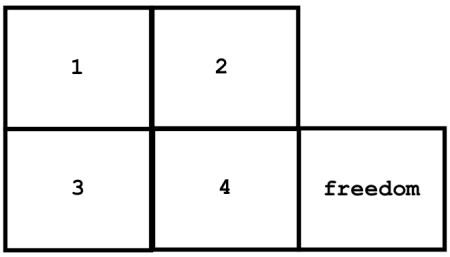

Written Homework 12 · MATH331

Markov Chain (chapter 11 in Grinstead)

Due on Thursday Apr 19 11:30AM

1. Markov Chain (Continue from Homework 11) Consider a rat in a maze with 4 cells, indexed as 1,2,3,4, and the outside (freedom), indexed by 5 (that can only be reached via cell 4). The rat starts initially in a given cell and then takes a move to another cell, continuing to do so until finally reaching freedom. We assume that at each move (transition) the rat, independent of the past, is equally likely to choose from among the neighboring cells (sharing an edge). eg. if it is in cell 4, then it is equal likely that the rat moves to cell 3, 2, or freedom.

   Please pay attention to my state labeling here $\{s_1, s_2, s_3, s_4, s_5\}$ where s_i corresponds to cell $i, i = 1, 2, 3, 4$ and s_5 corresponds to freedom

   Then corresponding transition matrix P is a 5 by 5 matrix:

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, P^{(2)} = P^2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{5}{12} & \frac{5}{12} & 0 & \frac{1}{6} \\ 0 & \frac{5}{12} & \frac{5}{12} & 0 & \frac{1}{6} \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

   You have already found $P^{(2)} = P^2$ in the previous homework.

   (a) The entry (i, j) in P means if the rat is in cell i now, then the probability that it moves to cell j in 1 step is p_{ij}.

$$p_{ij} = P(X_1 = j | X_0 = i) = P(X_{n+1} = j | X_n = i)$$

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   Now, write $p_{35}^{(2)}$ in $P^{(2)}$ as the product of the corresponding row and column of P, and Describe the probabilistic meaning of $p_{35}^{(2)}$ in terms of conditional probability

   (b) Find the distribution of the rat's positions after 2 moves.

(c) Find the following probability values:

i. The probability that the rat ends in cell 1 in 2 steps

ii. The probability that the rat ends in cell 2 in 2 steps

iii. The probability that the rat ends in cell 3 in 2 steps

iv. The probability that the rat ends in cell 4 in 2 steps

v. The probability that the rat ends in cell 5 in 2 steps

2. Consider a matrix

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

What conditions you need to enforce on p_{ij}'s so that P becomes a transition matrix for a discrete Markov chain with states $= 1, 2, 3$?

3. Grinstead's Chapter 11.1, Exercise 2

4. Grinstead's Chapter 11.1, Exercise 4

Exercises on Linear Algebra. This is for you to review some basic linear algebra

Given matrix $P = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1 & 0 \end{pmatrix}$

1. Find eigenvalues of P

2. Solve the linear system $P^T x = x$, where

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$