SOLITARY WAVE FAMILIES IN TWO NON-INTEGRABLE MODELS
USING REVERSIBLE SYSTEMS THEORY

by

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ABSTRACT

In this thesis, we apply a recently developed technique to comprehensively categorize all possible families of solitary wave solutions in two models of topical interest.

The models considered are:

- the Generalized Pochammer-Chree Equations, which govern the propagation of longitudinal waves in elastic rods,

  and

- generalized microstructure PDE.

Limited analytic results exist for the occurrence of one family of solitary wave solutions for each of these equations. Since, as mentioned above, solitary wave solutions often play a central role in the long-time evolution of an initial disturbance, we consider such solutions of both models here (via the normal form approach) within the framework of reversible systems theory.

Besides confirming the existence of the known family of solitary waves for each model, we find a continuum of delocalized solitary waves (or homoclinics to small-amplitude periodic orbits). On isolated curves in the relevant parameter region, the delocalized waves reduce to genuine embedded solitons. For both models, the new family of solutions occur in regions of parameter space distinct from the known solitary wave solutions and are thus entirely new.

Directions for future work, including the dynamics of each family of solitary waves using exponential asymptotics techniques, are also mentioned.

Dedicated To My Family

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CHAPTER ONE: INTRODUCTION

Solitary wave solutions of nonlinear models have become increasingly important, both as possible information carriers, as well as organizing centers for the solution dynamics in regimes where the initial conditions naturally break into stable pulses or pulse-trains. Standard techniques for investigating solitary waves of integrable nonlinear PDEs, such as the Inverse Scattering Transform, do not carry over to the non-integrable models which are of increasing relevance in modern applications. Other techniques which have been devised, such as variational ones, and exponential asymptotics methods, each yield results in certain regimes of the systems parameters.

In this thesis, we apply a recently developed technique to comprehensively categorize all possible families of solitary wave solutions in two models of topical interest.

The models considered are:

- the Generalized Pochammer-Chree Equations, which govern the propagation of longitudinal waves in elastic rods,

$$(u - u_{xx})_{tt} - \left(a_1 u + a_2 u^2 + a_3 u^3\right)_{xx} = 0 \tag{1.1}$$

  and

$$(u - u_{xx})_{tt} - \left(a_1 u + a_3 u^3 + a_5 u^5\right)_{xx} = 0 \tag{1.2}$$

- a generalized microstructure PDE.

$$v_{tt} - b v_{xx} - \frac{\mu}{2} \left(v^2\right)_{xx} - \delta \left(\beta v_{tt} - \gamma v_{xx}\right)_{xx} = 0 \tag{1.3}$$

Limited analytic results exist for the occurrence of one family of solitary wave solutions for each of these equations. Since, as mentioned above, solitary wave solutions often play a central role in the long-time evolution of an initial disturbance, we consider such solutions of both models here (via the normal form approach) within the framework of reversible systems theory.

Besides confirming the existence of the known family of solitary waves for each model, we find a continuum of delocalized solitary waves (or homoclinics to small-amplitude periodic orbits). On isolated curves in the relevant parameter region, the delocalized waves reduce to genuine embedded solitons. For both models, the new family of solutions occur in regions of parameter space distinct from the known solitary wave solutions and are thus entirely new.

Directions for future work, including the dynamics of each family of solitary waves using exponential asymptotics techniques, are also entioned

CHAPTER TWO: GENERALIZED POCHAMMER-CHREE EQUATIONS

2.1 Introduction

The propagation of longitudinal deformation waves in elastic rods is governed ([1], [2], [3]) by (1.1) and (1.2), corresponding to different constitutive relations.

References [1], [2], [3] also discuss the primary references, including derivations and applications of these equations in various fields. In addition, motivated by experimental and numerical results, there are derivations of special families of solitary wave solutions by the extended $Tanh$ method [1], and other ansatzen [3]. These extend earlier solitary wave solutions given by Bogolubsky [4] and Clarkson et. al [5] for special cases of (1.1) and (1.2). In addition, [2] generalizes the existence results in [6] for solitary waves of (1.1) and (1.2).

2.2 Solitary waves: local bifurcation

Solitary waves of (1.1) and (1.2) of the form $v(x,t) = \phi(x - ct) = \phi(z)$ satisfy the fourth-order traveling wave ODE

$$\phi_{zzzz} - q\phi_{zz} + p\phi = \mathcal{N}_{1,2}[\phi] \tag{2.1}$$

where

$$\mathcal{N}_1[\phi] = -\frac{1}{c^2}\left[3a_3\left(2\phi\phi_z^2 + \phi^2\phi_{zz}\right) + 2a_2\left(\phi_{zz}\phi_z + \phi_z^2\right)\right] \tag{2.2a}$$

$$\mathcal{N}_2[\phi] = -\frac{1}{c^2}\left[3a_3\left(2\phi\phi_z^2 + \phi^2\phi_{zz}\right) + 5a_5\left(4\phi^3\phi_z^2 + \phi^4\phi_{zz}\right)\right] \tag{2.2b}$$

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$$z \equiv x - ct \tag{2.3a}$$

$$p \equiv 0 \tag{2.3b}$$

$$q \equiv 1 - \frac{a_1}{c^2} \tag{2.3c}$$

Equation (2.1) is invariant under the transformation $z \mapsto -z$ and is thus a reversible system. In this section we shall use the theory of reversible systems to characterize the homoclinic orbits to the fixed point of (2.1), which correspond to pulses or solitary waves of (1.1) and (1.2) in various regions of the (p, q) plane.

The linearized system corresponding to (2.1)

$$\phi_{zzzz} - q\phi_{zz} + p\phi = 0 \tag{2.4}$$

has a fixed point

$$\phi = \phi_z = \phi_{zz} = \phi_{zzz} = 0 \tag{2.5}$$

Solutions $\phi = ke^{\lambda x}$ satisfy the characteristic equation $\lambda^4 - q\lambda^2 + p = 0$ from which one may deduce that the structure of the eigenvalues is distinct in two regions of (p, q)-space. Since $p = 0$ we have only two possible regions of eigenvalues. We denote C_0 as the positive q axis and C_1 the negative q-axis. First we shall consider the bounding curves C_0 and C_1 and their neighborhoods, then we shall discuss the possible occurrence and multiplicity of homoclinic orbits to (2.5), corresponding to pulse solitary waves of (1.1) and (1.2), in each region:

Near C_0 The eigenvalues have the structure $\lambda_{1-4} = 0, 0, \pm\lambda$, $(\lambda \in \mathbb{R})$ and the fixed point (2.5) is a saddle-focus.

Near C_1 Here the eigenvalues have the structure $\lambda_{1-4} = 0, 0, \pm i\omega$, $(\omega \in \mathbb{R})$. We will show by analysis of a four-dimensional normal form in Section 2.4 that there exists a sech^2 homoclinic orbit near C_1.

Having outlined the possible families of orbits homoclinic to the fixed point (2.5) of (2.4), corresponding to pulse solitary waves of (1.1) and (1.2), we now derive normal forms near the transition curves C_0 and C_1 to confirm the existence of regular or delocalized solitary waves in the corresponding regions of (p, q) parameter space.

