SOLITARY-WAVE FAMILIES OF THE GENERALIZED POCHAMMER-CHREE EQUATIONS

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ABSTRACT. The Generalized Pochammer-Chree Equations govern the propagation of longitudinal waves in elastic rods. Limited analytic results exist for the occurence of one family of solitary-wave solutions of these equations. Since solitary-wave solutions often play a central role in the long-time evolution of an initial disturbance, we consider such solutions here (via normal form approach) within the framework of reversible systems theory, confirming the existence of the known family of solitary waves. We find a continuum of delocalized solitary waves (or homoclinics to small-amplitude periodic orbits). On isolated curves in the relevant parameter region, the delocalized waves reduce to genuine embedded solitons. The new family of solutions occur in regions of parameter space distinct from the known solitary-wave solutions and are thus entirely new. Directions for future work are also mentioned.

1. INTRODUCTION

The propagation of longitudinal derformation waves in elastic rods is governed ([1], [6], [7]) by the Generalized Pochammer-Chree Equations:

$$(u-u_{xx})_{tt}-\left(a_1u+a_2u^2+a_3u^3\right)_{xx}=0 \tag{1.1}$$

and

$$(u-u_{xx})_{tt}-\left(a_1u+a_3u^3+a_5u^5\right)_{xx}=0 \tag{1.2}$$

corresponding to different constitutive relations.

References [1], [6], [7] also discuss the primary references, including derivations and applications of these equations in various fields. In addition, motivated by experimental and numerical results, there are derivations of special families of solitary wave solutions by the extended $Tanh$ method [1], and other ansatzen [7]. These extend earlier solitary wave solutions given by Bogolubsky [2] and Clarkson et. al [4] for special cases of (1.1) and (1.2). In addition, [6] generalizes the existence results in [5] for solitary waves of (1.1) and (1.2).

In this paper, we initiate a fresh approach to the solitary wave solutions of the Generalized Pochammer-Chree equations (1.1) and (1.2).

We invoke the theory of reversible systems and the method of normal forms to categorize the possible solitary waves of (1.1) and (1.2) much more completely than done so far. As we shall see, several families of solitary waves exist in various regions of parameter space. Our main focus here will be on delineating the possible occurence and multiplicity of solitary waves in different parameter regimes. Certain delicate questions relating to specific waves or wave families will form the basis of future work.

The remainder of this paper is organized as follows. In Section 2, we delineate the possible families of solitary waves in various parameter domains and on certain important curves using the theory of reversible systems. In Section 3 and 4, we next focus on the various transition curves and derive normal forms in their vicinity to confirm the existence of families of regular or delocalized solitary-wave solutions in their vicinity.

2. SOLITARY WAVES: LOCAL BIFURCATION

Solitary waves of (1.1) and (1.2) of the form $v(x,t)=\phi(x-ct)=\phi(z)$ satisfy the fourth-order travelling wave ODE

$$\phi_{zzzz}-q\phi_{zz}+p\phi=\mathcal{N}_{1,2}[\phi] \tag{2.1}$$

where

$$\mathcal{N}_1[\phi] = -\frac{1}{c^2}\left[3a_3\left(2\phi\phi_z^2+\phi^2\phi_{zz}\right)+2a_2\left(\phi_{zz}\phi_z+\phi_z^2\right)\right] \tag{2.2a}$$

$$\mathcal{N}_2[\phi] = -\frac{1}{c^2}\left[3a_3\left(2\phi\phi_z^2+\phi^2\phi_{zz}\right)+5a_5\left(4\phi^3\phi_z^2+\phi^4\phi_{zz}\right)\right] \tag{2.2b}$$

$$z \equiv x-ct \tag{2.3a}$$

$$p \equiv 0 \tag{2.3b}$$

$$q \equiv 1-\frac{a_1}{c^2} \tag{2.3c}$$

(2.3d)

Equation (2.1) is invariant under the transformation $z \mapsto -z$ and is thus a reversible system. In this section we shall use the theory of reversible systems to characterize the homoclinic orbits to the fixed point of (2.1), which correspond to pulses or solitary waves of the Micro-Structure PDE in various regions of the (p,q) plane.

