SOLITARY WAVE FAMILIES IN TWO NON-INTEGRABLE MODELS USING REVERSIBLE SYSTEMS THEORY

by

JONATHAN LETO
B.S. University of Central Florida, 2004

A thesis submitted in partial fulfillment of the requirements
for the degree of Masters in Mathematical Science
in the Department of Mathematics
in the College of Sciences
at the University of Central Florida
Orlando, Florida

Spring Term
2008

Major Professor: S. Roy Choudhury

ABSTRACT

In this thesis, we apply a recently developed technique to comprehensively categorize all possible families of solitary wave solutions in two models of topical interest.

The models considered are:

- the Generalized Pochhammer-Chree Equations, which govern the propagation of longitudinal waves in elastic rods,

  and

- a generalized microstructure PDE.

Limited analytic results exist for the occurrence of one family of solitary wave solutions for each of these equations. Since, as mentioned above, solitary wave solutions often play a central role in the long-time evolution of an initial disturbance, we consider such solutions of both models here (via the normal form approach) within the framework of reversible systems theory.

Besides confirming the existence of the known family of solitary waves for each model, we find a continuum of delocalized solitary waves (or homoclinics to small-amplitude periodic orbits). On isolated curves in the relevant parameter region, the delocalized waves reduce to genuine embedded solitons. For both models, the new family of solutions occur in regions of parameter space distinct from the known solitary wave solutions and are thus entirely new.

Directions for future work, including the dynamics of each family of solitary waves using exponential asymptotics techniques, are also mentioned.

Dedicated To My Family

ACKNOWLEDGMENTS

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER ONE: INTRODUCTION

Solitary wave solutions of nonlinear models have become increasingly important, both as possible information carriers, as well as organizing centers for the solution dynamics in regimes where the initial conditions naturally break into stable pulses or pulse-trains.

Since the numerical "re-discovery" of solitons in the Korteweg & de Vries equation $u_t + uu_x + \delta^2 u_{xxx} = 0$ [1] in 1965 there has been intense research in equations that admit soliton solutions. An analytic soliton solution to the KdV equation was found in 1967 [2] by quite unique means and at the time it was not clear whether the method was generally applicable.

A general principle for associating nonlinear evolution equations with the eigenvalues of linear operators was discovered in 1968 [3]. Soon after, solitons were found in an even more fundamental and canonical system, the nonlinear Schrödinger equations $i\Psi_t + \Psi_{xx} \pm \Psi\|\Psi\|^2 = 0$ [4]. These equations arise in diverse areas because they are canonical equations governing the modulation of the amplitude Ψ of a weakly nonlinear wave packets [5]. This led to a comprehensive theory, an extension of Fourier analysis for nonlinear systems, called the Inverse Scattering Transform [6].

These standard techniques for investigating solitary waves of integrable nonlinear PDEs do not carry over to the non-integrable models which are of increasing relevance in modern applications. Other techniques which have been devised, such as variational ones, and exponential asymptotics methods, each yield results in certain regimes of the systems parameters.

In this thesis, we apply a recently developed technique to comprehensively categorize all possible families of solitary wave solutions in two models of topical interest.

The models considered are:

- the Generalized Pochhammer-Chree Equations, which govern the propagation of longitudinal waves in elastic rods,

$$(u - u_{xx})_{tt} - \left(a_1 u + a_2 u^2 + a_3 u^3\right)_{xx} = 0 \tag{1.1}$$

and

$$(u - u_{xx})_{tt} - \left(a_1 u + a_3 u^3 + a_5 u^5\right)_{xx} = 0 \tag{1.2}$$

- a generalized microstructure PDE.

$$v_{tt} - b v_{xx} - \frac{\mu}{2}\left(v^2\right)_{xx} - \delta\left(\beta v_{tt} - \gamma v_{xx}\right)_{xx} = 0 \tag{1.3}$$

The phase space of the traveling-wave equation will be studied, specifically the homoclinic orbits, which correspond to solitary wave solutions in the original PDE. A homoclinic orbit is defined as any orbit which connects a fixed point to itself [7].

Limited analytic results exist for the occurrence of one family of solitary wave solutions for each of these equations. Since, as mentioned above, solitary wave solutions often play a central role in the long-time evolution of an initial disturbance, we consider such solutions of both models here (via the normal form approach) within the framework of reversible systems theory. Recently, an alternative approach using a Hamiltonian formulation has also been used to analyze the traveling wave ODE [8].

Besides confirming the existence of the known family of solitary waves for each model, we find a continuum of delocalized solitary waves (or homoclinics to small-amplitude periodic

orbits). On isolated curves in the relevant parameter region, the delocalized waves reduce to genuine embedded solitons. For the generalized Microstructure equation, the new family of solutions occur in regions of parameter space distinct from the known solitary wave solutions and are thus entirely new.

Directions for future work, including the dynamics of each family of solitary waves using exponential asymptotics techniques, are also mentioned.

CHAPTER TWO: GENERALIZED POCHHAMMER-CHREE EQUATIONS

2.1 INTRODUCTION

The propagation of longitudinal deformation waves in elastic rods is governed ([9], [10], [11]) by (1.1) and (1.2), corresponding to different constitutive relations.

References [9], [10], [11] also discuss the primary references, including derivations and applications of these equations in various fields. In addition, motivated by experimental and numerical results, there are derivations of special families of solitary wave solutions by the extended $Tanh$ method [9], and other ansatzen [11]. These extend earlier solitary wave solutions given by Bogolubsky [12] and Clarkson et. al [13] for special cases of (1.1) and (1.2). In addition, [10] generalizes the existence results in [14] for solitary waves of (1.1) and (1.2).

2.2 SOLITARY WAVES: LOCAL BIFURCATION

Solitary waves of (1.1) and (1.2) of the form $v(x, t) = \phi(x - ct) = \phi(z)$ satisfy the fourth-order traveling wave ODE

$$\phi_{zzzz} - q\phi_{zz} + p\phi = \mathcal{N}_{1,2}[\phi] \tag{2.1}$$

where

$$\mathcal{N}_1[\phi] = -\frac{1}{c^2}\left[3a_3\left(2\phi\phi_z^2 + \phi^2\phi_{zz}\right) + 2a_2\left(\phi_{zz}\phi_z + \phi_z^2\right)\right] \tag{2.2a}$$

$$\mathcal{N}_2[\phi] = -\frac{1}{c^2}\left[3a_3\left(2\phi\phi_z^2 + \phi^2\phi_{zz}\right) + 5a_5\left(4\phi^3\phi_z^2 + \phi^4\phi_{zz}\right)\right] \tag{2.2b}$$

4

$$z \equiv x - ct \tag{2.3a}$$

$$p \equiv 0 \tag{2.3b}$$

$$q \equiv 1 - \frac{a_1}{c^2} \tag{2.3c}$$

Equation (2.1) is invariant under the transformation $z \mapsto -z$ and is thus a reversible system. In this section we shall use the theory of reversible systems to characterize the homoclinic orbits to the fixed point of (2.1), which correspond to pulses or solitary waves of (1.1) and (1.2) in various regions of the (p, q) plane.

The linearized system corresponding to (2.1)

$$\phi_{zzzz} - q\phi_{zz} + p\phi = 0 \tag{2.4}$$

has a fixed point

$$\phi = \phi_z = \phi_{zz} = \phi_{zzz} = 0 \tag{2.5}$$

Solutions $\phi = ke^{\lambda x}$ satisfy the characteristic equation $\lambda^4 - q\lambda^2 + p = 0$ from which one may deduce that the structure of the eigenvalues is distinct in two regions of (p, q)-space. Since $p = 0$ we have only two possible regions of eigenvalues. We denote C_0 as the positive q axis and C_1 the negative q-axis. First we shall consider the bounding curves C_0 and C_1 and their neighborhoods, then we shall discuss the possible occurrence and multiplicity of homoclinic orbits to (2.5), corresponding to pulse solitary waves of (1.1) and (1.2), in each region:

Near C_0 The eigenvalues have the structure $\lambda_{1-4} = 0, 0, \pm\lambda$, $(\lambda \in \mathbb{R})$ and the fixed point (2.5) is a saddle-focus.

Near C_1 Here the eigenvalues have the structure $\lambda_{1-4} = 0, 0, \pm i\omega$, $(\omega \in \mathbb{R})$. We will show by analysis of a four-dimensional normal form in Section 2.4 that there exists a sech^2 homoclinic orbit near C_1.

Having outlined the possible families of orbits homoclinic to the fixed point (2.5) of (2.4), corresponding to pulse solitary waves of (1.1) and (1.2), we now derive normal forms near the transition curves C_0 and C_1 to confirm the existence of regular or delocalized solitary waves in the corresponding regions of (p, q) parameter space.

2.3 NORMAL FORM NEAR C_0: SOLITARY WAVE SOLUTIONS

Using (2.4), the curve C_0, corresponding to $\lambda = 0, 0, \pm\tilde{\lambda}$, is given by

$$C_0 : p = 0, q > 0 \tag{2.6}$$

Using (2.3c) implies

$$a_1 < c^2 \tag{2.7}$$

Denoting ϕ by y_1, (2.1) may be written as the two systems

$$\frac{dy_1}{dz} = y_2 \tag{2.8a}$$

$$\frac{dy_2}{dz} = y_3 \tag{2.8b}$$

$$\frac{dy_3}{dz} = y_4 \tag{2.8c}$$

$$\frac{dy_4}{dz} = qy_3 - py_1 - N_{1,2}(Y) \tag{2.8d}$$

where

$$\mathcal{N}_1(Y) = -\frac{1}{c^2}\left[3a_3\left(2y_1 y_2^2 + y_1^2 y_3\right) + 2a_2\left(y_3 y_2 + y_2^2\right)\right] \qquad (2.9\mathrm{a})$$

$$\mathcal{N}_2(Y) = -\frac{1}{c^2}\left[3a_3\left(2y_1 y_2^2 + y_1^2 y_3\right) + 5a_5\left(4y_1^3 y_2^2 + y_1^4 y_3\right)\right] \qquad (2.9\mathrm{b})$$

We wish to rewrite (2.8) as a first order reversible system in order to invoke the relevant theory [15].

To that end, defining $Y = \langle y_1, y_2, y_3, y_4\rangle^T$, equation (2.8) can be written

$$\frac{dY}{dz} = AY - G_{1,2}(Y,Y) \qquad (2.10)$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -p & 0 & q & 0 \end{pmatrix} \qquad (2.11)$$

$$G_{1,2}(Y,Y) = \langle 0, 0, 0, -\mathcal{N}_{1,2}(Y)\rangle^T \qquad (2.12)$$

The matrix A may be split into $A = L_0 + L_1$, where

$$L_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.13a}$$

$$L_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -p & 0 & q & 0 \end{pmatrix} \tag{2.13b}$$

We now derive a linear operator L_{pq} which is equivalent to $A = L_0 + L_1$ and in reversible form.

