# Solitary Wave Families in Two Non-Integrable Models Using Reversible Systems Theory

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# Overview

- Definitions
- Background
- The Generalized Pochhammer-Chree Equations
- A Generalized Microstructure Equation

# Reversible Dynamical System (Iooss & Adelmeyer)

Consider

$$\frac{dz}{dt} = F(z;\mu), z \in \mathbb{R}^n, \mu \in \mathbb{R} \tag{1}$$

where

$$F(0;0) = 0$$

. If there exists a **unitary map**

$$S : \mathbb{R}^n \mapsto \mathbb{R}^n, S \neq I$$

such that

$$F(Sz;\mu) = -SF(z;\mu)$$

for all $z$ and $\mu$ then (1) is a reversible system.

# Families of Solitary Waves

Here we will use the term solitary wave or "soliton" to mean a pulse-like solution to an evolution equation. For example,

$$A(z) = \ell \operatorname{sech}^2 kz \tag{2}$$

is a two-parameter family of solitary waves.
where $k$ and $\ell$ are parameters which determine the speed and the height of the wave.

# Normal Form Theory

After a nonlinear change of variables (Iooss & Adelmeyer) one may put the Center Manifold into Normal Form.

**Two-Dimensional Center Manifold**

- $\lambda_{1-4} = 0, 0, \pm\lambda, \lambda \in \mathbb{R}$
- $Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B)$

**Four-Dimensional Center Manifold**

- $\lambda_{1-4} = 0, 0, \pm i\omega, \omega \in \mathbb{R}$
- $Y = A\zeta_0 + B\zeta_1 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C})$

where $\zeta_0, \zeta_1, \zeta_+, \zeta_-$ are eigenvectors of the linearized operator.

# Properties of Bilinear Functions

A function

$$B : \mathbb{C}x\mathbb{C} \mapsto \mathbb{C}$$

satisfying the following axioms

$$\begin{aligned} B(x+y,z) &= B(x,z)+B(y,z) \\ B(\lambda x,y) &= \lambda B(x,y) \\ B(x,y+z) &= B(x,y)+B(x,z) \\ B(x,\lambda y) &= \lambda B(x,y) \end{aligned}$$

is called **bilinear** .
If $B(x,y)$ is bilinear, then $f(y) \equiv B(y,y)$ is invariant under the transformation $y \mapsto -y$ and thus is **reversible** .

# The Generalized Pochhammer-Chree Equations

The Generalized Pochhammer-Chree Equations govern the propagation of longitudinal waves in elastic rods.

- GPC1

$$(u - u_{xx})_{tt} - \left(a_1 u + a_2 u^2 + a_3 u^3\right)_{xx} = 0 \tag{4}$$

- GPC2

$$(u - u_{xx})_{tt} - \left(a_1 u + a_3 u^3 + a_5 u^5\right)_{xx} = 0 \tag{5}$$

# Travelling Wave ODE

Let $z = x - ct$ and $u(x,t) = \phi(z)$ to reduce (4) and (5) to the Travelling Wave ODE

$$\phi_{zzzz} - q\phi_{zz} + p\phi = \mathcal{N}_{1,2}[\phi] \tag{6}$$

where

$$\begin{aligned} p &\equiv 0 \\ q &\equiv 1 - \frac{a_1}{c^2} \end{aligned}$$

$$L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2 - p & 0 & q/3 & 0 \end{pmatrix} \tag{8a}$$

$$\begin{aligned} \mathcal{N}_1[\phi] &= -\frac{1}{c^2}\left[3a_3\left(2\phi\phi_z^2 + \phi^2\phi_{zz}\right) + 2a_2\left(\phi_{zz}\phi_z + \phi_z^2\right)\right] \\ \mathcal{N}_2[\phi] &= -\frac{1}{c^2}\left[3a_3\left(2\phi\phi_z^2 + \phi^2\phi_{zz}\right) + 5a_5\left(4\phi^3\phi_z^2 + \phi^4\phi_{zz}\right)\right] \end{aligned}$$

