SOLITARY-WAVE FAMILIES OF THE GENERALIZED POCHAMMER-CHREE PDE'S

JONATHAN LETO, S. ROY CHOUDHURY

1. Introduction

2. Solitary waves: local bifurcation

3. Calculations

Generalized Pochammer-Chree PDE (GPC 1/2 PDE) [?]

$$(u - u_{xx})_{tt} - \left(a_1 u + a_2 u^2 + a_3 u^3\right)_{xx} = 0 \tag{3.1}$$

$$(u - u_{xx})_{tt} - \left(a_1 u + a_3 u^3 + a_5 u^5\right)_{xx} = 0 \tag{3.2}$$

We now proceed to calculate the normal forms of the GPC PDE's.

If we let $z = x - ct$ in (3.1),(3.2) then the travelling wave ODE of the GPC PDE's is

$$\phi_{zzzz} - q\phi_{zz} + p\phi = \mathcal{N}_{1,2}[\phi] \tag{3.3}$$

where the notation $\mathcal{N}[\phi]$ means that the operator $\mathcal{N}$ operates on ϕ and all of it's derivatives, and where

$$\mathcal{N}_1[\phi] = -\frac{1}{c^2}\left[3a_3\left(2\phi\phi_z^2 + \phi^2\phi_{zz}\right) + 2a_2\left(\phi_z\phi_{zz} + \phi_z^2\right)\right] \tag{3.4a}$$

$$\mathcal{N}_2[\phi] = -\frac{1}{c^2}\left[3a_3\left(2\phi\phi_z^2 + \phi^2\phi_{zz}\right) + 5a_5\left(4\phi^3\phi_z^2 + \phi^4\phi_{zz}\right)\right] \tag{3.4b}$$

for the GPC 1 and GPC 2 PDE's, respectively.

Here we have

$$q = 1 - \frac{a_1}{c^2} \tag{3.5a}$$

$$p = 0 \tag{3.5b}$$

We emphasize that $p = 0$ because this corresponds to a 1-D subspace of the (p, q) parameter space in [?]. We denote C_0 as the positive q axis and C_1 the negative q-axis.

Solutions $\phi = ke^{\lambda x}$ satisfy the characteristic equation $\lambda^4 - q\lambda^2 + p = 0$. Since $p = 0$ for the GPC PDE's we have only two possible regions of eigenvalues. On C_0 the eigenvalues have the structure $\lambda_{1-4} = 0, 0, \pm\lambda$, $(\lambda \in \mathbb{R})$ while on C_1 we have $\lambda_{1-4} = 0, 0, \pm i\omega$, $(\omega \in \mathbb{R})$.

We now use the theory of Iooss and Adelmeyer [?] to write the GPC PDE's in reversible form: Let $Y = \langle y_1, y_2, y_3, y_4\rangle^T$ where $\langle y_1, y_2, y_3, y_4\rangle^T = \langle \phi_z, \phi_{zz}, \phi_{zzz}, \phi_{zzzz}\rangle^T$

We may now write (3.3) as a reversible first-order system

$$\frac{dY}{dz} = L_{pq}Y - F_{1,2}(Y, Y) \tag{3.6}$$

where

$$L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2 - p & 0 & q/3 & 0 \end{pmatrix} \tag{3.7}$$

Since $p = 0$ for the GPC PDE's, we have

$$\frac{dY}{dz} = L_{0q}Y - F_{1,2}(Y, Y) \tag{3.8}$$

where

$$F_{1,2}(Y,Y) = \langle 0,0,0,-\mathcal{N}_{1,2}\rangle^T \tag{3.9}$$

and

$$F_1(Y,Y) = \left\langle 0,0,0,\frac{1}{c^2}\left[3a_3\left(2y_1y_2^2+y_1^2y_2\right)+2a_2\left(y_2y_3+y_2^2\right)\right]\right\rangle^T \tag{3.10a}$$

$$F_2(Y,Y) = \left\langle 0,0,0,\frac{1}{c^2}\left[3a_3\left(2y_1y_2^2+y_1^2y_2\right)+5a_5\left(4y_1^3y_2^2+y_1^4y_3\right)\right]\right\rangle^T \tag{3.10b}$$

3.1. **Near C_0.** Near C_0 the dynamics reduce to a 2-D Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B) \tag{3.11}$$

with

$$\frac{dA}{dz} = B \tag{3.12a}$$

$$\frac{dB}{dz} = b\epsilon A + \tilde{c}A^2 \tag{3.12b}$$

and the eigenvectors are $\zeta_0 = \langle 1,0,-q/3,0\rangle^T$, $\zeta_1 = \langle 0,1,0,-2q/3\rangle^T$, ϵ determines the distance from C_0 . A simple dominant balance argument on the linearized equation yields $b = -\frac{1}{q}$.

To determine $\tilde{c}$ we calculate $\frac{dY}{dz}$ in two ways and match the $\mathcal{O}(A^2)$ terms. Expanding $\Psi(\epsilon, A, B)$ in series

$$\Psi(\epsilon, A, B) = \epsilon A\Psi_{10}^1 + \epsilon B\Psi_{01}^1 + A^2\Psi_{20}^0 + AB\Psi_{11}^0 + B^2\Psi_{02}^0 + \cdots \tag{3.13}$$

we find

$$\tilde{c}\zeta_1 = L_{0q}\Psi_{20}^0 - F_{1,2}(\zeta_0,\zeta_0) \tag{3.14}$$

We find that $F_1(\zeta_0,\zeta_0) = \left\langle 0,0,0,-\frac{q}{c^2}a_3\right\rangle^T$ and $F_2(\zeta_0,\zeta_0) = \left\langle 0,0,0,-\frac{q}{c^2}\left(a_3+\frac{5a_5}{3}\right)\right\rangle^T$ which implies $\tilde{c} = -\frac{a_3}{c^2}$ for the GPC 1 and $\tilde{c} = -\frac{1}{c^2}\left(a_3+\frac{5}{3}a_5\right)$ for the GPC 2.