The linearized system corresponding to (2.1)

$$\phi_{zzzz}-q\phi_{zz}+p\phi=0 \tag{2.4}$$

has a fixed point

$$\phi=\phi_z=\phi_{zz}=\phi_{zzz}=0 \tag{2.5}$$

Solutions $\phi = ke^{\lambda x}$ satisfy the characteristic equation $\lambda^4-q\lambda^2+p=0$ from which one may deduce that the structure of the eigenvalues is distinct in two regions of (p,q)-space. Since $p=0$ we have only two possible regions of eigenvalues. We denote C_0 as the positive q axis and C_1 the negative q-axis. First we shall consider the bounding curves C_0 and C_1 and their neighborhoods, then we shall discusss the possible occurence and multiplicities of homoclinic orbits to (2.5), corresponding to pulse solitary waves of (1.1) and (1.2), in each region:

Near C_0: The eigenvalues have the structure $\lambda_{1-4}=0,0,\pm\lambda$, $(\lambda\in\mathbb{R})$ and the fixed point (2.5) is a saddle-focus.

Near C_1: Here the eigenvalues have the structure $\lambda_{1-4}=0,0,\pm i\omega$, $(\omega\in\mathbb{R})$. We will show by analysis of a four-dimensional normal form in Section 4 that there exists a $sech^2$ homoclinic orbit near C_1.

Having outlined the possible families of orbits homoclinic to the fixed point (2.5) of (2.4), corresponding to pulse solitary waves of (1.1) and (1.2), we now derive normal forms near the transition curves C_0 and C_1 to confirm the existence of regular or delocalized solitary waves in the corresponding regions of (p,q) parameter space.

3. Normal form near C_0: solitary-wave solutions

Using (2.4), the curve C_0, corresponding to $\lambda=0,0,\pm\tilde{\lambda}$, is given by

$$C_0: p=0, q>0 \tag{3.1}$$

Using (2.3c) implies

$$a_1<c^2 \tag{3.2}$$

Denoting ϕ by y_1, (2.1) may be written as the two systems

$$\frac{dy_1}{dz} = y_2 \tag{3.3a}$$

$$\frac{dy_2}{dz} = y_3 \tag{3.3b}$$

$$\frac{dy_3}{dz} = y_4 \tag{3.3c}$$

$$\frac{dy_4}{dz} = qy_3-py_1-N_{1,2}(Y) \tag{3.3d}$$

where

$$\mathcal{N}_1(Y) = -\frac{1}{c^2}\left[3a_3\left(2y_1y_2^2 + y_1^2y_3\right) + 2a_2\left(y_3y_2 + y_2^2\right)\right] \tag{3.4a}$$

$$\mathcal{N}_2(Y) = -\frac{1}{c^2}\left[3a_3\left(2y_1y_2^2 + y_1^2y_3\right) + 5a_5\left(4y_1^3y_2^2 + y_1^4y_3\right)\right] \tag{3.4b}$$

We wish to rewrite this as a first order reversible system in order to invoke the relevant theory [3]. To that end, defining $Y = \langle y_1, y_2, y_3, y_4\rangle^T$ equation (3.3) may be written

$$\frac{dY}{dz} = L_{pq}Y - G_{1,2}(Y,Y) \tag{3.5}$$

where

$$L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2-p & 0 & q/3 & 0 \end{pmatrix} \tag{3.6}$$

Since $p = 0$ for (1.1) and (1.2), we have

$$\frac{dY}{dz} = L_{0q}Y - G_{1,2}(Y,Y) \tag{3.7}$$

where

$$G_{1,2}(Y,Y) = \langle 0,0,0,-\mathcal{N}_{1,2}(Y)\rangle^T \tag{3.8}$$

Next we calculate the normal form of (3.7) near C_0. The procedure is closely modeled on [3] and many intermediate steps may be found there.