# Reversible Form

Denoting $Y = \langle y_1, y_2, y_3, y_4 \rangle^T = \langle \phi, \phi_z, \phi_{zz}, \phi_{zzz} \rangle^T$ equation (6) can be written

$$\frac{dY}{dz} = L_{pq} Y - G_{1,2}(Y, Y) \tag{9}$$

where

$$L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2 - p & 0 & q/3 & 0 \end{pmatrix}$$

$$G_{1,2}(Y, Y) = \langle 0, 0, 0, -\mathcal{N}_{1,2}(Y, Y) \rangle^T$$

# Near $C_0$: Normal Form

The eigenvalues are $\lambda_{1,4} = 0, 0, \pm\lambda$, $\lambda \in \mathbb{R}$, We write the Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B) \tag{10}$$

with corresponding normal form

$$\frac{dA}{dz} = B \tag{11a}$$

$$\frac{dB}{dz} = b\epsilon A + \tilde{c}A^2 \tag{11b}$$

where $\epsilon$ measures the perturbation about $C_0$ and

$$\zeta_0 = \langle 1, 0, -q/3, 0 \rangle^T \tag{12a}$$

$$\zeta_1 = \langle 0, 1, 0, -2q/3 \rangle^T \tag{12b}$$

How do we determine the coefficients $b$ and $\tilde{c}$ ?

# Finding Coefficients Of The Normal Form

*When you follow two separate chains of thought, Watson, you will find some point of intersection which should approximate the truth.* - *Sir Arthur Conan Doyle*

By computing $\frac{dY}{dz}$ in two ways and comparing the coefficients of each term in the normal form, we find systems of equations which determine each coefficient. For example, to find $\tilde{c}$, we compare $\mathcal{O}(A^2)$ terms.

# Two Ways to Compute $\frac{dY}{dz}$

**Method 1**

Use the Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B)$$

in the reversible system

$$\frac{dY}{dz} = L_{pq}Y - G_{1,2}(Y, Y)$$

and repeatedly use the bilinear properties of $G_{1,2}$ to simplify.

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and repeatedly use the bilinear properties of $G_{1,2}$ to simplify.

**Method 2**

Differentiate the Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B)$$

and use the Normal Form

$$\begin{aligned} \frac{dA}{dz} &= B \\ \frac{dB}{dz} &= b\epsilon A + \tilde{c}A^2 \end{aligned}$$

to simplify all derivatives.

where

$$\Psi(\epsilon, A, B) = \epsilon A\Psi^1_{10} + \epsilon B\Psi^1_{01} + A^2\Psi^0_{20} + AB\Psi^0_{11} + B^2\Psi^0_{02} + \cdots$$

# Finding $\tilde{c}$

Matching $\mathcal{O}\left(A^2\right)$ terms in each method gives us the two systems of equations

## GPC 1

$$\tilde{c}\zeta_1 = L_{0q}\Psi_{20}^0 - G_1\left(\zeta_0, \zeta_0\right)$$

$$\begin{aligned}
0 &= x_2 \\
\tilde{c} &= \frac{q}{3}x_1 + x_3 \\
0 &= \frac{q}{3}x_2 + x_4 \implies x_4 = 0 \\
-\frac{2q}{3}\tilde{c} &= \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) + \frac{q}{3c^2}\left(3a_3 + 5a_5\right) \\
&= \frac{q}{3}\tilde{c} + \frac{q}{3c^2}a_3
\end{aligned}$$

$$\implies \tilde{c} = -\frac{a_3}{3c^2}$$

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## GPC 1

$$\tilde{c}\zeta_1 = L_{0q}\Psi_{20}^0 - G_1\left(\zeta_0, \zeta_0\right)$$

$$\begin{aligned} 0 &= x_2 \\ \tilde{c} &= \frac{q}{3}x_1 + x_3 \\ 0 &= \frac{q}{3}x_2 + x_4 \implies x_4 = 0 \\ -\frac{2q}{3}\tilde{c} &= \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) + \frac{q}{3c^2}\left(3a_3 + 5a_5\right) \\ &= \frac{q}{3}\tilde{c} + \frac{q}{3c^2}a_3 \end{aligned}$$

