

# Solitary Wave Families in Two Non-Integrable Models Using Reversible Systems Theory

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# Overview

- Definitions
- Background
- The Generalized Pochhammer-Chree Equations
- A Generalized Microstructure Equation

# Reversible Dynamical System (Iooss & Adelmeyer)

Consider

$$\frac{dz}{dt} = F(z; \mu), z \in \mathbb{R}^n, \mu \in \mathbb{R} \quad (1)$$

where

$$F(0; 0) = 0$$

. If there exists a **unitary map**

$$S : \mathbb{R}^n \mapsto \mathbb{R}^n, S \neq I$$

such that

$$F(Sz; \mu) = -SF(z; \mu)$$

for all  $z$  and  $\mu$  then (1) is a reversible system.

# Families of Solitary Waves

Here we will use the term solitary wave or "soliton" to mean a pulse-like solution to an evolution equation. For example,

$$A(z) = \ell \operatorname{sech}^2 kz \quad (2)$$

is a two-parameter family of solitary waves.

where  $k$  and  $\ell$  are parameters which determine the speed and the height of the wave.

# Normal Form Theory

After a nonlinear change of variables (Iooss & Adelmeyer) one may put the Center Manifold into Normal Form.

## Two-Dimensional Center Manifold

- $\lambda_{1-4} = 0, 0, \pm\lambda, \lambda \in \mathbb{R}$
- $Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B)$

## Four-Dimensional Center Manifold

- $\lambda_{1-4} = 0, 0, \pm i\omega, \omega \in \mathbb{R}$
- $Y = A\zeta_0 + B\zeta_1 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C})$

where  $\zeta_0, \zeta_1, \zeta_+, \zeta_-$  are eigenvectors of the linearized operator.

# Properties of Bilinear Functions

A function

$$B : \mathbb{C} \times \mathbb{C} \mapsto \mathbb{C}$$

satisfying the following axioms

$$B(x + y, z) = B(x, z) + B(y, z)$$

$$B(\lambda x, y) = \lambda B(x, y)$$

$$B(x, y + z) = B(x, y) + B(x, z)$$

$$B(x, \lambda y) = \lambda B(x, y)$$

is called **bilinear** .

If  $B(x, y)$  is bilinear, then  $f(y) \equiv B(y, y)$  is invariant under the transformation  $y \mapsto -y$  and thus is **reversible** .

# The Generalized Pochhammer-Chree Equations

The Generalized Pochhammer-Chree Equations govern the propagation of longitudinal waves in elastic rods.

- GPC1

$$(u - u_{xx})_{tt} - (a_1 u + a_2 u^2 + a_3 u^3)_{xx} = 0 \quad (4)$$

- GPC2

$$(u - u_{xx})_{tt} - (a_1 u + a_3 u^3 + a_5 u^5)_{xx} = 0 \quad (5)$$





## Reversible Form

Denoting  $Y = \langle y_1, y_2, y_3, y_4 \rangle^T = \langle \phi, \phi_z, \phi_{zz}, \phi_{zzz} \rangle^T$  equation (6) can be written

$$\frac{dY}{dz} = L_{pq} Y - G_{1,2}(Y, Y) \quad (9)$$

where

$$L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2 - p & 0 & q/3 & 0 \end{pmatrix}$$

$$G_{1,2}(Y, Y) = \langle 0, 0, 0, -\mathcal{N}_{1,2}(Y, Y) \rangle^T$$

## Near $C_0$ : Normal Form

The eigenvalues are  $\lambda_{1,4} = 0, 0, \pm\lambda$ ,  $\lambda \in \mathbb{R}$ , We write the Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B) \quad (10)$$

with corresponding normal form

$$\frac{dA}{dz} = B \quad (11a)$$

$$\frac{dB}{dz} = b\epsilon A + \tilde{c}A^2 \quad (11b)$$

where  $\epsilon$  measures the perturbation about  $C_0$  and

$$\zeta_0 = \langle 1, 0, -q/3, 0 \rangle^T \quad (12a)$$

$$\zeta_1 = \langle 0, 1, 0, -2q/3 \rangle^T \quad (12b)$$

How do we determine the coefficients  $b$  and  $\tilde{c}$  ?

# Finding Coefficients Of The Normal Form

*When you follow two separate chains of thought, Watson, you will find some point of intersection which should approximate the truth. - Sir Arthur Conan Doyle*

By computing  $\frac{dY}{dz}$  in two ways and comparing the coefficients of each term in the normal form, we find systems of equations which determine each coefficient. For example, to find  $\tilde{c}$ , we compare  $\mathcal{O}(A^2)$  terms.