3.1. **Near C_0.** Near C_0 the dynamics reduce to a 2-D Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B) \tag{3.9}$$

and the corresponding normal form is

$$\frac{dA}{dz} = B \tag{3.10a}$$

$$\frac{dB}{dz} = b\epsilon A + \tilde{c}A^2 \tag{3.10b}$$

Here,

$$\epsilon = \left(\frac{q^2}{9} - p\right) - \left(\frac{q}{3}\right)^2 = -p \tag{3.11}$$

measures the perturbation around C_0, and

$$\zeta_0 = \langle 1, 0, -q/3, 0\rangle^T \tag{3.12a}$$

$$\zeta_1 = \langle 0, 1, 0, -2q/3\rangle^T \tag{3.12b}$$

The linear eigenvalue of (3.10a) satisfies

$$\lambda^2 = b\epsilon \tag{3.13}$$

The characteristic equation of the linear part of (3.7) is

$$\lambda^4 - q\lambda^2 - \epsilon = 0 \tag{3.14}$$

Hence, the eigenvalues near zero (the Center Manifold) satisfy $\lambda^4 \ll \lambda^2$ and hence

$$\lambda^2 \sim -\frac{\epsilon}{q} \tag{3.15}$$

Matching (3.13) and (3.15) implies

$$b = -\frac{1}{q} \tag{3.16}$$

and only the nonlinear coefficient $\tilde{c}$ remains to be determined in the normal form (3.10a).

In order to determine $\tilde{c}$ (the coefficient of A^2 in (3.10a) we calculate $\frac{dY}{dz}$ in two ways and match the $\mathcal{O}(A^2)$ terms.

To this end, using the standard 'suspension' trick of treating the perturbation parameter ϵ as a variable, we expand the function Ψ in (3.9) as

$$\Psi(\epsilon, A, B) = \epsilon A\Psi^1_{10} + \epsilon B\Psi^1_{01} + A^2\Psi^0_{20} + AB\Psi^0_{11} + B^2\Psi^0_{02} + \cdots \tag{3.17}$$

where the subscripts denote powers of A and B, respectively, and the superscript denotes the power of ϵ.

In the first way of computing dY/dz, we take the z derivative of (3.9) (using (3.10a) and (4.10)). The coefficient of A^2 in the resulting expression is $\tilde{c}\zeta_1$. In the second way of computing dY/dz, use (3.9) and (4.10) in (3.5). The coefficient of A^2 in the resulting expression is $L_{0,q}\Psi^0_{20} - G_{1,2}(\zeta_0, \zeta_0)$. Hence

$$\tilde{c}\zeta_1 = L_{0q}\Psi^0_{20} - F_2(\zeta_0, \zeta_0) \tag{3.18}$$

Using (3.12a) and (3.8) and denoting $\Psi^0_{20} = \langle x_1, x_2, x_3, x_4 \rangle$ in (3.18) yields the equations

$$0 = x_2 \tag{3.19a}$$

$$\tilde{c} = \frac{q}{3}x_1 + x_3 \tag{3.19b}$$

$$0 = \frac{q}{3}x_2 + x_4 \implies x_4 = 0 \text{ using (3.19b)} \tag{3.19c}$$

and

$$-\frac{2q}{3}\tilde{c} = \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) + \frac{q}{3c^2}\left(3a_3 + 5a_5\right) = \frac{q}{3}\tilde{c} + \frac{q}{3c^2}\left(3a_3 + 5a_5\right) \text{ using (3.19b)} \tag{3.20}$$

Hence we obtain

$$\tilde{c} = -\frac{1}{3c^2}\left(3a_3 + 5a_5\right) \tag{3.21}$$

Therefore, the normal form near C_0 is

$$\frac{dA}{dz} = B \tag{3.22a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{a_3}{c^2}A^2 \tag{3.22b}$$

for (1.1) and

$$\frac{dA}{dz} = B \tag{3.23a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{1}{3c^2}\left(3a_3 + 5a_5\right)A^2 \tag{3.23b}$$

for (1.2).

