Chapter 11 of Grinstead

DISCRETE-TIME MARKOV CHAINS

- Topics
- State-transition matrix
- Network diagrams
- Examples: gambler's ruin
- Transient probabilities
- Steady-state probabilities

DISCRETE-TIME MARKOV CHAINS

A stochastic process $\{ X_n \}$ is called a Markov chain if
$\Pr\{ X_{n+1} = j \mid X_0 = k_0, \ldots, X_{n-1} = k_{n-1}, X_n = i \}$

$= \Pr\{ X_{n+1} = j \mid X_n = i \}$ ← transition probabilities

for every $i, j, k_0, \ldots, k_{n-1}$ and for every n.

- Discrete time means $n \in N = \{ 0, 1, 2, \ldots \}$.
- Markovian property means:
  The future behavior of the system depends only on the current state i and not on any of the previous states.
- eg: tomorrow's weather only depends on today, not any days before today.

PROCEDURE FOR SETTING UP A MARKOV CHAIN

- Check Markov property. i.e. current state only depends on previous state and NOT any other states.

- Define the state vector s = (s1, s2, . . . , sv) and list all the states. Number the states.
  eg: we have {R, N, S} as states, and label them as {R=1, N=2, S=3}

- Determine the state-transition matrix P = (pij).
  pij's in your matrix P

DISCRETE-TIME MARKOV CHAINS

EXAMPLE

$$\mathbf{P} = \begin{array}{c} \\ \mathrm{R} \\ \mathrm{N} \\ \mathrm{S} \end{array} \!\! \begin{array}{c} \begin{array}{ccc} \mathrm{R} & \mathrm{N} & \mathrm{S} \end{array} \\ \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \end{array} .$$

- state space S = { 1, 2, 3 } (set R = 1, N = 2, S = 3)
- transition matrix P (Note we use state space {1,2,3} to make the subscripts of the transition matrix coincide with conditional prob.)

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

where $p_{ij} = P\left(X_1 = j | X_0 = i\right)$

TRANSITION MATRIX MULTIPLICATION

BACK TO THE BOSTON WEATHER EXAMPLE

- state space S = { 1, 2, 3 } (set R = 1, N = 2, S = 3)
- transition matrix P $\begin{array}{cc} & \begin{array}{ccc} \mathrm{R} & \mathrm{N} & \mathrm{S} \end{array} \\ \begin{array}{c} \mathrm{R} \\ \mathrm{N} \\ \mathrm{S} \end{array} & \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \end{array}$

DISCRETE-TIME MARKOV CHAINS: BOSTON WEATHER

- state space S = { 1, 2, 3 } (set R = 1, N = 2, S = 3)
- transition matrix P

$$\begin{array}{c} \\ \mathrm{R} \\ \mathrm{N} \\ \mathrm{S} \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{R} & \mathrm{N} & \mathrm{S} \end{array} \\ \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \end{array}$$

- Question:
  if it is rainy today, then what is the probability that **<u>it is snowy two days from now?</u>**

DISCRETE-TIME MARKOV CHAINS: BOSTON WEATHER

- state space S = { 1, 2, 3 } (set R = 1, N = 2, S = 3)
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$$\begin{array}{c} \\ \mathrm{R} \\ \mathrm{N} \\ \mathrm{S} \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{R} & \mathrm{N} & \mathrm{S} \end{array} \\ \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \end{array}$$

- Question:
  if it is rainy today, then what is the probability that **it is snowy two days from now?**

- The disjoint union of the following three events:
  1) it is rainy tomorrow and snowy two days from now

  2) it is nice tomorrow and snowy two days from now,

  3) it is snowy tomorrow and snowy two days from now.

DISCRETE-TIME MARKOV CHAINS: BOSTON WEATHER

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- Question:
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- The disjoint union of the following three events:
  1) it is rainy tomorrow and snowy two days from now
  path: R->R->S
  2) it is nice tomorrow and snowy two days from now,
  path: R->N->S
  3) it is snowy tomorrow and snowy two days from now.
  path: R->S->S

DISCRETE-TIME MARKOV CHAINS: BOSTON WEATHER

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- Question:
  if it is rainy today, then what is the probability that **<u>it is snowy two days from now?</u>**

- The disjoint union of the following three events:
  1) it is rainy tomorrow and snowy two days from now
  path: R->R->S, probability = 1/2 * 1/4 = 1/8
  2) it is nice tomorrow and snowy two days from now,
  path: R->N->S, probability = 1/4 * 1/2 = 1/8
  3) it is snowy tomorrow and snowy two days from now.
  path: R->S->S, probability = 1/4 * 1/2 = 1/8

DISCRETE-TIME MARKOV CHAINS: BOSTON WEATHER

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- Question:
  if it is rainy today, then what is the probability that **<u>it is snowy two days from now?</u>**