2.3 Normal form near C_0: solitary wave solutions

Using (2.4), the curve C_0, corresponding to $\lambda = 0, 0, \pm\tilde{\lambda}$, is given by

$$C_0 : p = 0, q > 0 \tag{2.6}$$

Using (2.3c) implies

$$a_1 < c^2 \tag{2.7}$$

Denoting ϕ by y_1, (2.1) may be written as the two systems

$$\frac{dy_1}{dz} = y_2 \tag{2.8a}$$

$$\frac{dy_2}{dz} = y_3 \tag{2.8b}$$

$$\frac{dy_3}{dz} = y_4 \tag{2.8c}$$

$$\frac{dy_4}{dz} = qy_3 - py_1 - N_{1,2}(Y) \tag{2.8d}$$

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where

$$\mathcal{N}_1(Y) = -\frac{1}{c^2}\left[3a_3\left(2y_1y_2^2 + y_1^2y_3\right) + 2a_2\left(y_3y_2 + y_2^2\right)\right] \tag{2.9a}$$

$$\mathcal{N}_2(Y) = -\frac{1}{c^2}\left[3a_3\left(2y_1y_2^2 + y_1^2y_3\right) + 5a_5\left(4y_1^3y_2^2 + y_1^4y_3\right)\right] \tag{2.9b}$$

We wish to rewrite (2.8) as a first order reversible system in order to invoke the relevant theory [7].

To that end, defining $Y = \langle y_1, y_2, y_3, y_4 \rangle^T$ equation (2.8) can be written

$$\frac{dY}{dz} = AY - G_{1,2}(Y, Y) \tag{2.10}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -p & 0 & q & 0 \end{pmatrix} \tag{2.11}$$

$$G_{1,2}(Y, Y) = \langle 0, 0, 0, -\mathcal{N}_{1,2}(Y) \rangle^T \tag{2.12}$$

The matrix A may be split into $A = L_0 + L_1$, where

$$L_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.13a}$$

$$L_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -p & 0 & q & 0 \end{pmatrix} \tag{2.13b}$$

We now derive a linear operator L_{pq} which is equivalent to $A = L_0 + L_1$ and in reversible form.

Let $L_{pq} = L_0 + M$ where N must satifisfy the following properties:

- $ML_0^* = L_0^*M$: M commutes with L_0^*

- $SM = -MS$ or $[S, M] = 0$: S and M are antisymmetric with repect to each other

where $S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Since L_0^* commutes with the identity and powers of itself, we assume the form of M as

$$M = \alpha_1 I + \alpha_2 L_0^* + \alpha_3 L_0^2 + \alpha_4 L_0^{*3} \tag{2.14}$$

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Because we want M to be antisymmetric, we must have $\alpha_1 = \alpha_3 = 0$ since I and L_0^{*2} are symmetric. Therefore we have $M = \alpha_2 L_0^* + \alpha_4 L_0^{*3}$. We now impose the condition that the eigenvalues of L_{pq} must be identical, therefore we must have that the coefficients of the characteristic polynomials of $L_0 + L_1$ and $L_0 + M$ are identical. The characteristic polynomial of $L_0 + L_1$ is

$$\rho_1(\lambda) = det\left(L_0 + L_1 - \lambda I\right) = \lambda^4 - q\lambda^2 + p \tag{2.15}$$

The characteristic polynomial of $L_0 + M$ is

$$\rho_2(\lambda) = det\left(L_0 + M - \lambda I\right) = det\left(L_0 + \alpha_2 L_0^* + \alpha_4 L_0^{*3} - \lambda I\right) \tag{2.16}$$

After some algebra one finds that

$$\rho_2(\lambda) = \lambda^4 - 3\left(\alpha_2 + \alpha_4\right)\lambda^2 + \left(\alpha_2 + \alpha_4\right)^2 \tag{2.17}$$

This immediately gives us

$$p = \left(\alpha_2 + \alpha_4\right)^2 \tag{2.18a}$$

$$q = 3\left(\alpha_2 + \alpha_4\right) \tag{2.18b}$$

Noting that $\frac{q^2}{9} = p$, we choose $\alpha_4 = \frac{q^2}{9}$ which implies $\alpha_2 = \frac{q}{3}$. We now have that $p = \alpha_2^2$ and $q = 3\alpha_2$.

Now (2.8) may be written

$$\frac{dY}{dz} = L_{pq}Y - G_{1,2}(Y, Y) \tag{2.19}$$

where

$$L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2 - p & 0 & q/3 & 0 \end{pmatrix} \tag{2.20}$$

Since $p = 0$ for (1.1) and (1.2), we have

$$\frac{dY}{dz} = L_{0q}Y - G_{1,2}(Y,Y) \tag{2.21}$$

Next we calculate the normal form of (2.21) near C_0. The procedure is closely modeled on [7] and many intermediate steps may be found there.

2.3.1 Near C_0

Near C_0 the dynamics reduce to a two-dimensional Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B) \tag{2.22}$$

and the corresponding normal form is

$$\frac{dA}{dz} = B \tag{2.23a}$$

$$\frac{dB}{dz} = b\epsilon A + \tilde{c}A^2 \tag{2.23b}$$

Here,

$$\epsilon = \left(\frac{q^2}{9} - p\right) - \left(\frac{q}{3}\right)^2 = -p \tag{2.24}$$

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measures the perturbation around C_0, and

$$\zeta_0 \;=\; \langle 1, 0, -q/3, 0 \rangle^T \tag{2.25a}$$

$$\zeta_1 \;=\; \langle 0, 1, 0, -2q/3 \rangle^T \tag{2.25b}$$

The linear eigenvalue of (2.23) satisfies

$$\lambda^2 = b\epsilon \tag{2.26}$$

The characteristic equation of the linear part of (2.21) is

$$\lambda^4 - q\lambda^2 - \epsilon = 0 \tag{2.27}$$

Hence, the eigenvalues near zero (the Center Manifold) satisfy $\lambda^4 \ll \lambda^2$ and hence

$$\lambda^2 \sim -\frac{\epsilon}{q} \tag{2.28}$$

Matching (2.26) and (2.28) implies

$$b = -\frac{1}{q} \tag{2.29}$$

and only the nonlinear coefficient $\tilde{c}$ remains to be determined in the normal form (2.23).

