SOLITARY-WAVE FAMILIES OF THE GENERALIZED POCHAMMER-CHREE PDE'S

JONATHAN LETO, S. ROY CHOUDHURY

ABSTRACT. The Generalized Pochammer-Chree Equations Since solitary-wave solutions often play a central role in the long-time evolution of an initial disturbance, we consider such solutions here (via normal form approach) within the framework of reversible systems theory. On isolated curves in the relevant parameter region, the delocalized waves reduce to genuine embedded solitons. Directions for future work are also described.

1. INTRODUCTION

2. SOLITARY WAVES: LOCAL BIFURCATION

3. CALCULATIONS

Generalized Pochammer-Chree PDE (GPC 1/2 PDE) [2]

$$(u - u_{xx})_{tt} - \left(a_1 u + a_2 u^2 + a_3 u^3\right)_{xx} = 0 \tag{3.1}$$

$$(u - u_{xx})_{tt} - \left(a_1 u + a_3 u^3 + a_5 u^5\right)_{xx} = 0 \tag{3.2}$$

We now proceed to calculate the normal forms of the GPC PDE's.

If we let $z = x - ct$ in (3.1),(3.2) then the travelling wave ODE of the GPC PDE's is

$$\phi_{zzzz} - q\phi_{zz} + p\phi = \mathcal{N}_{1,2}[\phi] \tag{3.3}$$

where the notation $\mathcal{N}[\phi]$ means that the operator $\mathcal{N}$ operates on ϕ and all of it's derivatives, and where

$$\mathcal{N}_1[\phi] = -\frac{1}{c^2}\left[3a_3\left(2\phi\phi_z^2 + \phi^2\phi_{zz}\right) + 2a_2\left(\phi_z\phi_{zz} + \phi_z^2\right)\right] \tag{3.4a}$$

$$\mathcal{N}_2[\phi] = -\frac{1}{c^2}\left[3a_3\left(2\phi\phi_z^2 + \phi^2\phi_{zz}\right) + 5a_5\left(4\phi^3\phi_z^2 + \phi^4\phi_{zz}\right)\right] \tag{3.4b}$$

for the GPC 1 and GPC 2 PDE's, respectively.

Here we have

$$q = 1 - \frac{a_1}{c^2} \tag{3.5a}$$

$$p = 0 \tag{3.5b}$$

We emphasize that $p = 0$ because this corresponds to a 1-D subspace of the (p, q) parameter space in [2]. We denote C_0 as the positive q axis and C_1 the negative q-axis.

Solutions $\phi = ke^{\lambda x}$ satisfy the characteristic equation $\lambda^4 - q\lambda^2 + p = 0$. Since $p = 0$ for the GPC PDE's we have only two possible regions of eigenvalues. On C_0 the eigenvalues have the structure $\lambda_{1-4} = 0, 0, \pm\lambda$, $(\lambda \in \mathbb{R})$ while on C_1 we have $\lambda_{1-4} = 0, 0, \pm i\omega$, $(\omega \in \mathbb{R})$.

We now use the theory of Iooss and Adelmeyer [1] to write the GPC PDE's in reversible form: Let $Y = \langle y_1, y_2, y_3, y_4\rangle^T$ where $\langle y_1, y_2, y_3, y_4\rangle^T = \langle \phi_z, \phi_{zz}, \phi_{zzz}, \phi_{zzzz}\rangle^T$

We may now write (3.3) as a reversible first-order system

$$\frac{dY}{dz} = L_{pq}Y - F_{1,2}(Y, Y) \tag{3.6}$$

where

$$L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2 - p & 0 & q/3 & 0 \end{pmatrix} \tag{3.7}$$

