Chapter 11 of Grinstead

DISCRETE-TIME MARKOV CHAINS

- Topics
- State-transition matrix
- Network diagrams
- Examples: gambler's ruin
- Transient probabilities
- Steady-state probabilities

DISCRETE-TIME MARKOV CHAINS

A stochastic process $\{ X_n \}$ is called a Markov chain if

$$\Pr\{ X_{n+1} = j \mid X_0 = k_0, \ldots, X_{n-1} = k_{n-1}, X_n = i \}$$

$$= \Pr\{ X_{n+1} = j \mid X_n = i \} \quad \leftarrow \text{transition probabilities}$$

for every $i, j, k_0, \ldots, k_{n-1}$ and for every n.

- Discrete time means $n \in N = \{ 0, 1, 2, \ldots \}$.
- Markovian property means:
  The future behavior of the system depends only on the current state i and not on any of the previous states.
- eg: tomorrow's weather only depends on today, not any days before today.

DISCRETE-TIME MARKOV CHAINS

EXAMPLE REVISIT

- According to weather history records for Boston, we have the following information. (Note, the record only shows nice days, snow days and rain days)
- Boston never has two nice days in a row, and we assume tomorrow's weather only depends on today's weather.
    - If they have a nice day, they are just as likely to have snow as rain the next day.
    - If they have snow or rain, they have an even chance of having the same the next day.
    - If there is change from snow or rain, only half of the time is this a change to a nice day.

DISCRETE-TIME MARKOV CHAINS

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- The set of Time: t = {1,2,3 . . .} = {day 1, day 2, day 3, ….}

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- The set of Time: t = {1,2,3 . . .} = {day 1, day 2, day 3, ….}
- {Xn} is a sequence of random variables
  — where n is in t ={1,2,3,…}
  — each Xn takes values in the state space {R = 1, N = 2, S = 3}
  — the outcome of Xn+1 only depends on Xn

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  — the outcome of Xn+1 only depends on Xn
- Transition matrix of 1 step

$$\mathbf{P} = \begin{array}{c} \\ \mathrm{R} \\ \mathrm{N} \\ \mathrm{S} \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{R} & \mathrm{N} & \mathrm{S} \end{array} \\ \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \end{array}.$$

DISCRETE-TIME MARKOV CHAINS

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- state space { 1, 2, 3 } (set R = 1, N = 2, S = 3)
  p11 = P(X2=R | X1 = R) = 1/2, p32 = P(X2=N | X1 = S) = 1/4

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  p11 = P(X2=1 | X1 = 1) = 1/2, p32 = P(X2=2 | X1 = 3) = 1/4

DISCRETE-TIME MARKOV CHAINS

$$
\begin{array}{c c}
 & \begin{array}{ccc} \mathrm{R} & \mathrm{N} & \mathrm{S} \end{array} \\
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$$
P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}
$$

where $p_{ij} = P\left(X_2 = j | X_1 = i\right)$

DISCRETE-TIME MARKOV CHAINS

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$$p_{ij} = P\left(X_{n+1} = j | X_n = i\right)$$

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where $p_{ij} = P\left(X_2 = j | X_1 = i\right)$

for any 1 step from time n to time n+1,
we have this matrix

$$p_{ij} = P\left(X_{n+1} = j | X_n = i\right)$$

TRANSITION MATRIX MULTIPLICATION

TRANSITION MATRIX POWER — BOSTON WEATHER

- state space S = { 1, 2, 3 } (set R = 1, N = 2, S = 3)
- transition matrix P $\begin{array}{c c} & \begin{array}{ccc} \mathrm{R} & \mathrm{N} & \mathrm{S} \end{array} \\ \begin{array}{c} \mathrm{R} \\ \mathrm{N} \\ \mathrm{S} \end{array} & \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \end{array}$

TRANSITION MATRIX POWER — BOSTON WEATHER

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- Probability that Rain today, and snow 2 days from today.
- The disjoint union of the following three events:
  1) it is rainy tomorrow and snowy two days from now
  path: R->R->S, probability = 1/2 * 1/4 = 1/8
  2) it is nice tomorrow and snowy two days from now,
  path: R->N->S, probability = 1/4 * 1/2 = 1/8
  3) it is snowy tomorrow and snowy two days from now.
  path: R->S->S, probability = 1/4 * 1/2 = 1/8
- In total = 1/8 + 1/8 + 1/8 = 3/8 = (row(R) of P) dot (col(S) of P)

TRANSITION MATRIX POWER — BOSTON WEATHER

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-

power of transition matrix

$$P^2 = \begin{pmatrix} 7/16 & 3/16 & 3/8 \\ 3/8 & 1/4 & 3/8 \\ 3/8 & 3/16 & 7/16 \end{pmatrix}$$

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- **Meaning of $p_{12}^{(2)}$?**
  Starting from state S1, the probability of ending up with state S2 in 2 steps is $p_{12}^{(2)}$ = 3/16