Let $L_{pq} = L_0 + M$ where M must satifisfy the following properties:

- $ML_0^* = L_0^* M$ or $[M, L_0^*] = 0$: M commutes with L_0^*

- $SM = -MS$: S and M are antisymmetric with repect to each other

where $S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Since L_0^* commutes with the identity and powers of itself, we assume the form of M as

$$M = \alpha_1 I + \alpha_2 L_0^* + \alpha_3 L_0^2 + \alpha_4 L_0^{*3} \tag{2.14}$$

Because we want M to be antisymmetric, we must have $\alpha_1 = \alpha_3 = 0$ since I and L_0^{*2} are symmetric. Therefore we have $M = \alpha_2 L_0^* + \alpha_4 L_0^{*3}$. We now impose the condition that the eigenvalues of L_{pq} must be identical, therefore we must have that the coefficients of the characteristic polynomials of $L_0 + L_1$ and $L_0 + M$ are identical. The characteristic polynomial of $L_0 + L_1$ is

$$\rho_1(\lambda) = det\,(L_0 + L_1 - \lambda I) = \lambda^4 - q\lambda^2 + p \tag{2.15}$$

The characteristic polynomial of $L_0 + M$ is

$$\rho_2(\lambda) = det\,(L_0 + M - \lambda I) = det\,\left(L_0 + \alpha_2 L_0^* + \alpha_4 L_0^{*3} - \lambda I\right) \tag{2.16}$$

After some algebra one finds that

$$\rho_2(\lambda) = \lambda^4 - 3\,(\alpha_2 + \alpha_4)\,\lambda^2 + (\alpha_2 + \alpha_4)^2 \tag{2.17}$$

This immediately gives us

$$p = (\alpha_2 + \alpha_4)^2 \tag{2.18a}$$

$$q = 3\,(\alpha_2 + \alpha_4) \tag{2.18b}$$

Noting that $\frac{q^2}{9} = p$, we choose $\alpha_4 = \frac{q^2}{9} - p \equiv 0$ which implies $\alpha_2 = \frac{q}{3}$. We now have that $p = \alpha_2^2$ and $q = 3\alpha_2$.

Now (2.8) may be written

$$\frac{dY}{dz} = L_{pq}Y - G_{1,2}(Y,Y) \tag{2.19}$$

9

where

$$L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2 - p & 0 & q/3 & 0 \end{pmatrix} \tag{2.20}$$

Since $p = 0$ for (1.1) and (1.2), we have

$$\frac{dY}{dz} = L_{0q}Y - G_{1,2}(Y, Y) \tag{2.21}$$

Next we calculate the normal form of (2.21) near C_0. The procedure is closely modeled on [15] and many intermediate steps may be found there.

2.3.1 Near C_0

Near C_0 the dynamics reduce to a two-dimensional Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B) \tag{2.22}$$

and the corresponding normal form is

$$\frac{dA}{dz} = B \tag{2.23a}$$

$$\frac{dB}{dz} = b\epsilon A + \tilde{c}A^2 \tag{2.23b}$$

Here,

$$\epsilon = \left(\frac{q^2}{9} - p\right) - \left(\frac{q}{3}\right)^2 = -p \tag{2.24}$$

10

measures the perturbation around C_0, and

$$\zeta_0 = \langle 1, 0, -q/3, 0 \rangle^T \tag{2.25a}$$

$$\zeta_1 = \langle 0, 1, 0, -2q/3 \rangle^T \tag{2.25b}$$

The linear eigenvalue of (2.23) satisfies

$$\lambda^2 = b\epsilon \tag{2.26}$$

The characteristic equation of the linear part of (2.21) is

$$\lambda^4 - q\lambda^2 - \epsilon = 0 \tag{2.27}$$

Hence, the eigenvalues near zero (the Center Manifold) satisfy $\lambda^4 \ll \lambda^2$ and hence

$$\lambda^2 \sim -\frac{\epsilon}{q} \tag{2.28}$$

Matching (2.26) and (2.28) implies

$$b = -\frac{1}{q} \tag{2.29}$$

and only the nonlinear coefficient $\tilde{c}$ remains to be determined in the normal form (2.23).

In order to determine $\tilde{c}$ (the coefficient of A^2 in (2.23)) we calculate $\frac{dY}{dz}$ in two ways and match the $\mathcal{O}(A^2)$ terms. To this end, using the standard 'suspension' trick of treating the perturbation parameter ϵ as a variable, we expand the function Ψ in (2.22) as

$$\Psi(\epsilon, A, B) = \epsilon A \Psi_{10}^1 + \epsilon B \Psi_{01}^1 + A^2 \Psi_{20}^0 + AB \Psi_{11}^0 + B^2 \Psi_{02}^0 + \cdots \tag{2.30}$$

where the subscripts denote powers of A and B, respectively, and the superscript denotes the power of ϵ. In the first way of computing dY/dz, we take the z derivative of (2.22)

$$\frac{dY}{dz} = \frac{dA}{dz}\zeta_0 + \frac{dB}{dz}\zeta_1 + \frac{d\Psi}{dz} \tag{2.31}$$

Now, using (2.23) we have

$$\frac{dY}{dz} = B\zeta_0 + \left(b\epsilon A + \tilde{c}A^2\right)\zeta_1 + \frac{d\Psi}{dz} \tag{2.32}$$

And finally using (2.30) and (2.23) again we arrive at

$$\frac{dY}{dz} = B\zeta_0 + \left(b\epsilon A + \tilde{c}A^2\right)\zeta_1 + \left(b\epsilon^2 A + \epsilon\tilde{c}A^2\right) \tag{2.33}$$

$$+ 2AB\Psi_{20}^0 + \left(B^2 + b\epsilon A^2 + \tilde{c}A^3\right)\Psi_{11}^0$$

$$+ 2B\left(b\epsilon A + \tilde{c}A^2\right)\Psi_{02}^0 + \epsilon\left(b\epsilon + \tilde{c}A^2\right)\Psi_{01}^1 + \cdots$$

The coefficient of A^2 in (2.33) is $\tilde{c}\zeta_1$.

Now we alternately compute dY/dz by using (2.22) and (2.30) in (2.21)

$$\frac{dY}{dz} = \qquad\qquad L_{0q}\left(A\zeta_0 + B\zeta_1 + \Psi\right) - G_{1,2}\left(Y, Y\right) \tag{2.34}$$

$$= \qquad\qquad L_{0q}\left(A\zeta_0\right) + L_{0q}\left(B\zeta_1\right) + L_{0q}\left(\Psi\right) \tag{2.35}$$

$$- G_{1,2}\left(A\zeta_0 + B\zeta_1 + \Psi, A\zeta_0 + B\zeta_1 + \Psi\right)$$

$$= \qquad\qquad L_{0q}\left(A\zeta_0\right) + L_{0q}\left(B\zeta_1\right) + L_{0q}\left(\Psi\right) \tag{2.36}$$

$$- G_{1,2}\left(\zeta_0, \zeta_0\right)A^2 - G_{1,2}\left(\zeta_1, \zeta_1\right)B^2 - G_{1,2}\left(\Psi, \Psi\right)$$

where the linearity of L_{0q} and the bilinearity of $G_{1,2}$ have been used in (2.35) and (2.36) .

We have now found one of the $\mathcal{O}(A^2)$ terms, and must inspect $L_{0q}\left(\Psi\right)$ and $G_{1,2}\left(\Psi, \Psi\right)$, where we expect a $\mathcal{O}(A^2)$ term due to the expansion (2.30). Hence

$$L_{0q}\left(\Psi\right) = \epsilon A L_{0q}\Psi_{10}^1 + \epsilon B L_{0q}\Psi_{01}^1 + A^2 L_{0q}\Psi_{20}^0 + AB L_{0q}\Psi_{11}^0 + B^2 L_{0q}\Psi_{02}^0 + \cdots \tag{2.37}$$

and we find that $L_{0q}\Psi_{20}^0$ is another $\mathcal{O}(A^2)$ term. Lastly, we expand $G_{1,2}\left(\Psi, \Psi\right)$ to find

$$G_{1,2}\left(\Psi, \Psi\right) = G_{1,2}\left(\epsilon A\Psi_{10}^1 + \epsilon B\Psi_{01}^1 + A^2\Psi_{20}^0 \cdots, \epsilon A\Psi_{10}^1 + \epsilon B\Psi_{01}^1 + A^2\Psi_{20}^0 \cdots\right) \tag{2.38}$$

12

$$G_{1,2}\left(\Psi,\Psi\right) = \quad G_{1,2}\left(\epsilon A\Psi_{10}^1 + \epsilon B\Psi_{01}^1 + \cdots, \epsilon A\Psi_{10}^1 + \epsilon B\Psi_{01}^1 + \cdots\right) + \qquad (2.39)$$

$$G_{1,2}\left(A^2\Psi_{20}^0, A^2\Psi_{20}^0\right)$$

$$G_{1,2}\left(\Psi,\Psi\right) = \quad G_{1,2}\left(\epsilon A\Psi_{10}^1, \epsilon A\Psi_{10}^1\right) + G_{1,2}\left(\epsilon B\Psi_{01}^1, \epsilon B\Psi_{01}^1\right) + \qquad (2.40)$$

$$G_{1,2}\left(AB\Psi_{11}^0, AB\Psi_{11}^0\right) + G_{1,2}\left(A^2\Psi_{20}^0, A^2\Psi_{20}^0\right) +$$

$$G_{1,2}\left(B^2\Psi_{02}^0, B^2\Psi_{02}^0+\right) + \cdots$$

$$G_{1,2}\left(\Psi,\Psi\right) = \quad G_{1,2}\left(\Psi_{10}^1, \Psi_{10}^1\right)\epsilon^2 A^2 + G_{1,2}\left(\Psi_{01}^1, \Psi_{01}^1\right)\epsilon^2 B^2 + \qquad (2.41)$$

$$G_{1,2}\left(\Psi_{11}^0, \Psi_{11}^0\right)A^2 B^2 + G_{1,2}\left(\Psi_{20}^0, \Psi_{20}^0\right)A^4 +$$

$$G_{1,2}\left(\Psi_{02}^0, \Psi_{02}^0\right)B^4 + \cdots$$

where we have repeatedly used the fact that G_1 and G_2 are bilinear, to simplify sums inside

the arguments of each function. It now becomes clear that $G_{1,2}\left(\Psi,\Psi\right)$ does not contribute

any $\mathcal{O}(A^2)$ terms. So the $\mathcal{O}(A^2)$ terms of (2.34) are $L_{0,q}\Psi_{20}^0 - G_{1,2}\left(\zeta_0, \zeta_0\right)$.

Matching the $\mathcal{O}\left(A^2\right)$ terms in (2.33) and (2.34) results in the two systems of equations

$$\tilde{c}\zeta_1 = L_{0q}\Psi_{20}^0 - G_1\left(\zeta_0, \zeta_0\right) \qquad (2.42a)$$

$$\tilde{c}\zeta_1 = L_{0q}\Psi_{20}^0 - G_2\left(\zeta_0, \zeta_0\right) \qquad (2.42b)$$

for each of the operators G_1 and G_2.

