Chapter 11 of Grinstead

DISCRETE-TIME MARKOV CHAINS

- Topics
- State-transition matrix
- Network diagrams
- Examples: gambler's ruin
- Transient probabilities
- Steady-state probabilities

TRANSITION MATRIX MULTIPLICATION

now we have the original transition matrix P

$$\mathbf{P}^1 = \begin{array}{c} \\ \text{Rain} \\ \text{Nice} \\ \text{Snow} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Rain} & \text{Nice} & \text{Snow} \end{array} \\ \begin{pmatrix} .500 & .250 & .250 \\ .500 & .000 & .500 \\ .250 & .250 & .500 \end{pmatrix} \end{array}$$

by taking powers, we computed the 6th power of P

$$\mathbf{P}^6 = \begin{array}{c} \\ \text{Rain} \\ \text{Nice} \\ \text{Snow} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Rain} & \text{Nice} & \text{Snow} \end{array} \\ \begin{pmatrix} .400 & .200 & .400 \\ .400 & .200 & .400 \\ .400 & .200 & .400 \end{pmatrix} \end{array}$$

The 7th power of P is the same as the 6th power!

$$\begin{array}{c} \\ \text{Rain} \\ \text{Nice} \\ \text{Snow} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Rain} & \text{Nice} & \text{Snow} \end{array} \\ \begin{pmatrix} .400 & .200 & .400 \\ .400 & .200 & .400 \\ .400 & .200 & .400 \end{pmatrix} \end{array}$$

Then one can see that the nth power of P remains the same as P6 for any n>6

STABLE PROPERTY OF TRANSITION MATRIX

- “Thm” — (true theorem but we didn’t prove it yet)

  Let P be the transition matrix of a Markov chain. Then there exists a positive integer K, such that:
  For all n>k, the nth power of P stays the same.

- Or there is a unique stationary matrix S such that: SP = S.

FIND STATIONARY MATRIX

- Find the matrix S such that SP = S

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- Note that S is NOT investable!

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- We found S by taking powers before. It has the following form

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$$

- Note all the rows are the same!
  But taking power is computationally expensive! And
  N is not known

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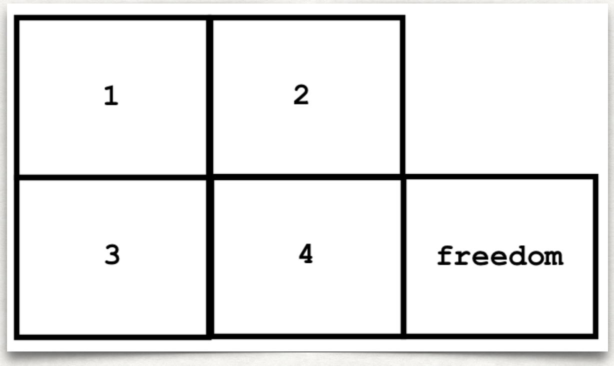
- Since all the rows are the same and sum of the row is 1, we can consider solving a linear system as the following:
- a vector s = (s1,s2,s3), then solve for s where sP = s, $\sum si = 1$

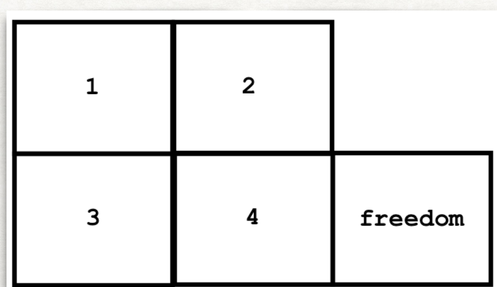
ABSORBING MARKOV CHAIN

ABSORBING MARKOV CHAIN

- Special types of Markov chains — absorbing Markov chain.
- The rat's example (homework)
  Once the rat achieves freedom, there is no way to leave the state freedom



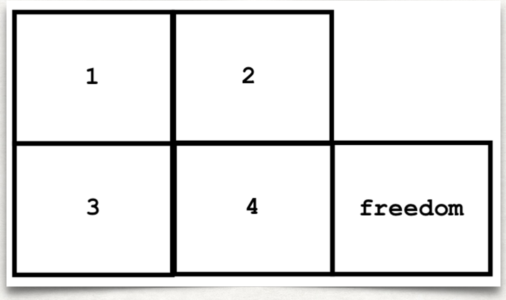
ABSORBING MARKOV CHAIN



$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- A state si of a Markov chain is called absorbing if it is impossible to leave it (p55 = 1 for the rat case)
- A Markov chain is **absorbing** if it has at least one absorbing state (freedom), and if from every state it is possible to go to an absorbing state (not necessarily in one step).
- In an absorbing Markov chain, a state which is not absorbing is called transient.