- The disjoint union of the following three events:
  1) it is rainy tomorrow and snowy two days from now
  path: R->R->S, probability = 1/2 * 1/4 = 1/8
  2) it is nice tomorrow and snowy two days from now,
  path: R->N->S, probability = 1/4 * 1/2 = 1/8
  3) it is snowy tomorrow and snowy two days from now.
  path: R->S->S, probability = 1/4 * 1/2 = 1/8

- In total = 1/8 + 1/8 + 1/8 = 3/8

DISCRETE-TIME MARKOV CHAINS: BOSTON WEATHER

- state space S = { 1, 2, 3 } (set R = 1, N = 2, S = 3)
- transition matrix P

$$
\begin{array}{c} \\ R \\ N \\ S \end{array}
\begin{array}{c}
\begin{array}{ccc} \mathrm{R} & \mathrm{N} & \mathrm{S} \end{array} \\
\begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}
\end{array}
$$

- Question:
if it is rainy today, then what is the probability that **<u>it is snowy two days from now?</u>**

- The disjoint union of the following three events:
1) it is rainy tomorrow and snowy two days from now
path: R->R->S, probability = 1/2 * 1/4 = 1/8
2) it is nice tomorrow and snowy two days from now,
path: R->N->S, probability = 1/4 * 1/2 = 1/8
3) it is snowy tomorrow and snowy two days from now.
path: R->S->S, probability = 1/4 * 1/2 = 1/8

- In total = 1/8 + 1/8 + 1/8 = 3/8 = (row(R) of P) dot (col(S) of P)

DISCRETE-TIME MARKOV CHAINS: BOSTON WEATHER

- state space S = { 1, 2, 3 } (set R = 1, N = 2, S = 3)
- transition matrix P $\begin{array}{c c} & \begin{array}{ccc} \mathrm{R} & \mathrm{N} & \mathrm{S} \end{array} \\ \begin{array}{c} \mathrm{R} \\ \mathrm{N} \\ \mathrm{S} \end{array} & \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \end{array}$
- Question:
  if it is rainy today, then what is the probability that **it is snowy two days from now? 3/8**
- **So, what is P^2? What does each entry in P^2 mean?**

DISCRETE-TIME MARKOV CHAINS: BOSTON WEATHER

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- Question:
  if it is rainy today, then what is the probability that <u>it is snowy two days from now? 3/8</u>
- <u>So we have</u>

$$P^2 = \begin{pmatrix} 7/16 & 3/16 & 3/8 \\ 3/8 & 1/4 & 3/8 \\ 3/8 & 3/16 & 7/16 \end{pmatrix}$$

<u>and each entry is denoted by</u> $p_{ij}^{(2)}$

DISCRETE-TIME MARKOV CHAINS: BOSTON WEATHER

- state space S = { 1, 2, 3 } (set R = 1, N = 2, S = 3)
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- Question:
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- <u>**So we have**</u>

$$P^2 = \begin{pmatrix} 7/16 & 3/16 & 3/8 \\ 3/8 & 1/4 & 3/8 \\ 3/8 & 3/16 & 7/16 \end{pmatrix}$$

- <u>**Meaning of $p_{12}^{(2)}$?**</u>

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- Question:
  if it is rainy today, then what is the probability that **it is snowy two days from now? 3/8**
- **So we have**

$$P^2 = \begin{pmatrix} 7/16 & 3/16 & 3/8 \\ 3/8 & 1/4 & 3/8 \\ 3/8 & 3/16 & 7/16 \end{pmatrix}$$

- **Meaning of $p_{12}^{(2)}$?**
  Starting from state S1 (rain),
  the probability of ending up with state S2 (nice)
  in 2 steps is
  $p_{12}^{(2)}$ = 3/16

DISCRETE-TIME MARKOV CHAINS: BOSTON WEATHER

- Question:

  the distribution of the states after 2 days?

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- Question:
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- First column
  if rain today, 7/16 chance rain in 2days
  if nice today, 3/8 chance rain in 2days
  if snow today, 3/8 chance rain in 2 days

DISCRETE-TIME MARKOV CHAINS: BOSTON WEATHER

- Question:
  the distribution of the states after 2 days?

$$P^2 = \begin{array}{c|ccc} & R & N & S \\ \hline R & 7/16 & 3/16 & 3/8 \\ N & 3/8 & 1/4 & 3/8 \\ S & 3/8 & 3/16 & 7/16 \end{array}$$

- First column
  if rain today, 7/16 chance rain in 2days
  if nice today, 3/8 chance rain in 2days
  if snow today, 3/8 chance rain in 2 days

- Assume 0th state is: given any day there is
  1/3 prob rain, 1/3 prob nice, 1/3 prob snow.

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- Question:
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  if rain today, 7/16 chance rain in 2days
  if nice today, 3/8 chance rain in 2days
  if snow today, 3/8 chance rain in 2 days

- Assume 0th state is: given any day there is
  1/3 prob rain, 1/3 prob nice, 1/3 prob snow.