Using (2.25) and (2.12) and denoting $\Psi_{20}^0 = \langle x_1, x_2, x_3, x_4\rangle^T$ in (2.42b) yields the equa-

tions

$$0 = x_2 \tag{2.43a}$$

$$\tilde{c} = \frac{q}{3}x_1 + x_3 \tag{2.43b}$$

$$0 = \frac{q}{3}x_2 + x_4 \implies x_4 = 0 \text{ using (2.43b)} \tag{2.43c}$$

and

$$-\frac{2q}{3}\tilde{c} = \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) + \frac{q}{3c^2}\left(3a_3 + 5a_5\right) = \frac{q}{3}\tilde{c} + \frac{q}{3c^2}\left(3a_3 + 5a_5\right) \text{ using (2.43b)} \tag{2.44}$$

Hence we obtain

$$\tilde{c} = -\frac{1}{3c^2}\left(3a_3 + 5a_5\right) \tag{2.45}$$

for (2.42b). Similarly for (2.42a), we use (2.25) and (2.12) and denote $\Psi_{20}^0 = \langle x_1, x_2, x_3, x_4 \rangle^T$ in which implies

$$0 = x_2 \tag{2.46a}$$

$$\tilde{c} = \frac{q}{3}x_1 + x_3 \tag{2.46b}$$

$$0 = \frac{q}{3}x_2 + x_4 \implies x_4 = 0 \text{ using (2.46b)} \tag{2.46c}$$

and

$$-\frac{2q}{3}\tilde{c} = \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) + \frac{q}{3c^2}\left(3a_3 + 5a_5\right) = \frac{q}{3}\tilde{c} + \frac{q}{3c^2}a_3 \text{ using (2.46b)} \tag{2.47}$$

which implies

$$\tilde{c} = -\frac{a_3}{3c^2} \tag{2.48}$$

for (2.42a).

Therefore, the normal form near C_0 is

$$\frac{dA}{dz} = B \tag{2.49a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{a_3}{c^2}A^2 \tag{2.49b}$$

for (1.1) and

$$\frac{dA}{dz} = B \tag{2.50a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{1}{3c^2}\left(3a_3 + 5a_5\right)A^2 \tag{2.50b}$$

for (1.2).

The normal form (2.49) admits a homoclinic solution (near C_0) of the form

$$A\left(z\right) = \ell\mathrm{sech}^2\left(kz\right) \tag{2.51}$$

To determine the coefficients k and l we first turn (2.49) into the single second order

equation for $A(z)$

$$\frac{d^2A}{dz^2} = -\frac{\epsilon}{q}A - \frac{a_3}{3c^2}A^2 \tag{2.52}$$

Now we use (2.51) in (2.52) which implies

$$\ell\left(-2k^2\mathrm{sech}^4(kz) + 4k^2\mathrm{sech}^2\left(kz\right)\tanh^2\left(kz\right)\right) = -\frac{\epsilon}{q}\mathrm{sech}^2(kz) - \frac{a_3}{c^2}\ell^2\mathrm{sech}^4(kz) \tag{2.53}$$

Noting the hyperbolic identities

$$\mathrm{sech}^2\left(z\right) - \mathrm{sech}^4\left(z\right) = \mathrm{sech}^2\left(z\right)\left(1 - \mathrm{sech}^2\left(z\right)\right)$$

$$= \mathrm{sech}^2\left(z\right)\tanh^2\left(z\right)$$

15

gives

$$-6k^2\text{sech}^4(kz) + 4k^2\text{sech}^2(kz) = -\frac{\epsilon}{q}\text{sech}^2(kz) - \frac{a_3}{c^2}\ell\text{sech}^4(kz) \tag{2.55}$$

Matching the $\mathcal{O}\left(\text{sech}^2(kz)\right)$ and $\mathcal{O}\left(\text{sech}^4(kz)\right)$ terms in (2.55) implies

$$k = \sqrt{\frac{-\epsilon}{4q}} \tag{2.56a}$$

$$\ell = \frac{6k^2c^2}{a_3} \tag{2.56b}$$

or simplifying

$$k = \sqrt{\frac{-\epsilon}{4q}} \tag{2.57a}$$

$$\ell = \frac{-3\epsilon c^2}{2qa_3} \tag{2.57b}$$

Similarly, the normal form (2.50) admits a homoclinic solution (near C_0) of the form

$$A(z) = \ell\text{sech}^2(kz) \tag{2.58}$$

which implies

$$\ell\left(-2k^2\text{sech}^4(kz) + 4k^2\text{sech}^2(kz)\tanh^2(kz)\right) = -\frac{\epsilon}{q}\text{sech}^2(kz) - \frac{1}{c^2}\left(3a_3 + 5a_5\right)\ell^2\text{sech}^4(kz) \tag{2.59}$$

Using the same hyperbolic identities as before we reduce this to

$$-6k^2\text{sech}^4(kz) + 4k^2\text{sech}^2(kz) = -\frac{\epsilon}{q}\text{sech}^2(kz) - \frac{\ell}{3c^2}\left(3a_3 + 5a_5\right)\text{sech}^4(kz) \tag{2.60}$$

Matching the $\mathcal{O}\left(\text{sech}^2(kz)\right)$ and $\mathcal{O}\left(\text{sech}^4(kz)\right)$ terms in (2.60) implies

$$k = \sqrt{\frac{-\epsilon}{4q}} \tag{2.61a}$$

$$\ell = \frac{18k^2\epsilon c^2}{(3a_3 + 5a_5)} \tag{2.61b}$$

16

or simplifying

$$k = \sqrt{\frac{-\epsilon}{4q}} \tag{2.62a}$$

$$\ell = \frac{-9\epsilon c^2}{2q\,(3a_3 + 5a_5)} \tag{2.62b}$$

Hence, since $\epsilon = -p$, and the curve C_0 corresponds to $p = 0, q > 0$, solitary waves of the

form (2.51) and (2.58) exist in the vicinity of C_0 for

$$p > 0, q > 0 \tag{2.63}$$

which implies that $a_1 < c^2$ (such that k is real.) As mentioned in section 2, one may show

the persistence of this homoclinic solution in the original traveling wave ODE (2.4). Thus,

we have demonstrated the existence of solitary waves of (1.1) and (1.2) for $p = 0^+, q > 0$.

Similarly, the curve C_1 corresponds to $p = 0, q < 0$, solitary waves of the form (2.51) and

(2.58) exist in the vicinity of C_1 for

$$p < 0, q < 0 \tag{2.64}$$

which implies $a_1 > c^2$.

Again, one may show the persistence of this homoclinic solution in the original traveling

wave ODE (2.4). Thus, we have demonstrated the existence of solitary waves of (1.1) and

(1.2) for $p = 0^-, q < 0$.

2.4   NORMAL FORM NEAR C_1: POSSIBLE SOLITARY WAVE

SOLUTIONS

Using (2.4), the curve C_1, corresponding to $\lambda = 0, 0 \pm i\omega$, is given by

$$C_1 : p = 0, q < 0 \tag{2.65}$$

Which implies

$$a_1 > c^2 \tag{2.66}$$

In order to investigate the possibility of a sech^2 homoclinic orbit in the neighborhood of C_1 and delocalized solitary waves, we next compute the normal form near C_1 following the procedure in [15].

Near C_1 the dynamics reduce to a four-dimensional Center Manifold [15]. Since all the eigenvalues are non-hyperbolic, the Center Manifold has the form (a nonlinear coordinate change [15])

$$Y = A\zeta_0 + B\zeta_0 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C}) \tag{2.67}$$

with a corresponding four-dimensional normal form

$$\frac{dA}{dz} = B \tag{2.68a}$$

$$\frac{dB}{dz} = \bar{\nu}A + b_*A^2 + c_* |C|^2 \tag{2.68b}$$

$$\frac{dC}{dz} = id_0C + i\bar{\nu}d_1C + id_2AC \tag{2.68c}$$

Here C is complex, $\bar{C}$ is the complex conjugate of C, $\epsilon, \zeta_0, \zeta_1$ are given previously and the two new complex eigenvectors co-spanning the Center Manifold are

$$\zeta_\pm = \left\langle 1, \lambda_\pm, 2q/3, \frac{\lambda_\pm}{3}q \right\rangle^T \tag{2.69}$$

Using (2.68b) and (2.23b) implies

$$\bar{\nu} = b\epsilon = -\frac{\epsilon}{q} \tag{2.70}$$

18

Also from the characteristic equation (2.27), the two non-zero (imaginary) roots are

$$\lambda^2 = \frac{q + \sqrt{q^2 + 4\epsilon}}{2} \approx q \text{ for } \epsilon \text{ small} \tag{2.71}$$

Hence

$$\lambda = \pm i\sqrt{-q}, q < 0 \tag{2.72}$$

Matching this to the linear part of (2.68c) (which corresponds to the imaginary eigenvalues), $\lambda = id_0 = i\sqrt{-q}$ or

$$d_0 = \sqrt{-q} \tag{2.73}$$

With a dominant balance argument on the characteristic equation (2.27) as $\lambda \to \pm i\sqrt{-q}$ we find

$$d_1 = \frac{\sqrt{-q}}{2q^2} \tag{2.74}$$

The remaining undetermined coefficients in the normal form are the coefficients b_*, c_* and d_2 which correspond to the $A^2, |C|^2$ and AC terms respectively. In order to determine them, we follow the same procedure as in Section 2.3 and compute dY/dz is two distinct ways. We expand the function Ψ as

$$\Psi(\epsilon, A, B, C, \bar{C}) = \epsilon A \Psi^1_{1000} + \epsilon B \Psi^1_{0100} + A^2 \Psi^0_{2000} + AB\Psi^0_{1100} + AC\Psi^0_{1010} + \epsilon C \Psi^1_{0010} + \cdots \tag{2.75}$$

with subscripts denoting powers of A, B, C and $\bar{C}$, respectively, and the superscript is the power of ϵ. In the first way, dY/dz is computed by taking the z derivative of (2.67) (using

(2.68) and (2.75))

$$\frac{dY}{dz} = \frac{dA}{dz}\zeta_0 + \frac{dB}{dz}\zeta_1 + \frac{dC}{dz}\zeta_+ + \frac{d\bar{C}}{dz}\zeta_- + \frac{d\Psi}{dz} \tag{2.76a}$$

$$= B\zeta_0 + \left(\bar{\nu}A + b_*A^2 + c_*\|C\|^2\right)\zeta_1 + \left(id_0 + id_1 C\bar{\nu} + id_2 AC\right)\zeta_+ + \tag{2.76b}$$

$$\left(-i\bar{C}d_9 - i\bar{C}d_1\bar{\nu} - iA\bar{C}d_2\right)\zeta_- + \frac{d\Psi}{dz}$$

Now, using (2.68) and (2.75) we expand $\frac{d\Psi}{dz}$ in the previous equation to find

$$\frac{dY}{dz} = \frac{dA}{dz}\zeta_0 + \frac{dB}{dz}\zeta_1 + \frac{dC}{dz}\zeta_+ + \frac{d\bar{C}}{dz}\zeta_- + \frac{d\Psi}{dz}$$

$$= B\zeta_0 + \left(\bar{\nu}A + b_*A^2 + c_*\|C\|^2\right)\zeta_1 + \left(id_0 + id_1 C\bar{\nu} + id_2 AC\right)\zeta_+ +$$

$$\left(-i\bar{C}d_9 - i\bar{C}d_1\bar{\nu} - iA\bar{C}d_2\right)\zeta_- +$$

$$\epsilon B\Psi^1_{1000} + \epsilon\left(\bar{\nu}A + b_*A^2 + c_*\|C\|^2\right)\Psi^1_{0100} +$$

$$\epsilon\left(id_0 C + id_1\bar{\nu}C + id_2 AC\right)\Psi^1_{0010} + \epsilon\left(-i\bar{C}d_9 - i\bar{C}d_1\bar{\nu} - iA\bar{C}d_2\right)\Psi^1_{0010} +$$

$$\left(B^2 + \left(\bar{\nu}A + b_*A^2 + c_*\|C\|^2\right)A\right)\Psi^0_{1100} +$$

$$\left(BC + A\left(id_0 C + id_1 C\bar{\nu} + id_2 AC\right)\right)\Psi^0_{1010} +$$

$$\left(B\bar{C} + A\left(-id_0 C - id_1\bar{C}\bar{\nu} - id_2 A\bar{C}\right)\right)\Psi^0_{1010} +$$

$$2AB\Psi^0_{2000} + 2B\left(\bar{\nu}A + b_*A^2 + c_*\|C\|^2\right) +$$

$$2C\left(id_0 C + id_1\bar{\nu}C + id_2 AC\right)\Psi^0_{0020} +$$

$$2\bar{C}\left(-id_0\bar{C} - id_1\bar{\nu}\bar{C} - id_2 A\bar{C}\right)\Psi^0_{0002} +$$

$$\left(\left(\bar{\nu}A + b_*A^2 + c_*\|C\|^2\right)C + \left(id_0 C + id_1\bar{\nu}C + id_2 AC\right)B\right)\Psi^0_{0110} +$$

$$\left(\left(\bar{\nu}A + b_*A^2 + c_*\|C\|^2\right)\bar{C} + \left(-id_0 C - id_1\bar{\nu}\bar{C} - id_2 A\bar{C}\right)B\right)\Psi^0_{0110} + \cdots$$

20

We will need the coefficients of $A^2, \|C\|^2, \epsilon C$ and AC terms in the previous equation, but first we compute $\frac{dY}{dz}$ in another way, so that we can match to coefficients of like terms.