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$$P^2 = \begin{pmatrix} 7/16 & 3/16 & 3/8 \\ 3/8 & 1/4 & 3/8 \\ 3/8 & 3/16 & 7/16 \end{pmatrix}$$

- **Meaning of $p_{12}^{(2)}$?**
  Starting from state S1, the probability of ending up with state S2 in 2 steps is $p_{12}^{(2)}$ = 3/16
- **$p_{12}^{(2)}$ = P(X3 = S2 | X1 = S1) = 3/16**

TRANSITION MATRIX POWER — BOSTON WEATHER

- Question:
  the distribution of the states after 2 days?

$$P^2 = \begin{array}{c} \\ R \\ N \\ S \end{array} \begin{array}{c} \begin{array}{ccc} R & N & S \end{array} \\ \begin{pmatrix} 7/16 & 3/16 & 3/8 \\ 3/8 & 1/4 & 3/8 \\ 3/8 & 3/16 & 7/16 \end{pmatrix} \end{array}$$

- First column
  if day1 is rainy , 7/16 chance rain in 2 days (7/16 chance day3 is rainy)
  if day1 is nice ,3/8 chance rain in 2 days (3/8 chance day3 is rainy)
  if day1 in snowy,3/8 chance rain in 2 days (3/8 chance day3 is rainy)

TRANSITION MATRIX POWER — BOSTON WEATHER

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- First column
  if day1 is rainy, 7/16 chance rain in 2 days (7/16 chance day3 is rainy)
  if day1 is nice, 3/8 chance rain in 2 days (3/8 chance day3 is rainy)
  if day1 in snowy, 3/8 chance rain in 2 days (3/8 chance day3 is rainy)

- Assume 0th state is: given any day there is
  1/3 chance rain, 1/3 chance nice, 1/3 chance snow.

TRANSITION MATRIX POWER — BOSTON WEATHER

- Question:
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- First column
  if day1 is rainy, 7/16 chance rain in 2 days (7/16 chance day3 is rainy)
  if day1 is nice, 3/8 chance rain in 2 days (3/8 chance day3 is rainy)
  if day1 in snowy, 3/8 chance rain in 2 days (3/8 chance day3 is rainy)

- Assume 0th state is: given any day there is
  1/3 chance rain, 1/3 chance nice, 1/3 chance snow.

- Then (1/3,1/3,1/3) dot (1st column) = prob of rain after 2 days
  similarly (1/3,1/3,1/3) dot (2nd column) = prob of nice after 2 days
  similarly (1/3,1/3,1/3) dot (3rd column) = prob of snow after 2 days

- So the probability distribution (discrete distribution) is:
  P(rain in 2 days) =19/48 , P(nice in 2 days) = 5/24 ,P(snow in 2 days) = 19/48

TRANSITION MATRIX POWER — BOSTON WEATHER

Theorem 11.2 Let $\mathbf{P}$ be the transition matrix of a Markov chain, and let $\mathbf{u}$ be the probability vector which represents the starting distribution. Then the probability that the chain is in state s_i after n steps is the ith entry in the vector

$$\mathbf{u}^{(n)} = \mathbf{u}\mathbf{P}^n .$$

- Note: u = (1/r, 1/r, …, 1/r) where r is the number of states
- The previous example, 3 states in total, r = 3
- so $u = (1/3, 1/3, 1/3)$

$$
u^{(2)} = (1/3, 1/3, 1/3) \cdot P^2 = (1/3, 1/3, 1/3) \cdot \begin{pmatrix} 7/16 & 3/16 & 3/8 \\ 3/8 & 1/4 & 3/8 \\ 3/8 & 3/16 & 7/16 \end{pmatrix}
$$

“ HIGHER POWERS OF TRANSITION MATRIX ”

now we have the original transition matrix P

$$\mathbf{P}^1 = \begin{array}{c} \\ \text{Rain} \\ \text{Nice} \\ \text{Snow} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Rain} & \text{Nice} & \text{Snow} \end{array} \\ \begin{pmatrix} .500 & .250 & .250 \\ .500 & .000 & .500 \\ .250 & .250 & .500 \end{pmatrix} \end{array}$$

DISCRETE-TIME MARKOV CHAINS: BOSTON WEATHER

$$\mathbf{P}^1 = \begin{array}{c} \\ \text{Rain} \\ \text{Nice} \\ \text{Snow} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Rain} & \text{Nice} & \text{Snow} \end{array} \\ \begin{pmatrix} .500 & .250 & .250 \\ .500 & .000 & .500 \\ .250 & .250 & .500 \end{pmatrix} \end{array}$$

$$\mathbf{P}^2 = \begin{array}{c} \\ \text{Rain} \\ \text{Nice} \\ \text{Snow} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Rain} & \text{Nice} & \text{Snow} \end{array} \\ \begin{pmatrix} .438 & .188 & .375 \\ .375 & .250 & .375 \\ .375 & .188 & .438 \end{pmatrix} \end{array}$$