$$\implies \tilde{c} = -\frac{a_3}{3c^2}$$

## GPC 2

$$\tilde{c}\zeta_1 = L_{0q}\Psi_{20}^0 - G_2\left(\zeta_0, \zeta_0\right)$$

$$\begin{aligned} 0 &= x_2 \\ \tilde{c} &= \frac{q}{3}x_1 + x_3 \\ 0 &= \frac{q}{3}x_2 + x_4 \implies x_4 = 0 \\ -\frac{2q}{3}\tilde{c} &= \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) + \frac{q}{3c^2}\left(3a_3 + 5a_5\right) \\ &= \frac{q}{3}\tilde{c} + \frac{q}{3c^2}\left(3a_3 + 5a_5\right) \end{aligned}$$

$$\implies \tilde{c} = -\frac{1}{3c^2}\left(3a_3 + 5a_5\right)$$

where $\Psi_{20}^0 = \langle x_1, x_2, x_3, x_4\rangle^T$.

# Normal Form Near $C_0$

Therefore the Normal Form near $C_0$ is

**GPC 1**

$$\begin{aligned} \frac{dA}{dz} &= B \\ \frac{dB}{dz} &= -\frac{\epsilon}{q}A - \frac{a_3}{c^2}A^2 \end{aligned}$$

**GPC 2**

$$\begin{aligned} \frac{dA}{dz} &= B \\ \frac{dB}{dz} &= -\frac{\epsilon}{q}A - \frac{1}{3c^2}\left(3a_3 + 5a_5\right)A^2 \end{aligned}$$

These equations admit homoclinic solutions near $C_0$ of the form

$$A(z) = \ell \operatorname{sech}^2(kz)$$

# Finding $k$ and $\ell$

To determine $k$ and $\ell$, we first write the Normal Form as a single second order equation. Then we use our expression for $A(z)$ and compare coefficients of $\mathcal{O}\left(\operatorname{sech}^2(kz)\right)$ and $\mathcal{O}\left(\operatorname{sech}^4(kz)\right)$ which implies

**GPC 1**

$$\begin{aligned} k &= \sqrt{\frac{-\epsilon}{4q}} \\ \ell &= \frac{-3\epsilon c^2}{2qa_3} \end{aligned}$$

**GPC 2**

$$\begin{aligned} k &= \sqrt{\frac{-\epsilon}{4q}} \\ \ell &= \frac{-9\epsilon c^2}{2q\left(3a_3 + 5a_5\right)} \end{aligned}$$

Hence, since $\epsilon = -p$, solitons of this form exist for $p = 0^+, q > 0$, which implies $a_1 < c^2$.

# Near $C_1$: Normal Form

The eigenvalues are $\lambda_{1,4} = 0, 0, \pm i\omega$, $\omega \in \mathbb{R}$, We write the Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C}) \qquad (20)$$

with corresponding normal form

$$\begin{aligned} \frac{dA}{dz} &= B \\ \frac{dB}{dz} &= \bar{\nu}\epsilon A + b_* A^2 + c_* \|C\|^2 \\ \frac{dC}{dz} &= id_0 C + i\bar{\nu} d_1 C + id_2 AC \end{aligned}$$

where the new eigenvectors co-spanning the four-dimensional Center Manifold are

$$\zeta_\pm = \langle 1, \lambda_\pm, 2q/3, \lambda_\pm q/3 \rangle^T$$

and $\lambda_\pm = \pm i\sqrt{-q}$, $q < 0$.

