

Solitary Wave Families in Two Non-Integrable Models Using Reversible Systems Theory

Jonathan Leto

Overview

- Definitions
- Background
- The Generalized Pochhammer-Chree Equations
- A Generalized Microstructure Equation

Reversible Dynamical System (Iooss & Adelmeyer)

Consider

$$\frac{dz}{dt} = F(z; \mu), z \in \mathbb{R}^n, \mu \in \mathbb{R} \quad (1)$$

where

$$F(0; 0) = 0$$

. If there exists a **unitary map**

$$S : \mathbb{R}^n \mapsto \mathbb{R}^n, S \neq I$$

such that

$$F(Sz; \mu) = -SF(z; \mu)$$

for all z and μ then (1) is a reversible system.

Families of Solitary Waves

Here we will use the term solitary wave or "soliton" to mean a pulse-like solution to an evolution equation. For example,

$$A(z) = \ell \operatorname{sech}^2 kz \quad (2)$$

is a two-parameter family of solitary waves.

where k and ℓ are parameters which determine the speed and the height of the wave.

Normal Form Theory

After a nonlinear change of variables (Iooss & Adelmeyer) one may put the Center Manifold into Normal Form.

Two-Dimensional Center Manifold

- $\lambda_{1-4} = 0, 0, \pm\lambda, \lambda \in \mathbb{R}$
- $Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B)$

Four-Dimensional Center Manifold

- $\lambda_{1-4} = 0, 0, \pm i\omega, \omega \in \mathbb{R}$
- $Y = A\zeta_0 + B\zeta_1 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C})$

where $\zeta_0, \zeta_1, \zeta_+, \zeta_-$ are eigenvectors of the linearized operator.

Properties of Bilinear Functions

A function

$$B : \mathbb{C} \times \mathbb{C} \mapsto \mathbb{C}$$

satisfying the following axioms

$$B(x + y, z) = B(x, z) + B(y, z)$$

$$B(\lambda x, y) = \lambda B(x, y)$$

$$B(x, y + z) = B(x, y) + B(x, z)$$

$$B(x, \lambda y) = \lambda B(x, y)$$

is called **bilinear** .

If $B(x, y)$ is bilinear, then $f(y) \equiv B(y, y)$ is invariant under the transformation $y \mapsto -y$ and thus is **reversible** .

The Generalized Pochhammer-Chree Equations

The Generalized Pochhammer-Chree Equations govern the propagation of longitudinal waves in elastic rods.

- GPC1

$$(u - u_{xx})_{tt} - (a_1 u + a_2 u^2 + a_3 u^3)_{xx} = 0 \quad (4)$$

- GPC2

$$(u - u_{xx})_{tt} - (a_1 u + a_3 u^3 + a_5 u^5)_{xx} = 0 \quad (5)$$

Reversible Form

Denoting $Y = \langle y_1, y_2, y_3, y_4 \rangle^T = \langle \phi, \phi_z, \phi_{zz}, \phi_{zzz} \rangle^T$ equation (6) can be written

$$\frac{dY}{dz} = L_{pq} Y - G_{1,2}(Y, Y) \quad (9)$$

where

$$L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2 - p & 0 & q/3 & 0 \end{pmatrix}$$

$$G_{1,2}(Y, Y) = \langle 0, 0, 0, -\mathcal{N}_{1,2}(Y, Y) \rangle^T$$

Near C_0 : Normal Form

The eigenvalues are $\lambda_{1,4} = 0, 0, \pm\lambda$, $\lambda \in \mathbb{R}$, We write the Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B) \quad (10)$$

with corresponding normal form

$$\frac{dA}{dz} = B \quad (11a)$$

$$\frac{dB}{dz} = b\epsilon A + \tilde{c}A^2 \quad (11b)$$

where ϵ measures the perturbation about C_0 and

$$\zeta_0 = \langle 1, 0, -q/3, 0 \rangle^T \quad (12a)$$

$$\zeta_1 = \langle 0, 1, 0, -2q/3 \rangle^T \quad (12b)$$

How do we determine the coefficients b and \tilde{c} ?

Finding Coefficients Of The Normal Form

When you follow two separate chains of thought, Watson, you will find some point of intersection which should approximate the truth. - Sir Arthur Conan Doyle

By computing $\frac{dY}{dz}$ in two ways and comparing the coefficients of each term in the normal form, we find systems of equations which determine each coefficient. For example, to find \tilde{c} , we compare $\mathcal{O}(A^2)$ terms.

