SOLITARY-WAVE FAMILIES OF THE MICRO-STRUCTURE PDE

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ABSTRACT. The Microstructure PDE is a model for ETC. Directions for future work are also described.

1. INTRODUCTION

The nonlinear dispersive equation [2]

$$v_{tt} - bv_{xx} - \frac{\mu}{2}\left(v^2\right)_{xx} - \delta\left(\beta v_{tt} - \gamma v_{xx}\right)_{xx} = 0 \tag{1.1}$$

is known as the Micro-Structure PDE.

2. SOLITARY WAVES; LOCAL BIFURCATIONS

Solitary waves of (1.1) of the form $u(x,t) = \phi(x - ct) = \phi(z)$ satisfy the fourth-order travelling wave ODE

$$\phi_{zzzz} - q\phi_{zz} + p\phi = \mathcal{N}[\phi] \tag{2.1}$$

where the notation $\mathcal{N}[\phi]$ means that the operator $\mathcal{N}$ operates on ϕ and all of it's derivatives, and

$$\mathcal{N}[\phi] = -\Delta_1\phi_z^2 - b\Delta_1\phi\phi_{zz} \tag{2.2}$$

where

$$\Delta_1 = \frac{\mu}{\delta\left(\beta c^2 - \gamma\right)} \tag{2.3a}$$

$$p = 0 \tag{2.3b}$$

The linearized system corresponding to (2.4)

$$\phi_{zzzz} - q\phi_{zz} + p\phi = 0 \tag{2.4}$$

has a fixed point $\phi = \phi_z = \phi_{zz} = \phi_{zzz} = 0$.

Solutions $\phi = ke^{\lambda x}$ satisfy the characteristic equation $\lambda^4 - q\lambda^2 + p = 0$ from which one may deduce that the structure of the eigenvalues is distinct in two regions of $p,q)$-space. Since $p = 0$ we have only two possible regions of eigenvalues. On C_0 the eigenvalues have the structure We denote C_0 as the positive q axis and C_1 the negative q-axis. $\lambda_{1-4} = 0, 0, \pm\lambda$, $(\lambda \in \mathbb{R})$ while on C_1 we have $\lambda_{1-4} = 0, 0, \pm i\omega$, $(\omega \in \mathbb{R})$.

2.1. Near C_0.

Near C_0 the dynamics reduce to a 2-D Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B) \tag{2.5}$$

with

$$\frac{dA}{dz} = B \tag{2.6a}$$

$$\frac{dB}{dz} = b\epsilon A + \tilde{c}A^2 \tag{2.6b}$$

and the eigenvectors are $\zeta_0 = \langle 1, 0, -q/3, 0\rangle^T$, $\zeta_1 = \langle 0, 1, 0, -2q/3\rangle^T$, ϵ determines the distance from C_0 . A simple dominant balance argument on the linearized equation yields $b = -\frac{1}{q}$. To determine $\tilde{c}$ we calculate $\frac{dY}{dz}$ in two ways and match the $\mathcal{O}(A^2)$ terms. Expanding $\Psi(\epsilon, A, B)$ in series

$$\Psi(\epsilon, A, B) = \epsilon A\Psi_{10}^1 + \epsilon B\Psi_{01}^1 + A^2\Psi_{20}^0 + AB\Psi_{11}^0 + B^2\Psi_{02}^0 + \cdots \tag{2.7}$$

we find

$$\tilde{c}\zeta_1 = L_{0q}\Psi^0_{20} - F_2(\zeta_0,\zeta_0) \tag{2.8}$$

We find that $F_2(\zeta_0,\zeta_0) = \left\langle 0,0,0,-\frac{b\Delta_1}{3}q\right\rangle^T$ which implies $\tilde{c} = -\frac{b\Delta_1}{3}$.
Therefore, the normal form for the MS PDE near C_0 is

$$\frac{dA}{dz} = B \tag{2.9a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{b\Delta_1}{3}A^2 \tag{2.9b}$$

2.2. **Near C_1.** Near C_1 the dynamics reduce to a 4-D Center Manifold

$$Y = A\zeta_0 + B\zeta_0 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C}) \tag{2.10}$$

