Chapter 11 of Grinstead

DISCRETE-TIME MARKOV CHAINS

- Markov Chains
- Transition matrices and their powers
- Ergodic
- Regular

ERGODIC AND REGULAR MARKOV CHAINS

ERGODIC AND REGULAR MARKOV CHAIN

- A Markov chain is called an **ergodic chain** if it is possible to go from every state to every state (not necessarily in one move). Note: in many books, ergodic Markov chains are called irreducible.
- A Markov chain is called a **regular chain** if some power of the transition matrix has only positive elements.

ERGODIC MARKOV CHAIN

- Consider the transition matrix of a Markov chain:

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$$

- Absorbing?Ergodic? regular?

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- Not absorbing — no stationary state

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- This is not regular.
- **So Ergodic does not imply regular!**

RECALL DEFINITIONS:

- ergodic chain : it is possible to go from every state to every state (not necessarily in one move).
- regular chain : some power of the transition matrix has **only** positive elements.
  meaning?
- absorbing chain:
  1) it has at least one absorbing state,
  2) from every state it is possible to go to an absorbing state (not necessarily in one step).

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- Relationship among them?
- We have seen that Ergodic does not imply regular!

BOSTON WEATHER EXAMPLE

- state space S = { 1, 2, 3 } (set R = 1, N = 2, S = 3)
- transition matrix P $\begin{array}{c} \\ R \\ N \\ S \end{array} \begin{array}{c} \begin{array}{ccc} R & N & S \end{array} \\ \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \end{array}$
- Absorbing? Ergodic? Regular?

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  power of transition matrix

$$P^2 = \begin{pmatrix} 7/16 & 3/16 & 3/8 \\ 3/8 & 1/4 & 3/8 \\ 3/8 & 3/16 & 7/16 \end{pmatrix}$$

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- All entries nonzero ———> Regular
- Meaning of being regular? difference between regular and ergodic?

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- We found its stationary matrix S. It has the following form

$$\begin{array}{c} \begin{array}{ccc} \text{Rain} & \text{Nice} & \text{Snow} \end{array} \\ \begin{pmatrix} .400 & .200 & .400 \\ .400 & .200 & .400 \\ .400 & .200 & .400 \end{pmatrix} \end{array}$$

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- Is it ergodic?

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- All entries are nonzero means you can go from one state to another.
- So we have: **regular implies ergodic!**

ERGODIC AND REGULAR MARKOV CHAINS

- Question:
- Can an absorbing Markov Chain be ergodic?

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- Question:
- Can an absorbing Markov Chain be ergodic?
- No.
- Why?

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- Absorption state cannot go anywhere else!

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- Question:
- Can an ergodic Markov Chain be absorbing?
- No.

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  1) it has at least one absorbing state,
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- Relationships:
- {set of **absorbing** chains} $\cap$ {set of **ergodic chains)** = $\emptyset$
- **regular implies ergodic** , **ergodic does not imply regular.**