

3. Consider the wave equation with a "mass" term  $u_{tt} = c^2 \Delta u - m^2 u = 0$  in  $\mathbb{R}^n \times \mathbb{R}$ , where  $c$  and  $m$  are positive constants.

(i) In dimension  $n = 1$ , show that the light cones  $ct = \pm x$  are characteristic lines for the equation.

The highest order derivative is of second order. We can write the equation as,

$$a_{(2)} \partial^2 u - m^2 u = 0,$$

where,

$$a_{(2)} = \begin{bmatrix} 1 & 0 \\ 0 & -c^2 \end{bmatrix}.$$

Consider an arbitrary hyper-plane through the origin given by  $\alpha t + \beta x = 0$ , where  $\hat{n} = (\alpha, \beta)$  is its normal vector and  $\alpha, \beta \in \mathbb{R}$  are constants.

$$\begin{aligned} a_{(2)} \cdot \hat{n}^2 &= a_{00} \hat{n}_0^2 + a_{11} \hat{n}_1^2 \\ &= \alpha^2 - c^2 \beta^2 \\ &= 0 \text{ for characteristic lines.} \end{aligned} \quad (*)$$

For light cones,  $x \pm ct = 0 \implies \alpha = \pm c, \beta = 1$ .

$\alpha^2 - c^2 \beta^2 = c^2 - c^2 = 0 \implies$  characteristic lines of the PDE are tangent to the light cone through  $(0, 0)$ .

The non-characteristic lines consists of

$$\alpha t + \beta x = 0 \text{ such that } \alpha \neq \pm c\beta$$

These lines are nowhere tangent to the light cone and  $t = 0$  is a straightforward choice for a non-characteristic hyperplane to specify the initial data.

(ii) Show that the energy  $E = \int_{-\infty}^{+\infty} (u_t^2 + c^2 u_x^2 + m^2 u^2) dx$  in dimension  $n = 1$  is conserved under a suitable decay condition on  $u$  as  $|x| \rightarrow \infty$ .

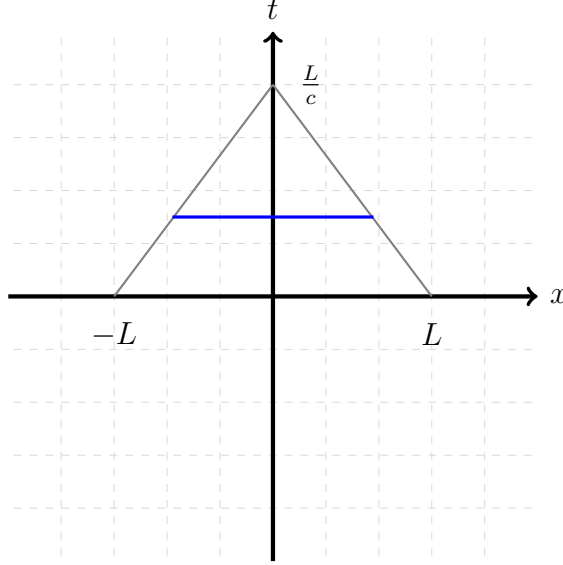
Route 1:

$$\begin{aligned} \frac{dE}{dt} &= 2 \int_{\mathbb{R}} (u_t \underbrace{u_{tt}}_{c^2 u_{xx} - m^2 u} + c^2 u_x u_{xt} + m^2 u u_t) dx \\ &= 2 \int_{\mathbb{R}} (c^2 \underbrace{(u_t u_{xx} + u_x u_{xt})}_{(u_t u_x)_x} - \underbrace{m^2 u u_t}_{m^2 u u_t}) dx \\ &= 2c^2 u_t u_x \Big|_{x=-\infty}^{\infty} \text{ (Gauss's theorem)} \\ &= 0 \text{ if } u_t u_x \text{ is even.} \end{aligned}$$

