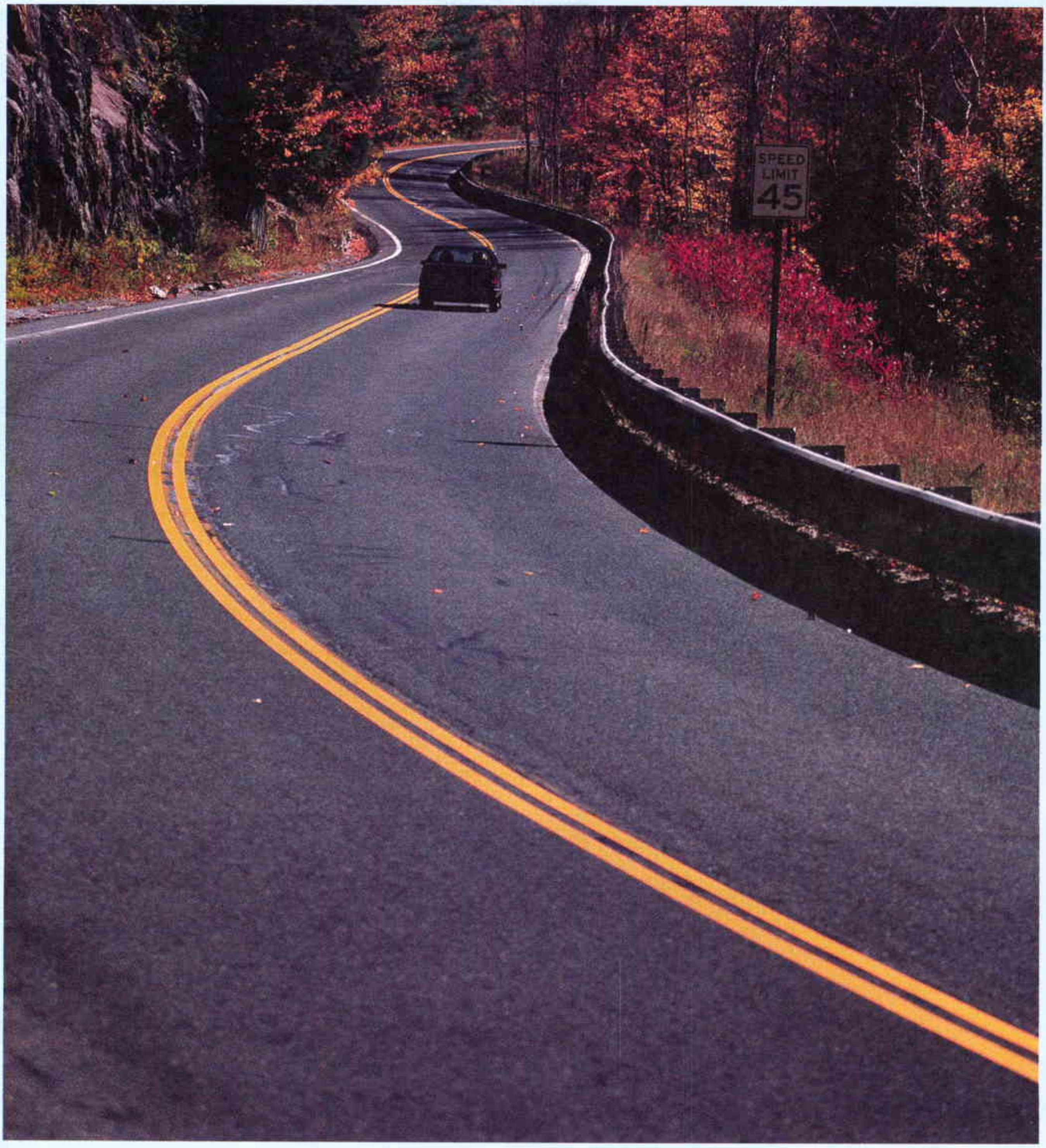


Photodisc/Media Bakery

Even if this car maintains a constant speed along the winding road, it accelerates laterally, and this acceleration must be considered in the design of the car, its tires, and the roadway itself.

2 KINEMATICS OF PARTICLES

CHAPTER OUTLINE

2/1 INTRODUCTION

Kinematics is the branch of dynamics which describes the motion of bodies without reference to the forces which either cause the motion or are generated as a result of the motion. Kinematics is often described as the "geometry of motion." Some engineering applications of kinematics include the design of cams, gears, linkages, and other machine elements to control or produce certain desired motions, and the calculation of flight trajectories for aircraft, rockets, and spacecraft. A thorough working knowledge of kinematics is a prerequisite to kinetics, which is the study of the relationships between motion and the corresponding forces which cause or accompany the motion.

Particle Motion

We begin our study of kinematics by first discussing in this chapter the motions of points or particles. A particle is a body whose physical dimensions are so small compared with the radius of curvature of its path that we may treat the motion of the particle as that of a point. For example, the wingspan of a jet transport flying between Los Angeles and New York is of no consequence compared with the radius of curvature of its flight path, and thus the treatment of the airplane as a particle or point is an acceptable approximation.

We can describe the motion of a particle in a number of ways, and the choice of the most convenient or appropriate way depends a great deal on experience and on how the data are given. Let us obtain an overview of the several methods developed in this chapter by referring to Fig. 2/1, which shows a particle P moving along some general path in space. If the particle is confined to a specified path, as with a bead sliding along a fixed wire, its motion is said to be *constrained*. If there are no physical guides, the motion is said to be *unconstrained*. A small rock tied to the end of a string and whirled in a circle undergoes constrained motion until the string breaks, after which instant its motion is unconstrained.

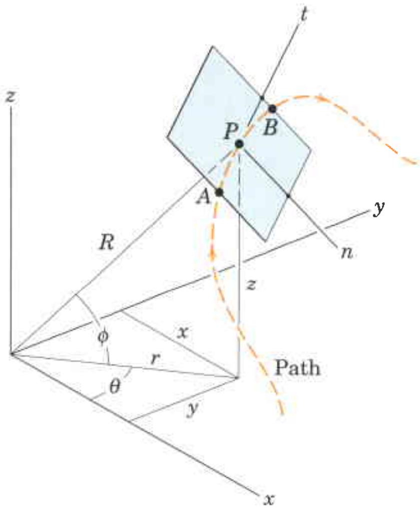


Figure 2/1

Choice of Coordinates

The position of particle P at any time t can be described by specifying its rectangular coordinates* x, y, z, its cylindrical coordinates r, θ, z, or its spherical coordinates R, θ, ϕ. The motion of P can also be described by measurements along the tangent t and normal n to the curve. The direction of n lies in the local plane of the curve.† These last two measurements are called *path variables*.

The motion of particles (or rigid bodies) can be described by using coordinates measured from fixed reference axes (*absolute-motion* analysis) or by using coordinates measured from moving reference axes (*relative-motion* analysis). Both descriptions will be developed and applied in the articles which follow.

With this conceptual picture of the description of particle motion in mind, we restrict our attention in the first part of this chapter to the case of *plane motion* where all movement occurs in or can be represented as occurring in a single plane. A large proportion of the motions of machines and structures in engineering can be represented as plane motion. Later, in Chapter 7, an introduction to three-dimensional motion is presented. We begin our discussion of plane motion with *rectilinear motion*, which is motion along a straight line, and follow it with a description of motion along a plane curve.

2/2 Rectilinear Motion

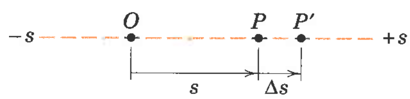


Figure 2/2

Consider a particle P moving along a straight line, Fig. 2/2. The position of P at any instant of time t can be specified by its distance s measured from some convenient reference point O fixed on the line. At time $t + \Delta t$ the particle has moved to P' and its coordinate becomes $s + \Delta s$. The change in the position coordinate during the interval Δt is called the *displacement* Δs of the particle. The displacement would be negative if the particle moved in the negative s-direction.

*Often called *Cartesian* coordinates, named after René Descartes (1596–1650), a French mathematician who was one of the inventors of analytic geometry.

†This plane is called the *osculating* plane, which comes from the Latin word *osculari* meaning "to kiss." The plane which contains P and the two points A and B, one on either side of P, becomes the osculating plane as the distances between the points approach zero.

Velocity and Acceleration

The average velocity of the particle during the interval Δt is the displacement divided by the time interval or $v_{av} = \Delta s/\Delta t$. As Δt becomes smaller and approaches zero in the limit, the average velocity approaches the *instantaneous velocity* of the particle, which is $v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$ or

$$\boxed{v = \frac{ds}{dt} = \dot{s}} \qquad \textbf{(2/1)}$$

Thus, the velocity is the time rate of change of the position coordinate s. The velocity is positive or negative depending on whether the corresponding displacement is positive or negative.

The average acceleration of the particle during the interval Δt is the change in its velocity divided by the time interval or $a_{av} = \Delta v/\Delta t$. As Δt becomes smaller and approaches zero in the limit, the average acceleration approaches the *instantaneous acceleration* of the particle, which is $a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$ or

$$\boxed{a = \frac{dv}{dt} = \dot{v}} \quad \text{or} \quad \boxed{a = \frac{d^2s}{dt^2} = \ddot{s}} \qquad \textbf{(2/2)}$$

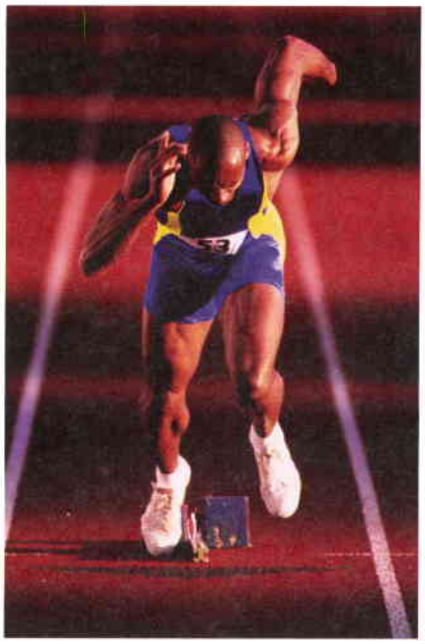
The acceleration is positive or negative depending on whether the velocity is increasing or decreasing. Note that the acceleration would be positive if the particle had a negative velocity which was becoming less negative. If the particle is slowing down, the particle is said to be *decelerating*.

Velocity and acceleration are actually vector quantities, as we will see for curvilinear motion beginning with Art. 2/3. For rectilinear motion in the present article, where the direction of the motion is that of the given straight-line path, the sense of the vector along the path is described by a plus or minus sign. In our treatment of curvilinear motion, we will account for the changes in direction of the velocity and acceleration vectors as well as their changes in magnitude.

By eliminating the time dt between Eq. 2/1 and the first of Eqs. 2/2, we obtain a differential equation relating displacement, velocity, and acceleration.* This equation is

$$\boxed{v\,dv = a\,ds} \quad \text{or} \quad \boxed{\dot{s}\,d\dot{s} = \ddot{s}\,ds} \qquad \textbf{(2/3)}$$

Equations 2/1, 2/2, and 2/3 are the differential equations for the rectilinear motion of a particle. Problems in rectilinear motion involving finite changes in the motion variables are solved by integration of these basic differential relations. The position coordinate s, the velocity v, and the acceleration a are algebraic quantities, so that their signs, positive or negative, must be carefully observed. Note that the positive directions for v and a are the same as the positive direction for s.



Jim Cummings/Taxi/Getty Images

This sprinter will undergo rectilinear acceleration until he reaches his terminal speed.

*Differential quantities can be multiplied and divided in exactly the same way as other algebraic quantities.

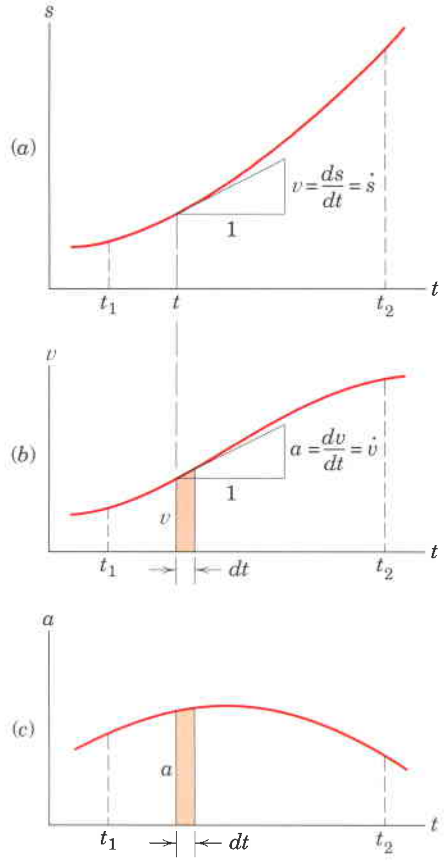


Figure 2/3

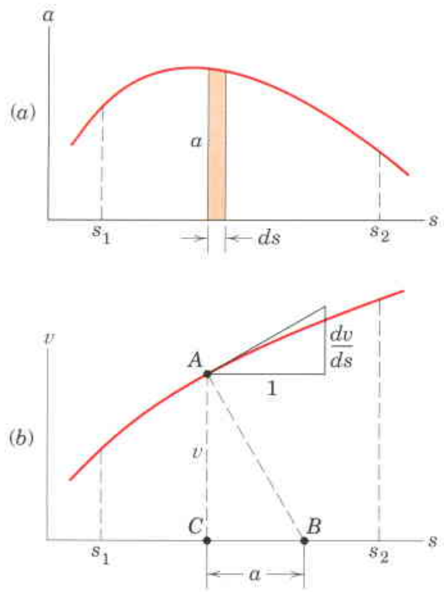


Figure 2/4

Graphical Interpretations

Interpretation of the differential equations governing rectilinear motion is considerably clarified by representing the relationships among s, v, a, and t graphically. Figure 2/3*a* is a schematic plot of the variation of s with t from time t_1 to time t_2 for some given rectilinear motion. By constructing the tangent to the curve at any time t, we obtain the slope, which is the velocity $v = ds/dt$. Thus, the velocity can be determined at all points on the curve and plotted against the corresponding time as shown in Fig. 2/3*b*. Similarly, the slope dv/dt of the v-t curve at any instant gives the acceleration at that instant, and the a-t curve can therefore be plotted as in Fig. 2/3*c*.

We now see from Fig. 2/3*b* that the area under the v-t curve during time dt is $v\,dt$, which from Eq. 2/1 is the displacement ds. Consequently, the net displacement of the particle during the interval from t_1 to t_2 is the corresponding area under the curve, which is

$$\int_{s_1}^{s_2} ds = \int_{t_1}^{t_2} v\,dt \qquad \text{or} \qquad s_2 - s_1 = (\text{area under } v\text{-}t \text{ curve})$$

Similarly, from Fig. 2/3*c* we see that the area under the a-t curve during time dt is $a\,dt$, which, from the first of Eqs. 2/2, is dv. Thus, the net change in velocity between t_1 and t_2 is the corresponding area under the curve, which is

$$\int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} a\,dt \qquad \text{or} \qquad v_2 - v_1 = (\text{area under } a\text{-}t \text{ curve})$$

Note two additional graphical relations. When the acceleration a is plotted as a function of the position coordinate s, Fig. 2/4*a*, the area under the curve during a displacement ds is $a\,ds$, which, from Eq. 2/3, is $v\,dv = d(v^2/2)$. Thus, the net area under the curve between position coordinates s_1 and s_2 is

$$\int_{v_1}^{v_2} v\,dv = \int_{s_1}^{s_2} a\,ds \qquad \text{or} \qquad \tfrac{1}{2}(v_2{}^2 - v_1{}^2) = (\text{area under } a\text{-}s \text{ curve})$$

When the velocity v is plotted as a function of the position coordinate s, Fig. 2/4*b*, the slope of the curve at any point A is dv/ds. By constructing the normal AB to the curve at this point, we see from the similar triangles that $\overline{CB}/v = dv/ds$. Thus, from Eq. 2/3, $\overline{CB} = v(dv/ds) = a$, the acceleration. It is necessary that the velocity and position coordinate axes have the same numerical scales so that the acceleration read on the position coordinate scale in meters (or feet), say, will represent the actual acceleration in meters (or feet) per second squared.

The graphical representations described are useful not only in visualizing the relationships among the several motion quantities but also in obtaining approximate results by graphical integration or differentiation. The latter case occurs when a lack of knowledge of the mathematical relationship prevents its expression as an explicit mathematical function which can be integrated or differentiated. Experimental data and motions which involve discontinuous relationships between the variables are frequently analyzed graphically.

Analytical Integration

If the position coordinate s is known for all values of the time t, then successive mathematical or graphical differentiation with respect to t gives the velocity v and acceleration a. In many problems, however, the functional relationship between position coordinate and time is unknown, and we must determine it by successive integration from the acceleration. Acceleration is determined by the forces which act on moving bodies and is computed from the equations of kinetics discussed in subsequent chapters. Depending on the nature of the forces, the acceleration may be specified as a function of time, velocity, or position coordinate, or as a combined function of these quantities. The procedure for integrating the differential equation in each case is indicated as follows.

(a) Constant Acceleration. When a is constant, the first of Eqs. 2/2 and 2/3 can be integrated directly. For simplicity with $s = s_0$, $v = v_0$, and $t = 0$ designated at the beginning of the interval, then for a time interval t the integrated equations become

$$\int_{v_0}^{v} dv = a\int_0^t dt \qquad \text{or} \qquad v = v_0 + at$$

$$\int_{v_0}^{v} v\,dv = a\int_{s_0}^{s} ds \qquad \text{or} \qquad v^2 = {v_0}^2 + 2a(s - s_0)$$

Substitution of the integrated expression for v into Eq. 2/1 and integration with respect to t give

$$\int_{s_0}^{s} ds = \int_0^t (v_0 + at)\,dt \qquad \text{or} \qquad s = s_0 + v_0 t + \tfrac{1}{2}at^2$$

These relations are necessarily restricted to the special case where the acceleration is constant. The integration limits depend on the initial and final conditions, which for a given problem may be different from those used here. It may be more convenient, for instance, to begin the integration at some specified time t_1 rather than at time $t = 0$.

Caution: The foregoing equations have been integrated for constant acceleration only. A common mistake is to use these equations for problems involving variable acceleration, where they do not apply.

(b) Acceleration Given as a Function of Time, $a = f(t)$. Substitution of the function into the first of Eqs. 2/2 gives $f(t) = dv/dt$. Multiplying by dt separates the variables and permits integration. Thus,

$$\int_{v_0}^{v} dv = \int_0^t f(t)\,dt \qquad \text{or} \qquad v = v_0 + \int_0^t f(t)\,dt$$

From this integrated expression for v as a function of t, the position coordinate s is obtained by integrating Eq. 2/1, which, in form, would be

$$\int_{s_0}^{s} ds = \int_0^t v\, dt \qquad \text{or} \qquad s = s_0 + \int_0^t v\, dt$$

If the indefinite integral is employed, the end conditions are used to establish the constants of integration. The results are identical with those obtained by using the definite integral.

If desired, the displacement s can be obtained by a direct solution of the second-order differential equation $\ddot{s} = f(t)$ obtained by substitution of $f(t)$ into the second of Eqs. 2/2.

(c) Acceleration Given as a Function of Velocity, $a = f(v)$. Substitution of the function into the first of Eqs. 2/2 gives $f(v) = dv/dt$, which permits separating the variables and integrating. Thus,

$$t = \int_0^t dt = \int_{v_0}^{v} \frac{dv}{f(v)}$$

This result gives t as a function of v. Then it would be necessary to solve for v as a function of t so that Eq. 2/1 can be integrated to obtain the position coordinate s as a function of t.

Another approach is to substitute the function $a = f(v)$ into the first of Eqs. 2/3, giving $v\, dv = f(v)\, ds$. The variables can now be separated and the equation integrated in the form

$$\int_{v_0}^{v} \frac{v\, dv}{f(v)} = \int_{s_0}^{s} ds \qquad \text{or} \qquad s = s_0 + \int_{v_0}^{v} \frac{v\, dv}{f(v)}$$

Note that this equation gives s in terms of v without explicit reference to t.

(d) Acceleration Given as a Function of Displacement, $a = f(s)$. Substituting the function into Eq. 2/3 and integrating give the form

$$\int_{v_0}^{v} v\, dv = \int_{s_0}^{s} f(s)\, ds \qquad \text{or} \qquad v^2 = v_0^{\,2} + 2\int_{s_0}^{s} f(s)\, ds$$

Next we solve for v to give $v = g(s)$, a function of s. Now we can substitute ds/dt for v, separate variables, and integrate in the form

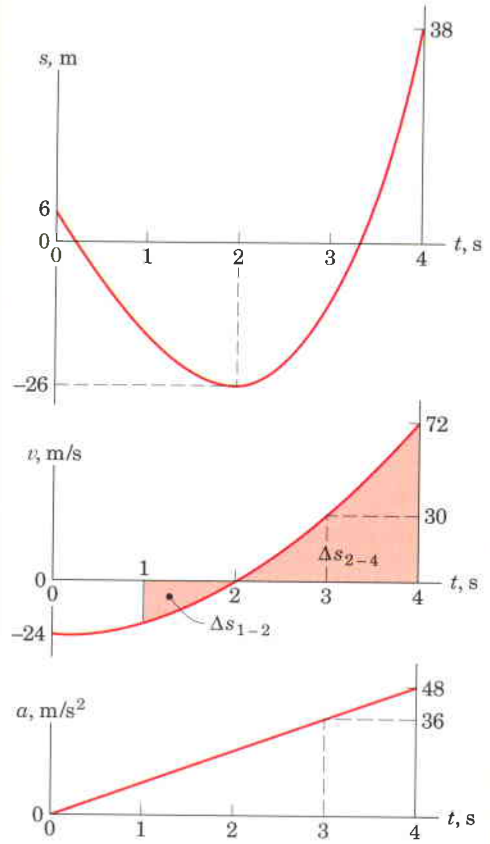
$$\int_{s_0}^{s} \frac{ds}{g(s)} = \int_0^t dt \qquad \text{or} \qquad t = \int_{s_0}^{s} \frac{ds}{g(s)}$$

which gives t as a function of s. Finally, we can rearrange to obtain s as a function of t.

In each of the foregoing cases when the acceleration varies according to some functional relationship, the possibility of solving the equations by direct mathematical integration will depend on the form of the function. In cases where the integration is excessively awkward or difficult, integration by graphical, numerical, or computer methods can be utilized.

Sample Problem 2/1

The position coordinate of a particle which is confined to move along a straight line is given by $s = 2t^3 - 24t + 6$, where s is measured in meters from a convenient origin and t is in seconds. Determine (a) the time required for the particle to reach a velocity of 72 m/s from its initial condition at $t = 0$, (b) the acceleration of the particle when $v = 30$ m/s, and (c) the net displacement of the particle during the interval from $t = 1$ s to $t = 4$ s.



Solution. The velocity and acceleration are obtained by successive differentiation of s with respect to the time. Thus,

$[v = \dot{s}]$ $\qquad v = 6t^2 - 24$ m/s

$[a = \dot{v}]$ $\qquad a = 12t$ m/s^2

(a) Substituting $v = 72$ m/s into the expression for v gives us $72 = 6t^2 - 24$, from which $t = \pm 4$ s. The negative root describes a mathematical solution for t before ① the initiation of motion, so this root is of no physical interest. Thus, the desired result is

$$t = 4 \text{ s} \qquad \textit{Ans.}$$

(b) Substituting $v = 30$ m/s into the expression for v gives $30 = 6t^2 - 24$, from which the positive root is $t = 3$ s, and the corresponding acceleration is

$$a = 12(3) = 36 \text{ m/s}^2 \qquad \textit{Ans.}$$

(c) The net displacement during the specified interval is

$$\Delta s = s_4 - s_1 \qquad \text{or}$$

$$\Delta s = [2(4^3) - 24(4) + 6] - [2(1^3) - 24(1) + 6]$$

$$= 54 \text{ m} \qquad \textit{Ans.}$$

② which represents the net advancement of the particle along the s-axis from the position it occupied at $t = 1$ s to its position at $t = 4$ s.

To help visualize the motion, the values of s, v, and a are plotted against the time t as shown. Because the area under the v-t curve represents displacement, ③ we see that the net displacement from $t = 1$ s to $t = 4$ s is the positive area Δs_{2-4} less the negative area Δs_{1-2}.

Helpful Hints

① Be alert to the proper choice of sign when taking a square root. When the situation calls for only one answer, the positive root is not always the one you may need.

② Note carefully the distinction between italic s for the position coordinate and the vertical s for seconds.

③ Note from the graphs that the values for v are the slopes ($\dot{s}$) of the s-t curve and that the values for a are the slopes ($\dot{v}$) of the v-t curve. *Suggestion:* Integrate $v\,dt$ for each of the two intervals and check the answer for Δs. Show that the total distance traveled during the interval $t = 1$ s to $t = 4$ s is 74 m.

Sample Problem 2/2

A particle moves along the x-axis with an initial velocity $v_x = 50$ ft/sec at the origin when $t = 0$. For the first 4 seconds it has no acceleration, and thereafter it is acted on by a retarding force which gives it a constant acceleration $a_x = -10$ ft/sec^2. Calculate the velocity and the x-coordinate of the particle for the conditions of $t = 8$ sec and $t = 12$ sec and find the maximum positive x-coordinate reached by the particle. ①

Helpful Hints

① Learn to be flexible with symbols. The position coordinate x is just as valid as s.

Solution. The velocity of the particle after $t = 4$ sec is computed from

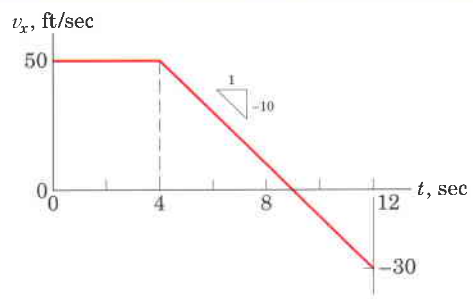
② $$\left[\int dv = \int a\,dt\right] \qquad \int_{50}^{v_x} dv_x = -10\int_4^t dt \qquad v_x = 90 - 10t \text{ ft/sec}$$

and is plotted as shown. At the specified times, the velocities are

$$t = 8 \text{ sec}, \qquad v_x = 90 - 10(8) = 10 \text{ ft/sec}$$

$$t = 12 \text{ sec}, \qquad v_x = 90 - 10(12) = -30 \text{ ft/sec} \qquad \textit{Ans.}$$

② Note that we integrate to a general time t and then substitute specific values.



The x-coordinate of the particle at any time greater than 4 seconds is the distance traveled during the first 4 seconds plus the distance traveled after the discontinuity in acceleration occurred. Thus,

$$\left[\int ds = \int v\,dt\right] \qquad x = 50(4) + \int_4^t (90 - 10t)\,dt = -5t^2 + 90t - 80 \text{ ft}$$

For the two specified times,

$$t = 8 \text{ sec}, \qquad x = -5(8^2) + 90(8) - 80 = 320 \text{ ft}$$

$$t = 12 \text{ sec}, \qquad x = -5(12^2) + 90(12) - 80 = 280 \text{ ft} \qquad \textit{Ans.}$$

The x-coordinate for $t = 12$ sec is less than that for $t = 8$ sec since the motion is in the negative x-direction after $t = 9$ sec. The maximum positive x-coordinate is, then, the value of x for $t = 9$ sec which is

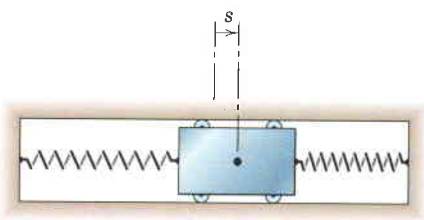
$$x_{max} = -5(9^2) + 90(9) - 80 = 325 \text{ ft} \qquad \textit{Ans.}$$

③ These displacements are seen to be the net positive areas under the v-t graph up to the values of t in question.

③ Show that the total distance traveled by the particle in the 12 sec is 370 ft.

Sample Problem 2/3

The spring-mounted slider moves in the horizontal guide with negligible friction and has a velocity v_0 in the s-direction as it crosses the mid-position where $s = 0$ and $t = 0$. The two springs together exert a retarding force to the motion of the slider, which gives it an acceleration proportional to the displacement but oppositely directed and equal to $a = -k^2 s$, where k is constant. (The constant is arbitrarily squared for later convenience in the form of the expressions.) Determine the expressions for the displacement s and velocity v as functions of the time t.



Solution I. Since the acceleration is specified in terms of the displacement, the differential relation $v\,dv = a\,ds$ may be integrated. Thus,

① $$\int v\,dv = \int -k^2 s\,ds + C_1 \text{ a constant,} \quad \text{or} \quad \frac{v^2}{2} = -\frac{k^2 s^2}{2} + C_1$$

When $s = 0$, $v = v_0$, so that $C_1 = v_0{}^2/2$, and the velocity becomes

$$v = +\sqrt{v_0{}^2 - k^2 s^2}$$

The plus sign of the radical is taken when v is positive (in the plus s-direction). This last expression may be integrated by substituting $v = ds/dt$. Thus,

② $$\int \frac{ds}{\sqrt{v_0{}^2 - k^2 s^2}} = \int dt + C_2 \text{ a constant,} \quad \text{or} \quad \frac{1}{k}\sin^{-1}\frac{ks}{v_0} = t + C_2$$

With the requirement of $t = 0$ when $s = 0$, the constant of integration becomes $C_2 = 0$, and we may solve the equation for s so that

$$s = \frac{v_0}{k}\sin kt \qquad \textit{Ans.}$$

The velocity is $v = \dot{s}$, which gives

$$v = v_0 \cos kt \qquad \textit{Ans.}$$

Solution II. Since $a = \ddot{s}$, the given relation may be written at once as

$$\ddot{s} + k^2 s = 0$$

This is an ordinary linear differential equation of second order for which the solution is well known and is

$$s = A \sin Kt + B \cos Kt$$

where A, B, and K are constants. Substitution of this expression into the differential equation shows that it satisfies the equation, provided that $K = k$. The velocity is $v = \dot{s}$, which becomes

$$v = Ak \cos kt - Bk \sin kt$$

The initial condition $v = v_0$ when $t = 0$ requires that $A = v_0/k$, and the condition $s = 0$ when $t = 0$ gives $B = 0$. Thus, the solution is

③ $$s = \frac{v_0}{k}\sin kt \quad \text{and} \quad v = v_0 \cos kt \qquad \textit{Ans.}$$

Helpful Hints

① We have used an indefinite integral here and evaluated the constant of integration. For practice, obtain the same results by using the definite integral with the appropriate limits.

② Again try the definite integral here as above.

③ This motion is called *simple harmonic motion* and is characteristic of all oscillations where the restoring force, and hence the acceleration, is proportional to the displacement but opposite in sign.

Sample Problem 2/4

A freighter is moving at a speed of 8 knots when its engines are suddenly ① stopped. If it takes 10 minutes for the freighter to reduce its speed to 4 knots, determine and plot the distance s in nautical miles moved by the ship and its speed v in knots as functions of the time t during this interval. The deceleration of the ship is proportional to the square of its speed, so that $a = -kv^2$.

Solution. The speeds and the time are given, so we may substitute the expression for acceleration directly into the basic definition $a = dv/dt$ and integrate. Thus,

$$-kv^2 = \frac{dv}{dt} \qquad \frac{dv}{v^2} = -k\,dt \qquad \int_8^v \frac{dv}{v^2} = -k\int_0^t dt$$

② $$-\frac{1}{v} + \frac{1}{8} = -kt \qquad v = \frac{8}{1 + 8kt}$$

Now we substitute the end limits of $v = 4$ knots and $t = \frac{10}{60} = \frac{1}{6}$ hour and get

$$4 = \frac{8}{1 + 8k(1/6)} \qquad k = \frac{3}{4}\text{ mi}^{-1} \qquad v = \frac{8}{1 + 6t} \qquad \textit{Ans.}$$

The speed is plotted against the time as shown.

The distance is obtained by substituting the expression for v into the definition $v = ds/dt$ and integrating. Thus,

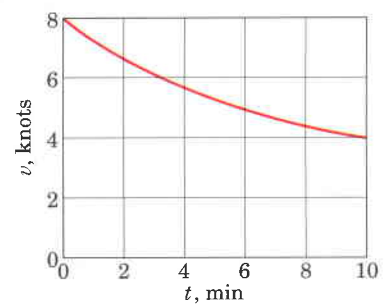
$$\frac{8}{1 + 6t} = \frac{ds}{dt} \qquad \int_0^t \frac{8\,dt}{1 + 6t} = \int_0^s ds \qquad s = \frac{4}{3}\ln(1 + 6t) \qquad \textit{Ans.}$$

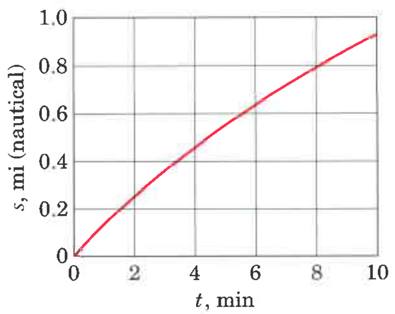
The distance s is also plotted against the time as shown, and we see that the ship has moved through a distance $s = \frac{4}{3}\ln(1 + \frac{6}{6}) = \frac{4}{3}\ln 2 = 0.924$ mi (nautical) during the 10 minutes.

Helpful Hints

① Recall that one knot is the speed of one nautical mile (6076 ft) per hour. Work directly in the units of nautical miles and hours.

② We choose to integrate to a general value of v and its corresponding time t so that we may obtain the variation of v with t.

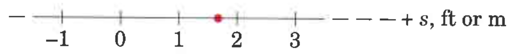




PROBLEMS

Introductory Problems

Problems 2/1 through 2/7 treat the motion of a particle which moves along the s-axis shown in the figure.



Problems 2/1–2/7

2/1 The velocity of a particle is given by $v = 25t^2 - 80t - 200$, where v is in feet per second and t is in seconds. Plot the velocity v and acceleration a versus time for the first 6 seconds of motion and evaluate the velocity when a is zero.

Ans. $v = -264$ ft/sec

2/2 The position of a particle is given by $s = 2t^3 - 40t^2 + 200t - 50$, where s is in meters and t is in seconds. Plot the position, velocity, and acceleration as functions of time for the first 12 seconds of motion. Determine the time at which the velocity is zero.

2/3 The velocity of a particle which moves along the s-axis is given by $v = 2 - 4t + 5t^{3/2}$, where t is in seconds and v is in meters per second. Evaluate the position s, velocity v, and acceleration a when $t = 3$ s. The particle is at the position $s_0 = 3$ m when $t = 0$.

Ans. $s = 22.2$ m, $v = 15.98$ m/s, $a = 8.99$ m/s^2

2/4 The displacement of a particle which moves along the s-axis is given by $s = (-2 + 3t)e^{-0.5t}$, where s is in meters and t is in seconds. Plot the displacement, velocity, and acceleration versus time for the first 20 seconds of motion. Determine the time at which the acceleration is zero.

2/5 The acceleration of a particle is given by $a = 2t - 10$, where a is in meters per second squared and t is in seconds. Determine the velocity and displacement as functions of time. The initial displacement at $t = 0$ is $s_0 = -4$ m, and the initial velocity is $v_0 = 3$ m/s.

Ans. $v = 3 - 10t + t^2$ (m/s)
$s = -4 + 3t - 5t^2 + \frac{1}{3}t^3$ (m)

2/6 The acceleration of a particle is given by $a = -ks^2$, where a is in meters per second squared, k is a constant, and s is in meters. Determine the velocity of the particle as a function of its position s. Evaluate your expression for $s = 5$ m if $k = 0.1$ m^{-1}s^{-2} and the initial conditions at time $t = 0$ are $s_0 = 3$ m and $v_0 = 10$ m/s.

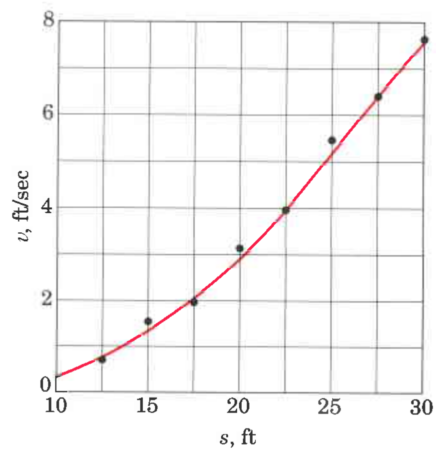
2/7 The acceleration of a particle which is moving along a straight line is given by $a = -k\sqrt{v}$, where a is in meters per second squared, k is a constant, and v is the velocity in meters per second. Determine the velocity as a function of both time t and position s. Evaluate your expressions for $t = 2$ s and at $s = 3$ m if $k = 0.2$ m$^{1/2}$s$^{-3/2}$ and the initial conditions at time $t = 0$ are $s_0 = 1$ m and $v_0 = 7$ m/s.

Ans. $v = (v_0{}^{1/2} - \frac{1}{2}kt)^2$, $v = [v_0{}^{3/2} - \frac{3}{2}k(s - s_0)]^{2/3}$
$v = 5.98$ m/s at $t = 2$ s, $v = 6.85$ m/s at $s = 3$ m

2/8 The velocity of a particle moving in a straight line is decreasing at the rate of 3 m/s per meter of displacement at an instant when the velocity is 10 m/s. Determine the acceleration a of the particle at this instant.

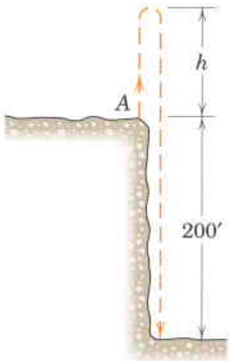
2/9 Experimental data for the motion of a particle along a straight line yield measured values of the velocity v for various position coordinates s. A smooth curve is drawn through the points as shown in the graph. Determine the acceleration of the particle when $s = 20$ ft.

Ans. $a = 1.2$ ft/sec^2



Problem 2/9

2/10 A ball is thrown vertically up with a velocity of 80 ft/sec at the edge of a 200-ft cliff. Calculate the height h to which the ball rises and the total time t after release for the ball to reach the bottom of the cliff. Neglect air resistance and take the downward acceleration to be 32.2 ft/sec^2.



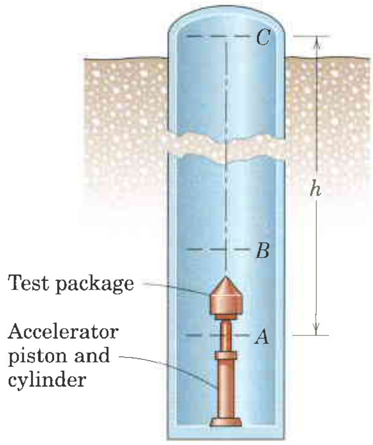
Problem 2/10

2/11 A rocket is fired vertically up from rest. If it is designed to maintain a constant upward acceleration of $1.5g$, calculate the time t required for it to reach an altitude of 30 km and its velocity at that position.
Ans. $t = 63.9$ s, $v = 940$ m/s

2/12 A car comes to a complete stop from an initial speed of 50 mi/hr in a distance of 100 ft. With the same constant acceleration, what would be the stopping distance s from an initial speed of 70 mi/hr?

2/13 Calculate the constant acceleration a in g's which the catapult of an aircraft carrier must provide to produce a launch velocity of 180 mi/hr in a distance of 300 ft. Assume that the carrier is at anchor.
Ans. $a = 3.61g$

2/14 To test the effects of "weightlessness" for short periods of time, a test facility is designed which accelerates a test package vertically up from A to B by means of a gas-activated piston and allows it to ascend and descend from B to C to B under free-fall conditions. The test chamber consists of a deep well and is evacuated to eliminate any appreciable air resistance. If a constant acceleration of $40g$ from A to B is provided by the piston and if the total test time for the "weightless" condition from B to C to B is 10 s, calculate the required working height h of the chamber. Upon returning to B, the test package is recovered in a basket filled with polystyrene pellets inserted in the line of fall.



"Zero-g" test facility

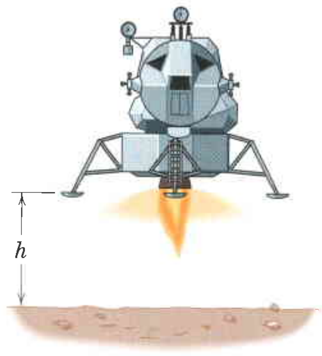
Problem 2/14

2/15 The pilot of a jet transport brings the engines to full takeoff power before releasing the brakes as the aircraft is standing on the runway. The jet thrust remains constant, and the aircraft has a near-constant acceleration of $0.4g$. If the takeoff speed is 200 km/h, calculate the distance s and time t from rest to takeoff.
Ans. $s = 393$ m, $t = 14.16$ s

2/16 A jet aircraft with a landing speed of 200 km/h has a maximum of 600 m of available runway after touchdown in which to reduce its ground speed to 30 km/h. Compute the average acceleration a required of the aircraft during braking.

2/17 A particle traveling in a straight line encounters a retarding force which causes its velocity to decrease according to $v = 20e^{-t/10}$ ft/sec, where t is the time in seconds during which the force acts. Determine the acceleration a of the particle when $t = 10$ sec and find the corresponding distance s which the particle has moved during the 10-second interval. Plot v as a function of t for the first 10 seconds.
Ans. $a = -0.736$ ft/sec^2, $s = 126.4$ ft

2/18 In the final stages of a moon landing, the lunar module descends under retrothrust of its descent engine to within $h = 5$ m of the lunar surface where it has a downward velocity of 2 m/s. If the descent engine is cut off abruptly at this point, compute the impact velocity of the landing gear with the moon. Lunar gravity is $\frac{1}{6}$ of the earth's gravity.



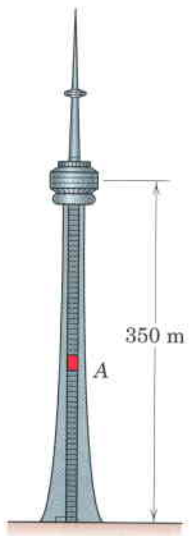
Problem 2/18

2/19 A particle moves along the s-direction with constant acceleration. The displacement, measured from a convenient position, is 2 m at time $t = 0$ and is zero when $t = 10$ s. If the velocity of the particle is momentarily zero when $t = 6$ s, determine the acceleration a and the velocity v when $t = 10$ s.

Ans. $a = 0.2\ \text{m/s}^2$, $v = 0.8$ m/s

Representative Problems

2/20 The main elevator A of the CN Tower in Toronto rises about 350 m and for most of its run has a constant speed of 22 km/h. Assume that both the acceleration and deceleration have a constant magnitude of $\frac{1}{4}g$ and determine the time duration t of the elevator run.

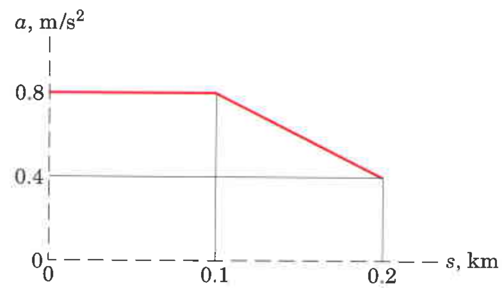


Problem 2/20

2/21 A particle oscillates along a straight line with a sinusoidally varying velocity in millimeters per second given by $v = 16 \sin \pi t/6$, where t is in seconds. If the displacement of the particle is 8 mm when $t = 0$, determine its maximum displacement s_{max} and plot s versus t for one complete cycle.

