be a linear function of t since $u_{tt} = 0$ and

$$
c^2u_{xx} - m^2u = 0 \implies A''(x) = \frac{m^2}{c^2}A(x) \implies A(x) = c_1e^{\frac{m}{c}x} + c_2e^{-\frac{m}{c}x}, c_1, c_2 \in \mathbb{R}
$$

In order to avoid blowup as $|x| \to \pm\infty$, $A(x)$ must be a constant (e^x and e^{-x} cannot be joined together to avoid the blowup without producing a cusp at $c = 0$).

$$\implies u^2|_{x=-L+ct}^{L-ct} = 0$$

$$\implies e'(t) \leq 0 \tag{a}$$

$$e(0) = \int_{-L}^{L} (u_1^2 + c^2 u_{0x}^2 + m^2 u_0^2) dx = 0 \tag{b}$$

$$e(t) \geq 0 \text{ since the integrand is } \geq 0. \tag{c}$$

(a), (b), and (c) $\implies e'(t) = 0 \implies e(t) = e(0) = 0 \implies u_t, u_x, u = 0$ inside the interval.

Hence u is zero inside the light cone. Since u_0 and u_1 are not supported inside $[-L, L]$, the signal has to travel faster than speed c to create an influence inside the light cone. Our result forbids it.

(iv) In dimensions $n \geq 1$, find all non-characteristic hyperplanes and show that they do not depend on m. Explain whether any of these hyperplanes are tangent to the light cone.

The highest order derivative is of second order. We can write the equation as,

$$a_{(2)}\partial^2 u - m^2 u = 0,$$

where,

$$a_{(2)} = \text{diag}[1, -c^2 \underbrace{(1, \cdots, 1)}_{n \text{ times}}]$$

Consider an arbitrary hyper-plane through the origin given by $\alpha t + \beta.x = 0$, where $\hat{n} = (\alpha, \beta)$ is its normal vector and $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^n$ are constants.

$$\begin{aligned} a_{(2)}.\hat{n}^2 &= a_{00}\hat{n_0}^2 + \sum_{i=1}^{n} \overbrace{a_{ii}}^{-c^2} \hat{n_i}^2 \\ &= \alpha^2 - c^2|\beta|^2 \\ &= 0 \text{ for characteristic hyperplanes.} \end{aligned} \tag{*}$$

Or $\alpha = \pm c|\beta|$. For light cones, $ct \pm |x| = 0$. Its normal vector,

$$\begin{aligned} n_{lc} &= \nabla_{(t,x)}(ct \pm |x|) \\ &= (c, \pm\hat{x}) \end{aligned}$$

For a tangent plane to the light cone at (t, x) having the normal vector $\hat{n}_{lc}, \alpha = c, \beta = \hat{x}$ or $\hat{\beta} = 1 \implies \alpha = \pm c|\beta|$. Clearly, tangent planes to light cones at the origin are characteristic. Every other plane through the origin with $\alpha \neq \pm c|\beta|$ are non-charecteristic.

(v) Construct a formal power series solution in t. Use Cauchy-Kovalevskaya theory to state sufficient conditions under which the power series converges for $t \geq 0$. Prove that the power series solution is the unique solution for any given analytic initial data u_0, u_1.

For $u_{tt} = c^2\Delta u - m^2 u = 0, (t, x) \in \mathbb{R}^n \times \mathbb{R}$, we are looking for a solution of the form,

$$u(t, x) = u_0(x) + tu_1(x) + t^2 u_2(x) + \cdots = \sum_{k=0}^{\infty} t^k u_k(x).$$

The coefficients are given by

$$\begin{aligned}
u_0(x) &= u|_{t=0} = u_0(x) \\
u_1(x) &= u_t|_{t=0} = u_1(x) \\
u_2(x) &= \frac{1}{2}u_{tt}|_{t=0} = \frac{1}{2}(c^2\Delta - m^2)u|_{t=0} = \frac{1}{2}\left(c^2\Delta u_0 - m^2 u_0\right) \\
u_3(x) &= \frac{1}{3!}u_{ttt}|_{t=0} = \frac{1}{3!}(c^2\Delta - m^2)u_t|_{t=0} = \frac{1}{3!}\left(c^2\Delta u_1 - m^2 u_1\right) \\
&\vdots
\end{aligned}$$

$$u(t,x) = \sum_{k=0}^{\infty}\left[\frac{t^{2k}}{(2k)!}(c^2\Delta - m^2)^k u_0(x) + \frac{t^{2k+1}}{(2k+1)!}(c^2\Delta - m^2)^k u_1(x)\right]$$