- Then (1/3,1/3,1/3) dot (1st column) = prob of rain after 2 days
  similarly (1/3,1/3,1/3) dot (2nd column) = prob of nice after 2 days
  similarly (1/3,1/3,1/3) dot (3rd column) = prob of snow after 2 days

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- Question:
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- First column
  if rain today, 7/16 chance rain in 2days
  if nice today, 3/8 chance rain in 2days
  if snow today, 3/8 chance rain in 2 days

- Assume 0th state is: given any day there is
  1/3 prob rain, 1/3 prob nice, 1/3 prob snow.

- Then (1/3,1/3,1/3) dot (1st column) = prob of rain after 2 days
  similarly (1/3,1/3,1/3) dot (2nd column) = prob of nice after 2 days
  similarly (1/3,1/3,1/3) dot (3rd column) = prob of snow after 2 days

- So the probability distribution (discrete distribution) is:
  P(rain in 2 days) =19/48 , P(nice in 2 days) = 5/24
  P(snow in 2 days) = 19/48

DISCRETE-TIME MARKOV CHAINS: BOSTON WEATHER

Theorem 11.2 Let $\mathbf{P}$ be the transition matrix of a Markov chain, and let $\mathbf{u}$ be the probability vector which represents the starting distribution. Then the probability that the chain is in state s_i after n steps is the ith entry in the vector

$$\mathbf{u}^{(n)} = \mathbf{u}\mathbf{P}^n \ .$$

- Note: u = (1/r, 1/r, …, 1/r) where r is the number of states
- The previous example, 3 states, so $u = (1/3, 1/3, 1/3)$

$$u^{(2)} = (1/3, 1/3, 1/3) \cdot P^2 = (1/3, 1/3, 1/3) \cdot \begin{pmatrix} 7/16 & 3/16 & 3/8 \\ 3/8 & 1/4 & 3/8 \\ 3/8 & 3/16 & 7/16 \end{pmatrix}$$

“

HIGHER POWERS OF TRANSITION MATRIX

”

now we have the original transition matrix P

$$
\mathbf{P}^1 = \begin{array}{c} \\ \text{Rain} \\ \text{Nice} \\ \text{Snow} \end{array}
\begin{array}{c}
\begin{array}{ccc} \text{Rain} & \text{Nice} & \text{Snow} \end{array} \\
\begin{pmatrix} .500 & .250 & .250 \\ .500 & .000 & .500 \\ .250 & .250 & .500 \end{pmatrix}
\end{array}
$$

now we have the original transition matrix P

$$\mathbf{P}^1 = \begin{array}{c} \\ \text{Rain} \\ \text{Nice} \\ \text{Snow} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Rain} & \text{Nice} & \text{Snow} \end{array} \\ \begin{pmatrix} .500 & .250 & .250 \\ .500 & .000 & .500 \\ .250 & .250 & .500 \end{pmatrix} \end{array}$$

by taking powers, we computed till 6th power of P

$$\mathbf{P}^6 = \begin{array}{c} \\ \text{Rain} \\ \text{Nice} \\ \text{Snow} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Rain} & \text{Nice} & \text{Snow} \end{array} \\ \begin{pmatrix} .400 & .200 & .400 \\ .400 & .200 & .400 \\ .400 & .200 & .400 \end{pmatrix} \end{array}$$

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What's the 7th power of P?

now we have the original transition matrix P

$$\mathbf{P}^1 = \begin{array}{c|ccc} & \text{Rain} & \text{Nice} & \text{Snow} \\ \hline \text{Rain} & .500 & .250 & .250 \\ \text{Nice} & .500 & .000 & .500 \\ \text{Snow} & .250 & .250 & .500 \end{array}$$

by taking powers, we computed the 6th power of P

$$\mathbf{P}^6 = \begin{array}{c|ccc} & \text{Rain} & \text{Nice} & \text{Snow} \\ \hline \text{Rain} & .400 & .200 & .400 \\ \text{Nice} & .400 & .200 & .400 \\ \text{Snow} & .400 & .200 & .400 \end{array}$$

The 7th power of P is the same as the 6th power!

$$\begin{array}{c|ccc} & \text{Rain} & \text{Nice} & \text{Snow} \\ \hline \text{Rain} & .400 & .200 & .400 \\ \text{Nice} & .400 & .200 & .400 \\ \text{Snow} & .400 & .200 & .400 \end{array}$$

now we have the original transition matrix P

$$\mathbf{P}^1 = \begin{array}{c|ccc} & \text{Rain} & \text{Nice} & \text{Snow} \\ \hline \text{Rain} & .500 & .250 & .250 \\ \text{Nice} & .500 & .000 & .500 \\ \text{Snow} & .250 & .250 & .500 \end{array}$$

by taking powers, we computed the 6th power of P

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The 7th power of P is the same as the 6th power!

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Then one can see that the nth power of P remains the same as P6 for any n>6