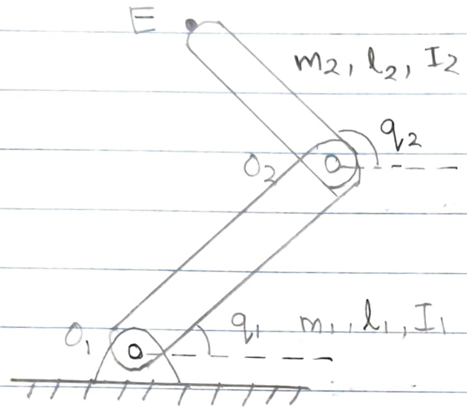
Consider the 2R elbow manipulator with two links of masses m_1 & m_2 lengths l_1 & l_2 and moments of inertia I_1 & I_2 respectively. as shown in the figure below:

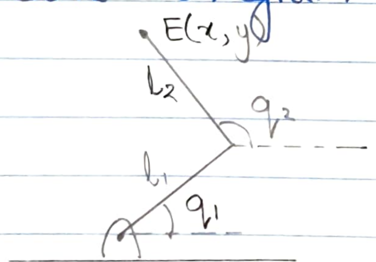


; $q_1, q_2 \rightarrow$ angles
E - End effector.

→ It is open series chain.

⇒ Assume that there are motors connected to O_1 and O_2 providing torques τ_1 and τ_2 controlling the angles q_1 & q_2 as desired.

→ let us consider forward kinematics for a FBD of above diagram



The x and y co-ordinates of the above system can be represented as

$$x = l_1 \cos q_1 + l_2 \cos q_2$$
$$y = l_1 \sin q_1 + l_2 \sin q_2 \quad \text{———} \quad (I)$$

Now, Differentiating the above equation (I) w.r.t Time we get,

$$\dot{x} = -l_1 \sin q_1 \dot{q}_1 - l_2 \sin q_2 \dot{q}_2$$
$$\dot{y} = l_1 \cos q_1 \cdot \dot{q}_1 + l_2 \cos q_2 \dot{q}_2$$

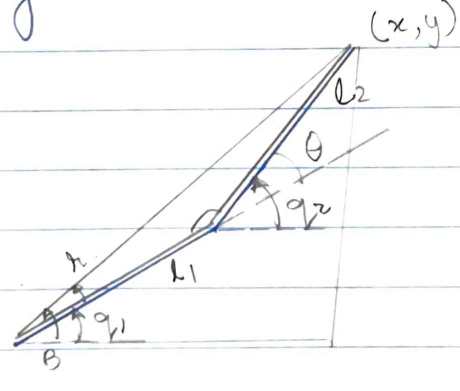
Writing it in matrix form gives,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 \sin q_1 & -l_2 \sin q_2 \\ l_1 \cos q_1 & l_2 \cos q_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \text{———} \text{(II)}$$

→ The above equation can be called equations representing velocity kinematics.

→ We can say that any input/work done on $\dot{q}_1$ & $\dot{q}_2$ can produce effects on $\dot{x}$ & $\dot{y}$.

Applying Inverse Kinematics approach:



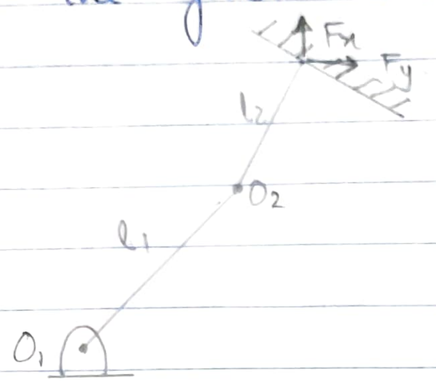
Using cosine rule

$$\theta = \cos^{-1}\left(\frac{x^2+y^2-l_1^2-l_2^2}{2l_1 l_2}\right)$$

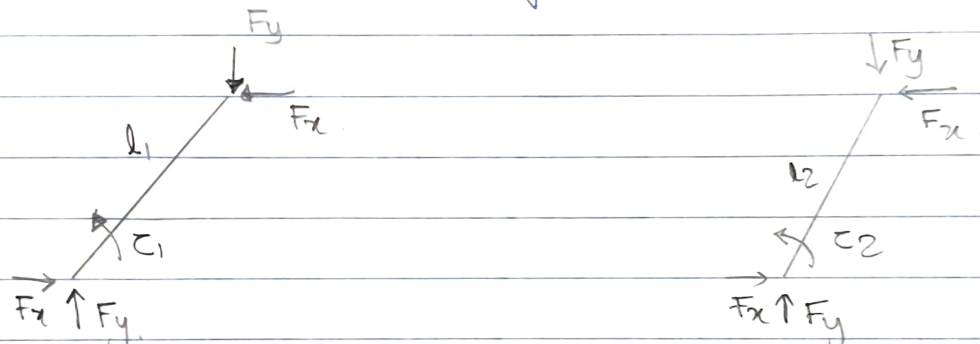
Now, $q_1 = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{l_2 \sin\theta}{l_1 + l_2 \cos\theta}\right)$