The normal form (3.22) admits a homoclinic solution (near C_0) of the form

$$A(z) = \ell sech^2(kz) \tag{3.24}$$

with

$$k = \sqrt{\frac{-\epsilon}{4q}} \tag{3.25a}$$

$$\ell = \frac{-3\epsilon c^2}{2qa_3} \tag{3.25b}$$

Similarly, the normal form (3.23) admits a homoclinic solution (near C_0) of the form

$$A(z) = \ell sech^2(kz) \tag{3.26}$$

with

$$k = \sqrt{\frac{-\epsilon}{4q}} \tag{3.27a}$$

$$\ell = \frac{-3\epsilon c^2}{2q\,(3a_3+5a_5)} \tag{3.27b}$$

Hence, since $\epsilon = -p$, and the curve C_0 corresponds to $p = 0, q > 0$, solitary waves of the form (3.24) exist in the vicinity of C_0 for

$$p > 0, q > 0 \tag{3.28}$$

which implies that $a_1 < c^2$ (such that k in (3.27a) is real.) As mentioned in section 2, one may show the persistence of this homoclinic solution in the original travelling wave ODE (2.4). Thus, we have demonstrated the existence of solitary waves of (1.2) for $p = 0^+, q > 0$.

Similarly, the curve C_1 corresponds to $p = 0, q < 0$, solitary waves of the form (3.24) exist in the vicinity of C_1 for

$$p < 0, q < 0 \tag{3.29}$$

which implies $a_1 > c^2$.

Again, one may show the persistence of this homoclinic solution in the original travelling wave ODE (2.4). Thus, we have demonstrated the existence of solitary waves of (1.2) for $p = 0^-, q < 0$.

4. Normal form near C_1: possible solitary-wave solutions

Using (2.4), the curve C_1, corresponding to $\lambda = 0, 0 \pm i\omega$, is given by

$$C_1 : p = 0, q < 0 \tag{4.1}$$

Which implies

$$a_1 > c^2 \tag{4.2}$$

In order to investigate the possibility of a $sech^2$ homoclinic orbit in the neighborhood of C_1 and delocalized solitary waves, we next compute the normal form near C_1 following the procedure in [3].

Near C_1 the dynamics reduce to a 4-D Center Manifold [3] Since all the eigenvalues are non-hyperbolic, the Center Manifold has the form (a nonlinear coordinate change [3])

$$Y = A\zeta_0 + B\zeta_0 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C}) \tag{4.3}$$

with a corresponding four-dimensional normal form

$$\frac{dA}{dz} = B \tag{4.4a}$$

$$\frac{dB}{dz} = \bar{\nu}A + b_* A^2 + c_* \left|C\right|^2 \tag{4.4b}$$

$$\frac{dC}{dz} = id_0 C + i\bar{\nu} d_1 C + id_2 AC \tag{4.4c}$$

Here C is complex, $\bar{C}$ is the complex conjugate of C, $\epsilon, \zeta_0, \zeta_1$ are given previously and the two new complex eigenvectors co-spanning the Center Manifold are

$$\zeta_\pm = \left\langle 1, \lambda_\pm, 2q/3, \frac{\lambda_\pm}{3} q \right\rangle^T \tag{4.5}$$

Using (4.4c) and (3.10a)

$$\bar{\nu} = b\epsilon = -\frac{\epsilon}{q} \tag{4.6}$$

Also from the characteristic equation (3.14), the two non-zero (imaginary) roots are

$$\lambda^2 = \frac{q+\sqrt{q^2+4\epsilon}}{2} \approx q \text{ for } \epsilon \text{ small} \tag{4.7}$$

Hence

$$\lambda = \pm i\sqrt{-q}, q < 0 \tag{4.8}$$

Matching this to the linear part of (4.4c) (which corresponds to the imaginary eigenvalues), $\lambda = id_0 = i\sqrt{-q}$ or

$$d_0 = \sqrt{-q} \tag{4.9}$$

If we do a dominant balance argument after the change of variable $\epsilon = \sqrt{-3\alpha}$ on the characteristic equation as $\lambda \to 0$ then we find $d_1 = \frac{\sqrt{-3\alpha}}{18\alpha^2}$. Using $\alpha = q/3$ we find $d_1 = \frac{\sqrt{-q}}{2q^2}$