Ans. $s_{max} = 69.1$ mm

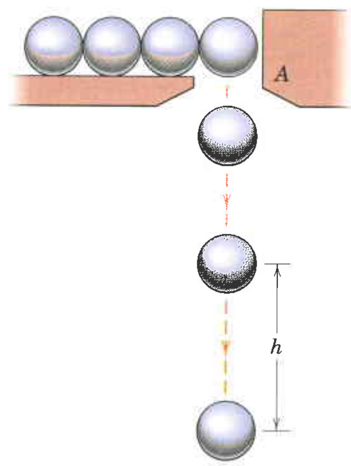
2/22 A vehicle enters a test section of straight road at $s = 0$ with a speed of 40 km/h. It then undergoes an acceleration which varies with displacement as shown. Determine the velocity v of the vehicle as it passes the position $s = 0.2$ km.



Problem 2/22

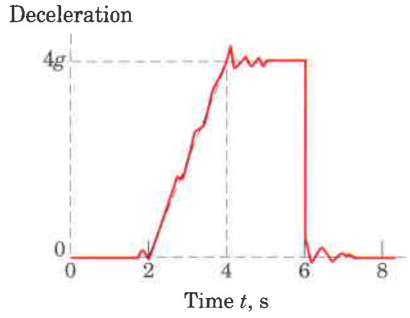
2/23 Small steel balls fall from rest through the opening at A at the steady rate of two per second. Find the vertical separation h of two consecutive balls when the lower one has dropped 3 meters. Neglect air resistance.

Ans. $h = 2.61$ m



Problem 2/23

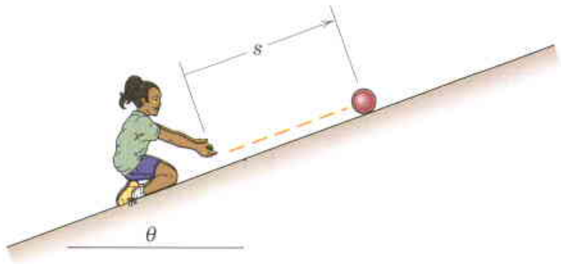
2/24 A retarding force acts on a particle moving initially with a velocity of 100 m/s and gives it a deceleration as recorded by the oscilloscope record shown. Approximate the velocity of the particle at $t = 4$ s and at $t = 8$ s.



Problem 2/24

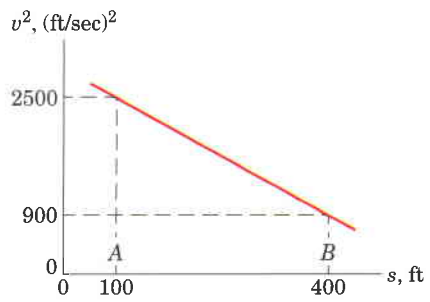
2/25 A girl rolls a ball up an incline and allows it to return to her. For the angle θ and ball involved, the acceleration of the ball along the incline is constant at $0.25g$, directed down the incline. If the ball is released with a speed of 4 m/s, determine the distance s it moves up the incline before reversing its direction and the total time t required for the ball to return to the child's hand.

Ans. $s = 3.26$ m, $t = 3.26$ s



Problem 2/25

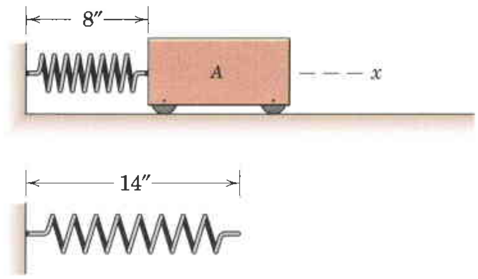
2/26 A body moves in a straight line with a velocity whose square decreases linearly with the displacement between two points A and B, which are 300 ft apart as shown. Determine the displacement Δs of the body during the last 2 seconds before arrival at B.



Problem 2/26

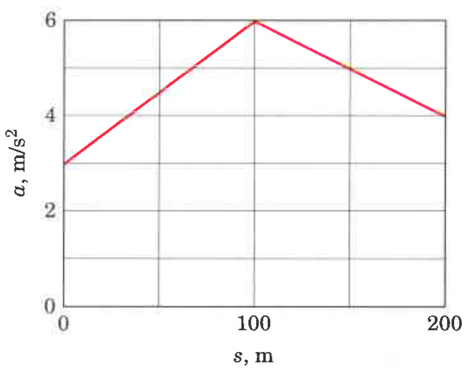
2/27 The 14-in. spring is compressed to an 8-in. length, where it is released from rest and accelerates the sliding block A. The acceleration has an initial value of 400 ft/sec^2 and then decreases linearly with the x-movement of the block, reaching zero when the spring regains its original 14-in. length. Calculate the time t for the block to go (a) 3 in. and (b) 6 in.

Ans. (a) $t = 0.0370$ sec, (b) $t = 0.0555$ sec



Problem 2/27

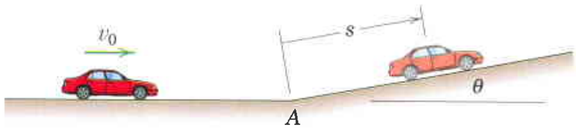
2/28 A motorcycle starts from rest with an initial acceleration of 3 m/s^2, and the acceleration then changes with distance s as shown. Determine the velocity v of the motorcycle when $s = 200$ m. At this point also determine the value of the derivative $\dfrac{dv}{ds}$.



Problem 2/28

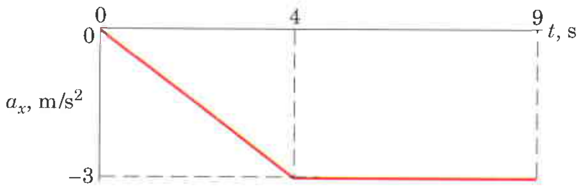
2/29 The car is traveling at a constant speed $v_0 = 100$ km/h on the level portion of the road. When the 6-percent ($\tan\theta = 6/100$) incline is encountered, the driver does not change the throttle setting and consequently the car decelerates at the constant rate $g \sin\theta$. Determine the speed of the car (*a*) 10 seconds after passing point *A* and (*b*) when $s = 100$ m.

Ans. (*a*) $v = 21.9$ m/s, (*b*) $v = 25.6$ m/s



Problem 2/29

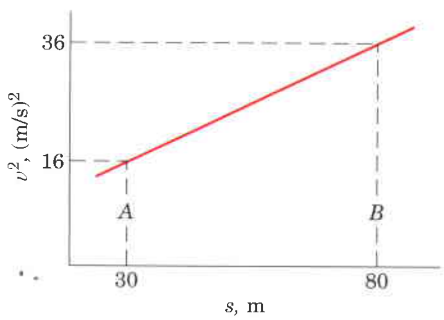
2/30 A particle moving along the positive *x*-direction with an initial velocity of 12 m/s is subjected to a retarding force that gives it a negative acceleration which varies linearly with time for the first 4 seconds as shown. For the next 5 seconds the force is constant and the acceleration remains constant. Plot the velocity of the particle during the 9 seconds and specify its value at $t = 4$ s. Also find the distance Δx traveled by the particle from its position at $t = 0$ to the point where it reverses its direction.



Problem 2/30

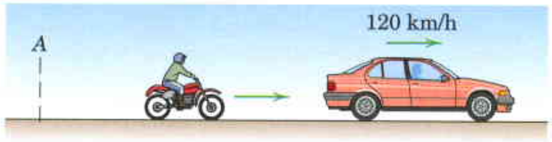
2/31 A body which moves in a straight line between two points *A* and *B* a distance of 50 m apart has a velocity whose square increases linearly with the distance traveled, as shown on the graph. Determine the displacement Δs of the body during the last 2 seconds before arrival at *B*.

Ans. $\Delta s = 11.6$ m



Problem 2/31

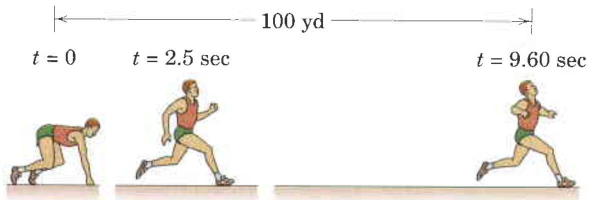
2/32 A motorcycle patrolman starts from rest at *A* two seconds after a car, speeding at the constant rate of 120 km/h, passes point *A*. If the patrolman accelerates at the rate of 6 m/s^2 until he reaches his maximum permissible speed of 150 km/h, which he maintains, calculate the distance *s* from point *A* to the point at which he overtakes the car.



Problem 2/32

2/33 A sprinter reaches his maximum speed v_{max} in 2.5 seconds from rest with constant acceleration. He then maintains that speed and finishes the 100 yards in the overall time of 9.60 seconds. Determine his maximum speed v_{max}.

Ans. $v_{max} = 35.9$ ft/sec



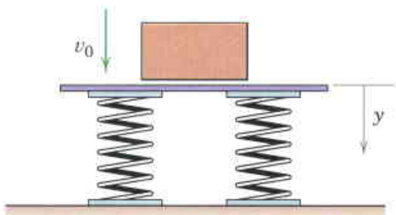
Problem 2/33

2/34 A vacuum-propelled capsule for a high-speed tube transportation system of the future is being designed for operation between two stations *A* and *B*, which are 10 km apart. If the acceleration and deceleration are to have a limiting magnitude of $0.6g$ and if velocities are to be limited to 400 km/h, determine the minimum time *t* for the capsule to make the 10-km trip.

Problem 2/34

2/35 The body falling with speed v_0 strikes and maintains contact with the platform supported by a nest of springs. The acceleration of the body after impact is $a = g - cy$, where c is a positive constant and y is measured from the original platform position. If the maximum compression of the springs is observed to be y_m, determine the constant c.

$$Ans.\ c = \frac{v_0^2 + 2gy_m}{y_m^2}$$



Problem 2/35

2/36 Particle 1 is subjected to an acceleration $a = -kv$, particle 2 is subjected to $a = -kt$, and particle 3 is subjected to $a = -ks$. All three particles start at the origin $s = 0$ with an initial velocity $v_0 = 10$ m/s at time $t = 0$, and the magnitude of k is 0.1 for all three particles (note that the units of k vary from case to case). Plot the position, velocity, and acceleration versus time for each particle over the range $0 \le t \le 10$ s.

2/37 A self-propelled vehicle of mass m whose engine delivers constant power P has an acceleration $a = P/(mv)$ where all frictional resistance is neglected. Determine expressions for the distance s traveled and the corresponding time t required by the vehicle to increase its speed from v_1 to v_2.

$$Ans.\ s = \frac{m}{3P}(v_2^3 - v_1^3)$$

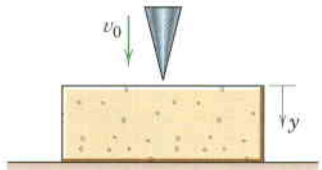
$$t = \frac{m}{2P}(v_2^2 - v_1^2)$$

2/38 A certain lake is proposed as a landing area for large jet aircraft. The touchdown speed of 100 mi/hr upon contact with the water is to be reduced to 20 mi/hr in a distance of 1500 ft. If the deceleration is proportional to the square of the velocity of the aircraft through the water, $a = -Kv^2$, find the value of the design parameter K, which would be a measure of the size and shape of the landing gear vanes that plow through the water. Also find the time t elapsed during the specified interval.

2/39 A particle moving along a straight line decelerates according to $a = -kv$, where k is a constant and v is velocity. If its initial velocity at time $t = 0$ is $v_0 = 4$ m/s and its velocity at time $t = 2$ s is $v = 1$ m/s, determine the time T and corresponding distance D for the particle speed to be reduced to one-tenth of its initial value.

Ans. $T = 3.32$ s, $D = 5.19$ m

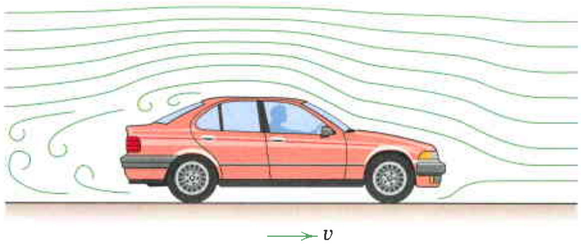
2/40 The cone falling with a speed v_0 strikes and penetrates the block of packing material. The acceleration of the cone after impact is $a = g - cy^2$, where c is a positive constant and y is the penetration distance. If the maximum penetration depth is observed to be y_m, determine the constant c.



Problem 2/40

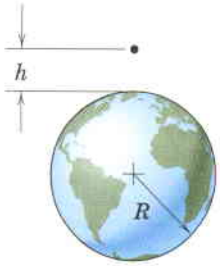
2/41 The aerodynamic resistance to motion of a car is nearly proportional to the square of its velocity. Additional frictional resistance is constant, so that the acceleration of the car when coasting may be written $a = -C_1 - C_2v^2$, where C_1 and C_2 are constants which depend on the mechanical configuration of the car. If the car has an initial velocity v_0 when the engine is disengaged, derive an expression for the distance D required for the car to coast to a stop.

$$Ans.\ D = \frac{1}{2C_2}\ln\left(1 + \frac{C_2}{C_1}v_0^2\right)$$



Problem 2/41

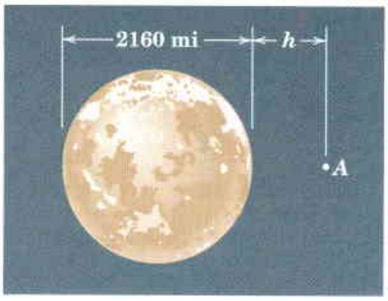
2/42 Compute the impact speed of a body released from rest at an altitude $h = 500$ mi. (*a*) Assume a constant gravitational acceleration $g_0 = 32.2$ ft/sec^2 and (*b*) account for the variation of g with altitude (refer to Art. 1/5). Neglect the effects of atmospheric drag.



Problem 2/42

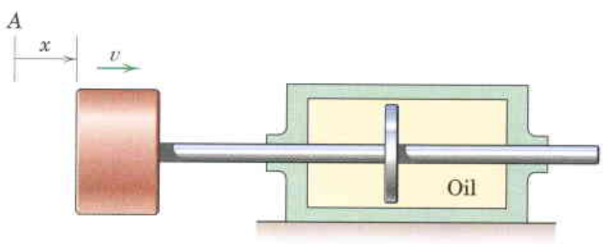
2/43 Compute the impact speed of body A which is released from rest at an altitude $h = 750$ mi above the surface of the moon. (a) First assume a constant gravitational acceleration $g_{m_0} = 5.32$ ft/sec^2 and (b) then account for the variation of g_m with altitude (refer to Art. 1/5).

Ans. (a) $v = 6490$ ft/sec, (b) $v = 4990$ ft/sec



Problem 2/43

2/44 The horizontal motion of the plunger and shaft is arrested by the resistance of the attached disk which moves through the oil bath. If the velocity of the plunger is v_0 in the position A where $x = 0$ and $t = 0$, and if the deceleration is proportional to v so that $a = -kv$, derive expressions for the velocity v and position coordinate x in terms of the time t. Also express v in terms of x.



Problem 2/44

2/45 A small object is released from rest in a tank of oil. The downward acceleration of the object is $g - kv$, where g is the constant acceleration due to gravity, k is a constant which depends on the viscosity of the oil and shape of the object, and v is the downward velocity of the object. Derive expressions for the velocity v and vertical drop y as functions of the time t after release.

$$\textit{Ans. } v = \frac{g}{k}(1 - e^{-kt})$$

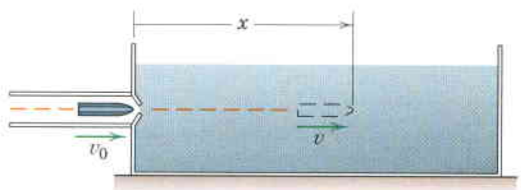
$$y = \frac{g}{k}\left[t - \frac{1}{k}(1 - e^{-kt})\right]$$

2/46 On its takeoff roll, the airplane starts from rest and accelerates according to $a = a_0 - kv^2$, where a_0 is the constant acceleration resulting from the engine thrust and $-kv^2$ is the acceleration due to aerodynamic drag. If $a_0 = 2$ m/s^2, $k = 0.00004$ m^{-1}, and v is in meters per second, determine the design length of runway required for the airplane to reach the takeoff speed of 250 km/h if the drag term is (a) excluded and (b) included.

Problem 2/46

2/47 A test projectile is fired horizontally into a viscous liquid with a velocity of v_0. The retarding force is proportional to the square of the velocity, so that the acceleration becomes $a = -kv^2$. Derive expressions for the distance D traveled in the liquid and the corresponding time t required to reduce the velocity to $v_0/2$. Neglect any vertical motion.

Ans. $D = 0.693/k$, $t = \dfrac{1}{kv_0}$

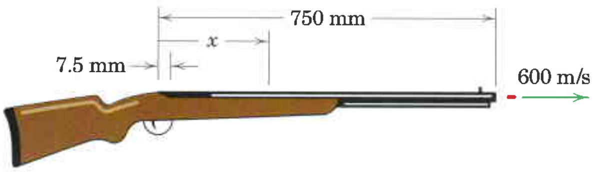


Problem 2/47

2/48 A car starts from rest and accelerates at a constant rate until it reaches 60 mi/hr in a distance of 200 ft, at which time the clutch is disengaged. The car then slows down to a velocity of 30 mi/hr in an additional distance of 400 ft with a deceleration which is proportional to its velocity. Find the time t for the car to travel the 600 ft.

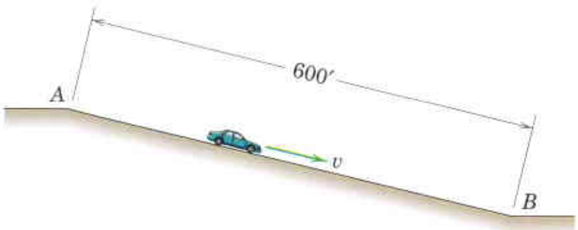
2/49 To a close approximation the pressure behind a rifle bullet varies inversely with the position x of the bullet along the barrel. Thus the acceleration of the bullet may be written as $a = k/x$ where k is a constant. If the bullet starts from rest at x = 7.5 mm and if the muzzle velocity of the bullet is 600 m/s at the end of the 750-mm barrel, compute the acceleration of the bullet as it passes the midpoint of the barrel at x = 375 mm.

Ans. a = 104.2 km/s^2



Problem 2/49

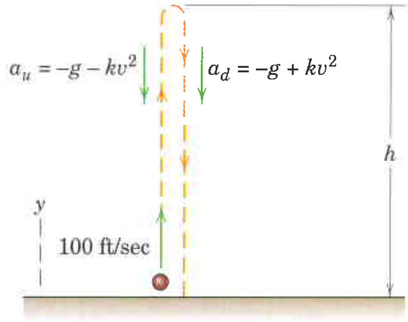
2/50 The driver of a car, which is initially at rest at the top A of the grade, releases the brakes and coasts down the grade with an acceleration in feet per second squared given by $a = 3.22 - 0.004v^2$, where v is the velocity in feet per second. Determine the velocity v_B at the bottom B of the grade.



Problem 2/50

2/51 When the effect of aerodynamic drag is included, the y-acceleration of a baseball moving vertically upward is $a_u = -g - kv^2$, while the acceleration when the ball is moving downward is $a_d = -g + kv^2$, where k is a positive constant and v is the speed in feet per second. If the ball is thrown upward at 100 ft/sec from essentially ground level, compute its maximum height h and its speed v_f upon impact with the ground. Take k to be 0.002 ft^{-1} and assume that g is constant.

Ans. h = 120.8 ft, v_f = 78.5 ft/sec

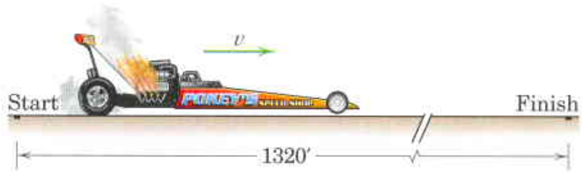


Problem 2/51

2/52 For the baseball of Prob. 2/51 thrown upward with an initial speed of 100 ft/sec, determine the time t_u from ground to apex and the time t_d from apex to ground.

2/53 The acceleration of the drag racer is modeled as $a = c_1 - c_2v^2$, where the v^2-term accounts for aerodynamic drag and where c_1 and c_2 are positive constants. If c_2 is known (from wind-tunnel tests) to be $5(10^{-5})$ ft^{-1}, determine c_1 if the final speed is 190 mi/hr. A drag race is a 1/4-mile straight run from a standing start.

Ans. c_1 = 31.4 ft/sec^2

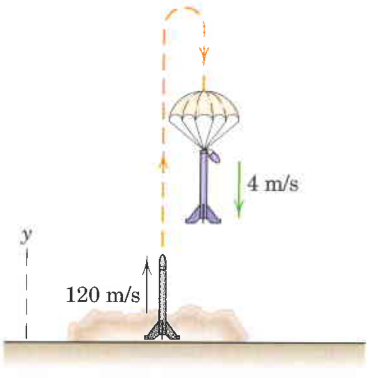


Problem 2/53

2/54 Use the value for c_1 cited in the answer to Prob. 2/53 and determine the time t required for the drag racer described in that problem to complete the 1/4-mile run.

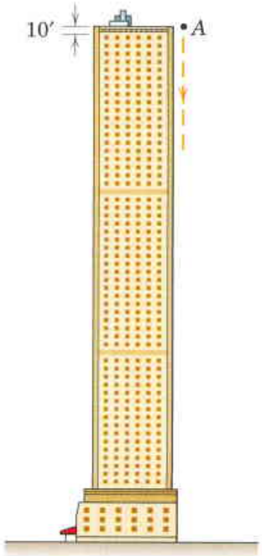
2/55 The fuel of a model rocket is burned so quickly that one may assume that the rocket acquires its burnout velocity of 120 m/s while essentially still at ground level. The rocket then coasts vertically upward to the trajectory apex. With the inclusion of aerodynamic drag, the y-acceleration is $a_y = -g - 0.0005v^2$ during this motion, where the units are meters and seconds. At apex a parachute pops out of the nose cone, and the rocket quickly acquires a constant downward speed of 4 m/s. Estimate the flight time t.

Ans. t = 147.7 s



Problem 2/55

2/56 The stories of a tall building are uniformly 10 feet in height. A ball A is dropped from the rooftop position shown. Determine the times required for it to pass the 10 feet of the first, tenth, and one-hundredth stories (counted from the top). Neglect aerodynamic drag.



Problem 2/56

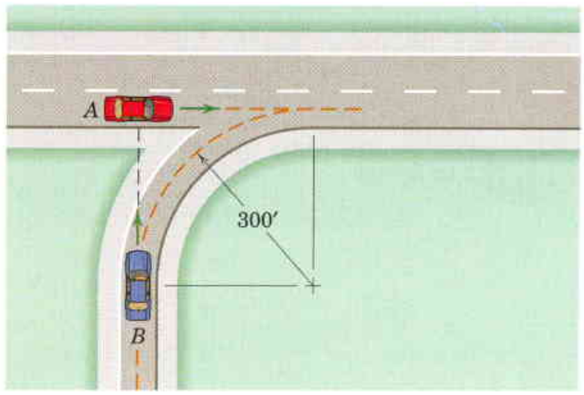
2/57 Repeat Prob. 2/56, except now include the effects of aerodynamic drag. The drag force causes an acceleration component in ft/sec^2 of $0.005v^2$ in the direction opposite the velocity vector, where v is in ft/sec.

Ans. $t_1 = 0.795$ sec, $t_{10} = 0.1592$ sec
$t_{100} = 0.1246$ sec

2/58 A particle which moves along the x-axis is subjected to an accelerating force which increases linearly with time and a retarding force which increases directly with displacement. The resulting acceleration is $a = Kt - k^2x$, where K and k are positive constants and where both x and $v = \dot{x}$ are zero when the time $t = 0$. Determine the displacement x as a function of t.

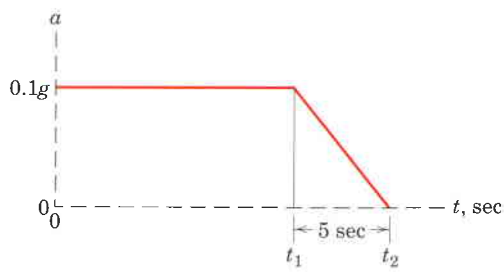
2/59 Car A travels at a constant speed of 65 mi/hr. When in the position shown at time $t = 0$, car B has a speed of 25 mi/hr and accelerates at a constant rate of $0.1g$ along its path until it reaches a speed of 65 mi/hr, after which it travels at that constant speed. What is the steady-state position of car A with respect to car B?

Ans. A is ahead of B by 706 ft



Problem 2/59

2/60 Repeat Prob. 2/59, except that car B, rather than possessing a constant acceleration, now accelerates as shown in the accompanying plot. Time t_2 is the time at which the speed of car B reaches 65 mi/hr. After time t_2, the speed remains constant. Compare your result with that stated for Prob. 2/59.



Problem 2/60

2/3 Plane Curvilinear Motion

We now treat the motion of a particle along a curved path which lies in a single plane. This motion is a special case of the more general three-dimensional motion introduced in Art. 2/1 and illustrated in Fig. 2/1. If we let the plane of motion be the x-y plane, for instance, then the coordinates z and ϕ of Fig. 2/1 are both zero, and R becomes the same as r. As mentioned previously, the vast majority of the motions of points or particles encountered in engineering practice can be represented as plane motion.

Before pursuing the description of plane curvilinear motion in any specific set of coordinates, we will first use vector analysis to describe the motion, since the results will be independent of any particular coordinate system. What follows in this article constitutes one of the most basic concepts in dynamics, namely, the *time derivative of a vector*. Much analysis in dynamics utilizes the time rates of change of vector quantities. You are therefore well advised to master this topic at the outset because you will have frequent occasion to use it.

Consider now the continuous motion of a particle along a plane curve as represented in Fig. 2/5. At time t the particle is at position A, which is located by the *position vector* $\mathbf{r}$ measured from some convenient fixed origin O. If both the magnitude and direction of $\mathbf{r}$ are known at time t, then the position of the particle is completely specified. At time $t + \Delta t$, the particle is at A', located by the position vector $\mathbf{r} + \Delta\mathbf{r}$. We note, of course, that this combination is vector addition and not scalar addition. The *displacement* of the particle during time Δt is the vector $\Delta\mathbf{r}$ which represents the vector change of position and is clearly independent of the choice of origin. If an origin were chosen at some different location, the position vector $\mathbf{r}$ would be changed, but $\Delta\mathbf{r}$ would be unchanged. The *distance* actually traveled by the particle as it moves along the path from A to A' is the scalar length Δs measured along the path. Thus, we distinguish between the vector displacement $\Delta\mathbf{r}$ and the scalar distance Δs.

Velocity

The *average velocity* of the particle between A and A' is defined as $\mathbf{v}_{av} = \Delta\mathbf{r}/\Delta t$, which is a vector whose direction is that of $\Delta\mathbf{r}$ and whose magnitude is the magnitude of $\Delta\mathbf{r}$ divided by Δt. The average speed of

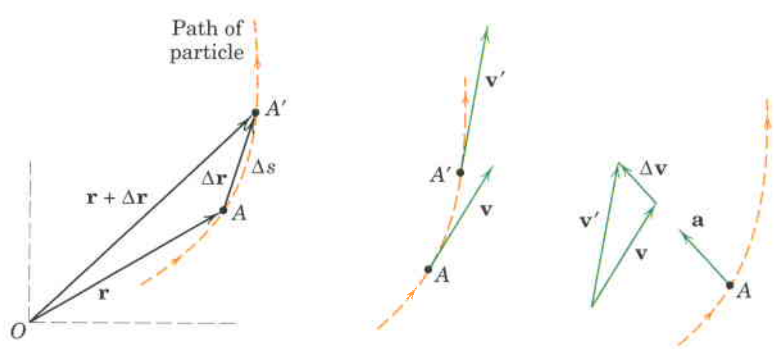


Figure 2/5

the particle between A and A' is the scalar quotient $\Delta s/\Delta t$. Clearly, the magnitude of the average velocity and the speed approach one another as the interval Δt decreases and A and A' become closer together.

The *instantaneous velocity* $\mathbf{v}$ of the particle is defined as the limiting value of the average velocity as the time interval approaches zero. Thus,

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t}$$

We observe that the direction of $\Delta\mathbf{r}$ approaches that of the tangent to the path as Δt approaches zero and, thus, the velocity $\mathbf{v}$ is always a vector tangent to the path.

We now extend the basic definition of the derivative of a scalar quantity to include a vector quantity and write

$$\boxed{\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}} \tag{2/4}$$

The derivative of a vector is itself a vector having both a magnitude and a direction. The magnitude of $\mathbf{v}$ is called the *speed* and is the scalar

$$v = |\mathbf{v}| = \frac{ds}{dt} = \dot{s}$$

At this point we make a careful distinction between the *magnitude of the derivative* and the *derivative of the magnitude*. The magnitude of the derivative can be written in any one of the several ways $|d\mathbf{r}/dt| = |\dot{\mathbf{r}}| = \dot{s} = |\mathbf{v}| = v$ and represents the magnitude of the velocity, or the speed, of the particle. On the other hand, the derivative of the magnitude is written $d|\mathbf{r}|/dt = dr/dt = \dot{r}$, and represents the rate at which the length of the position vector $\mathbf{r}$ is changing. Thus, these two derivatives have two entirely different meanings, and we must be extremely careful to distinguish between them in our thinking and in our notation. For this and other reasons, you are urged to adopt a consistent notation for handwritten work for all vector quantities to distinguish them from scalar quantities. For simplicity the underline $\underline{v}$ is recommended. Other handwritten symbols such as $\vec{v}$, $\underset{\sim}{v}$, and $\hat{v}$ are sometimes used.

With the concept of velocity as a vector established, we return to Fig. 2/5 and denote the velocity of the particle at A by the tangent vector $\mathbf{v}$ and the velocity at A' by the tangent $\mathbf{v}'$. Clearly, there is a vector change in the velocity during the time Δt. The velocity $\mathbf{v}$ at A plus (vectorially) the change $\Delta\mathbf{v}$ must equal the velocity at A', so we can write $\mathbf{v}' - \mathbf{v} = \Delta\mathbf{v}$. Inspection of the vector diagram shows that $\Delta\mathbf{v}$ depends both on the change in magnitude (length) of $\mathbf{v}$ and on the change in direction of $\mathbf{v}$. These two changes are fundamental characteristics of the derivative of a vector.

Acceleration

The *average acceleration* of the particle between A and A' is defined as $\Delta\mathbf{v}/\Delta t$, which is a vector whose direction is that of $\Delta\mathbf{v}$. The magnitude of this average acceleration is the magnitude of $\Delta\mathbf{v}$ divided by Δt.

The *instantaneous acceleration* **a** of the particle is defined as the limiting value of the average acceleration as the time interval approaches zero. Thus,

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t}$$

By definition of the derivative, then, we write

$$\boxed{\mathbf{a} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}}} \qquad \textbf{(2/5)}$$

As the interval Δt becomes smaller and approaches zero, the direction of the change $\Delta\mathbf{v}$ approaches that of the differential change $d\mathbf{v}$ and, thus, of **a**. The acceleration **a**, then, includes the effects of both the change in magnitude of **v** and the change of direction of **v**. It is apparent, in general, that the direction of the acceleration of a particle in curvilinear motion is neither tangent to the path nor normal to the path. We do observe, however, that the acceleration component which is normal to the path points toward the center of curvature of the path.

Visualization of Motion

A further approach to the visualization of acceleration is shown in Fig. 2/6, where the position vectors to three arbitrary positions on the path of the particle are shown for illustrative purpose. There is a velocity vector tangent to the path corresponding to each position vector, and the relation is $\mathbf{v} = \dot{\mathbf{r}}$. If these velocity vectors are now plotted from some arbitrary point C, a curve, called the *hodograph*, is formed. The derivatives of these velocity vectors will be the acceleration vectors $\mathbf{a} = \dot{\mathbf{v}}$ which are tangent to the hodograph. We see that the acceleration has the same relation to the velocity as the velocity has to the position vector.

The geometric portrayal of the derivatives of the position vector **r** and velocity vector **v** in Fig. 2/5 can be used to describe the derivative of any vector quantity with respect to t or with respect to any other scalar variable. Now that we have used the definitions of velocity and acceleration to introduce the concept of the derivative of a vector, it is important to establish the rules for differentiating vector quantities. These rules

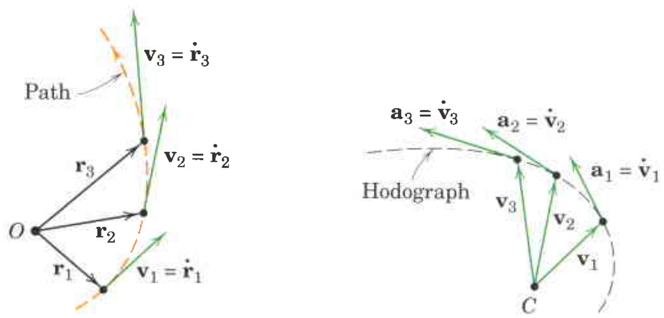


Figure 2/6

are the same as for the differentiation of scalar quantities, except for the case of the cross product where the order of the terms must be preserved. These rules are covered in Art. C/7 of Appendix C and should be reviewed at this point.

Three different coordinate systems are commonly used for describing the vector relationships for curvilinear motion of a particle in a plane: rectangular coordinates, normal and tangential coordinates, and polar coordinates. An important lesson to be learned from the study of these coordinate systems is the proper choice of a reference system for a given problem. This choice is usually revealed by the manner in which the motion is generated or by the form in which the data are specified. Each of the three coordinate systems will now be developed and illustrated.

2/4 Rectangular Coordinates (x-y)

This system of coordinates is particularly useful for describing motions where the x- and y-components of acceleration are independently generated or determined. The resulting curvilinear motion is then obtained by a vector combination of the x- and y-components of the position vector, the velocity, and the acceleration.

Vector Representation

The particle path of Fig. 2/5 is shown again in Fig. 2/7 along with x- and y-axes. The position vector $\mathbf{r}$, the velocity $\mathbf{v}$, and the acceleration $\mathbf{a}$ of the particle as developed in Art. 2/3 are represented in Fig. 2/7 together with their x- and y-components. With the aid of the unit vectors $\mathbf{i}$ and $\mathbf{j}$, we can write the vectors $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$ in terms of their x- and y-components. Thus,

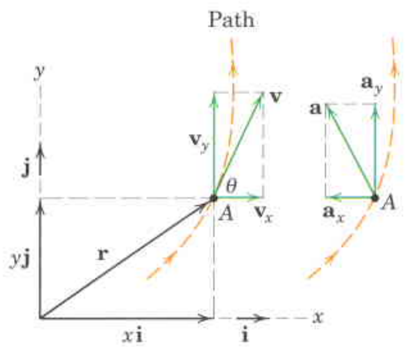


Figure 2/7

$$\boxed{\begin{aligned} \mathbf{r} &= x\mathbf{i} + y\mathbf{j} \\ \mathbf{v} = \dot{\mathbf{r}} &= \dot{x}\mathbf{i} + \dot{y}\mathbf{j} \\ \mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} &= \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} \end{aligned}} \tag{2/6}$$

As we differentiate with respect to time, we observe that the time derivatives of the unit vectors are zero because their magnitudes and directions remain constant. The scalar values of the components of $\mathbf{v}$ and $\mathbf{a}$ are merely $v_x = \dot{x}$, $v_y = \dot{y}$ and $a_x = \dot{v}_x = \ddot{x}$, $a_y = \dot{v}_y = \ddot{y}$. (As drawn in Fig. 2/7, a_x is in the negative x-direction, so that $\ddot{x}$ would be a negative number.)

As observed previously, the direction of the velocity is always tangent to the path, and from the figure it is clear that

$$v^2 = v_x{}^2 + v_y{}^2 \qquad v = \sqrt{v_x{}^2 + v_y{}^2} \qquad \tan\theta = \frac{v_y}{v_x}$$

$$a^2 = a_x{}^2 + a_y{}^2 \qquad a = \sqrt{a_x{}^2 + a_y{}^2}$$

If the angle θ is measured counterclockwise from the x-axis to $\mathbf{v}$ for the configuration of axes shown, then we can also observe that $dy/dx = \tan\theta = v_y/v_x$.

If the coordinates x and y are known independently as functions of time, $x = f_1(t)$ and $y = f_2(t)$, then for any value of the time we can combine them to obtain $\mathbf{r}$. Similarly, we combine their first derivatives $\dot{x}$ and $\dot{y}$ to obtain $\mathbf{v}$ and their second derivatives $\ddot{x}$ and $\ddot{y}$ to obtain $\mathbf{a}$. On the other hand, if the acceleration components a_x and a_y are given as functions of the time, we can integrate each one separately with respect to time, once to obtain v_x and v_y and again to obtain $x = f_1(t)$ and $y = f_2(t)$. Elimination of the time t between these last two parametric equations gives the equation of the curved path $y = f(x)$.

From the foregoing discussion we can see that the rectangular-coordinate representation of curvilinear motion is merely the superposition of the components of two simultaneous rectilinear motions in the x- and y-directions. Therefore, everything covered in Art. 2/2 on rectilinear motion can be applied separately to the x-motion and to the y-motion.

Projectile Motion

An important application of two-dimensional kinematic theory is the problem of projectile motion. For a first treatment of the subject, we neglect aerodynamic drag and the curvature and rotation of the earth, and we assume that the altitude change is small enough so that the acceleration due to gravity can be considered constant. With these assumptions, rectangular coordinates are useful for the trajectory analysis.

For the axes shown in Fig. 2/8, the acceleration components are

$$a_x = 0 \qquad a_y = -g$$

Integration of these accelerations follows the results obtained previously in Art. 2/2*a* for constant acceleration and yields

$$\begin{aligned} v_x &= (v_x)_0 & v_y &= (v_y)_0 - gt \\ x &= x_0 + (v_x)_0 t & y &= y_0 + (v_y)_0 t - \tfrac{1}{2}gt^2 \end{aligned}$$
$$v_y{}^2 = (v_y)_0{}^2 - 2g(y - y_0)$$

In all these expressions, the subscript zero denotes initial conditions, frequently taken as those at launch where, for the case illustrated,

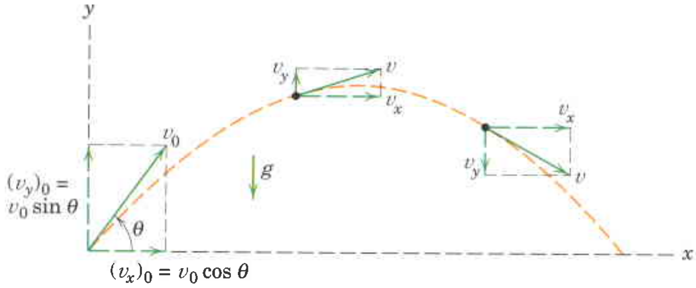
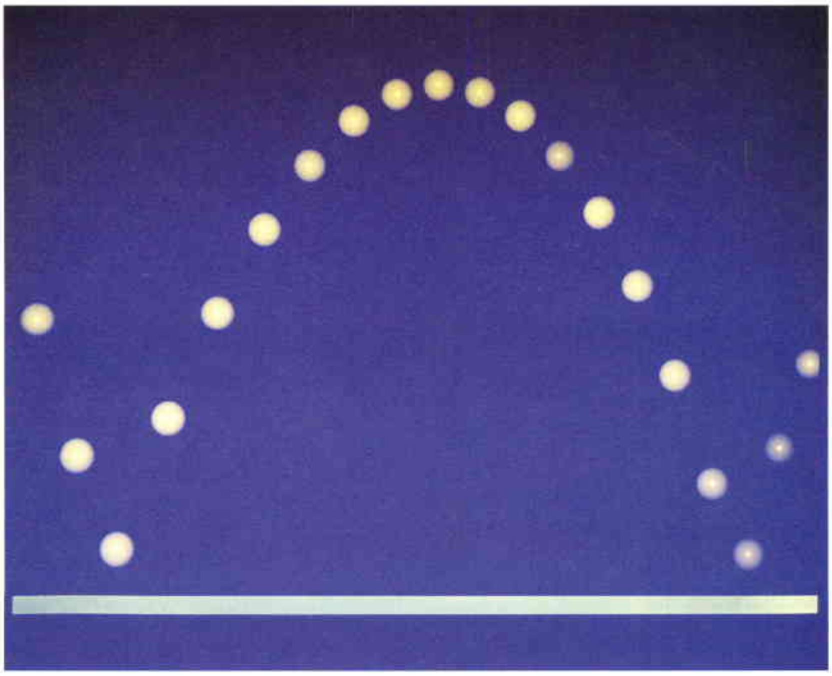


Figure 2/8

$x_0 = y_0 = 0$. Note that the quantity g is taken to be positive throughout this text.

We can see that the x- and y-motions are independent for the simple projectile conditions under consideration. Elimination of the time t between the x- and y-displacement equations shows the path to be parabolic (see Sample Problem 2/6). If we were to introduce a drag force which depends on the speed squared (for example), then the x- and y-motions would be coupled (interdependent), and the trajectory would be nonparabolic.

When the projectile motion involves large velocities and high altitudes, to obtain accurate results we must account for the shape of the projectile, the variation of g with altitude, the variation of the air density with altitude, and the rotation of the earth. These factors introduce considerable complexity into the motion equations, and numerical integration of the acceleration equations is usually necessary.

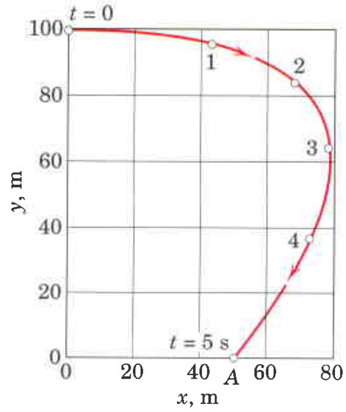


Herman Eisenbeiss/Photo Researchers, Inc.

This stroboscopic photograph of a bouncing ping-pong ball suggests not only the parabolic nature of the path, but also the fact that the speed is lower near the apex.

Sample Problem 2/5

The curvilinear motion of a particle is defined by $v_x = 50 - 16t$ and $y = 100 - 4t^2$, where v_x is in meters per second, y is in meters, and t is in seconds. It is also known that $x = 0$ when $t = 0$. Plot the path of the particle and determine its velocity and acceleration when the position $y = 0$ is reached.



Solution. The x-coordinate is obtained by integrating the expression for v_x, and the x-component of the acceleration is obtained by differentiating v_x. Thus,

$$\left[\int dx = \int v_x\, dt\right] \qquad \int_0^x dx = \int_0^t (50 - 16t)\, dt \qquad x = 50t - 8t^2 \text{ m}$$

$$[a_x = \dot{v}_x] \qquad a_x = \frac{d}{dt}(50 - 16t) \qquad a_x = -16 \text{ m/s}^2$$

The y-components of velocity and acceleration are

$$[v_y = \dot{y}] \qquad v_y = \frac{d}{dt}(100 - 4t^2) \qquad v_y = -8t \text{ m/s}$$

$$[a_y = \dot{v}_y] \qquad a_y = \frac{d}{dt}(-8t) \qquad a_y = -8 \text{ m/s}^2$$

We now calculate corresponding values of x and y for various values of t and plot x against y to obtain the path as shown.

