

SOLITARY-WAVE FAMILIES OF THE MICRO-STRUCTURE PDE

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ABSTRACT. The Microstructure PDE is a model for ETC. Directions for future work are also described.

1. INTRODUCTION

The nonlinear dispersive equation [2]

$$(1.1) \quad v_{tt} - bv_{xx} - \frac{\mu}{2} (v^2)_{xx} - \delta (\beta v_{tt} - \gamma v_{xx})_{xx} = 0$$

is known as the Micro-Structure PDE.

2. SOLITARY WAVES; LOCAL BIFURCATIONS

Solitary waves of (1.1) of the form $u(x, t) = \phi(x - ct) = \phi(z)$ satisfy the fourth-order travelling wave ODE

$$(2.1) \quad \phi_{zzzz} - q\phi_{zz} + p\phi = \mathcal{N}[\phi]$$

where the notation $\mathcal{N}[\phi]$ means that the operator \mathcal{N} operates on ϕ and all of its derivatives, and

$$(2.2) \quad \mathcal{N}[\phi] = -\Delta_1 \phi_z^2 - b\Delta_1 \phi \phi_{zz}$$

where

$$(2.3a) \quad \Delta_1 = \frac{\mu}{\delta(\beta c^2 - \gamma)}$$

$$(2.3b) \quad p = 0$$

The linearized system corresponding to (2.4)

$$(2.4) \quad \phi_{zzzz} - q\phi_{zz} + p\phi = 0$$

has a fixed point $\phi = \phi_z = \phi_{zz} = \phi_{zzz} = 0$.

Solutions $\phi = ke^{\lambda x}$ satisfy the characteristic equation $\lambda^4 - q\lambda^2 + p = 0$. Since $p = 0$ for the MS PDE we have only two possible regions of eigenvalues. On C_0 the eigenvalues have the structure $\lambda_{1-4} = 0, 0, \pm\lambda$, ($\lambda \in \mathbb{R}$) while on C_1 we have $\lambda_{1-4} = 0, 0, \pm i\omega$, ($\omega \in \mathbb{R}$).

3. CALCULATIONS

We now proceed to calculate the normal forms of (1.1)

We denote C_0 as the positive q axis and C_1 the negative q -axis.

We now use the theory of Iooss and Adelmeyer [1] to write the Microstructure PDE in reversible form:

Let $Y = \langle y_1, y_2, y_3, y_4 \rangle^T$ where $\langle y_1, y_2, y_3, y_4 \rangle^T = \langle \phi_z, \phi_{zz}, \phi_{zzz}, \phi_{zzzz} \rangle^T$

We may now write (2.4) as a reversible first-order system

$$(3.1) \quad \frac{dY}{dz} = L_{pq}Y - F_2(Y, Y)$$

where $F_2(Y, Y) = \langle 0, 0, 0, -\mathcal{N}(Y) \rangle^T$ and

$$(3.2) \quad L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2 - p & 0 & q/3 & 0 \end{pmatrix}$$

Since $p = 0$ for the Microstructure PDE, we have

$$(3.3) \quad \frac{dY}{dz} = L_{0q}Y - F_2(Y, Y)$$

where

$$(3.4) \quad F_2(Y, Y) = \langle 0, 0, 0, \Delta_1 y_2^2 + b\Delta_1 y_1 y_3 \rangle^T$$

3.1. Near C_0 . Near C_0 the dynamics reduce to a 2-D Center Manifold

$$(3.5) \quad Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B)$$

with

$$(3.6a) \quad \frac{dA}{dz} = B$$

$$(3.6b) \quad \frac{dB}{dz} = b\epsilon A + \tilde{c}A^2$$

and the eigenvectors are $\zeta_0 = \langle 1, 0, -q/3, 0 \rangle^T$, $\zeta_1 = \langle 0, 1, 0, -2q/3 \rangle^T$, ϵ determines the distance from C_0 . A simple dominant balance argument on the linearized equation yields $b = -\frac{1}{q}$.

To determine \tilde{c} we calculate $\frac{dY}{dz}$ in two ways and match the $\mathcal{O}(A^2)$ terms. Expanding $\Psi(\epsilon, A, B)$ in series

$$(3.7) \quad \Psi(\epsilon, A, B) = \epsilon A \Psi_{10}^1 + \epsilon B \Psi_{01}^1 + A^2 \Psi_{20}^0 + AB \Psi_{11}^0 + B^2 \Psi_{02}^0 + \dots$$

we find

$$(3.8) \quad \tilde{c}\zeta_1 = L_{0q}\Psi_{20}^0 - F_2(\zeta_0, \zeta_0)$$

We find that $F_2(\zeta_0, \zeta_0) = \langle 0, 0, 0, -\frac{b\Delta_1}{3}q \rangle^T$ which implies $\tilde{c} = -\frac{b\Delta_1}{3}$.