In the second way, we compute dY/dz by using (2.67) and (2.75) in (2.19) (with $p = 0$ on C_1 as given in (2.65)) which implies

$$
\begin{aligned}
\frac{dY}{dz} &= L_{0q}\left(A\zeta_0 + B\zeta_0 + C\zeta_+\bar{C}\zeta_- + \Psi\right) - G_{1,2}\left(Y,Y\right) \\
&= L_{0q}\zeta_0 A + L_{0q}\zeta_0 B + L_{0q}\zeta_0 B + L_{0q}\zeta_+ C + L_{0q}\zeta_-\bar{C} + L_{0q}\Psi - \\
&\quad G_{1,2}\left(A\zeta_0 + B\zeta_0 + C\zeta_+\bar{C}\zeta_- + \Psi, A\zeta_0 + B\zeta_0 + C\zeta_+\bar{C}\zeta_- + \Psi\right) \\
&= L_{0q}\zeta_0 A + L_{0q}\zeta_0 B + L_{0q}\zeta_0 B + L_{0q}\zeta_+ C + L_{0q}\zeta_-\bar{C} + \\
&\quad L_{0q}\Psi^1_{1010}\epsilon A + L_{0q}\Psi^1_{0100}\epsilon B + L_{0q}\Psi^0_{2000}A^2 + L_{0q}\Psi^0_{1010}AC + L_{0q}\Psi^1_{0010}\epsilon C - \\
&\quad G_{1,2}\left(A\zeta_0 + B\zeta_0 + C\zeta_+\bar{C}\zeta_- + \Psi, A\zeta_0 + B\zeta_0 + C\zeta_+\bar{C}\zeta_- + \Psi\right) + \cdots \\
&= L_{0q}\zeta_0 A + L_{0q}\zeta_0 B + L_{0q}\zeta_0 B + L_{0q}\zeta_+ C + L_{0q}\zeta_-\bar{C} + \\
&\quad L_{0q}\Psi^1_{1010}\epsilon A + L_{0q}\Psi^1_{0100}\epsilon B + L_{0q}\Psi^0_{2000}A^2 + L_{0q}\Psi^0_{1010}AC + L_{0q}\Psi^1_{0010}\epsilon C - \\
&\quad G_{1,2}\left(\zeta_0, \zeta_0\right)A^2 + G_{1,2}\left(\zeta_0, \zeta_0\right)B^2 + G_{1,2}\left(\zeta_0, \zeta_+\right)C^2 + G_{1,2}\left(\zeta_-, \zeta_-\right)\bar{C}^2 \\
&\quad +2G_{1,2}\left(\zeta_0, \zeta_0\right)AB + 2G_{1,2}\left(\zeta_0, \zeta_+\right)BC + 2G_{1,2}\left(\zeta_0, \zeta_+\right)AC + 2G_{1,2}\left(\zeta_0, \zeta_-\right)A\bar{C} + \\
&\quad 2G_{1,2}\left(\zeta_0, \zeta_-\right)B\bar{C} + 2G_{1,2}\left(\zeta_-, \zeta_+\right)\|C\|^2 + G_{1,2}\left(\bar{C}\zeta_-, \Psi\right) + G_{1,2}\left(A\zeta_0, \Psi\right) + G_{1,2}\left(B\zeta_0, \Psi\right) + \\
&\quad G_{1,2}\left(C\zeta_+, \Psi\right) + G_{1,2}\left(\Psi, \Psi\right) + \cdots
\end{aligned}
$$

We note that any term of the form $G_{1,2}\left(., \Psi\right)$ is of order ≥ 3 because Ψ has no terms of order less than 2. What we mean by order is the sum of the indices in $\Psi^\epsilon_{ABC\bar{C}}$. For example, $\epsilon C\Psi^1_{0010}$ is of order 2. In particular, $G(\Psi, \Psi)$ has terms of order 4 and higher. Since we are only in search of terms of order 2 (A^2, ϵC $\|C\|^2$ and AC), we need not look inside any of

terms of the form $G_{1,2}(., \Psi)$ and hence we can now compare the two expansions of $\frac{dY}{dz}$.

Matching the coefficients of A^2, $\epsilon C \|C\|^2$ and AC in the two separate expressions for dY/dz yields the following two systems of equations:

$$\mathcal{O}(A^2): \qquad b_* \zeta_1 \qquad = L_{0q} \Psi^0_{2000} - G_{1,2}(\zeta_0, \zeta_0) \qquad \text{(2.79a)}$$

$$\mathcal{O}(|C|^2): \qquad c_* \zeta_1 \qquad = L_{0q} \Psi^0_{0011} - 2 G_{1,2}(\zeta_+, \zeta_-) \qquad \text{(2.79b)}$$

$$\mathcal{O}(\epsilon C): \quad -\frac{i}{q}(d_1 \zeta_+ + d_0 \Psi^1_{0010}) \quad = L_{0q} \Psi^1_{0010} \qquad \text{(2.79c)}$$

$$\mathcal{O}(AC): \quad i d_2 \zeta_+ + i d_0 \Psi^0_{1010} \quad = L_{0q} \Psi^0_{1010} - 2 G_{1,2}(\zeta_0, \zeta_+) \qquad \text{(2.79d)}$$

where we have repeatedly used the fact that $G_1(Y, Y)$ and $G_2(Y, Y)$ are symmetric bilinear forms. We now detail how one calculates $G_1(X, Y)$. For example, if $B(X, X)$ is a bilinear form and

$$B(X, X) = \left\langle 0, 0, 0, x_2^2 + x_1 x_3 \right\rangle^T$$

then having $B(X, Y)$ symmetric and linear with respect to each X and Y is satisfied by

$$B(X, Y) = \left\langle 0, 0, 0, x_2 y_2 + \frac{1}{2}(x_1 y_3 + y_1 x_3) \right\rangle^T$$

In general, if there is a term of the form $f(Y, Y) = y_1^m y_2^n$ then $f(X, Y) = \frac{1}{2}(x_1^m y_2^n + y_1^m x_2^n)$ for $n, m \geq 1$. If $f(Y, Y) = y_n^m$ then $f(X, Y) = \frac{1}{2}(x_n^{m-1} y_n + y_n^{m-1} x_n)$ where $m \geq 1$.

Using these properties, we can now evaluate the terms $G_{1,2}(\zeta_0, \zeta_0)$, $G_{1,2}(\zeta_+, \zeta_-)$ and $G_{1,2}(\zeta_0, \zeta_+)$ in the above equations. We have

$$G_1(Y, Y) = \left\langle 0, 0, 0, -\mathcal{N}(Y, Y) \right\rangle^T \qquad \text{(2.80a)}$$

$$= \left\langle 0, 0, 0, \frac{1}{c^2}(2 a_2 y_2 y_3 + 2 a_2 y_2^2 + 6 a_3 y_1 y_2^2 + 3 a_3 y_1^2 y_3) \right\rangle^T \qquad \text{(2.80b)}$$

so using the above properties to calculate $G_1(X, Y)$ we find

$$
\begin{aligned}
G_1(X, Y) &= \langle 0, 0, 0, -\mathcal{N}(X, Y) \rangle^T \\
&= \left\langle 0, 0, 0, \frac{1}{c^2} \left(a_2 \left(x_2 y_3 + y_2 x_3 + 2 x_2 y_2 \right) + 3 a_3 \left(x_1 y_2^2 + y_1 x_2^2 \right) + \frac{3 a_3}{2} \left(x_1^2 y_3 + y_1^2 x_3 \right) \right) \right\rangle^T
\end{aligned}
$$

We now evaluate the above equation for $X = \zeta_+, Y = \zeta_-$ where

$$
\zeta_\pm = \left\langle 1, \lambda_\pm, 2\frac{q}{3}, \lambda_\pm \frac{q}{3} \right\rangle^T \tag{2.82}
$$

which implies

$$
\begin{aligned}
G_1(\zeta_+, \zeta_-) &= \langle 0, 0, 0, -\mathcal{N}(X, Y) \rangle^T \\
&= \left\langle 0, 0, 0, \frac{1}{c^2} \left(2 a_2 \left(\frac{q}{3} \left(\lambda_- + \lambda_+ \right) + 2 \lambda_- \lambda_+ \right) + 3 a_3 \left(\lambda_-^2 + \lambda_+^2 + \frac{2q}{3} \right) \right) \right\rangle^T
\end{aligned}
$$

Using the facts that $\lambda_-^2 = q, \lambda_+^2 = q, \lambda_+ \lambda_- = -q$ and $\lambda_- + \lambda_+ = 0$ we finally reach the simplified form

$$
G_1(\zeta_+, \zeta_-) = \left\langle 0, 0, 0, \frac{q}{c^2} \left(8 a_3 - 4 a_2 \right) \right\rangle^T \tag{2.84}
$$

Using this in (2.79b) we get the following system of equations which determine c_*:

$$
0 = x_2 \tag{2.85a}
$$

$$
c_* = \frac{q}{3} x_1 + x_3 \tag{2.85b}
$$

$$
0 = \frac{q}{3} x_2 + x_4 \tag{2.85c}
$$

$$
-2\frac{q}{3} c_* = \frac{q}{3} \left(\frac{q}{3} x_1 + x_3 \right) - 2\frac{q}{c^2} \left(8 a_3 - 4 a_2 \right) \tag{2.85d}
$$

Using (2.85b) in (2.85d) we find

$$
c_* = \frac{8}{c^2} \left(2 a_3 - a_2 \right) \tag{2.86}
$$

for (1.1).