Therefore, the normal form for the GPC 1 PDE near C_0 is

$$\frac{dA}{dz} = B \tag{3.15a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{a_3}{c^2}A^2 \tag{3.15b}$$

and the normal form for the GPC 2 PDE near C_0 is

$$\frac{dA}{dz} = B \tag{3.16a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{1}{c^2}\left(a_3+\frac{5}{3}a_5\right)A^2 \tag{3.16b}$$

3.2. **Near C_1.** Near C_1 the dynamics reduce to a 4-D Center Manifold

$$Y = A\zeta_0 + B\zeta_0 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C}) \tag{3.17}$$

with

$$\frac{dA}{dz} = B \tag{3.18a}$$

$$\frac{dB}{dz} = \bar{\nu}A + b_*A^2 + c_*|C|^2 \tag{3.18b}$$

$$\frac{dC}{dz} = id_0C + i\bar{\nu}d_1C + id_2AC \tag{3.18c}$$

where $\zeta_\pm = \left\langle 1,\lambda_\pm,2q/3,\frac{\lambda_\pm}{3}q\right\rangle^T$ where

$\bar{\nu} = -\frac{\epsilon}{q}$, $b_* = -\frac{b\Delta 1}{3}$ and $\lambda_\pm = \pm i\sqrt{-q}$. Comparing to the linearized equations gives $d_0 = \sqrt{-q}$.
DONE

ACTUALLY DO THIS: If we do a dominant balance argument after the change of variable $\tilde{\epsilon} = \sqrt{-3\mu}$ on the characteristic equation as $\lambda \to 0$ then we find $d_1 = \frac{\sqrt{-3\mu}}{18\mu^2}$. Using $\mu = q/3$ we find $d_1 = \frac{\sqrt{-q}}{2q^2}$

To determine b_*, c_* and d_2 we expand

$$\Psi(\epsilon, A, B, C, \bar{C}) = \epsilon A\Psi^1_{1000} + \epsilon B\Psi^1_{0100} + A^2\Psi^0_{2000} + AB\Psi^0_{1100} + AC\Psi^0_{1010} + \epsilon C\Psi^1_{0010} + \cdots \tag{3.19}$$

Now we use (3.18b), (3.18c), (3.18c) in (3.19) when we calculate $\frac{dY}{dz}$. Matching coefficients yields

$$\mathcal{O}(A^2): \quad b_*\zeta_1 = L_{0q}\Psi^0_{2000} - F_2(\zeta_0, \zeta_0) \tag{3.20a}$$

$$\mathcal{O}(|C|^2): \quad c_*\zeta_1 = L_{0q}\Psi^0_{0011} - 2F_2(\zeta_+, \zeta_-) \tag{3.20b}$$

$$\mathcal{O}(\epsilon C): \quad -\frac{i}{q}\left(d_1\zeta_+ + d_0\Psi^1_{0010}\right) = L_{0q}\Psi^1_{0010} - F_2(\Psi^1_{0010}, \Psi^1_{0010}) \tag{3.20c}$$

$$\mathcal{O}(AC): \quad id_2\zeta_+ + id_0\Psi^0_{1010} = L_{0q}\Psi^0_{1010} - 2F_2(\zeta_0, \zeta_+) \tag{3.20d}$$

(3.20e)

where we have used the fact that F_2 is a symmetric bilinear form. Equation (3.20b) is decoupled and yields $c_* = 2\Delta_1\left(\frac{1}{q} + \frac{2b}{3}\right)$ The only coefficient left to determine is d_2 which we shall compute now.

Put $\Psi^0_{1010} = \langle x_1, x_2, x_3, x_4\rangle^T$ into (3.20e) implies

$$id_2 + id_0x_1 = x_2 \tag{3.21a}$$

$$-d_0d_2 + id_0x_2 = \frac{q}{3}x_1 + x_3 \tag{3.21b}$$

$$\frac{2iq}{3}d_2 + id_0x_3 = \frac{q}{3}x_2 + x_4 \tag{3.21c}$$

$$-\frac{q}{3}d_0d_2 + id_0x_4 = \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) - \frac{2bq\Delta_1}{3} \tag{3.21d}$$

Using (3.21a) in (3.21b) , (3.21b) in (3.21d) and using these in (3.21c) yields $d_2 = \frac{b\Delta_1}{3\sqrt{-q}}$.

Therefore the normal form for the MS PDE near C_1 is

$$\frac{dA}{dz} = B \tag{3.22a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{b\Delta_1}{3}A^2 + 2\Delta_1\left(\frac{1}{q} + \frac{2b}{3}\right)|C|^2 \tag{3.22b}$$

$$\frac{dC}{dz} = i\sqrt{-q}C - i\frac{\epsilon\sqrt{-q}}{q^3}C + i\frac{b\Delta_1}{3\sqrt{-q}}AC \tag{3.22c}$$

4. Analysis of Normal Forms

References

[1] Iooss G., Adelmeyer, M., *Topics in Bifurcation Theory and Applications*, World Scientific, Singapore, 1998.

[2] S. Roy Choudhury, *Solitary-wave families of the Ostrovsky equation: An approach via reversible systems theory and normal forms* , Elsevier, (2007), pp. 1468–1479.