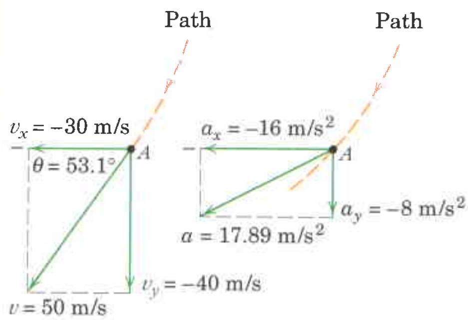
When $y = 0$, $0 = 100 - 4t^2$, so $t = 5$ s. For this value of the time, we have

$$v_x = 50 - 16(5) = -30 \text{ m/s}$$

$$v_y = -8(5) = -40 \text{ m/s}$$

$$v = \sqrt{(-30)^2 + (-40)^2} = 50 \text{ m/s}$$

$$a = \sqrt{(-16)^2 + (-8)^2} = 17.89 \text{ m/s}^2$$



The velocity and acceleration components and their resultants are shown on the separate diagrams for point A, where $y = 0$. Thus, for this condition we may write

$$\mathbf{v} = -30\mathbf{i} - 40\mathbf{j} \text{ m/s} \qquad \textit{Ans.}$$

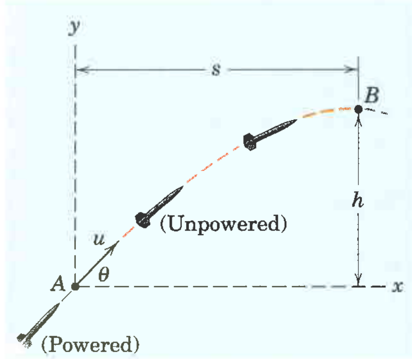
$$\mathbf{a} = -16\mathbf{i} - 8\mathbf{j} \text{ m/s}^2 \qquad \textit{Ans.}$$

Helpful Hint

We observe that the velocity vector lies along the tangent to the path as it should, but that the acceleration vector is not tangent to the path. Note especially that the acceleration vector has a component that points toward the inside of the curved path. We concluded from our diagram in Fig. 2/5 that it is impossible for the acceleration to have a component that points toward the outside of the curve.

Sample Problem 2/6

A rocket has expended all its fuel when it reaches position A, where it has a velocity of $\mathbf{u}$ at an angle θ with respect to the horizontal. It then begins unpowered flight and attains a maximum added height h at position B after traveling a horizontal distance s from A. Determine the expressions for h and s, the time t of flight from A to B, and the equation of the path. For the interval concerned, assume a flat earth with a constant gravitational acceleration g and neglect any atmospheric resistance.



Solution. Since all motion components are directly expressible in terms of horizontal and vertical coordinates, a rectangular set of axes x-y will be employed. With the neglect of atmospheric resistance, $a_x = 0$ and $a_y = -g$, and the resulting motion is a direct superposition of two rectilinear motions with constant acceleration. Thus, ①

$$[dx = v_x\, dt] \qquad x = \int_0^t u \cos\theta\, dt \qquad x = ut\cos\theta$$

$$[dv_y = a_y\, dt] \qquad \int_{u\sin\theta}^{v_y} dv_y = \int_0^t (-g)\, dt \qquad v_y = u\sin\theta - gt$$

$$[dy = v_y\, dt] \qquad y = \int_0^t (u\sin\theta - gt)\, dt \qquad y = ut\sin\theta - \tfrac{1}{2}gt^2$$

Position B is reached when $v_y = 0$, which occurs for $0 = u\sin\theta - gt$ or

$$t = (u\sin\theta)/g \qquad \textit{Ans.}$$

Substitution of this value for the time into the expression for y gives the maximum added altitude

$$h = u\left(\frac{u\sin\theta}{g}\right)\sin\theta - \frac{1}{2}g\left(\frac{u\sin\theta}{g}\right)^2 \qquad h = \frac{u^2\sin^2\theta}{2g} \qquad \textit{Ans.}$$

The horizontal distance is seen to be

② $$s = u\left(\frac{u\sin\theta}{g}\right)\cos\theta \qquad s = \frac{u^2\sin 2\theta}{2g} \qquad \textit{Ans.}$$

which is clearly a maximum when $\theta = 45°$. The equation of the path is obtained by eliminating t from the expressions for x and y, which gives

$$y = x\tan\theta - \frac{gx^2}{2u^2}\sec^2\theta \qquad \textit{Ans.}$$

③ This equation describes a vertical parabola as indicated in the figure.

Helpful Hints

① Note that this problem is simply the description of projectile motion neglecting atmospheric resistance.

② We see that the total range and time of flight for a projectile fired above a horizontal plane would be twice the respective values of s and t given here.

③ If atmospheric resistance were to be accounted for, the dependency of the acceleration components on the velocity would have to be established before an integration of the equations could be carried out. This becomes a much more difficult problem.

PROBLEMS

(In the following problems where motion as a projectile in air is involved, neglect air resistance unless otherwise stated and use $g = 9.81$ m/s^2 or $g = 32.2$ ft/sec^2.)

Introductory Problems

2/61 At time $t = 10$ s, the velocity of a particle moving in the x-y plane is $\mathbf{v} = +\,0.1\mathbf{i} + 2\mathbf{j}$ m/s. By time $t = 10.1$ s, its velocity has become $-0.1\mathbf{i} + 1.8\mathbf{j}$ m/s. Determine the magnitude a_{av} of its average acceleration during this interval and the angle θ made by the average acceleration with the positive x-axis.

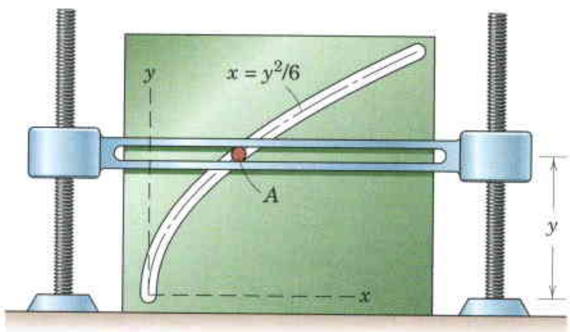
Ans. $a_{av} = 2.83$ m/s^2, $\theta = 225°$

2/62 A particle which moves with curvilinear motion has coordinates in millimeters which vary with the time t in seconds according to $x = 3t^2 - 4t$ and $y = 4t^2 - \frac{1}{3}t^3$. Determine the magnitudes of the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ and the angles which these vectors make with the x-axis when $t = 2$ s.

2/63 A particle which moves in two-dimensional motion has coordinates given in inches by $x = t^2 - 4t + 20$ and $y = 3 \sin 2t$, where the time t is in seconds. Determine the magnitudes of the velocity $\mathbf{v}$ and the acceleration $\mathbf{a}$ and the angle θ between these two vectors at time $t = 3$ sec.

Ans. $v = 6.10$ in./sec, $a = 3.90$ in./sec^2
$\theta = 11.67°$

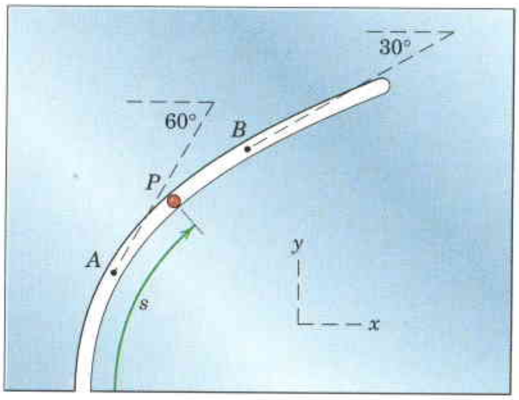
2/64 For a certain interval of motion the pin A is forced to move in the fixed parabolic slot by the horizontal slotted arm which is elevated in the y-direction at the constant rate of 3 in./sec. All measurements are in inches and seconds. Calculate the velocity v and acceleration a of pin A when $x = 6$ in.



Problem 2/64

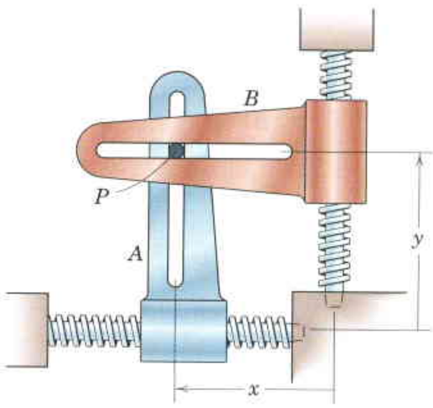
2/65 The particle P moves along the curved slot, a portion of which is shown. Its distance in meters measured along the slot is given by $s = t^2/4$, where t is in seconds. The particle is at A when $t = 2.00$ s and at B when $t = 2.20$ s. Determine the magnitude a_{av} of the average acceleration of P between A and B. Also express the acceleration as a vector $\mathbf{a}_{av}$ using unit vectors $\mathbf{i}$ and $\mathbf{j}$.

Ans. $a_{av} = 2.76$ m/s^2
$\mathbf{a}_{av} = 2.26\mathbf{i} - 1.580\mathbf{j}$ m/s^2



Problem 2/65

2/66 The x- and y-motions of guides A and B with right-angle slots control the curvilinear motion of the connecting pin P, which slides in both slots. For a short interval, the motions are governed by $x = 20 + \frac{1}{4}t^2$ and $y = 15 - \frac{1}{6}t^3$, where x and y are in millimeters and t is in seconds. Calculate the magnitudes of the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ of the pin for $t = 2$ s. Sketch the direction of the path and indicate its curvature for this instant.



Problem 2/66

2/67 The position vector of a point which moves in the x-y plane is given by

$$\mathbf{r} = \left(\frac{2}{3}t^3 - \frac{3}{2}t^2\right)\mathbf{i} + \frac{t^4}{12}\mathbf{j}$$

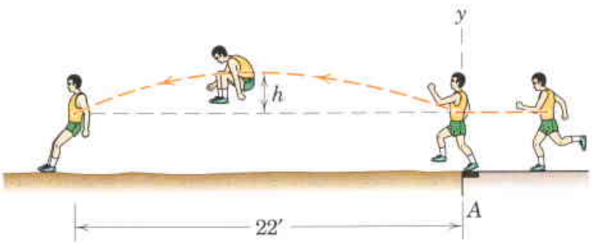
where $\mathbf{r}$ is in meters and t is in seconds. Determine the angle between the velocity $\mathbf{v}$ and the acceleration $\mathbf{a}$ when (a) $t = 2$ s and (b) $t = 3$ s.

Ans. (a) $\theta = 14.47°$, (b) $\theta = 0$

2/68 The rectangular coordinates of a particle moving in the x-y plane are given by $x = 3 \cos 4t$ and $y = 2 \sin 4t$, where the time t is in seconds and x and y are in feet. Sketch the position $\mathbf{r}$, velocity $\mathbf{v}$, and acceleration $\mathbf{a}$ at time $t = 1.4$ sec and determine the angles θ_1 between $\mathbf{v}$ and $\mathbf{a}$ and θ_2 between $\mathbf{r}$ and $\mathbf{a}$.

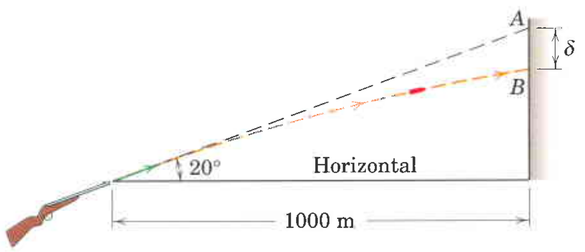
2/69 A long jumper approaches his takeoff board A with a horizontal velocity of 30 ft/sec. Determine the vertical component v_y of the velocity of his center of gravity at takeoff for him to make the jump shown. What is the vertical rise h of his center of gravity?

Ans. $v_y = 11.81$ ft/sec, $h = 2.16$ ft



Problem 2/69

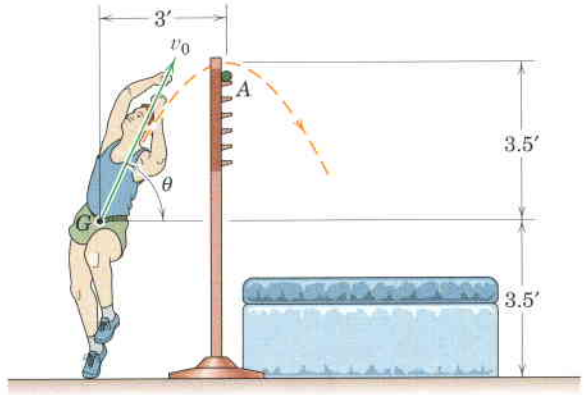
2/70 If the barrel of the rifle shown is aimed at point A, compute the distance δ below A to the point B where the bullet strikes. The muzzle velocity is 600 m/s.



Problem 2/70

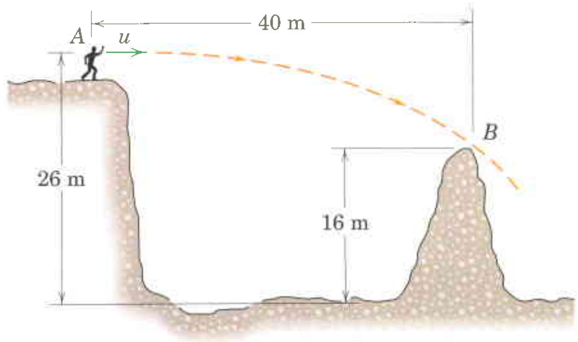
2/71 The center of mass G of a high jumper follows the trajectory shown. Determine the component v_0, measured in the vertical plane of the figure, of his takeoff velocity and angle θ if the apex of the trajectory just clears the bar at A. (In general, must the mass center G of the jumper clear the bar during a successful jump?)

Ans. $v_0 = 16.33$ ft/sec, $\theta = 66.8°$



Problem 2/71

2/72 With what minimum horizontal velocity u can a boy throw a rock at A and have it just clear the obstruction at B?



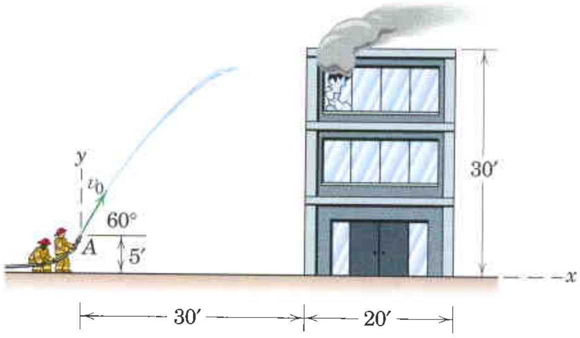
Problem 2/72

Representative Problems

2/73 Prove the well-known result that, for a given launch speed v_0, the launch angle $\theta = 45°$ yields the maximum horizontal range R. Determine the maximum range. (Note that this result does not hold when aerodynamic drag is included in the analysis.)

Ans. $R_{max} = \dfrac{v_0^2}{g}$

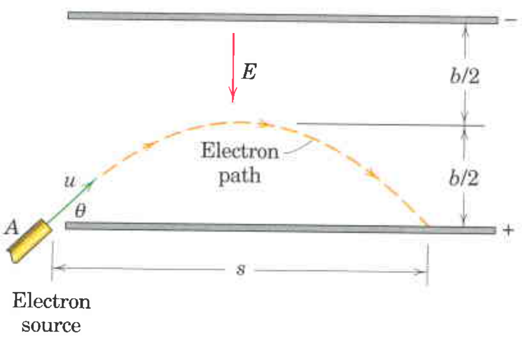
2/74 Water issues from the nozzle at A, which is 5 ft above the ground. Determine the coordinates of the point of impact of the stream if the initial water speed is (a) $v_0 = 45$ ft/sec and (b) $v_0 = 60$ ft/sec.



Problem 2/74

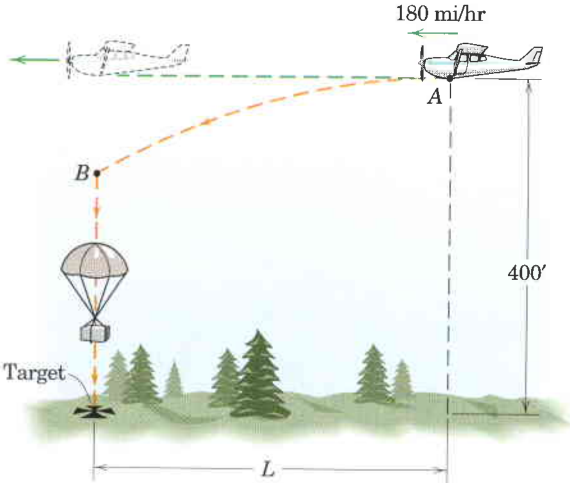
2/75 Electrons are emitted at A with a velocity u at the angle θ into the space between two charged plates. The electric field between the plates is in the direction E and repels the electrons approaching the upper plate. The field produces an acceleration of the electrons in the E-direction of eE/m, where e is the electron charge and m is its mass. Determine the field strength E which will permit the electrons to cross one-half of the gap between the plates. Also find the distance s.

Ans. $E = \dfrac{mu^2 \sin^2\theta}{eb}$, $s = 2b \cot\theta$



Problem 2/75

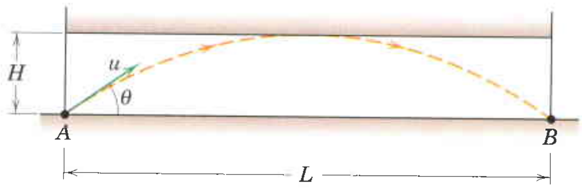
2/76 A small airplane flying horizontally with a speed of 180 mi/hr at an altitude of 400 ft above a remote valley drops an emergency medical package at A. The package has a parachute which deploys at B and allows the package to descend vertically at the constant rate of 6 ft/sec. If the drop is designed so that the package is to reach the ground 37 seconds after release at A, determine the horizontal lead L so that the package hits the target. Neglect atmospheric resistance from A to B.



Problem 2/76

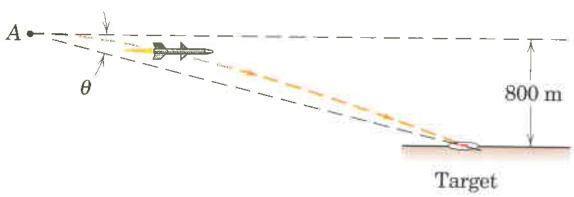
2/77 A projectile is fired with a velocity $\mathbf{u}$ at the entrance A to a horizontal tunnel of length L and height H. Determine the minimum value of u and the corresponding value of the angle θ for which the projectile will reach B at the other end of the tunnel without touching the top of the tunnel.

Ans. $u = \sqrt{2gH}\sqrt{1 + (\frac{L}{4H})^2}$, $\theta = \tan^{-1}(4H/L)$



Problem 2/77

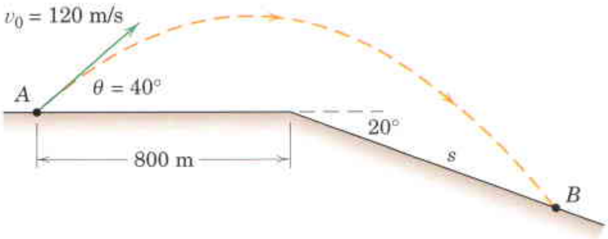
2/78 A rocket is released at point A from a jet aircraft flying horizontally at 1000 km/h at an altitude of 800 m. If the rocket thrust remains horizontal and gives the rocket a horizontal acceleration of $0.5g$, determine the angle θ from the horizontal to the line of sight to the target.



Problem 2/78

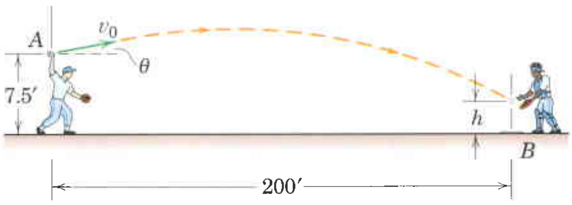
2/79 A projectile is launched from point A with the initial conditions shown in the figure. Determine the slant distance s which locates the point B of impact. Calculate the time of flight t.

Ans. $s = 1057$ m, $t = 19.50$ s



Problem 2/79

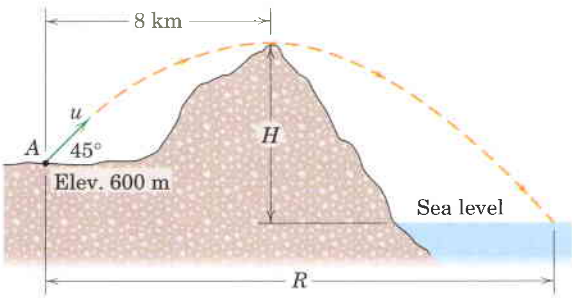
2/80 An outfielder experiments with two different trajectories for throwing to home plate from the position shown: (*a*) $v_0 = 140$ ft/sec with $\theta = 8°$ and (*b*) $v_0 = 120$ ft/sec with $\theta = 12°$. For each set of initial conditions, determine the time t required for the baseball to reach home plate and the altitude h as the ball crosses the plate.



Problem 2/80

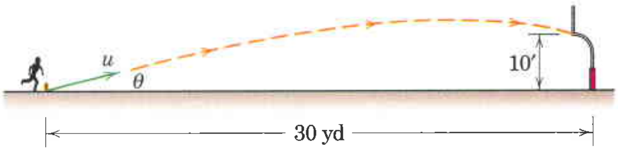
2/81 A long-range artillery rifle at A is aimed at an angle of 45° with the horizontal, and its shell is just able to clear the mountain peak at the top of its trajectory. Determine the magnitude u of the muzzle velocity, the height H of the mountain above sea level, and the range R to the sea.

Ans. $u = 396$ m/s, $H = 4600$ m, $R = 16.58$ km



Problem 2/81

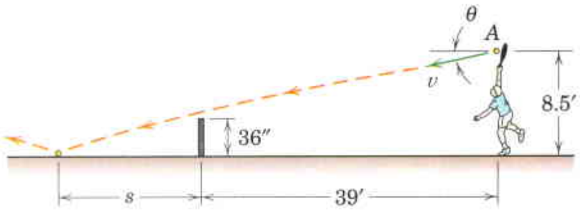
2/82 A football player attempts a 30-yd field goal. If he is able to impart a velocity u of 100 ft/sec to the ball, compute the minimum angle θ for which the ball will clear the crossbar of the goal. (*Hint:* Let $m = \tan\theta$.)



Problem 2/82

2/83 If the tennis player serves the ball horizontally ($\theta = 0$), calculate its velocity v if the center of the ball clears the 36-in. net by 6 in. Also find the distance s from the net to the point where the ball hits the court surface. Neglect air resistance and the effect of ball spin.

Ans. $v = 70.0$ ft/sec, $s = 11.85$ ft

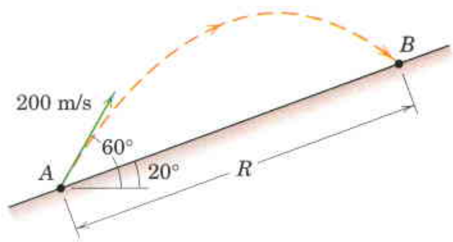


Problem 2/83

2/84 If the tennis player shown in Prob. 2/83 serves the ball with a velocity v of 80 mi/hr at the angle $\theta = 5°$, calculate the vertical clearance h of the center of the ball above the net and the distance s from the net where the ball hits the court surface. Neglect air resistance and the effect of ball spin.

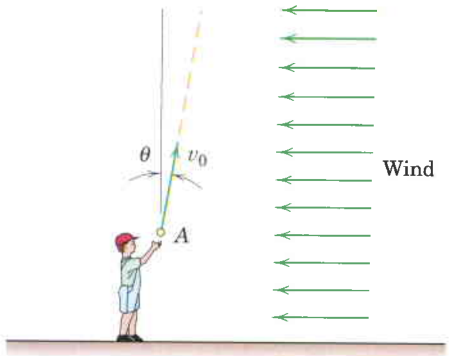
2/85 A projectile is launched with an initial speed of 200 m/s at an angle of 60° with respect to the horizontal. Compute the range R as measured up the incline.

Ans. $R = 2970$ m



Problem 2/85

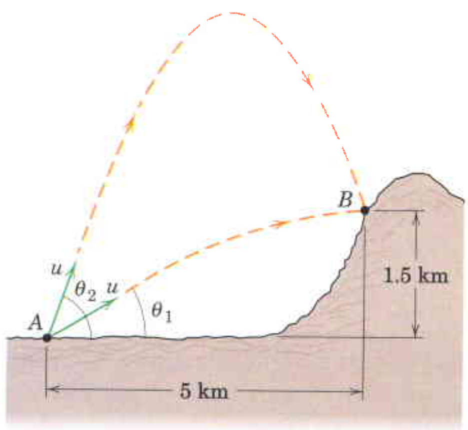
2/86 A boy throws a ball upward with a speed $v_0 = 12$ m/s. The wind imparts a horizontal acceleration of 0.4 m/s^2 to the left. At what angle θ must the ball be thrown so that it returns to the point of release? Assume that the wind does not affect the vertical motion.



Problem 2/86

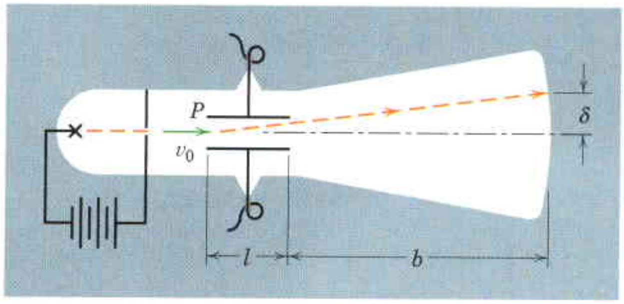
2/87 The muzzle velocity of a long-range rifle at A is $u = 400$ m/s. Determine the two angles of elevation θ which will permit the projectile to hit the mountain target B.

Ans. $\theta_1 = 26.1°$, $\theta_2 = 80.6°$



Problem 2/87

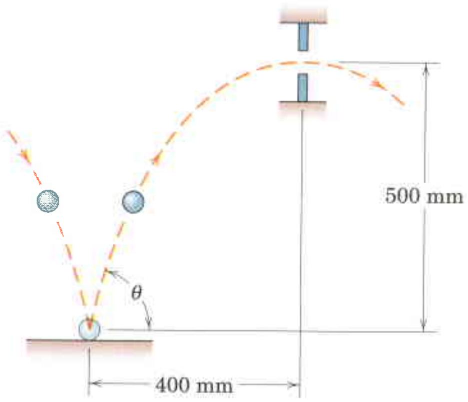
2/88 In the cathode-ray tube, electrons traveling horizontally from their source with the velocity v_0 are deflected by an electric field E due to the voltage gradient across the plates P. The deflecting force causes an acceleration in the vertical direction on the sketch equal to eE/m, where e is the electron charge and m is its mass. When clear of the plates, the electrons travel in straight lines. Determine the expression for the deflection δ for the tube and plate dimensions shown.



Problem 2/88

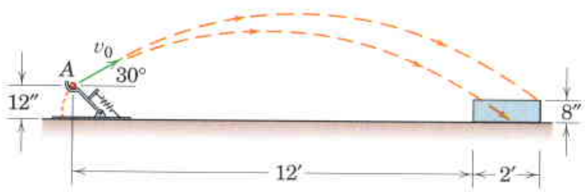
2/89 To meet design criteria, small ball bearings must bounce through an opening of limited size at the top of their trajectory when rebounding from a heavy plate as shown. Calculate the angle θ made by the rebound velocity with the horizontal and the velocity v of the balls as they pass through the opening.

Ans. $\theta = 68.2°$, $v = 1.253$ m/s



Problem 2/89

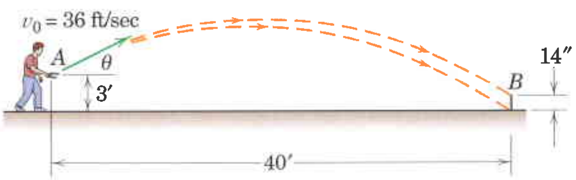
2/90 A team of engineering students is designing a catapult to launch a small ball at A so that it lands in the box. If it is known that the initial velocity vector makes a 30° angle with the horizontal, determine the range of launch speeds v_0 for which the ball will land inside the box.



Problem 2/90

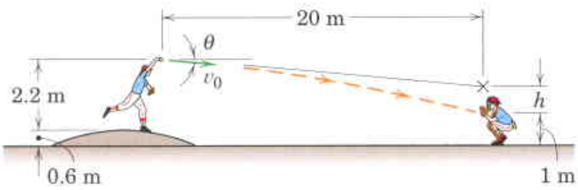
2/91 A horseshoe player releases the horseshoe at A with an initial speed $v_0 = 36$ ft/sec. Determine the range for the launch angle θ for which the shoe will strike the 14-in. vertical stake.

Ans. $31.0° \leq \theta \leq 34.3°$
or $53.1° \leq \theta \leq 54.7°$



Problem 2/91

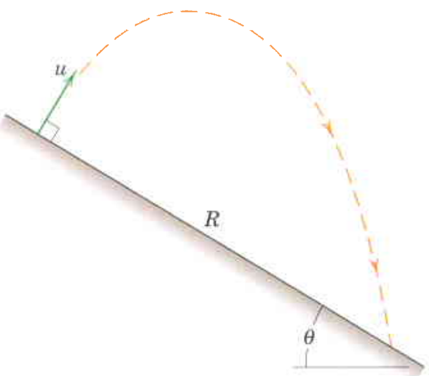
2/92 Determine the location h of the spot toward which the pitcher must throw if the ball is to hit the catcher's mitt. The ball is released with a speed of 40 m/s.



Problem 2/92

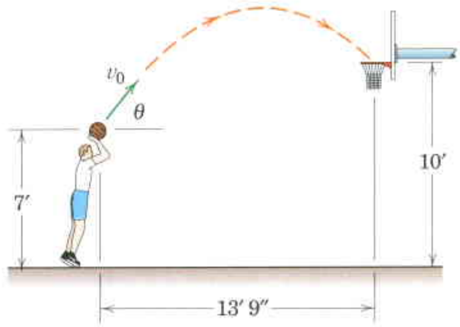
2/93 A projectile is fired with a velocity u at right angles to the slope, which is inclined at an angle θ with the horizontal. Derive an expression for the distance R to the point of impact.

$$Ans.\ R = \frac{2u^2}{g} \tan\theta \sec\theta$$



Problem 2/93

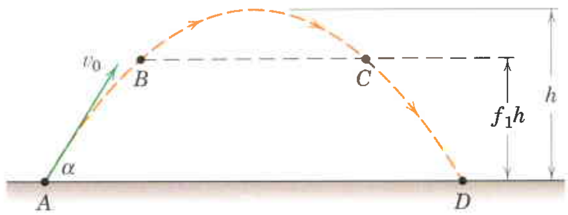
2/94 The basketball player likes to release his foul shots at an angle $\theta = 50°$ to the horizontal as shown. What initial speed v_0 will cause the ball to pass through the center of the rim?



Problem 2/94

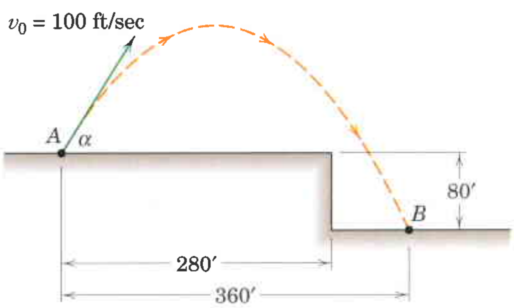
2/95 A projectile is launched from point A and lands on the same level at D. Its maximum altitude is h. Determine and plot the fraction f_2 of the total flight time that the projectile is above the level f_1h, where f_1 is a fraction which can vary from zero to 1. State the value of f_2 for $f_1 = \frac{3}{4}$.

Ans. $f_2 = \sqrt{1 - f_1}$, $f_2 = \frac{1}{2}$



Problem 2/95

2/96 A projectile is launched from point A with an initial speed $v_0 = 100$ ft/sec. Determine the minimum value of the launch angle α for which the projectile will land at point B.



Problem 2/96

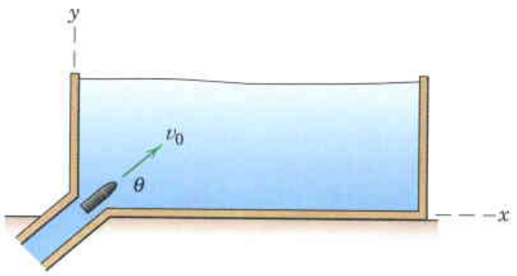
▶**2/97** A projectile is ejected into an experimental fluid at time $t = 0$. The initial speed is v_0 and the angle to the horizontal is θ. The drag on the projectile results in an acceleration term $\mathbf{a}_D = -k\mathbf{v}$, where k is a constant and $\mathbf{v}$ is the velocity of the projectile. Determine the x- and y-components of both the velocity and displacement as functions of time. What is the terminal velocity? Include the effects of gravitational acceleration.

Ans. $v_x = (v_0 \cos\theta)e^{-kt}$, $x = \dfrac{v_0 \cos\theta}{k}(1 - e^{-kt})$

$$v_y = \left(v_0 \sin\theta + \frac{g}{k}\right)e^{-kt} - \frac{g}{k}$$

$$y = \frac{1}{k}\left(v_0 \sin\theta + \frac{g}{k}\right)(1 - e^{-kt}) - \frac{g}{k}t$$

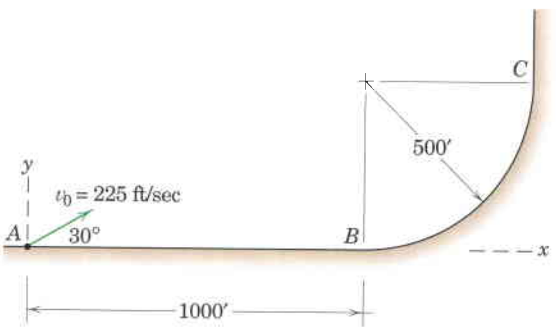
$$v_x \to 0,\ v_y \to -\frac{g}{k}$$



Problem 2/97

▶**2/98** A projectile is launched from point A with the initial conditions shown in the figure. Determine the x- and y-coordinates of the point of impact.

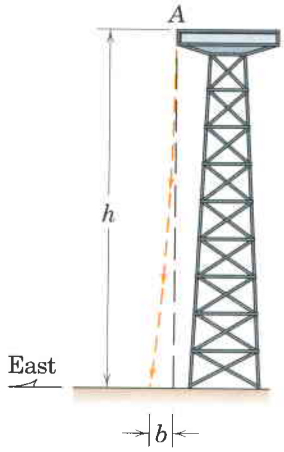
Ans. $x = 1242$ ft, $y = 62.7$ ft



Problem 2/98

▶**2/99** An object which is released from rest from the top A of a tower of height h will appear not to fall straight down due to the effect of the earth's rotation. It may be shown that the object has an eastward horizontal acceleration relative to the horizontal surface of the earth equal to $2v_y\omega \cos\gamma$, where v_y is the free-fall downward velocity, ω is the angular velocity of the earth, and γ is the latitude, north or south. Determine the deflection b if $h = 1000$ ft and $\gamma = 30°$ north. From Table D/3, $\omega = 0.7292(10^{-4})$ rad/sec and from Fig. 1/1, $g = 32.13$ ft/sec^2.

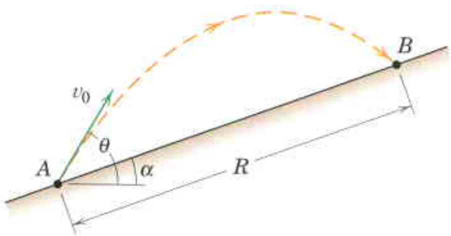
Ans. $b = 3.99$ in.



Problem 2/99

▶**2/100** A projectile is launched with speed v_0 from point A. Determine the launch angle θ which results in the maximum range R up the incline of angle α (where $0 \le \alpha \le 90°$). Evaluate your results for $\alpha = 0$, 30°, and 45°.

Ans. $\theta = \dfrac{90° + \alpha}{2}$, $\theta = 45°, 60°, 67.5°$



Problem 2/100

2/5 Normal and Tangential Coordinates (n-t)

As we mentioned in Art. 2/1, one of the common descriptions of curvilinear motion uses path variables, which are measurements made along the tangent t and normal n to the path of the particle. These coordinates provide a very natural description for curvilinear motion and are frequently the most direct and convenient coordinates to use. The n- and t-coordinates are considered to move along the path with the particle, as seen in Fig. 2/9 where the particle advances from A to B to C. The positive direction for n at any position is always taken toward the center of curvature of the path. As seen from Fig. 2/9, the positive n-direction will shift from one side of the curve to the other side if the curvature changes direction.

Figure 2/9

Velocity and Acceleration

We now use the coordinates n and t to describe the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ which were introduced in Art. 2/3 for the curvilinear motion of a particle. For this purpose, we introduce unit vectors $\mathbf{e}_n$ in the n-direction and $\mathbf{e}_t$ in the t-direction, as shown in Fig. 2/10a for the position of the particle at point A on its path. During a differential increment of time dt, the particle moves a differential distance ds along the curve from A to A'. With the radius of curvature of the path at this position designated by ρ, we see that $ds = \rho\, d\beta$, where β is in radians. It is unnecessary to consider the differential change in ρ between A and A' because a higher-order term would be introduced which disappears in the limit. Thus, the magnitude of the velocity can be written $v = ds/dt = \rho\, d\beta/dt$, and we can write the velocity as the vector

$$\boxed{\mathbf{v} = v\mathbf{e}_t = \rho\dot{\beta}\mathbf{e}_t} \qquad \textbf{(2/7)}$$

The acceleration $\mathbf{a}$ of the particle was defined in Art. 2/3 as $\mathbf{a} = d\mathbf{v}/dt$, and we observed from Fig. 2/5 that the acceleration is a vector which reflects both the change in magnitude and the change in direction of $\mathbf{v}$. We now differentiate $\mathbf{v}$ in Eq. 2/7 by applying the ordinary rule for the differentiation of the product of a scalar and a vector* and get

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d(v\mathbf{e}_t)}{dt} = v\dot{\mathbf{e}}_t + \dot{v}\mathbf{e}_t \qquad \textbf{(2/8)}$$

where the unit vector $\mathbf{e}_t$ now has a nonzero derivative because its direction changes.

To find $\dot{\mathbf{e}}_t$ we analyze the change in $\mathbf{e}_t$ during a differential increment of motion as the particle moves from A to A' in Fig. 2/10a. The unit vector $\mathbf{e}_t$ correspondingly changes to $\mathbf{e}_t'$, and the vector difference $d\mathbf{e}_t$ is shown in part b of the figure. The vector $d\mathbf{e}_t$ in the limit has a magnitude equal to the length of the arc $|\mathbf{e}_t|\, d\beta = d\beta$ obtained by swinging the unit vector $\mathbf{e}_t$ through the angle $d\beta$ expressed in radians.

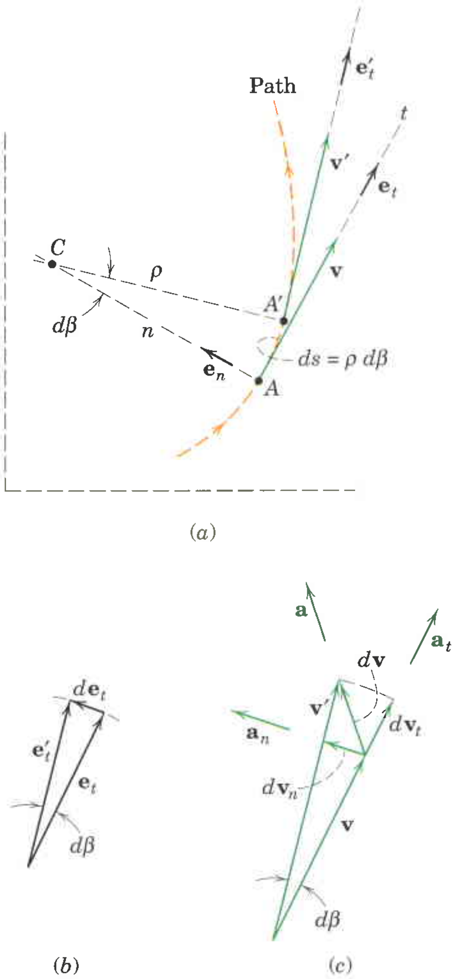


Figure 2/10

*See Art. C/7 of Appendix C.

The direction of $d\mathbf{e}_t$ is given by $\mathbf{e}_n$. Thus, we can write $d\mathbf{e}_t = \mathbf{e}_n\, d\beta$. Dividing by $d\beta$ gives

$$\frac{d\mathbf{e}_t}{d\beta} = \mathbf{e}_n$$

Dividing by dt gives $d\mathbf{e}_t/dt = (d\beta/dt)\mathbf{e}_n$, which can be written

$$\boxed{\dot{\mathbf{e}}_t = \dot{\beta}\mathbf{e}_n} \qquad \textbf{(2/9)}$$

With the substitution of Eq. 2/9 and $\dot{\beta}$ from the relation $v = \rho\dot{\beta}$, Eq. 2/8 for the acceleration becomes

$$\boxed{\mathbf{a} = \frac{v^2}{\rho}\mathbf{e}_n + \dot{v}\mathbf{e}_t} \qquad \textbf{(2/10)}$$

where

$$a_n = \frac{v^2}{\rho} = \rho\dot{\beta}^2 = v\dot{\beta}$$

$$a_t = \dot{v} = \ddot{s}$$

$$a = \sqrt{{a_n}^2 + {a_t}^2}$$

We may also note that $a_t = \dot{v} = d(\rho\dot{\beta})/dt = \rho\ddot{\beta} + \dot{\rho}\dot{\beta}$. This relation, however, finds little use because we seldom have reason to compute $\dot{\rho}$.

Geometric Interpretation

Full understanding of Eq. 2/10 comes only when we clearly see the geometry of the physical changes it describes. Figure 2/10*c* shows the velocity vector $\mathbf{v}$ when the particle is at A and $\mathbf{v}'$ when it is at A'. The vector change in the velocity is $d\mathbf{v}$, which establishes the direction of the acceleration $\mathbf{a}$. The n-component of $d\mathbf{v}$ is labeled $d\mathbf{v}_n$, and in the limit its magnitude equals the length of the arc generated by swinging the vector $\mathbf{v}$ as a radius through the angle $d\beta$. Thus, $|d\mathbf{v}_n| = v\, d\beta$ and the n-component of acceleration is $a_n = |d\mathbf{v}_n|/dt = v(d\beta/dt) = v\dot{\beta}$ as before. The t-component of $d\mathbf{v}$ is labeled $d\mathbf{v}_t$, and its magnitude is simply the change dv in the magnitude or length of the velocity vector. Therefore, the t-component of acceleration is $a_t = dv/dt = \dot{v} = \ddot{s}$ as before. The acceleration vectors resulting from the corresponding vector changes in velocity are shown in Fig. 2/10*c*.

It is especially important to observe that the normal component of acceleration a_n is *always directed toward the center of curvature C*. The tangential component of acceleration, on the other hand, will be in the positive t-direction of motion if the speed v is increasing and in the negative t-direction if the speed is decreasing. In Fig. 2/11 are shown schematic representations of the variation in the acceleration vector for a particle moving from A to B with (a) increasing speed and (b) decreasing speed. At an inflection point on the curve, the normal acceleration v^2/ρ goes to zero because ρ becomes infinite.

