Normal Forms

Bilinear

Travelling Wave

Solitary Wave Families in Two Non-Integrable Models Using Reversible Systems Theory

Jonathan Leto

Overview

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Definitions

- Background
- Literature Review
- Method of Solution
- The Generalized Pochammer-Chree Equations
- A Generalized Microstructure Equation

Reversible

System

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Reversible Dynamical System (looss & Adelmeyer)

Consider

$$\frac{dz}{dt} = F(z; \mu), z \in \mathbb{R}^n, \mu \in \mathbb{R}$$
 (1)

where

$$F\left(0;0\right) =0$$

. If there exists a unitary map

$$S: \mathbb{R}^n \mapsto \mathbb{R}^n, S \neq I$$

such that

$$F(Sz; \mu) = -SF(z; \mu)$$

for all z and μ then (1) is a reversible system.

Solitary Wave

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Families of Solitary Waves

Here we will use the term solitary wave or "soliton" to mean a pulse-like solution to an evolution equation. For example,

$$A(z) = \ell \operatorname{sech}^2 kz \tag{2}$$

is a two-parameter family of solitary waves.

where k and ℓ are parameters which determine the speed and the height of the wave.

Travelling Wave

Normal Form Theory

After a nonlinear change of variables (looss & Adelmeyer) one may put the Center Manifold into Normal Form.

- Two-Dimensional Center Manifold
 - $\lambda_{1-4} = 0, 0, \pm \lambda, \lambda \in \mathbb{R}$
 - $Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B)$
- Four-Dimensional Center Manifold
 - $\lambda_{1-4} = 0, 0, \pm i\omega, \omega \in \mathbb{R}$
 - $Y = A\zeta_0 + B\zeta_1 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C})$

where $\zeta_0, \zeta_1, \zeta_+, \zeta_-$ are eigenvectors of the linearized operator.

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Bilinear Functions

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Properties of Bilinear Functions

A function

$$B: \mathbb{C}x\mathbb{C} \mapsto \mathbb{C}$$

satisfying the following axioms

$$B(x+y,z) = B(x,z) + B(y,z)$$
 (3a)

$$B(\lambda x, y) = \lambda B(x, y) \tag{3b}$$

$$B(x, y + z) = B(x, y) + B(x, z)$$
 (3c)

$$B(x, \lambda y) = \lambda B(x, y) \tag{3d}$$

is called bilinear .

If B(x,y) is bilinear, then $f(y) \equiv B(y,y)$ is invariant under the transformation $y \mapsto -y$ and thus is **reversible**.

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The Generalized Pochammer-Chree Equations

The Generalized Pochammer-Chree Equations govern the propagation of longitudinal waves in elastic rods.

GPC1

$$(u - u_{xx})_{tt} - (a_1 u + a_2 u^2 + a_3 u^3)_{xx} = 0$$
 (4)

GPC2

$$(u - u_{xx})_{tt} - (a_1 u + a_3 u^3 + a_5 u^5)_{xx} = 0$$
 (5)

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Travelling Wave ODE

Let z=x-ct and $u(x,t)=\phi(z)$ to reduce (4) and (5) to the Travelling Wave ODE

$$\phi_{zzzz} - q\phi_{zz} + p\phi = \mathcal{N}_{1,2}[\phi] \tag{6}$$

where

$$p \equiv 0$$

$$q \equiv 1 - \frac{a_1}{c^2}$$

$$L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2 - p & 0 & q/3 & 0 \end{pmatrix}$$
 (8a)

$$\mathcal{N}_{1} [\phi] = -\frac{1}{c^{2}} \left[3a_{3} \left(2\phi\phi_{z}^{2} + \phi^{2}\phi_{zz} \right) + 2a_{2} \left(\phi_{zz}\phi_{z} + \phi_{z}^{2} \right) \right]$$

$$\mathcal{N}_{2} [\phi] = -\frac{1}{c^{2}} \left[3a_{3} \left(2\phi\phi_{z}^{2} + \phi^{2}\phi_{zz} \right) + 5a_{5} \left(4\phi^{3}\phi_{z}^{2} + \phi^{4}\phi_{zz} \right) \right]$$

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Reversible Form

Denoting $Y = \langle y1, y2, y3, y4 \rangle^T = \langle \phi, \phi_z, \phi_{zz}, \phi_{zzz} \rangle^T$ equation (6) can be written

$$\frac{dY}{dz} = L_{pq}Y - G_{1,2}(Y,Y) \tag{9}$$

where

$$L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2 - p & 0 & q/3 & 0 \end{pmatrix}$$

$$\textit{G}_{1,2}(\textit{Y},\textit{Y}) = \left\langle 0,0,0,-\mathcal{N}_{1,2}\left(\textit{Y},\textit{Y}\right)\right\rangle^{\textit{T}}$$

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Near C_0

The eigenvalues are $\lambda_{1,4} = 0, 0, \pm \lambda, \lambda \in \mathbb{R}$, We write the Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B) \tag{10}$$

with corresponding normal form

$$\frac{dA}{dz} = B \tag{11a}$$

$$\frac{dB}{dz} = b\epsilon A + \tilde{c}A^2 \tag{11b}$$

where ϵ measures the perturbation about C_0 and

$$\zeta_0 = \langle 1, 0, -q/3, 0 \rangle^T$$

$$\zeta_1 = \langle 0, 1, 0, -2q/3 \rangle^T$$
(12a)

$$\zeta_1 = \langle 0, 1, 0, -2q/3 \rangle' \tag{12b}$$

How do we determine the coefficients b and c_* ?

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A Generalized Microstructure PDF

One dimensional wave propagation in microstructured solids has recently been modeled [?] by an equation

$$v_{tt} - bv_{xx} - \frac{\mu}{2} \left(v^2 \right)_{xx} - \delta \left(\beta v_{tt} - \gamma v_{xx} \right)_{xx} = 0$$
 (13)

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Travelling Wave ODE

Let z=x-ct and $u(x,t)=\phi(z)$ to reduce to the Travelling Wave ODE

$$\phi_{zzzz} - q\phi_{zz} + p\phi = \mathcal{N}[\phi] \tag{14}$$

where

$$\mathcal{N}\left[\phi\right] = -\Delta_1 \phi_z^2 - b\Delta_1 \phi \phi_{zz}$$

$$\Delta_1 = \frac{\mu}{\delta \left(\beta c^2 - \gamma\right)}$$
(15)

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Results: The Generalized Pochammer-Chree Equations

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Results: A Generalized Microstructure PDE