The PDE ($u_{tt} = c^2\Delta u - m^2 u$) is in the Cauchy-Kovalevskaya (CK) form

$$\partial_z^k u = \sum_{l=0}^{k-1}\tilde{a}_{l,k-1}(y,z,u,\partial u,\cdots,\partial^{k-1}u)\partial_z^l\partial_y^{k-l}u + \tilde{b}(y,z,u,\partial u,\cdots,\partial^{k-1}u) \qquad (*)$$

with

- $k=2, z=t, y=x$
- $\tilde{b} = -m^2 u$ (analytic), $\quad \tilde{a}_{(0,2)} = -c^2$.
- highest order (second order) time derivative isolated in the left hand side with no time derivatives of order ≥ 1 on the right hand side
- the analytic initial data, u_0 and u_1 , are specified on a non-characteristic hypersurface, namely $t=0$.

The CK theorem guarantees the existence of a unique convergent power series solution of u with in t in some open ball in $\mathbb{R}$ cetered at 0.

To establish uniqueness, assume two solutions u and $\tilde{u}$ with

$$u|_{t=0} = u|_{t=0} = u_0(x) \text{ and}$$

$$\tilde{u}_t|_{t=0} = \tilde{u}_t|_{t=0} = u_1(x).$$

$w = u - \tilde{u}$ is also a solution since the PDE is linear.

$$w|_{t=0} = w_t|_{t=0} = 0$$

From energy conservation, we have

$$E(t) = \int_{\mathbb{R}}(w_t^2 + c^2|\nabla w|^2 + m^2 w^2)dx = E(0) = 0.$$

$$\implies w_t = |\nabla w| = w = 0 \implies u = \tilde{u}$$

Hence our power series solution is the unique solution of the PDE for the given initial data.

(vi) Consider the case $m=0$ and observe that the power series solution does not contain n explicitly. Since the support properties of solutions in $n=2$ versus $n=3$ are very different, how is the power series compatible with Huygen's principle?

When $m = 0$, the power series becomes,

$$u(t,x) = \sum_{k=0}^{\infty} \left[\frac{t^{2k}}{(2k)!} \Delta^k u_0(x) + \frac{t^{2k+1}}{(2k+1)!} \Delta^k u_1(x) \right], \text{ with } c = 1.$$

n is in the Laplacian,

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^{n-1}} \Delta_{S^{n-1}}$$

The time asymetric solution ($u_0 = 0, u_1 = \delta(r)$) is

$$u(t,r) = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \Delta^k \delta(r).$$

$$u(t,r) = \begin{cases} \frac{1}{2\pi\sqrt{t^2-r^2}} \chi(t-r), n = 2 \\ \frac{1}{4\pi r} \delta(t-r), n = 3 \end{cases}$$

$$\begin{aligned} u(t,r) &= \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \Delta^k \delta(r) \\ &= \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} \right)^k \delta(r) \\ &= \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \sum_{l=0}^{k} \frac{k!}{(k-l)!l!} \frac{(n-1)^l}{r^l} \left(\frac{\partial}{\partial r} \right)^{2(k-l)} \left(\frac{\partial}{\partial r} \right)^{l} \delta(r) \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^{2k+1} \frac{k! t^{2k+1}}{(2k+1)!(k-l)!l!} \frac{(n-1)^l}{r^l} \underbrace{\left(\frac{\partial}{\partial r} \right)^{2k-l} \delta(r)}_{\delta^{(2k-l)}(r)} \end{aligned}$$

Case 1, $t = r$ (on the light cone):

Case 2, $t = r + \epsilon, epsilon \in \mathbb{R}$ (inside the light cone):

Case 3, $t = r - \epsilon, epsilon \in \mathbb{R}$ (outside the light cone):