SOLITARY-WAVE FAMILIES OF THE GENERALIZED POCHAMMER-CHREE PDE'S

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ABSTRACT. The Generalized Pochammer-Chree Equations Since solitary-wave solutions often play a central role in the long-time evolution of an initial disturbance, we consider such solutions here (via normal form approach) within the framework of reversible systems theory. On isolated curves in the relevant parameter region, the delocalized waves reduce to genuine embedded solitons. Directions for future work are also described.

1. INTRODUCTION

The nonlinear dispersive equations [2]

$$(u - u_{xx})_{tt} - \left(a_1 u + a_2 u^2 + a_3 u^3\right)_{xx} = 0 \tag{1.1a}$$

$$(u - u_{xx})_{tt} - \left(a_1 u + a_3 u^3 + a_5 u^5\right)_{xx} = 0 \tag{1.1b}$$

are known as the Generalized Pochammer-Chree Equation 1 and 2, respectively.

We use the theory of reversible systems and the method of normal forms to categorize the possible solitary waves of (1.1). As we shall see, several families of solitary waves exist in various regions of parameter space. Our main focus here will be on delineating the possible occurence and multiplicity of solitary waves in different parameter regimes. Certain delicate questions relating to specific waves or wave families will form the basis of future work.

The remainder of this paper is organized as follows. In Section 2, we delineate the possible families of solitary waves in various parameter domains and on certain important curves using the theory of reversible systems. In Section 3 and 4, we next focus on the various transition curves and derive normal forms in their vicinity to confirm the existence of families of regular or delocalized solitary-wave solutions in their vicinity.

2. SOLITARY WAVES: LOCAL BIFURCATION

Solitary waves of (1.1) of the form $v(x,t) = \phi(x - ct) = \phi(z)$ satisfy the fourth-order travelling wave ODE

$$\phi_{zzzz} - q\phi_{zz} + p\phi = \mathcal{N}_{1,2}[\phi] \tag{2.1}$$

where

$$\mathcal{N}_1[\phi] = -\frac{1}{c^2}\left[3a_3\left(2\phi\phi_z^2 + \phi^2\phi_{zz}\right) + 2a_2\left(\phi_{zz}\phi_z + \phi_z^2\right)\right] \tag{2.2a}$$

$$\mathcal{N}_2[\phi] = -\frac{1}{c^2}\left[3a_3\left(2\phi\phi_z^2 + \phi^2\phi_{zz}\right) + 5a_5\left(4\phi^3\phi_z^2 + \phi^4\phi_{zz}\right)\right] \tag{2.2b}$$

$$z \equiv x - ct \tag{2.3a}$$

$$p \equiv 0 \tag{2.3b}$$

$$q \equiv 1 - \frac{a_1}{c^2} \tag{2.3c}$$

(2.3d)

Equation (2.1) is invariant under the transformation $z \mapsto -z$ and is thus a reversible system. In this section we shall use the theory of reversible systems to characterize the homoclinic orbits to the fixed point of (2.1), which correspond to pulses or solitary waves of the Micro-Structure PDE in various regions of the (p, q) plane.

The linearized system corresponding to (2.1)

$$\phi_{zzzz} - q\phi_{zz} + p\phi = 0 \tag{2.4}$$

has a fixed point

$$\phi = \phi_z = \phi_{zz} = \phi_{zzz} = 0 \tag{2.5}$$

Solutions $\phi = ke^{\lambda x}$ satisfy the characteristic equation $\lambda^4 - q\lambda^2 + p = 0$ from which one may deduce that the structure of the eigenvalues is distinct in two regions of (p, q)-space. Since $p = 0$ we have only two possible regions of eigenvalues. We denote C_0 as the positive q axis and C_1 the negative q-axis. First we shall consider the bounding curves C_0 and C_1 and their neighborhoods, then we shall discusss the possible occurence and multiplicities of homoclinic orbits to (2.5), corresponding to pulse solitary waves of (1.1), in each region:

Near C_0: The eigenvalues have the structure $\lambda_{1-4} = 0, 0, \pm\lambda$, $(\lambda \in \mathbb{R})$ and the fixed point (2.5) is a saddle-focus.