Route 2:

$$\begin{aligned} \frac{dE}{dt} &= 2 \int_{\mathbb{R}} (u_t \underbrace{u_{tt}}_0 + c^2 \underbrace{u_x u_{xt}}_{2 \frac{\partial}{\partial t} u_x^2} + m^2 \underbrace{u u_t}_{2 \frac{\partial}{\partial t} u^2}) dx \\ &= \frac{d}{dt} \int_{\mathbb{R}} (c^2 u_x^2 + m^2 u^2) dx \end{aligned}$$

- (iii) Suppose the initial data  $u|_{t=0} = u_0(x)$  and  $u_t|_{t=0} = u_1(x)$  vanish inside an interval  $-L \leq x \leq L, L > 0$ . Show that the energy contained inside the forward light cone  $e(t) = \int_{-L+ct}^{L-ct} (u_t^2 + c^2 u_x^2 + m^2 u^2) dx \geq 0$  obeys the inequality  $e'(t) \leq (2c |u_t u_x| - u_t^2 - c^2 u_x^2 - m^2 u^2)|_{-L+ct}^{L-ct} \leq 0$  for all  $0 \leq t \leq L/c$ . Explain why this implies that disturbances (caused by a change in initial data) cannot propagate faster than speed  $c$ .



$$\begin{aligned}
 e'(t) &= \frac{d}{dt} \int_{-L+ct}^{L-ct} (u_t^2 + c^2 u_x^2 + m^2 u^2) dx \\
 &= \int_{-L+ct}^{L-ct} \underbrace{\frac{\partial}{\partial t} (u_t^2 + c^2 u_x^2 + m^2 u^2)}_{2c^2 (u_t u_x)_x} dx + \overbrace{\frac{d}{dt} (L-ct)}^{-c} (u_t^2 + c^2 u_x^2 + m^2 u^2)|_{x=L-ct} \\
 &\quad - \underbrace{\frac{d}{dt} (-L+ct)}_c (u_t^2 + c^2 u_x^2 + m^2 u^2)|_{x=-L+ct} \text{ (Leibniz integral rule)} \\
 &= 2c^2 \underbrace{(u_t u_x)|_{x=-L+ct}^{L-ct}}_{\text{1D Gauss's theorem}} - c \{ (u_t^2 + c^2 u_x^2 + m^2 u^2)|_{x=-L+ct} + (u_t^2 + c^2 u_x^2 + m^2 u^2)|_{x=L-ct} \} \\
 &\quad - c \{ (u_t^2 + c^2 u_x^2 + m^2 u^2)|_{x=-L+ct} + (u_t^2 + c^2 u_x^2 + m^2 u^2)|_{x=L-ct} \} \\
 &\leq -c \{ (u_t^2 + c^2 u_x^2 + m^2 u^2)|_{x=-L+ct} - (u_t^2 + c^2 u_x^2 + m^2 u^2)|_{x=L-ct} \} \\
 &= c (u_t^2 + c^2 u_x^2 + m^2 u^2)|_{x=-L+ct}^{L-ct}
 \end{aligned}$$

So

$$\begin{aligned}
 e'(t) &\leq c \overbrace{(2cu_t u_x - u_t^2 - c^2 u_x^2 - m^2 u^2)}^{-(u_t - cu_x)^2 \leq 0} |_{x=-L+ct}^{L-ct} \\
 &= -m^2 cu^2 |_{x=-L+ct}^{L-ct}
 \end{aligned}$$

If we look for separable solutions of the form  $u(t, x) = A(x)B(t)$ ,  $B$  will be a linear function of  $t$  since  $u_{tt} = 0$  and

$$c^2 u_{xx} - m^2 u = 0 \implies A''(x) = \frac{m^2}{c^2} A(x) \implies A(x) = c_1 e^{\frac{m}{c}x} + c_2 e^{-\frac{m}{c}x}, c_1, c_2 \in \mathbb{R}$$

In order to avoid blowup as  $|x| \rightarrow \pm\infty$ ,  $A(x)$  must be a constant ( $e^x$  and  $e^{-x}$  cannot be joined together to avoid the blowup without producing a cusp at  $c = 0$ ).

$$\implies u^2|_{x=-L+ct}^{L-ct} = 0$$

$$\implies e'(t) \leq 0 \quad (a)$$

$$e(0) = \int_{-L}^L (u_1^2 + c^2 u_{0x}^2 + m^2 u_0^2) dx = 0 \quad (b)$$

$$e(t) \geq 0 \text{ since the integrand is } \geq 0. \quad (c)$$

(a), (b), and (c)  $\implies e'(t) = 0 \implies e(t) = e(0) = 0 \implies u_t, u_x, u = 0$  inside the interval.