Therefore, the normal form for the MS PDE near C_0 is

$$(3.9a) \quad \frac{dA}{dz} = B$$

$$(3.9b) \quad \frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{b\Delta_1}{3}A^2$$

3.2. Near C_1 . Near C_1 the dynamics reduce to a 4-D Center Manifold

$$(3.10) \quad Y = A\zeta_0 + B\zeta_0 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C})$$

with

$$(3.11a) \quad \frac{dA}{dz} = B$$

$$(3.11b) \quad \frac{dB}{dz} = \bar{\nu}A + b_*A^2 + c_*|C|^2$$

$$(3.11c) \quad \frac{dC}{dz} = id_0C + i\bar{\nu}d_1C + id_2AC$$

where $\zeta_{\pm} = \langle 1, \lambda_{\pm}, 2q/3, \frac{\lambda_{\pm}}{3}q \rangle^T$ where

$\bar{\nu} = -\frac{\epsilon}{q}$, $b_* = -\frac{b\Delta_1}{3}$ and $\lambda_{\pm} = \pm i\sqrt{-q}$. Comparing to the linearized equations gives $d_0 = \sqrt{-q}$.

If we do a dominant balance argument after the change of variable $\tilde{\epsilon} = \sqrt{-3\mu}$ on the characteristic equation as $\lambda \rightarrow 0$ then we find $d_1 = \frac{\sqrt{-3\mu}}{18\mu^2}$. Using $\mu = q/3$ we find $d_1 = \frac{\sqrt{-q}}{2q^2}$.

To determine b_* , c_* and d_2 we expand

$$(3.12) \quad \Psi(\epsilon, A, B, C, \bar{C}) = \epsilon A \Psi_{1000}^1 + \epsilon B \Psi_{0100}^1 + A^2 \Psi_{2000}^0 + AB \Psi_{1100}^0 + AC \Psi_{1010}^0 + \epsilon C \Psi_{0010}^1 + \dots$$

Now we use (3.11b), (3.11c), (3.11c) in (3.12) when we calculate $\frac{dY}{dz}$. Matching coefficients yields

$$\begin{aligned}
(3.13a) \quad \mathcal{O}(A^2) : \quad b_* \zeta_1 &= L_{0q} \Psi_{2000}^0 - F_2(\zeta_0, \zeta_0) \\
(3.13b) \quad \mathcal{O}(|C|^2) : \quad c_* \zeta_1 &= L_{0q} \Psi_{0011}^0 - 2F_2(\zeta_+, \zeta_-) \\
(3.13c) \quad \mathcal{O}(\epsilon C) : \quad -\frac{i}{q} (d_1 \zeta_+ + d_0 \Psi_{0010}^1) &= L_{0q} \Psi_{0010}^1 - F_2(\Psi_{0010}^1, \Psi_{0010}^1) \\
(3.13d) \quad \mathcal{O}(AC) : \quad i d_2 \zeta_+ + i d_0 \Psi_{1010}^0 &= L_{0q} \Psi_{1010}^0 - 2F_2(\zeta_0, \zeta_+) \\
(3.13e)
\end{aligned}$$

where we have used the fact that F_2 is a symmetric bilinear form. Equation (3.13b) is decoupled and yields $c_* = 2\Delta_1 \left(\frac{1}{q} + \frac{2b}{3} \right)$. The only coefficient left to determine is d_2 which we shall compute now.

Put $\Psi_{1010}^0 = \langle x_1, x_2, x_3, x_4 \rangle^T$ into (3.13e) implies

$$\begin{aligned}
(3.14a) \quad i d_2 + i d_0 x_1 &= x_2 \\
(3.14b) \quad -d_0 d_2 + i d_0 x_2 &= \frac{q}{3} x_1 + x_3 \\
(3.14c) \quad \frac{2iq}{3} d_2 + i d_0 x_3 &= \frac{q}{3} x_2 + x_4 \\
(3.14d) \quad -\frac{q}{3} d_0 d_2 + i d_0 x_4 &= \frac{q}{3} \left(\frac{q}{3} x_1 + x_3 \right) - \frac{2bq\Delta_1}{3}
\end{aligned}$$

Using (3.14a) in (3.14b), (3.14b) in (3.14d) and using these in (3.14c) yields $d_2 = \frac{b\Delta_1}{3\sqrt{-q}}$.

Therefore the normal form for the MS PDE near C_1 is

$$\begin{aligned}
(3.15a) \quad \frac{dA}{dz} &= B \\
(3.15b) \quad \frac{dB}{dz} &= -\frac{\epsilon}{q} A - \frac{b\Delta_1}{3} A^2 + 2\Delta_1 \left(\frac{1}{q} + \frac{2b}{3} \right) |C|^2 \\
(3.15c) \quad \frac{dC}{dz} &= i\sqrt{-q}C - i\frac{\epsilon\sqrt{-q}}{q^3}C + i\frac{b\Delta_1}{3\sqrt{-q}}AC
\end{aligned}$$

4. ANALYSIS OF NORMAL FORMS

REFERENCES

- [1] IOOSS G., ADELMAYER, M., *Topics in Bifurcation Theory and Applications*, World Scientific, Singapore, 1998.
- [2] S. ROY CHOUDHURY, *Solitary-wave families of the Ostrovsky equation: An approach via reversible systems theory and normal forms*, Elsevier, (2007), pp. 1468–1479.