# Two Ways to Compute $\frac{dY}{dz}$

## Method 1

Use the Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B)$$

in the reversible system

$$\frac{dY}{dz} = L_{pq} Y - G_{1,2}(Y, Y)$$

and repeatedly use the bilinear  
properties of  $G_{1,2}$  to simplify.

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where

$$\Psi(\epsilon, A, B) = \epsilon A \Psi_{10}^1 + \epsilon B \Psi_{01}^1 + A^2 \Psi_{20}^0 + AB \Psi_{11}^0 + B^2 \Psi_{02}^0 + \dots$$

## Method 2

Differentiate the Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B)$$

and use the Normal Form

$$\begin{aligned} \frac{dA}{dz} &= B \\ \frac{dB}{dz} &= b\epsilon A + \tilde{c}A^2 \end{aligned}$$

to simplify all derivatives.

# Finding $\tilde{c}$

Matching  $\mathcal{O}(A^2)$  terms in each method gives us the two systems of equations

## GPC 1

$$\tilde{c}\zeta_1 = L_{0q}\psi_{20}^0 - G_1(\zeta_0, \zeta_0)$$

$$0 = x_2$$

$$\tilde{c} = \frac{q}{3}x_1 + x_3$$

$$0 = \frac{q}{3}x_2 + x_4 \implies x_4 = 0$$

$$-\frac{2q}{3}\tilde{c} = \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) + \frac{q}{3c^2}(3a_3 + 5a_5)$$

$$= \frac{q}{3}\tilde{c} + \frac{q}{3c^2}a_3$$

$$\implies \tilde{c} = -\frac{a_3}{3c^2}$$

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$$\implies \tilde{c} = -\frac{a_3}{3c^2}$$

## GPC 2

$$\tilde{c}\zeta_1 = L_{0q}\psi_{20}^0 - G_2(\zeta_0, \zeta_0)$$

$$0 = x_2$$

$$\tilde{c} = \frac{q}{3}x_1 + x_3$$

$$0 = \frac{q}{3}x_2 + x_4 \implies x_4 = 0$$

$$-\frac{2q}{3}\tilde{c} = \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) + \frac{q}{3c^2}(3a_3 + 5a_5)$$

$$= \frac{q}{3}\tilde{c} + \frac{q}{3c^2}(3a_3 + 5a_5)$$

$$\implies \tilde{c} = -\frac{1}{3c^2}(3a_3 + 5a_5)$$

where  $\Psi_{20}^0 = \langle x_1, x_2, x_3, x_4 \rangle^T$ .

# Normal Form Near $C_0$

Therefore the Normal Form near  $C_0$  is

## GPC 1

$$\begin{aligned}\frac{dA}{dz} &= B \\ \frac{dB}{dz} &= -\frac{\epsilon}{q}A - \frac{a_3}{c^2}A^2\end{aligned}$$

## GPC 2

$$\begin{aligned}\frac{dA}{dz} &= B \\ \frac{dB}{dz} &= -\frac{\epsilon}{q}A - \frac{1}{3c^2}(3a_3 + 5a_5)A^2\end{aligned}$$

These equations admit homoclinic solutions near  $C_0$  of the form

$$A(z) = \ell \operatorname{sech}^2(kz)$$



## Finding $k$ and $\ell$

To determine  $k$  and  $\ell$ , we first write the Normal Form as a single second order equation. Then we use our expression for  $A(z)$  and compare coefficients of  $\mathcal{O}(\operatorname{sech}^2(kz))$  and  $\mathcal{O}(\operatorname{sech}^4(kz))$  which implies

### GPC 1

$$\begin{aligned} k &= \sqrt{\frac{-\epsilon}{4q}} \\ \ell &= \frac{-3\epsilon c^2}{2qa_3} \end{aligned}$$

### GPC 2

$$\begin{aligned} k &= \sqrt{\frac{-\epsilon}{4q}} \\ \ell &= \frac{-9\epsilon c^2}{2q(3a_3 + 5a_5)} \end{aligned}$$

Hence, since  $\epsilon = -p$ , solitons of this form exist for  $p = 0^+$ ,  $q > 0$ , which implies  $a_1 < c^2$ .