In the same way we for (1.2) we calculate the components of $G_2(Y, Y)$

$$G_2(Y, Y) = \langle 0, 0, 0, \mathcal{N}_2(Y, Y) \rangle^T \tag{2.87}$$

$$= \left\langle 0, 0, 0, \frac{1}{c^2} \left(3a_3 \left(2y_1 y_2^2 + y_1^2 y_3 \right) + 5a_5 \left(4y_1^3 y_2^2 + y_1^4 y_3 \right) \right) \right\rangle^T \tag{2.88}$$

We now calculate $G_2(\zeta_+, \zeta_-)$ where $\zeta_\pm = \left\langle 1, \lambda_\pm, 2\frac{q}{3}, \lambda_\pm \frac{q}{3} \right\rangle^T$

$$
\begin{aligned}
G_2(\zeta_+, \zeta_-) &= \left\langle 0, 0, 0, \frac{1}{c^2} \left(3a_3 \left(\lambda_-^2 + \lambda_+^2 + \frac{2q}{3} \right) + 5a_5 \left(2\lambda_-^2 + 2\lambda_+^2 + \frac{2q}{3} \right) \right) \right\rangle^T \\
&= \left\langle 0, 0, 0, \frac{1}{c^2} \left(3a_3 \left(2q + \frac{2q}{3} \right) + 5a_5 \left(4q + \frac{2q}{3} \right) \right) \right\rangle^T \\
&= \left\langle 0, 0, 0, \frac{q}{c^2} \left(8a_3 + \frac{70}{3} a_5 \right) \right\rangle^T
\end{aligned}
$$

Using this value of $G_2(\zeta_+, \zeta_-)$ in (2.79b) we find

$$0 = x_2 \tag{2.90a}$$

$$c_* = \frac{q}{3} x_1 + x_3 \tag{2.90b}$$

$$0 = \frac{q}{3} x_2 + x_4 \tag{2.90c}$$

$$-2\frac{q}{3} c_* = \frac{q}{3} \left(\frac{q}{3} x_1 + x_3 \right) - 2\frac{q}{c^2} \left(8a_3 + \frac{70}{3} a_5 \right) \tag{2.90d}$$

Using (2.90b) in (2.90d) we find

$$c_* = \frac{1}{c^2} \left(16a_3 + \frac{140}{3} a_5 \right) \tag{2.91}$$

for (1.2). The only coefficient left to determine is d_2 which we shall compute now.

Using $\Psi^0_{1010} = \langle x_1, x_2, x_3, x_4 \rangle^T$ in (2.79d) implies

$$id_2 + id_0 x_1 = x_2 \tag{2.92a}$$

$$-d_0 d_2 + id_0 x_2 = \frac{q}{3} x_1 + x_3 \tag{2.92b}$$

$$\frac{2iq}{3} d_2 + id_0 x_3 = \frac{q}{3} x_2 + x_4 \tag{2.92c}$$

$$-\frac{q}{3} d_0 d_2 + id_0 x_4 = \frac{q}{3}\left(\frac{q}{3} x_1 + x_3\right) - \frac{2q}{c^2}\left(\frac{7}{2} a_3 - \frac{i}{3} d_0 a_2\right) \tag{2.92d}$$

for (1.1) and

$$id_2 + id_0 x_1 = x_2 \tag{2.93a}$$

$$-d_0 d_2 + id_0 x_2 = \frac{q}{3} x_1 + x_3 \tag{2.93b}$$

$$\frac{2iq}{3} d_2 + id_0 x_3 = \frac{q}{3} x_2 + x_4 \tag{2.93c}$$

$$-\frac{q}{3} d_0 d_2 + id_0 x_4 = \frac{q}{3}\left(\frac{q}{3} x_1 + x_3\right) - \frac{2q}{c^2}\left(\frac{7}{2} a_3 + \frac{32}{3} a_5\right) \tag{2.93d}$$

for (1.2)

Using (2.92a) in (2.92b), (2.92b) in (2.92d) and using these in (2.92c) yields $d_2 =$ $\frac{1}{c^2}\left(\frac{7}{2\sqrt{-q}} a_3 - \frac{i}{3} a_2\right)$ for (1.1). Similarly using (2.93a) in (2.93b), (2.93b) in (2.93d) and using these in (2.93c) yields $d_2 = \frac{1}{\sqrt{-q}c^2}\left(\frac{7}{2} a_3 + \frac{32}{3} a_5\right)$ for (1.2).

Therefore the normal form near C_1 is

$$\frac{dA}{dz} = B \tag{2.94a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q} A - \frac{a_3}{c^2} A^2 + \frac{1}{c^2}\left(\frac{7}{2\sqrt{-q}} a_3 - \frac{i}{3} a_2\right) |C|^2 \tag{2.94b}$$

$$\frac{dC}{dz} = i\sqrt{-q} C - i\frac{\sqrt{-q}}{q^3} C\epsilon + i\frac{1}{c^2}\left(\frac{7}{2\sqrt{-q}} a_3 - \frac{i}{3} a_2\right) AC \tag{2.94c}$$

for (1.1) and

$$\frac{dA}{dz} = B \tag{2.95a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{(3a_3 + 5a_5)}{3c^2}A^2 + \frac{1}{c^2}\left(16a_3 + \frac{140}{3}a_5\right)|C|^2 \tag{2.95b}$$

$$\frac{dC}{dz} = i\sqrt{-q}C - i\frac{\sqrt{-q}}{q^3}C\epsilon + i\frac{1}{\sqrt{-q}c^2}\left(\frac{7}{2}a_3 + \frac{32}{3}a_5\right)AC \tag{2.95c}$$

for (1.2), where we have used the fact that b_* in the four-dimensional Center Manifold is the same as $\tilde{c}$ in the two-dimensional manifold, since we recover the two-dimensional manifold when $C \equiv 0$.

The dynamics inherent in (2.94), (2.95) may be elucidated following the discussions of [15], [16], [17] and [18]. The two first integrals of (2.68) are

$$K = |C|^2 \tag{2.96}$$

and

$$H = B^2 - \frac{2}{3}b_*A^3 - \bar{\nu}A^2 - 2c_*KA \tag{2.97}$$

Here, the appropriate coefficients $b_*, \bar{\nu}$ and c_*, derived above, apply for (1.1) and (1.2). Also, c_* should be real, or a_2 must be zero in (1.1) for the following energy arguments to apply. As a typical case, consider the level curve $H = 0$ of the energy-like first integral function H. In the (A, B) phase plane, this will compromise a homoclinic orbit. The intersection of $H = 0$ with the A axis occurs for $\frac{2}{3}b_*A^2 - \bar{\nu}A - 2c_*K = 0$ or

$$A_{\mp} = \frac{3}{4b_*}\left[\bar{\nu} \pm \sqrt{\bar{\nu}^2 + \frac{16b_*c_*K}{3}}\right] \tag{2.98}$$

Note that $A_+ > 0, A_- < 0$ for $b_*c_* > 0$ and $b_* < 0$ as relevant for us. A general homoclinic orbit, homoclinic to A_+, is sketched in Figure 1 where the flow direction is deduced from

26

(2.94a) and (2.95a) for (1.1) and (1.2), respectively. For $K = |C|^2 = 0$, the orbit is homoclinic to $A_+ = 0$. For small non-zero $|K|$, $A_+ \sim -2c_* K/\bar{\nu}$, meaning that oscillations at infinity are then very small in this case. For $K = 0$ this corresponds to an *orbit homoclinic to* 0 for the normal form. This is indeed valid for the normal form taken at any order. However this solution does not exist mathematically for the full original system, even though one may compute its expansion in powers of the bifurcation parameter up to any order (see [17] and [18]). This is an example of the famous challenging problem of asymptotics beyond any orders. Other solutions found on the normal form mainly persist under the perturbation from higher order terms provided by the original system [16]. These solutions are delocalized waves and their existence in Region 2 is guaranteed by the general theory for reversible systems in [17] and [18]. Also, as mentioned in Section 2.2, genuine solitary waves are found on isolated curves in Region 2 of Figure 1 on which the oscillation amplitudes vanish. Since these are embedded in the sea of delocalized solitary waves and in the continuous spectrum, they are referred to as embedded solitons [19]. These will further be investigated in Region 2 subsequently using a mix of exponential asymptotics and numerical shooting.
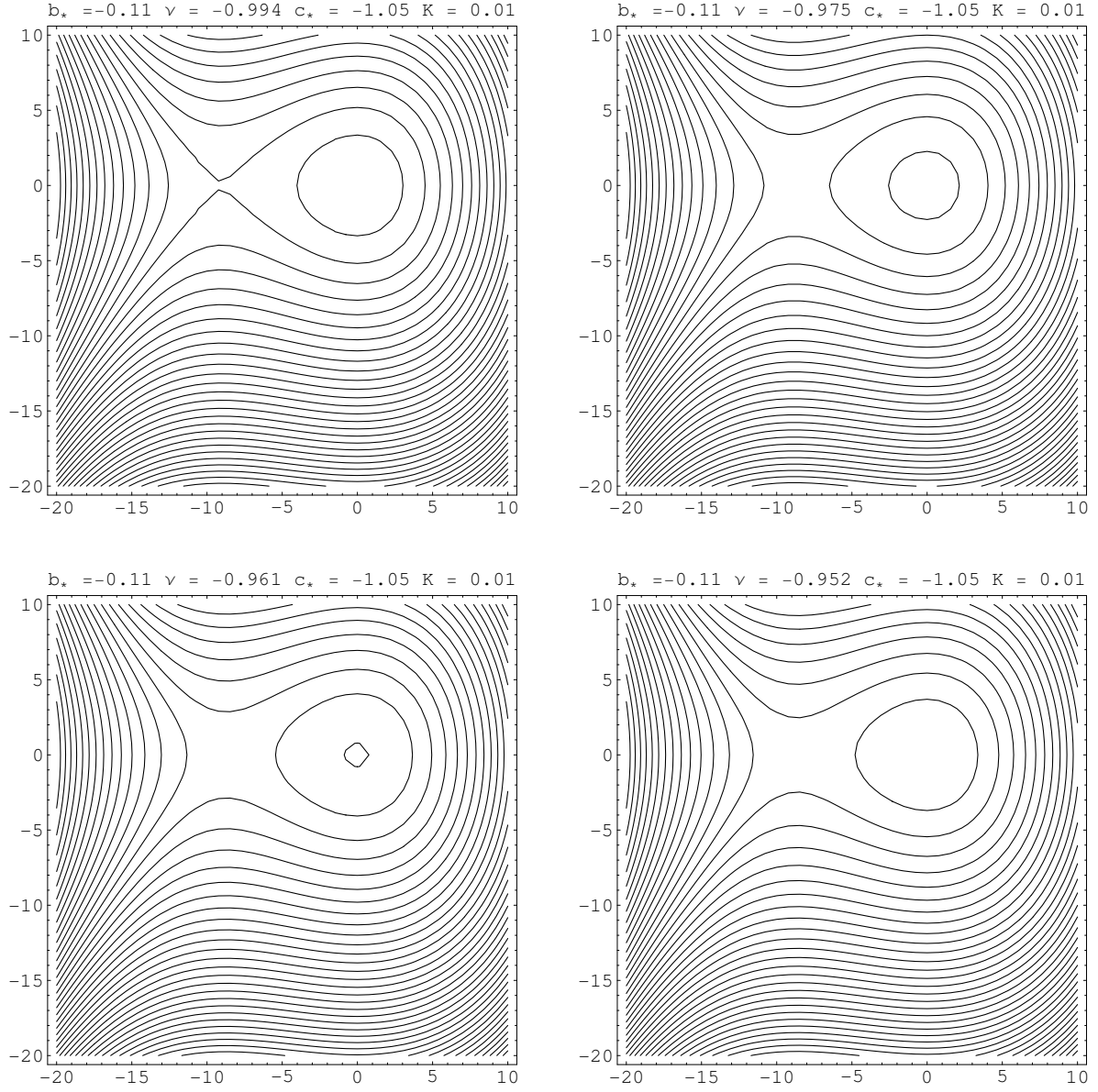
Figure 2.1: Level curves of (2.97) corresponding to various values of H.