and $q_2 = q_1 + \theta = \cos^{-1}\left(\frac{x^2+y^2-l_1^2-l_2^2}{2l_1 l_2}\right) + \tan^{-1}\left(\frac{l_2 \sin\theta}{l_1 + l_2\cos\theta}\right)$ —— (III)

Now, for determining the forces acting on wall consider the given FBD.



Let us consider F.B.D of link 1 & 2 individually,



→ We consider the system to be in static equilibrium. From the diagrams, we can say that

$\Sigma M_{O_1} = 0$ $\quad\quad\quad\quad$ $\Sigma M_{O_2} = 0$

$\tau_1 = -F_x l_1 \sin q_1 + F_y l_1 \cos q_1$ $\quad\quad$ $\tau_2 = -F_y l_2 \sin q_2 + F_y l_2 \cos q_2$

$$\therefore \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -l_1 sq_1 & +l_1 cq_1 \\ -l_2 sq_2 & +l_2 cq_2 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix} \quad \text{——— (IV)}$$

Further, if we consider the Dynamics of system then we have to use Lagrange's Equations.

Lagrangian: $L = K - V$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i' \quad ; \; i = 1, 2, 3 \ldots n.$$

Q_i' = generalized forces.

— (V)

$$K = \underbrace{\frac{1}{2}\left(\frac{1}{3} m_1 l_1^2\right) \dot{q}_1^2}_{\text{pure rotation of Link 1}} + \underbrace{\frac{1}{2}\left(\frac{1}{12} m_2 l_2^2\right) \dot{q}_2^2}_{\text{rotation of Link 2 about its center of mass}} + \underbrace{\frac{1}{2} m_2 V_{c_2}^2}_{\text{translation of c.m of link 2}}$$

$$V_{c_2}^2 = (l_1 \dot{q}_1)^2 + (l_2/2\, \dot{q}_2)^2 + 2 l_1 \dot{q}_1\, l_2/2\, \dot{q}_2 \cos(q_2 - q_1)$$

→ considering gravity,

$$V = m_1 g\, l_1/2 \sin q_1 + m_2 g\,(l_1 \sin q_1 + l_2/2 \sin q_2).$$

$$\Rightarrow \frac{1}{3} m_1 l_1^2 \ddot{q}_1 + m_2 l_1^2 \ddot{q}_1 + m_2 \frac{l_1 l_2}{2} \ddot{q}_2 \cos(q_2 - q_1) - m_2 \frac{l_1 l_2}{2} \dot{q}_2 (\dot{q}_2 - \dot{q}_1) \times \sin(q_2 - q_1)$$

$$+ m_1 g \frac{l_1}{2} \cos q_2 + m_2 g l_1 \cos q_1 = \tau_1$$

$$\Rightarrow \frac{1}{3} m_2 l_2^2 \ddot{q}_2 + m_2 \frac{l_2^2}{4} \ddot{q}_2 + m_2 \frac{l_1 l_2}{2} \ddot{q} \cos(q_2 - q_1) - m_2 \frac{l_1 l_2}{2} \dot{q}_1 (\dot{q}_2 - \dot{q}_1) \times \sin(q_2 - q_1)$$

$$+ m_2 g \frac{l_2}{2} \sin q_2 = \tau_2$$

— (VI)

Next, we know that (N) is valid for any end-effector

$F_x = kx$ or $F_x = k_x(x - x_0)$

$F_y = ky$ $\quad$ $F_y = k_y(y - y_0)$

$\therefore$ From (i), we get

$$F_x = k(l_1 cq_1 + l_2 cq_2)$$
$$F_y = k(l_1 sq_1 + l_2 sq_2)$$

and from (4), we get

$$k(l_1 sq_1 + l_2 sq_2)\, l_2 cq_2 - k(l_1 cq_1 + l_2 cq_2)\, l_2 sq_2 = \tau_{2S}$$

$$k(l_1 sq_1 + l_2 sq_2)\, l_1 cq_1 - k(l_1 cq_1 + l_2 cq_2)\, l_1 sq_1 = \tau_{1S}$$

VII