In order to determine $\tilde{c}$ (the coefficient of A^2 in (2.23)) we calculate $\frac{dY}{dz}$ in two ways and match the $\mathcal{O}(A^2)$ terms. To this end, using the standard 'suspension' trick of treating the perturbation parameter ϵ as a variable, we expand the function Ψ in (2.22) as

$$\Psi(\epsilon, A, B) = \epsilon A \Psi^1_{10} + \epsilon B \Psi^1_{01} + A^2 \Psi^0_{20} + AB \Psi^0_{11} + B^2 \Psi^0_{02} + \cdots \tag{2.30}$$

where the subscripts denote powers of A and B, respectively, and the superscript denotes the power of ϵ. In the first way of computing dY/dz, we take the z derivative of (2.22) (using

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(2.23) and (2.30)). The coefficient of A^2 in the resulting expression is $\tilde{c}\zeta_1$. In the second way of computing dY/dz, we use (2.22) and (2.30) in (2.19). The coefficient of A^2 in the resulting expression is $L_{0,q}\Psi_{20}^0 - G_{1,2}(\zeta_0, \zeta_0)$. Hence

$$\tilde{c}\zeta_1 = L_{0q}\Psi_{20}^0 - F_2(\zeta_0, \zeta_0) \tag{2.31}$$

Using (2.25) and (2.12) and denoting $\Psi_{20}^0 = \langle x_1, x_2, x_3, x_4 \rangle$ in (2.31) yields the equations

$$0 = x_2 \tag{2.32a}$$

$$\tilde{c} = \frac{q}{3}x_1 + x_3 \tag{2.32b}$$

$$0 = \frac{q}{3}x_2 + x_4 \implies x_4 = 0 \text{ using (2.32b)} \tag{2.32c}$$

and

$$-\frac{2q}{3}\tilde{c} = \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) + \frac{q}{3c^2}(3a_3 + 5a_5) = \frac{q}{3}\tilde{c} + \frac{q}{3c^2}(3a_3 + 5a_5) \text{ using (2.32b)} \tag{2.33}$$

Hence we obtain

$$\tilde{c} = -\frac{1}{3c^2}(3a_3 + 5a_5) \tag{2.34}$$

Therefore, the normal form near C_0 is

$$\frac{dA}{dz} = B \tag{2.35a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{a_3}{c^2}A^2 \tag{2.35b}$$

for (1.1) and

$$\frac{dA}{dz} = B \tag{2.36a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{1}{3c^2}(3a_3 + 5a_5)A^2 \tag{2.36b}$$

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for (1.2).

The normal form (2.35) admits a homoclinic solution (near C_0) of the form

$$A(z) = \ell \operatorname{sech}^2(kz) \tag{2.37}$$

with

$$k = \sqrt{\frac{-\epsilon}{4q}} \tag{2.38a}$$

$$\ell = \frac{-3\epsilon c^2}{2qa_3} \tag{2.38b}$$

Similarly, the normal form (2.36) admits a homoclinic solution (near C_0) of the form

$$A(z) = \ell \operatorname{sech}^2(kz) \tag{2.39}$$

with

$$k = \sqrt{\frac{-\epsilon}{4q}} \tag{2.40a}$$

$$\ell = \frac{-3\epsilon c^2}{2q(3a_3 + 5a_5)} \tag{2.40b}$$

Hence, since $\epsilon = -p$, and the curve C_0 corresponds to $p = 0, q > 0$, solitary waves of the form (2.37) exist in the vicinity of C_0 for

$$p > 0, q > 0 \tag{2.41}$$

which implies that $a_1 < c^2$ (such that k in (2.40a) is real.) As mentioned in section 2, one may show the persistence of this homoclinic solution in the original traveling wave ODE (2.4). Thus, we have demonstrated the existence of solitary waves of (1.2) for $p = 0^+, q > 0$.

Similarly, the curve C_1 corresponds to $p = 0, q < 0$, solitary waves of the form (2.37) exist in the vicinity of C_1 for

$$p < 0, q < 0 \tag{2.42}$$

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which implies $a_1 > c^2$.

Again, one may show the persistence of this homoclinic solution in the original traveling wave ODE (2.4). Thus, we have demonstrated the existence of solitary waves of (1.2) for $p = 0^-, q < 0$.

2.4   Normal form near C_1: possible solitary wave solutions

Using (2.4), the curve C_1, corresponding to $\lambda = 0, 0 \pm i\omega$, is given by

$$C_1 : p = 0, q < 0 \tag{2.43}$$

Which implies

$$a_1 > c^2 \tag{2.44}$$

In order to investigate the possibility of a sech^2 homoclinic orbit in the neighborhood of C_1 and delocalized solitary waves, we next compute the normal form near C_1 following the procedure in [7].

Near C_1 the dynamics reduce to a four-dimensional Center Manifold [7]. Since all the eigenvalues are non-hyperbolic, the Center Manifold has the form (a nonlinear coordinate change [7])

$$Y = A\zeta_0 + B\zeta_0 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C}) \tag{2.45}$$

with a corresponding four-dimensional normal form

$$\frac{dA}{dz} = B \tag{2.46a}$$

$$\frac{dB}{dz} = \bar{\nu}A + b_*A^2 + c_* |C|^2 \tag{2.46b}$$

$$\frac{dC}{dz} = id_0C + i\bar{\nu}d_1C + id_2AC \tag{2.46c}$$

Here C is complex, $\bar{C}$ is the complex conjugate of C, $\epsilon, \zeta_0, \zeta_1$ are given previously and the

two new complex eigenvectors co-spanning the Center Manifold are

$$\zeta_\pm = \left\langle 1, \lambda_\pm, 2q/3, \frac{\lambda_\pm}{3}q \right\rangle^T \tag{2.47}$$

Using (2.46b) and (2.23b) implies

$$\bar{\nu} = b\epsilon = -\frac{\epsilon}{q} \tag{2.48}$$

Also from the characteristic equation (2.27), the two non-zero (imaginary) roots are

$$\lambda^2 = \frac{q + \sqrt{q^2 + 4\epsilon}}{2} \approx q \text{ for } \epsilon \text{ small} \tag{2.49}$$

Hence

$$\lambda = \pm i\sqrt{-q}, q < 0 \tag{2.50}$$

Matching this to the linear part of (2.46c) (which corresponds to the imaginary eigenval-

ues), $\lambda = id_0 = i\sqrt{-q}$ or

$$d_0 = \sqrt{-q} \tag{2.51}$$

With a dominant balance argument on the characteristic equation (2.27) as $\lambda \to \pm i\sqrt{-q}$

we find

$$d_1 = \frac{\sqrt{-q}}{2q^2} \tag{2.52}$$

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The remaining undetermined coefficients in the normal form are the coefficients b_*, c_* and d_2 which correspond to the $A^2, |C|^2$ and AC terms respectively. In order to determine them, we follow the same procedure as in Section 2.3 and compute dY/dz is two distinct ways. We expand the function Ψ as

$$\Psi(\epsilon, A, B, C, \bar{C}) = \epsilon A \Psi^1_{1000} + \epsilon B \Psi^1_{0100} + A^2 \Psi^0_{2000} + AB \Psi^0_{1100} + AC \Psi^0_{1010} + \epsilon C \Psi^1_{0010} + \cdots \quad (2.53)$$

with subscripts denoting powers of A, B, C and $\bar{C}$, respectively, and the superscript is the power of ϵ. In the first way, dY/dz is computed by taking the z derivative of (2.45) (using (2.46) and (2.53)) and read off the coefficients of $A^2, \|C\|^2, C\epsilon$ and AC terms. In the second way, dY/dz is computed using (2.45) and (2.53) in (2.19) (with $p = 0$ on C_1 as given in (2.43)) and the coefficients of A, B, C and $\bar{C}$ are once again read off. Equating the coefficients of the corresponding terms in the two separate expressions for dY/dz yields the following two systems of equations:

