Practice Quiz 1 Solutions

1) Julia is having an outdoor wedding ceremony tomorrow. In recent years, it has rained on average 50 days per year. Unfortunately, the meteorologist has predicted rain for her wedding day. When it rains, the meteorologist will have correctly predicted it 80 percent of the time. When it does not rain, the meteorologist will have incorrectly predicted rain 30 percent of the time. Given this information, what is the probability that it rains on Julia's wedding day?

Solution: Let R^+ denote the event that it rains on Julia's wedding day, and P^+ denote the event that the meteorologist predicted rain on Julia's wedding day. Then, by Bayes' rule,

$$p(R^+|P^+) = \frac{p(P^+|R^+)p(R^+)}{p(P^+)}$$

We can decompose $p(P^+)$ into $p(P^+|R^+)p(R^+) + p(P^+|R^-)p(R^-)$ so the formula becomes

$$p(R^+|P^+) = \frac{p(P^+|R^+)p(R^+)}{p(P^+|R^+)p(R^+) + p(P^+|R^-)p(R^-)}$$

where R^- denotes the event that it does not rain on Julia's wedding day. From the problem description, $p(P^+|R^+) = 4/5$ and $p(P^+|R^-) = 3/10$. Further, since it rains on average 50 days in a year and there are 365 days in a year, our prior probability that it rains is $p(R^+) = 50/365$, which implies $p(R^-) = 315/365$. Plugging these numbers into the formula,

$$\begin{aligned} p(R^+|P^+) &= \frac{(\frac{4}{5})(\frac{50}{365})}{(\frac{4}{5})(\frac{50}{365}) + (\frac{3}{10})(\frac{315}{365})} \\ &= \frac{80}{269} \approx 0.297 \end{aligned}$$

2) Suppose we have two hypotheses, H_0 and H_1. Assuming our prior places equal weight on H_0 and H_1, which of the following statements is false?

a) If the posterior probability of H_0 is less than .05, the p-value under H_0 will also be less than .05.

False. A simple but extreme counterexample: Suppose the prior probability $P(H_0) = 0$. Then regardless of the how well H_0 explains the data (which increases the p-value), the posterior probability of H_0 will also be zero.

b) If the p-value is less than .05, the probability that we see data at least as extreme as our observed data is less than .05, given that H_0 is true.

True. This is the definition of a p-value and is very important to remember. When the p-value is low, either a rare event occurred or H_0 is false.

c) If the cost of making a type-I error is the same as the cost of making a type II error and the posterior probability of H_1 is greater than the posterior probability of H_0, we should reject H_0.

True. If the cost of making a wrong decision is the same regardless of which decision we make, we should choose the hypothesis with the highest posterior probability. This corresponds to a $\{0, 1\}$ loss function, which you will learn about later in the course.

3) Suppose 20 people are randomly sampled from the population and their sex is recorded. Which of the following best represents the likelihood of the number of males observed k?

Solution: The likelihood corresponds to the probability of observing the data, given the parameters of the data model. In this case, since 20 people are randomly (and hence independently) sampled from the population, we can model the number of males we see as coming from a binomial distribution with parameter p, the true proportion of males in the population.

Hence, since we observe exactly k males out of 20 people in the data, the phrase that best represents the likelihood is "the probability of observing exactly k males in 20 samples, given p, the true population proportion of males."

4) Which of the following statements is consistent with both Bayesian and frequentist interpretations of probability?

a) Probability can be represented by a degree of belief, which changes as more data are collected.

No, this statement is only consistent with a Bayesian interpretation of probability. Bayesians view probability as a degree of belief, which can be quantified by indifference to a wager.

b) Probability can be represented by the long-run frequency of an event divided by the number of trials.

No, this statement is only consistent with a Frequentist interpretation of probability.

c) Probability is the tendency of an experiment to produce a certain outcome, even if it is performed only once.

No, this statement is too narrow for a Bayesian. Probabilities are found outside the context of experiments.

d) Probability is a measure of the likelihood that an event will occur.

Yes, probability is a measure of the likelihood that an event will occur. This likelihood can be treated as a degree of belief by a Bayesian or a long-run frequency proportion by a frequentist.

5) You are told that either a coin has either a strong tails bias (p = 0.2), a weak tails bias (p = 0.4), no bias (p = 0.5), a weak heads bias (p = 0.6), or a strong heads bias (p = 0.8). You assign a prior probability of 1/2 that the coin is fair and distribute the remaining 1/2 prior probability equally over the other four possible scenarios. You flip the coin three times and it comes up heads all three times. What is the posterior probability that the coin is biased towards heads?

Solution: Let D represent the data, in this case the coin coming up heads three times in three flips. We can model the number of heads as coming from a binomial distribution with probability p. As such, the likelihood of the data given p is p^3.

Using Bayes' rule, we can find the probability that the coin is biased towards heads by computing the following:

$$P(p = 0.6 \text{ or } p = 0.8|D) = \frac{P(D|p = 0.6)P(p = 0.6) + P(D|p = 0.8)P(p = 0.8)}{P(D)}$$

Since the prior $P(p = 0.5) = 0.5$ and the remaining probability is distributed equally over the other four scenarios, $P(p = 0.2) = P(p = 0.4) = P(p = 0.6) = P(p = 0.8) = 0.125$. Plugging the likelihood and prior probabilities into the formula above, we get

$$\begin{aligned} P(p = 0.6 \text{ or } p = 0.8|D) &= \frac{(0.6)^3(0.125) + (0.8)^3(0.125)}{(0.6)^3(0.125) + (0.8)^3(0.125) + (0.4)^3(0.125) + (0.2)^3(0.125) + (0.5)^3(0.5)} \\ &\approx 0.56 \end{aligned}$$