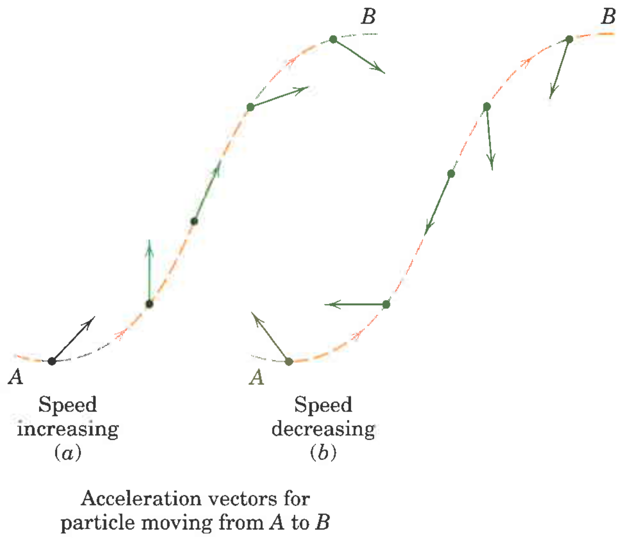


Figure 2/11

Circular Motion

Circular motion is an important special case of plane curvilinear motion where the radius of curvature ρ becomes the constant radius r of the circle and the angle β is replaced by the angle θ measured from any convenient radial reference to OP, Fig. 2/12. The velocity and the acceleration components for the circular motion of the particle P become

$$\boxed{\begin{aligned} v &= r\dot{\theta} \\ a_n &= v^2/r = r\dot{\theta}^2 = v\dot{\theta} \\ a_t &= \dot{v} = r\ddot{\theta} \end{aligned}} \qquad \textbf{(2/11)}$$

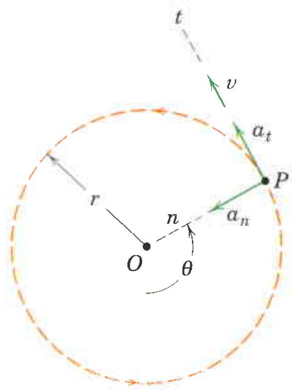


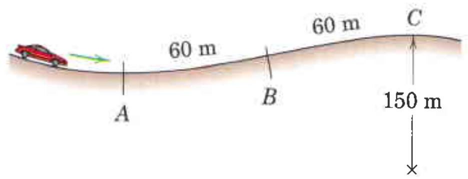
Figure 2/12

We find repeated use for Eqs. 2/10 and 2/11 in dynamics, so these relations and the principles behind them should be mastered.

An example of uniform circular motion is this car moving with constant speed around a skidpad, which is a circular roadway with a diameter of about 200 feet.

Sample Problem 2/7

To anticipate the dip and hump in the road, the driver of a car applies her brakes to produce a uniform deceleration. Her speed is 100 km/h at the bottom A of the dip and 50 km/h at the top C of the hump, which is 120 m along the road from A. If the passengers experience a total acceleration of 3 m/s^2 at A and if the radius of curvature of the hump at C is 150 m, calculate (*a*) the radius of curvature ρ at A, (*b*) the acceleration at the inflection point B, and (*c*) the total acceleration at C.



Solution. The dimensions of the car are small compared with those of the ① path, so we will treat the car as a particle. The velocities are

$$v_A = \left(100\,\frac{\text{km}}{\text{h}}\right)\left(\frac{1\text{ h}}{3600\text{ s}}\right)\left(1000\,\frac{\text{m}}{\text{km}}\right) = 27.8 \text{ m/s}$$

$$v_C = 50\,\frac{1000}{3600} = 13.89 \text{ m/s}$$

We find the constant deceleration along the path from

$$\left[\int v\,dv = \int a_t\,ds\right] \qquad \int_{v_A}^{v_C} v\,dv = a_t\int_0^s ds$$

$$a_t = \frac{1}{2s}(v_C{}^2 - v_A{}^2) = \frac{(13.89)^2 - (27.8)^2}{2(120)} = -2.41 \text{ m/s}^2$$

(a) Condition at A. With the total acceleration given and a_t determined, we can easily compute a_n and hence ρ from

$[a^2 = a_n{}^2 + a_t{}^2]$ $\qquad a_n{}^2 = 3^2 - (2.41)^2 = 3.19 \qquad a_n = 1.785 \text{ m/s}^2$

$[a_n = v^2/\rho]$ $\qquad \rho = v^2/a_n = (27.8)^2/1.785 = 432 \text{ m}$ *Ans.*

(b) Condition at B. Since the radius of curvature is infinite at the inflection point, $a_n = 0$ and

$$a = a_t = -2.41 \text{ m/s}^2$$ *Ans.*

(c) Condition at C. The normal acceleration becomes

$[a_n = v^2/\rho]$ $\qquad a_n = (13.89)^2/150 = 1.286 \text{ m/s}^2$

With unit vectors $\mathbf{e}_n$ and $\mathbf{e}_t$ in the n- and t-directions, the acceleration may be written

$$\mathbf{a} = 1.286\mathbf{e}_n - 2.41\mathbf{e}_t \text{ m/s}^2$$

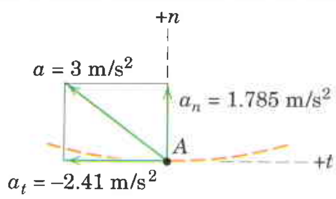
where the magnitude of $\mathbf{a}$ is

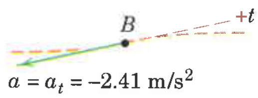
$[a = \sqrt{a_n{}^2 + a_t{}^2}]$ $\qquad a = \sqrt{(1.286)^2 + (-2.41)^2} = 2.73 \text{ m/s}^2$ *Ans.*

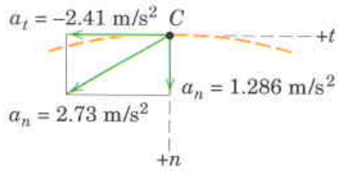
The acceleration vectors representing the conditions at each of the three points are shown for clarification.

Helpful Hint

① Actually, the radius of curvature to the road differs by about 1 m from that to the path followed by the center of mass of the passengers, but we have neglected this relatively small difference.

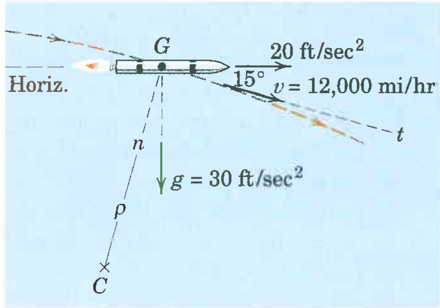






Sample Problem 2/8

A certain rocket maintains a horizontal attitude of its axis during the powered phase of its flight at high altitude. The thrust imparts a horizontal component of acceleration of 20 ft/sec^2, and the downward acceleration component is the acceleration due to gravity at that altitude, which is $g = 30$ ft/sec^2. At the instant represented, the velocity of the mass center G of the rocket along the 15° direction of its trajectory is 12,000 mi/hr. For this position determine (a) the radius of curvature of the flight trajectory, (b) the rate at which the speed v is increasing, (c) the angular rate $\dot{\beta}$ of the radial line from G to the center of curvature C, and (d) the vector expression for the total acceleration $\mathbf{a}$ of the rocket.



Solution. We observe that the radius of curvature appears in the expression for the normal component of acceleration, so we use n- and t-coordinates to describe the motion of G. The n- and t-components of the total acceleration are obtained by resolving the given horizontal and vertical accelerations into their n- and t-components and then combining. From the figure we get ①

$$a_n = 30 \cos 15° - 20 \sin 15° = 23.8 \text{ ft/sec}^2$$

$$a_t = 30 \sin 15° + 20 \cos 15° = 27.1 \text{ ft/sec}^2$$

(a) We may now compute the radius of curvature from

② $[a_n = v^2/\rho]$ $\quad \rho = \dfrac{v^2}{a_n} = \dfrac{[(12{,}000)(44/30)]^2}{23.8} = 13.01(10^6) \text{ ft}$ *Ans.*

(b) The rate at which v is increasing is simply the t-component of acceleration.

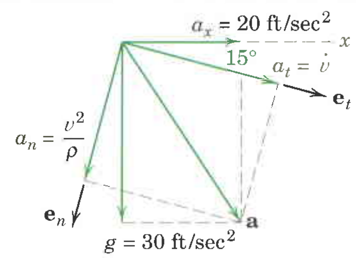
$[\dot{v} = a_t]$ $\quad \dot{v} = 27.1 \text{ ft/sec}^2$ *Ans.*

(c) The angular rate $\dot{\beta}$ of line GC depends on v and ρ and is given by

$[v = \rho\dot{\beta}]$ $\quad \dot{\beta} = v/\rho = \dfrac{12{,}000(44/30)}{13.01(10^6)} = 13.53(10^{-4}) \text{ rad/sec}$ *Ans.*

(d) With unit vectors $\mathbf{e}_n$ and $\mathbf{e}_t$ for the n- and t-directions, respectively, the total acceleration becomes

$$\mathbf{a} = 23.8\mathbf{e}_n + 27.1\mathbf{e}_t \text{ ft/sec}^2$$ *Ans.*



Helpful Hints

① Alternatively, we could find the resultant acceleration and then resolve it into n- and t-components.

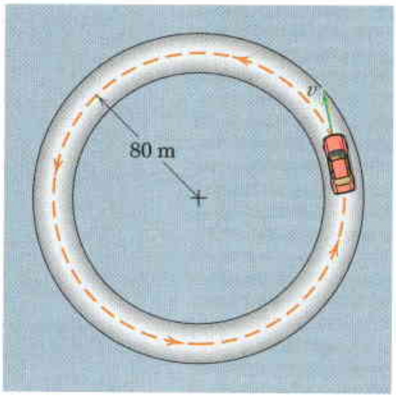
② To convert from mi/hr to ft/sec, multiply by $\dfrac{5280 \text{ ft/mi}}{3600 \text{ sec/hr}} = \dfrac{44 \text{ ft/sec}}{30 \text{ mi/hr}}$ which is easily remembered, as 30 mi/hr is the same as 44 ft/sec.

PROBLEMS

Introductory Problems

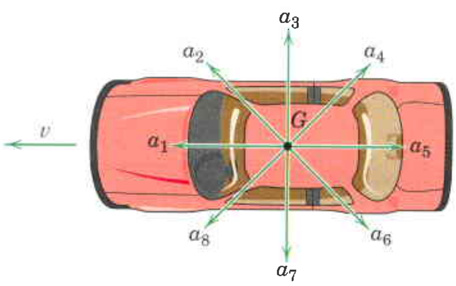
2/101 A test car starts from rest on a horizontal circular track of 80-m radius and increases its speed at a uniform rate to reach 100 km/h in 10 seconds. Determine the magnitude a of the total acceleration of the car 8 seconds after the start.

Ans. $a = 6.77\ \text{m/s}^2$



Problem 2/101

2/102 The car moves on a horizontal surface without any slippage of its tires. For each of the eight horizontal acceleration vectors, describe in words the instantaneous motion of the car. The car velocity is directed to the left as shown for all cases.

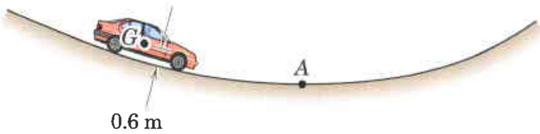


Problem 2/102

2/103 A particle moves in a circular path of 0.3-m radius. Calculate the magnitude a of the acceleration of the particle (*a*) if its speed is constant at 0.6 m/s and (*b*) if its speed is 0.6 m/s but is increasing at the rate of 0.9 m/s each second.

Ans. (*a*) $a = 1.2\ \text{m/s}^2$, (*b*) $a = 1.5\ \text{m/s}^2$

2/104 The car passes through a dip in the road at A with a constant speed which gives its mass center G an acceleration equal to $0.5g$. If the radius of curvature of the road at A is 100 m, and if the distance from the road to the mass center G of the car is 0.6 m, determine the speed v of the car.



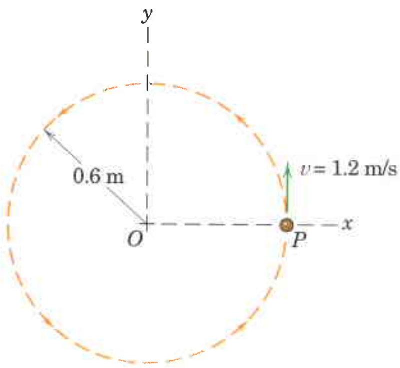
Problem 2/104

2/105 The car travels at a constant speed from the bottom A of the dip to the top B of the hump. If the radius of curvature of the road at A is $\rho_A = 120$ m and the car acceleration at A is $0.4g$, determine the car speed v. If the acceleration at B must be limited to $0.25g$, determine the minimum radius of curvature ρ_B of the road at B.

Ans. $v = 21.6$ m/s, $\rho_B = 190.4$ m

Problem 2/105

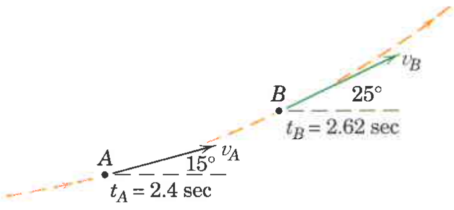
2/106 The particle P moves in the circular path shown. Sketch the acceleration vector **a** and determine its magnitude a for the following cases: (*a*) the speed v is 1.2 m/s and is constant, (*b*) the speed is 1.2 m/s and is increasing at the rate of 2.4 m/s each second, and (*c*) the speed is 1.2 m/s and is decreasing at the rate of 4.8 m/s each second. In each case the particle is in the position shown in the figure.



Problem 2/106

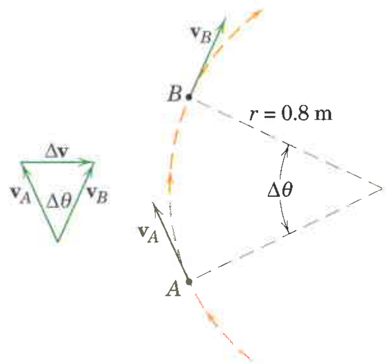
2/107 A particle moves along the curved path shown. The particle has a speed $v_A = 12$ ft/sec at time t_A and a speed $v_B = 14$ ft/sec at time t_B. Determine the average values of the normal and tangential accelerations of the particle between points A and B.

Ans. $a_n = 10.31$ ft/sec^2
$a_t = 9.09$ ft/sec^2



Problem 2/107

2/108 A particle moves on a circular path of radius $r = 0.8$ m with a constant speed of 2 m/s. The velocity undergoes a vector change $\Delta\mathbf{v}$ from A to B. Express the magnitude of $\Delta\mathbf{v}$ in terms of v and $\Delta\theta$ and divide it by the time interval Δt between A and B to obtain the magnitude of the average acceleration of the particle for (a) $\Delta\theta = 30°$, (b) $\Delta\theta = 15°$, and (c) $\Delta\theta = 5°$. In each case, determine the percentage difference from the instantaneous value of acceleration.

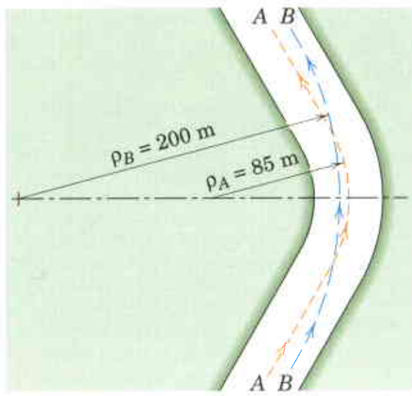


Problem 2/108

Representative Problems

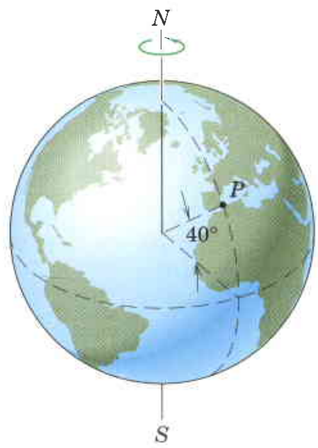
2/109 The figure shows two possible paths for negotiating an unbanked turn on a horizontal portion of a race course. Path A-A follows the centerline of the road and has a radius of curvature $\rho_A = 85$ m, while path B-B uses the width of the road to good advantage in increasing the radius of curvature to $\rho_B = 200$ m. If the drivers limit their speeds in their curves so that the lateral acceleration does not exceed $0.8g$, determine the maximum speed for each path.

Ans. $v_A = 25.8$ m/s, $v_B = 39.6$ m/s



Problem 2/109

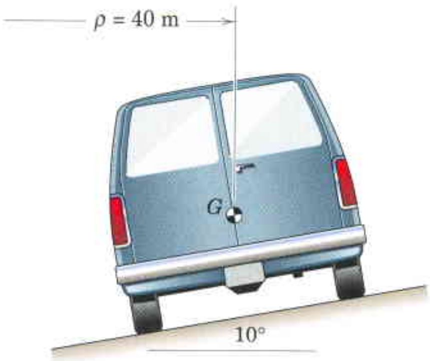
2/110 Consider the polar axis of the earth to be fixed in space and compute the magnitude of the acceleration $\mathbf{a}$ of a point P on the earth's surface at latitude 40° north. The mean diameter of the earth is 12 742 km and its angular velocity is $0.729(10^{-4})$ rad/s.



Problem 2/110

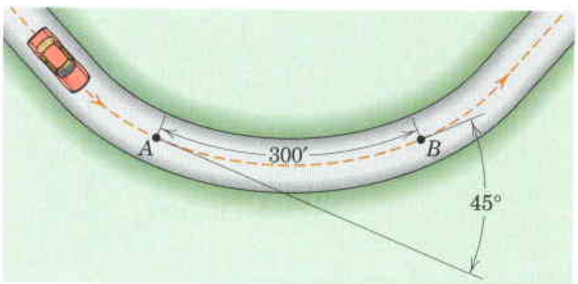
2/111 A minivan starts from rest on the road whose constant radius of curvature is 40 m and whose bank angle is 10°. The motion occurs in a horizontal plane. If the constant forward acceleration of the minivan is 1.8 m/s^2, determine the magnitude a of its total acceleration 5 seconds after starting.

Ans. $a = 2.71$ m/s^2



Problem 2/111

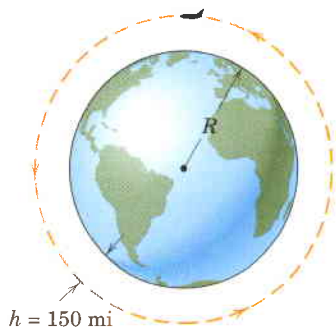
2/112 A car rounds a turn of constant curvature between A and B with a steady speed of 45 mi/hr. If an accelerometer were mounted in the car, what magnitude of acceleration would it record between A and B?



Problem 2/112

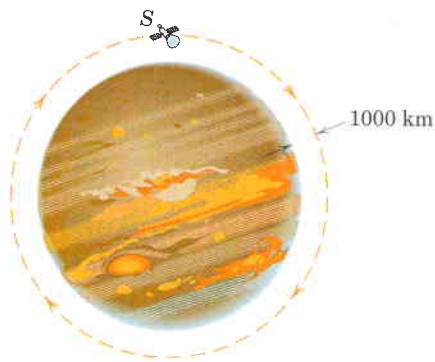
2/113 A space shuttle which moves in a circular orbit around the earth at a height h = 150 mi above its surface must have a speed of 17,369 mi/hr. Calculate the gravitational acceleration g for this altitude. The mean radius of the earth is 3959 mi. (Check your answer by computing g from the gravitational law $g = g_0\left(\frac{R}{R+h}\right)^2$, where g_0 = 32.22 ft/sec^2 from Table D/2 in Appendix D.)

Ans. $a_n = g = 29.91$ ft/sec^2



Problem 2/113

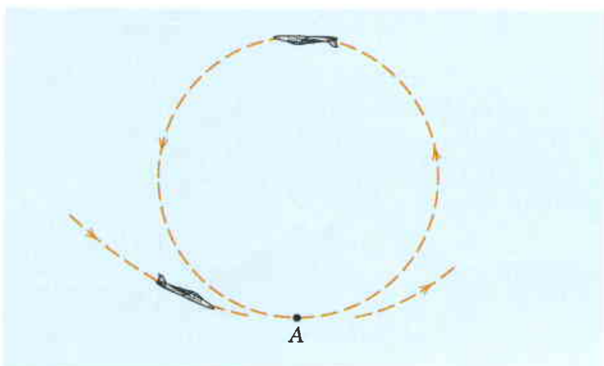
2/114 A spacecraft S is orbiting Jupiter in a circular path 1000 km above the surface with a constant speed. Using the gravitational law, calculate the magnitude v of its orbital velocity with respect to Jupiter. The diameter of Jupiter is 142 984 km and its surface-level gravitational acceleration is 24.85 m/s^2.



Problem 2/114

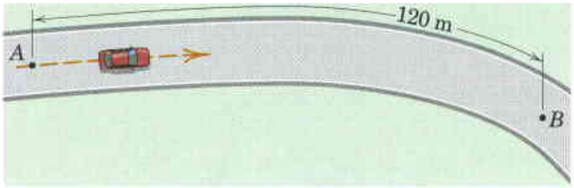
2/115 At the bottom A of the vertical inside loop, the magnitude of the total acceleration of the airplane is $3g$. If the airspeed is 800 km/h and is increasing at the rate of 20 km/h per second, calculate the radius of curvature ρ of the path at A.

Ans. ρ = 1709 m



Problem 2/115

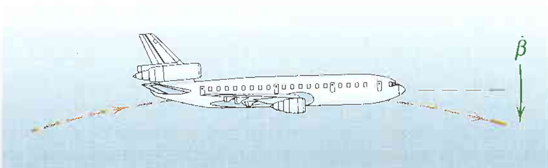
2/116 A car travels along the level curved road with a speed which is decreasing at the constant rate of 0.6 m/s each second. The speed of the car as it passes point A is 16 m/s. Calculate the magnitude of the total acceleration of the car as it passes point B which is 120 m along the road from A. The radius of curvature of the road at B is 60 m.



Problem 2/116

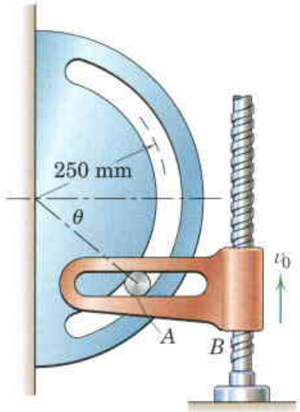
2/117 To simulate a condition of "weightlessness" in its cabin, a jet transport traveling at 800 km/h moves on a sustained vertical curve as shown. At what rate $\dot{\beta}$ in degrees per second should the pilot drop his longitudinal line of sight to effect the desired condition? The maneuver takes place at a mean altitude of 8 km, and the gravitational acceleration may be taken as 9.79 m/s^2.

Ans. $\dot{\beta} = 2.52$ deg/s



Problem 2/117

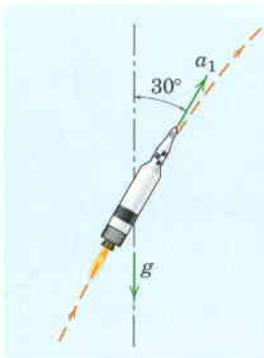
2/118 In the design of a timing mechanism, the motion of the pin A in the fixed circular slot is controlled by the guide B, which is being elevated by its lead screw with a constant upward velocity $v_0 = 2$ m/s for an interval of its motion. Calculate both the normal and tangential components of acceleration of pin A as it passes the position for which $\theta = 30°$.



Problem 2/118

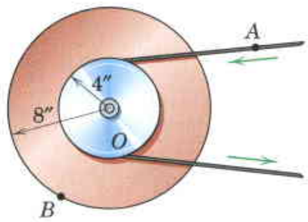
2/119 A rocket traveling above the atmosphere at an altitude of 500 km would have a free-fall acceleration $g = 8.43$ m/s^2 in the absence of forces other than gravitational attraction. Because of thrust, however, the rocket has an additional acceleration component a_1 of 8.80 m/s^2 tangent to its trajectory, which makes an angle of 30° with the vertical at the instant considered. If the velocity v of the rocket is 30 000 km/h at this position, compute the radius of curvature ρ of the trajectory and the rate at which v is changing with time.

Ans. $\rho = 16\ 480$ km, $\dot{v} = 1.499$ m/s^2



Problem 2/119

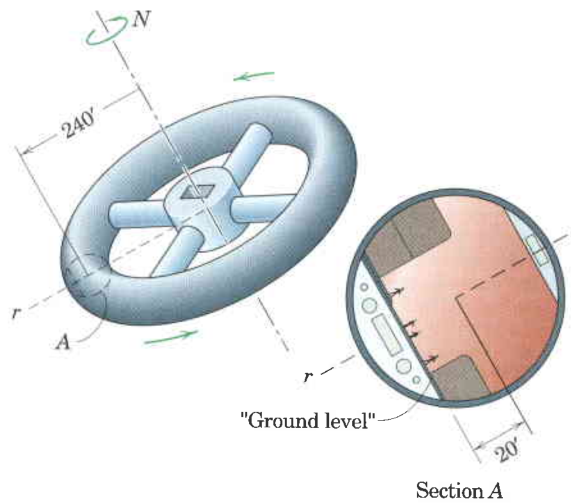
2/120 The wheel and attached pulley rotate about the fixed shaft at O and are driven by the belt shown. At a certain instant the velocity and acceleration of a point A on the belt are 2 ft/sec and 6 ft/sec^2, respectively, both in the direction shown. Calculate the magnitude of the total acceleration of point B on the wheel for this instant. Observe that the linear motion of point A on the belt and the tangential motion of a point on the 4-in.-radius circle are identical and that $\dot{\theta}$ and $\ddot{\theta}$ for the radial lines to all points on the wheel are the same.



Problem 2/120

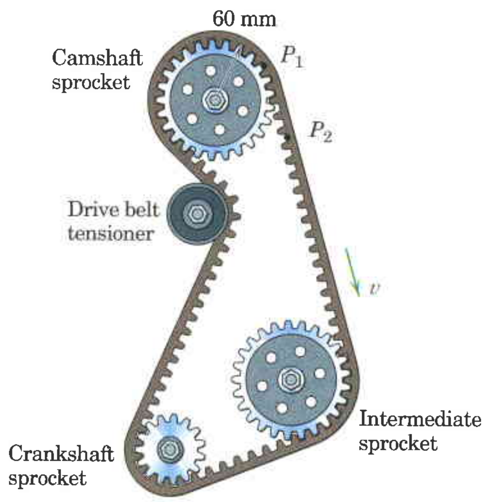
2/121 The preliminary design for a "small" space station to orbit the earth in a circular path consists of a ring (torus) with a circular cross section as shown. The living space within the torus is shown in section A, where the "ground level" is 20 ft from the center of the section. Calculate the angular speed N in revolutions per minute required to simulate standard gravity at the surface of the earth (32.17 ft/sec^2). Recall that you would be unaware of a gravitational field if you were in a nonrotating spacecraft in a circular orbit around the earth.

Ans. $N = 3.36$ rev/min



Problem 2/121

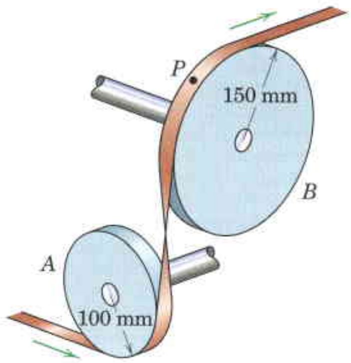
2/122 The design of a camshaft-drive system of a four-cylinder automobile engine is shown. As the engine is revved up, the belt speed v changes uniformly from 3 m/s to 6 m/s over a two-second interval. Calculate the magnitudes of the accelerations of points P_1 and P_2 halfway through this time interval.



Problem 2/122

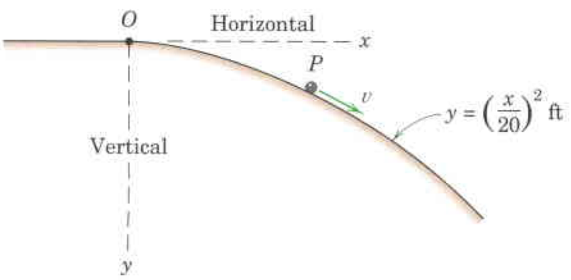
2/123 The direction of motion of a flat tape in a numerical-control device is changed by the two pulleys A and B shown. If the speed of the tape increases uniformly from 2 m/s to 18 m/s while 8 meters of tape pass over the pulleys, calculate the magnitude of the acceleration of point P on the tape in contact with pulley B at the instant when the tape speed is 3 m/s.

Ans. $a = 63.2 \text{ m/s}^2$



Problem 2/123

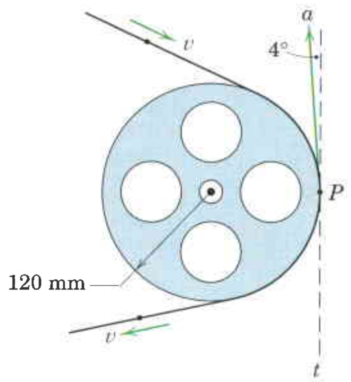
2/124 A small particle P starts from point O with a negligible speed and increases its speed to a value $v = \sqrt{2gy}$, where y is the vertical drop from O. When $x = 50$ ft, determine the n-component of acceleration of the particle. (See Art. C/10 for the radius of curvature.)



Problem 2/124

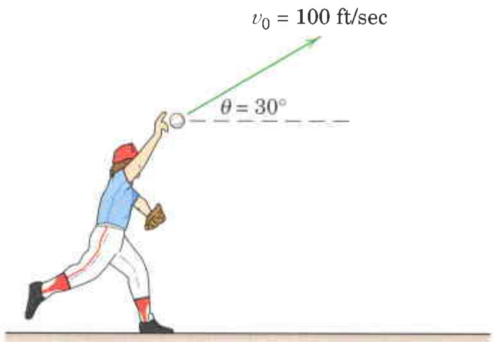
2/125 Magnetic tape runs over the idler pulley in a computer as shown. If the total acceleration of a point P on the tape in contact with the pulley makes an angle of 4° with the tangent to the tape at time $t = 0$ when the velocity v of the tape is 4 m/s, determine the time t required to bring the pulley to a stop with constant deceleration. Assume no slipping between the pulley and the tape.

Ans. $t = 2.10(10^{-3})$ s



Problem 2/125

2/126 A baseball player releases a ball with the initial conditions shown in the figure. Determine the radius of curvature of the trajectory (a) just after release and (b) at the apex. For each case, compute the time rate of change of the speed.

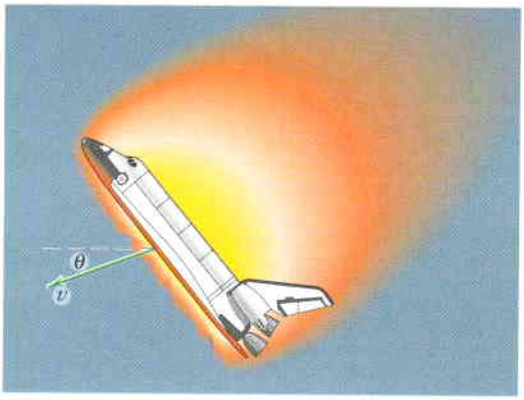


Problem 2/126

2/127 For the baseball of Prob. 2/126, determine the radius of curvature ρ of the path and the time rate of change $\dot{v}$ of the speed at times $t = 1$ sec and $t = 2.5$ sec, where $t = 0$ is the time of release from the player's hand.

Ans. (*a*) $\rho = 248$ ft, $\dot{v} = -6.48$ ft/sec^2
(*b*) $\rho = 278$ ft, $\dot{v} = 10.70$ ft/sec^2

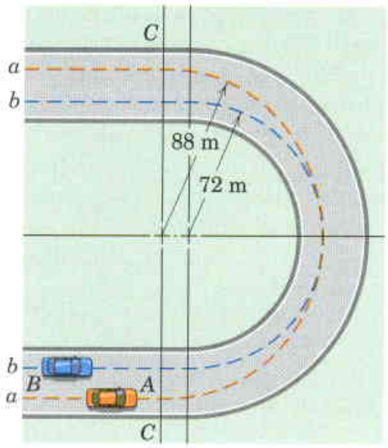
2/128 At a certain point in the reentry of the space shuttle into the earth's atmosphere, the total acceleration of the shuttle may be represented by two components. One component is the gravitational acceleration $g = 9.66$ m/s^2 at this altitude. The second component equals 12.90 m/s^2 due to atmospheric resistance and is directed opposite to the velocity. The shuttle is at an altitude of 48.2 km and has reduced its orbital velocity of 28 300 km/h to 15 450 km/h in the direction $\theta = 1.50°$. For this instant, calculate the radius of curvature ρ of the path and the rate $\dot{v}$ at which the speed is changing.



Problem 2/128

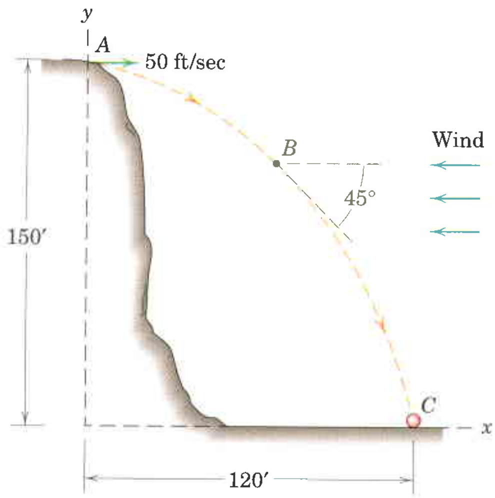
2/129 Race car A follows path a-a while race car B follows path b-b on the unbanked track. If each car has a constant speed limited to that corresponding to a lateral (normal) acceleration of $0.8g$, determine the times t_A and t_B for both cars to negotiate the turn as delimited by the line C-C.

Ans. $t_A = 10.52$ s, $t_B = 10.86$ s



Problem 2/129

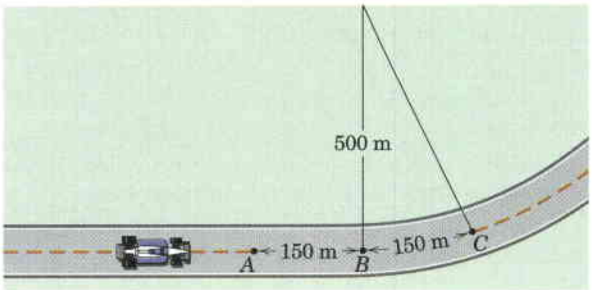
2/130 A ball is thrown horizontally from the top of a 150-ft cliff at A with a speed of 50 ft/sec and lands at point C. Because of a strong horizontal wind, the ball has a constant acceleration in the negative x-direction. Determine the radius of curvature ρ of the path of the ball at B where its trajectory makes an angle of 45° with the horizontal. Neglect any effect of air resistance in the vertical direction.



Problem 2/130

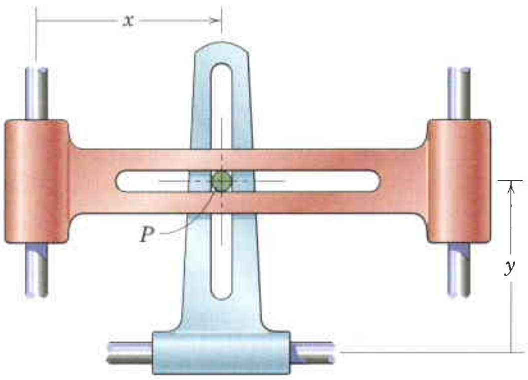
2/131 A race driver traveling at a speed of 250 km/h on the straightaway applies his brakes at point A and reduces his speed at a uniform rate to 200 km/h at C in a distance of 150 + 150 = 300 m. Calculate the magnitude of the total acceleration of the race car an instant after it passes point B.

Ans. $a = 8.42\ \text{m/s}^2$



Problem 2/131

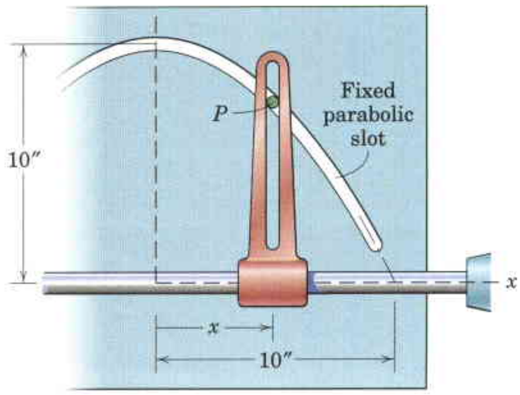
2/132 During a short interval the slotted guides are designed to move according to $x = 16 - 12t + 4t^2$ and $y = 2 + 15t - 3t^2$, where x and y are in millimeters and t is in seconds. At the instant when $t = 2$ s, determine the radius of curvature ρ of the path of the constrained pin P.



Problem 2/132

2/133 In the design of a control mechanism, the vertical slotted guide is moving with a constant velocity $\dot{x} = 15$ in./sec during the interval of motion from $x = -8$ in. to $x = +8$ in. For the instant when $x = 6$ in., calculate the n- and t-components of acceleration of the pin P, which is confined to move in the parabolic slot. From these results, determine the radius of curvature ρ of the path at this position. Verify your result by computing ρ from the expression cited in Appendix C/10.

Ans. $\rho = 19.06$ in.



Problem 2/133

▶2/134 In a handling test, a car is driven through the slalom course shown. It is assumed that the car path is sinusoidal and that the maximum lateral acceleration is $0.7g$. If the testers wish to design a slalom through which the maximum speed is 80 km/h, what cone spacing L should be used?

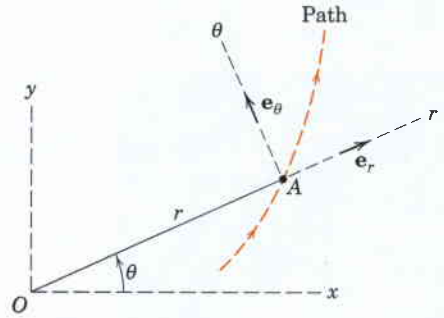
Ans. $L = 46.1$ m

Problem 2/134

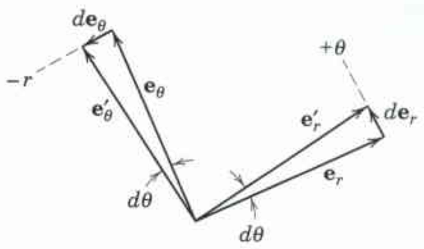
2/6 Polar Coordinates (r-θ)

We now consider the third description of plane curvilinear motion, namely, polar coordinates where the particle is located by the radial distance r from a fixed point and by an angular measurement θ to the radial line. Polar coordinates are particularly useful when a motion is constrained through the control of a radial distance and an angular position or when an unconstrained motion is observed by measurements of a radial distance and an angular position.

Figure 2/13a shows the polar coordinates r and θ which locate a particle traveling on a curved path. An arbitrary fixed line, such as the x-axis, is used as a reference for the measurement of θ. Unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are established in the positive r- and θ-directions, respectively. The position vector $\mathbf{r}$ to the particle at A has a magnitude equal to the radial distance r and a direction specified by the unit vector $\mathbf{e}_r$. Thus, we express the location of the particle at A by the vector



(a)



(b)

Figure 2/13

$$\mathbf{r} = r\mathbf{e}_r$$

Time Derivatives of the Unit Vectors

To differentiate this relation with respect to time to obtain $\mathbf{v} = \dot{\mathbf{r}}$ and $\mathbf{a} = \dot{\mathbf{v}}$, we need expressions for the time derivatives of both unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$. We obtain $\dot{\mathbf{e}}_r$ and $\dot{\mathbf{e}}_\theta$ in exactly the same way we derived $\dot{\mathbf{e}}_t$ in the preceding article. During time dt the coordinate directions rotate through the angle $d\theta$, and the unit vectors also rotate through the same angle from $\mathbf{e}_r$ and $\mathbf{e}_\theta$ to $\mathbf{e}'_r$ and $\mathbf{e}'_\theta$, as shown in Fig. 2/13b. We note that the vector change $d\mathbf{e}_r$ is in the plus θ-direction and that $d\mathbf{e}_\theta$ is in the minus r-direction. Because their magnitudes in the limit are equal to the unit vector as radius times the angle $d\theta$ in radians, we can write them as $d\mathbf{e}_r = \mathbf{e}_\theta\, d\theta$ and $d\mathbf{e}_\theta = -\mathbf{e}_r\, d\theta$. If we divide these equations by $d\theta$, we have

$$\frac{d\mathbf{e}_r}{d\theta} = \mathbf{e}_\theta \qquad \text{and} \qquad \frac{d\mathbf{e}_\theta}{d\theta} = -\mathbf{e}_r$$

If, on the other hand, we divide them by dt, we have $d\mathbf{e}_r/dt = (d\theta/dt)\mathbf{e}_\theta$ and $d\mathbf{e}_\theta/dt = -(d\theta/dt)\mathbf{e}_r$, or simply

$$\boxed{\dot{\mathbf{e}}_r = \dot{\theta}\,\mathbf{e}_\theta \qquad \text{and} \qquad \dot{\mathbf{e}}_\theta = -\dot{\theta}\,\mathbf{e}_r} \tag{2/12}$$

Velocity

We are now ready to differentiate $\mathbf{r} = r\mathbf{e}_r$ with respect to time. Using the rule for differentiating the product of a scalar and a vector gives

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r$$

With the substitution of $\dot{\mathbf{e}}_r$ from Eq. 2/12, the vector expression for the velocity becomes

$$\boxed{\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta} \tag{2/13}$$

where
$$v_r = \dot{r}$$
$$v_\theta = r\dot{\theta}$$
$$v = \sqrt{v_r^{\,2} + v_\theta^{\,2}}$$

The r-component of $\mathbf{v}$ is merely the rate at which the vector $\mathbf{r}$ stretches. The θ-component of $\mathbf{v}$ is due to the rotation of $\mathbf{r}$.

Acceleration

We now differentiate the expression for $\mathbf{v}$ to obtain the acceleration $\mathbf{a} = \dot{\mathbf{v}}$. Note that the derivative of $r\dot{\theta}\mathbf{e}_\theta$ will produce three terms, since all three factors are variable. Thus,

$$\mathbf{a} = \dot{\mathbf{v}} = (\ddot{r}\mathbf{e}_r + \dot{r}\dot{\mathbf{e}}_r) + (\dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta + r\dot{\theta}\dot{\mathbf{e}}_\theta)$$

Substitution of $\dot{\mathbf{e}}_r$ and $\dot{\mathbf{e}}_\theta$ from Eq. 2/12 and collecting terms give

$$\boxed{\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta} \qquad \textbf{(2/14)}$$

where
$$a_r = \ddot{r} - r\dot{\theta}^2$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$
$$a = \sqrt{a_r^{\,2} + a_\theta^{\,2}}$$

We can write the θ-component alternatively as

$$a_\theta = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

which can be verified easily by carrying out the differentiation. This form for a_θ will be useful when we treat the angular momentum of particles in the next chapter.

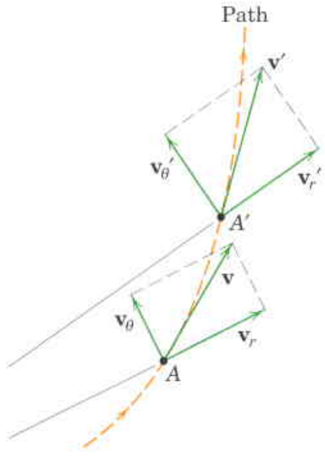
Geometric Interpretation

The terms in Eq. 2/14 can be best understood when the geometry of the physical changes can be clearly seen. For this purpose, Fig. 2/14a is developed to show the velocity vectors and their r- and θ-components at position A and at position A' after an infinitesimal movement. Each of these components undergoes a change in magnitude and direction as shown in Fig. 2/14b. In this figure we see the following changes:

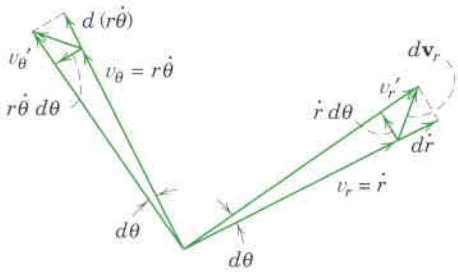
(a) Magnitude Change of $\mathbf{v}_r$. This change is simply the increase in length of v_r or $dv_r = d\dot{r}$, and the corresponding acceleration term is $d\dot{r}/dt = \ddot{r}$ in the positive r-direction.