ABSORBING MARKOV CHAIN



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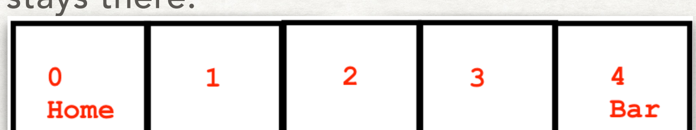
ABSORBING MARKOV CHAIN

- Example: Drunkard's Walk
- A man walks along a four-block stretch of Park Avenue.
  If he is at corner 1, 2, or 3, then he walks to the left or right with equal probability. He continues until he reaches corner 4, which is a bar, or corner 0, which is his home. If he reaches either home or the bar, he stays there.



- Questions:
  Transition matrix?
  States?
  Absorbing state(s)?
  Absorbing Markov Chain?

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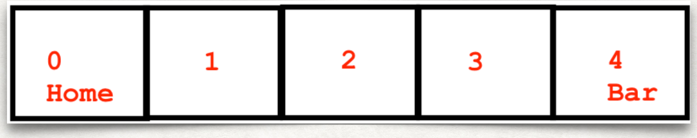

- states = {0,1,2,3,4}

ABSORBING MARKOV CHAIN

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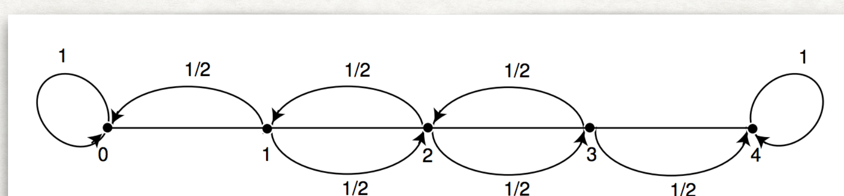

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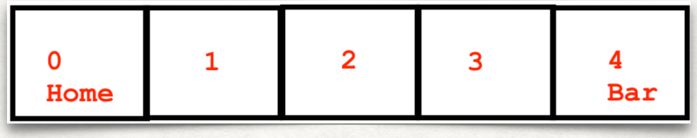

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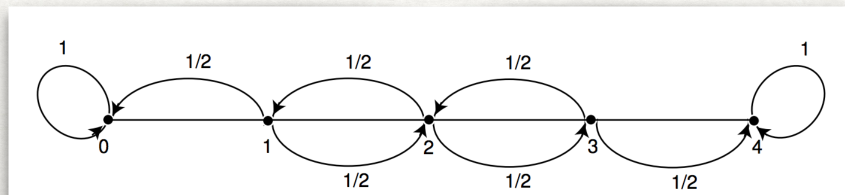

ABSORBING MARKOV CHAIN

- Example: Drunkard's Walk
- Questions:
  Transition matrix?
  States?
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$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 0 & \;\;1 & \;\;2 & \;\;3 & \;\;4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

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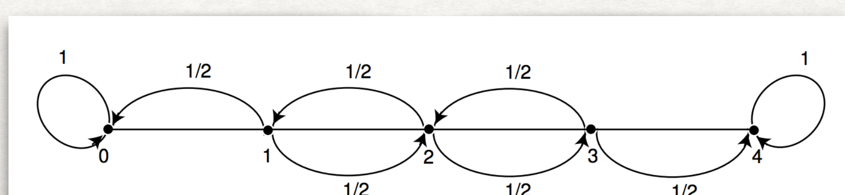