CHAPTER THREE: A MICROSTRUCTURE PDE

3.1   INTRODUCTION

One dimensional wave propagation in microstructured solids is currently a topic of great interest. This phenomenon has recently been modeled [20] by an equation

$$v_{tt} - bv_{xx} - \frac{\mu}{2} \left(v^2\right)_{xx} - \delta \left(\beta v_{tt} - \gamma v_{xx}\right)_{xx} = 0 \qquad (3.1)$$

with complicated dispersive and nonlinear terms. Here b, μ, β, δ and γ are dimensionless parameters, v denotes the macrodeformation, and x and t denote space and time coordinates respectively.

Equation (3.1) is derived, using the so-called Mindlin Model, in [20], [21], [22]. It is non-integrable. However, analytic conditions for the existence of solitary waves of (3.1) have been derived in [22] and [20]. These references also numerically construct asymmetric solitary wave solutions of the form $v(x - ct)$ of (3.1). More recently ([23],[24]) pulse trains in (3.1) have been numerically constructed.

3.2   SOLITARY WAVES; LOCAL BIFURCATIONS

Solitary waves of (3.1) of the form $v(x,t) = \phi(x - ct) = \phi(z)$ satisfy the fourth-order traveling wave ODE

$$\phi_{zzzz} - q\phi_{zz} + p\phi = \mathcal{N}[\phi] \qquad (3.2)$$

where

$$\mathcal{N}[\phi] = -\Delta_1 \phi_z^2 - b\Delta_1 \phi\phi_{zz} \tag{3.3}$$

and

$$z \equiv x - ct \tag{3.4a}$$

$$p \equiv 0 \tag{3.4b}$$

$$q \equiv \frac{c^2 - b}{\delta(\beta c^2 - \gamma)} \tag{3.4c}$$

$$\Delta_1 \equiv \frac{\mu}{\delta(\beta c^2 - \gamma)} \tag{3.4d}$$

Equation (3.2) is invariant under the transformation $z \mapsto -z$ and is thus a reversible system. In this section we shall use the theory of reversible systems to characterize the homoclinic orbits to the fixed point of (3.2), which correspond to pulses or solitary waves of (3.1) in various regions of the (p, q) plane.

The linearized system corresponding to (3.2)

$$\phi_{zzzz} - q\phi_{zz} + p\phi = 0 \tag{3.5}$$

has a fixed point

$$\phi = \phi_z = \phi_{zz} = \phi_{zzz} = 0 \tag{3.6}$$

Solutions $\phi = ke^{\lambda x}$ satisfy the characteristic equation $\lambda^4 - q\lambda^2 + p = 0$ from which one may deduce that the structure of the eigenvalues is distinct in two regions of (p, q)-space. Since $p = 0$ we have only two possible regions of eigenvalues. We denote C_0 as the positive q axis and C_1 the negative q-axis. First we shall consider the bounding curves C_0 and C_1

30

and their neighborhoods, then we shall discuss the possible occurrence and multiplicity of homoclinic orbits to (3.6), corresponding to pulse solitary waves of (3.1), in each region:

Near C_0 The eigenvalues have the structure $\lambda_{1-4} = 0, 0, \pm\lambda$, $(\lambda \in \mathbb{R})$ and the fixed point (2.5) is a saddle-focus.

Near C_1 Here the eigenvalues have the structure $\lambda_{1-4} = 0, 0, \pm i\omega$, $(\omega \in \mathbb{R})$. We will show by analysis of a four-dimensional normal form in Section 4 that there exists a sech^2 homoclinic orbit near C_1.

Having outlined the possible families of orbits homoclinic to the fixed point (3.6) of (3.5), corresponding to pulse solitary waves of (3.1), we now derive normal forms near the transition curves C_0 and C_1 to confirm the existence of regular or delocalized solitary waves in the corresponding regions of (p, q) parameter space.

3.3   NORMAL FORM NEAR C_0: SOLITARY WAVE SOLUTIONS

Using (3.5), the curve C_0, corresponding to $\lambda = 0, 0, \pm\tilde{\lambda}$, is given by

$$C_0 : p = 0, q > 0 \tag{3.7}$$

Using (3.4c) implies

$$\frac{c^2 - b}{\delta\left(\beta c^2 - \gamma\right)} > 0 \tag{3.8}$$

Denoting ϕ by y_1, equation (3.2) may be written as the system

$$\frac{dy_1}{dz} = y_2 \tag{3.9a}$$

$$\frac{dy_2}{dz} = y_3 \tag{3.9b}$$

$$\frac{dy_3}{dz} = y_4 \tag{3.9c}$$

$$\frac{dy_4}{dz} = qy_3 - py_1 - \left(\Delta_1 y_2^2 + b\Delta_1 y_1 y_3\right) \tag{3.9d}$$

We wish to rewrite this as a first order reversible system in order to invoke the relevant theory [15]. To that end, defining $Y = \langle y_1, y_2, y_3, y_4 \rangle^T$, equation (3.9) may be written

$$\frac{dY}{dz} = L_{pq}Y - F_2(Y, Y) \tag{3.10}$$

where

$$L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2 - p & 0 & q/3 & 0 \end{pmatrix} \tag{3.11}$$

Since $p = 0$ for (3.1), we have

$$\frac{dY}{dz} = L_{0q}Y - F_2(Y, Y) \tag{3.12}$$

where

$$F_2(Y, Y) = \langle 0, 0, 0, \Delta_1 y_2^2 + b\Delta_1 y_1 y_3 \rangle^T \tag{3.13}$$

Next we calculate the normal form of (3.12) near C_0. The procedure is closely modeled on [15] and many intermediate steps may be found there.

3.3.1 Near C_0

Near C_0 the dynamics reduce to a two-dimensional Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B) \tag{3.14}$$

and the corresponding normal form is

$$\frac{dA}{dz} = B \tag{3.15a}$$

$$\frac{dB}{dz} = b\epsilon A + \tilde{c}A^2 \tag{3.15b}$$

Here,

$$\epsilon = \left(\frac{q^2}{9} - p\right) - \left(\frac{q}{3}\right)^2 = -p \tag{3.16}$$

measures the perturbation around C_0, and

$$\zeta_0 = \langle 1, 0, -q/3, 0 \rangle^T \tag{3.17a}$$

$$\zeta_1 = \langle 0, 1, 0, -2q/3 \rangle^T \tag{3.17b}$$

The linear eigenvalue of (3.15) satisfies

$$\lambda^2 = b\epsilon \tag{3.18}$$

The characteristic equation of the linear part of (3.12) is

$$\lambda^4 - q\lambda^2 - \epsilon = 0 \tag{3.19}$$

Hence, the eigenvalues near zero (the Center Manifold) satisfy $\lambda^4 \ll \lambda^2$ and hence

$$\lambda^2 \sim -\frac{\epsilon}{q} \tag{3.20}$$

Matching (3.18) and (3.20)

$$b = -\frac{1}{q} \tag{3.21}$$

and only the nonlinear coefficient $\tilde{c}$ remains to be determined in the normal form (3.15).

In order to determine $\tilde{c}$ (the coefficient of A^2 in (3.15)) we calculate $\frac{dY}{dz}$ in two ways and match the $\mathcal{O}(A^2)$ terms. To this end, using the standard 'suspension' trick of treating the perturbation parameter ϵ as a variable, we expand the function Ψ in (3.14) as

$$\Psi(\epsilon, A, B) = \epsilon A \Psi_{10}^1 + \epsilon B \Psi_{01}^1 + A^2 \Psi_{20}^0 + AB \Psi_{11}^0 + B^2 \Psi_{02}^0 + \cdots \tag{3.22}$$

where the subscripts denote powers of A and B, respectively, and the superscript denotes the power of ϵ.

In the first way of computing dY/dz, we take the z derivative of (2.22)

$$\frac{dY}{dz} = B\zeta_0 + \left(b\epsilon A + \tilde{c}A^2\right)\zeta_1 + \frac{d\Psi}{dz} \tag{3.23}$$

And finally using (3.22) and (3.15) again we arrive at

$$\frac{dY}{dz} = B\zeta_0 + \left(b\epsilon A + \tilde{c}A^2\right)\zeta_1 + \left(b\epsilon^2 A + \epsilon\tilde{c}A^2\right) \tag{3.24}$$

$$+ 2AB\Psi_{20}^0 + \left(B^2 + b\epsilon A^2 + \tilde{c}A^3\right)\Psi_{11}^0$$

$$+ 2B\left(b\epsilon A + \tilde{c}A^2\right)\Psi_{02}^0 + \cdots$$

The coefficient of A^2 in the resulting expression is $\tilde{c}\zeta_1$. In the second way of computing dY/dz, we use (3.14) and (3.22) in (3.12). The coefficient of A^2 in the resulting expression is $L_{0,q}\Psi_{20}^0 - F_2(\zeta_0, \zeta_0)$, which leads to

$$\tilde{c}\zeta_1 = L_{0q}\Psi_{20}^0 - F_2(\zeta_0, \zeta_0) \tag{3.25}$$

Using (3.17) and (3.13) and denoting $\Psi^0_{20} = \langle x_1, x_2, x_3, x_4 \rangle$ in (3.25) yields the equations

$$0 \;=\; x_2 \tag{3.26a}$$

$$\tilde{c} \;=\; \frac{q}{3} x_1 + x_3 \tag{3.26b}$$

$$0 \;=\; \frac{q}{3} x_2 + x_4 \implies x_4 = 0 \text{ using (3.26b)} \tag{3.26c}$$

and

$$-\frac{2q}{3}\tilde{c} = \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) + \frac{2q}{3} = \frac{q}{3}\tilde{c} + \frac{b\Delta_1}{3} \text{ using (3.26b)} \tag{3.27}$$

Hence we obtain

$$\tilde{c} = -\frac{b\Delta_1}{3} \tag{3.28}$$

Therefore, the normal form for (3.1) near C_0 is

$$\frac{dA}{dz} \;=\; B \tag{3.29a}$$

$$\frac{dB}{dz} \;=\; -\frac{\epsilon}{q}A - \frac{b\Delta_1}{3}A^2 \tag{3.29b}$$

The normal form (3.29) admits a homoclinic solution (near C_0) of the form

$$A(z) = \ell \operatorname{sech}^2(kz) \tag{3.30}$$

To determine k and ℓ we put (3.30) into the equivalent second order equation for (3.29), $\frac{dA^2}{dz^2} = -\frac{\epsilon}{q}A - \frac{b\Delta_1}{3}A^2$. which implies

$$\ell\left(-2k^2\operatorname{sech}^4(kz) + 4k^2\operatorname{sech}^2(kz)\tanh^2(kz)\right) = -\frac{\epsilon}{q}\ell\operatorname{sech}^2(kz) - \frac{b\Delta_1}{3}\ell^2\operatorname{sech}^4(kz)$$

$$-2k\left(-2k\left(\operatorname{sech}^2(kz) - \operatorname{sech}^4(kz)\right) + k\operatorname{sech}^4(kz)\right) = -\frac{\epsilon}{q}\operatorname{sech}^2(kz) - \frac{b\Delta_1}{3}\ell\operatorname{sech}^4(kz)$$

where we have used the same hyperbolic identities as in Chapter 2. Matching the $\mathcal{O}(\mathrm{sech}^2(kz))$

and $\mathcal{O}(\mathrm{sech}^2(kz))$ terms on each side of the preceding equation implies

$$4k^2 = \frac{-\epsilon}{4q} \implies k = \sqrt{\frac{-\epsilon}{4q}} \tag{3.32a}$$

$$2k^2 = \frac{b\Delta_1}{3}\ell \implies \ell = \frac{6k^2}{b\Delta_1} \tag{3.32b}$$

Hence, since $\epsilon = -p$, and the curve C_0 corresponds to $p = 0, q > 0$, solitary waves of the

form (3.30) exist in the vicinity of C_0 for

$$p > 0, q > 0 \tag{3.33}$$

which implies that $\frac{c^2-b}{\delta(\beta c^2-\gamma)} > 0$ (such that k is real.) As mentioned in section 2, one may

show the persistence of this homoclinic solution in the original traveling wave ODE (3.5).