## Near $C_1$ : Normal Form

The eigenvalues are  $\lambda_{1,4} = 0, 0, \pm i\omega$ ,  $\omega \in \mathbb{R}$ , We write the Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C}) \quad (20)$$

with corresponding normal form

$$\begin{aligned} \frac{dA}{dz} &= B \\ \frac{dB}{dz} &= \bar{\nu}\epsilon A + b_* A^2 + c_* \|C\|^2 \\ \frac{dC}{dz} &= id_0 C + i\bar{\nu}d_1 C + id_2 AC \end{aligned}$$

where the new eigenvectors co-spanning the four-dimensional Center Manifold are

$$\lambda_{\pm} = \langle 1, \lambda_{\pm}, 2q/3, \lambda_{\pm}q/3 \rangle^T$$

and  $\lambda_{\pm} = \pm i\sqrt{-q}$ ,  $q < 0$ .

# Compute $\frac{dY}{dz}$ by both methods

Matching the coefficients of  $A^2$ ,  $\epsilon C \|C\|^2$  and  $AC$  in the two separate expressions for  $dY/dz$  yields the following two systems of equations:

$$\begin{aligned}
 \mathcal{O}(A^2) : \quad & b_* \zeta_1 &= L_{0q} \Psi_{2000}^0 - G_{1,2}(\zeta_0, \zeta_0) \\
 \mathcal{O}(|C|^2) : \quad & c_* \zeta_1 &= L_{0q} \Psi_{0011}^0 - 2G_{1,2}(\zeta_+, \zeta_-) \\
 \mathcal{O}(\epsilon C) : \quad & -\frac{i}{q} (d_1 \zeta_+ + d_0 \Psi_{0010}^1) &= L_{0q} \Psi_{0010}^1 \\
 \mathcal{O}(AC) : \quad & id_2 \zeta_+ + id_0 \Psi_{1010}^0 &= L_{0q} \Psi_{1010}^0 - 2G_{1,2}(\zeta_0, \zeta_+)
 \end{aligned}$$

## Normal Form near $C_1$ : GPC 1

The resolution of the preceding equations yields the coefficients in the Normal Form for the GPC 1 equation:

$$\frac{dA}{dz} = B$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A + \frac{a_3}{c^2}A^2 + \frac{1}{c^2} \left( \frac{7}{2\sqrt{-q}}a_3 - \frac{i}{3}a_2 \right) |C|^2$$

$$\frac{dC}{dz} = i\sqrt{-q}C - i\frac{\sqrt{-q}}{q^3}C\epsilon + i\frac{1}{c^2} \left( \frac{7}{2\sqrt{-q}}a_3 - \frac{i}{3}a_2 \right) AC$$

where  $q \equiv 1 - \frac{a_1}{c^2}$

## Normal Form near $C_1$ : GPC 2

The resolution of the preceding equations yields the coefficients in the Normal Form for the GPC 2 equation:

$$\frac{dA}{dz} = B$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A + \frac{1}{3c^2}(3a_3 + 5a_5)A^2 + \frac{1}{c^2}\left(16a_3 + \frac{140}{3}a_5\right)|C|^2$$

$$\frac{dC}{dz} = i\sqrt{-q}C - i\frac{\sqrt{-q}}{q^3}C\epsilon + i\frac{1}{\sqrt{-q}c^2}\left(\frac{7}{2}a_3 + \frac{32}{3}a_5\right)AC$$

where  $q \equiv 1 - \frac{a_1}{c^2}$

# Dynamics Near $C_1$

The two first integrals of the four-dimensional Normal Form are

$$K = \|C\|^2$$

and

$$H = B^2 - \frac{2}{3}b_*A^2 - \bar{\nu}A^2 - 2c_*KA$$

In the  $(A, B)$  phase plane, the level curve  $H = 0$  comprises a homoclinic orbit. The intersection of  $H = 0$  with the  $A$  axis occurs for

$$A_{\mp} = \frac{3}{4b_*} \left[ \bar{\nu} \pm \sqrt{\bar{\nu}^2 + \frac{16b_*c_*K}{3}} \right]$$

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# Homoclinic Orbits for Various Values of H