ABSORBING MARKOV CHAIN

- Canonical Form of Absorbing Markov chain

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- For an absorbing Markov chain, renumber the states so that the transient states come first.
- If there are r absorbing states and t transient states, the transition matrix will have the following canonical form

$$P = \begin{array}{rc} & \begin{array}{cc}\text{TR.} & \text{ABS.}\end{array} \\ \begin{array}{r}\text{TR.}\\ \text{ABS.}\end{array} & \left(\begin{array}{c|c} Q & R \\ \hline 0 & I \end{array}\right) \end{array}$$

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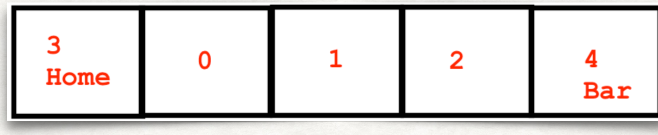
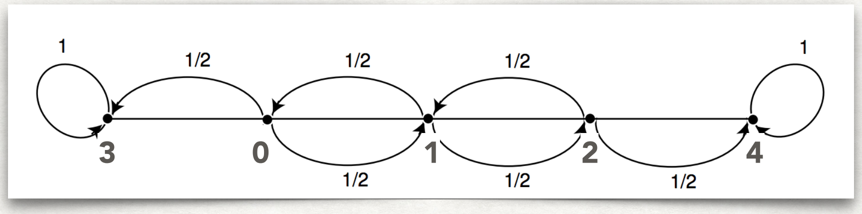
- The first t states are transient and the last r states are absorbing.
- I is an r-by-r identity matrix, 0 is an r-by-t zero matrix, R is a nonzero t-by-r matrix, and Q is an t-by-t matrix.

ABSORBING MARKOV CHAIN

- Example: Drunkard's Walk
- Relabel states





- So the transition matrix changed into

$$
\begin{array}{cc}
 & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
P = & \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\end{array}
$$

ABSORBING MARKOV CHAIN

- Canonical Form of Absorbing Markov chain

Recall that the entry $p_{ij}^{(n)}$ of the matrix P^n is the probability of being in the state s_j after n steps, when the chain is started in state s_i.

P^n is of the form

$$
P^n = \begin{array}{cc}
 & \begin{array}{cc} \text{TR.} & \text{ABS.} \end{array} \\
\begin{array}{c} \text{TR.} \\ \text{ABS.} \end{array} & \left(\begin{array}{c|c} Q^n & ? \\ \hline 0 & I \end{array}\right)
\end{array}
$$

ABSORBING MARKOV CHAIN

- Probability of Absorption
- Theorem.
  In an absorbing Markov chain, the probability that the process will be absorbed is 1 (i.e Q^n→ 0 as n → ∞)

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ABSORBING MARKOV CHAIN

- The Fundamental Matrix

Theorem. *For an absorbing Markov chain the matrix $I - Q$ has an inverse N and $N = I + Q + Q^2 + \cdots$.*

For an absorbing Markov chain P, the matrix $N = (I - Q)^{-1}$ is called the *fundamental matrix* for P. The entry n_{ij} of N gives the expected number of times that the process is in the transient state s_j if it is started in the transient state s_i.

ABSORBING MARKOV CHAIN

- The Fundamental Matrix
- Drunkard's example

$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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- Canonical form:

$$P = \begin{array}{c} \\ \text{TR.} \\ \text{ABS.} \end{array} \begin{array}{c} \begin{array}{cc} \text{TR.} & \text{ABS.} \end{array} \\ \left(\begin{array}{c|c} Q & R \\ \hline 0 & I \end{array}\right) \end{array}$$

ABSORBING MARKOV CHAIN

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- So we have

$$Q = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix},$$

$$I - Q = \begin{pmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/2 & 1 \end{pmatrix}.$$

ABSORBING MARKOV CHAIN

- The Fundamental Matrix

$$N = (I - Q)^{-1} = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{pmatrix} \end{array}$$

- Time to Absorption
- Theorem. Let ti be the expected number of steps before the chain is absorbed, given that the chain starts in state si, and let t be the column vector whose ith entry is ti. Then t = N c ,

  where c is a column vector all of whose entries are 1

ABSORBING MARKOV CHAIN

- Absorption Probabilities

Let b_{ij} be the probability that an absorbing chain will be absorbed in the absorbing state s_j if it starts in the transient state s_i. Let B be the matrix with entries b_{ij}. Then B is an t-by-r matrix, and

$$B = NR \ ,$$

where N is the fundamental matrix and R is as in the canonical form.