Thus, we have demonstrated the existence of solitary waves of (3.1) for $p = 0^+, q > 0$.

3.4   NORMAL FORM NEAR C_1: POSSIBLE SOLITARY WAVE

SOLUTIONS

Using (3.5), the curve C_1, corresponding to $\lambda = 0, 0 \pm i\omega$, is given by

$$C_1 : p = 0, q < 0 \tag{3.34}$$

Which implies

$$\frac{c^2 - b}{\delta\left(\beta c^2 - \gamma\right)} < 0 \tag{3.35}$$

In order to investigate the possibility of a sech^2 homoclinic orbit in the neighborhood of

C_1 and delocalized solitary waves, we next compute the normal form near C_1 following the

procedure in [15].

Near C_1 the dynamics reduce to a four-dimensional Center Manifold [15]. Since all the eigenvalues are non-hyperbolic, the Center Manifold has the form (a nonlinear coordinate change [15])

$$Y = A\zeta_0 + B\zeta_0 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C}) \tag{3.36}$$

with a corresponding four-dimensional normal form

$$\frac{dA}{dz} = B \tag{3.37a}$$

$$\frac{dB}{dz} = \bar{\nu}A + b_*A^2 + c_* |C|^2 \tag{3.37b}$$

$$\frac{dC}{dz} = id_0C + i\bar{\nu}d_1C + id_2AC \tag{3.37c}$$

Here C is complex, $\bar{C}$ is the complex conjugate of C, $\epsilon, \zeta_0, \zeta_1$ are given previously and the two new complex eigenvectors co-spanning the Center Manifold are

$$\zeta_\pm = \left\langle 1, \lambda_\pm, 2q/3, \frac{\lambda_\pm}{3}q \right\rangle^T \tag{3.38}$$

Using (3.37b) and (3.15b)

$$\bar{\nu} = b\epsilon = -\frac{\epsilon}{q} \tag{3.39}$$

Also from the characteristic equation (3.19), the two non-zero (imaginary) roots are

$$\lambda^2 = \frac{q + \sqrt{q^2 + 4\epsilon}}{2} \approx q \text{ for } \epsilon \text{ small} \tag{3.40}$$

Hence

$$\lambda = \pm i\sqrt{-q}, q < 0 \tag{3.41}$$

37

Matching this to the linear part of (3.37c) (which corresponds to the imaginary eigenvalues), $\lambda = id_0 = i\sqrt{-q}$ or

$$d_0 = \sqrt{-q} \tag{3.42}$$

With a dominant balance argument on the characteristic equation (3.19) as $\lambda \to \pm i\sqrt{-q}$ we find

$$d_1 = \frac{\sqrt{-q}}{2q^2} \tag{3.43}$$

The remaining undetermined coefficients in the normal form are the coefficients b_*, c_* and d_2 which correspond to the $A^2, |C|^2$ and AC terms respectively. In order to determine them, we follow the same procedure as in Section 3 and compute dY/dz is two distinct ways. We expand the function Ψ as

$$\Psi(\epsilon, A, B, C, \bar{C}) = \epsilon A \Psi^1_{1000} + \epsilon B \Psi^1_{0100} + A^2 \Psi^0_{2000} + AB \Psi^0_{1100} + AC \Psi^0_{1010} + \epsilon C \Psi^1_{0010} + \cdots \tag{3.44}$$

with subscripts denoting powers of A, B, C and $\bar{C}$, respectively, and the superscript is the power of ϵ. In the first way, dY/dz is computed by taking the z derivative of (3.36) (using (3.37) and (3.44)) and read off the coefficients of $A^2, \|C\|^2, C\epsilon$ and AC terms. In the second way, dY/dz is computed using (3.36) and (3.44) in (3.12) (with $p = 0$ on C_1 as given in (3.34)) and the coefficients of A, B, C and $\bar{C}$ are once again read off. Equating the coefficients of the corresponding terms in the two separate expressions for dY/dz yields the

following equations:

$$\mathcal{O}(A^2): \qquad b_*\zeta_1 \qquad = L_{0q}\Psi^0_{2000} - F_2(\zeta_0, \zeta_0) \tag{3.45a}$$

$$\mathcal{O}(|C|^2): \qquad c_*\zeta_1 \qquad = L_{0q}\Psi^0_{0011} - 2F_2(\zeta_+, \zeta_-) \tag{3.45b}$$

$$\mathcal{O}(\epsilon C): \quad -\frac{i}{q}\left(d_1\zeta_+ + d_0\Psi^1_{0010}\right) \quad = L_{0q}\Psi^1_{0010} \tag{3.45c}$$

$$\mathcal{O}(AC): \qquad id_2\zeta_+ + id_0\Psi^0_{1010} \qquad = L_{0q}\Psi^0_{1010} - 2F_2(\zeta_0, \zeta_+) \tag{3.45d}$$

where we have used the fact that $F_2(Y,Y)$ is a symmetric bilinear form. We would like to evaluate $F_2(\zeta_+, \zeta_-)$, so we first exhibit the form of $F_2(X,Y)$, (which is linear and symmetric with respect to both arguments)

$$F_2(X,Y) = \left\langle 0, 0, 0, \Delta_1 x_2 y_2 + \frac{b}{2}\Delta_1\left(x_1 y_2 + y_1 x_3\right)\right\rangle^T \tag{3.46}$$

We now use this to evaluate $F_2(X,Y)$

$$F_2(\zeta_+, \zeta_-) \;=\; \left\langle 0, 0, 0, \Delta_1(-q) + \frac{b}{2}\Delta_1\left(\frac{2q}{3} + \frac{2q}{3}\right)\right\rangle^T \tag{3.47a}$$

$$\;=\; \left\langle 0, 0, 0, q\Delta_1\left(\frac{2b}{3} - 1\right)\right\rangle^T \tag{3.47b}$$

Using this in (3.45b) implies

$$0 \;=\; x_2 \tag{3.48a}$$

$$c_* \;=\; \frac{q}{3}x_1 + x_3 \tag{3.48b}$$

$$0 \;=\; \frac{q}{3}x_2 + x_4 \tag{3.48c}$$

$$-\frac{2q}{3}c_* \;=\; \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) - 2q\Delta_1\left(\frac{2b}{3} - 1\right) \tag{3.48d}$$

Using the second equation in the fourth implies

$$c_* = 2\Delta_1\left(\frac{2b}{3} - 1\right) \tag{3.49}$$

The only coefficient left to determine is d_2 which we shall compute now. First, we will need to compute $F_2(\zeta_0, \zeta_+)$ from $F_2(X, Y)$ which gives

$$
\begin{aligned}
F_2(\zeta_0, \zeta_+) &= \left\langle 0, 0, 0, 0 + \frac{b}{2}\Delta_1\left(\frac{2q}{3} - \frac{q}{3}\right) \right\rangle^T \\
&= \left\langle 0, 0, 0, \frac{b}{6}q\Delta_1 \right\rangle^T
\end{aligned}
$$

Using $\Psi^0_{1010} = \langle x_1, x_2, x_3, x_4 \rangle^T$ in (3.45d) implies

$$
id_2 + id_0 x_1 = x_2 \tag{3.51a}
$$

$$
-d_0 d_2 + id_0 x_2 = \frac{q}{3}x_1 + x_3 \tag{3.51b}
$$

$$
\frac{2iq}{3}d_2 + id_0 x_3 = \frac{q}{3}x_2 + x_4 \tag{3.51c}
$$

$$
-\frac{q}{3}d_0 d_2 + id_0 x_4 = \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) - \frac{bq\Delta_1}{3} \tag{3.51d}
$$

Using (3.51a) in (3.51b) we find

$$
-d_0 d_2 + id_0 \left(id_2 + id_0 x_1\right) = \frac{q}{3}x_1 + x_3 \tag{3.52a}
$$

$$
-2d_0 d_2 + qx_1 = \frac{q}{3}x_1 + x_3 \tag{3.52b}
$$

$$
\implies x_3 = \frac{2q}{3}x_1 - 2d_0 d_2 \tag{3.52c}
$$

Similarly, using (3.51b) in (3.51d) we find

$$
x_4 = \frac{q}{3}x_2 + ib\Delta_1 \frac{q}{3d_0} \tag{3.53}
$$

Using the preceding two equations and (3.51a) in (3.51c) implies

$$\frac{2iq}{d\,_2} + id_0\left(\frac{2q}{3}x_1 - 2d_0d_2\right) = \frac{q}{3}\left(id_2 + id_0x_1\right) + \frac{q}{3}\left(id_2 + id_0x_1\right) + ib\Delta_1\frac{q}{3d_0} \quad (3.54a)$$

$$\frac{2iq}{3}d_2 - 2id_0^2d_2 = i\frac{q}{3}d_2 + i\frac{q}{3}d_2 + ib\Delta_1\frac{q}{3d_0} \quad (3.54b)$$

$$-2d_0^2d_2 = \frac{b\Delta_1 q}{3d_0} \quad (3.54c)$$

$$2qd_2 = \frac{b\Delta_1 q}{3d_0} \quad (3.54d)$$

$$d_2 = \frac{b\Delta_1}{6\sqrt{-q}} \quad (3.54e)$$

since $d_0^2 = -q$.

