Practice Quiz 3 Solutions

Question 1:

Fill in the blank: Under a **0/1** loss function, the summary statistic that minimizes the posterior expected loss is the ________________ of the posterior.

Solution:

1. Mean
   Not correct. The mean is the summary statistic that minimizes the posterior expected loss under the ***quadratic*** *loss function.*

2. Median
   Not correct. The median is the summary statistic that minimizes the posterior expected loss under the ***linear*** *loss function.*

3. **Mode**
   Correct. The mode is the summary statistic that minimizes the posterior expected loss under the ***0/1*** *loss function.*

Question 2

True or *False*: If the posterior distribution is **normally distributed**, the estimate that minimizes posterior expected loss is the same, regardless of whether the loss function is 0/1, linear, or quadratic.

Solution:

- The answer is **True**. For a normal distribution, its mean, median and mode are all the same, since it is symmetric, and unimodel.

Question 3

Which of the following statements is *false*?

Solution:

- A Bayes factor of greater than 100 suggests strong evidence in favor of one of the hypotheses.

  This statement is correct. By Jeffreys' scale, a Bayes Factor of over 100 suggests strong or decisive evidence.

- The Bayes factor represents the ratio of the marginal likelihoods of observing the data under the two hypotheses.

  This statement is correct. By definition, the Bayes factor is $\frac{P(\text{data} \mid H_2)}{P(\text{data} \mid H_1)}$.

- The Bayes factor is not sensitive to the choice of prior probability on hypotheses.

  This statement is correct. By definition, the Bayes factor is defined to be the ratio of the marginal likelihoods of the observed data under the two hypotheses. The prior probability on hypotheses will only affect the prior odd $O[H_1 : H_2] = \dfrac{P(H_1)}{P(H_2)}$*, and therefore, the posterior odd* $PO[H_1 : H_2] = BF[H_1 : H_2] \times O[H_1 : H_2]$.

  However, the Bayes Factor may be sensitive to the prior distribution on parameters under each hypothesis, such as the joint prior distribution $p(\mu, \sigma)$ *when comparing two means.*

- **A Bayes factor of less than 0.01 suggests that the evidence in favor of one of the hypotheses is barely worth mentioning**.

  This statement is false, and therefore, this is the correct answer. A Bayes factor $BF[H_1 : H_2]$ *of less than 0.01 is equivalent to the Bayes factor* $BF[H_2 : H_1]$ *greater than* $1/0.01 = 100$*. This will yield a strong evidence in favor of one of the hypotheses.*

Question 4

In the 2004 North Carolina Birth Survey data set, we only have 133 data points for matured mothers. Suppose we want to infer the weight gain of matured mothers in North Carolina in 2004 using these data, we would like to conduct a Bayesian inference. We assume the prior distribution of the mean μ and the precision ϕ of weight gain for matured mothers follows a Normal-Gamma distribution:

$$\mathsf{NormalGamma}(m_0 = 30, n_0 = 100, s_0^2 = 10^2, v_0 = 99).$$

With 133 observations in the data set, the mean of weight gain of matured mothers is about 28 pounds, the standard deviation of the weight gain of matured mothers is about 13 pounds. We can assume the data are normally distributed. What is the joint posterior distribution of μ and ϕ?

Solution:

- The Normal-Gamma distribution is the conjugate prior of the Normal distribution. The posterior distribution is again the Normal-Gamma distribution. We can update the parameters using

$$\begin{aligned} m_n &= \frac{n_0 m_0 + n\bar{Y}}{n + n_0} \\ n_n &= n_0 + n \\ v_n &= v_0 + n \\ s_n &= \frac{1}{v_n}\left[s_0^2 v_0 + s^2(n-1) + \frac{n_0 n}{n_n}(\bar{Y} - m_0)^2\right]. \end{aligned}$$

From the question, $m_0 = 30,\ n_0 = 100,\ s_0^2 = 10^2,\ v_0 = 99$, and $n = 133,\ \bar{Y} = 28,\ s^2 = 13^2 = 169$. Plugging into the above formulas, we get

$$\begin{aligned} m_n &= \frac{n_0 m_0 + n\bar{Y}}{n + n_0} = \frac{100 \times 30 + 133 \times 28}{100 + 133} \approx 28.86, \\ n_n &= n_0 + n = 100 + 133 = 233, \\ v_n &= v_0 + n = 99 + 133 = 232, \\ s_n^2 &= \frac{1}{v_n}\left[s_0^2 v_0 + s^2(n-1) + \frac{n_0 n}{n_n}(\bar{Y} - m_0)^2\right] \\ &= \frac{1}{232}\left[10^2 \times 99 + 13^2 \times (133 - 1) + \frac{100 \times 133}{233}(28 - 30)^2\right] \approx 139.81. \end{aligned}$$

The Gamma distribution is the posterior distribution of the precision ϕ. The Student's t-distribution is the marginal posterior distribution of μ. These two distributions are not the joint posterior distribution of both μ and ϕ. Hence, the correct answer is $\textsf{NormalGamma}(28.86, 233, 139.81, 232)$.

Question 5

Forty-four sixth graders were randomly selected from a school district. Then, they were divided into 22 matched pairs, each pair having equal IQs. One member of each pair was randomly selected to receive special training. Then, all of the students were given an IQ test. Results showed that the students given special training obtained IQ test scores **on**

average 1 point **higher than** the paired students without training. The standard deviation s of the **difference** of the IQ test scores is about 3.6. Let μ be the **difference** of the IQ test scores between the two groups. We conduct a Bayesian inference to compare the following hypotheses:

$$H_1 : \mu = 0, \qquad H_2 : \mu \neq 0.$$

For hypothesis H_2, we impose a Normal prior on μ conditioning on σ^2, and the reference prior on σ^2 for both H_1 and H_2:

$$\begin{aligned} \mu \mid \sigma^2, H_2 &\sim \mathsf{N}(0, \sigma^2/n_0), \\ p(\sigma^2) &\propto 1/\sigma^2, \end{aligned}$$

with $n_0 = 20$. What is the Bayes factor of H_1 over H_2?

Hint: the Bayes factor formula is given as follows

$$BF[H_1 : H_2] = \left(\frac{n+n_0}{n_0}\right)^{1/2} \left(\frac{t^2 \frac{n_0}{n+n_0} + \nu}{t^2 + \nu}\right)^{(\nu+1)/2}.$$

Solution:

- From the data given in the question, we can calculate the t-statistics $t = \frac{|\bar{Y}|}{s/\sqrt{n}} = \frac{1}{3.6/\sqrt{22}} \approx 1.303$, and the degrees of freedom $\nu = n - 1 = 22 - 1 = 21$.

  We also have the values $n_0 = 20,\ \overline{n = 22},\ \nu = n - 1 = 21$. By the Bayes factor formula, we get

$$BF[H_1 : H_2] = \left(\frac{22+20}{20}\right)^{1/2} \left(\frac{1.303^2 \frac{20}{22+20} + 21}{1.303^2 + 21}\right)^{(21+1)/2} \approx 0.93.$$