$$\mathcal{O}(A^2): \qquad b_* \zeta_1 \qquad = L_{0q} \Psi^0_{2000} - G_{1,2}(\zeta_0, \zeta_0) \qquad (2.54\text{a})$$

$$\mathcal{O}(|C|^2): \qquad c_* \zeta_1 \qquad = L_{0q} \Psi^0_{0011} - 2G_{1,2}(\zeta_+, \zeta_-) \qquad (2.54\text{b})$$

$$\mathcal{O}(\epsilon C): \quad -\tfrac{i}{q}\left(d_1 \zeta_+ + d_0 \Psi^1_{0010}\right) = L_{0q} \Psi^1_{0010} - G_{1,2}(\Psi^1_{0010}, \Psi^1_{0010}) \qquad (2.54\text{c})$$

$$\mathcal{O}(AC): \quad id_2 \zeta_+ + id_0 \Psi^0_{1010} \quad = L_{0q} \Psi^0_{1010} - 2G_{1,2}(\zeta_0, \zeta_+) \qquad (2.54\text{d})$$

where we have used the fact that G_1 and G_2 are symmetric bilinear forms. Equation (2.54b) is decoupled and yields $c_* = \frac{8}{c^2}(2a_3 - a_2)$ and $c_* = \frac{1}{c^2}\left(16a_3 + \frac{140}{3}a_5\right)$ for (1.2). The only coefficient left to determine is d_2 which we shall compute now.

Using $\Psi^0_{1010} = \langle x_1, x_2, x_3, x_4 \rangle^T$ in (2.54d) implies

$$id_2 + id_0 x_1 = x_2 \tag{2.55a}$$

$$-d_0 d_2 + id_0 x_2 = \frac{q}{3} x_1 + x_3 \tag{2.55b}$$

$$\frac{2iq}{3} d_2 + id_0 x_3 = \frac{q}{3} x_2 + x_4 \tag{2.55c}$$

$$-\frac{q}{3} d_0 d_2 + id_0 x_4 = \frac{q}{3}\left(\frac{q}{3} x_1 + x_3\right) - \frac{2q}{c^2}\left(\frac{7}{2} a_3 - \frac{i}{3} d_0 a_2\right) \tag{2.55d}$$

for (1.1) and

$$id_2 + id_0 x_1 = x_2 \tag{2.56a}$$

$$-d_0 d_2 + id_0 x_2 = \frac{q}{3} x_1 + x_3 \tag{2.56b}$$

$$\frac{2iq}{3} d_2 + id_0 x_3 = \frac{q}{3} x_2 + x_4 \tag{2.56c}$$

$$-\frac{q}{3} d_0 d_2 + id_0 x_4 = \frac{q}{3}\left(\frac{q}{3} x_1 + x_3\right) - \frac{2q}{c^2}\left(\frac{7}{2} a_3 + \frac{32}{3} a_5\right) \tag{2.56d}$$

for (1.2)

Using (2.55a) in (2.55b), (2.55b) in (2.55d) and using these in (2.55c) yields $d_2 = \frac{1}{c^2}\left(\frac{7}{2\sqrt{-q}} a_3 - \frac{i}{3} a_2\right)$ for (1.1). Similarly using (2.56a) in (2.56b), (2.56b) in (2.56d) and using these in (2.56c) yields $d_2 = \frac{1}{\sqrt{-q}c^2}\left(\frac{7}{2} a_3 + \frac{32}{3} a_5\right)$ for (1.2).

Therefore the normal form near C_1 is

$$\frac{dA}{dz} = B \tag{2.57a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q} A - b_* A^2 + \frac{1}{c^2}\left(\frac{7}{2\sqrt{-q}} a_3 - \frac{i}{3} a_2\right)|C|^2 \tag{2.57b}$$

$$\frac{dC}{dz} = i\sqrt{-q}C - i\frac{\sqrt{-q}}{q^3} C\epsilon + i\frac{1}{c^2}\left(\frac{7}{2\sqrt{-q}} a_3 - \frac{i}{3} a_2\right) AC \tag{2.57c}$$

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for (1.1) and

$$\frac{dA}{dz} = B \tag{2.58a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - b_*A^2 + \frac{1}{c^2}\left(16a_3 + \frac{140}{3}a_5\right)|C|^2 \tag{2.58b}$$

$$\frac{dC}{dz} = i\sqrt{-q}C - i\frac{\sqrt{-q}}{q^3}C\epsilon + i\frac{1}{\sqrt{-q}c^2}\left(\frac{7}{2}a_3 + \frac{32}{3}a_5\right)AC \tag{2.58c}$$

for (1.2).

The dynamics inherent in (2.57), (2.58) may be elucidated following the discussions of [7], [8], [9] and [10]. The two first integrals of (2.46) are

$$K = |C|^2 \tag{2.59}$$

and

$$H = B^2 - \frac{2}{3}b_*A^3 - \bar{\nu}A^2 - 2c_*KA \tag{2.60}$$

Here, the appropriate coefficients $b_*, \bar{\nu}$ and c_*, derived above, apply for (1.1) and (1.2). Also, c_* should be real, or a_2 must be zero in (1.1) for the following energy arguments to apply. As a typical case, consider the level curve $H = 0$ of the energy-like first integral function H. In the (A, B) phase plane, this will compromise a homoclinic orbit. The intersection of $H = 0$ with the A axis occurs for $\frac{2}{3}b_*A^2 - \bar{\nu}A - 2c_*K = 0$ or

$$A_{\mp} = \frac{3}{4b_*}\left[\bar{\nu} \pm \sqrt{\bar{\nu}^2 + \frac{16b_*c_*K}{3}}\right] \tag{2.61}$$

Note that $A_+ > 0, A_- < 0$ for $b_*c_* > 0$ and $b_* < 0$ as relevant for us. A general homoclinic orbit, homoclinic to A_+, is sketched in Figure 1 where the flow direction is deduced from (2.57a) and (2.58a) for (1.1) and (1.2), respectively. For $K = |C|^2 = 0$, the orbit is homoclinic