Two Ways to Compute $\frac{dY}{dz}$

Method 1

Use the Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B)$$

in the reversible system

$$\frac{dY}{dz} = L_{pq} Y - G_{1,2}(Y, Y)$$

and repeatedly use the bilinear
properties of $G_{1,2}$ to simplify.

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where

$$\Psi(\epsilon, A, B) = \epsilon A \Psi_{10}^1 + \epsilon B \Psi_{01}^1 + A^2 \Psi_{20}^0 + AB \Psi_{11}^0 + B^2 \Psi_{02}^0 + \dots$$

Method 2

Differentiate the Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B)$$

and use the Normal Form

$$\begin{aligned} \frac{dA}{dz} &= B \\ \frac{dB}{dz} &= b\epsilon A + \tilde{c}A^2 \end{aligned}$$

to simplify all derivatives.

Finding \tilde{c}

Matching $\mathcal{O}(A^2)$ terms in each method gives us the two systems of equations

GPC 1

$$\tilde{c}\zeta_1 = L_{0q}\psi_{20}^0 - G_1(\zeta_0, \zeta_0)$$

$$0 = x_2$$

$$\tilde{c} = \frac{q}{3}x_1 + x_3$$

$$0 = \frac{q}{3}x_2 + x_4 \implies x_4 = 0$$

$$-\frac{2q}{3}\tilde{c} = \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) + \frac{q}{3c^2}(3a_3 + 5a_5)$$

$$= \frac{q}{3}\tilde{c} + \frac{q}{3c^2}a_3$$

$$\implies \tilde{c} = -\frac{a_3}{3c^2}$$

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$$\implies \tilde{c} = -\frac{a_3}{3c^2}$$

GPC 2

$$\tilde{c}\zeta_1 = L_{0q}\psi_{20}^0 - G_2(\zeta_0, \zeta_0)$$

$$0 = x_2$$

$$\tilde{c} = \frac{q}{3}x_1 + x_3$$

$$0 = \frac{q}{3}x_2 + x_4 \implies x_4 = 0$$

$$-\frac{2q}{3}\tilde{c} = \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) + \frac{q}{3c^2}(3a_3 + 5a_5)$$

$$= \frac{q}{3}\tilde{c} + \frac{q}{3c^2}(3a_3 + 5a_5)$$

$$\implies \tilde{c} = -\frac{1}{3c^2}(3a_3 + 5a_5)$$

where $\Psi_{20}^0 = \langle x_1, x_2, x_3, x_4 \rangle^T$.

Normal Form Near C_0

Therefore the Normal Form near C_0 is

GPC 1

$$\begin{aligned}\frac{dA}{dz} &= B \\ \frac{dB}{dz} &= -\frac{\epsilon}{q}A - \frac{a_3}{c^2}A^2\end{aligned}$$

GPC 2

$$\begin{aligned}\frac{dA}{dz} &= B \\ \frac{dB}{dz} &= -\frac{\epsilon}{q}A - \frac{1}{3c^2}(3a_3 + 5a_5)A^2\end{aligned}$$

These equations admit homoclinic solutions near C_0 of the form

$$A(z) = \ell \operatorname{sech}^2(kz)$$

Finding k and ℓ

To determine k and ℓ , we first write the Normal Form as a single second order equation. Then we use our expression for $A(z)$ and compare coefficients of $\mathcal{O}(\operatorname{sech}^2(kz))$ and $\mathcal{O}(\operatorname{sech}^4(kz))$ which implies

GPC 1

$$\begin{aligned} k &= \sqrt{\frac{-\epsilon}{4q}} \\ \ell &= \frac{-3\epsilon c^2}{2qa_3} \end{aligned}$$

GPC 2

$$\begin{aligned} k &= \sqrt{\frac{-\epsilon}{4q}} \\ \ell &= \frac{-9\epsilon c^2}{2q(3a_3 + 5a_5)} \end{aligned}$$

Hence, since $\epsilon = -p$, solitons of this form exist for $p = 0^+$, $q > 0$, which implies $a_1 < c^2$.

