# Practice Final

**Please show your work.**

1. Which of the following matrices are transition matrices for ergodic Markov chains? Justify your answer.

$$P_1 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \quad P_2 = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix} \quad P_3 = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix} \quad P_4 = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

2. Consider the Markov chain with general 2 by 2 transition matrix $P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$

   (a) Under what conditions is P absorbing? Justify your answer.

   (b) Under what conditions is P ergodic? Justify your answer

3. $\{X_1, X_2, \ldots X_5\}$ is a sequence of IID follows a exponential distribution with parameter 1. Let $A_5 = \frac{\sum_{i=1}^5 X_i}{5}$. Find the following values

   (a) $E(X_2)$

   (b) $E(X_2 - 3X_4)$

   (c) $E(A_5)$

   (d) $E(A_5 - X_1)$

   (e) $Var(A_5)$

   (f) $std(A_5)$

   (g) Use the Central Limit Theorem to find a normal distribution $N(\mu, \sigma^2)$ (i.e. to find $\mu$ and $\sigma$) to approximate the distribution of $A_5$.

4. Let $U, V, W$ be random numbers chosen independently from the interval $[1, 2]$ uniformly. Find the cumulative distribution and density for the random variable $Y = max(U, V, W)$

5. $X$ follows a Normal Distribution with mean 0 and standard deviation 2. Calculate the following values

   (a) $P(|X| > 2)$

   (b) $P(X < -2)$

   (c) $P(X > 2)$

(d) $P(e^X < e^4)$

(e) $E(X^2)$

6. $X$ follows a Normal Distribution with mean 1 and standard deviation 1. Calculate the following values

   (a) $P(X^2 > 1)$

   (b) $P(X < 0)$

   (c) $P(X = 0)$

   (d) $P(X > 0)$

   (e) $E((X - 1)^2)$

7. Use Hypothesis Testing to answer the question. Suppose that ordinary aspirin has been found effective against headaches 60 percent of the time, and that a drug company claims that its new aspirin (with special additive) is more effective. They test this claim as follows: they give the aspirin to 500 people to take when they have a headache, and 350 of them are cured.
   Does the experiment provide enough evidence to conclude that the new aspirin is more effective than the ordinary aspirin, assuming the confidence level is $\alpha = 5\%$?

8. Maximum Likelihood Estimates. Suppose you observe the following data set $\mathbf{x}^{(0)} = (2,3), \mathbf{x}^{(1)} = (-1, -2), \mathbf{x}^{(2)} = (-1, 1)$. For any vector $\mathbf{x}$, we denote the first component of $\mathbf{x}$ by $x_1$ and the second component by $x_2$. Suppose that the data is drawn from the same two-dimensional probability distribution with pdf $f_X$, that is, $\mathbf{x}^{(i)} \overset{iid}{\sim} f_X$, where

$$f_X(\mathbf{x}) = \lambda_1^2 x_1 x_2 \exp\left\{-\lambda_0(x_1^2 + x_2^2)\right\}.$$

You should assume that $\lambda_1, \lambda_0 > 0$ and that $f_X$ is supported on the nonnegative quandrant of $\mathbb{R}^2$ (i.e. $f_X$ is zero when either component is negative). **What are the values for $\lambda_0$ and $\lambda_1$ that maximize the likelihood of the observed data? Support your answer with full and rigorous analytic derivations.**

9. Maximum Likelihood Estimates. Suppose you observe the following data set $x_1 = 2, x_2 = 3, x_3 = -2, x_4 = -3$. Suppose that the data is drawn from a normal distribution $N(\mu, \sigma^2)$ **What are the values for $\mu$ and $\sigma^2$ that maximize the likelihood of the observed data? Support your answer with full and rigorous analytic derivations.**

10. Determine local max/min of the function

$$f(x, y) = x^2 + y^2 - xy^2 + \frac{1}{3}x^3$$

where $y \in (-\infty, \infty)$, $x \in (-\infty, \infty)$

11. Given the following matrices

$$P_1 = \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{pmatrix} \quad P_2 = \begin{pmatrix} 1/3 & 2/3 \\ 1 & 0 \end{pmatrix} \quad P_3 = \begin{pmatrix} 0 & 1 \\ 1/3 & 2/3 \end{pmatrix} \quad P_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}$$

$$P_5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad P_6 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

(a) Which one(s) are transition matrices for some Markov chain? Justify your answer.

(b) Which one(s) are transition matrices for some absorbing Markov chain? Justify your answer.

(c) Which one(s) are transition matrices for some ergodic Markov chain? Justify your answer.

(d) For the one(s) which are absorbing, find their time to absorption and probability to be absorbed.

12. A man walks along a 2-block stretch of Park Avenue. If he is at A or B, then he walks to the left or right with equal probability. He continues until he reaches home or a bar. If he reaches either home or the bar, he stays there.



(a) The man's moves can be described using a Markov chain. Find the state space of his moves.

(b) Find the 1-step transition matrix $P$ of his moves.

(c) In 2 steps, determine the probability distribution of the man's location

(d) Find the probability that the man will eventually stay at home.

(e) Find the probability that the man will eventually stay at the bar.

(f) Find the probability that the man will eventually stay at block A .

(g) Find the probability that the man will eventually stay at block B.

13. I store the $funny\_weather.txt$ file in a folder called data, which is in the same directory as my running python file. If I open $funny\_weather.txt$ , it looks like the following

```
R, 0.50, 0.25, 0.25
N, 0.50, 0.00, 0.50
S, 0.25, 0.25, 0.50
```

Determine which of the following code(s) will load the file properly as either a 3 by 3 matrix or a 3 by 3 ndarray. There may be multiple answers.

(a)

```
1  import numpy as np
2  weather = np.loadtxt('funny_weather.txt')
```

(b)

```
5  import numpy as np
6  weather = np.loadtxt('funny_weather.txt',usecols=[1,2,3])
```

(c)

```
10  import numpy as np
11  weather = np.loadtxt('funny_weather.txt',delimiter=',',skiprows=1)
```

(d)

```
14  import numpy as np
15  weather = np.loadtxt('funny_weather.txt',delimiter=',',usecols=[1,2,3])
```

(e) none of the above

14. In python, if I run the following code, what is the output?(Assume I have imported numpy as np)

```
1  arr = np.array([
2  [1,2,3],
3  [2,3,4],
4  ])
5
6  print(np.any(arr>3))
7
```

15. In python, I define arr as:

$$arr = np.arange(9).reshape((3,3))$$

which of the following code(s) will print a **True** (i.e. a boolean type True)? There may be multiple answers.

(a) $print(arr > 1)$

(b) $print(np.all(arr > 1))$

(c) $print(np.any(arr == 1))$

(d) $print(np.all(arr < 0))$

(e) none of the above