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to $A_+ = 0$. For small non-zero $|K|$, $A_+ \sim -2c_*K/\bar{\nu}$, meaning that oscillations at infinity are then very small in this case. For $K = 0$ this corresponds to an *orbit homoclinic to* 0 for the normal form. This is indeed valid for the normal form taken at any order. However this solution does not exist mathematically for the full original system, even though one may compute its expansion in powers of the bifurcation parameter up to any order (see [9] and [10]). This is an example of the famous challenging problem of asymptotics beyond any orders. Other solutions found on the normal form mainly persist under the perturbation from higher order terms provided by the original system [8]. These solutions are delocalized waves and their existence in Region 2 is guaranteed by the general theory for reversible systems in [9] and [10]. Also, as mentioned in Section 2.2, genuine solitary waves are found on isolated curves in Region 2 of Figure 1 on which the oscillation amplitudes vanish. Since these are embedded in the sea of delocalized solitary waves and in the continuous spectrum, they are referred to as embedded solitons [11]. These will further be investigated in Region 2 subsequently using a mix of exponential asymptotics and numerical shooting.
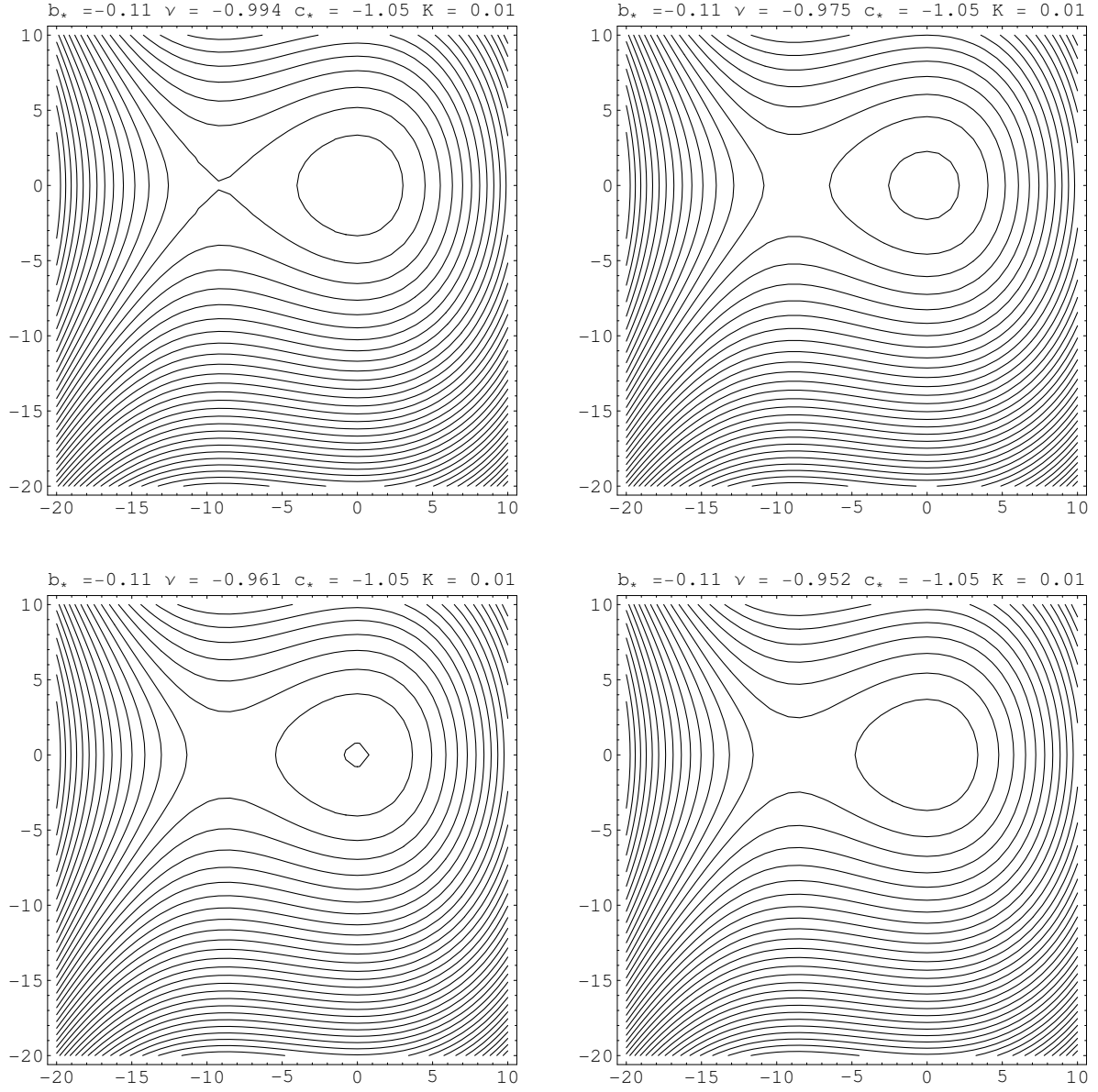
Figure 2.1: Level curves of (2.60) corresponding to various values of H.

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CHAPTER THREE: A MICROSTRUCTURE PDE

3.1   Introduction

One dimensional wave propagation in microstructured solids is currently a topic of great interest. This phenomenon has recently been modeled [12] by an equation

$$v_{tt} - bv_{xx} - \frac{\mu}{2}\left(v^2\right)_{xx} - \delta\left(\beta v_{tt} - \gamma v_{xx}\right)_{xx} = 0 \tag{3.1}$$

with complicated dispersive and nonlinear terms. Here b, μ, β, δ and γ are dimensionless parameters, v denotes the macrodeformation, and x and t denote space and time coordinates respectively.

Equation (3.1) is derived, using the so-called Mindlin Model, in [12], [13], [14]. It is non-integrable. However, analytic conditions for the existence of solitary waves of (3.1) have been derived in [14] and [12]. These references also numerically construct asymmetric solitary wave solutions of the form $v\left(x - ct\right)$ of (3.1). More recently ([15],[16]) pulse trains in (3.1) have been numerically constructed.

3.2   Solitary waves; local bifurcations

Solitary waves of (3.1) of the form $v(x,t) = \phi\left(x - ct\right) = \phi\left(z\right)$ satisfy the fourth-order traveling wave ODE

$$\phi_{zzzz} - q\phi_{zz} + p\phi = \mathcal{N}[\phi] \tag{3.2}$$

where

$$\mathcal{N}[\phi] = -\Delta_1 \phi_z^2 - b\Delta_1 \phi \phi_{zz} \tag{3.3}$$

and

$$z \equiv x - ct \tag{3.4a}$$

$$p \equiv 0 \tag{3.4b}$$

$$q \equiv \frac{c^2 - b}{\delta(\beta c^2 - \gamma)} \tag{3.4c}$$

$$\Delta_1 \equiv \frac{\mu}{\delta(\beta c^2 - \gamma)} \tag{3.4d}$$

Equation (3.2) is invariant under the transformation $z \mapsto -z$ and is thus a reversible system. In this section we shall use the theory of reversible systems to characterize the homoclinic orbits to the fixed point of (3.2), which correspond to pulses or solitary waves of (3.1) in various regions of the (p, q) plane.

The linearized system corresponding to (3.2)

$$\phi_{zzzz} - q\phi_{zz} + p\phi = 0 \tag{3.5}$$

has a fixed point

$$\phi = \phi_z = \phi_{zz} = \phi_{zzz} = 0 \tag{3.6}$$

Solutions $\phi = ke^{\lambda x}$ satisfy the characteristic equation $\lambda^4 - q\lambda^2 + p = 0$ from which one may deduce that the structure of the eigenvalues is distinct in two regions of (p, q)-space. Since $p = 0$ we have only two possible regions of eigenvalues. We denote C_0 as the positive q axis and C_1 the negative q-axis. First we shall consider the bounding curves C_0 and C_1

and their neighborhoods, then we shall discuss the possible occurrence and multiplicity of homoclinic orbits to (3.6), corresponding to pulse solitary waves of (3.1), in each region:

Near C_0 The eigenvalues have the structure $\lambda_{1-4} = 0, 0, \pm\lambda$, $(\lambda \in \mathbb{R})$ and the fixed point (2.5) is a saddle-focus.

