Solitary Wave Families in Two Non-Integrable Models Using Reversible Systems Theory

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Overview

- Definitions
- Background
- The Generalized Pochammer-Chree Equations
- A Generalized Microstructure Equation

Reversible Dynamical System (Iooss & Adelmeyer)

Consider

$$\frac{dz}{dt} = F(z;\mu), z \in \mathbb{R}^n, \mu \in \mathbb{R} \tag{1}$$

where

$$F(0;0) = 0$$

. If there exists a **unitary map**

$$S : \mathbb{R}^n \mapsto \mathbb{R}^n, S \neq I$$

such that

$$F(Sz;\mu) = -SF(z;\mu)$$

for all z and μ then (1) is a reversible system.

Families of Solitary Waves

Here we will use the term solitary wave or "soliton" to mean a pulse-like solution to an evolution equation. For example,

$$A(z) = \ell \mathrm{sech}^2 kz \tag{2}$$

is a two-parameter family of solitary waves.
where k and ℓ are parameters which determine the speed and the height of the wave.

Normal Form Theory

After a nonlinear change of variables (Iooss & Adelmeyer) one may put the Center Manifold into Normal Form.

Two-Dimenional Center Manifold

- $\lambda_{1-4} = 0, 0, \pm\lambda, \lambda \in \mathbb{R}$
- $Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B)$

Four-Dimenional Center Manifold

- $\lambda_{1-4} = 0, 0, \pm i\omega, \omega \in \mathbb{R}$
- $Y = A\zeta_0 + B\zeta_1 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C})$

where $\zeta_0, \zeta_1, \zeta_+, \zeta_-$ are eigenvectors of the linearized operator.

Properties of Bilinear Functions

A function

$$B : \mathbb{C}x\mathbb{C} \mapsto \mathbb{C}$$

satisfying the following axioms

$$\begin{aligned} B(x+y,z) &= B(x,z)+B(y,z) \\ B(\lambda x,y) &= \lambda B(x,y) \\ B(x,y+z) &= B(x,y)+B(x,z) \\ B(x,\lambda y) &= \lambda B(x,y) \end{aligned}$$

is called **bilinear** .
If $B(x,y)$ is bilinear, then $f(y) \equiv B(y,y)$ is invariant under the transformation $y \mapsto -y$ and thus is **reversible** .

The Generalized Pochammer-Chree Equations

The Generalized Pochammer-Chree Equations govern the propagation of longitudinal waves in elastic rods.

- GPC1

$$(u - u_{xx})_{tt} - \left(a_1 u + a_2 u^2 + a_3 u^3\right)_{xx} = 0 \tag{4}$$

- GPC2

$$(u - u_{xx})_{tt} - \left(a_1 u + a_3 u^3 + a_5 u^5\right)_{xx} = 0 \tag{5}$$

Travelling Wave ODE

Let $z = x - ct$ and $u(x,t) = \phi(z)$ to reduce (4) and (5) to the Travelling Wave ODE

$$\phi_{zzzz} - q\phi_{zz} + p\phi = \mathcal{N}_{1,2}[\phi] \tag{6}$$

where

$$\begin{aligned} p &\equiv 0 \\ q &\equiv 1 - \frac{a_1}{c^2} \end{aligned}$$

$$L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2 - p & 0 & q/3 & 0 \end{pmatrix} \tag{8a}$$

$$\begin{aligned} \mathcal{N}_1[\phi] &= -\frac{1}{c^2}\left[3a_3\left(2\phi\phi_z^2 + \phi^2\phi_{zz}\right) + 2a_2\left(\phi_{zz}\phi_z + \phi_z^2\right)\right] \\ \mathcal{N}_2[\phi] &= -\frac{1}{c^2}\left[3a_3\left(2\phi\phi_z^2 + \phi^2\phi_{zz}\right) + 5a_5\left(4\phi^3\phi_z^2 + \phi^4\phi_{zz}\right)\right] \end{aligned}$$

Reversible Form

Denoting $Y = \langle y_1, y_2, y_3, y_4 \rangle^T = \langle \phi, \phi_z, \phi_{zz}, \phi_{zzz} \rangle^T$ equation (6) can be written

$$\frac{dY}{dz} = L_{pq} Y - G_{1,2}(Y, Y) \tag{9}$$

where

$$L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2 - p & 0 & q/3 & 0 \end{pmatrix}$$

$$G_{1,2}(Y, Y) = \langle 0, 0, 0, -\mathcal{N}_{1,2}(Y, Y) \rangle^T$$

Near C_0: Normal Form

The eigenvalues are $\lambda_{1,4} = 0, 0, \pm\lambda$, $\lambda \in \mathbb{R}$, We write the Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B) \tag{10}$$

with corresponding normal form

$$\frac{dA}{dz} = B \tag{11a}$$

$$\frac{dB}{dz} = b\epsilon A + \tilde{c}A^2 \tag{11b}$$

where ϵ measures the perturbation about C_0 and

$$\zeta_0 = \langle 1, 0, -q/3, 0\rangle^T \tag{12a}$$

$$\zeta_1 = \langle 0, 1, 0, -2q/3\rangle^T \tag{12b}$$

How do we determine the coefficients b and $\tilde{c}$?