Overview

Definitions

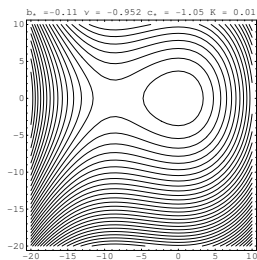
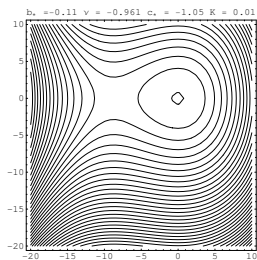
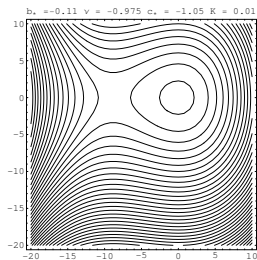
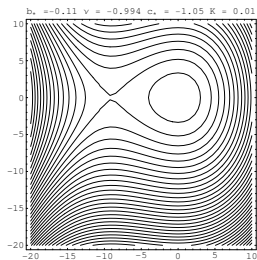
Reversible  
System  
Solitary Wave

Background

Normal Forms  
Bilinear  
Functions

GPC

MS



# A Generalized Microstructure PDE

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One dimensional wave propagation in microstructured solids has recently been modeled by an equation

$$v_{tt} - bv_{xx} - \frac{\mu}{2} (v^2)_{xx} - \delta (\beta v_{tt} - \gamma v_{xx})_{xx} = 0 \quad (25)$$



# Travelling Wave ODE

Let  $z = x - ct$  and  $u(x, t) = \phi(z)$  to reduce (25) to the Travelling Wave ODE

$$\phi_{zzzz} - q\phi_{zz} + p\phi = \mathcal{N}[\phi] \quad (26)$$

where

$$\mathcal{N}[\phi] = -\Delta_1 \phi_z^2 - b\Delta_1 \phi \phi_{zz} \quad (27)$$

$$z \equiv x - ct$$

$$p \equiv 0$$

$$q \equiv \frac{c^2 - b}{\delta(\beta c^2 - \gamma)}$$

$$\Delta_1 \equiv \frac{\mu}{\delta(\beta c^2 - \gamma)}$$

# Normal Form near $C_0$

With the same kind analysis we find

$$\begin{aligned}\frac{dA}{dz} &= B \\ \frac{dB}{dz} &= -\frac{\epsilon}{q}A - \frac{b\Delta_1}{3}A^2\end{aligned}$$

# Solitary Waves Near $C_0$

The Normal Form admits a homoclinic solution near  $C_0$  of the form

$$A(z) = \ell \operatorname{sech}^2(kz)$$

where

$$\begin{aligned} k &= \sqrt{\frac{-\epsilon}{4q}} \\ \ell &= \frac{6k^2}{b\Delta_1} \end{aligned}$$

Since  $\epsilon = -p$ , and the curve  $C_0$  corresponds to  $p = 0, q > 0$ , solitary waves exist near  $C_0$  for  $p > 0, q > 0$ , which implies that  $\frac{c^2 - b}{\delta(\beta c^2 - \gamma)} > 0$ .

# Normal Form near $C_1$

Near  $C_1$  the Normal Form for (25) is

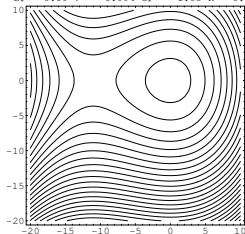
$$\frac{dA}{dz} = B$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{b\Delta_1}{3}A^2 + 2\Delta_1 \left( \frac{2b}{3} - 1 \right) |C|^2$$

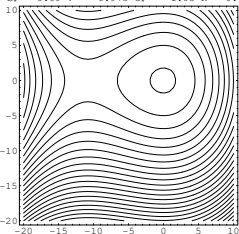
$$\frac{dC}{dz} = i\sqrt{-q}C - i\frac{\sqrt{-q}}{q^3}C\epsilon + i\frac{b\Delta_1}{6\sqrt{-q}}AC$$

# Homoclinic Orbits for Various Values of $H$

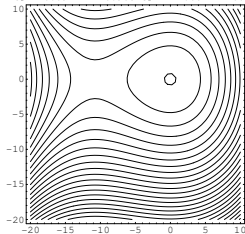
$b_* = -0.09$   $v = -0.994$   $c_* = -1.05$   $K = 0.01$



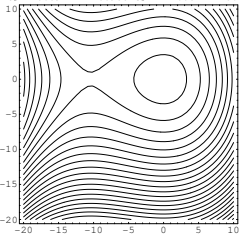
$b_* = -0.09$   $v = -0.975$   $c_* = -1.05$   $K = 0.01$



$b_* = -0.09$   $v = -0.968$   $c_* = -1.05$   $K = 0.01$



$b_* = -0.09$   $v = -0.938$   $c_* = -1.05$   $K = 0.01$



# Open Problems

Study embedded solitons using a mix of

- exponential asymptotics
- numerical shooting

# Thanks

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