# Compute $\frac{dY}{dz}$ by both methods

Matching the coefficients of $A^2$, $\epsilon C$ $\|C\|^2$ and $AC$ in the two separate expressions for $dY/dz$ yields the following two systems of equations:

$$
\begin{aligned}
\mathcal{O}(A^2): &\quad b_*\zeta_1 &&= L_{0q}\Psi^0_{2000} - G_{1,2}(\zeta_0,\zeta_0)\\
\mathcal{O}(|C|^2): &\quad c_*\zeta_1 &&= L_{0q}\Psi^0_{0011} - 2G_{1,2}(\zeta_+,\zeta_-)\\
\mathcal{O}(\epsilon C): &\quad -\tfrac{i}{q}\left(d_1\zeta_+ + d_0\Psi^1_{0010}\right) &&= L_{0q}\Psi^1_{0010}\\
\mathcal{O}(AC): &\quad id_2\zeta_+ + id_0\Psi^0_{1010} &&= L_{0q}\Psi^0_{1010} - 2G_{1,2}(\zeta_0,\zeta_+)
\end{aligned}
$$

# Normal Form near $C_1$ : GPC 1

The resolution of the preceding equations yields the coefficients in the Normal Form for the GPC 1 equation:

$$
\begin{aligned}
\frac{dA}{dz} &= B \\
\frac{dB}{dz} &= -\frac{\epsilon}{q}A + \frac{a_3}{c^2}A^2 + \frac{1}{c^2}\left(\frac{7}{2\sqrt{-q}}a_3 - \frac{i}{3}a_2\right)|C|^2 \\
\frac{dC}{dz} &= i\sqrt{-q}C - i\frac{\sqrt{-q}}{q^3}C\epsilon + i\frac{1}{c^2}\left(\frac{7}{2\sqrt{-q}}a_3 - \frac{i}{3}a_2\right)AC
\end{aligned}
$$

where $q \equiv 1 - \frac{a_1}{c^2}$

# Normal Form near $C_1$: GPC 2

The resolution of the preceding equations yields the coefficients in the Normal Form for the GPC 2 equation:

$$
\begin{aligned}
\frac{dA}{dz} &= B \\
\frac{dB}{dz} &= -\frac{\epsilon}{q}A + \frac{1}{3c^2}\left(3a_3 + 5a_5\right)A^2 + \frac{1}{c^2}\left(16a_3 + \frac{140}{3}a_5\right)|C|^2 \\
\frac{dC}{dz} &= i\sqrt{-q}C - i\frac{\sqrt{-q}}{q^3}C\epsilon + i\frac{1}{\sqrt{-q}c^2}\left(\frac{7}{2}a_3 + \frac{32}{3}a_5\right)AC
\end{aligned}
$$

where $q \equiv 1 - \frac{a_1}{c^2}$

# Dynamics Near $C_1$

The two first integrals of the four-dimensional Normal Form are

$$K = \|C\|^2$$

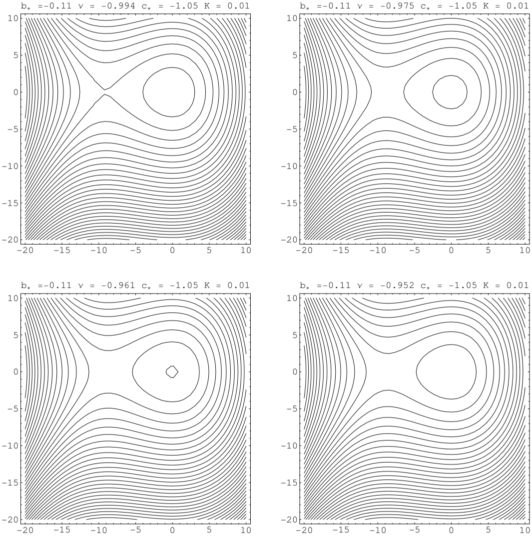
and

$$H = B^2 - \frac{2}{3} b_* A^2 - \bar{\nu} A^2 - 2c_* KA$$

In the $(A, B)$ phase plane, the level curve $H = 0$ compromises a homoclinic orbit. The intersection of $H = 0$ with the $A$ axis occurs for

$$A_{\mp} = \frac{3}{4b_*} \left[ \bar{\nu} \pm \sqrt{\bar{\nu}^2 + \frac{16 b_* c_* K}{3}} \right]$$