Near C_1 : Normal Form

The eigenvalues are $\lambda_{1,4} = 0, 0, \pm i\omega$, $\omega \in \mathbb{R}$, We write the Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C}) \quad (20)$$

with corresponding normal form

$$\begin{aligned} \frac{dA}{dz} &= B \\ \frac{dB}{dz} &= \bar{\nu}\epsilon A + b_* A^2 + c_* \|C\|^2 \\ \frac{dC}{dz} &= id_0 C + i\bar{\nu}d_1 C + id_2 AC \end{aligned}$$

where the new eigenvectors co-spanning the four-dimensional Center Manifold are

$$\lambda_{\pm} = \langle 1, \lambda_{\pm}, 2q/3, \lambda_{\pm}q/3 \rangle^T$$

and $\lambda_{\pm} = \pm i\sqrt{-q}$, $q < 0$.

Compute $\frac{dY}{dz}$ by both methods

Matching the coefficients of A^2 , $\epsilon C \|C\|^2$ and AC in the two separate expressions for dY/dz yields the following two systems of equations:

$$\begin{aligned}
 \mathcal{O}(A^2) : \quad & b_* \zeta_1 & = L_{0q} \Psi_{2000}^0 - G_{1,2}(\zeta_0, \zeta_0) \\
 \mathcal{O}(|C|^2) : \quad & c_* \zeta_1 & = L_{0q} \Psi_{0011}^0 - 2G_{1,2}(\zeta_+, \zeta_-) \\
 \mathcal{O}(\epsilon C) : \quad & -\frac{i}{q} (d_1 \zeta_+ + d_0 \Psi_{0010}^1) & = L_{0q} \Psi_{0010}^1 \\
 \mathcal{O}(AC) : \quad & id_2 \zeta_+ + id_0 \Psi_{1010}^0 & = L_{0q} \Psi_{1010}^0 - 2G_{1,2}(\zeta_0, \zeta_+)
 \end{aligned}$$

Normal Form near C_1 : GPC 1

The resolution of the preceding equations yields the coefficients in the Normal Form for the GPC 1 equation:

$$\frac{dA}{dz} = B$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A + \frac{a_3}{c^2}A^2 + \frac{1}{c^2} \left(\frac{7}{2\sqrt{-q}}a_3 - \frac{i}{3}a_2 \right) |C|^2$$

$$\frac{dC}{dz} = i\sqrt{-q}C - i\frac{\sqrt{-q}}{q^3}C\epsilon + i\frac{1}{c^2} \left(\frac{7}{2\sqrt{-q}}a_3 - \frac{i}{3}a_2 \right) AC$$

where $q \equiv 1 - \frac{a_1}{c^2}$

Normal Form near C_1 : GPC 2

The resolution of the preceding equations yields the coefficients in the Normal Form for the GPC 2 equation:

$$\frac{dA}{dz} = B$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A + \frac{1}{3c^2}(3a_3 + 5a_5)A^2 + \frac{1}{c^2}\left(16a_3 + \frac{140}{3}a_5\right)|C|^2$$

$$\frac{dC}{dz} = i\sqrt{-q}C - i\frac{\sqrt{-q}}{q^3}C\epsilon + i\frac{1}{\sqrt{-q}c^2}\left(\frac{7}{2}a_3 + \frac{32}{3}a_5\right)AC$$

where $q \equiv 1 - \frac{a_1}{c^2}$

Dynamics Near C_1

The two first integrals of the four-dimensional Normal Form are

$$K = \|C\|^2$$

and

$$H = B^2 - \frac{2}{3}b_*A^2 - \bar{\nu}A^2 - 2c_*KA$$

In the (A, B) phase plane, the level curve $H = 0$ comprises a homoclinic orbit. The intersection of $H = 0$ with the A axis occurs for

$$A_{\mp} = \frac{3}{4b_*} \left[\bar{\nu} \pm \sqrt{\bar{\nu}^2 + \frac{16b_*c_*K}{3}} \right]$$

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Homoclinic Orbits for Various Values of H