(b) Direction Change of $\mathbf{v}_r$. The magnitude of this change is seen from the figure to be $v_r\, d\theta = \dot{r}\, d\theta$, and its contribution to the acceleration becomes $\dot{r}\, d\theta/dt = \dot{r}\dot{\theta}$ which is in the positive θ-direction.

(c) Magnitude Change of $\mathbf{v}_\theta$. This term is the change in length of $\mathbf{v}_\theta$ or $d(r\dot{\theta})$, and its contribution to the acceleration is $d(r\dot{\theta})/dt = r\ddot{\theta} + \dot{r}\dot{\theta}$ and is in the positive θ-direction.



(a)



(b)

Figure 2/14

(d) Direction Change of $\mathbf{v}_\theta$. The magnitude of this change is $v_\theta\, d\theta = r\dot{\theta}\, d\theta$, and the corresponding acceleration term is observed to be $r\dot{\theta}(d\theta/dt) = r\dot{\theta}^2$ in the negative r-direction.

Collecting terms gives $a_r = \ddot{r} - r\dot{\theta}^2$ and $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$ as obtained previously. We see that the term $\ddot{r}$ is the acceleration which the particle would have along the radius in the absence of a change in θ. The term $-r\dot{\theta}^2$ is the normal component of acceleration if r were constant, as in circular motion. The term $r\ddot{\theta}$ is the tangential acceleration which the particle would have if r were constant, but is only a part of the acceleration due to the change in magnitude of $\mathbf{v}_\theta$ when r is variable. Finally, the term $2\dot{r}\dot{\theta}$ is composed of two effects. The first effect comes from that portion of the change in magnitude $d(r\dot{\theta})$ of v_θ due to the change in r, and the second effect comes from the change in direction of $\mathbf{v}_r$. The term $2\dot{r}\dot{\theta}$ represents, therefore, a combination of changes and is not so easily perceived as are the other acceleration terms.

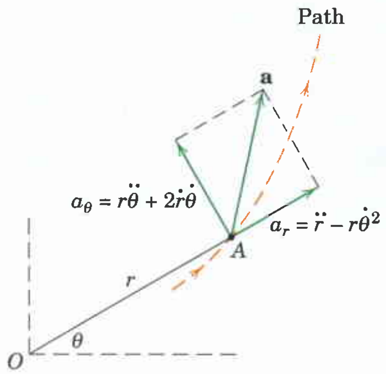


Figure 2/15

Note the difference between the vector change $d\mathbf{v}_r$ in $\mathbf{v}_r$ and the change dv_r in the magnitude of v_r. Similarly, the vector change $d\mathbf{v}_\theta$ is not the same as the change dv_θ in the magnitude of v_θ. When we divide these changes by dt to obtain expressions for the derivatives, we see clearly that the magnitude of the derivative $|d\mathbf{v}_r/dt|$ and the derivative of the magnitude dv_r/dt are *not* the same. Note also that a_r is not $\dot{v}_r$ and that a_θ is not $\dot{v}_\theta$.

The total acceleration $\mathbf{a}$ and its components are represented in Fig. 2/15. If $\mathbf{a}$ has a component normal to the path, we know from our analysis of n- and t-components in Art. 2/5 that the sense of the n-component *must* be toward the center of curvature.

Circular Motion

For motion in a circular path with r constant, the components of Eqs. 2/13 and 2/14 become simply

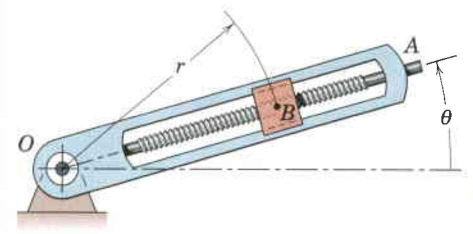
$$v_r = 0 \qquad v_\theta = r\dot{\theta}$$
$$a_r = -r\dot{\theta}^2 \qquad a_\theta = r\ddot{\theta}$$

This description is the same as that obtained with n- and t-components, where the θ- and t-directions coincide but the positive r-direction is in the negative n-direction. Thus, $a_r = -a_n$ for circular motion centered at the origin of the polar coordinates.

The expressions for a_r and a_θ in scalar form can also be obtained by direct differentiation of the coordinate relations $x = r\cos\theta$ and $y = r\sin\theta$ to obtain $a_x = \ddot{x}$ and $a_y = \ddot{y}$. Each of these rectangular components of acceleration can then be resolved into r- and θ-components which, when combined, will yield the expressions of Eq. 2/14.

Sample Problem 2/9

Rotation of the radially slotted arm is governed by $\theta = 0.2t + 0.02t^3$, where θ is in radians and t is in seconds. Simultaneously, the power screw in the arm engages the slider B and controls its distance from O according to $r = 0.2 + 0.04t^2$, where r is in meters and t is in seconds. Calculate the magnitudes of the velocity and acceleration of the slider for the instant when $t = 3$ s.



Solution. The coordinates and their time derivatives which appear in the expressions for velocity and acceleration in polar coordinates are obtained first and evaluated for $t = 3$ s. ①

$$r = 0.2 + 0.04t^2 \qquad r_3 = 0.2 + 0.04(3^2) = 0.56 \text{ m}$$

$$\dot{r} = 0.08t \qquad \dot{r}_3 = 0.08(3) = 0.24 \text{ m/s}$$

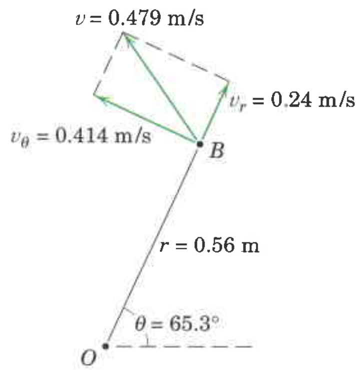
$$\ddot{r} = 0.08 \qquad \ddot{r}_3 = 0.08 \text{ m/s}^2$$

$$\theta = 0.2t + 0.02t^3 \qquad \theta_3 = 0.2(3) + 0.02(3^3) = 1.14 \text{ rad}$$

$$\text{or } \theta_3 = 1.14(180/\pi) = 65.3°$$

$$\dot{\theta} = 0.2 + 0.06t^2 \qquad \dot{\theta}_3 = 0.2 + 0.06(3^2) = 0.74 \text{ rad/s}$$

$$\ddot{\theta} = 0.12t \qquad \ddot{\theta}_3 = 0.12(3) = 0.36 \text{ rad/s}^2$$



The velocity components are obtained from Eq. 2/13 and for $t = 3$ s are

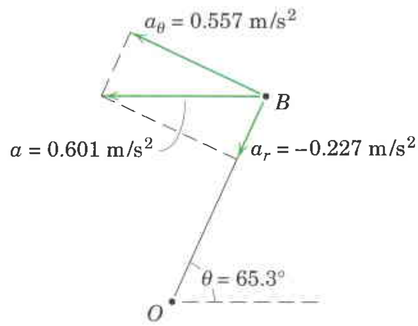
$$[v_r = \dot{r}] \qquad v_r = 0.24 \text{ m/s}$$

$$[v_\theta = r\dot{\theta}] \qquad v_\theta = 0.56(0.74) = 0.414 \text{ m/s}$$

$$[v = \sqrt{v_r^2 + v_\theta^2}] \qquad v = \sqrt{(0.24)^2 + (0.414)^2} = 0.479 \text{ m/s} \qquad \textit{Ans.}$$

The velocity and its components are shown for the specified position of the arm.

The acceleration components are obtained from Eq. 2/14 and for $t = 3$ s are



$$[a_r = \ddot{r} - r\dot{\theta}^2] \qquad a_r = 0.08 - 0.56(0.74)^2 = -0.227 \text{ m/s}^2$$

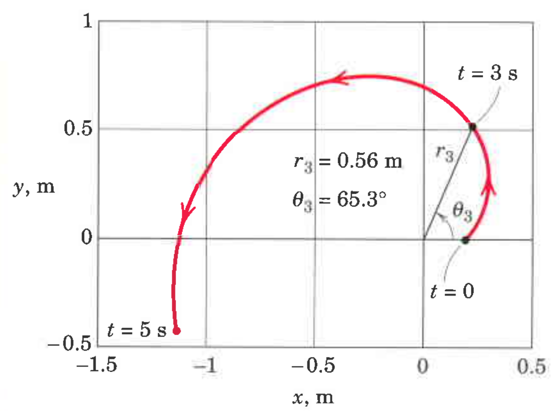
$$[a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}] \qquad a_\theta = 0.56(0.36) + 2(0.24)(0.74) = 0.557 \text{ m/s}^2$$

$$[a = \sqrt{a_r^2 + a_\theta^2}] \qquad a = \sqrt{(-0.227)^2 + (0.557)^2} = 0.601 \text{ m/s}^2 \qquad \textit{Ans.}$$

The acceleration and its components are also shown for the 65.3° position of the arm.

Plotted in the final figure is the path of the slider B over the time interval $0 \le t \le 5$ s. This plot is generated by varying t in the given expressions for r and θ. Conversion from polar to rectangular coordinates is given by

$$x = r\cos\theta \qquad y = r\sin\theta$$

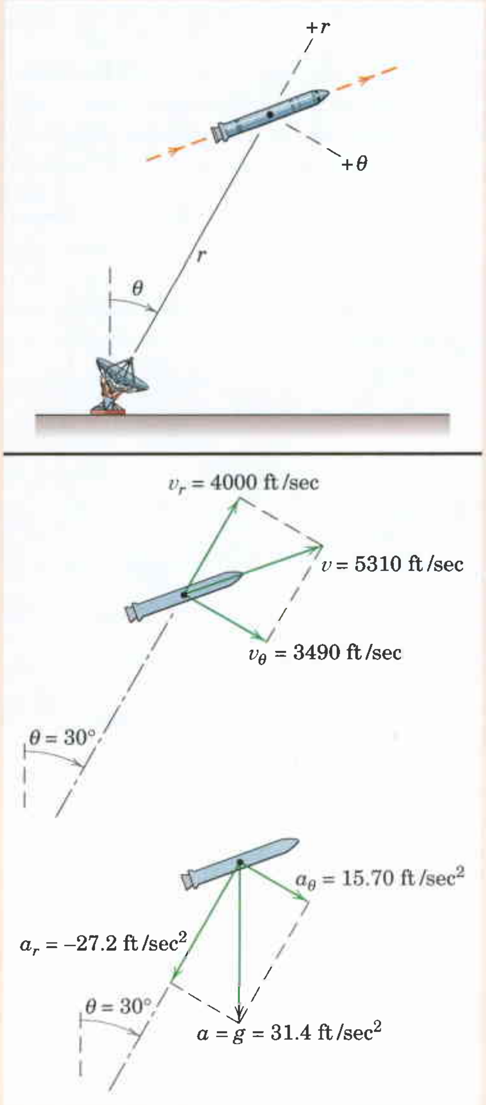


Helpful Hint

① We see that this problem is an example of constrained motion where the center B of the slider is mechanically constrained by the rotation of the slotted arm and by engagement with the turning screw.

Sample Problem 2/10

A tracking radar lies in the vertical plane of the path of a rocket which is coasting in unpowered flight above the atmosphere. For the instant when $\theta = 30°$, the tracking data give $r = 25(10^4)$ ft, $\dot{r} = 4000$ ft/sec, and $\dot{\theta} = 0.80$ deg/sec. The acceleration of the rocket is due only to gravitational attraction and for its particular altitude is 31.4 ft/sec² vertically down. For these conditions determine the velocity v of the rocket and the values of $\ddot{r}$ and $\ddot{\theta}$.



Solution. The components of velocity from Eq. 2/13 are

$[v_r = \dot{r}]$ $\qquad v_r = 4000$ ft/sec

① $[v_\theta = r\dot{\theta}]$ $\qquad v_\theta = 25(10^4)(0.80)\left(\dfrac{\pi}{180}\right) = 3490$ ft/sec

$[v = \sqrt{v_r^2 + v_\theta^2}]$ $\qquad v = \sqrt{(4000)^2 + (3490)^2} = 5310$ ft/sec $\qquad$ *Ans.*

Since the total acceleration of the rocket is $g = 31.4$ ft/sec² down, we can easily find its r- and θ-components for the given position. As shown in the figure, they are

② $$a_r = -31.4 \cos 30° = -27.2 \text{ ft/sec}^2$$

$$a_\theta = 31.4 \sin 30° = 15.70 \text{ ft/sec}^2$$

We now equate these values to the polar-coordinate expressions for a_r and a_θ which contain the unknowns $\ddot{r}$ and $\ddot{\theta}$. Thus, from Eq. 2/14

③ $[a_r = \ddot{r} - r\dot{\theta}^2]$ $\qquad -27.2 = \ddot{r} - 25(10^4)\left(0.80\,\dfrac{\pi}{180}\right)^2$

$\ddot{r} = 21.5$ ft/sec² $\qquad$ *Ans.*

$[a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}]$ $\qquad 15.70 = 25(10^4)\ddot{\theta} + 2(4000)\left(0.80\,\dfrac{\pi}{180}\right)$

$\ddot{\theta} = -3.84(10^{-4})$ rad/sec² $\qquad$ *Ans.*

Helpful Hints

① We observe that the angle θ in polar coordinates need not always be taken positive in a counterclockwise sense.

② Note that the r-component of acceleration is in the negative r-direction, so it carries a minus sign.

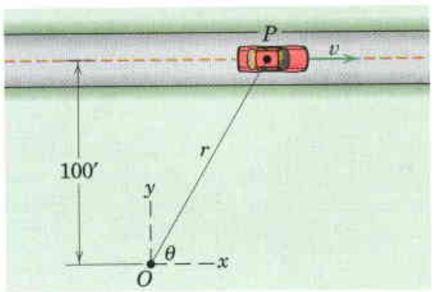
③ We must be careful to convert $\dot{\theta}$ from deg/sec to rad/sec.

PROBLEMS

Introductory Problems

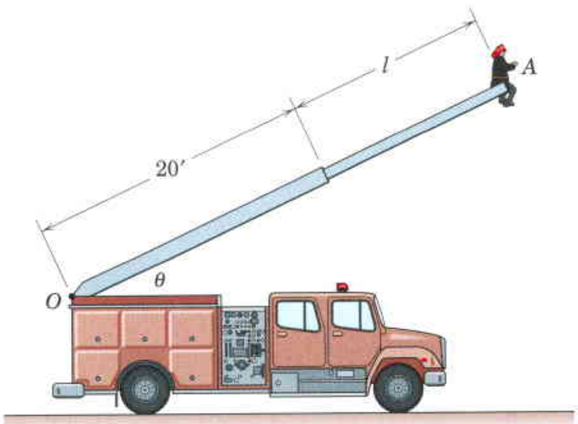
2/135 A car P travels along a straight road with a constant speed $v = 65$ mi/hr. At the instant when the angle $\theta = 60°$, determine the values of $\dot{r}$ in ft/sec and $\dot{\theta}$ in deg/sec.

Ans. $\dot{r} = 47.7$ ft/sec, $\dot{\theta} = -41.0$ deg/sec



Problem 2/135

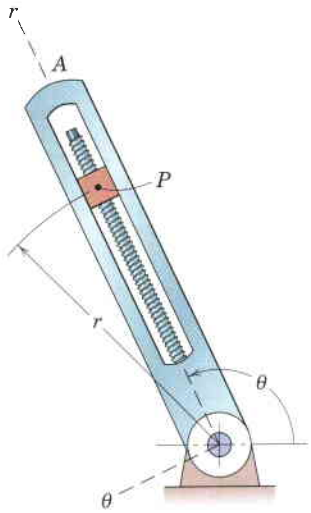
2/136 The ladder of a fire truck is designed to be extended at the constant rate $\dot{l} = 6$ in./sec and to be elevated at the constant rate $\dot{\theta} = 2$ deg/sec. As the position $\theta = 50°$ and $l = 15$ ft is reached, determine the magnitudes of the velocity **v** and the acceleration **a** of the fireman at A.



Problem 2/136

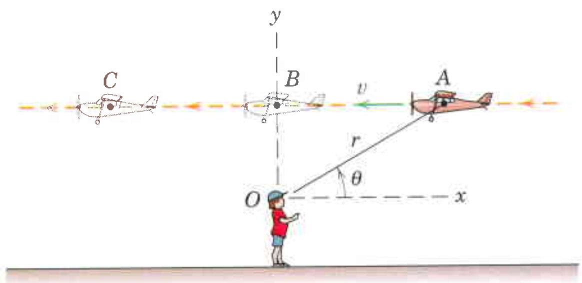
2/137 Motion of the sliding block P in the rotating radial slot is controlled by the power screw as shown. For the instant represented, $\dot{\theta} = 0.1$ rad/s, $\ddot{\theta} = -0.4$ rad/s^2, and $r = 300$ mm. Also, the screw turns at a constant speed giving $\dot{r} = 40$ mm/s. For this instant, determine the magnitudes of the velocity **v** and acceleration **a** of P. Sketch **v** and **a** if $\theta = 120°$.

Ans. $v = 50$ mm/s, $a = 5$ mm/s^2



Problem 2/137

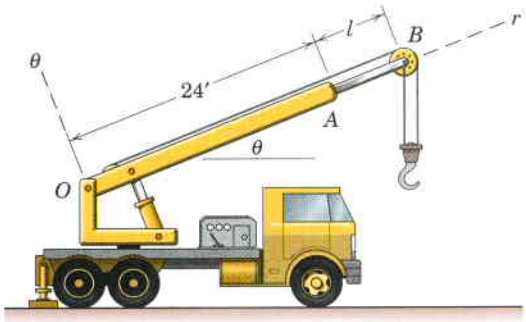
2/138 A model airplane flies over an observer O with constant speed in a straight line as shown. Determine the signs (plus, minus, or zero) for r, $\dot{r}$, $\ddot{r}$, θ, $\dot{\theta}$, and $\ddot{\theta}$ for each of the positions A, B, and C.



Problem 2/138

2/139 The boom OAB pivots about point O, while section AB simultaneously extends from within section OA. Determine the velocity and acceleration of the center B of the pulley for the following conditions: $\theta = 20°$, $\dot{\theta} = 5$ deg/sec, $\ddot{\theta} = 2$ deg/sec^2, $l = 7$ ft, $\dot{l} = 1.5$ ft/sec, $\ddot{l} = -4$ ft/sec^2. The quantities $\dot{l}$ and $\ddot{l}$ are the first and second time derivatives, respectively, of the length l of section AB.

Ans. $\mathbf{v} = 1.5\mathbf{e}_r + 2.71\mathbf{e}_\theta$ ft/sec
$\mathbf{a} = -4.24\mathbf{e}_r + 1.344\mathbf{e}_\theta$ ft/sec^2



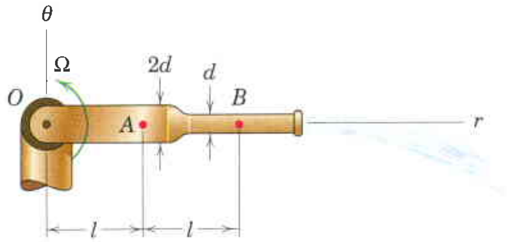
Problem 2/139

2/140 A particle moving along a plane curve has a position vector $\mathbf{r}$, a velocity $\mathbf{v}$, and an acceleration $\mathbf{a}$. Unit vectors in the r- and θ-directions are $\mathbf{e}_r$ and $\mathbf{e}_\theta$, respectively, and both r and θ are changing with time. Explain why each of the following statements is correctly marked as an inequality.

$\dot{\mathbf{r}} \neq v$	$\ddot{\mathbf{r}} \neq a$	$\dot{\mathbf{r}} \neq \dot{r}\mathbf{e}_r$
$\dot{r} \neq v$	$\ddot{r} \neq a$	$\ddot{\mathbf{r}} \neq \ddot{r}\mathbf{e}_r$
$\dot{r} \neq \mathbf{v}$	$\ddot{r} \neq \mathbf{a}$	$\dot{\mathbf{r}} \neq r\dot{\theta}\mathbf{e}_\theta$

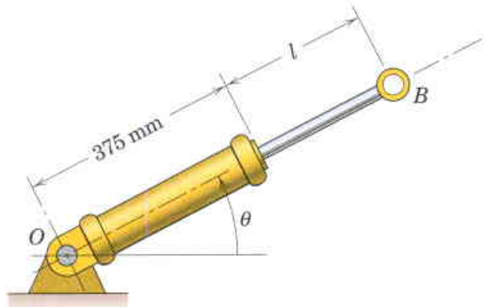
2/141 The nozzle shown rotates with constant angular speed Ω about a fixed horizontal axis through point O. Because of the change in diameter by a factor of 2, the water speed relative to the nozzle at A is v, while that at B is $4v$. The water speeds at both A and B are constant. Determine the velocity and acceleration of a water particle as it passes (*a*) point A and (*b*) point B.

Ans. (*a*) $\mathbf{v}_A = v\mathbf{e}_r + l\Omega\mathbf{e}_\theta$
$\mathbf{a}_A = -l\Omega^2\mathbf{e}_r + 2v\Omega\mathbf{e}_\theta$
(*b*) $\mathbf{v}_B = 4v\mathbf{e}_r + 2l\Omega\mathbf{e}_\theta$
$\mathbf{a}_B = -2l\Omega^2\mathbf{e}_r + 8v\Omega\mathbf{e}_\theta$



Problem 2/141

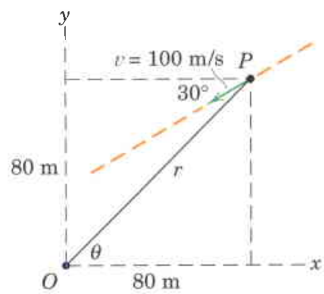
2/142 As the hydraulic cylinder rotates around O, the exposed length l of the piston rod P is controlled by the action of oil pressure in the cylinder. If the cylinder rotates at the constant rate $\dot{\theta} = 60$ deg/s and l is decreasing at the constant rate of 150 mm/s, calculate the magnitudes of the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ of end B when $l = 125$ mm.



Problem 2/142

2/143 As it passes the position shown, the particle P has a constant speed $v = 100$ m/s along the straight line shown. Determine the corresponding values of $\dot{r}$, $\dot{\theta}$, $\ddot{r}$, and $\ddot{\theta}$.

Ans. $\dot{r} = -96.6$ m/s, $\dot{\theta} = 0.229$ rad/s
$\ddot{r} = 5.92$ m/s^2, $\ddot{\theta} = -0.391$ rad/s^2

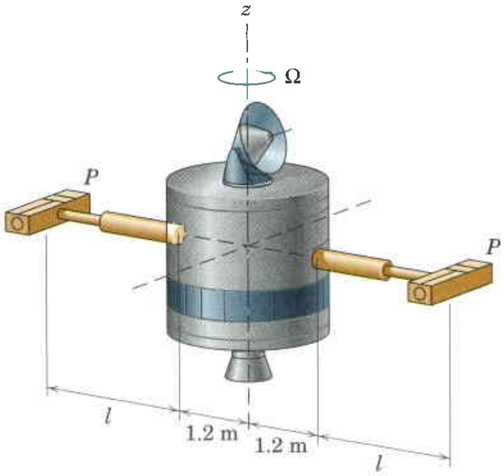


Problem 2/143

2/144 Repeat Prob. 2/143 but now the speed of the particle P is decreasing at the rate of 20 m/s^2 as it moves along the indicated straight path.

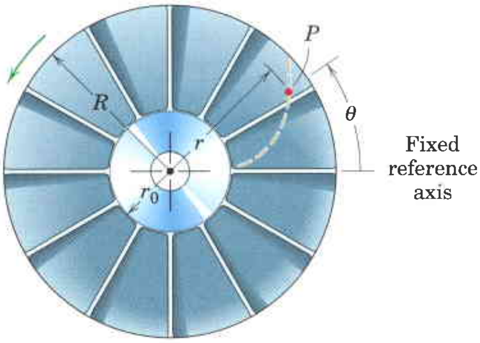
2/145 An internal mechanism is used to maintain a constant angular rate Ω = 0.05 rad/s about the z-axis of the spacecraft as the telescopic booms are extended at a constant rate. The length l is varied from essentially zero to 3 m. The maximum acceleration to which the sensitive experiment modules P may be subjected is 0.011 m/s^2. Determine the maximum allowable boom extension rate $\dot{l}$.

Ans. $\dot{l}$ = 32.8 mm/s



Problem 2/145

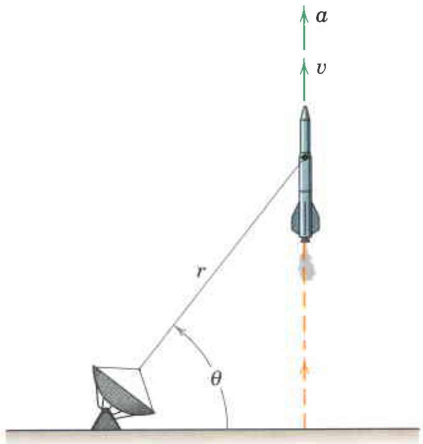
2/146 The radial position of a fluid particle P in a certain centrifugal pump with radial vanes is approximated by $r = r_0 \cosh Kt$, where t is time and $K = \dot{\theta}$ is the constant angular rate at which the impeller turns. Determine the expression for the magnitude of the total acceleration of the particle just prior to leaving the vane in terms of r_0, R, and K.



Problem 2/146

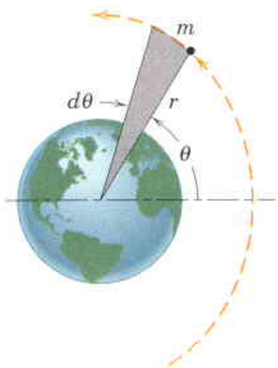
2/147 The rocket is fired vertically and tracked by the radar station shown. When θ reaches 60°, other corresponding measurements give the values r = 30,000 ft, $\ddot{r}$ = 70 ft/sec^2, and $\dot{\theta}$ = 0.02 rad/sec. Calculate the magnitudes of the velocity and acceleration of the rocket at this position.

Ans. v = 1200 ft/sec, a = 67.0 ft/sec^2



Problem 2/147

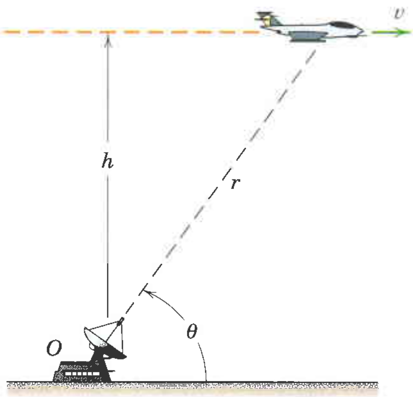
2/148 A satellite m moves in an elliptical orbit around the earth. There is no force on the satellite in the θ-direction, so that a_θ = 0. Prove Kepler's second law of planetary motion, which says that the radial line r sweeps through equal areas in equal times. The area dA swept by the radial line during time dt is shaded in the figure.



Problem 2/148

2/149 A jet plane flying at a constant speed v at an altitude $h = 10$ km is being tracked by radar located at O directly below the line of flight. If the angle θ is decreasing at the rate of 0.020 rad/s when $\theta = 60°$, determine the value of $\ddot{r}$ at this instant and the magnitude of the velocity $\mathbf{v}$ of the plane.

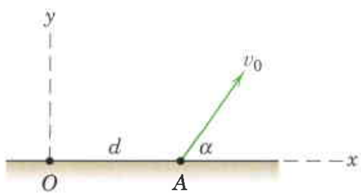
Ans. $\ddot{r} = 4.62\ \text{m/s}^2$, $v = 960$ km/h



Problem 2/149

Representative Problems

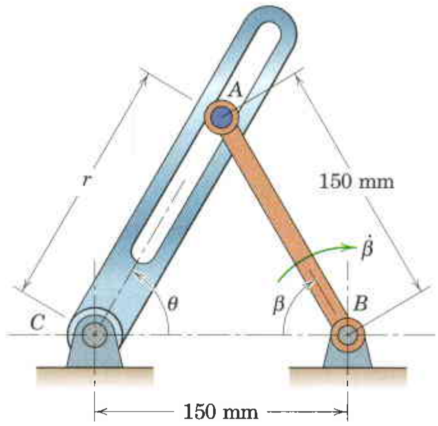
2/150 A projectile is launched from point A with the initial conditions shown. With the conventional definitions of r- and θ-coordinates relative to the Oxy coordinate system, determine r, θ $\dot{r}$, $\dot{\theta}$, $\ddot{r}$, and $\ddot{\theta}$ at the instant just after launch. Neglect aerodynamic drag.



Problem 2/150

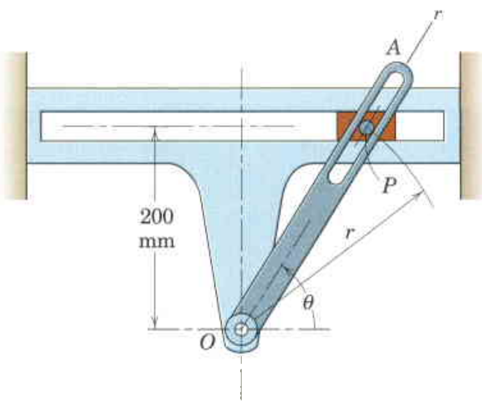
2/151 Link AB rotates through a limited range of the angle β, and its end A causes the slotted link AC to rotate also. For the instant represented where $\beta = 60°$ and $\dot{\beta} = 0.6$ rad/s constant, determine the corresponding values of $\dot{r}$, $\ddot{r}$, $\dot{\theta}$, and $\ddot{\theta}$. Make use of Eqs. 2/13 and 2/14.

Ans. $\dot{r} = 77.9$ mm/s, $\ddot{r} = -13.5\ \text{mm/s}^2$
$\dot{\theta} = -0.3$ rad/s, $\ddot{\theta} = 0$



Problem 2/151

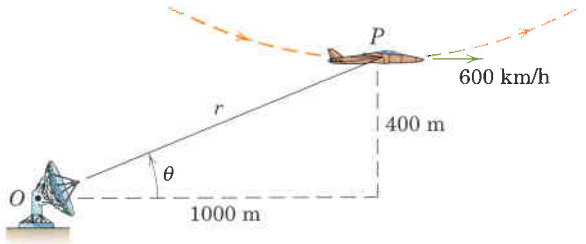
2/152 The fixed horizontal guide carries a slider and pin P whose motion is controlled by the rotating slotted arm OA. If the arm is revolving about O at the constant rate $\dot{\theta} = 2$ rad/s for an interval of its designed motion, determine the magnitudes of the velocity and acceleration of the slider in the slot for the instant when $\theta = 60°$. Also find the r-components of the velocity and acceleration.



Problem 2/152

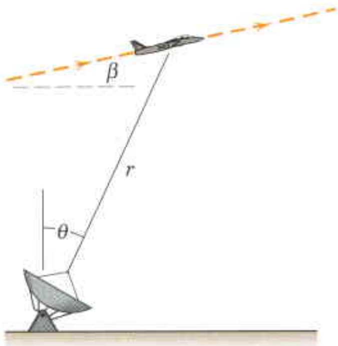
2/153 At the bottom of a loop in the vertical (r-θ) plane at an altitude of 400 m, the airplane P has a horizontal velocity of 600 km/h and no horizontal acceleration. The radius of curvature of the loop is 1200 m. For the radar tracking at O, determine the recorded values of $\ddot{r}$ and $\ddot{\theta}$ for this instant.

Ans. $\ddot{r} = 12.15$ m/s^2, $\ddot{\theta} = 0.0365$ rad/s^2



Problem 2/153

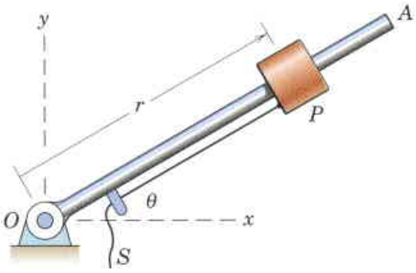
2/154 An aircraft flying in a straight line at a climb angle β to the horizontal is tracked by radar located directly below the line of flight. At a certain instant, the following data are recorded: $r = 12{,}000$ ft, $\dot{r} = 360$ ft/sec, $\ddot{r} = 19.60$ ft/sec^2, $\theta = 30°$, and $\dot{\theta} = 2.20$ deg/sec. For this instant, determine the aircraft altitude h, velocity v, angle of climb β, $\ddot{\theta}$, and acceleration a.



Problem 2/154

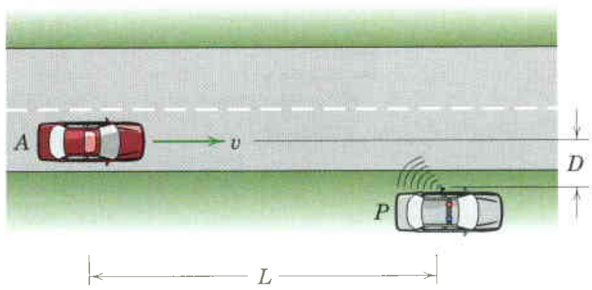
2/155 The slider P can be moved inward by means of the string S as the bar OA rotates about the pivot O. The angular position of the bar is given by $\theta = 0.4 + 0.12t + 0.06t^3$, where θ is in radians and t is in seconds. The position of the slider is given by $r = 0.8 - 0.1t - 0.05t^2$, where r is in meters and t is in seconds. Determine and sketch the velocity and acceleration of the slider at time $t = 2$ s. Find the angles α and β which **v** and **a** make with the positive x-axis.

Ans. $\mathbf{v} = -0.3\mathbf{e}_r + 0.336\mathbf{e}_\theta$ m/s
$\mathbf{a} = -0.382\mathbf{e}_r - 0.216\mathbf{e}_\theta$ m/s^2
$\alpha = 195.9°$, $\beta = -86.4°$



Problem 2/155

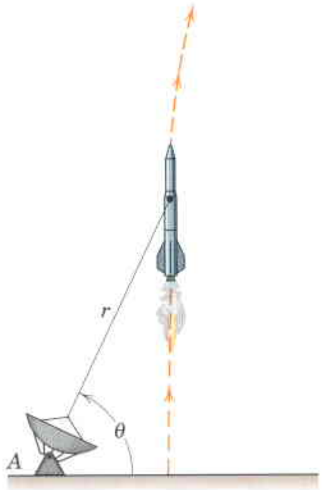
2/156 Car A is moving with constant speed v on the straight and level highway. The police officer in the stationary car P attempts to measure the speed v with radar. If the radar measures "line-of-sight" velocity, what velocity v' will the officer observe? Evaluate your general expression for the values $v = 70$ mi/hr, $L = 500$ ft, and $D = 20$ ft, and draw any appropriate conclusions.



Problem 2/156

2/157 A rocket follows a trajectory in the vertical plane and is tracked by radar from point A. At a certain instant, the radar measurements give $r = 35{,}000$ ft, $\dot{r} = 1600$ ft/sec, $\dot{\theta} = 0$, and $\ddot{\theta} = -0.00720$ rad/sec^2. Sketch the position of the rocket for this instant and determine the radius of curvature ρ of the trajectory at this position of the rocket.

Ans. $\rho = 10.16(10^3)$ ft

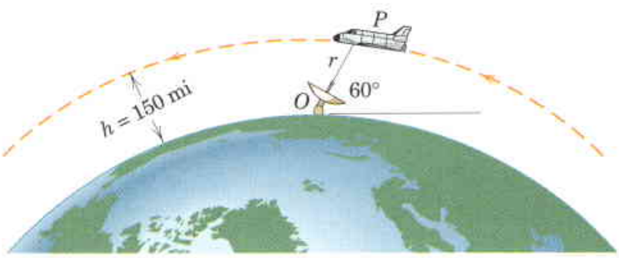


Problem 2/157

2/158 At a given instant, a particle has the following position, velocity, and acceleration components relative to a fixed x-y coordinate system: $x = 4$ m, $y = 2$ m, $\dot{x} = 2\sqrt{3}$ m/s, $\dot{y} = -2$ m/s, $\ddot{x} = -5$ m/s^2, $\ddot{y} = 5$ m/s^2. Determine the following properties associated with polar coordinates: θ, $\dot{\theta}$, $\ddot{\theta}$, r, $\dot{r}$, $\ddot{r}$. Sketch the geometry of your solution as you proceed.

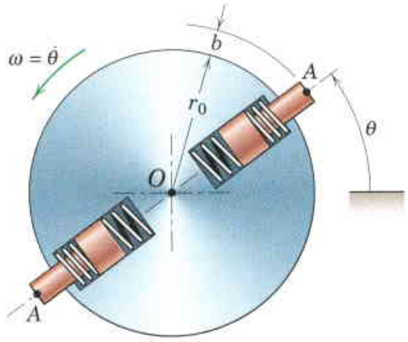
2/159 At the instant depicted in the figure, the radar station at O measures the range rate of the space shuttle P to be $\dot{r} = -12{,}272$ ft/sec, with O considered fixed. If it is known that the shuttle is in a circular orbit at an altitude $h = 150$ mi, determine the orbital speed of the shuttle from this information.

Ans. $v = 25{,}474$ ft/sec



Problem 2/159

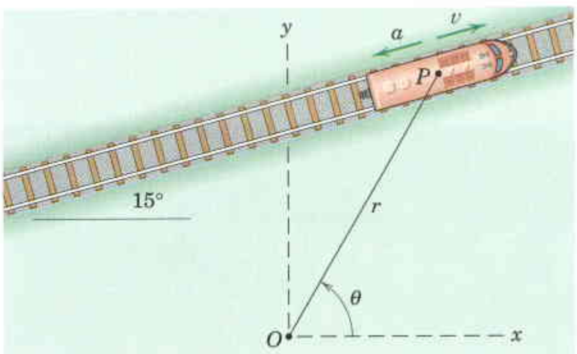
2/160 The circular disk rotates about its center O with a constant angular velocity $\omega = \dot{\theta}$ and carries the two spring-loaded plungers shown. The distance b which each plunger protrudes from the rim of the disk varies according to $b = b_0 \sin 2\pi nt$, where b_0 is the maximum protrusion, n is the constant frequency of oscillation of the plungers in the radial slots, and t is the time. Determine the maximum magnitudes of the r- and θ-components of the acceleration of the ends A of the plungers during their motion.



Problem 2/160

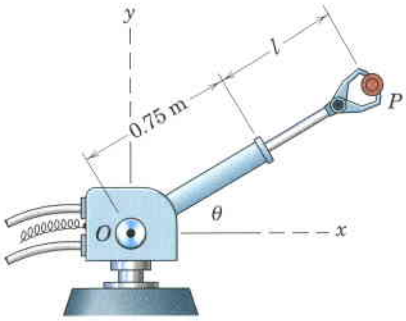
2/161 A locomotive is traveling on the straight and level track with a speed $v = 90$ km/h and a deceleration $a = 0.5$ m/s^2 as shown. Relative to the fixed observer at O, determine the quantities $\dot{r}$, $\ddot{r}$, $\dot{\theta}$, and $\ddot{\theta}$ at the instant when $\theta = 60°$ and $r = 400$ m.

Ans. $\dot{r} = 17.68$ m/s, $\dot{\theta} = -0.0442$ rad/s
$\ddot{r} = 0.428$ m/s^2, $\ddot{\theta} = 0.00479$ rad/s^2



Problem 2/161

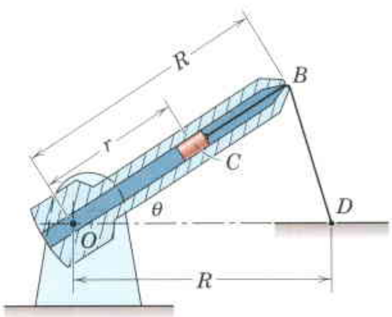
2/162 The robot arm is elevating and extending simultaneously. At a given instant, $\theta = 30°$, $\dot{\theta} = 10$ deg/s = constant, $l = 0.5$ m, $\dot{l} = 0.2$ m/s, and $\ddot{l} = -0.3$ m/s^2. Compute the magnitudes of the velocity **v** and acceleration **a** of the gripped part P. In addition, express **v** and **a** in terms of the unit vectors **i** and **j**.



Problem 2/162

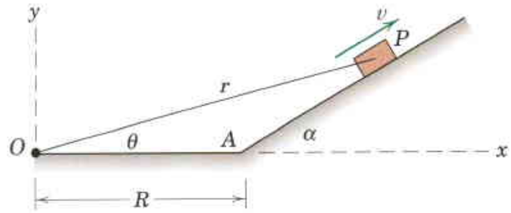
2/163 The slotted arm is pivoted at O and carries the slider C. The position of C in the slot is governed by the cord which is fastened at D and remains taut. The arm turns counterclockwise with a constant angular rate $\dot{\theta} = 4$ rad/sec during an interval of its motion. The length DBC of the cord equals R, which makes $r = 0$ when $\theta = 0$. Determine the magnitude a of the acceleration of the slider at the position for which $\theta = 30°$. The distance R is 15 in.

Ans. $a = 489$ in./sec^2



Problem 2/163

2/164 The small block P starts from rest at time $t = 0$ at point A and moves up the incline with constant acceleration a. Determine $\dot{r}$ as a function of time.

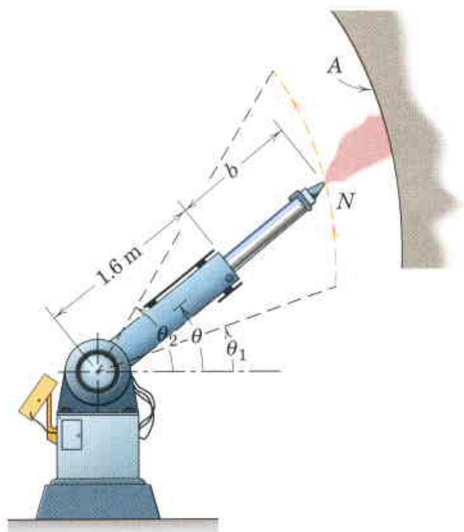


Problem 2/164

2/165 For the conditions of Prob. 2/164, determine $\dot{\theta}$ as a function of time.

$$Ans.\ \dot{\theta} = \frac{Rat \sin \alpha}{R^2 + Rat^2 \cos \alpha + \frac{1}{4}a^2t^4}$$

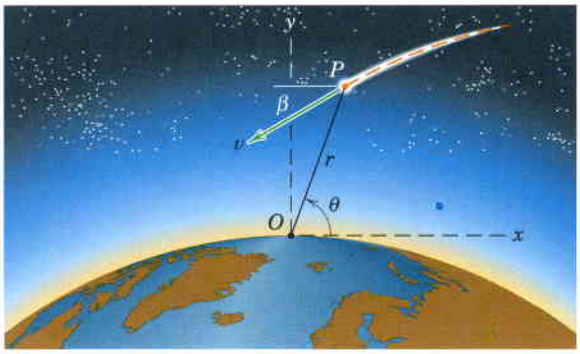
2/166 The paint-spraying robot is programmed to paint a production line of curved surfaces A (seen on edge). The length of the telescoping arm is controlled according to $b = 0.3 \sin (\pi t/2)$, where b is in meters and t is in seconds. Simultaneously, the arm is programmed to rotate according to $\theta = \pi/4 + (\pi/8) \sin (\pi t/2)$ radians. Calculate the magnitude v of the velocity of the nozzle N and the magnitude a of the acceleration of N for $t = 1$ s and for $t = 2$ s.