The remaining undetermined coefficients in the normal form are the coefficients b_*, c_* and d_2 which correspond to the $A^2, |C|^2$ and AC terms respectively. In order to determine them, we follow the same procedure as in Section 3 and compute dY/dz is two distinct ways. We expand the function Ψ as

$$\Psi(\epsilon, A, B, C, \bar{C}) = \epsilon A\Psi^1_{1000} + \epsilon B\Psi^1_{0100} + A^2\Psi^0_{2000} + AB\Psi^0_{1100} + AC\Psi^0_{1010} + \epsilon C\Psi^1_{0010} + \cdots \tag{4.10}$$

with subscripts denoting powers of A, B, C and $\bar{C}$, respectively, and the superscript is the power of ϵ. In the first way of computing dY/dz is computed by taking the z derivative of (4.3) (using (4.4a) and (4.10)) and read off the coefficients of $A^2, \|C\|^2, C\epsilon$ and AC terms.

In the second way, dY/dz is computed using (4.3) and (4.10) in (3.5) (with $p = 0$ on C_1 as given in (4.1)) and the coefficients of A, B, C and $\bar{C}$ are once again read off.

Equating the coefficients of the corresponding terms in the two separate expressions for dY/dz yields the following equations:

$$\mathcal{O}(A^2): \quad b_*\zeta_1 = L_{0q}\Psi^0_{2000} - G_{1,2}(\zeta_0, \zeta_0) \tag{4.11a}$$

$$\mathcal{O}(|C|^2): \quad c_*\zeta_1 = L_{0q}\Psi^0_{0011} - 2G_{1,2}(\zeta_+, \zeta_-) \tag{4.11b}$$

$$\mathcal{O}(\epsilon C): \quad -\frac{i}{q}\left(d_1\zeta_+ + d_0\Psi^1_{0010}\right) = L_{0q}\Psi^1_{0010} - G_{1,2}(\Psi^1_{0010}, \Psi^1_{0010}) \tag{4.11c}$$

$$\mathcal{O}(AC): \quad id_2\zeta_+ + id_0\Psi^0_{1010} = L_{0q}\Psi^0_{1010} - 2G_{1,2}(\zeta_0, \zeta_+) \tag{4.11d}$$

where we have used the fact that $G_{1,2}$ is a symmetric bilinear form. Equation (4.11b) is decoupled and yields $c_* = \frac{8}{c^2}(2a_3 - a_2)$ for (1.1) and $c_* = \frac{1}{c^2}\left(16a_3 + \frac{140}{3}a_5\right)$ for (1.2). The only coefficient left to determine is d_2 which we shall compute now.

Using $\Psi^0_{1010} = \langle x_1, x_2, x_3, x_4\rangle^T$ in (4.11d) implies

$$id_2 + id_0x_1 = x_2 \tag{4.12a}$$

$$-d_0d_2 + id_0x_2 = \frac{q}{3}x_1 + x_3 \tag{4.12b}$$

$$\frac{2iq}{3}d_2 + id_0x_3 = \frac{q}{3}x_2 + x_4 \tag{4.12c}$$

$$-\frac{q}{3}d_0d_2 + id_0x_4 = \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) - \frac{2q}{c^2}\left(\frac{7}{2}a_3 - \frac{i}{3}d_0a_2\right) \tag{4.12d}$$

for (1.1) and

$$id_2 + id_0x_1 = x_2 \tag{4.13a}$$

$$-d_0d_2 + id_0x_2 = \frac{q}{3}x_1 + x_3 \tag{4.13b}$$

$$\frac{2iq}{3}d_2 + id_0x_3 = \frac{q}{3}x_2 + x_4 \tag{4.13c}$$

$$-\frac{q}{3}d_0d_2 + id_0x_4 = \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) - \frac{2q}{c^2}\left(\frac{7}{2}a_3 + \frac{32}{3}a_5\right) \tag{4.13d}$$

for (1.2)

Using (4.13a) in (4.13b) , (4.13b) in (4.13d) and using these in (4.13c) yields $d_2 = \frac{1}{c^2}\left(\frac{7}{2\sqrt{-q}}a_3 - \frac{i}{3}a_2\right)$ for (1.1) and $d_2 = \frac{1}{\sqrt{-q}c^2}\left(\frac{7}{2}a_3 + \frac{32}{3}a_5\right)$ for (1.2).