Overview

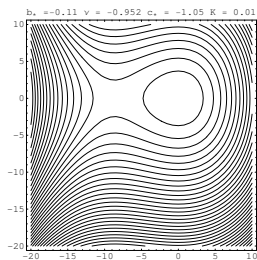
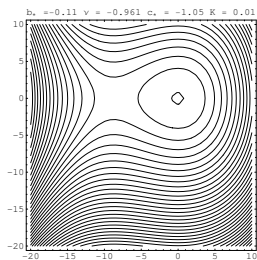
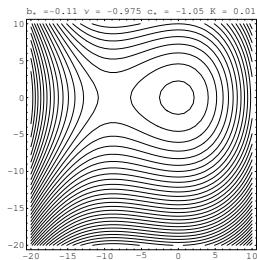
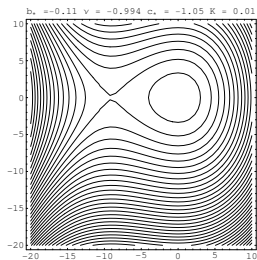
Definitions

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Solitary Wave

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GPC

MS



A Generalized Microstructure PDE

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One dimensional wave propagation in microstructured solids has recently been modeled by an equation

$$v_{tt} - bv_{xx} - \frac{\mu}{2} (v^2)_{xx} - \delta (\beta v_{tt} - \gamma v_{xx})_{xx} = 0 \quad (25)$$

Travelling Wave ODE

Let $z = x - ct$ and $u(x, t) = \phi(z)$ to reduce (25) to the Travelling Wave ODE

$$\phi_{zzzz} - q\phi_{zz} + p\phi = \mathcal{N}[\phi] \quad (26)$$

where

$$\mathcal{N}[\phi] = -\Delta_1 \phi_z^2 - b\Delta_1 \phi \phi_{zz} \quad (27)$$

$$z \equiv x - ct$$

$$p \equiv 0$$

$$q \equiv \frac{c^2 - b}{\delta(\beta c^2 - \gamma)}$$

$$\Delta_1 \equiv \frac{\mu}{\delta(\beta c^2 - \gamma)}$$

Normal Form near C_0

With the same kind analysis we find

$$\begin{aligned}\frac{dA}{dz} &= B \\ \frac{dB}{dz} &= -\frac{\epsilon}{q}A - \frac{b\Delta_1}{3}A^2\end{aligned}$$

Solitary Waves Near C_0

The Normal Form admits a homoclinic solution near C_0 of the form

$$A(z) = \ell \operatorname{sech}^2(kz)$$

where

$$\begin{aligned} k &= \sqrt{\frac{-\epsilon}{4q}} \\ \ell &= \frac{6k^2}{b\Delta_1} \end{aligned}$$

Since $\epsilon = -p$, and the curve C_0 corresponds to $p = 0, q > 0$, solitary waves exist near C_0 for $p > 0, q > 0$, which implies that $\frac{c^2 - b}{\delta(\beta c^2 - \gamma)} > 0$.

Normal Form near C_1

Near C_1 the Normal Form for (25) is

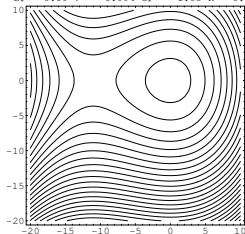
$$\frac{dA}{dz} = B$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{b\Delta_1}{3}A^2 + 2\Delta_1 \left(\frac{2b}{3} - 1 \right) |C|^2$$

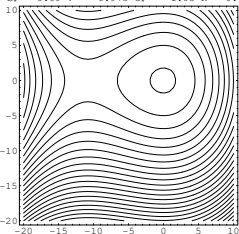
$$\frac{dC}{dz} = i\sqrt{-q}C - i\frac{\sqrt{-q}}{q^3}C\epsilon + i\frac{b\Delta_1}{6\sqrt{-q}}AC$$

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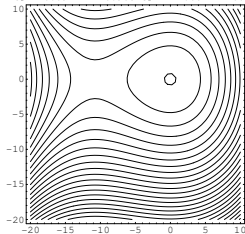
$b_* = -0.09$ $v = -0.994$ $c_* = -1.05$ $K = 0.01$



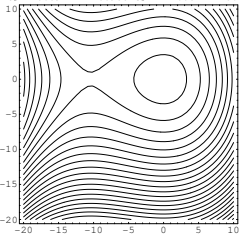
$b_* = -0.09$ $v = -0.975$ $c_* = -1.05$ $K = 0.01$



$b_* = -0.09$ $v = -0.968$ $c_* = -1.05$ $K = 0.01$



$b_* = -0.09$ $v = -0.938$ $c_* = -1.05$ $K = 0.01$



Open Problems

Study embedded solitons using a mix of

- exponential asymptotics
- numerical shooting

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