December Mu Alpha Theta Problems

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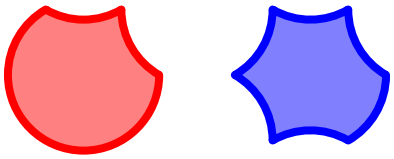
1 Hex Jigsaw

For this problem, we define two two-dimensional shapes, which we name Tile A and Tile B.

Both Tile A and Tile B are constructed from the circumcircles of regular hexagons, except that some arcs of the circle are reflected over some sides.



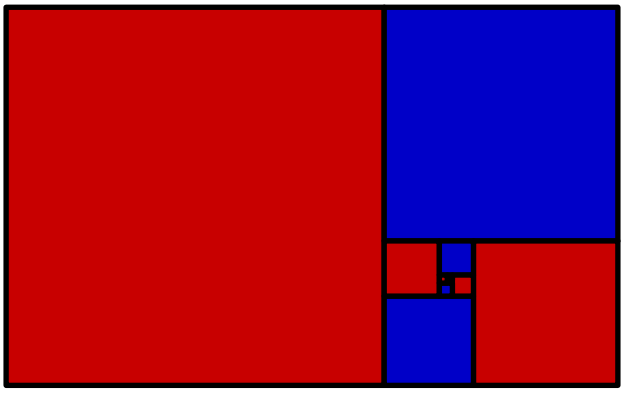
These two tiles are used to form a complete, periodic tiling of the plane (see `https://en.wikipedia.org/wiki/Tessellation`).

(a) Given that such a tiling exists, prove that the ratio of the number of Tile A and Tile B used in the tessellation must be a certain number, and find that number. (5 points)

(b) Prove such a tiling exists. (7 points)

2 Shiny Rectangles

Define $\phi = \frac{1+\sqrt{5}}{2}$, or the Golden Ratio. A $1 \times \phi$ rectangle is colored red and blue with the following rule: The largest square which shares a side with the rectangle is colored red. Then, the largest square which completely shares a side with the remaining portion of the rectangle is colored blue. This repeats, alternating colors.



(a) Find the ratio of the red area to the blue area. (5 points)

(b) Instead of a $1 \times \phi$ rectangle, a $1 \times N$ rectangle is used. This rectangle has the property that there are infinitely such squares which the rectangle is divided into, and that the red and blue areas are equal. Find any such value of N with proof that it works. (5 points)

3 Useless Calculator

Pichu's calculator is a bit lacking in functionality; the only buttons on the calculator are the 0-9 and Enter buttons, plus, minus, multiplication and division buttons, as well as two buttons, such that when a is displayed on the screen, return 10^a and $\log_{10} a$ respectively. He also has a sheet of paper where he can copy down any numbers that appear on the calculator, and input them back into the calculator.

(a) Pichu wants to calculate a^b, where both a and b are arbitrary positive reals. How does he do this? (3 points)

(b) Pichu also wants to calculate $\sin(x)$ where x is a real number strictly between 0 and 2π. How can he create a number on the calculator such that the result is within ± 0.0001 of the actual value? (3 points)

4 Circular Centroid

A unit circle is drawn, centered at $(0, 0)$ on the coordinate plane.

(a) Three points are chosen uniformly at random on this circle. Find the expected coordinates of the centroid of the triangle formed by these three points. (In other words, find the expected values of the x-coordinate and y-coordinate of the triangle's centroid.) (1 point)

(b) Two points are chosen uniformly at random on this circle. Find the expected coordinates of the centroid of the triangle formed by these two points and the point $(1, 0)$. (3 points)

(c) Three points are chosen uniformly at random from the arc of the circle in the first quadrant. Find the expected coordinates of the centroid of the triangle formed by these three points. (4 points)

(d) Three points are chosen uniformly at random inside the sector of the circle in the first quadrant. Find the expected coordinates of the centroid of the triangle formed by these three points. (8 points)

5 Polynomial Peril

Define the polynomial $P(x)$ such that P is the monic polynomial with least degree which is divisible by all of the elements in

$$\bigcup_{n=0}^{11} \bigcup_{k=1}^{12} \{(x - e^{\frac{n\pi i}{6}})^k - 1\}$$

over the complex numbers.

Find the degree of P. (12 points)