Near C_1 Here the eigenvalues have the structure $\lambda_{1-4} = 0, 0, \pm i\omega$, $(\omega \in \mathbb{R})$. We will show by analysis of a four-dimensional normal form in Section 4 that there exists a sech^2 homoclinic orbit near C_1.

Having outlined the possible families of orbits homoclinic to the fixed point (3.6) of (3.5), corresponding to pulse solitary waves of (3.1), we now derive normal forms near the transition curves C_0 and C_1 to confirm the existence of regular or delocalized solitary waves in the corresponding regions of (p, q) parameter space.

3.3 Normal form near C_0: solitary wave solutions

Using (3.5), the curve C_0, corresponding to $\lambda = 0, 0, \pm\tilde{\lambda}$, is given by

$$C_0 : p = 0, q > 0 \tag{3.7}$$

Using (3.4c) implies

$$\frac{c^2 - b}{\delta \left(\beta c^2 - \gamma\right)} > 0 \tag{3.8}$$

Denoting ϕ by y_1, equation (3.2) may be written as the system

$$\frac{dy_1}{dz} = y_2 \tag{3.9a}$$

$$\frac{dy_2}{dz} = y_3 \tag{3.9b}$$

$$\frac{dy_3}{dz} = y_4 \tag{3.9c}$$

$$\frac{dy_4}{dz} = qy_3 - py_1 - \left(\Delta_1 y_2^2 + b\Delta_1 y_1 y_3\right) \tag{3.9d}$$

We wish to rewrite this as a first order reversible system in order to invoke the relevant

theory [7]. To that end, defining $Y = \langle y_1, y_2, y_3, y_4 \rangle^T$, equation (3.9) may be written

$$\frac{dY}{dz} = L_{pq}Y - F_2(Y, Y) \tag{3.10}$$

where

$$L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2 - p & 0 & q/3 & 0 \end{pmatrix} \tag{3.11}$$

Since $p = 0$ for (3.1), we have

$$\frac{dY}{dz} = L_{0q}Y - F_2(Y, Y) \tag{3.12}$$

where

$$F_2(Y, Y) = \left\langle 0, 0, 0, \Delta_1 y_2^2 + b\Delta_1 y_1 y_3 \right\rangle^T \tag{3.13}$$

Next we calculate the normal form of (3.12) near C_0. The procedure is closely modeled

on [7] and many intermediate steps may be found there.

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3.3.1 Near C_0

Near C_0 the dynamics reduce to a two-dimensional Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B) \tag{3.14}$$

and the corresponding normal form is

$$\frac{dA}{dz} = B \tag{3.15a}$$

$$\frac{dB}{dz} = b\epsilon A + \tilde{c}A^2 \tag{3.15b}$$

Here,

$$\epsilon = \left(\frac{q^2}{9} - p\right) - \left(\frac{q}{3}\right)^2 = -p \tag{3.16}$$

measures the perturbation around C_0, and

$$\zeta_0 = \langle 1, 0, -q/3, 0 \rangle^T \tag{3.17a}$$

$$\zeta_1 = \langle 0, 1, 0, -2q/3 \rangle^T \tag{3.17b}$$

The linear eigenvalue of (3.15) satisfies

$$\lambda^2 = b\epsilon \tag{3.18}$$

The characteristic equation of the linear part of (3.12) is

$$\lambda^4 - q\lambda^2 - \epsilon = 0 \tag{3.19}$$

Hence, the eigenvalues near zero (the Center Manifold) satisfy $\lambda^4 \ll \lambda^2$ and hence

$$\lambda^2 \sim -\frac{\epsilon}{q} \tag{3.20}$$

Matching (3.18) and (3.20)

$$b = -\frac{1}{q} \tag{3.21}$$

and only the nonlinear coefficient $\tilde{c}$ remains to be determined in the normal form (3.15).

In order to determine $\tilde{c}$ (the coefficient of A^2 in (3.15)) we calculate $\frac{dY}{dz}$ in two ways and match the $\mathcal{O}(A^2)$ terms. To this end, using the standard 'suspension' trick of treating the perturbation parameter ϵ as a variable, we expand the function Ψ in (3.14) as

$$\Psi(\epsilon, A, B) = \epsilon A \Psi_{10}^1 + \epsilon B \Psi_{01}^1 + A^2 \Psi_{20}^0 + AB \Psi_{11}^0 + B^2 \Psi_{02}^0 + \cdots \tag{3.22}$$

where the subscripts denote powers of A and B, respectively, and the superscript denotes the power of ϵ.

In the first way of computing dY/dz, we take the z derivative of (3.14) (using (3.15) and (3.41)). The coefficient of A^2 in the resulting expression is $\tilde{c}\zeta_1$. In the second way of computing dY/dz, we use (3.14) and (3.41) in (3.12). The coefficient of A^2 in the resulting expression is $L_{0,q}\Psi_{20}^0 - F_2(\zeta_0, \zeta_0)$, which leads to

$$\tilde{c}\zeta_1 = L_{0q}\Psi_{20}^0 - F_2(\zeta_0, \zeta_0) \tag{3.23}$$

Using (3.17) and (3.13) and denoting $\Psi_{20}^0 = \langle x_1, x_2, x_3, x_4 \rangle$ in (3.23) yields the equations

$$0 = x_2 \tag{3.24a}$$

$$\tilde{c} = \frac{q}{3}x_1 + x_3 \tag{3.24b}$$

$$0 = \frac{q}{3}x_2 + x_4 \implies x_4 = 0 \text{ using (3.24b)} \tag{3.24c}$$

and

$$-\frac{2q}{3}\tilde{c} = \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) + \frac{2q}{3} = \frac{q}{3}\tilde{c} + \frac{b\Delta_1}{3} \text{ using (3.24b)} \tag{3.25}$$

Hence we obtain

$$\tilde{c} = -\frac{b\Delta_1}{3} \tag{3.26}$$

Therefore, the normal form for (3.1) near C_0 is

$$\frac{dA}{dz} = B \tag{3.27a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{b\Delta_1}{3}A^2 \tag{3.27b}$$

The normal form (3.15) admits a homoclinic solution (near C_0) of the form

$$A(z) = \ell \operatorname{sech}^2(kz) \tag{3.28}$$

with

$$k = \sqrt{\frac{-\epsilon}{4q}} \tag{3.29a}$$

$$\ell = \frac{6k^2}{b\Delta_1} \tag{3.29b}$$

Hence, since $\epsilon = -p$, and the curve C_0 corresponds to $p = 0, q > 0$, solitary waves of the

form (3.28) exist in the vicinity of C_0 for

$$p > 0, q > 0 \tag{3.30}$$

which implies that $\frac{c^2 - b}{\delta(\beta c^2 - \gamma)} > 0$ (such that k in (3.29a) is real.) As mentioned in section 2,

one may show the persistence of this homoclinic solution in the original traveling wave ODE

(3.5). Thus, we have demonstrated the existence of solitary waves of (3.1) for $p = 0^+, q > 0$.

