Practice Quiz 2 Solutions

1) Which of the following statements is true?

   a) The likelihood is a mixture between the prior and posterior.

      No, the likelihood is completely independent of the prior. The likelihood is the probability of observing the data given the model parameters, whereas the prior represents a subjective distribution imposed on those model parameters prior to observing the data.

   b) The prior is a mixture between the likelihood and posterior.

      No, the prior is completely independent of the likelihood.

   c) The posterior is a mixture between the prior and likelihood.

      Yes, the posterior is proportional to the likelihood times the prior, which means that it is influenced by both of them.

2) Which of the following distributions would be a good choice of prior to use if you wanted to determine if a coin is fair when you have a **strong** belief that the coin is fair? (Assume a model where we call heads a success and tails a failure).

   Solution: Since a Beta(a, b) distribution corresponds to observing data with prior mean $a/(a+b)$ and prior sample size $a+b$, we seek an answer with $a/a+b=0.5$ and $a+b$ large. Of the answer choices given, a Beta$(50, 50)$ distribution corresponds to prior mean 0.5 with a larger sample size than any other answer.

3) If Amy is trying to make inferences about the average number of customers that visit Macy's between noon and 1 p.m., which of the following distributions represents a conjugate prior?

   Solution: The number of customers that visit Macy's between noon and 1 p.m. can only take on integer values and has no clearly defined upper bound. As such, a Poisson distribution with parameter λ would be a good model for customer arrivals. The corresponding conjugate prior for λ, the average number of customers that visit Macy's in the time period, is a Gamma distribution, which we learned from the lectures.

4) Suppose that you sample 24 M&Ms from a bag and find that 3 of them are yellow. Assuming that you place a uniform Beta(1,1) prior on the proportion of yellow M&Ms p, what is the posterior probability that p ¡ 0.2?

   Solution: The Beta distribution is conjugate to the Binomial distribution, the update rule is that the posterior Beta will have parameter values $\alpha + x$, $\beta + n - x$ where α and β are the parameters of the prior beta. In this case, $x = 3$ and $n = 24$. Hence, the posterior distribution of p is a Beta(4,22) distribution. To find the posterior probability that $p < 0.2$, we use the following command in R:

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pbeta(q = 0.2, shape1 = 4, shape2 = 22)
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This gives a posterior probability of approximately 0.766.

5) Suppose you are given a coin and told that the coin is either biased towards heads (p = 0.6) or biased towards tails (p = 0.4). Since you have no prior knowledge about the bias of the coin, you place a prior probability of 0.5 on the outcome that the coin is biased towards heads. You flip the coin twice and it comes up tails both times. What is the posterior probability that your next two flips will be heads?

Solution: To solve this problem, we first need to find the posterior probability that the coin is biased towards heads. Using Bayes' rule, we find that

$$P(p = 0.6|\{T, T\}) = \frac{(0.4^2)(0.5)}{(0.4^2)(0.5) + (0.6^2)(0.5)} = \frac{4}{13}$$

To find the posterior probability that the next two flips will be heads, we note that

$$\begin{aligned} P(\{T, T, H, H\}|\{T, T\}) &= P(\{T, T, H, H\}|p = 0.6, \{T, T\})P(p = 0.6|\{T, T\}) \\ &\quad + P(\{T, T, H, H\}|p = 0.4, \{T, T\})P(p = 0.4|\{T, T\}) \\ &= (0.6)^2(4/13) + (0.4)^2(9/13) = 72/325 \approx 0.222 \end{aligned}$$