Solitary Wave Families in Two Non-Integrable Models Using Reversible Systems Theory

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Overview

- Definitions
- Background
- Literature Review
- Method of Solution
- The Generalized Pochammer-Chree Equations
- A Generalized Microstructure Equation

Reversible Dynamical System (Iooss & Adelmeyer)

Consider

$$\frac{dz}{dt} = F(z;\mu), z \in \mathbb{R}^n, \mu \in \mathbb{R} \tag{1}$$

where

$$F(0;0) = 0$$

. If there exists a **unitary map**

$$S : \mathbb{R}^n \mapsto \mathbb{R}^n, S \neq I$$

such that

$$F(Sz;\mu) = -SF(z;\mu)$$

for all z and μ then (1) is a reversible system.

Families of Solitary Waves

Here we will use the term solitary wave or "soliton" to mean a pulse-like solution to an evolution equation. For example,

$$A(z) = \ell \operatorname{sech}^2 kz \tag{2}$$

is a two-parameter family of solitary waves.
where k and ℓ are parameters which determine the speed and the height of the wave.

Normal Form Theory

After a nonlinear change of variables (Iooss & Adelmeyer) one may put the Center Manifold into Normal Form.

- Two-Dimenional Center Manifold
  - $\lambda_{1-4} = 0, 0, \pm\lambda, \lambda \in \mathbb{R}$
  - $Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B)$
- Four-Dimenional Center Manifold
  - $\lambda_{1-4} = 0, 0, \pm i\omega, \omega \in \mathbb{R}$
  - $Y = A\zeta_0 + B\zeta_1 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C})$

where $\zeta_0, \zeta_1, \zeta_+, \zeta_-$ are eigenvectors of the linearized operator.

Properties of Bilinear Functions

A function

$$B : \mathbb{C}x\mathbb{C} \mapsto \mathbb{C}$$

satisfying the following axioms

$$\begin{aligned} B(x+y,z) &= B(x,z)+B(y,z) && \text{(3a)} \\ B(\lambda x,y) &= \lambda B(x,y) && \text{(3b)} \\ B(x,y+z) &= B(x,y)+B(x,z) && \text{(3c)} \\ B(x,\lambda y) &= \lambda B(x,y) && \text{(3d)} \end{aligned}$$

is called **bilinear** .

If $B(x,y)$ is bilinear, then $f(y) \equiv B(y,y)$ is invariant under the transformation $y \mapsto -y$ and thus is **reversible** .

The Generalized Pochammer-Chree Equations

The Generalized Pochammer-Chree Equations govern the propagation of longitudinal waves in elastic rods.

- GPC1

$$(u - u_{xx})_{tt} - \left(a_1 u + a_2 u^2 + a_3 u^3\right)_{xx} = 0 \tag{4}$$

- GPC2

$$(u - u_{xx})_{tt} - \left(a_1 u + a_3 u^3 + a_5 u^5\right)_{xx} = 0 \tag{5}$$

Travelling Wave ODE

Let $z = x - ct$ and $u(x,t) = \phi(z)$ to reduce (4) and (5) to the Travelling Wave ODE

$$\phi_{zzzz} - q\phi_{zz} + p\phi = \mathcal{N}_{1,2}[\phi] \tag{6}$$

where

$$\begin{aligned} p &\equiv 0 \\ q &\equiv 1 - \frac{a_1}{c^2} \end{aligned}$$

$$L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2 - p & 0 & q/3 & 0 \end{pmatrix} \tag{8a}$$

$$\begin{aligned} \mathcal{N}_1[\phi] &= -\frac{1}{c^2}\left[3a_3\left(2\phi\phi_z^2 + \phi^2\phi_{zz}\right) + 2a_2\left(\phi_{zz}\phi_z + \phi_z^2\right)\right] \\ \mathcal{N}_2[\phi] &= -\frac{1}{c^2}\left[3a_3\left(2\phi\phi_z^2 + \phi^2\phi_{zz}\right) + 5a_5\left(4\phi^3\phi_z^2 + \phi^4\phi_{zz}\right)\right] \end{aligned}$$

Reversible Form

Denoting $Y = \langle y1, y2, y3, y4 \rangle^T = \langle \phi, \phi_z, \phi_{zz}, \phi_{zzz} \rangle^T$ equation (6) can be written

$$\frac{dY}{dz} = L_{pq} Y - G_{1,2}(Y, Y) \tag{9}$$

where

$$L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2 - p & 0 & q/3 & 0 \end{pmatrix}$$

$$G_{1,2}(Y, Y) = \langle 0, 0, 0, -\mathcal{N}_{1,2}(Y, Y) \rangle^T$$

Near C_0

The eigenvalues are $\lambda_{1,4} = 0, 0, \pm\lambda,\ \lambda \in \mathbb{R}$, We write the Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B) \tag{10}$$

with corresponding normal form

$$\frac{dA}{dz} = B \tag{11a}$$

$$\frac{dB}{dz} = b\epsilon A + \tilde{c}A^2 \tag{11b}$$

where ϵ measures the perturbation about C_0 and

$$\zeta_0 = \langle 1, 0, -q/3, 0\rangle^T \tag{12a}$$

$$\zeta_1 = \langle 0, 1, 0, -2q/3\rangle^T \tag{12b}$$

How do we determine the coefficients b and c_* ?

A Generalized Microstructure PDE

One dimensional wave propagation in microstructured solids has recently been modeled [?] by an equation

$$v_{tt} - bv_{xx} - \frac{\mu}{2}\left(v^2\right)_{xx} - \delta\left(\beta v_{tt} - \gamma v_{xx}\right)_{xx} = 0 \tag{13}$$

Travelling Wave ODE

Let $z = x - ct$ and $u(x,t) = \phi(z)$ to reduce to the Travelling Wave ODE

$$\phi_{zzzz} - q\phi_{zz} + p\phi = \mathcal{N}[\phi] \tag{14}$$

where

$$\mathcal{N}\left[\phi\right] = -\Delta_1\phi_z^2 - b\Delta_1\phi\phi_{zz} \tag{15}$$

$$\Delta_1 = \frac{\mu}{\delta\left(\beta c^2 - \gamma\right)}$$

Results: The Generalized Pochammer-Chree Equations

(16)

Results: A Generalized Microstructure PDE