Since $p = 0$ for the GPC PDE's, we have

$$\frac{dY}{dz} = L_{0q} Y - F_{1,2}(Y, Y) \tag{3.8}$$

where

$$F_{1,2}(Y, Y) = \langle 0, 0, 0, -\mathcal{N}_{1,2} \rangle^T \tag{3.9}$$

and

$$F_1(Y, Y) = \left\langle 0, 0, 0, \frac{1}{c^2} \left[3a_3 \left(2y_1 y_2^2 + y_1^2 y_2 \right) + 2a_2 \left(y_2 y_3 + y_2^2 \right) \right] \right\rangle^T \tag{3.10a}$$

$$F_2(Y, Y) = \left\langle 0, 0, 0, \frac{1}{c^2} \left[3a_3 \left(2y_1 y_2^2 + y_1^2 y_2 \right) + 5a_5 \left(4y_1^3 y_2^2 + y_1^4 y_3 \right) \right] \right\rangle^T \tag{3.10b}$$

3.1. **Near C_0.** Near C_0 the dynamics reduce to a 2-D Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B) \tag{3.11}$$

with

$$\frac{dA}{dz} = B \tag{3.12a}$$

$$\frac{dB}{dz} = b\epsilon A + \tilde{c} A^2 \tag{3.12b}$$

and the eigenvectors are $\zeta_0 = \langle 1, 0, -q/3, 0 \rangle^T$, $\zeta_1 = \langle 0, 1, 0, -2q/3 \rangle^T$, ϵ determines the distance from C_0. A simple dominant balance argument on the linearized equation yields $b = -\frac{1}{q}$.

To determine $\tilde{c}$ we calculate $\frac{dY}{dz}$ in two ways and match the $\mathcal{O}(A^2)$ terms. Expanding $\Psi(\epsilon, A, B)$ in series

$$\Psi(\epsilon, A, B) = \epsilon A \Psi_{10}^1 + \epsilon B \Psi_{01}^1 + A^2 \Psi_{20}^0 + AB \Psi_{11}^0 + B^2 \Psi_{02}^0 + \cdots \tag{3.13}$$

we find

$$\tilde{c}\zeta_1 = L_{0q} \Psi_{20}^0 - F_{1,2}(\zeta_0, \zeta_0) \tag{3.14}$$

We find that $F_1(\zeta_0, \zeta_0) = \left\langle 0, 0, 0, -\frac{q}{c^2} a_3 \right\rangle^T$ and $F_2(\zeta_0, \zeta_0) = \left\langle 0, 0, 0, -\frac{q}{c^2} \left(a_3 + \frac{5a_5}{3} \right) \right\rangle^T$ which implies $\tilde{c} = -\frac{a_3}{c^2}$ for the GPC 1 and $\tilde{c} = -\frac{1}{c^2} \left(a_3 + \frac{5}{3} a_5 \right)$ for the GPC 2.
Therefore, the normal form for the GPC 1 PDE near C_0 is

$$\frac{dA}{dz} = B \tag{3.15a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q} A - \frac{a_3}{c^2} A^2 \tag{3.15b}$$

and the normal form for the GPC 2 PDE near C_0 is

$$\frac{dA}{dz} = B \tag{3.16a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q} A - \frac{1}{c^2} \left(a_3 + \frac{5}{3} a_5 \right) A^2 \tag{3.16b}$$

3.2. **Near C_1.** Near C_1 the dynamics reduce to a 4-D Center Manifold

$$Y = A\zeta_0 + B\zeta_0 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C}) \tag{3.17}$$

with

$$\frac{dA}{dz} = B \tag{3.18a}$$

$$\frac{dB}{dz} = \bar{\nu}A + b_* A^2 + c_* |C|^2 \tag{3.18b}$$

$$\frac{dC}{dz} = id_0 C + i\bar{\nu}d_1 C + id_2 AC \tag{3.18c}$$

where $\zeta_\pm = \left\langle 1, \lambda_\pm, 2q/3, \frac{\lambda_\pm}{3}q \right\rangle^T$ where
$\bar{\nu} = -\frac{\epsilon}{q}$, $b_* = -\frac{b\Delta 1}{3}$ and $\lambda_\pm = \pm i\sqrt{-q}$. Comparing to the linearized equations gives $d_0 = \sqrt{-q}$.
DONE