Hence  $u$  is zero inside the light cone. Since  $u_0$  and  $u_1$  are not supported inside  $[-L, L]$ , the signal has to travel faster than speed  $c$  to create an influence inside the light cone. Our result forbids it.

(iv) In dimensions  $n \geq 1$ , find all non-characteristic hyperplanes and show that they do not depend on  $m$ . Explain whether any of these hyperplanes are tangent to the light cone.

The highest order derivative is of second order. We can write the equation as,

$$a_{(2)} \partial^2 u - m^2 u = 0,$$

where,

$$a_{(2)} = \text{diag}[1, \underbrace{-c^2(1, \dots, 1)}_{n \text{ times}}]$$

Consider an arbitrary hyper-plane through the origin given by  $\alpha t + \beta \cdot x = 0$ , where  $\hat{n} = (\alpha, \beta)$  is its normal vector and  $\alpha \in \mathbb{R}$  and  $\beta \in \mathbb{R}^n$  are constants.

$$\begin{aligned} a_{(2)} \cdot \hat{n}^2 &= a_{00} \hat{n}_0^2 + \sum_{i=1}^n \underbrace{-c^2}_{a_{ii}} \hat{n}_i^2 \\ &= \alpha^2 - c^2 |\beta|^2 \\ &= 0 \text{ for characteristic hyperplanes.} \end{aligned} \quad (*)$$

Or  $\alpha = \pm c|\beta|$ . For light cones,  $ct \pm |x| = 0$ . Its normal vector,

$$\begin{aligned} n_{lc} &= \nabla_{(t,x)}(ct \pm |x|) \\ &= (c, \pm \hat{x}) \end{aligned}$$

For a tangent plane to the light cone at  $(t, x)$  having the normal vector  $\hat{n}_{lc}$ ,  $\alpha = c, \beta = \hat{x}$  or  $\hat{\beta} = 1 \implies \alpha = \pm c|\beta|$ . Clearly, tangent planes to light cones at the origin are characteristic. Every other plane through the origin with  $\alpha \neq \pm c|\beta|$  are non-characteristic.

(v) Construct a formal power series solution in  $t$ . Use Cauchy-Kovalevskaya theory to state sufficient conditions under which the power series converges for  $t \geq 0$ . Prove that the power series solution is the unique solution for any given analytic initial data  $u_0, u_1$ .

For  $u_{tt} = c^2 \Delta u - m^2 u = 0$ ,  $(t, x) \in \mathbb{R}^n \times \mathbb{R}$ , we are looking for a solution of the form,

$$u(t, x) = u_0(x) + t u_1(x) + t^2 u_2(x) + \dots = \sum_{k=0}^{\infty} t^k u_k(x).$$

The coefficients are given by

$$\begin{aligned}
u_0(x) &= u|_{t=0} = u_0(x) \\
u_1(x) &= u_t|_{t=0} = u_1(x) \\
u_2(x) &= \frac{1}{2}u_{tt}|_{t=0} = \frac{1}{2}(c^2\Delta - m^2)u|_{t=0} = \frac{1}{2}(c^2\Delta u_0 - m^2u_0) \\
u_3(x) &= \frac{1}{3!}u_{ttt}|_{t=0} = \frac{1}{3!}(c^2\Delta - m^2)u_t|_{t=0} = \frac{1}{3!}(c^2\Delta u_1 - m^2u_1) \\
&\vdots \\
u(t, x) &= \sum_{k=0}^{\infty} \left[ \frac{t^{2k}}{(2k)!} (c^2\Delta - m^2)^k u_0(x) + \frac{t^{2k+1}}{(2k+1)!} (c^2\Delta - m^2)^k u_1(x) \right]
\end{aligned}$$