ABSORBING MARKOV CHAIN

- Drunkard's example: expected number of steps before the the drunkard reaches home or bar if it starts at block 1,2, or 3

$$N = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{pmatrix} \end{array} .$$

$$\begin{aligned} t = Nc &= \begin{pmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} . \end{aligned}$$

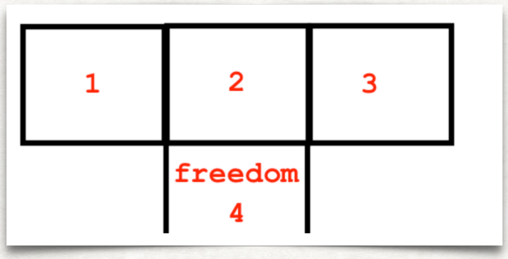
ABSORBING MARKOV CHAIN

- Drunkard's example: probability that the drunkard reaches home if it starts from block 1 is 3/4

$$\begin{aligned} B = NR &= \begin{pmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \\ 0 & 1/2 \end{pmatrix} \\ &= \begin{array}{c c} & \begin{array}{cc} 0 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix} \end{array} . \end{aligned}$$

ABSORBING MARKOV CHAIN

Example from the lab: rat's moving to freedom



$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Consider a rat in a maze with 3 cells, indexed as 1,2,3 and the outside (freedom), indexed by 4 (that can only be reached via cell 3). The rat starts initially in a given cell and then takes a move to another cell, continuing to do so until finally reaching freedom.
- Question:
  Is there an absorbing state? If so, which one?
  Is this an absorbing Markov chain?

ABSORBING MARKOV CHAIN

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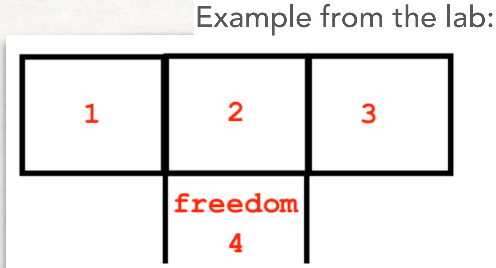

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- Absorbing state: state 4!
  Note that p44 = 1.
  Even if p12 = 1, it is NOT absorbing!

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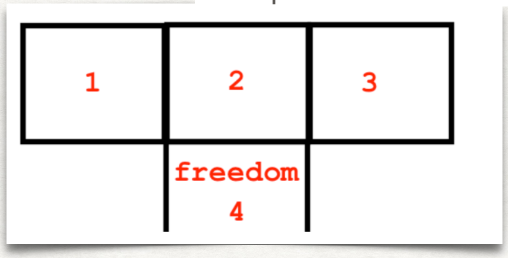

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- Absorbing state: state 4!
  Note that p44 = 1.
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- An absorbing Markov chain needs two ingredients:
  1) an absorbing state,
  2) every other transient state can move to the absorbing state

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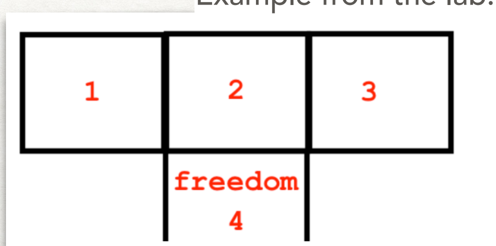

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Question:
  1) If I drop the rat in cell 1, 2 and 3 respectively, how many steps it takes (on average) for the rat to achieve freedom?
  2) Probability that the rat reaches freedom if it starts from 1,2 or 3?

MORE EXAMPLES

ABSORBING MARKOV CHAIN

- Determine if the following matrix represent an Absorbing Markov Chain. If so, find absorption time and probability

No
$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

No
$$P_3 = \begin{pmatrix} 1/3 & 2/3 & 0 & 0 \\ 2/3 & 1/3 & 0 & 0 \\ 0 & 0 & 2/3 & 1/3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

No
$$P_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

No
$$P_4 = \begin{pmatrix} 0 & 1/3 & 1/3 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Yes
$$P_5 = \begin{pmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$