Therefore the normal form near C_1 is

$$\frac{dA}{dz} = B \tag{4.14a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - b_*A^2 + \frac{1}{c^2}\left(\frac{7}{2\sqrt{-q}}a_3 - \frac{i}{3}a_2\right)|C|^2 \tag{4.14b}$$

$$\frac{dC}{dz} = i\sqrt{-q}C - i\frac{\sqrt{-q}}{q^3}C\epsilon + i\frac{1}{c^2}\left(\frac{7}{2\sqrt{-q}}a_3 - \frac{i}{3}a_2\right)AC \tag{4.14c}$$

for (1.1) and

$$\frac{dA}{dz} = B \tag{4.15a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - b_*A^2 + \frac{1}{c^2}\left(16a_3 + \frac{140}{3}a_5\right)|C|^2 \tag{4.15b}$$

$$\frac{dC}{dz} = i\sqrt{-q}C - i\frac{\sqrt{-q}}{q^3}C\epsilon + i\frac{1}{\sqrt{-q}c^2}\left(\frac{7}{2}a_3 + \frac{32}{3}a_5\right)AC \tag{4.15c}$$

for (1.2).

The dynamics inherent in (1.1), (1.2) may be elucidated following the discussions of [?],[?],[?] and [?]. The two first integrals of (4.4a) are

$$K = |C|^2 \tag{4.16}$$

and

$$H = B^2 - \frac{2}{3}b_* - \bar{\nu}A^2 - 2c_*KA_* \tag{4.17}$$

Here, the appropriate coefficients $b_*, \bar{\nu}$ and c_*, derived above, apply for (1.1) and (1.2). Also, c_* should be real, or a_2 must be zero in (1.1) for the following energy arguments to apply.

As a typical case, consider the level curve $H = 0$ of the energy-like first integral function H. In the (A, B) phase plane, this will compromise a homoclinic orbit. The intersection of $H = 0$ with the A axis occours for $\frac{2}{3}b_*A^2 - \bar{\nu}A - 2c_*K = 0$ or

$$A_{\mp} = \frac{3}{4b_*}\left[\bar{\nu} \pm \sqrt{\bar{\nu}^2 + \frac{16b_*c_*K}{3}}\right] \tag{4.18}$$

Note that $A_+ > 0, A_- < 0$ for $b_*c_* > 0$ and $b_* < 0$ as relevant for us. A general homoclinic orbit, homoclinic to A_+, is sketched in FIGURE where the flow direction is deduced from (4.14b) and (4.15b) for (1.1) and (1.2), respectively. For $K = |C|^2 = 0$, the orbit is homoclinic to $A_+ = 0$. For small non-zero $|K|$, $A_+ \sim -2c_*K/\bar{\nu}$, meaming that oscillations at infinity are then very small in this case. For $K = 0$ this corresponds to an orbit to 0 for the normal form. This is indeed valid for the normal form taken at any order. However this solution does not exist mathematically for the full original system, even though one may compute its expansion in powers of the bifurcation parameter up to any order (see [?] and [?]). This is an example of the famous challenging problem of asymptotics beyond any orders. Other solutions found on the normal form mainly persist under the perturbation from higher order terms provided by the original system [?]. These solutions are delocalized waves and their existence in Region 2 is guaranteed by the general theory for reversible systems in [?] and [?]. Also, as mentioned in Section 2, genuine solitary waves are found on isolated curves oin Region 2 of FIGURE on which the oscillation amplitudes vanish. Since these are embedded in the sea of delocalized solitary waves and in the continuous spectrum, they are referred to as embedded solitons [?]. These will further be investigated in Region 2 subsequently using a mix of exponential asymptotics and numerical shooting.

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