Problem 2/166

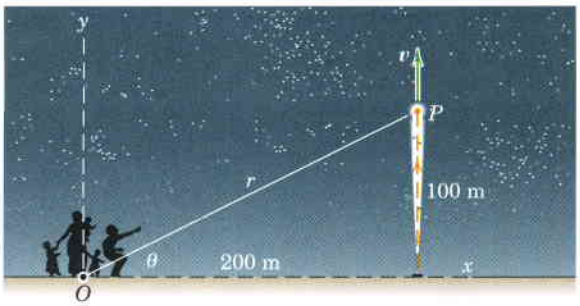
2/167 A meteor P is tracked by a radar observatory on the earth at O. When the meteor is directly overhead ($\theta = 90°$), the following observations are recorded: $r = 80$ km, $\dot{r} = -20$ km/s, and $\dot{\theta} = 0.4$ rad/s. (*a*) Determine the speed v of the meteor and the angle β which its velocity vector makes with the horizontal. Neglect any effects due to the earth's rotation. (*b*) Repeat with all given quantities remaining the same, except that $\theta = 75°$.

Ans. (*a*) $v = 37.7$ km/s, $\beta = 32.0°$
(*b*) $v = 37.7$ km/s, $\beta = 17.01°$



Problem 2/167

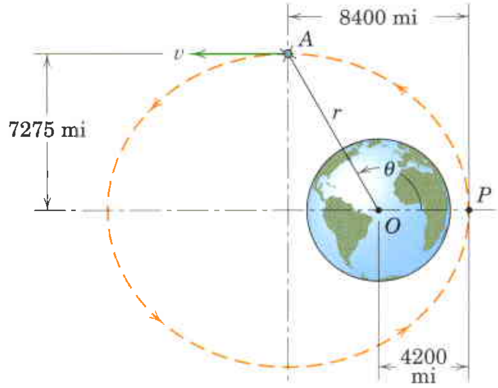
2/168 A fireworks shell P fired in a vertical trajectory has a y-acceleration given by $a_y = -g - kv^2$, where the latter term is due to aerodynamic drag. If the speed of the shell is 15 m/s at the instant shown, determine the corresponding values of r, $\dot{r}$, $\ddot{r}$, θ, $\dot{\theta}$, and $\ddot{\theta}$. The drag parameter k has a constant value of $0.01\ \text{m}^{-1}$.



Problem 2/168

2/169 An earth satellite traveling in the elliptical orbit shown has a velocity $v = 12{,}149$ mi/hr as it passes the end of the semiminor axis at A. The acceleration of the satellite at A is due to gravitational attraction and is $32.23[3959/8400]^2 = 7.159$ ft/sec^2 directed from A to O. For position A calculate the values of $\dot{r}$, $\ddot{r}$, $\dot{\theta}$, and $\ddot{\theta}$.

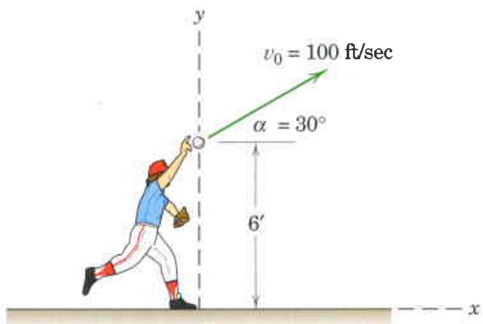
Ans. $\dot{r} = 8910$ ft/sec
$\ddot{r} = -1.790$ ft/sec^2
$\dot{\theta} = 3.48(10^{-4})$ rad/sec
$\ddot{\theta} = -1.398(10^{-7})$ rad/sec^2



Problem 2/169

▶2/170 The baseball player of Prob. 2/126 is repeated here with additional information supplied. At time $t = 0$, the ball is thrown with an initial speed of 100 ft/sec at an angle of 30° to the horizontal. Determine the quantities r, $\dot{r}$, $\ddot{r}$, θ, $\dot{\theta}$, and $\ddot{\theta}$, all relative to the x-y coordinate system shown, at time $t = 0.5$ sec.

Ans. $r = 51.0$ ft, $\dot{r} = 91.4$ ft/sec
$\ddot{r} = -11.35$ ft/sec^2, $\theta = 31.9°$
$\dot{\theta} = -0.334$ rad/sec, $\ddot{\theta} = 0.660$ rad/sec^2



Problem 2/170

2/7 Space Curvilinear Motion

The general case of three-dimensional motion of a particle along a space curve was introduced in Art. 2/1 and illustrated in Fig. 2/1. Three coordinate systems, rectangular (x-y-z), cylindrical (r-θ-z), and spherical (R-θ-ϕ), are commonly used to describe this motion. These systems are indicated in Fig. 2/16, which also shows the unit vectors for the three coordinate systems.*

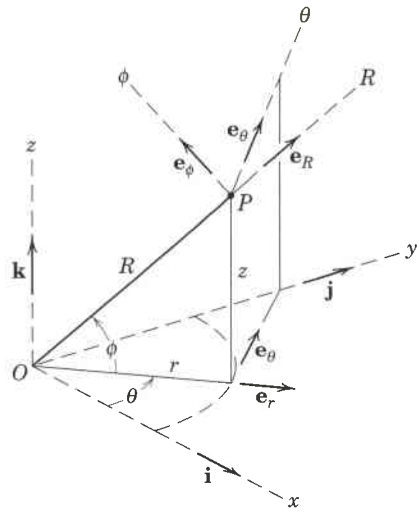


Figure 2/16

Before describing the use of these coordinate systems, we note that a path-variable description, using n- and t-coordinates, which we developed in Art. 2/5, can be applied in the osculating plane shown in Fig. 2/1. We defined this plane as the plane which contains the curve at the location in question. We see that the velocity $\mathbf{v}$, which is along the tangent t to the curve, lies in the osculating plane. The acceleration $\mathbf{a}$ also lies in the osculating plane. As in the case of plane motion, it has a component $a_t = \dot{v}$ tangent to the path due to the change in magnitude of the velocity and a component $a_n = v^2/\rho$ normal to the curve due to the change in direction of the velocity. As before, ρ is the radius of curvature of the path at the point in question and is measured in the osculating plane. This description of motion, which is natural and direct for many plane-motion problems, is awkward to use for space motion because the osculating plane continually shifts its orientation. We will confine our attention, therefore, to the three fixed coordinate systems shown in Fig. 2/16.

Rectangular Coordinates (x-y-z)

The extension from two to three dimensions offers no particular difficulty. We merely add the z-coordinate and its two time derivatives to the two-dimensional expressions of Eqs. 2/6 so that the position vector $\mathbf{R}$, the velocity $\mathbf{v}$, and the acceleration $\mathbf{a}$ become

$$\boxed{\begin{aligned} \mathbf{R} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\ \mathbf{v} = \dot{\mathbf{R}} &= \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \\ \mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{R}} &= \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k} \end{aligned}} \qquad (2/15)$$

Note that in three dimensions we are using $\mathbf{R}$ in place of $\mathbf{r}$ for the position vector.

Cylindrical Coordinates (r-θ-z)

If we understand the polar-coordinate description of plane motion, then there should be no difficulty with cylindrical coordinates because all that is required is the addition of the z-coordinate and its two time derivatives. The position vector $\mathbf{R}$ to the particle for cylindrical coordinates is simply

$$\mathbf{R} = r\mathbf{e}_r + z\mathbf{k}$$

*In a variation of spherical coordinates commonly used, angle ϕ is replaced by its complement.

In place of Eq. 2/13 for plane motion, we can write the velocity as

$$\boxed{\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{k}} \tag{2/16}$$

where

$$\begin{aligned} v_r &= r \\ v_\theta &= r\dot{\theta} \\ v_z &= \dot{z} \\ v &= \sqrt{v_r^{\,2} + v_\theta^{\,2} + v_z^{\,2}} \end{aligned}$$

Similarly, the acceleration is written by adding the z-component to Eq. 2/14, which gives us

$$\boxed{\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta + \ddot{z}\mathbf{k}} \tag{2/17}$$

where

$$\begin{aligned} a_r &= \ddot{r} - r\dot{\theta}^2 \\ a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) \\ a_z &= \ddot{z} \\ a &= \sqrt{a_r^{\,2} + a_\theta^{\,2} + a_z^{\,2}} \end{aligned}$$

Whereas the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ have nonzero time derivatives due to the changes in their directions, we note that the unit vector $\mathbf{k}$ in the z-direction remains fixed in direction and therefore has a zero time derivative.

Spherical Coordinates (R-θ-ϕ)

Spherical coordinates R, θ, ϕ are utilized when a radial distance and two angles are utilized to specify the position of a particle, as in the case of radar measurements, for example. Derivation of the expression for the velocity $\mathbf{v}$ is easily obtained, but the expression for the acceleration $\mathbf{a}$ is more complex because of the added geometry. Consequently, only the results will be cited here.* First we designate unit vectors $\mathbf{e}_R$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ as shown in Fig. 2/16. Note that the unit vector $\mathbf{e}_R$ is in the direction in which the particle P would move if R increases but θ and ϕ are held constant. The unit vector $\mathbf{e}_\theta$ is in the direction in which P would move if θ increases while R and ϕ are held constant. Finally, the unit vector $\mathbf{e}_\phi$ is in the direction in which P would move if ϕ increases while R and θ are held constant. The resulting expressions for $\mathbf{v}$ and $\mathbf{a}$ are

$$\boxed{\mathbf{v} = v_R\mathbf{e}_R + v_\theta\mathbf{e}_\theta + v_\phi\mathbf{e}_\phi} \tag{2/18}$$

where

$$\begin{aligned} v_R &= \dot{R} \\ v_\theta &= R\dot{\theta}\cos\phi \\ v_\phi &= R\dot{\phi} \end{aligned}$$

*For a complete derivation of $\mathbf{v}$ and $\mathbf{a}$ in spherical coordinates, see the first author's book *Dynamics*, 2nd edition, 1971, or SI Version, 1975 (John Wiley & Sons, Inc.).

and

$$\boxed{\mathbf{a} = a_R\mathbf{e}_R + a_\theta\mathbf{e}_\theta + a_\phi\mathbf{e}_\phi} \qquad \textbf{(2/19)}$$

where

$$a_R = \ddot{R} - R\dot{\phi}^2 - R\dot{\theta}^2\cos^2\phi$$

$$a_\theta = \frac{\cos\phi}{R}\frac{d}{dt}(R^2\dot{\theta}) - 2R\dot{\theta}\dot{\phi}\sin\phi$$

$$a_\phi = \frac{1}{R}\frac{d}{dt}(R^2\dot{\phi}) + R\dot{\theta}^2\sin\phi\cos\phi$$

Linear algebraic transformations between any two of the three coordinate-system expressions for velocity or acceleration can be developed. These transformations make it possible to express the motion component in rectangular coordinates, for example, if the components are known in spherical coordinates, or vice versa.* These transformations are easily handled with the aid of matrix algebra and a simple computer program.

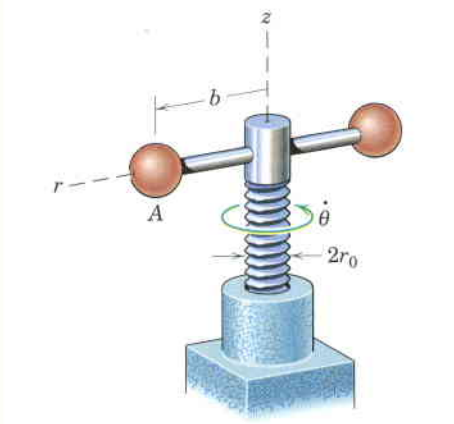
Dennis Macdonald/Index Stock

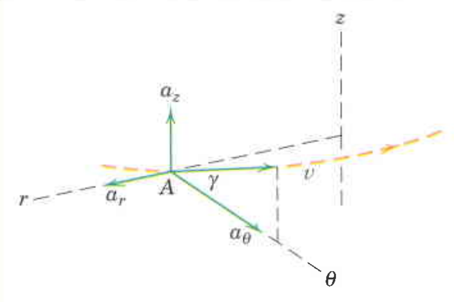
The track of this amusement-park ride has a helical shape.

*These coordinate transformations are developed and illustrated in the first author's book *Dynamics*, 2nd edition, 1971, or SI Version, 1975 (John Wiley & Sons, Inc.).

Sample Problem 2/11

The power screw starts from rest and is given a rotational speed $\dot{\theta}$ which increases uniformly with time t according to $\dot{\theta} = kt$, where k is a constant. Determine the expressions for the velocity v and acceleration a of the center of ball A when the screw has turned through one complete revolution from rest. The lead of the screw (advancement per revolution) is L.





Solution. The center of ball A moves in a helix on the cylindrical surface of radius b, and the cylindrical coordinates r, θ, z are clearly indicated.

Integrating the given relation for $\dot{\theta}$ gives $\theta = \Delta\theta = \int \dot{\theta}\, dt = \frac{1}{2}kt^2$. For one revolution from rest we have

$$2\pi = \frac{1}{2}kt^2$$

giving

$$t = 2\sqrt{\pi/k}$$

Thus, the angular rate at one revolution is

$$\dot{\theta} = kt = k(2\sqrt{\pi/k}) = 2\sqrt{\pi k}$$

① The helix angle γ of the path followed by the center of the ball governs the relation between the θ- and z-components of velocity and is given by $\tan\gamma = L/(2\pi b)$. Now from the figure we see that $v_\theta = v\cos\gamma$. Substituting $v_\theta = r\dot{\theta} = b\dot{\theta}$ ② from Eq. 2/16 gives $v = v_\theta/\cos\gamma = b\dot{\theta}/\cos\gamma$. With $\cos\gamma$ obtained from $\tan\gamma$ and with $\dot{\theta} = 2\sqrt{\pi k}$, we have for the one-revolution position

$$v = 2b\sqrt{\pi k}\,\frac{\sqrt{L^2 + 4\pi^2 b^2}}{2\pi b} = \sqrt{\frac{k}{\pi}}\sqrt{L^2 + 4\pi^2 b^2} \qquad \textit{Ans.}$$

The acceleration components from Eq. 2/17 become

③ $[a_r = \ddot{r} - r\dot{\theta}^2]$ $\qquad a_r = 0 - b(2\sqrt{\pi k})^2 = -4b\pi k$

$[a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}]$ $\qquad a_\theta = bk + 2(0)(2\sqrt{\pi k}) = bk$

$[a_z = \ddot{z} = \dot{v}_z]$

$$a_z = \frac{d}{dt}(v_z) = \frac{d}{dt}(v_\theta \tan\gamma) = \frac{d}{dt}(b\dot{\theta}\tan\gamma)$$
$$= (b\tan\gamma)\ddot{\theta} = b\frac{L}{2\pi b}k = \frac{kL}{2\pi}$$

Now we combine the components to give the magnitude of the total acceleration, which becomes

$$a = \sqrt{(-4b\pi k)^2 + (bk)^2 + \left(\frac{kL}{2\pi}\right)^2}$$
$$= bk\sqrt{(1 + 16\pi^2) + L^2/(4\pi^2 b^2)} \qquad \textit{Ans.}$$

Helpful Hints

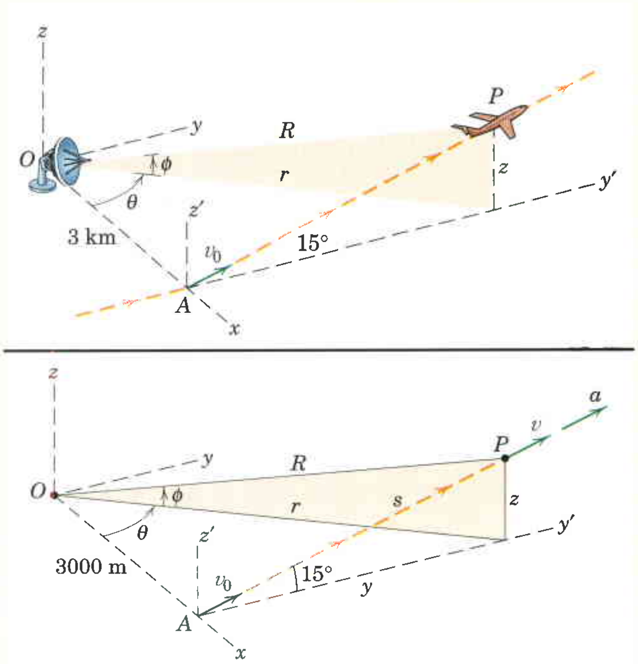
① We must be careful to divide the lead L by the circumference $2\pi b$ and not the diameter $2b$ to obtain $\tan\gamma$. If in doubt, unwrap one turn of the helix traced by the center of the ball.

② Sketch a right triangle and recall that for $\tan\beta = a/b$ the cosine of β becomes $b/\sqrt{a^2 + b^2}$.

③ The negative sign for a_r is consistent with our previous knowledge that the normal component of acceleration is directed toward the center of curvature.

Sample Problem 2/12

An aircraft P takes off at A with a velocity v_0 of 250 km/h and climbs in the vertical y'-z' plane at the constant 15° angle with an acceleration along its flight path of 0.8 m/s^2. Flight progress is monitored by radar at point O. (a) Resolve the velocity of P into cylindrical-coordinate components 60 seconds after takeoff and find $\dot{r}$, $\dot{\theta}$, and $\dot{z}$ for that instant. (b) Resolve the velocity of the aircraft P into spherical-coordinate components 60 seconds after takeoff and find $\dot{R}$, $\dot{\theta}$, and $\dot{\phi}$ for that instant.



(a)

Solution. (a) The accompanying figure shows the velocity and acceleration vectors in the y'-z' plane. The takeoff speed is

$$v_0 = \frac{250}{3.6} = 69.4 \text{ m/s}$$

and the speed after 60 seconds is

$$v = v_0 + at = 69.4 + 0.8(60) = 117.4 \text{ m/s}$$

The distance s traveled after takeoff is

$$s = s_0 + v_0 t + \frac{1}{2}at^2 = 0 + 69.4(60) + \frac{1}{2}(0.8)(60)^2 = 5610 \text{ m}$$

The y-coordinate and associated angle θ are

$$y = 5610 \cos 15° = 5420 \text{ m}$$

$$\theta = \tan^{-1}\frac{5420}{3000} = 61.0°$$

From the figure (b) of x-y projections, we have

$$r = \sqrt{3000^2 + 5420^2} = 6190 \text{ m}$$

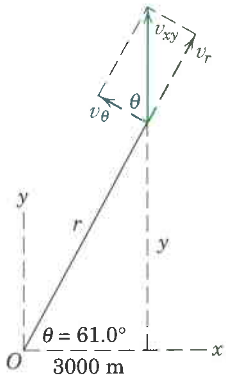
$$v_{xy} = v \cos 15° = 117.4 \cos 15° = 113.4 \text{ m/s}$$

$$v_r = \dot{r} = v_{xy} \sin\theta = 113.4 \sin 61.0° = 99.2 \text{ m/s} \qquad \textit{Ans.}$$

$$v_\theta = r\dot{\theta} = v_{xy}\cos\theta = 113.4 \cos 61.0° = 55.0 \text{ m/s}$$

So $$\dot{\theta} = \frac{55.0}{6190} = 8.88(10^{-3}) \text{ rad/s} \qquad \textit{Ans.}$$

Finally $$\dot{z} = v_z = v \sin 15° = 117.4 \sin 15° = 30.4 \text{ m/s} \qquad \textit{Ans.}$$



(b)

(b) Refer to the accompanying figure (c), which shows the x-y plane and various velocity components projected into the vertical plane containing r and R. Note that

$$z = y \tan 15° = 5420 \tan 15° = 1451 \text{ m}$$

$$\phi = \tan^{-1}\frac{z}{r} = \tan^{-1}\frac{1451}{6190} = 13.19°$$

$$R = \sqrt{r^2 + z^2} = \sqrt{6190^2 + 1451^2} = 6360 \text{ m}$$

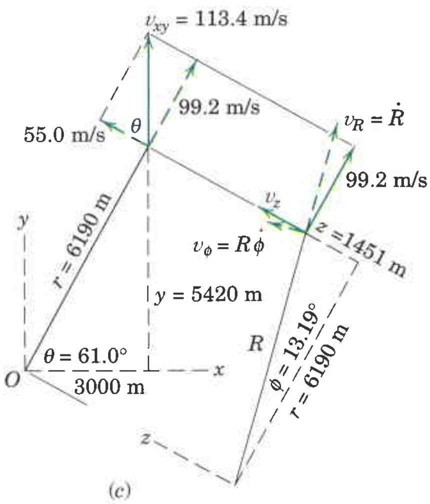
From the figure,

$$v_R = \dot{R} = 99.2 \cos 13.19° + 30.4 \sin 13.19° = 103.6 \text{ m/s} \qquad \textit{Ans.}$$

$$\dot{\theta} = 8.88(10^{-3}) \text{ rad/s, as in part } (a) \qquad \textit{Ans.}$$

$$v_\phi = R\dot{\phi} = 30.4 \cos 13.19° - 99.2 \sin 13.19° = 6.95 \text{ m/s}$$

$$\dot{\phi} = \frac{6.95}{6360} = 1.093(10^{-3}) \text{ rad/s} \qquad \textit{Ans.}$$



(c)

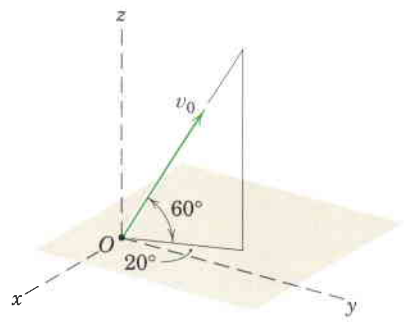
PROBLEMS

Introductory Problems

2/171 The rectangular coordinates of a particle are given in millimeters as functions of time t in seconds by $x = 30 \cos 2t$, $y = 40 \sin 2t$, and $z = 20t + 3t^2$. Determine the angle θ_1 between the position vector $\mathbf{r}$ and the velocity $\mathbf{v}$ and the angle θ_2 between the position vector $\mathbf{r}$ and the acceleration $\mathbf{a}$, both at time $t = 2$ s.

Ans. $\theta_1 = 60.8°$, $\theta_2 = 122.4°$

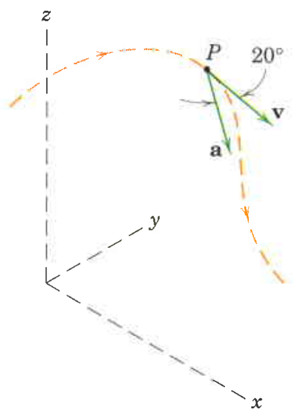
2/172 A projectile is launched from point O with an initial velocity of magnitude $v_0 = 600$ ft/sec, directed as shown in the figure. Compute the x-, y-, and z-components of position, velocity, and acceleration 20 seconds after launch. Neglect aerodynamic drag.



Problem 2/172

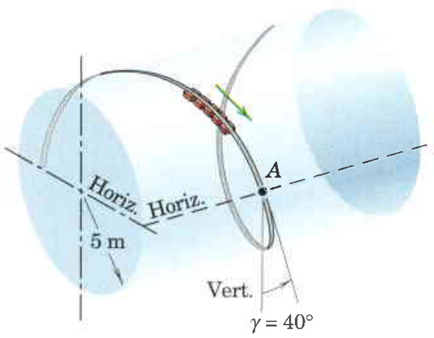
2/173 The particle P moves along the space curve and has a velocity $\mathbf{v} = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ m/s for the instant shown. At the same instant the particle has an acceleration $\mathbf{a}$ whose magnitude is 8 m/s^2. Calculate the radius of curvature ρ of the path for this position and the rate $\dot{v}$ at which the magnitude of the velocity is increasing.

Ans. $\rho = 7.67$ m, $\dot{v} = 7.52$ m/s^2



Problem 2/173

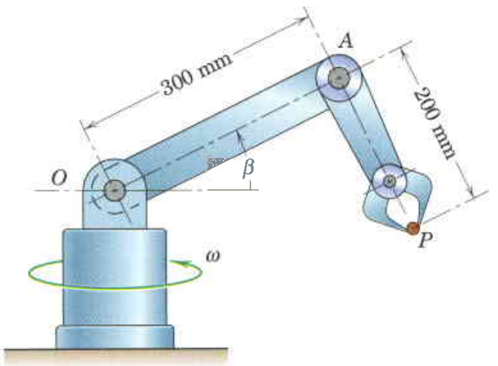
2/174 An amusement ride called the "corkscrew" takes the passengers through the upside-down curve of a horizontal cylindrical helix. The velocity of the cars as they pass position A is 15 m/s, and the component of their acceleration measured along the tangent to the path is $g \cos \gamma$ at this point. The effective radius of the cylindrical helix is 5 m, and the helix angle is $\gamma = 40°$. Compute the magnitude of the acceleration of the passengers as they pass position A.



Problem 2/174

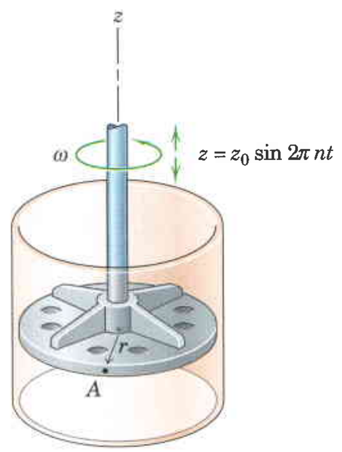
2/175 An industrial robot is being used to position a small part P. Calculate the magnitude of the acceleration $\mathbf{a}$ of P for the instant when $\beta = 30°$ if $\dot{\beta} = 10$ degrees per second and $\ddot{\beta} = 20$ degrees per second squared at this same instant. The base of the robot is revolving at the constant rate $\omega = 40$ degrees per second. During the motion arms AO and AP remain perpendicular.

Ans. $a = 219\ \text{mm/s}^2$



Problem 2/175

2/176 The rotating element in a mixing chamber is given a periodic axial movement $z = z_0 \sin 2\pi nt$ while it is rotating at the constant angular velocity $\dot{\theta} = \omega$. Determine the expression for the maximum magnitude of the acceleration of a point A on the rim of radius r. The frequency n of vertical oscillation is constant.

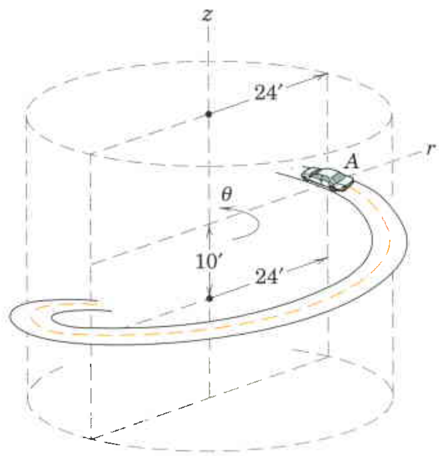


Problem 2/176

Representative Problems

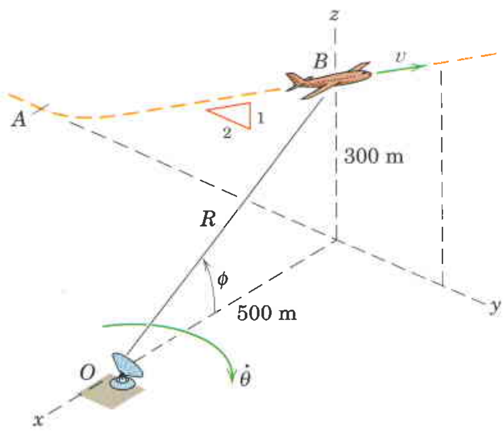
2/177 The car A is ascending a parking-garage ramp in the form of a cylindrical helix of 24-ft radius rising 10 ft for each half turn. At the position shown the car has a speed of 15 mi/hr, which is decreasing at the rate of 2 mi/hr per second. Determine the r-, θ-, and z-components of the acceleration of the car.

Ans. $a_r = -19.82\ \text{ft/sec}^2$
$a_\theta = -2.91\ \text{ft/sec}^2$
$a_z = -0.386\ \text{ft/sec}^2$



Problem 2/177

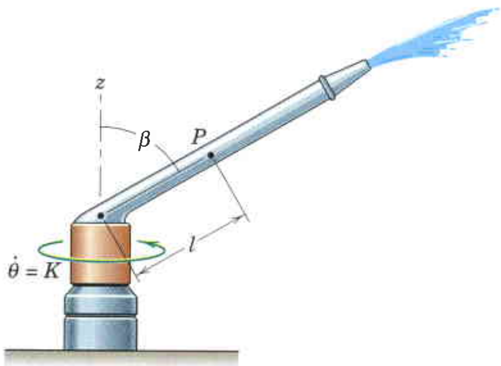
2/178 An aircraft takes off at A and climbs at a steady angle with slope of 1 to 2 in the vertical y-z plane at a constant speed $v = 400$ km/h. The aircraft is tracked by radar at O. For the position B, determine the values of $\dot{R}$, $\dot{\theta}$, and $\dot{\phi}$.



Problem 2/178

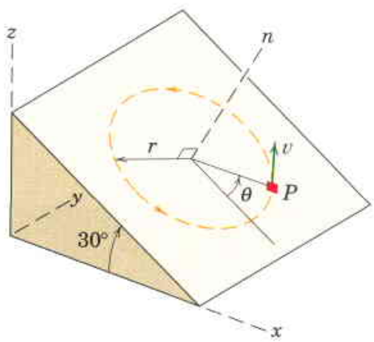
2/179 The rotating nozzle sprays a large circular area and turns with the constant angular rate $\dot{\theta} = K$. Particles of water move along the tube at the constant rate $\dot{l} = c$ relative to the tube. Write expressions for the magnitudes of the velocity and acceleration of a water particle P for a given position l in the rotating tube.

Ans. $v = \sqrt{c^2 + K^2l^2 \sin^2 \beta}$
$a = K \sin \beta \sqrt{K^2l^2 + 4c^2}$



Problem 2/179

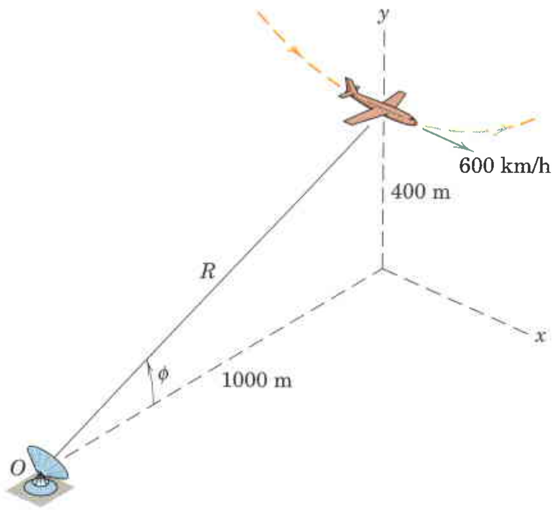
2/180 The small block P travels with constant speed v in the circular path of radius r on the inclined surface. If $\theta = 0$ at time $t = 0$, determine the x-, y-, and z-components of velocity and acceleration as functions of time.



Problem 2/180

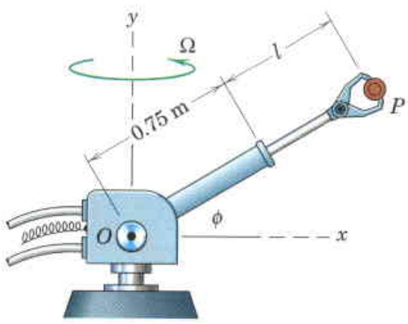
2/181 At the bottom of a vertical loop in the x-y plane at an altitude of 400 m, the airplane has a speed of 600 km/h with no horizontal x-acceleration. The radius of curvature of the loop at the bottom is 1200 m. For the radar tracking at O, determine the recorded values of $\ddot{R}$ and $\ddot{\phi}$ for this instant.

Ans. $\ddot{R} = 34.4$ m/s^2, $\ddot{\phi} = 0.01038$ rad/s^2



Problem 2/181

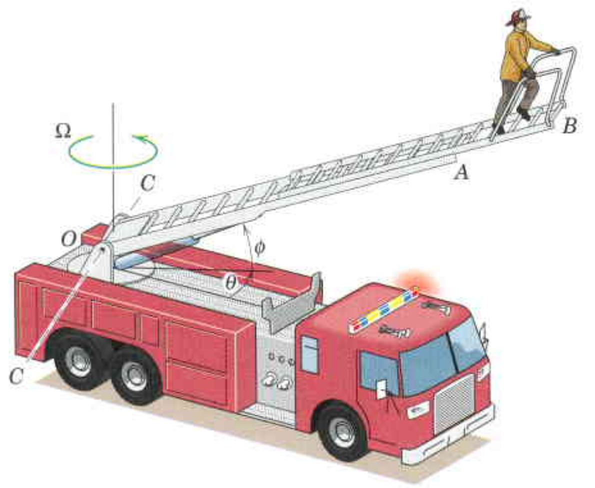
2/182 The robotic device of Prob. 2/162 now rotates about a fixed vertical axis while its arm extends and elevates. At a given instant, $\phi = 30°$, $\dot{\phi} = 10$ deg/s = constant, $l = 0.5$ m, $\dot{l} = 0.2$ m/s, $\ddot{l} = -0.3$ m/s^2, and $\Omega = 20$ deg/s = constant. Determine the magnitudes of the velocity $\mathbf{v}$ and the acceleration $\mathbf{a}$ of the gripped part P.



Problem 2/182

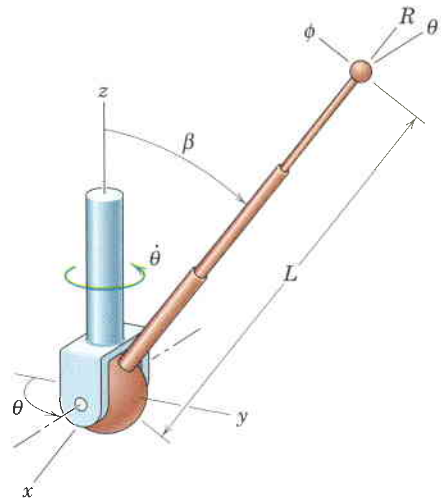
2/183 The base structure of the firetruck ladder rotates about a vertical axis through O with a constant angular velocity $\Omega = 10$ deg/s. At the same time, the ladder unit OB elevates at a constant rate $\dot{\phi} = 7$ deg/s, and section AB of the ladder extends from within section OA at the constant rate of 0.5 m/s. At the instant under consideration, $\phi = 30°$, $\overline{OA} = 9$ m, and $\overline{AB} = 6$ m. Determine the magnitudes of the velocity and acceleration of the end B of the ladder.

Ans. $v = 2.96$ m/s, $a = 0.672$ m/s^2



Problem 2/183

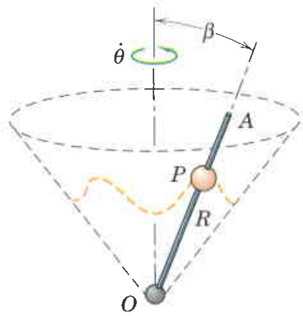
2/184 In a design test of the actuating mechanism for a telescoping antenna on a spacecraft, the supporting shaft rotates about the fixed z-axis with an angular rate $\dot{\theta}$. Determine the R-, θ-, and ϕ-components of the acceleration $\mathbf{a}$ of the end of the antenna at the instant when $L = 1.2$ m and $\beta = 45°$ if the rates $\dot{\theta} = 2$ rad/s, $\dot{\beta} = \frac{3}{2}$ rad/s, and $\dot{L} = 0.9$ m/s are constant during the motion.



Problem 2/184

2/185 The rod OA is held at the constant angle $\beta = 30°$ while it rotates about the vertical with a constant angular rate $\dot{\theta} = 120$ rev/min. Simultaneously, the sliding ball P oscillates along the rod with its distance in millimeters from the fixed pivot O given by $R = 200 + 50 \sin 2\pi nt$, where the frequency n of oscillation along the rod is a constant 2 cycles per second and where t is the time in seconds. Calculate the magnitude of the acceleration of P for an instant when its velocity along the rod from O toward A is a maximum.

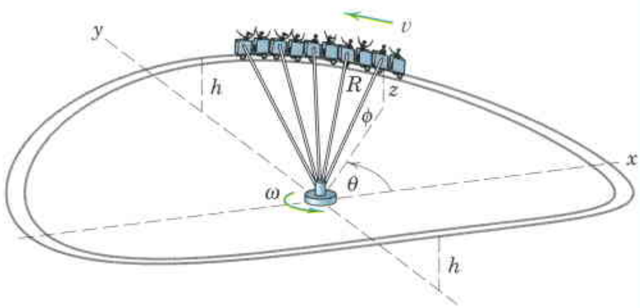
Ans. $a = 17.66$ m/s^2



Problem 2/185

►2/186 In the design of an amusement-park ride, the cars are attached to arms of length R which are hinged to a central rotating collar which drives the assembly about the vertical axis with a constant angular rate $\omega = \dot{\theta}$. The cars rise and fall with the track according to the relation $z = (h/2)(1 - \cos 2\theta)$. Find the R-, θ-, and ϕ-components of the velocity $\mathbf{v}$ of each car as it passes the position $\theta = \pi/4$ rad.

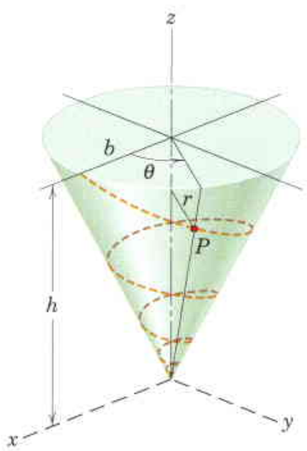
Ans. $v_R = 0$, $v_\theta = R\omega\sqrt{1 - (h/2R)^2}$
$v_\phi = h\omega/\sqrt{1 - (h/2R)^2}$



Problem 2/186

►2/187 The particle P moves down the spiral path which is wrapped around the surface of a right circular cone of base radius b and altitude h. The angle γ between the tangent to the curve at any point and a horizontal tangent to the cone at this point is constant. Also the motion of the particle is controlled so that $\dot{\theta}$ is constant. Determine the expression for the radial acceleration a_r of the particle for any value of θ.

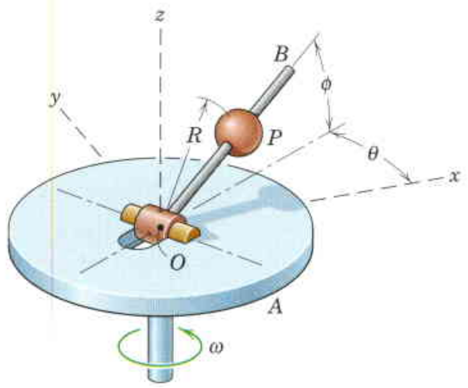
Ans. $a_r = b\dot{\theta}^2(\tan^2\gamma \sin^2\beta - 1)e^{-\theta \tan\gamma \sin\beta}$
where $\beta = \tan^{-1}(b/h)$



Problem 2/187

►2/188 The disk A rotates about the vertical z-axis with a constant speed $\omega = \dot{\theta} = \pi/3$ rad/s. Simultaneously, the hinged arm OB is elevated at the constant rate $\dot{\phi} = 2\pi/3$ rad/s. At time $t = 0$, both $\theta = 0$ and $\phi = 0$. The angle θ is measured from the fixed reference x-axis. The small sphere P slides out along the rod according to $R = 50 + 200t^2$, where R is in millimeters and t is in seconds. Determine the magnitude of the total acceleration $\mathbf{a}$ of P when $t = \frac{1}{2}$ s.

Ans. $a = 0.904$ m/s^2



Problem 2/188

2/8 Relative Motion (Translating Axes)

In the previous articles of this chapter, we have described particle motion using coordinates referred to fixed reference axes. The displacements, velocities, and accelerations so determined are termed *absolute*. It is not always possible or convenient, however, to use a fixed set of axes to describe or to measure motion. In addition, there are many engineering problems for which the analysis of motion is simplified by using measurements made with respect to a moving reference system. These measurements, when combined with the absolute motion of the moving coordinate system, enable us to determine the absolute motion in question. This approach is called a *relative-motion* analysis.

Courtesy of Blue Angels/U.S. Navy

Relative motion is a critical issue for the pilots of these Navy Blue Angel aircraft, even when the aircraft are not rotating.

Choice of Coordinate System

The motion of the moving coordinate system is specified with respect to a fixed coordinate system. Strictly speaking, in Newtonian mechanics, this fixed system is the primary inertial system, which is assumed to have no motion in space. For engineering purposes, the fixed system may be taken as any system whose absolute motion is negligible for the problem at hand. For most earthbound engineering problems, it is sufficiently precise to take for the fixed reference system a set of axes attached to the earth, in which case we neglect the motion of the earth. For the motion of satellites around the earth, a nonrotating coordinate system is chosen with its origin on the axis of rotation of the earth. For interplanetary travel, a nonrotating coordinate system fixed to the sun would be used. Thus, the choice of the fixed system depends on the type of problem involved.

We will confine our attention in this article to moving reference systems which translate but do not rotate. Motion measured in rotating systems will be discussed in Art. 5/7 of Chapter 5 on rigid-body kinematics, where this approach finds special but important application. We will also confine our attention here to relative-motion analysis for plane motion.

Vector Representation

Now consider two particles A and B which may have separate curvilinear motions in a given plane or in parallel planes, Fig. 2/17. We will arbitrarily attach the origin of a set of translating (nonrotating) axes x-y to particle B and observe the motion of A from our moving position on B. The position vector of A as measured relative to the frame x-y is $\mathbf{r}_{A/B} = x\mathbf{i} + y\mathbf{j}$, where the subscript notation "A/B" means "A relative to B" or "A with respect to B." The unit vectors along the x- and y-axes are $\mathbf{i}$ and $\mathbf{j}$, and x and y are the coordinates of A measured in the x-y frame. The absolute position of B is defined by the vector $\mathbf{r}_B$ measured from the origin of the fixed axes X-Y. The absolute position of A is seen, therefore, to be determined by the vector

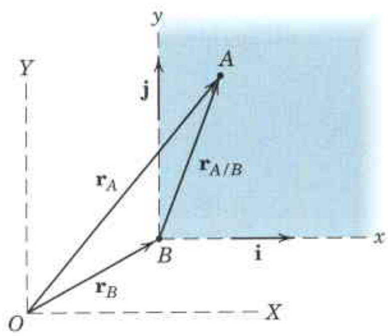


Figure 2/17

$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B}$$

We now differentiate this vector equation once with respect to time to obtain velocities and twice to obtain accelerations. Thus,

$$\dot{\mathbf{r}}_A = \dot{\mathbf{r}}_B + \dot{\mathbf{r}}_{A/B} \quad \text{or} \quad \boxed{\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}} \tag{2/20}$$

$$\ddot{\mathbf{r}}_A = \ddot{\mathbf{r}}_B + \ddot{\mathbf{r}}_{A/B} \quad \text{or} \quad \boxed{\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}} \tag{2/21}$$

In Eq. 2/20 the velocity which we observe A to have from our position at B attached to the moving axes x-y is $\dot{\mathbf{r}}_{A/B} = \mathbf{v}_{A/B} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}$. This term is the velocity of A with respect to B. Similarly, in Eq. 2/21 the acceleration which we observe A to have from our nonrotating position on B is $\ddot{\mathbf{r}}_{A/B} = \dot{\mathbf{v}}_{A/B} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}$. This term is the acceleration of A with respect to B. We note that the unit vectors $\mathbf{i}$ and $\mathbf{j}$ have zero derivatives because their directions as well as their magnitudes remain unchanged. (Later when we discuss rotating reference axes, we must account for the derivatives of the unit vectors when they change direction.)