Finding Coefficients Of The Normal Form

When you follow two separate chains of thought, Watson, you will find some point of intersection which should approximate the truth. - Sir Arthur Conan Doyle

By computing $\frac{dY}{dz}$ in two ways and comparing the coefficients of each term in the normal form, we find systems of equations which determine each coefficient. For example, to find $\tilde{c}$, we compare $\mathcal{O}(A^2)$ terms.

Two Ways to Compute $\frac{dY}{dz}$

Method 1
Use the Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B)$$

in the reversible system

$$\frac{dY}{dz} = L_{pq}Y - G_{1,2}(Y, Y)$$

and repetedly use the bilinear properties of $G_{1,2}$ to simplify.

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Method 2
Differentiate the Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B)$$

and use the Normal Form

$$\begin{aligned}\frac{dA}{dz} &= B\\ \frac{dB}{dz} &= b\epsilon A + \tilde{c}A^2\end{aligned}$$

to simplify all derivatives.

where

$$\Psi(\epsilon, A, B) = \epsilon A\Psi^1_{10} + \epsilon B\Psi^1_{01} + A^2\Psi^0_{20} + AB\Psi^0_{11} + B^2\Psi^0_{02} + \cdots$$

Finding $\tilde{c}$

Matching $\mathcal{O}\left(A^2\right)$ terms in each method gives us the two systems of equations

GPC 1

$$\tilde{c}\zeta_1 = L_{0q}\Psi_{20}^0 - G_1\left(\zeta_0, \zeta_0\right)$$

Let $\Psi_{20}^0 = \langle x_1, x_2, x_3, x_4\rangle^T$ which implies

$$\begin{aligned}
0 &= x_2 \\
\tilde{c} &= \frac{q}{3}x_1 + x_3 \\
0 &= \frac{q}{3}x_2 + x_4 \implies x_4 = 0 \\
-\frac{2q}{3}\tilde{c} &= \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) + \frac{q}{3c^2}\left(3a_3 + 5a_5\right) \\
&= \frac{q}{3}\tilde{c} + \frac{q}{3c^2}a_3
\end{aligned}$$

$$\implies \tilde{c} = -\frac{a_3}{3c^2}$$

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GPC 2

$$\tilde{c}\zeta_1 = L_{0q}\Psi^0_{20} - G_2\left(\zeta_0, \zeta_0\right)$$

Letting $\Psi^0_{20} = \langle x_1, x_2, x_3, x_4\rangle^T$ yields the equations

$$\begin{aligned}
0 &= x_2 \\
\tilde{c} &= \frac{q}{3}x_1 + x_3 \\
0 &= \frac{q}{3}x_2 + x_4 \implies x_4 = 0 \\
-\frac{2q}{3}\tilde{c} &= \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) + \frac{q}{3c^2}\left(3a_3 + 5a_5\right) \\
&= \frac{q}{3}\tilde{c} + \frac{q}{3c^2}\left(3a_3 + 5a_5\right)
\end{aligned}$$

$$\implies \tilde{c} = -\frac{1}{3c^2}\left(3a_3 + 5a_5\right)$$

Normal Form Near C_0

Therefore the Normal Form near C_0 is

GPC 1

$$\begin{aligned} \frac{dA}{dz} &= B \\ \frac{dB}{dz} &= -\frac{\epsilon}{q}A - \frac{a_3}{c^2}A^2 \end{aligned}$$

GPC 2

$$\begin{aligned} \frac{dA}{dz} &= B \\ \frac{dB}{dz} &= -\frac{\epsilon}{q}A - \frac{1}{3c^2}\left(3a_3 + 5a_5\right)A^2 \end{aligned}$$

These equations admit homoclinic solutions near C_0 of the form

$$A(z) = \ell \operatorname{sech}^2(kz)$$

Finding k and ℓ

To determine k and ℓ, we write the Normal Form as a single second order equation, use our expression for $A(z)$ and note the hyperbolic identity $\operatorname{sech}^2(z) - \operatorname{sech}^4(z) = \operatorname{sech}^2(z)\tanh^2(z)$ which implies

GPC 1

$$\begin{aligned} k &= \sqrt{\frac{-\epsilon}{4q}} \\ \ell &= \frac{-3\epsilon c^2}{2qa_3} \end{aligned}$$

GPC 2

$$\begin{aligned} k &= \sqrt{\frac{-\epsilon}{4q}} \\ \ell &= \frac{-9\epsilon c^2}{2q\left(3a_3 + 5a_5\right)} \end{aligned}$$

A Generalized Microstructure PDE

One dimensional wave propagation in microstructured solids has recently been modeled [?] by an equation

$$v_{tt} - bv_{xx} - \frac{\mu}{2}\left(v^2\right)_{xx} - \delta\left(\beta v_{tt} - \gamma v_{xx}\right)_{xx} = 0 \qquad (20)$$

Travelling Wave ODE

Let $z = x - ct$ and $u(x,t) = \phi(z)$ to reduce to the Travelling Wave ODE

$$\phi_{zzzz} - q\phi_{zz} + p\phi = \mathcal{N}[\phi] \tag{21}$$

where

$$\mathcal{N}[\phi] = -\Delta_1 \phi_z^2 - b\Delta_1 \phi\phi_{zz} \tag{22}$$

$$\Delta_1 = \frac{\mu}{\delta\left(\beta c^2 - \gamma\right)}$$

Results: The Generalized Pochammer-Chree Equations

(23)

Results: A Generalized Microstructure PDE