with

$$\frac{dA}{dz} = B \tag{2.11a}$$

$$\frac{dB}{dz} = \bar{\nu}A + b_*A^2 + c_*\left|C\right|^2 \tag{2.11b}$$

$$\frac{dC}{dz} = id_0C + i\bar{\nu}d_1C + id_2AC \tag{2.11c}$$

where $\zeta_\pm = \left\langle 1, \lambda_\pm, 2q/3, \frac{\lambda_\pm}{3}q\right\rangle^T$ where
$\bar{\nu} = -\frac{\epsilon}{q}$, $b_* = -\frac{b\Delta 1}{3}$ and $\lambda_\pm = \pm i\sqrt{-q}$. Comparing to the linearized equations gives $d_0 = \sqrt{-q}$.
If we do a dominant balance argument after the change of variable $\tilde{\epsilon} = \sqrt{-3\mu}$ on the characteristic equation as $\lambda \to 0$ then we find $d_1 = \frac{\sqrt{-3\mu}}{18\mu^2}$. Using $\mu = q/3$ we find $d_1 = \frac{\sqrt{-q}}{2q^2}$
To determine b_*, c_* and d_2 we expand

$$\Psi(\epsilon, A, B, C, \bar{C}) = \epsilon A\Psi^1_{1000} + \epsilon B\Psi^1_{0100} + A^2\Psi^0_{2000} + AB\Psi^0_{1100} + AC\Psi^0_{1010} + \epsilon C\Psi^1_{0010} + \cdots \tag{2.12}$$

Now we use (4.2b), (4.2c), (4.2c) in (4.3) when we calculate $\frac{dY}{dz}$. Matching coefficients yields

$$\mathcal{O}(A^2): \quad b_*\zeta_1 = L_{0q}\Psi^0_{2000} - F_2(\zeta_0,\zeta_0) \tag{2.13a}$$

$$\mathcal{O}(\left|C\right|^2): \quad c_*\zeta_1 = L_{0q}\Psi^0_{0011} - 2F_2(\zeta_+,\zeta_-) \tag{2.13b}$$

$$\mathcal{O}(\epsilon C): \quad -\tfrac{i}{q}\left(d_1\zeta_+ + d_0\Psi^1_{0010}\right) = L_{0q}\Psi^1_{0010} - F_2(\Psi^1_{0010},\Psi^1_{0010}) \tag{2.13c}$$

$$\mathcal{O}(AC): \quad id_2\zeta_+ + id_0\Psi^0_{1010} = L_{0q}\Psi^0_{1010} - 2F_2(\zeta_0,\zeta_+) \tag{2.13d}$$

(2.13e)

where we have used the fact that F_2 is a symmetric bilinear form. Equation (4.4b) is decoupled and yields $c_* = 2\Delta_1\left(\frac{1}{q} + \frac{2b}{3}\right)$ The only coefficient left to determine is d_2 which we shall compute now.
Put $\Psi^0_{1010} = \langle x_1, x_2, x_3, x_4\rangle^T$ into (4.4e) implies

$$id_2 + id_0x_1 = x_2 \tag{2.14a}$$

$$-d_0d_2 + id_0x_2 = \frac{q}{3}x_1 + x_3 \tag{2.14b}$$

$$\frac{2iq}{3}d_2 + id_0x_3 = \frac{q}{3}x_2 + x_4 \tag{2.14c}$$

$$-\frac{q}{3}d_0d_2 + id_0x_4 = \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) - \frac{2bq\Delta_1}{3} \tag{2.14d}$$

Using (4.5a) in (4.5b) , (4.5b) in (4.5d) and using these in (4.5c) yields $d_2 = \frac{b\Delta_1}{3\sqrt{-q}}$.
Therefore the normal form for the MS PDE near C_1 is

$$\frac{dA}{dz} = B \tag{2.15a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{b\Delta_1}{3}A^2 + 2\Delta_1\left(\frac{1}{q}+\frac{2b}{3}\right)|C|^2 \tag{2.15b}$$

$$\frac{dC}{dz} = i\sqrt{-q}C - i\frac{\epsilon\sqrt{-q}}{q^3}C + i\frac{b\Delta_1}{3\sqrt{-q}}AC \tag{2.15c}$$

3. Calculations

We now proceed to calculate the normal forms of (1.1)

We now use the theory of Iooss and Adelmeyer [1] to write the Microstructure PDE in reversible form: Let $Y = \langle y_1, y_2, y_3, y_4\rangle^T$ where $\langle y_1, y_2, y_3, y_4\rangle^T = \langle \phi_z, \phi_{zz}, \phi_{zzz}, \phi_{zzzz}\rangle^T$

We may now write (2.4) as a reversible first-order system

$$\frac{dY}{dz} = L_{pq}Y - F_2(Y,Y) \tag{3.1}$$

where $F_2(Y,Y) = \langle 0,0,0,-\mathcal{N}(Y)\rangle^T$ and

$$L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2-p & 0 & q/3 & 0 \end{pmatrix} \tag{3.2}$$

Since $p=0$ for the Microstructure PDE, we have

$$\frac{dY}{dz} = L_{0q}Y - F_2(Y,Y) \tag{3.3}$$

where

$$F_2(Y,Y) = \left\langle 0,0,0,\Delta_1 y_2^2 + b\Delta_1 y_1 y_3\right\rangle^T \tag{3.4}$$