Equation 2/20 (or 2/21) states that the absolute velocity (or acceleration) of A equals the absolute velocity (or acceleration) of B plus, vectorially, the velocity (or acceleration) of A relative to B. The relative term is the velocity (or acceleration) measurement which an observer attached to the moving coordinate system x-y would make. We can express the relative-motion terms in whatever coordinate system is convenient—rectangular, normal and tangential, or polar—and the formulations in the preceding articles can be used for this purpose. The appropriate fixed system of the previous articles becomes the moving system in the present article.

Additional Considerations

The selection of the moving point B for attachment of the reference coordinate system is arbitrary. As shown in Fig. 2/18, point A could be used just as well for the attachment of the moving system, in which case the three corresponding relative-motion equations for position, velocity, and acceleration are

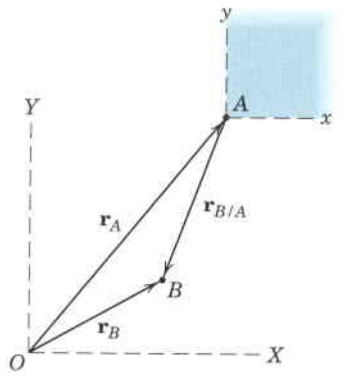


Figure 2/18

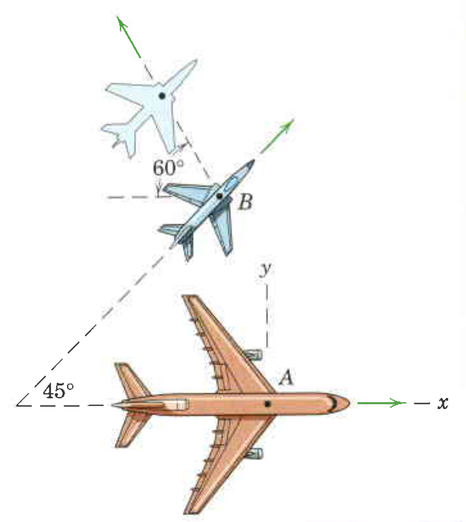
$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A} \qquad \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \qquad \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

It is seen, therefore, that $\mathbf{r}_{B/A} = -\mathbf{r}_{A/B}$, $\mathbf{v}_{B/A} = -\mathbf{v}_{A/B}$, and $\mathbf{a}_{B/A} = -\mathbf{a}_{A/B}$.

In relative-motion analysis, it is important to realize that the acceleration of a particle as observed in a translating system x-y is the same as that observed in a fixed system X-Y if the moving system has a constant velocity. This conclusion broadens the application of Newton's second law of motion (Chapter 3). We conclude, consequently, that a set of axes which has a constant absolute velocity may be used in place of a "fixed" system for the determination of accelerations. A translating reference system which has no acceleration is called an *inertial system*.

Sample Problem 2/13

Passengers in the jet transport A flying east at a speed of 800 km/h observe a second jet plane B that passes under the transport in horizontal flight. Although the nose of B is pointed in the 45° northeast direction, plane B appears to the passengers in A to be moving away from the transport at the 60° angle as shown. Determine the true velocity of B.



Solution. The moving reference axes x-y are attached to A, from which the relative observations are made. We write, therefore,

① $$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

Next we identify the knowns and unknowns. The velocity $\mathbf{v}_A$ is given in both magnitude and direction. The 60° direction of $\mathbf{v}_{B/A}$, the velocity which B appears to
② have to the moving observers in A, is known, and the true velocity of B is in the 45° direction in which it is heading. The two remaining unknowns are the magni-
③ tudes of $\mathbf{v}_B$ and $\mathbf{v}_{B/A}$. We may solve the vector equation in any one of three ways.

(I) Graphical. We start the vector sum at some point P by drawing $\mathbf{v}_A$ to a convenient scale and then construct a line through the tip of $\mathbf{v}_A$ with the known direction of $\mathbf{v}_{B/A}$. The known direction of $\mathbf{v}_B$ is then drawn through P, and the intersection C yields the unique solution enabling us to complete the vector triangle and scale off the unknown magnitudes, which are found to be

$$v_{B/A} = 586 \text{ km/h} \qquad \text{and} \qquad v_B = 717 \text{ km/h} \qquad \textit{Ans.}$$

(II) Trigonometric. A sketch of the vector triangle is made to reveal the trigonometry, which gives

④ $$\frac{v_B}{\sin 60°} = \frac{v_A}{\sin 75°} \qquad v_B = 800\frac{\sin 60°}{\sin 75°} = 717 \text{ km/h} \qquad \textit{Ans.}$$

(III) Vector Algebra. Using unit vectors $\mathbf{i}$ and $\mathbf{j}$, we express the velocities in vector form as

$$\mathbf{v}_A = 800\mathbf{i} \text{ km/h} \qquad \mathbf{v}_B = (v_B \cos 45°)\mathbf{i} + (v_B \sin 45°)\mathbf{j}$$

$$\mathbf{v}_{B/A} = (v_{B/A} \cos 60°)(-\mathbf{i}) + (v_{B/A} \sin 60°)\mathbf{j}$$

Substituting these relations into the relative-velocity equation and solving separately for the $\mathbf{i}$ and $\mathbf{j}$ terms give

$$(\mathbf{i}\text{-terms}) \qquad v_B \cos 45° = 800 - v_{B/A} \cos 60°$$

$$(\mathbf{j}\text{-terms}) \qquad v_B \sin 45° = v_{B/A} \sin 60°$$

⑤ Solving simultaneously yields the unknown velocity magnitudes

$$v_{B/A} = 586 \text{ km/h} \qquad \text{and} \qquad v_B = 717 \text{ km/h} \qquad \textit{Ans.}$$

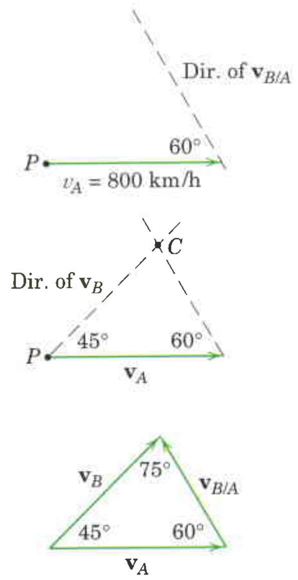
It is worth noting the solution of this problem from the viewpoint of an observer in B. With reference axes attached to B, we would write $\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$. The apparent velocity of A as observed by B is then $\mathbf{v}_{A/B}$, which is the negative of $\mathbf{v}_{B/A}$.

Helpful Hints

① We treat each airplane as a particle.

② We assume no side slip due to cross wind.

③ Students should become familiar with all three solutions.

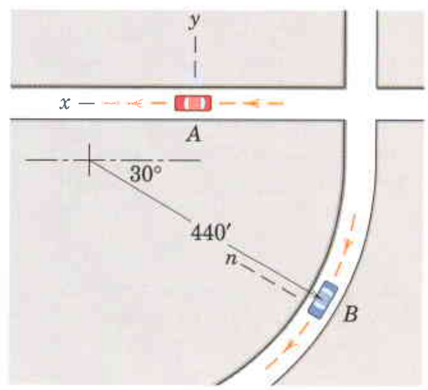


④ We must be prepared to recognize the appropriate trigonometric relation, which here is the law of sines.

⑤ We can see that the graphical or trigonometric solution is shorter than the vector algebra solution in this particular problem.

Sample Problem 2/14

Car A is accelerating in the direction of its motion at the rate of 3 ft/sec^2. Car B is rounding a curve of 440-ft radius at a constant speed of 30 mi/hr. Determine the velocity and acceleration which car B appears to have to an observer in car A if car A has reached a speed of 45 mi/hr for the positions represented.



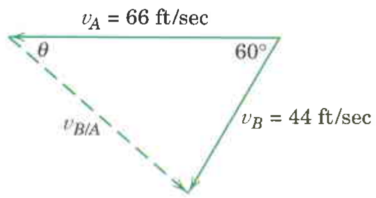
Solution. We choose nonrotating reference axes attached to car A since the motion of B with respect to A is desired.

Velocity. The relative-velocity equation is

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

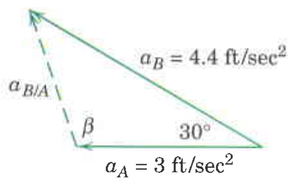
and the velocities of A and B for the position considered have the magnitudes

$$v_A = 45\frac{5280}{60^2} = 45\frac{44}{30} = 66 \text{ ft/sec} \qquad v_B = 30\frac{44}{30} = 44 \text{ ft/sec}$$



The triangle of velocity vectors is drawn in the sequence required by the equation, and application of the law of cosines and the law of sines gives

① $$v_{B/A} = 58.2 \text{ ft/sec} \qquad \theta = 40.9° \qquad \textit{Ans.}$$



Acceleration. The relative-acceleration equation is

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

The acceleration of A is given, and the acceleration of B is normal to the curve in the n-direction and has the magnitude

$$[a_n = v^2/\rho] \qquad a_B = (44)^2/440 = 4.4 \text{ ft/sec}^2$$

The triangle of acceleration vectors is drawn in the sequence required by the equation as illustrated. Solving for the x- and y-components of $\mathbf{a}_{B/A}$ gives us

$$(a_{B/A})_x = 4.4 \cos 30° - 3 = 0.810 \text{ ft/sec}^2$$

$$(a_{B/A})_y = 4.4 \sin 30° = 2.2 \text{ ft/sec}^2$$

$$\text{from which } a_{B/A} = \sqrt{(0.810)^2 + (2.2)^2} = 2.34 \text{ ft/sec}^2 \qquad \textit{Ans.}$$

The direction of $\mathbf{a}_{B/A}$ may be specified by the angle β which, by the law of sines, becomes

② $$\frac{4.4}{\sin\beta} = \frac{2.34}{\sin 30°} \qquad \beta = \sin^{-1}\left(\frac{4.4}{2.34}0.5\right) = 110.2° \qquad \textit{Ans.}$$

Helpful Hints

① Alternatively, we could use either a graphical or a vector algebraic solution.

② Be careful to choose between the two values 69.8° and 180 − 69.8 = 110.2°.

Suggestion: To gain familiarity with the manipulation of vector equations, it is suggested that the student rewrite the relative-motion equations in the form $\mathbf{v}_{B/A} = \mathbf{v}_B - \mathbf{v}_A$ and $\mathbf{a}_{B/A} = \mathbf{a}_B - \mathbf{a}_A$ and redraw the vector polygons to conform with these alternative relations.

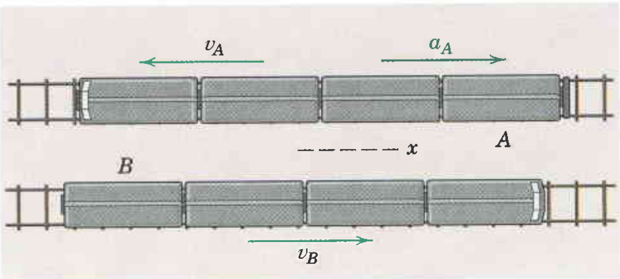
Caution: So far we are only prepared to handle motion relative to *nonrotating* axes. If we had attached the reference axes rigidly to car B, they would rotate with the car, and we would find that the velocity and acceleration terms relative to the rotating axes are *not* the negative of those measured from the nonrotating axes moving with A. Rotating axes are treated in Art. 5/7.

PROBLEMS

Introductory Problems

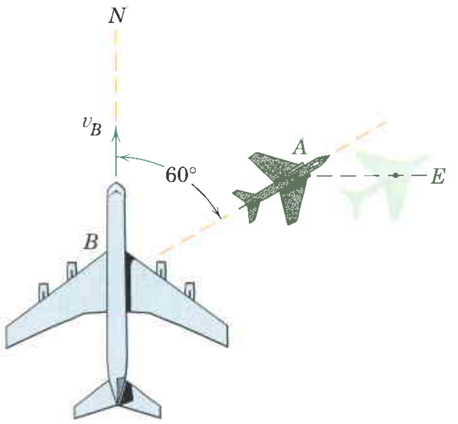
2/189 Rapid-transit trains A and B travel on parallel tracks. Train A has a speed of 80 km/h and is slowing at the rate of 2 m/s^2, while train B has a constant speed of 40 km/h. Determine the velocity and acceleration of train B relative to train A.

Ans. $\mathbf{v}_{B/A} = 120\mathbf{i}$ km/h, $\mathbf{a}_{B/A} = -2\mathbf{i}$ m/s^2



Problem 2/189

2/190 The jet transport B is flying north with a velocity $v_B = 600$ km/h when a smaller aircraft A passes underneath the transport headed in the 60° direction shown. To passengers in B, however, A appears to be flying sideways and moving east. Determine the actual velocity of A and the velocity which A appears to have relative to B.



Problem 2/190

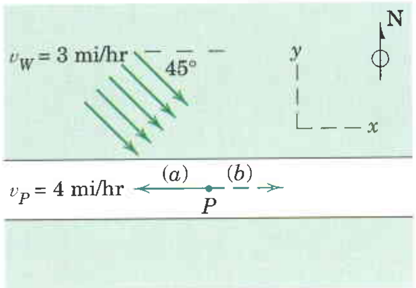
2/191 A woman P walks on an east-west street at a speed of 4 mi/hr. The wind blows out of the northwest as shown at a speed of 3 mi/hr. Determine the velocity of the wind relative to the woman if she (a) walks west and (b) walks east on the street. Express your results both in terms of unit vectors $\mathbf{i}$ and $\mathbf{j}$ and as magnitudes and compass directions.

Ans. (a) $\mathbf{v}_{w/p} = 6.12\mathbf{i} - 2.12\mathbf{j}$ mi/hr

$v_{w/p} = 6.48$ mi/hr at 19.11° south of east

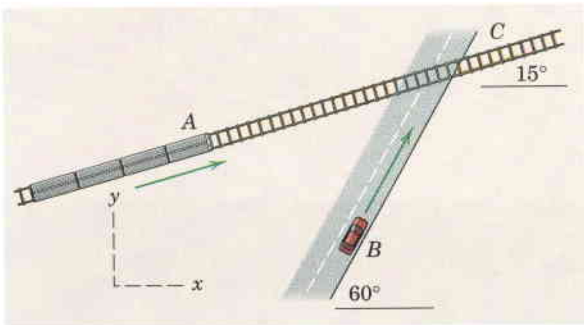
(b) $\mathbf{v}_{w/p} = -1.879\mathbf{i} - 2.12\mathbf{j}$ mi/hr

$v_{w/p} = 2.83$ mi/hr at 48.5° south of west



Problem 2/191

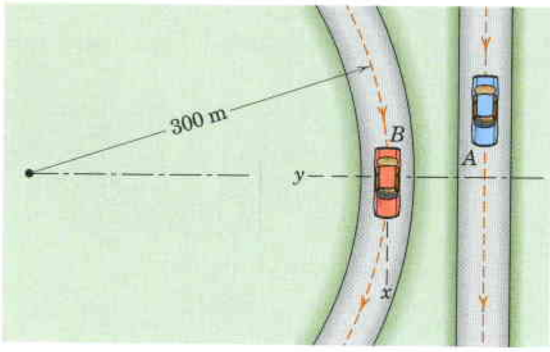
2/192 Train A travels with a constant speed $v_A = 120$ km/h along the straight and level track. The driver of car B, anticipating the railway grade crossing C, decreases the car speed of 90 km/h at the rate of 3 m/s^2. Determine the velocity and acceleration of the train relative to the car.



Problem 2/192

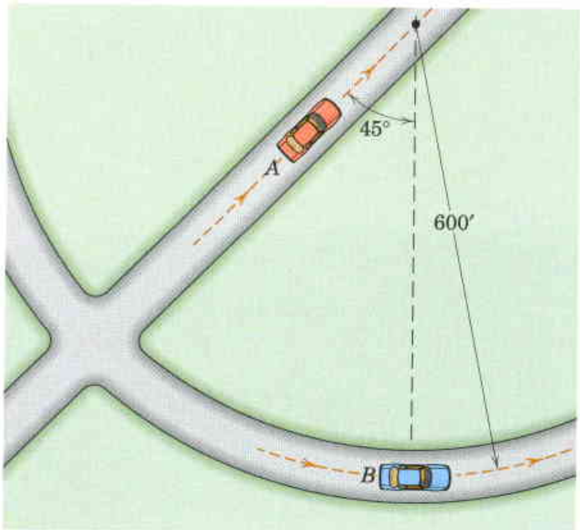
2/193 For the instant represented, car A has a speed of 100 km/h, which is increasing at the rate of 8 km/h each second. Simultaneously, car B also has a speed of 100 km/h as it rounds the turn and is slowing down at the rate of 8 km/h each second. Determine the acceleration that car B appears to have to an observer in car A.

Ans. $\mathbf{a}_{B/A} = -4.44\mathbf{i} + 2.57\mathbf{j}$ m/s^2



Problem 2/193

2/194 For the instant represented, car A has an acceleration in the direction of its motion, and car B has a speed of 45 mi/hr which is increasing. If the acceleration of B as observed from A is zero for this instant, determine the acceleration of A and the rate at which the speed of B is changing.



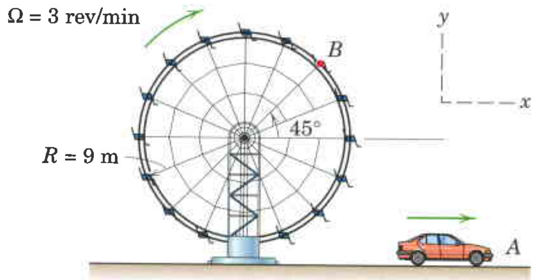
Problem 2/194

Representative Problems

2/195 The car A has a forward speed of 18 km/h and is accelerating at 3 m/s^2. Determine the velocity and acceleration of the car relative to observer B, who rides in a nonrotating chair on the Ferris wheel. The angular rate $\Omega = 3$ rev/min of the Ferris wheel is constant.

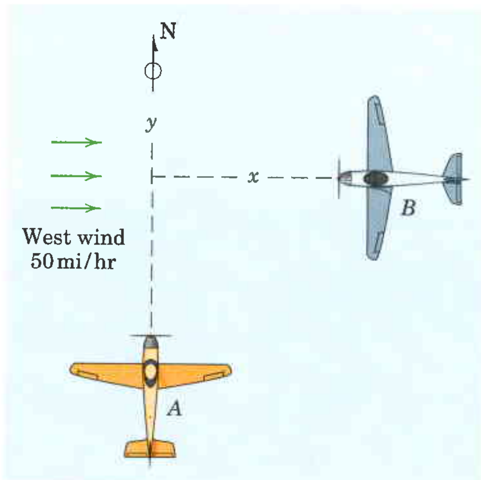
Ans. $\mathbf{v}_{A/B} = 3.00\mathbf{i} + 1.999\mathbf{j}$ m/s

$\mathbf{a}_{A/B} = 3.63\mathbf{i} + 0.628\mathbf{j}$ m/s^2



Problem 2/195

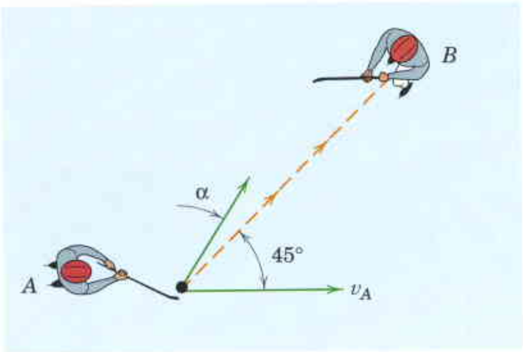
2/196 The small airplane A initially flying north with a ground speed of 150 mi/hr encounters a 50 mi/hr west wind (blowing east). Airplane B flying west with an airspeed of 180 mi/hr passes A at nearly the same altitude. Determine the magnitude and direction of the velocity which A appears to have to the pilot of B.



Problem 2/196

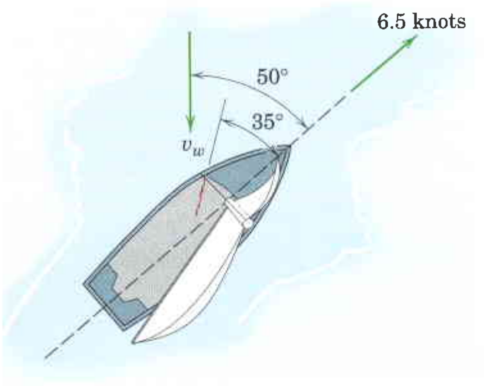
2/197 Hockey player A carries the puck on his stick and moves in the direction shown with a speed $v_A = 4$ m/s. In passing the puck to his stationary teammate B, by what angle α should the direction of his shot trail the line of sight if he launches the puck with a speed of 7 m/s relative to himself?

Ans. $\alpha = 23.8°$



Problem 2/197

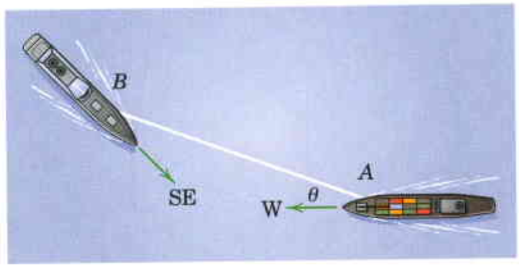
2/198 A sailboat moving in the direction shown is tacking to windward against a north wind. The log registers a hull speed of 6.5 knots. A "telltale" (light string tied to the rigging) indicates that the direction of the apparent wind is 35° from the centerline of the boat. What is the true wind velocity v_w?



Problem 2/198

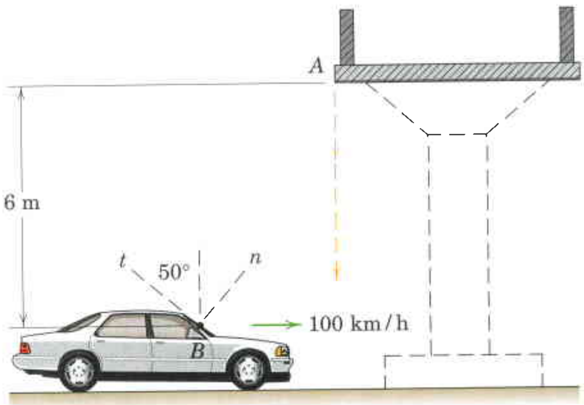
2/199 Ship A is headed west at a speed of 15 knots, and ship B is headed southeast. The relative bearing θ of B with respect to A is 20° and is unchanging. If the distance between A and B is 10 nautical miles at 2:00 P.M., when would collision occur if neither ship altered course?

Ans. 2:24 P.M.



Problem 2/199

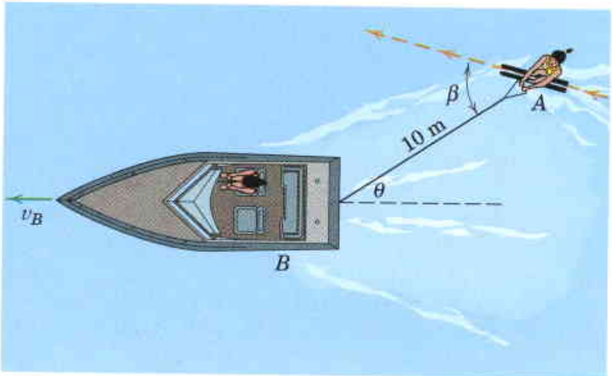
2/200 A drop of water falls with no initial speed from point A of a highway overpass. After dropping 6 m, it strikes the windshield at point B of a car which is traveling at a speed of 100 km/h on the horizontal road. If the windshield is inclined 50° from the vertical as shown, determine the angle θ relative to the normal n to the windshield at which the water drop strikes.



Problem 2/200

2/201 To increase his speed, the water skier A cuts across the wake of the tow boat B, which has a velocity of 60 km/h. At the instant when $\theta = 30°$, the actual path of the skier makes an angle $\beta = 50°$ with the tow rope. For this position determine the velocity v_A of the skier and the value of $\dot{\theta}$.

Ans. $v_A = 80.8$ km/h, $\dot{\theta} = 0.887$ rad/s

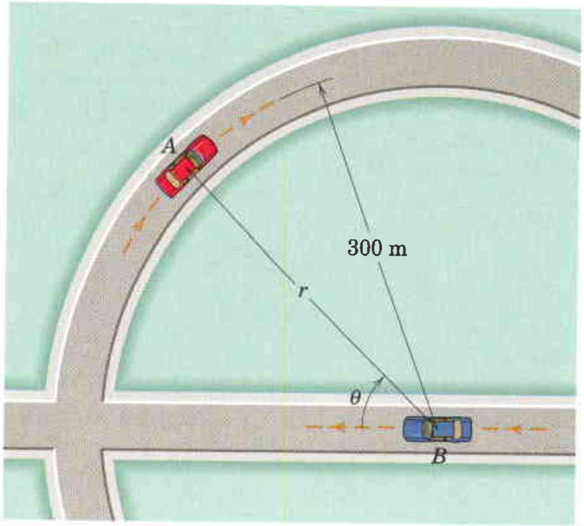


Problem 2/201

2/202 An earth satellite is put into a circular polar orbit at an altitude of 240 km, which requires an orbital velocity of 27 940 km/h with respect to the center of the earth considered fixed in space. In going from south to north, when the satellite passes over an observer on the equator, in which direction does the satellite appear to be moving? The equatorial radius of the earth is 6378 km, and the angular velocity of the earth is $0.729(10^{-4})$ rad/s.

2/203 Car A is traveling at the constant speed of 60 km/h as it rounds the circular curve of 300-m radius and at the instant represented is at the position $\theta = 45°$. Car B is traveling at the constant speed of 80 km/h and passes the center of the circle at this same instant. Car A is located with respect to car B by polar coordinates r and θ with the pole moving with B. For this instant determine $v_{A/B}$ and the values of $\dot{r}$ and $\dot{\theta}$ as measured by an observer in car B.

Ans. $v_{A/B} = 36.0$ m/s
$\dot{r} = -15.71$ m/s, $\dot{\theta} = 0.1079$ rad/s

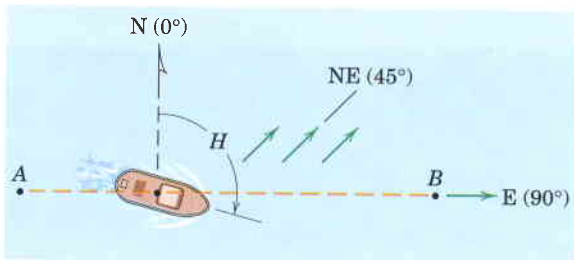


Problem 2/203

2/204 For the conditions of Prob. 2/203, determine the values of $\ddot{r}$ and $\ddot{\theta}$ as measured by an observer in car B at the instant represented. Use the results for $\dot{r}$ and $\dot{\theta}$ cited in the answers for that problem.

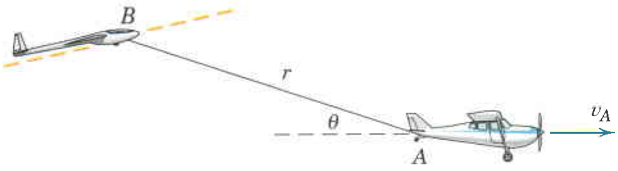
2/205 The captain of a small ship capable of making a speed of 6 knots through still water desires to set a course which will take the boat due east from A to B a distance of 10 nautical miles. To allow for a steady 2-knot current running northeast, determine his necessary compass heading H, measured clockwise from the north to the nearest degree. Also determine the time t of the trip. (Recall that 1 knot is 1 nautical mile per hour.)

Ans. $H = 104°$, $t = 1$ hr 23 min



Problem 2/205

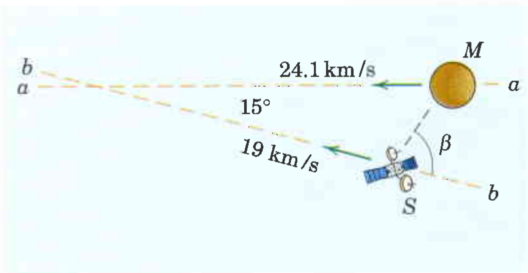
2/206 Airplane A is flying horizontally with a constant speed of 200 km/h and is towing the glider B, which is gaining altitude. If the tow cable has a length r = 60 m and θ is increasing at the constant rate of 5 degrees per second, determine the magnitudes of the velocity **v** and acceleration **a** of the glider for the instant when $\theta = 15°$.



Problem 2/206

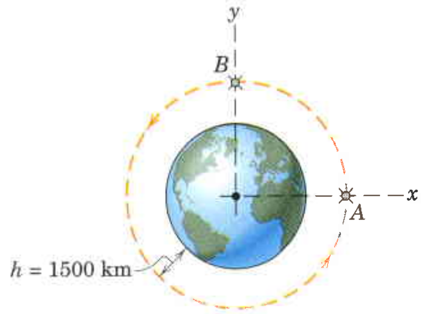
2/207 The spacecraft S approaches the planet Mars along a trajectory b–b in the orbital plane of Mars with an absolute velocity of 19 km/s. Mars has a velocity of 24.1 km/s along its trajectory a–a. Determine the angle β between the line of sight S–M and the trajectory b–b when Mars appears from the spacecraft to be approaching it head on.

Ans. $\beta = 55.6°$



Problem 2/207

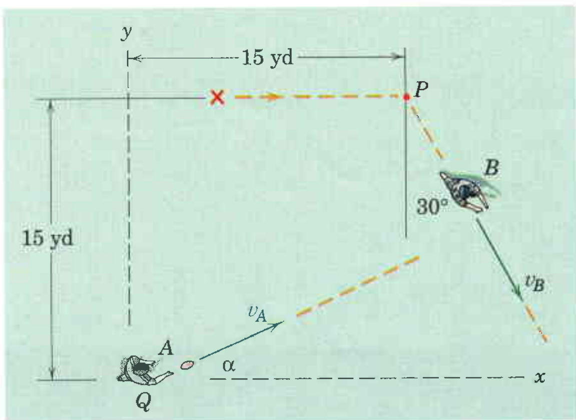
2/208 Satellites A and B are in a circular orbit of altitude h = 1500 km. Determine the magnitude of the acceleration of satellite B relative to a nonrotating observer in satellite A. Use $g_0 = 9.825$ m/s^2 for the surface-level gravitational acceleration and R = 6371 km for the radius of the earth.



Problem 2/208

2/209 After starting from the position marked with the "x", a football receiver B runs the slant-in pattern shown, making a cut at P and thereafter running with a constant speed v_B = 7 yd/sec in the direction shown. The quarterback releases the ball with a horizontal velocity of 100 ft/sec at the instant the receiver passes point P. Determine the angle α at which the quarterback must throw the ball, and the velocity of the ball relative to the receiver when the ball is caught. Neglect any vertical motion of the ball.

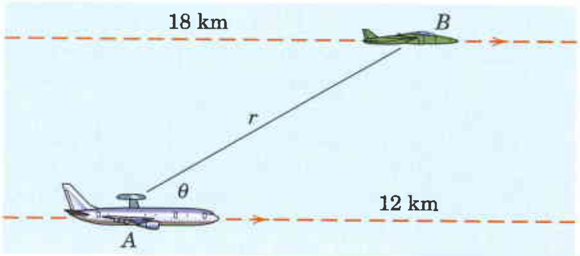
Ans. $\alpha = 33.3°$, $\mathbf{v}_{A/B} = 73.1\mathbf{i} + 73.1\mathbf{j}$ ft/sec



Problem 2/209

▶2/210 The aircraft A with radar detection equipment is flying horizontally at an altitude of 12 km and is increasing its speed at the rate of 1.2 m/s each second. Its radar locks onto an aircraft B flying in the same direction and in the same vertical plane at an altitude of 18 km. If A has a speed of 1000 km/h at the instant when $\theta = 30°$, determine the values of $\ddot{r}$ and $\ddot{\theta}$ at this same instant if B has a constant speed of 1500 km/h.

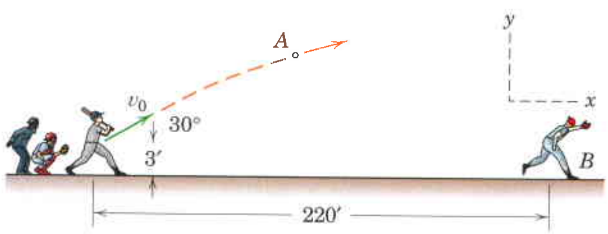
Ans. $\ddot{r} = -0.637$ m/s^2
$\ddot{\theta} = 1.660(10^{-4})$ rad/s^2



Problem 2/210

▶2/211 A batter hits the baseball A with an initial velocity of $v_0 = 100$ ft/sec directly toward fielder B at an angle of 30° to the horizontal; the initial position of the ball is 3 ft above ground level. Fielder B requires $\frac{1}{4}$ sec to judge where the ball should be caught and begins moving to that position with constant speed. Because of great experience, fielder B chooses his running speed so that he arrives at the "catch position" simultaneously with the baseball. The catch position is the field location at which the ball altitude is 7 ft. Determine the velocity of the ball relative to the fielder at the instant the catch is made.

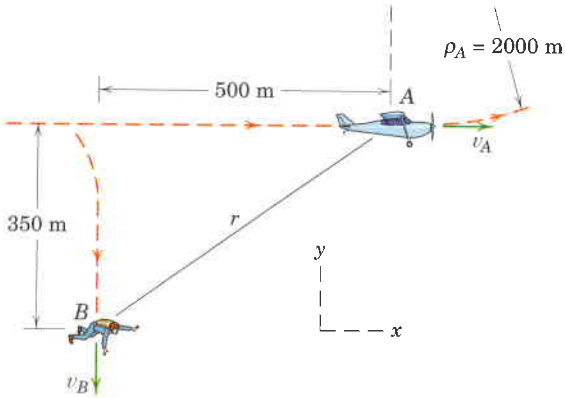
Ans. $\mathbf{v}_{A/B} = 71.5\mathbf{i} - 47.4\mathbf{j}$ ft/sec



Problem 2/211

▶2/212 At a certain instant after jumping from the airplane A, a skydiver B is in the position shown and has reached a terminal (constant) speed $v_B = 50$ m/s. The airplane has the same constant speed $v_A = 50$ m/s, and after a period of level flight is just beginning to follow the circular path shown of radius $\rho_A = 2000$ m. (*a*) Determine the velocity and acceleration of the airplane relative to the skydiver. (*b*) Determine the time rate of change of the speed v_r of the airplane and the radius of curvature ρ_r of its path, both as observed by the nonrotating skydiver.

Ans. (*a*) $\mathbf{v}_{A/B} = 50\mathbf{i} + 50\mathbf{j}$ m/s
$\mathbf{a}_{A/B} = 1.250\mathbf{j}$ m/s^2
(*b*) $\dot{v}_r = 0.884$ m/s^2, $\rho_r = 5660$ m



Problem 2/212

2/9 Constrained Motion of Connected Particles

Sometimes the motions of particles are interrelated because of the constraints imposed by interconnecting members. In such cases it is necessary to account for these constraints in order to determine the respective motions of the particles.

One Degree of Freedom

Consider first the very simple system of two interconnected particles A and B shown in Fig. 2/19. It should be quite evident by inspection that the horizontal motion of A is twice the vertical motion of B. Nevertheless we will use this example to illustrate the method of analysis which applies to more complex situations where the results cannot be easily obtained by inspection. The motion of B is clearly the same as that of the center of its pulley, so we establish position coordinates y and x measured from a convenient fixed datum. The total length of the cable is

$$L = x + \frac{\pi r_2}{2} + 2y + \pi r_1 + b$$

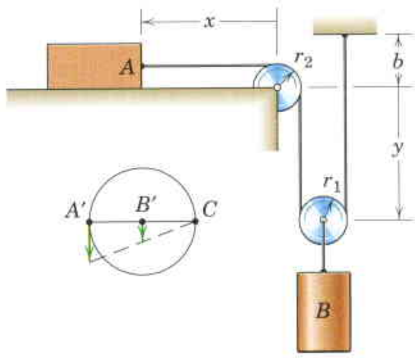


Figure 2/19

With L, r_2, r_1, and b all constant, the first and second time derivatives of the equation give

$$0 = \dot{x} + 2\dot{y} \qquad \text{or} \qquad 0 = v_A + 2v_B$$
$$0 = \ddot{x} + 2\ddot{y} \qquad \text{or} \qquad 0 = a_A + 2a_B$$

The velocity and acceleration constraint equations indicate that, for the coordinates selected, the velocity of A must have a sign which is opposite to that of the velocity of B, and similarly for the accelerations. The constraint equations are valid for the motion of the system in either direction. We emphasize that $v_A = \dot{x}$ is positive to the left and that $v_B = \dot{y}$ is positive down.

Because the results do not depend on the lengths or pulley radii, we should be able to analyze the motion without considering them. In the lower-left portion of Fig. 2/19 is shown an enlarged view of the horizontal diameter $A'B'C'$ of the lower pulley at an instant of time. Clearly, A' and A have the same motion magnitudes, as do B and B'. During an infinitesimal motion of A', it is easy to see from the triangle that B' moves half as far as A' because point C as a point on the fixed portion of the cable momentarily has no motion. Thus, with differentiation by time in mind, we can obtain the velocity and acceleration magnitude relationships by inspection. The pulley, in effect, is a wheel which rolls on the fixed vertical cable. (The kinematics of a rolling wheel will be treated more extensively in Chapter 5 on rigid-body motion.) The system of Fig. 2/19 is said to have *one degree of freedom* since only one variable, either x or y, is needed to specify the positions of all parts of the system.

Two Degrees of Freedom

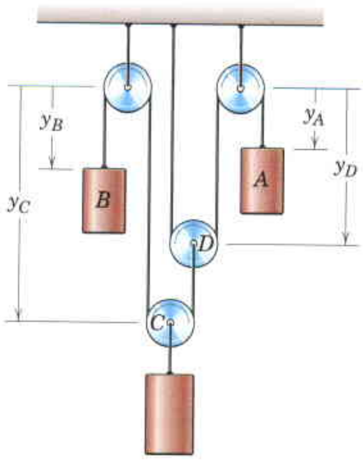


Figure 2/20

The system with *two degrees of freedom* is shown in Fig. 2/20. Here the positions of the lower cylinder and pulley C depend on the separate specifications of the two coordinates y_A and y_B. The lengths of the cables attached to cylinders A and B can be written, respectively, as

$$L_A = y_A + 2y_D + \text{constant}$$

$$L_B = y_B + y_C + (y_C - y_D) + \text{constant}$$

and their time derivatives are

$$0 = \dot{y}_A + 2\dot{y}_D \qquad \text{and} \qquad 0 = \dot{y}_B + 2\dot{y}_C - \dot{y}_D$$

$$0 = \ddot{y}_A + 2\ddot{y}_D \qquad \text{and} \qquad 0 = \ddot{y}_B + 2\ddot{y}_C - \ddot{y}_D$$

Eliminating the terms in $\dot{y}_D$ and $\ddot{y}_D$ gives

$$\dot{y}_A + 2\dot{y}_B + 4\dot{y}_C = 0 \qquad \text{or} \qquad v_A + 2v_B + 4v_C = 0$$

$$\ddot{y}_A + 2\ddot{y}_B + 4\ddot{y}_C = 0 \qquad \text{or} \qquad a_A + 2a_B + 4a_C = 0$$

It is clearly impossible for the signs of all three terms to be positive simultaneously. So, for example, if both A and B have downward (positive) velocities, then C will have an upward (negative) velocity.

These results can also be found by inspection of the motions of the two pulleys at C and D. For an increment dy_A (with y_B held fixed), the center of D moves up an amount $dy_A/2$, which causes an upward movement $dy_A/4$ of the center of C. For an increment dy_B (with y_A held fixed), the center of C moves up a distance $dy_B/2$. A combination of the two movements gives an upward movement

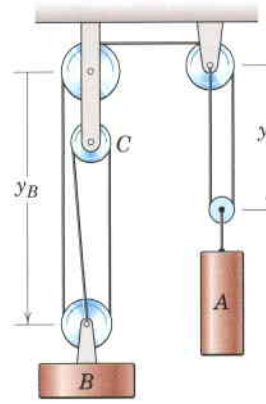
$$-dy_C = \frac{dy_A}{4} + \frac{dy_B}{2}$$

so that $-v_C = v_A/4 + v_B/2$ as before. Visualization of the actual geometry of the motion is an important ability.

A second type of constraint where the direction of the connecting member changes with the motion is illustrated in the second of the two sample problems which follow.

Sample Problem 2/15

In the pulley configuration shown, cylinder A has a downward velocity of 0.3 m/s. Determine the velocity of B. Solve in two ways.



Solution (I). The centers of the pulleys at A and B are located by the coordinates y_A and y_B measured from fixed positions. The total constant length of cable in the pulley system is

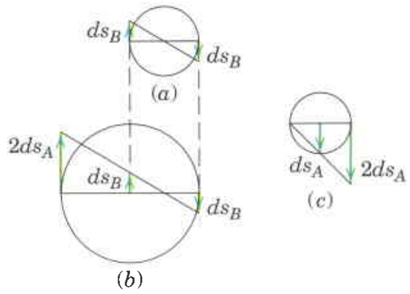
$$L = 3y_B + 2y_A + \text{constants}$$

where the constants account for the fixed lengths of cable in contact with the circumferences of the pulleys and the constant vertical separation between the two upper left-hand pulleys. Differentiation with time gives ①

$$0 = 3\dot{y}_B + 2\dot{y}_A$$

Substitution of $v_A = \dot{y}_A = 0.3$ m/s and $v_B = \dot{y}_B$ gives

② $$0 = 3(v_B) + 2(0.3) \qquad \text{or} \qquad v_B = -0.2 \text{ m/s} \qquad \textit{Ans.}$$



Solution (II). An enlarged diagram of the pulleys at A, B, and C is shown. During a differential movement ds_A of the center of pulley A, the left end of its horizontal diameter has no motion since it is attached to the fixed part of the cable. Therefore, the right-hand end has a movement of $2ds_A$ as shown. This movement is transmitted to the left-hand end of the horizontal diameter of the pulley at B. Further, from pulley C with its fixed center, we see that the displacements on each side are equal and opposite. Thus, for pulley B, the right-hand end of the diameter has a downward displacement equal to the upward displacement ds_B of its center. By inspection of the geometry, we conclude that

$$2ds_A = 3ds_B \qquad \text{or} \qquad ds_B = \tfrac{2}{3}ds_A$$

Dividing by dt gives

$$|v_B| = \tfrac{2}{3}v_A = \tfrac{2}{3}(0.3) = 0.2 \text{ m/s (upward)} \qquad \textit{Ans.}$$

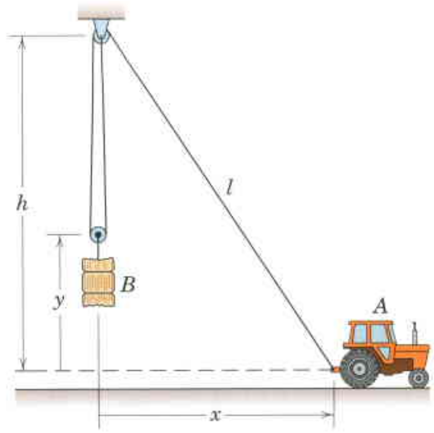
Helpful Hints

① We neglect the small angularity of the cables between B and C.

② The negative sign indicates that the velocity of B is *upward*.

Sample Problem 2/16

The tractor A is used to hoist the bale B with the pulley arrangement shown. If A has a forward velocity v_A, determine an expression for the upward velocity v_B of the bale in terms of x.