ACTUALLY DO THIS: If we do a dominant balance argument after the change of variable $\tilde{\epsilon} = \sqrt{-3\mu}$ on the characteristic equation as $\lambda \to 0$ then we find $d_1 = \frac{\sqrt{-3\mu}}{18\mu^2}$. Using $\mu = q/3$ we find $d_1 = \frac{\sqrt{-q}}{2q^2}$

To determine b_*, c_* and d_2 we expand

$$\Psi(\epsilon, A, B, C, \bar{C}) = \epsilon A\Psi^1_{1000} + \epsilon B\Psi^1_{0100} + A^2\Psi^0_{2000} + AB\Psi^0_{1100} + AC\Psi^0_{1010} + \epsilon C\Psi^1_{0010} + \cdots \tag{3.19}$$

Now we use (3.18b), (3.18c), (3.18c) in (3.19) when we calculate $\frac{dY}{dz}$. Matching coefficients yields

$$\mathcal{O}(A^2): \quad b_*\zeta_1 = L_{0q}\Psi^0_{2000} - F_2(\zeta_0, \zeta_0) \tag{3.20a}$$

$$\mathcal{O}(|C|^2): \quad c_*\zeta_1 = L_{0q}\Psi^0_{0011} - 2F_2(\zeta_+, \zeta_-) \tag{3.20b}$$

$$\mathcal{O}(\epsilon C): \quad -\frac{i}{q}\left(d_1\zeta_+ + d_0\Psi^1_{0010}\right) = L_{0q}\Psi^1_{0010} - F_2(\Psi^1_{0010}, \Psi^1_{0010}) \tag{3.20c}$$

$$\mathcal{O}(AC): \quad id_2\zeta_+ + id_0\Psi^0_{1010} = L_{0q}\Psi^0_{1010} - 2F_2(\zeta_0, \zeta_+) \tag{3.20d}$$

(3.20e)

where we have used the fact that F_2 is a symmetric bilinear form. Equation (3.20b) is decoupled and yields $c_* = 2\Delta_1\left(\frac{1}{q} + \frac{2b}{3}\right)$ The only coefficient left to determine is d_2 which we shall compute now.

Put $\Psi^0_{1010} = \langle x_1, x_2, x_3, x_4\rangle^T$ into (3.20e) implies

$$id_2 + id_0 x_1 = x_2 \tag{3.21a}$$

$$-d_0 d_2 + id_0 x_2 = \frac{q}{3}x_1 + x_3 \tag{3.21b}$$

$$\frac{2iq}{3}d_2 + id_0 x_3 = \frac{q}{3}x_2 + x_4 \tag{3.21c}$$

$$-\frac{q}{3}d_0 d_2 + id_0 x_4 = \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) - \frac{2bq\Delta_1}{3} \tag{3.21d}$$

Using (3.21a) in (3.21b) , (3.21b) in (3.21d) and using these in (3.21c) yields $d_2 = \frac{b\Delta_1}{3\sqrt{-q}}$.

Therefore the normal form for the MS PDE near C_1 is

$$\frac{dA}{dz} = B \tag{3.22a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{b\Delta_1}{3}A^2 + 2\Delta_1\left(\frac{1}{q} + \frac{2b}{3}\right)|C|^2 \tag{3.22b}$$

$$\frac{dC}{dz} = i\sqrt{-q}C - i\frac{\epsilon\sqrt{-q}}{q^3}C + i\frac{b\Delta_1}{3\sqrt{-q}}AC \tag{3.22c}$$

4. Analysis of Normal Forms

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References

[1] Iooss G., Adelmeyer, M., *Topics in Bifurcation Theory and Applications*, World Scientific, Singapore, 1998.

[2] S. Roy Choudhury, *Solitary-wave families of the Ostrovsky equation: An approach via reversible systems theory and normal forms* , Elsevier, (2007), pp. 1468–1479.