we can also compute higher powers of P
(see the right hand side)

$$\mathbf{P}^3 = \begin{array}{c} \\ \text{Rain} \\ \text{Nice} \\ \text{Snow} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Rain} & \text{Nice} & \text{Snow} \end{array} \\ \begin{pmatrix} .406 & .203 & .391 \\ .406 & .188 & .406 \\ .391 & .203 & .406 \end{pmatrix} \end{array}$$

$$\mathbf{P}^4 = \begin{array}{c} \\ \text{Rain} \\ \text{Nice} \\ \text{Snow} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Rain} & \text{Nice} & \text{Snow} \end{array} \\ \begin{pmatrix} .402 & .199 & .398 \\ .398 & .203 & .398 \\ .398 & .199 & .402 \end{pmatrix} \end{array}$$

$$\mathbf{P}^5 = \begin{array}{c} \\ \text{Rain} \\ \text{Nice} \\ \text{Snow} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Rain} & \text{Nice} & \text{Snow} \end{array} \\ \begin{pmatrix} .400 & .200 & .399 \\ .400 & .199 & .400 \\ .399 & .200 & .400 \end{pmatrix} \end{array}$$

$$\mathbf{P}^6 = \begin{array}{c} \\ \text{Rain} \\ \text{Nice} \\ \text{Snow} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Rain} & \text{Nice} & \text{Snow} \end{array} \\ \begin{pmatrix} .400 & .200 & .400 \\ .400 & .200 & .400 \\ .400 & .200 & .400 \end{pmatrix} \end{array}$$

……

now we have the original transition matrix P

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by taking powers, we computed till 6th power of P

$$\mathbf{P}^6 = \begin{array}{c} \\ \text{Rain} \\ \text{Nice} \\ \text{Snow} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Rain} & \text{Nice} & \text{Snow} \end{array} \\ \begin{pmatrix} .400 & .200 & .400 \\ .400 & .200 & .400 \\ .400 & .200 & .400 \end{pmatrix} \end{array}$$

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What's the 7th power of P?

now we have the original transition matrix P

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The 7th power of P is the same as the 6th power!

$$\begin{array}{cc} & \begin{array}{ccc}\text{Rain} & \text{Nice} & \text{Snow}\end{array} \\ \begin{array}{c}\text{Rain}\\ \text{Nice}\\ \text{Snow}\end{array} & \begin{pmatrix} .400 & .200 & .400 \\ .400 & .200 & .400 \\ .400 & .200 & .400 \end{pmatrix}\end{array}$$

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Then one can see that the nth power of P remains the same as P6 for any n>6

The 7th power of P is the same as the 6th power!

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STABLE PROPERTY OF TRANSITION MATRIX

- "Thm" — (true theorem but we didn't prove it yet)

  Let P be the transition matrix of a Markov chain. Then there exists a positive integer K, such that:
  For all n>k, the nth power of P stays the same.

STABLE PROPERTY OF TRANSITION MATRIX

- "Thm" — (true theorem but we didn't prove it yet)

  Let P be the transition matrix of a Markov chain. Then there exists a positive integer K, such that:
  For all n>k, the nth power of P stays the same.

- Or there is a unique stationary matrix S such that: SP = S.

FIND STATIONARY MATRIX

- Find the matrix S such that SP = S

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\begin{array}{c} \begin{array}{ccc} \text{Rain} & \text{Nice} & \text{Snow} \end{array} \\
\begin{pmatrix} .500 & .250 & .250 \\ .500 & .000 & .500 \\ .250 & .250 & .500 \end{pmatrix} \end{array}
$$

FIND STATIONARY MATRIX

- Find the matrix S such that SP = S
- Note that S is NOT investable!

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- Find the matrix S such that SP = S
- Note that S is NOT investable!

$$\mathbf{P}^1 = \begin{array}{c} \\ \text{Rain} \\ \text{Nice} \\ \text{Snow} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Rain} & \text{Nice} & \text{Snow} \end{array} \\ \begin{pmatrix} .500 & .250 & .250 \\ .500 & .000 & .500 \\ .250 & .250 & .500 \end{pmatrix} \end{array}$$

- We found S by taking powers before. It has the following form

$$\begin{array}{c} \begin{array}{ccc} \text{Rain} & \text{Nice} & \text{Snow} \end{array} \\ \begin{pmatrix} .400 & .200 & .400 \\ .400 & .200 & .400 \\ .400 & .200 & .400 \end{pmatrix} \end{array}$$

FIND STATIONARY MATRIX

- Find the matrix S such that SP = S
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- Note all the rows are the same!
  But taking power is computationally expensive! And
  N is not known

FIND STATIONARY MATRIX

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- Since all the rows are the same and sum of the row is 1, we can consider solving a linear system as the following:
- a vector s = (s1,s2,s3), then solve for s where sP = s, ∑si = 1