The PDE ( $u_{tt} = c^2\Delta u - m^2u$ ) is in the Cauchy-Kovalevskaya (CK) form

$$\partial_z^k u = \sum_{l=0}^{k-1} \tilde{a}_{l,k-1}(y, z, u, \partial u, \dots, \partial^{k-1}u) \partial_z^l \partial_y^{k-l} u + \tilde{b}(y, z, u, \partial u, \dots, \partial^{k-1}u) \quad (*)$$

with

- $k = 2, z = t, y = x$
- $\tilde{b} = -m^2u$  (analytic),  $\tilde{a}_{(0,2)} = -c^2$ .
- highest order (second order) time derivative isolated in the left hand side with no time derivatives of order  $\geq 1$  on the right hand side
- the analytic initial data,  $u_0$  and  $u_1$ , are specified on a non-characteristic hypersurface, namely  $t = 0$ .

The CK theorem guarantees the existence of a unique convergent power series solution of  $u$  with in  $t$  in some open ball in  $\mathbb{R}$  cetered at 0.

To establish uniqueness, assume two solutions  $u$  and  $\tilde{u}$  with

$$\begin{aligned}
u|_{t=0} &= \tilde{u}|_{t=0} = u_0(x) \text{ and} \\
u_t|_{t=0} &= \tilde{u}_t|_{t=0} = u_1(x).
\end{aligned}$$

$w = u - \tilde{u}$  is also a solution since the PDE is linear.

$$w|_{t=0} = w_t|_{t=0} = 0$$

From energy conservation, we have

$$\begin{aligned}
E(t) &= \int_{\mathbb{R}} (w_t^2 + c^2|\nabla w|^2 + m^2w^2)dx = E(0) = 0. \\
\implies w_t &= |\nabla w| = w = 0 \implies u = \tilde{u}
\end{aligned}$$

Hence our power series solution is the unique solution of the PDE for the given initial data.

- (vi) Consider the case  $m = 0$  and observe that the power series solution does not contain  $n$  explicitly. Since the support properties of solutions in  $n = 2$  versus  $n = 3$  are very different, how is the power series compatible with Huygen's principle?

When  $m = 0$ , the power series becomes,

$$u(t, x) = \sum_{k=0}^{\infty} \left[ \frac{t^{2k}}{(2k)!} \Delta^k u_0(x) + \frac{t^{2k+1}}{(2k+1)!} \Delta^k u_1(x) \right], \text{ with } c = 1.$$

$n$  is in the Laplacian,

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^{n-1}} \Delta_{S^{n-1}}$$

The time asymmetric solution ( $u_0 = 0, u_1 = \delta(r)$ ) is

$$u(t, r) = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \Delta^k \delta(r).$$

$$u(t, r) = \begin{cases} \frac{1}{2\pi\sqrt{t^2-r^2}} \chi(t-r), n=2 \\ \frac{1}{4\pi r} \delta(t-r), n=3 \end{cases}$$

$$\begin{aligned} u(t, r) &= \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \Delta^k \delta(r) \\ &= \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \left( \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} \right)^k \delta(r) \\ &= \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \sum_{l=0}^k \frac{k!}{(k-l)!l!} \frac{(n-1)^l}{r^l} \left( \frac{\partial}{\partial r} \right)^{2(k-l)} \left( \frac{\partial}{\partial r} \right)^l \delta(r) \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^{2k+1} \frac{k! t^{2k+1}}{(2k+1)!(k-l)!l!} \frac{(n-1)^l}{r^l} \underbrace{\left( \frac{\partial}{\partial r} \right)^{2k-l}}_{\delta^{(2k-l)}(r)} \delta(r) \end{aligned}$$

Case 1,  $t = r$  (on the light cone):

Case 2,  $t = r + \epsilon, \epsilon \in \mathbb{R}$  (inside the light cone):

Case 3,  $t = r - \epsilon, \epsilon \in \mathbb{R}$  (outside the light cone):