3.4   Normal form near C_1: possible solitary wave solutions

Using (3.5), the curve C_1, corresponding to $\lambda = 0, 0 \pm i\omega$, is given by

$$C_1 : p = 0, q < 0 \tag{3.31}$$

Which implies

$$\frac{c^2 - b}{\delta \left(\beta c^2 - \gamma\right)} < 0 \tag{3.32}$$

In order to investigate the possibility of a sech^2 homoclinic orbit in the neighborhood of C_1 and delocalized solitary waves, we next compute the normal form near C_1 following the procedure in [7].

Near C_1 the dynamics reduce to a four-dimensional Center Manifold [7]. Since all the eigenvalues are non-hyperbolic, the Center Manifold has the form (a nonlinear coordinate change [7])

$$Y = A\zeta_0 + B\zeta_0 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C}) \tag{3.33}$$

with a corresponding four-dimensional normal form

$$\frac{dA}{dz} = B \tag{3.34a}$$

$$\frac{dB}{dz} = \bar{\nu}A + b_* A^2 + c_* |C|^2 \tag{3.34b}$$

$$\frac{dC}{dz} = id_0 C + i\bar{\nu}d_1 C + id_2 AC \tag{3.34c}$$

Here C is complex, $\bar{C}$ is the complex conjugate of C, $\epsilon, \zeta_0, \zeta_1$ are given previously and the two new complex eigenvectors co-spanning the Center Manifold are

$$\zeta_\pm = \left\langle 1, \lambda_\pm, 2q/3, \frac{\lambda_\pm}{3}q \right\rangle^T \tag{3.35}$$

Using (3.34b) and (3.15b)

$$\bar{\nu} = b\epsilon = -\frac{\epsilon}{q} \tag{3.36}$$

Also from the characteristic equation (3.19), the two non-zero (imaginary) roots are

$$\lambda^2 = \frac{q + \sqrt{q^2 + 4\epsilon}}{2} \approx q \text{ for } \epsilon \text{ small} \tag{3.37}$$

Hence

$$\lambda = \pm i\sqrt{-q}, q < 0 \tag{3.38}$$

Matching this to the linear part of (3.34c) (which corresponds to the imaginary eigenvalues), $\lambda = id_0 = i\sqrt{-q}$ or

$$d_0 = \sqrt{-q} \tag{3.39}$$

With a dominant balance argument on the characteristic equation (3.19) as $\lambda \to \pm i\sqrt{-q}$ we find

$$d_1 = \frac{\sqrt{-q}}{2q^2} \tag{3.40}$$

The remaining undetermined coefficients in the normal form are the coefficients b_*, c_* and d_2 which correspond to the $A^2, |C|^2$ and AC terms respectively. In order to determine them, we follow the same procedure as in Section 3 and compute dY/dz is two distinct ways. We expand the function Ψ as

$$\Psi(\epsilon, A, B, C, \bar{C}) = \epsilon A\Psi^1_{1000} + \epsilon B\Psi^1_{0100} + A^2\Psi^0_{2000} + AB\Psi^0_{1100} + AC\Psi^0_{1010} + \epsilon C\Psi^1_{0010} + \cdots \tag{3.41}$$

with subscripts denoting powers of A, B, C and $\bar{C}$, respectively, and the superscript is the power of ϵ. In the first way, dY/dz is computed by taking the z derivative of (3.33) (using (3.34) and (3.41)) and read off the coefficients of $A^2, \|C\|^2, C\epsilon$ and AC terms. In

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the second way, dY/dz is computed using (3.33) and (3.41) in (3.12) (with $p = 0$ on C_1 as given in (3.31)) and the coefficients of A, B, C and $\bar{C}$ are once again read off. Equating the coefficients of the corresponding terms in the two separate expressions for dY/dz yields the following equations:

$$\mathcal{O}(A^2): \qquad b_*\zeta_1 \qquad = L_{0q}\Psi^0_{2000} - F_2(\zeta_0, \zeta_0) \qquad (3.42a)$$

$$\mathcal{O}(|C|^2): \qquad c_*\zeta_1 \qquad = L_{0q}\Psi^0_{0011} - 2F_2(\zeta_+, \zeta_-) \qquad (3.42b)$$

$$\mathcal{O}(\epsilon C): \quad -\frac{i}{q}\left(d_1\zeta_+ + d_0\Psi^1_{0010}\right) = L_{0q}\Psi^1_{0010} - F_2(\Psi^1_{0010}, \Psi^1_{0010}) \qquad (3.42c)$$

$$\mathcal{O}(AC): \quad id_2\zeta_+ + id_0\Psi^0_{1010} \quad = L_{0q}\Psi^0_{1010} - 2F_2(\zeta_0, \zeta_+) \qquad (3.42d)$$

where we have used the fact that F_2 is a symmetric bilinear form. Equation (3.42b) is decoupled and yields $c_* = 2\Delta_1\left(\frac{2b}{3} - 1\right)$. The only coefficient left to determine is d_2 which we shall compute now.

Using $\Psi^0_{1010} = \langle x_1, x_2, x_3, x_4\rangle^T$ in (3.42d) implies

$$id_2 + id_0x_1 \quad = \quad x_2 \qquad (3.43a)$$

$$-d_0d_2 + id_0x_2 \quad = \quad \frac{q}{3}x_1 + x_3 \qquad (3.43b)$$

$$\frac{2iq}{3}d_2 + id_0x_3 \quad = \quad \frac{q}{3}x_2 + x_4 \qquad (3.43c)$$

$$-\frac{q}{3}d_0d_2 + id_0x_4 \quad = \quad \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) - \frac{2bq\Delta_1}{3} \qquad (3.43d)$$

Using (3.43a) in (3.43b) , (3.43b) in (3.43d) and using these in (3.43c) yields $d_2 = \frac{b\Delta_1}{3\sqrt{-q}}$.