Solution. We designate the position of the tractor by the coordinate x and the position of the bale by the coordinate y, both measured from a fixed reference. The total constant length of the cable is

$$L = 2(h - y) + l = 2(h - y) + \sqrt{h^2 + x^2}$$

① Differentiation with time yields

$$0 = -2\dot{y} + \frac{x\dot{x}}{\sqrt{h^2 + x^2}}$$

Substituting $v_A = \dot{x}$ and $v_B = \dot{y}$ gives

$$v_B = \frac{1}{2}\frac{xv_A}{\sqrt{h^2 + x^2}} \qquad \textit{Ans.}$$

Helpful Hint

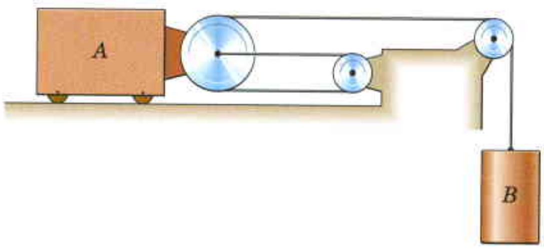
① Differentiation of the relation for a right triangle occurs frequently in mechanics.

PROBLEMS

Introductory Problems

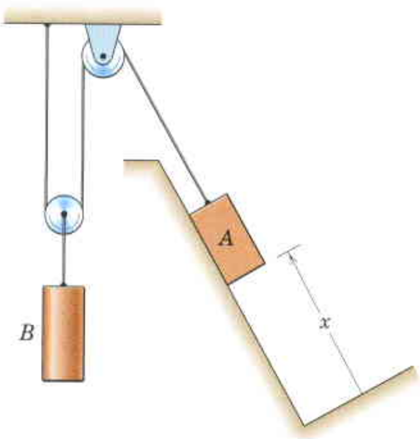
2/213 If block A has a velocity of 3.6 ft/sec to the right, determine the velocity of cylinder B.

Ans. $v_B = 10.8$ ft/sec down



Problem 2/213

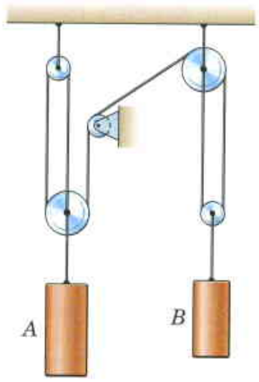
2/214 If the velocity $\dot{x}$ of block A up the incline is increasing at the rate of 0.044 m/s each second, determine the acceleration of B.



Problem 2/214

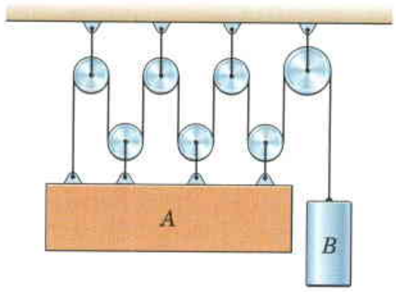
2/215 At a certain instant, cylinder A has a downward velocity of 0.8 m/s and an upward acceleration of 2 m/s^2. Determine the corresponding velocity and acceleration of cylinder B.

Ans. $v_B = 1.2$ m/s up
$a_B = 3$ m/s^2 down



Problem 2/215

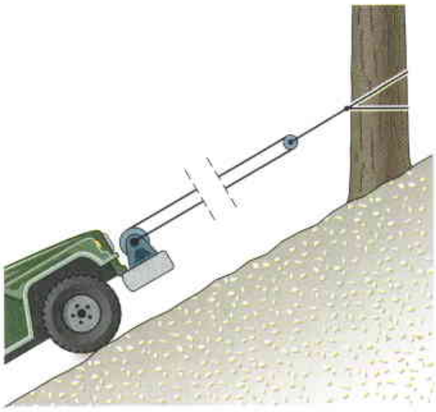
2/216 Determine the constraint equation which relates the accelerations of bodies A and B. Assume that the upper surface of A remains horizontal.



Problem 2/216

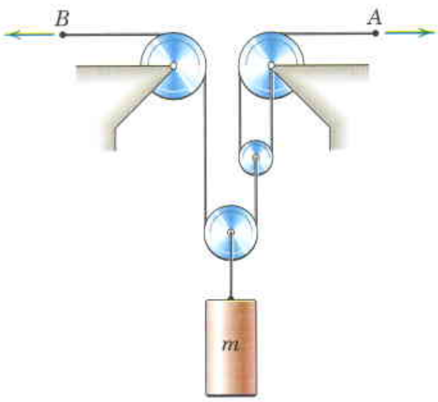
2/217 A truck equipped with a power winch on its front end pulls itself up a steep incline with the cable and pulley arrangement shown. If the cable is wound up on the drum at the constant rate of 40 mm/s, how long does it take for the truck to move 4 m up the incline?

Ans. $t = 3$ min 20 s



Problem 2/217

2/218 For the pulley system shown, each of the cables at A and B is given a velocity of 2 m/s in the direction of the arrow. Determine the upward velocity v of the load m.

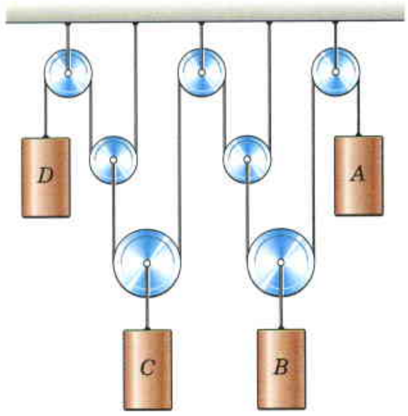


Problem 2/218

Representative Problems

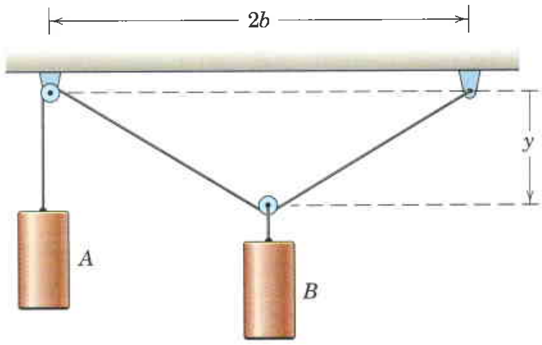
2/219 Determine the relationship which governs the velocities of the four cylinders. Express all velocities as positive down. How many degrees of freedom are there?

Ans. $4v_A + 8v_B + 4v_C + v_D = 0$
3 degrees of freedom



Problem 2/219

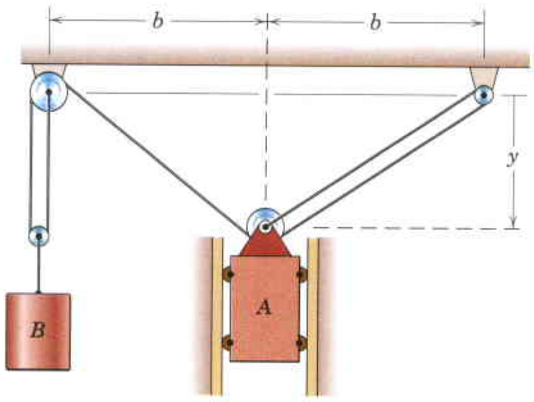
2/220 For a given value of y, determine the upward velocity of A in terms of the downward velocity of B. Neglect the diameters of the pulleys.



Problem 2/220

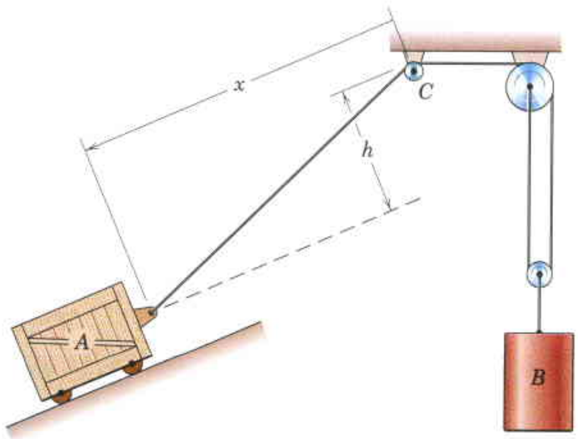
2/221 Neglect the diameters of the small pulleys and establish the relationship between the velocity of A and the velocity of B for a given value of y.

$$Ans.\ v_B = -\frac{3yv_A}{2\sqrt{y^2 + b^2}}$$



Problem 2/221

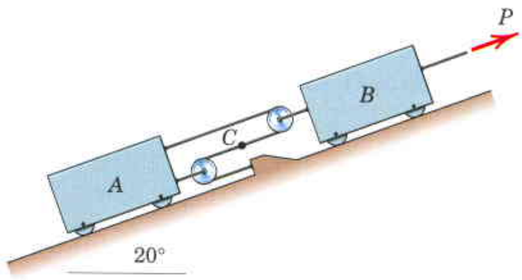
2/222 Determine an expression for the velocity v_A of the cart A down the incline in terms of the upward velocity v_B of cylinder B.



Problem 2/222

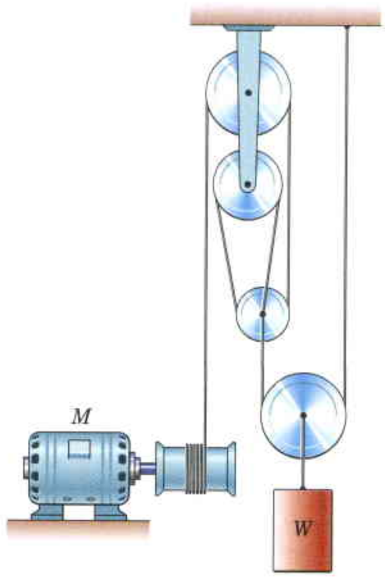
2/223 Under the action of force P, the constant acceleration of block B is 6 ft/sec^2 up the incline. For the instant when the velocity of B is 3 ft/sec up the incline, determine the velocity of B relative to A, the acceleration of B relative to A, and the absolute velocity of point C of the cable.

Ans. $v_{B/A} = 1$ ft/sec, $a_{B/A} = 2$ ft/sec^2
$v_C = 4$ ft/sec



Problem 2/223

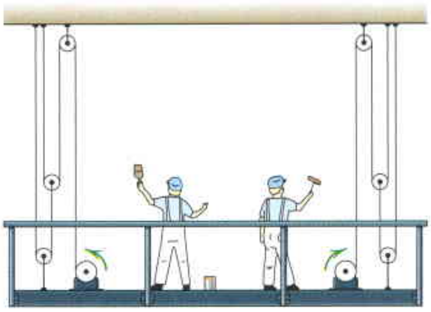
2/224 Determine the vertical rise h of the load W during 10 seconds if the hoisting drum draws in cable at the constant rate of 180 mm/s.



Problem 2/224

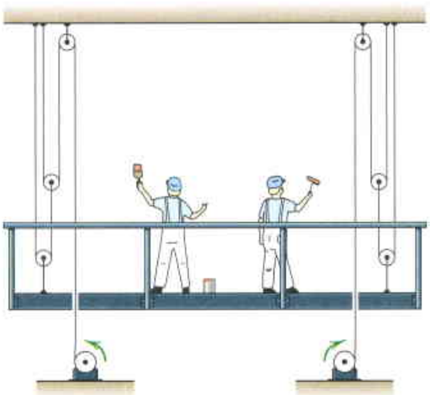
2/225 The power winches on the industrial scaffold enable it to be raised or lowered. For rotation in the sense indicated, the scaffold is being raised. If each drum has a diameter of 200 mm and turns at the rate of 40 rev/min, determine the upward velocity v of the scaffold.

Ans. $v = 83.8$ mm/s



Problem 2/225

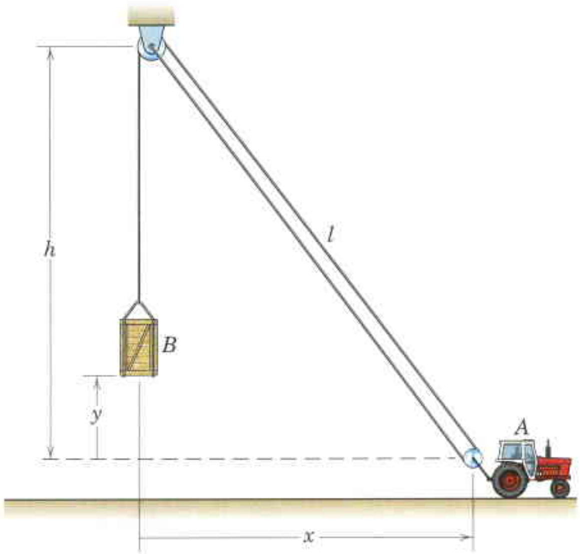
2/226 The scaffold of Prob. 2/225 is modified here by placing the power winches on the ground instead of on the scaffold. Other conditions remain the same. Calculate the upward velocity v of the scaffold.



Problem 2/226

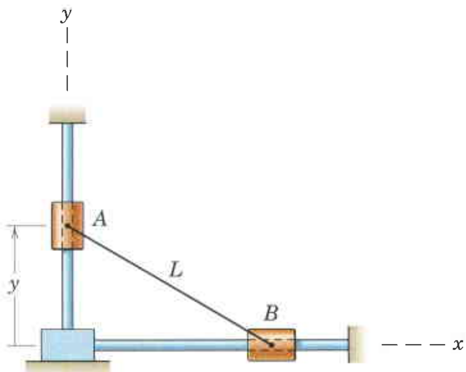
2/227 In order to speed up the hoisting of bales depicted in Sample Problem 2/16, the pulley arrangement is altered as shown here. If the tractor A has a forward velocity v_A, determine an expression for the upward velocity v_B of the bale in terms of x. Neglect the small distance between the tractor and its pulley so that both have essentially the same motion. Compare your results with those for Sample Problem 2/16.

Ans. $v_B = \dfrac{2xv_A}{\sqrt{x^2 + h^2}}$



Problem 2/227

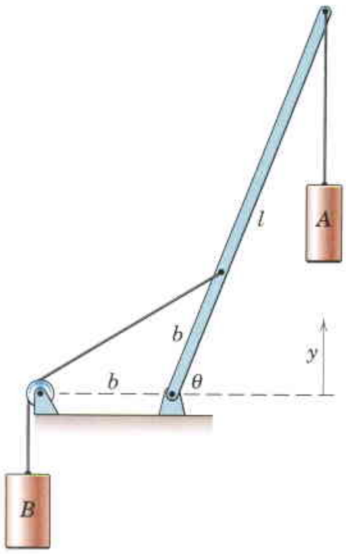
2/228 Collars A and B slide along the fixed right-angle rods and are connected by a cord of length L. Determine the acceleration a_x of collar B as a function of y if collar A is given a constant upward velocity v_A.



Problem 2/228

2/229 If load B has a downward velocity v_B, determine the upward component $(v_A)_y$ of the velocity of A in terms of b, the boom length l, and the angle θ. Assume that the cable supporting A remains vertical.

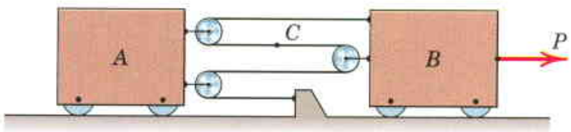
Ans. $(v_A)_y = \dfrac{l\sqrt{2(1+\cos\theta)}}{b\tan\theta}v_B$



Problem 2/229

▶2/230 Under the action of force P, the constant acceleration of block B is 3 m/s^2 to the right. At the instant when the velocity of B is 2 m/s to the right, determine the velocity of B relative to A, the acceleration of B relative to A, and the absolute velocity of point C of the cable.

Ans. $v_{B/A} = 0.5$ m/s, $a_{B/A} = 0.75$ m/s^2
$v_C = 1$ m/s, all to the right



Problem 2/230

2/10 Chapter Review

In Chapter 2 we have developed and illustrated the basic methods for describing particle motion. The concepts developed in this chapter form the basis for much of dynamics, and it is important to review and master this material before proceeding to the following chapters.

By far the most important concept in Chapter 2 is the time derivative of a vector. The time derivative of a vector depends on direction change as well as magnitude change. As we proceed in our study of dynamics, we will need to examine the time derivatives of vectors other than position and velocity vectors, and the principles and procedures developed in Chapter 2 will be useful for this purpose.

Categories of Motion

The following categories of motion have been examined in this chapter:

1. Rectilinear motion (one coordinate)
2. Plane curvilinear motion (two coordinates)
3. Space curvilinear motion (three coordinates)

In general, the geometry of a given problem enables us to identify the category readily. One exception to this categorization is encountered when only the magnitudes of the motion quantities measured along the path are of interest. In this event, we can use the single distance coordinate measured along the curved path, together with its scalar time derivatives giving the speed $|\dot{s}|$ and the tangential acceleration $\ddot{s}$.

Plane motion is easier to generate and control, particularly in machinery, than space motion, and thus a large fraction of our motion problems come under the plane curvilinear or rectilinear categories.

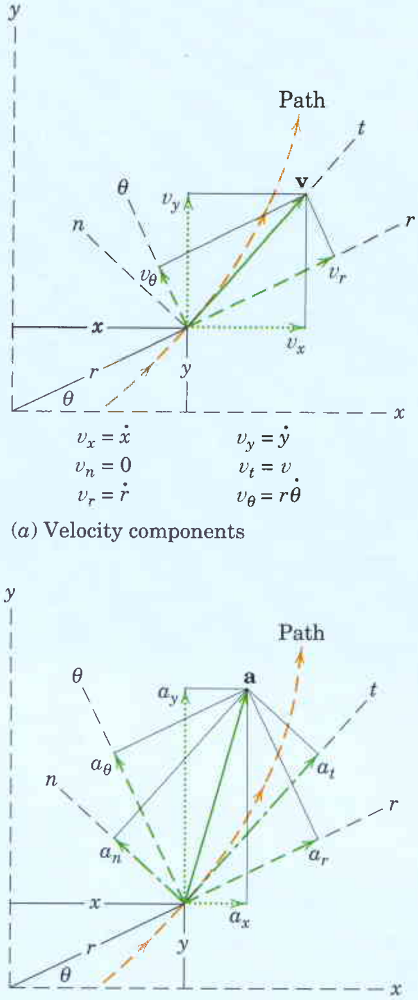
Use of Fixed Axes

We commonly describe motion or make motion measurements with respect to fixed reference axes (absolute motion) and moving axes (relative motion). The acceptable choice of the fixed axes depends on the problem. Axes attached to the surface of the earth are sufficiently "fixed" for most engineering problems, although important exceptions include earth–satellite and interplanetary motion, accurate projectile trajectories, navigation, and other problems. The equations of relative motion discussed in Chapter 2 are restricted to translating reference axes.

Choice of Coordinates

The choice of coordinates is of prime importance. We have developed the description of motion using the following coordinates:

1. Rectangular (Cartesian) coordinates (x-y) and (x-y-z)
2. Normal and tangential coordinates (n-t)
3. Polar coordinates (r-θ)



$v_x = \dot{x}$ $\quad$ $v_y = \dot{y}$

$v_n = 0$ $\quad$ $v_t = v$

$v_r = \dot{r}$ $\quad$ $v_\theta = r\dot{\theta}$

(*a*) Velocity components

$a_x = \ddot{x}$ $\quad$ $a_y = \ddot{y}$

$a_n = v^2/\rho$ $\quad$ $a_t = \dot{v}$

$a_r = \ddot{r} - r\dot{\theta}^2$ $\quad$ $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

(*b*) Acceleration components

Figure 2/21

4. Cylindrical coordinates (r-θ-z)
5. Spherical coordinates (R-θ-ϕ)

When the coordinates are not specified, the appropriate choice usually depends on how the motion is generated or measured. Thus, for a particle which slides radially along a rotating rod, polar coordinates are the natural ones to use. Radar tracking calls for polar or spherical coordinates. When measurements are made along a curved path, normal and tangential coordinates are indicated. An x-y plotter clearly involves rectangular coordinates.

Figure 2/21 is a composite representation of the x-y, n-t, and r-θ coordinate descriptions of the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ for curvilinear motion in a plane. It is frequently essential to transpose motion description from one set of coordinates to another, and Fig. 2/21 contains the information necessary for that transition.

Approximations

Making appropriate approximations is one of the most important abilities you can acquire. The assumption of constant acceleration is valid when the forces which cause the acceleration do not vary appreciably. When motion data are acquired experimentally, we must utilize the nonexact data to acquire the best possible description, often with the aid of graphical or numerical approximations.

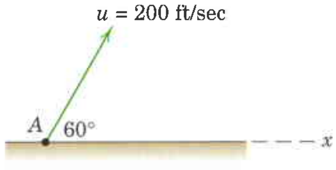
Choice of Mathematical Method

We frequently have a choice of solution using scalar algebra, vector algebra, trigonometric geometry, or graphical geometry. All of these methods have been illustrated, and all are important to learn. The choice of method will depend on the geometry of the problem, how the motion data are given, and the accuracy desired. Mechanics by its very nature is geometric, so you are encouraged to develop facility in sketching vector relationships, both as an aid to the disclosure of appropriate geometric and trigonometric relations and as a means of solving vector equations graphically. Geometric portrayal is the most direct representation of the vast majority of mechanics problems.

REVIEW PROBLEMS

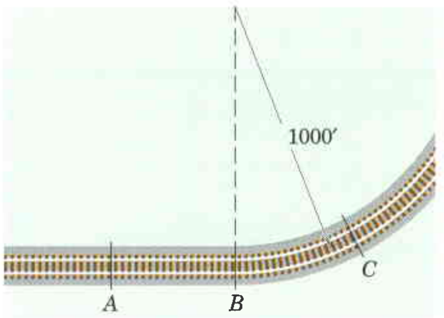
2/231 At time $t = 0$ a small ball is projected from point A with a velocity of 200 ft/sec at the 60° angle. Neglect atmospheric resistance and determine the two times t_1 and t_2 when the velocity of the ball makes an angle of 45° with the horizontal x-axis.

Ans. $t_1 = 2.27$ sec, $t_2 = 8.48$ sec



Problem 2/231

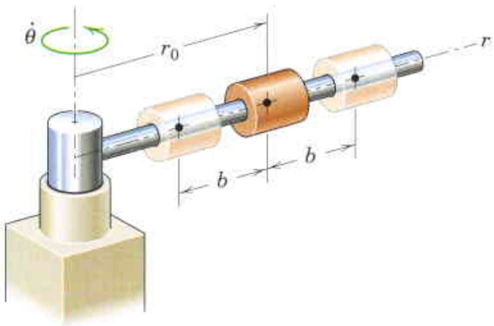
2/232 An inexperienced designer of a roadbed for a new high-speed train proposes to join a straight section of track to a circular section of 1000-ft radius as shown. For a train that would travel at a constant speed of 90 mi/hr, plot the magnitude of its acceleration as a function of distance along the track between points A and C and explain why this design is unacceptable.



Problem 2/232

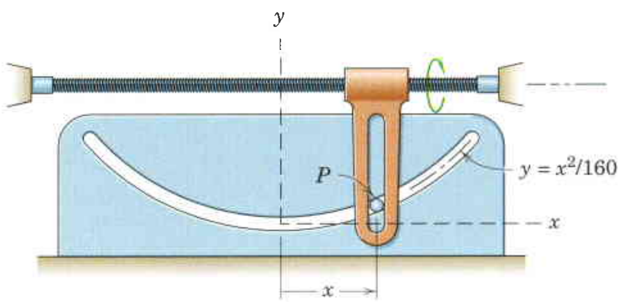
2/233 The small cylinder is made to move along the rotating rod with a motion between $r = r_0 + b$ and $r = r_0 - b$ given by $r = r_0 + b \sin \frac{2\pi t}{\tau}$, where t is the time counted from the instant the cylinder passes the position $r = r_0$ and τ is the period of the oscillation (time for one complete oscillation). Simultaneously, the rod rotates about the vertical at the constant angular rate $\dot{\theta}$. Determine the value of r for which the radial (r-direction) acceleration is zero.

$$Ans.\ r = r_0 \frac{1}{1 + \left(\frac{\tau\dot{\theta}}{2\pi}\right)^2}$$



Problem 2/233

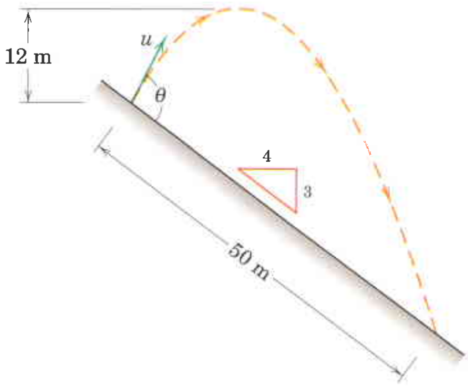
2/234 For a certain interval of motion, the pin P is forced to move in the fixed parabolic slot by the vertical slotted guide, which moves in the x-direction at the constant rate of 20 mm/s. All measurements are in millimeters and seconds. Calculate the magnitudes of the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ of pin P when $x = 60$ mm.



Problem 2/234

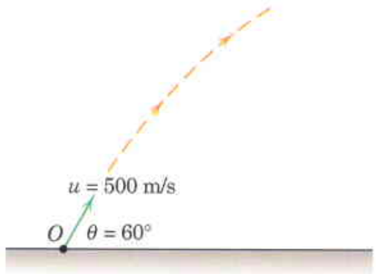
2/235 A stone is thrown down the slope as shown. Determine the magnitude u and direction θ of its initial velocity so that the stone will rise 12 m and still have a range of 50 m down the slope.

Ans. $u = 17.74$ m/s, $\theta = 96.7°$



Problem 2/235

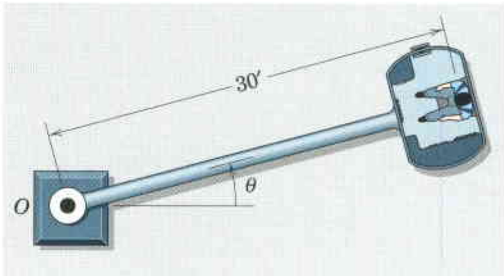
2/236 A small projectile is fired from point O with an initial velocity $u = 500$ m/s at the angle of 60° from the horizontal as shown. Neglect atmospheric resistance and any change in g and compute the radius of curvature ρ of the path of the projectile 30 seconds after the firing.



Problem 2/236

2/237 The angular displacement of the centrifuge is given by $\theta = 4[t + 30e^{-0.03t} - 30]$ rad, where t is in seconds and $t = 0$ is the startup time. If the person loses consciousness at an acceleration level of $10g$, determine the time t at which this would occur. Verify that the tangential acceleration is negligible as the normal acceleration approaches $10g$.

Ans. $t = 53.5$ sec



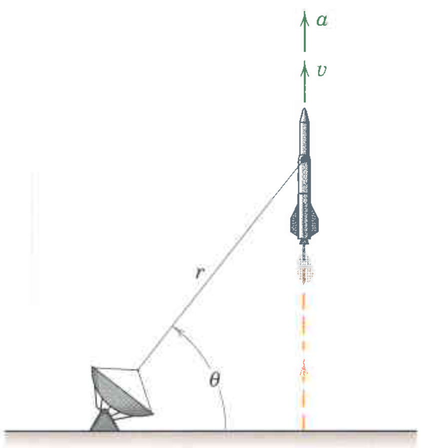
Problem 2/237

2/238 As part of a training exercise, the pilot of aircraft A adjusts her airspeed (speed relative to the wind) to 220 km/h while in the level portion of the approach path and thereafter holds her absolute speed constant as she negotiates the 10° glide path. The absolute speed of the aircraft carrier is 30 km/h and that of the wind is 48 km/h. What will be the angle β of the glide path with respect to the horizontal as seen by an observer on the ship?

Problem 2/238

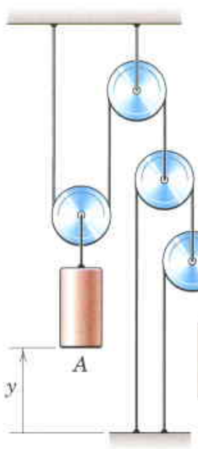
2/239 The vertically-fired rocket and tracking radar of Prob. 2/147 are shown again here. At the instant when $\theta = 60°$, measurements give $\dot{\theta} = 0.03$ rad/sec and $r = 25{,}000$ ft, and the vertical acceleration of the rocket is found to be $a = 64$ ft/sec^2. For this instant determine the values of $\ddot{r}$ and $\ddot{\theta}$.

Ans. $\ddot{r} = 77.9$ ft/sec^2
$\ddot{\theta} = -1.838(10^{-3})$ rad/sec^2



Problem 2/239

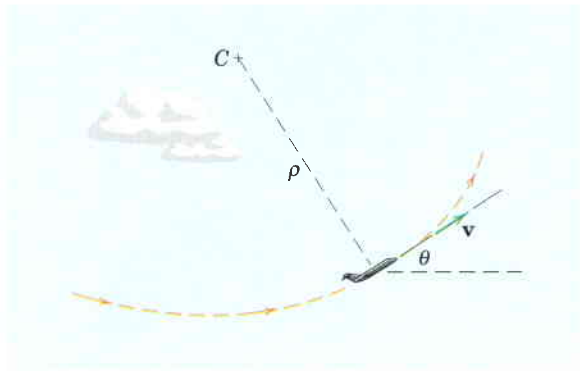
2/240 The vertical displacement of cylinder A in meters is given by $y = t^2/4$ where t is in seconds. Calculate the downward acceleration a_B of cylinder B. Identify the number of degrees of freedom.



Problem 2/240

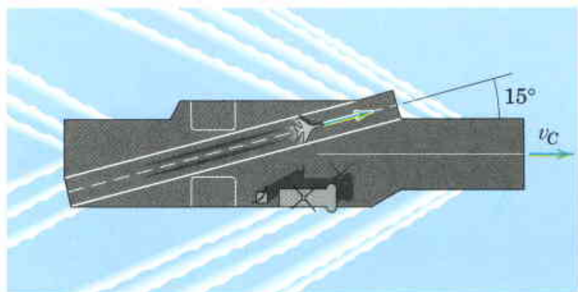
2/241 A jet aircraft pulls up into a vertical curve as shown. As it passes the position where $\theta = 30°$, its speed is 1000 km/h and is decreasing at the rate of 15 km/h per second. If the radius of curvature ρ of the flight path is 1.5 km at this point, calculate the corresponding horizontal and vertical components, $\ddot{x}$ and $\ddot{y}$, of the acceleration of the aircraft.

Ans. $\ddot{x} = -29.3$ m/s^2
$\ddot{y} = 42.5$ m/s^2



Problem 2/241

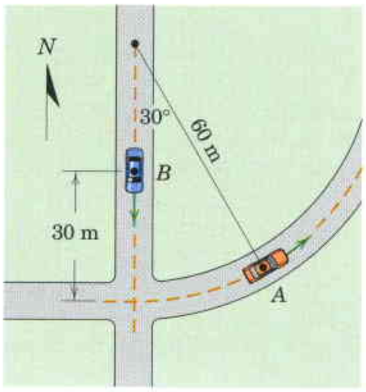
2/242 The launching catapult of the aircraft carrier gives the jet fighter a constant acceleration of 50 m/s^2 from rest relative to the flight deck and launches the aircraft in a distance of 100 m measured along the angled takeoff ramp. If the carrier is moving at a steady 30 knots (1 knot = 1.852 km/h), determine the magnitude v of the actual velocity of the fighter when it is launched.



Problem 2/242

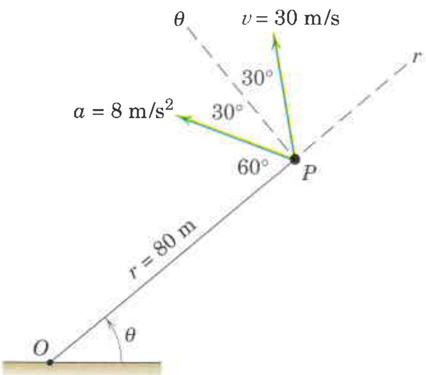
2/243 Car A negotiates a curve of 60-m radius at a constant speed of 50 km/h. When A passes the position shown, car B is 30 m from the intersection and is accelerating south toward the intersection at the rate of 1.5 m/s^2. Determine the acceleration which A appears to have when observed by an occupant of B at this instant.

Ans. $a_{A/B} = 4.58$ m/s^2, $\beta = 20.6°$ west of north



Problem 2/243

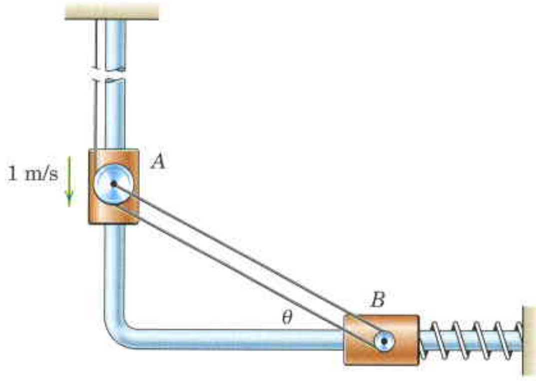
2/244 At the instant depicted, assume that the particle P, which moves on a curved path, is 80 m from the pole O and has the velocity v and acceleration a as indicated. Determine the instantaneous values of $\dot{r}$, $\ddot{r}$, $\dot{\theta}$, $\ddot{\theta}$, the n- and t-components of acceleration, and the radius of curvature ρ.



Problem 2/244

2/245 Cylinder A has a constant downward speed of 1 m/s. Compute the velocity of cylinder B for (*a*) $\theta = 45°$, (*b*) $\theta = 30°$, and (*c*) $\theta = 15°$. The spring is in tension throughout the motion range of interest, and the pulleys are connected by the cable of fixed length.

Ans. (*a*) $v_B = 0.293$ m/s
(*b*) $v_B = 0$
(*c*) $v_B = -0.250$ m/s

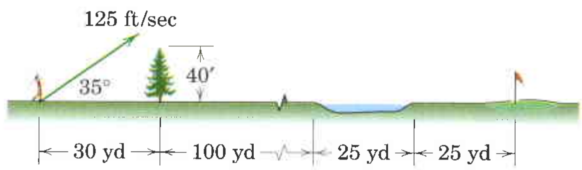


Problem 2/245

2/246 A particle has the following position, velocity, and acceleration components: $x = 50$ ft, $y = 25$ ft, $\dot{x} = -10$ ft/sec, $\dot{y} = 10$ ft/sec, $\ddot{x} = -10$ ft/sec^2, and $\ddot{y} = 5$ ft/sec^2. Determine the following quantities: v, a, $\mathbf{e}_t$, $\mathbf{e}_n$, a_t, $\mathbf{a}_t$, a_n, $\mathbf{a}_n$, ρ, $\mathbf{e}_r$, $\mathbf{e}_\theta$, v_r, $\mathbf{v}_r$, v_θ, $\mathbf{v}_\theta$, a_r, $\mathbf{a}_r$, a_θ, $\mathbf{a}_\theta$, r, $\dot{r}$, $\ddot{r}$, θ, $\dot{\theta}$, and $\ddot{\theta}$. Express all vectors in terms of $\mathbf{i}$ and $\mathbf{j}$, and graph all vectors on one set of x-y axes as you proceed.

2/247 Just after being struck by the club, a golf ball has a velocity of 125 ft/sec directed at 35° to the horizontal as shown. Determine the location of the point of impact.

Ans. $R = 152.0$ yd

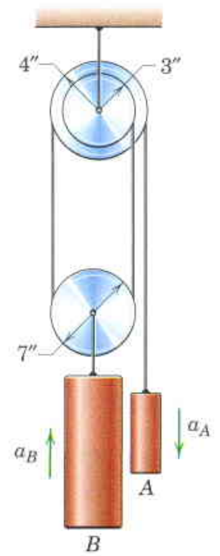


Problem 2/247

2/248 A rocket fired vertically up from the north pole achieves a velocity of 27 000 km/h at an altitude of 350 km when its fuel is exhausted. Calculate the additional vertical height h reached by the rocket before it starts its descent back to the earth. The coasting phase of its flight occurs above the atmosphere. Consult Fig. 1/1 in choosing the appropriate value of gravitational acceleration and use the mean radius of the earth from Table D/2. (*Note:* Launching from the earth's pole avoids considering the effect of the earth's rotation.)

2/249 In the differential pulley hoist shown, the two upper pulleys are fastened together to form an integral unit. The cable is wrapped around the smaller pulley with its end secured to the pulley so that it cannot slip. Determine the upward acceleration a_B of cylinder B if cylinder A has a downward acceleration of 2 ft/sec^2. (*Suggestion:* Analyze geometrically the consequences of a differential movement of cylinder A.)

Ans. $a_B = 0.25$ ft/sec^2



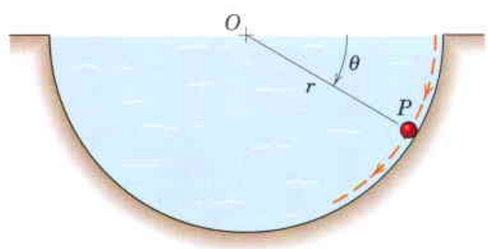
Problem 2/249

*Computer-Oriented Problems

***2/250** A baseball is dropped from an altitude $h = 200$ ft and is found to be traveling at 85 ft/sec when it strikes the ground. In addition to gravitational acceleration, which may be assumed constant, air resistance causes a deceleration component of magnitude kv^2, where v is the speed and k is a constant. Determine the value of the coefficient k. Plot the speed of the baseball as a function of altitude y. If the baseball were dropped from a high altitude, but one at which g may still be assumed constant, what would be the terminal velocity v_t? (The *terminal velocity* is that speed at which the acceleration of gravity and that due to air resistance are equal and opposite, so that the baseball drops at a constant speed.) If the baseball were dropped from $h = 200$ ft, at what speed v' would it strike the ground if air resistance were neglected?

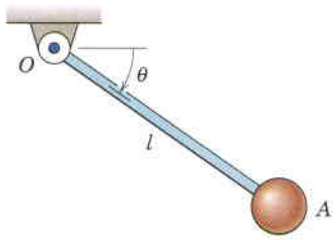
***2/251** At time $t = 0$, the 1.8-lb particle P is given an initial velocity $v_0 = 1$ ft/sec at the position $\theta = 0$ and subsequently slides along the circular path of radius $r = 1.5$ ft. Because of the viscous fluid and the effect of gravitational acceleration, the tangential acceleration is $a_t = g\cos\theta - \frac{k}{m}v$, where the constant $k = 0.2$ lb-sec/ft is a drag parameter. Determine and plot both θ and $\dot{\theta}$ as functions of the time t over the range $0 \le t \le 5$ sec. Determine the maximum values of θ and $\dot{\theta}$ and the corresponding values of t. Also determine the first time at which $\theta = 90°$.

Ans. $\theta_{max} = 110.4°$ at $t = 0.802$ sec
$\dot{\theta}_{max} = 3.79$ rad/sec at $t = 0.324$ sec
$\theta = 90°$ at $t = 0.526$ sec



Problem 2/251

***2/252** If all frictional effects are neglected, the expression for the angular acceleration of the simple pendulum is $\ddot{\theta} = \frac{g}{l}\cos\theta$, where g is the acceleration of gravity and l is the length of the rod OA. If the pendulum has a clockwise angular velocity $\dot{\theta} = 2$ rad/s when $\theta = 0$ at $t = 0$, determine the time t' at which the pendulum passes the vertical position $\theta = 90°$. The pendulum length is $l = 0.6$ m. Also plot the time t versus the angle θ.

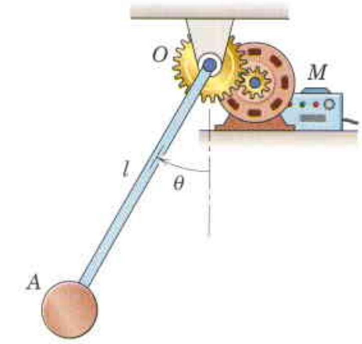


Problem 2/252

***2/253** A ship with a total displacement of 16 000 metric tons (1 metric ton = 1000 kg) starts from rest in still water under a constant propeller thrust $T = 250$ kN. The ship develops a total resistance to motion through the water given by $R = 4.50v^2$, where R is in kilonewtons and v is in meters per second. The acceleration of the ship is $a = (T - R)/m$, where m equals the mass of the ship in metric tons. Plot the speed v of the ship in knots as a function of the distance s in nautical miles which the ship goes for the first 5 nautical miles from rest. Find the speed after the ship has gone 1 nautical mile. What is the maximum speed which the ship can reach?

Ans. $v_{1\text{ mi}} = 11.66$ knots
$v_{max} = 14.49$ knots

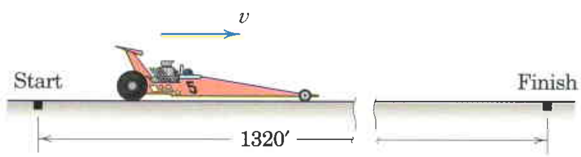
***2/254** By means of the control unit M, the pendulum OA is given an oscillatory motion about the vertical given by $\theta = \theta_0 \sin\sqrt{\frac{g}{l}}\,t$, where θ_0 is the maximum angular displacement in radians, g is the acceleration of gravity, l is the pendulum length, and t is the time in seconds measured from an instant when OA is vertical. Determine and plot the magnitude a of the acceleration of A as a function of time and as a function of θ over the first quarter cycle of motion. Determine the minimum and maximum values of a and the corresponding values of t and θ. Use the values $\theta_0 = \pi/3$ radians, $l = 0.8$ m, and $g = 9.81$ m/s^2. (*Note:* The prescribed motion is not precisely that of a freely swinging pendulum for large amplitudes.)



Problem 2/254

***2/255** The acceleration of the drag racer is modeled by $a = c_1 - c_2v^2$, where the v^2-term accounts for aerodynamic drag and where c_1 and c_2 are positive constants. If c_1 is known to be 30 ft/sec^2, determine c_2 if the racer completes the $\frac{1}{4}$-mi run in 9.4 sec. Then plot the velocity and displacement as functions of time. A drag race is a $\frac{1}{4}$-mi straight run from a standing start.

Ans. $c_2 = 9.28(10^{-6})$ ft^{-1}

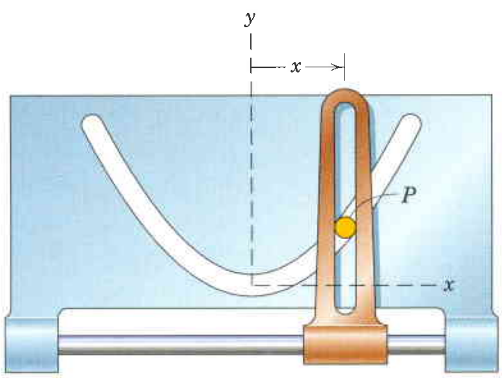


Problem 2/255

***2/256** A bullet with a muzzle velocity of 2000 ft/sec is fired vertically upward and reaches a maximum height of 1 mi. Air resistance causes an additional component of downward acceleration kv^2 proportional to the square of the velocity v. Take g to be constant at 32.2 ft/sec^2 and calculate the coefficient k.

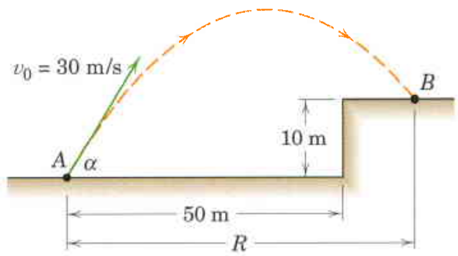
***2/257** The guide with the vertical slot is given a horizontal oscillatory motion according to $x = 4 \sin 2t$, where x is in inches and t is in seconds. The oscillation causes the pin P to move in the fixed parabolic slot whose shape is given by $y = x^2/4$, with y also in inches. Plot the magnitude v of the velocity of the pin as a function of time during the interval required for pin P to go from the center to the extremity $x = 4$ in. Find and locate the maximum value of v and verify your results analytically.

Ans. $v_{max} = 10$ in./sec at $t = 0.330$ sec
$x = 2.45$ in.



Problem 2/257

***2/258** A projectile is launched from point A with speed $v_0 = 30$ m/s. Determine the value of the launch angle α which maximizes the range R indicated in the figure. Determine the corresponding value of R.



Problem 2/258