# Homoclinic Orbits for Various Values of H



# A Generalized Microstructure PDE

One dimensional wave propagation in microstructured solids has recently been modeled by an equation

$$v_{tt} - bv_{xx} - \frac{\mu}{2}\left(v^2\right)_{xx} - \delta\left(\beta v_{tt} - \gamma v_{xx}\right)_{xx} = 0 \tag{25}$$

# Travelling Wave ODE

Let $z = x - ct$ and $u(x,t) = \phi(z)$ to reduce (25) to the Travelling Wave ODE

$$\phi_{zzzz} - q\phi_{zz} + p\phi = \mathcal{N}[\phi] \tag{26}$$

where

$$\mathcal{N}[\phi] = -\Delta_1 \phi_z^2 - b\Delta_1 \phi\phi_{zz} \tag{27}$$

$$\begin{aligned} z &\equiv x - ct \\ p &\equiv 0 \\ q &\equiv \frac{c^2 - b}{\delta\left(\beta c^2 - \gamma\right)} \\ \Delta_1 &\equiv \frac{\mu}{\delta\left(\beta c^2 - \gamma\right)} \end{aligned}$$

# Normal Form near $C_0$

With the same kind analysis we find

$$\begin{aligned}\frac{dA}{dz} &= B\\ \frac{dB}{dz} &= -\frac{\epsilon}{q}A - \frac{b\Delta_1}{3}A^2\end{aligned}$$

# Solitary Waves Near $C_0$

The Normal Form admits a homoclinic solution near $C_0$ of the form

$$A(z) = \ell \operatorname{sech}^2(kz)$$

where

$$\begin{aligned} k &= \sqrt{\frac{-\epsilon}{4q}} \\ \ell &= \frac{6k^2}{b\Delta_1} \end{aligned}$$

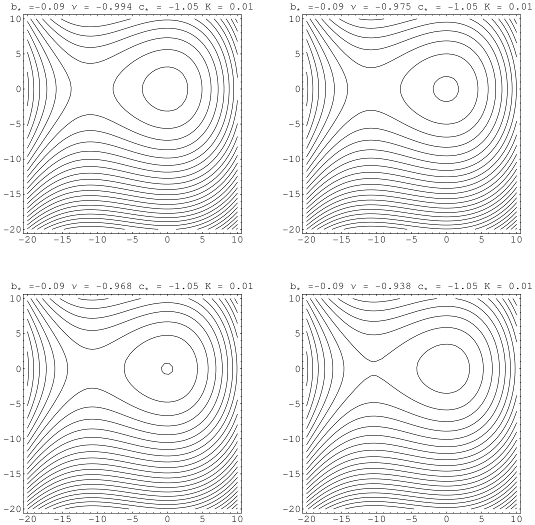
Since $\epsilon = -p$, and the curve $C_0$ corresponds to $p = 0, q > 0$, solitary waves exist near $C_0$ for $p > 0, q > 0$, which implies that $\frac{c^2-b}{\delta(\beta c^2-\gamma)} > 0$.

# Normal Form near $C_1$

Near $C_1$ the Normal Form for (25) is

$$
\begin{aligned}
\frac{dA}{dz} &= B \\
\frac{dB}{dz} &= -\frac{\epsilon}{q}A - \frac{b\Delta_1}{3}A^2 + 2\Delta_1\left(\frac{2b}{3} - 1\right)|C|^2 \\
\frac{dC}{dz} &= i\sqrt{-q}\,C - i\frac{\sqrt{-q}}{q^3}C\epsilon + i\frac{b\Delta_1}{6\sqrt{-q}}AC
\end{aligned}
$$

# Homoclinic Orbits for Various Values of H



# Open Problems

Study embedded solitons using a mix of

- exponential asymptotics
- numerical shooting

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