4. Normal form near C_1: possible solitary-wave solutions

Near C_1 the dynamics reduce to a 4-D Center Manifold

$$Y = A\zeta_0 + B\zeta_0 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C}) \tag{4.1}$$

with

$$\frac{dA}{dz} = B \tag{4.2a}$$

$$\frac{dB}{dz} = \bar{\nu}A + b_*A^2 + c_*|C|^2 \tag{4.2b}$$

$$\frac{dC}{dz} = id_0C + i\bar{\nu}d_1C + id_2AC \tag{4.2c}$$

where $\zeta_\pm = \left\langle 1, \lambda_\pm, 2q/3, \frac{\lambda_\pm}{3}q\right\rangle^T$ where
$\bar{\nu} = -\frac{\epsilon}{q}$, $b_* = -\frac{b\Delta 1}{3}$ and $\lambda_\pm = \pm i\sqrt{-q}$. Comparing to the linearized equations gives $d_0 = \sqrt{-q}$.

If we do a dominant balance argument after the change of variable $\tilde{\epsilon} = \sqrt{-3\mu}$ on the characteristic equation as $\lambda \to 0$ then we find $d_1 = \frac{\sqrt{-3\mu}}{18\mu^2}$. Using $\mu = q/3$ we find $d_1 = \frac{\sqrt{-q}}{2q^2}$

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$$\Psi(\epsilon, A, B, C, \bar{C}) = \epsilon A\Psi^1_{1000} + \epsilon B\Psi^1_{0100} + A^2\Psi^0_{2000} + AB\Psi^0_{1100} + AC\Psi^0_{1010} + \epsilon C\Psi^1_{0010} + \cdots \tag{4.3}$$

Now we use (4.2b), (4.2c), (4.2c) in (4.3) when we calculate $\frac{dY}{dz}$. Matching coefficients yields

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\begin{aligned}
&(4.4a) && \mathcal{O}(A^2): && b_*\zeta_1 && = L_{0q}\Psi^0_{2000} - F_2(\zeta_0,\zeta_0)\\
&(4.4b) && \mathcal{O}(|C|^2): && c_*\zeta_1 && = L_{0q}\Psi^0_{0011} - 2F_2(\zeta_+,\zeta_-)\\
&(4.4c) && \mathcal{O}(\epsilon C): && -\tfrac{i}{q}\left(d_1\zeta_+ + d_0\Psi^1_{0010}\right) && = L_{0q}\Psi^1_{0010} - F_2(\Psi^1_{0010},\Psi^1_{0010})\\
&(4.4d) && \mathcal{O}(AC): && id_2\zeta_+ + id_0\Psi^0_{1010} && = L_{0q}\Psi^0_{1010} - 2F_2(\zeta_0,\zeta_+)\\
&(4.4e) && && &&
\end{aligned}
$$

where we have used the fact that F_2 is a symmetric bilinear form. Equation (4.4b) is decoupled and yields $c_* = 2\Delta_1\left(\frac{1}{q}+\frac{2b}{3}\right)$ The only coefficient left to determine is d_2 which we shall compute now.

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&(4.5c) && \frac{2iq}{3}d_2 + id_0x_3 &&= \frac{q}{3}x_2 + x_4\\
&(4.5d) && -\frac{q}{3}d_0d_2 + id_0x_4 &&= \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) - \frac{2bq\Delta_1}{3}
\end{aligned}
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Using (4.5a) in (4.5b) , (4.5b) in (4.5d) and using these in (4.5c) yields $d_2 = \frac{b\Delta_1}{3\sqrt{-q}}$.

Therefore the normal form for the MS PDE near C_1 is

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\begin{aligned}
&(4.6a) && \frac{dA}{dz} &&= B\\
&(4.6b) && \frac{dB}{dz} &&= -\frac{\epsilon}{q}A - \frac{b\Delta_1}{3}A^2 + 2\Delta_1\left(\frac{1}{q}+\frac{2b}{3}\right)|C|^2\\
&(4.6c) && \frac{dC}{dz} &&= i\sqrt{-q}C - i\frac{\epsilon\sqrt{-q}}{q^3}C + i\frac{b\Delta_1}{3\sqrt{-q}}AC
\end{aligned}
$$

References

[1] Iooss G., Adelmeyer, M., *Topics in Bifurcation Theory and Applications*, World Scientific, Singapore, 1998.

[2] S. Roy Choudhury, *Solitary-wave families of the Ostrovsky equation: An approach via reversible systems theory and normal forms* , Elsevier, (2007), pp. 1468–1479.