Therefore the normal form for (3.1) near C_1 is

$$\frac{dA}{dz} = B \quad (3.55a)$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{b\Delta_1}{3}A^2 + 2\Delta_1\left(\frac{2b}{3} - 1\right)|C|^2 \quad (3.55b)$$

$$\frac{dC}{dz} = i\sqrt{-q}C - i\frac{\sqrt{-q}}{q^3}C\epsilon + i\frac{b\Delta_1}{6\sqrt{-q}}AC \quad (3.55c)$$

The dynamics inherent in (3.55) may be elucidated following the discussions of [15], [16], [17] and [18]. The two first integrals of (3.37) are

$$K = |C|^2 \quad (3.56)$$

and

$$H = B^2 - \frac{2}{3}b_*A^3 - \bar{\nu}A^2 - 2c_*KA \quad (3.57)$$

Also, c_* should be real for the following energy arguments to apply. As a typical case, consider the level curve $H = 0$ of the energy-like first integral function H. In the (A, B)

41

phase plane, this will compromise a homoclinic orbit. The intersection of $H = 0$ with the A axis occurs for $\frac{2}{3}b_*A^2 - \bar{\nu}A - 2c_*K = 0$ or

$$A_{\mp} = \frac{3}{4b_*}\left[\bar{\nu} \pm \sqrt{\bar{\nu}^2 + \frac{16b_*c_*K}{3}}\right] \tag{3.58}$$

Note that $A_+ > 0$, $A_- < 0$ for $b_*c_* > 0$ and $b_* < 0$ as relevant for us. A general homoclinic orbit, homoclinic to A_+, is sketched in Figure 1 where the flow direction is deduced from (3.55a). For $K = |C|^2 = 0$, the orbit is homoclinic to $A_+ = 0$. For small non-zero $|K|$, $A_+ \sim -2c_*K/\bar{\nu}$, meaning that oscillations at infinity are then very small in this case. For $K = 0$ this corresponds to an *orbit homoclinic to* 0 for the normal form. This is indeed valid for the normal form taken at any order. However this solution does not exist mathematically for the full original system, even though one may compute its expansion in powers of the bifurcation parameter up to any order (see [17] and [18]). This is an example of the famous challenging problem of asymptotics beyond any orders. Other solutions found on the normal form mainly persist under the perturbation from higher order terms provided by the original system [16]. These solutions are delocalized waves and their existence in Region 2 is guaranteed by the general theory for reversible systems in [17] and [18]. Also, as mentioned in Section 2, genuine solitary waves are found on isolated curves in Region 2 of Figure 1 on which the oscillation amplitudes vanish. Since these are embedded in the sea of delocalized solitary waves and in the continuous spectrum, they are referred to as embedded solitons [19]. These will further be investigated in Region 2 subsequently using a mix of exponential asymptotics and numerical shooting.
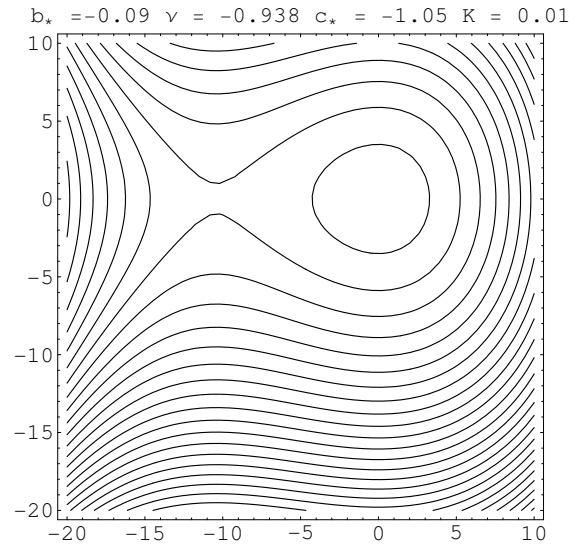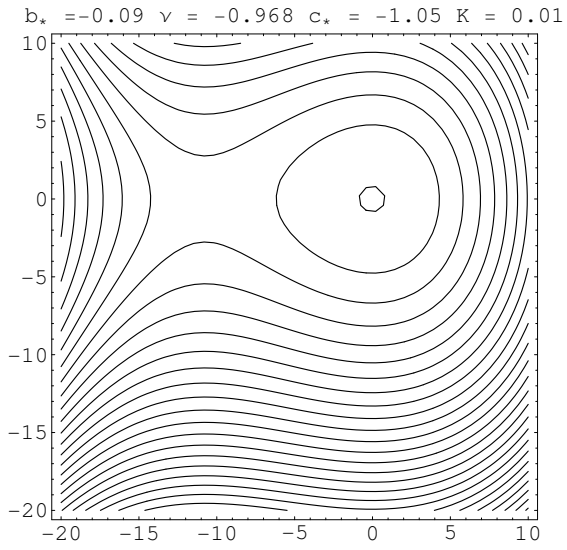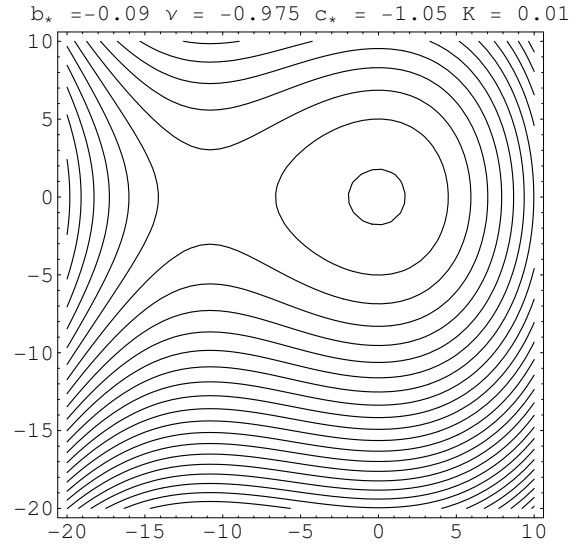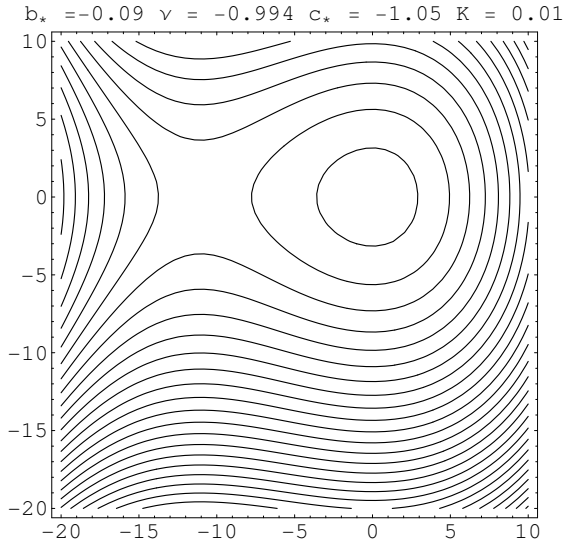
Figure 3.1: Level curves of (3.57) corresponding to various values of H.

CHAPTER FOUR: RESULTS

In this thesis, we apply a recently developed technique to comprehensively categorize all possible families of solitary wave solutions in two models of topical interest.

The models considered are:

- the Generalized Pochhammer-Chree Equations, which govern the propagation of longitudinal waves in elastic rods,

  and

- a generalized microstructure PDE.

Limited analytic results exist for the occurrence of one family of solitary wave solutions for the Microstructure equation and results using a Hamiltonian formulation have recently been found in the Pochammer-Chree equations. Since, as mentioned above, solitary wave solutions often play a central role in the long-time evolution of an initial disturbance, we consider such solutions of both models here (via the normal form approach) within the framework of reversible systems theory.

Besides confirming the existence of the known family of solitary waves for each model, of the form

$$A(z) = \ell \text{sech}^2(kz) \tag{4.1}$$

we find a continuum of delocalized solitary waves (or homoclinics to small-amplitude periodic orbits). On isolated curves in the relevant parameter region, the delocalized waves reduce

to genuine embedded solitons.

For the Microstructure equation , the new family of solutions occur in regions of parameter space distinct from the known solitary wave solutions and are thus entirely new.

Directions for future work, including the dynamics of each family of solitary waves using exponential asymptotics techniques, are also mentioned.

LIST OF REFERENCES

[1] Zabusky, N.J., & Kruskal, M.D. Interactions of 'solitons' in a collisionless plasma and the recurrence of initial states. *Phys. Rev. Lett.*, 15:240–243, 1965.

[2] C.S. Gardner, J.M. Greene, M.D. Kruskal, R.M. Miura. Method for Solving the Kortewege-deVries Equation. *Phys. Rev. Lett*, pages 1095–1097, 1967.

[3] Peter D. Lax. Integrals of Nonlinear Equations of Evolution and Solitary Waves. *Comm. Pure Appl. Math.*, 21:467–490, 1968.

[4] Zakharov, V.E., Shabat, A.B. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Soviet Physics JETP*, 34:62–69, 1972.

[5] P.G. Drazin & R.S. Johnson. *Solitons: an introduction.* Cambridge University Press, Cambridge, 1989.

[6] Ablowitz, M.J., Kaup, D.J., Newell, A.C & Segur, H. The Inverse Scattering Transform - Fourier analysis for nonlinear problem. *Stud. Appl. Math.*, 53:2490315, 1974.

[7] Steven H. Strogatz. *Nonlinear Dynamics and Chaos.* Westview, Cambridge, MA, 2000.

[8] J. Li, L. Zhang. Bifurcations of traveling wave solutions in generalized Pochhammer-Chree Equations. *Chaos, Solitons and Fractals*, 14:581593, 2002.

[9] B. Li, Y. Chen and H. Zhang. Travelling Wave Solutions for Generalized Pochammer-Chree Equations. *Z. Naturforsch.*, 57a:874–882, 2002.

[10] X. Runzhang. Initial boundary value problem of two classes of evolution equations. In *Fifth IMACS Conf. on Nonlinear Evolution Equations*, April 2007.

[11] Z. Weiguo and W. X. Ma. Explicit Solitary-Wave Solutions to Generalized Pochammer-Chree Equations. *Appl. Math and Mech.*, 20:666–674, 1999.

[12] I.L. Bogolubsky. Some Examples of Inelastic Soliton Interaction. *Comp. Phys.*, 31:149–155, 1977.

[13] P.A. Clarkson, R.J. Le Veque and R. Saxton. Solitary-wave Interactions in Elastic Rods. *Stud. App. Math.*, 75:95–122, 1986.

[14] R. Saxton. Existence of Solutions for a Finite Nonlinearly Hyperelastic Rod. *J. Math. Anal. Appl.*, 105:59–75, 1985.

[15] Iooss G., Adelmeyer, M. *Topics in Bifurcation Theory and Applications.* World Scientific, Singapore, 1998.

[16] Iooss G, Kirchgassner K. Water waves for small surface tension: an approach via normal form. *Proc Roy Soc Edinburgh A*, 112:62–88, 1992.

[17] Lombardi E. Homoclinic orbits to small periodic orbits for a class of reversible systems. *Proc Roy Soc Edinburgh A*, 126:1035–54, 1996.

[18] Lombardi E. Homoclinic orbits to exponentially small periodic orbits for a class of reversible systems: Application to water waves. *Arch Rat Mech Anal*, 137:227–304, 1997.

[19] Champneys A.R., Malomed B.A., Yang J., Kaup D.J. Embedded Solitons: solitary waves in resonance with the linear spectrum. *Physica D*, 340:152–153, 2001.

[20] A. Salupere, K. Tamm, J. Engelbrecht. Interaction of solitary deformation waves in microstructured solids. In *Fifth IMACS Conf. on Nonlinear Evolution Equations*, April 2007.

[21] J. Janno, J. Engelbrecht. Solitary waves in nonlinear microstructured materials. *J. Phys. A: Math. Gen.*, 38:5159–5172, 2005.

[22] J. Janno, J. Engelbrecht. An inverse solitary wave problem related to microstructured materials. *Inverse Problems*, 21:2019–2034, 2005.

[23] J. Engelbrecht, F. Pastrone. Waves in microstructured solids with nonlinearities in microscale. *Phil. Mag.*, 85:4127–41, 2005.

[24] J. Engelbrecht, A. Berezovski, A. Salupere. Nonlinear deformation waves in solids and dispersion. *Wave Motion*, 2007.