Therefore the normal form for (3.1) near C_1 is

$$\frac{dA}{dz} = B \tag{3.44a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{b\Delta_1}{3}A^2 + 2\Delta_1\left(\frac{2b}{3} - 1\right)|C|^2 \tag{3.44b}$$

$$\frac{dC}{dz} = i\sqrt{-q}C - i\frac{\sqrt{-q}}{q^3}C\epsilon + i\frac{b\Delta_1}{3\sqrt{-q}}AC \tag{3.44c}$$

The dynamics inherent in (3.44) may be elucidated following the discussions of [7], [8], [9] and [10]. The two first integrals of (3.34) are

$$K = |C|^2 \tag{3.45}$$

and

$$H = B^2 - \frac{2}{3}b_*A^3 - \bar{\nu}A^2 - 2c_*KA \tag{3.46}$$

Also, c_* should be real for the following energy arguments to apply. As a typical case, consider the level curve $H = 0$ of the energy-like first integral function H. In the (A, B) phase plane, this will compromise a homoclinic orbit. The intersection of $H = 0$ with the A axis occurs for $\frac{2}{3}b_*A^2 - \bar{\nu}A - 2c_*K = 0$ or

$$A_\mp = \frac{3}{4b_*}\left[\bar{\nu} \pm \sqrt{\bar{\nu}^2 + \frac{16b_*c_*K}{3}}\right] \tag{3.47}$$

Note that $A_+ > 0, A_- < 0$ for $b_*c_* > 0$ and $b_* < 0$ as relevant for us. A general homoclinic orbit, homoclinic to A_+, is sketched in Figure 1 where the flow direction is deduced from (3.44a). For $K = |C|^2 = 0$, the orbit is homoclinic to $A_+ = 0$. For small non-zero $|K|$, $A_+ \sim -2c_*K/\bar{\nu}$, meaning that oscillations at infinity are then very small in this case. For $K = 0$ this corresponds to an *orbit homoclinic to* 0 for the normal form. This is indeed valid for the

normal form taken at any order. However this solution does not exist mathematically for the full original system, even though one may compute its expansion in powers of the bifurcation parameter up to any order (see [9] and [10]). This is an example of the famous challenging problem of asymptotics beyond any orders. Other solutions found on the normal form mainly persist under the perturbation from higher order terms provided by the original system [8]. These solutions are delocalized waves and their existence in Region 2 is guaranteed by the general theory for reversible systems in [9] and [10]. Also, as mentioned in Section 2, genuine solitary waves are found on isolated curves in Region 2 of Figure 1 on which the oscillation amplitudes vanish. Since these are embedded in the sea of delocalized solitary waves and in the continuous spectrum, they are referred to as embedded solitons [11]. These will further be investigated in Region 2 subsequently using a mix of exponential asymptotics and numerical shooting.
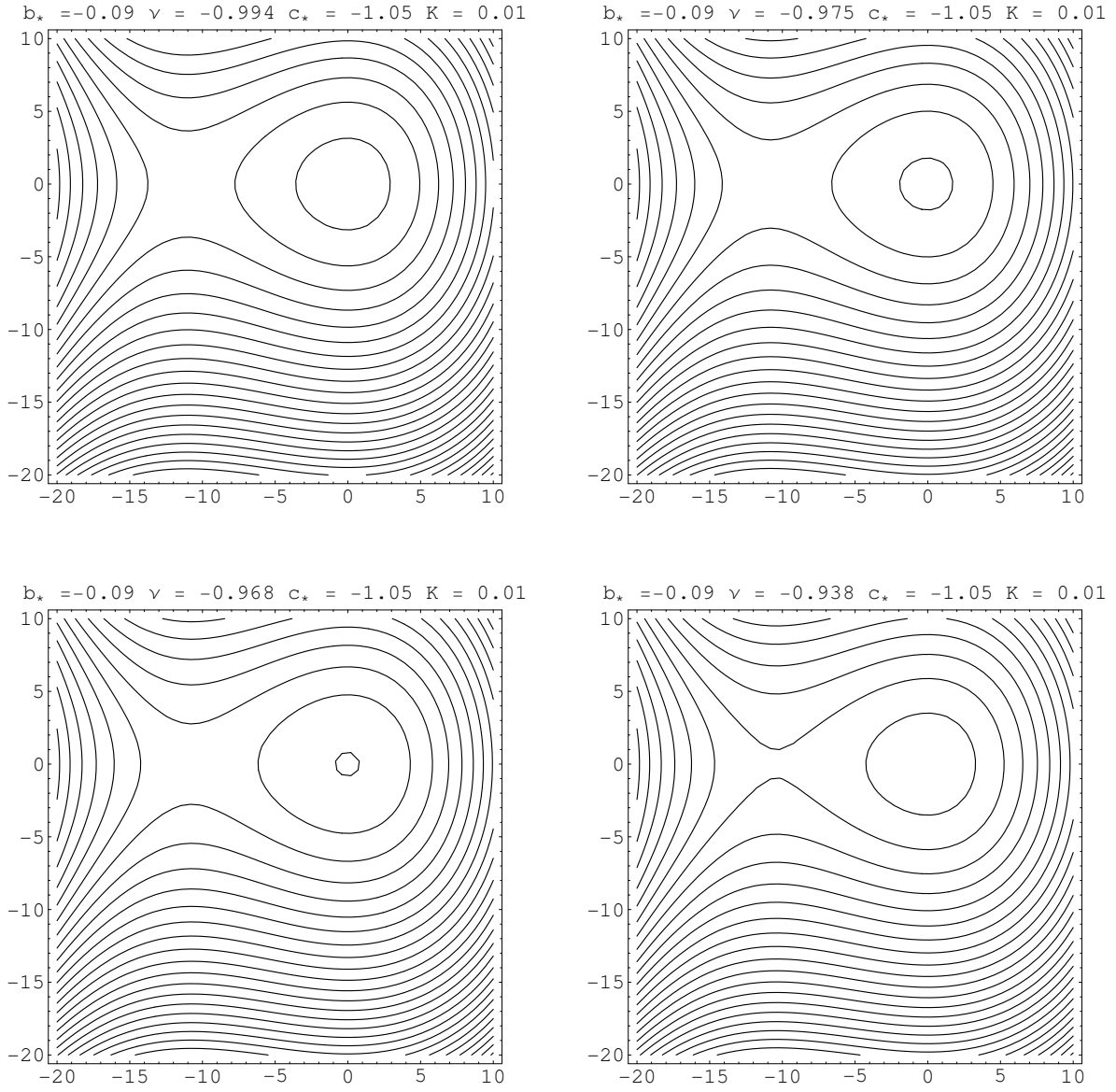
Figure 3.1: Level curves of (3.46) corresponding to various values of H.

CHAPTER FOUR: RESULTS

In this thesis, we apply a recently developed technique to comprehensively categorize all possible families of solitary wave solutions in two models of topical interest.

The models considered are:

- the Generalized Pochammer-Chree Equations, which govern the propagation of longitudinal waves in elastic rods,

  and

- generalized microstructure PDE.

Limited analytic results exist for the occurrence of one family of solitary wave solutions for each of these equations. Since, as mentioned above, solitary wave solutions often play a central role in the long-time evolution of an initial disturbance, we consider such solutions of both models here (via the normal form approach) within the framework of reversible systems theory.

Besides confirming the existence of the known family of solitary waves for each model, we find a continuum of delocalized solitary waves (or homoclinics to small-amplitude periodic orbits). On isolated curves in the relevant parameter region, the delocalized waves reduce to genuine embedded solitons. For both models, the new family of solutions occur in regions of parameter space distinct from the known solitary wave solutions and are thus entirely new.

Directions for future work, including the dynamics of each family of solitary waves using exponential asymptotics techniques, are also mentioned.

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