Near C_1: Here the eigenvalues have the structure $\lambda_{1-4} = 0, 0, \pm i\omega$, $(\omega \in \mathbb{R})$. We will show by analysis of a four-dimensional normal form in Section 4 that there exists a $sech^2$ homoclinic orbit near C_1.

Having outlined the possible families of orbits homoclinic to the fixed point (2.5) of (2.4), corresponding to pulse solitary waves of (1.1), we now derive normal forms near the transition curves C_0 and C_1 to confirm the existence of regular or delocalized solitary waves in the corresponding regions of (p, q) parameter space.

3. Calculations

Using (2.4), the curve C_0, corresponding to $\lambda = 0, 0, \pm\tilde{\lambda}$, is given by

$$C_0 : p = 0, q > 0 \tag{3.1}$$

Using (2.3c) implies

$$stuff > 0 \tag{3.2}$$

Denoting ϕ by y_1, (2.1) may be written as the system

$$\frac{dy_1}{dz} = y_2 \tag{3.3a}$$

$$\frac{dy_2}{dz} = y_3 \tag{3.3b}$$

$$\frac{dy_3}{dz} = y_4 \tag{3.3c}$$

$$\frac{dy_4}{dz} = qy_3 - py_1 - \left(\Delta_1 y_2^2 + b\Delta_1 y_1 y_3\right) \tag{3.3d}$$

We wish to rewrite this as a first order reversible system in order to invoke the relevant theory [1]. To that end, defining $Y = \langle y_1, y_2, y_3, y_4\rangle^T$ (3.3) may be written

$$\frac{dY}{dz} = L_{pq}Y - F_2(Y, Y) \tag{3.4}$$

where

$$L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2 - p & 0 & q/3 & 0 \end{pmatrix} \tag{3.5}$$

Since $p = 0$ for (1.1), we have

$$\frac{dY}{dz} = L_{0q}Y - F_2(Y, Y) \tag{3.6}$$

where

$$F_2(Y, Y) = \langle 0, 0, 0, XXX\rangle^T \tag{3.7}$$

Next we calculate the normal form of (3.6) near C_0. The procedure is closely modeled on [1] and many intermediate steps may be found there.

3.1. **Near C_0.** HERE

Here we have

$$q = 1 - \frac{a_1}{c^2} \tag{3.8a}$$

$$p = 0 \tag{3.8b}$$

We emphasize that $p = 0$ because this corresponds to a 1-D subspace of the (p, q) parameter space in [2]. We denote C_0 as the positive q axis and C_1 the negative q-axis.

Solutions $\phi = ke^{\lambda x}$ satisfy the characteristic equation $\lambda^4 - q\lambda^2 + p = 0$. Since $p = 0$ for the GPC PDE's we have only two possible regions of eigenvalues. On C_0 the eigenvalues have the structure $\lambda_{1-4} = 0, 0, \pm\lambda$, $(\lambda \in \mathbb{R})$ while on C_1 we have $\lambda_{1-4} = 0, 0, \pm i\omega$, $(\omega \in \mathbb{R})$.

We now use the theory of Iooss and Adelmeyer [1] to write the GPC PDE's in reversible form: Let $Y = \langle y_1, y_2, y_3, y_4 \rangle^T$ where $\langle y_1, y_2, y_3, y_4 \rangle^T = \langle \phi_z, \phi_{zz}, \phi_{zzz}, \phi_{zzzz} \rangle^T$

We may now write (2.1) as a reversible first-order system

$$\frac{dY}{dz} = L_{pq}Y - F_{1,2}(Y, Y) \tag{3.9}$$

where

$$L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2 - p & 0 & q/3 & 0 \end{pmatrix} \tag{3.10}$$

Since $p = 0$ for the GPC PDE's, we have

$$\frac{dY}{dz} = L_{0q}Y - F_{1,2}(Y, Y) \tag{3.11}$$

where

$$F_{1,2}(Y, Y) = \langle 0, 0, 0, -\mathcal{N}_{1,2} \rangle^T \tag{3.12}$$

and

$$F_1(Y, Y) = \left\langle 0, 0, 0, \frac{1}{c^2}\left[3a_3\left(2y_1y_2^2 + y_1^2y_2\right) + 2a_2\left(y_2y_3 + y_2^2\right)\right] \right\rangle^T \tag{3.13a}$$

$$F_2(Y, Y) = \left\langle 0, 0, 0, \frac{1}{c^2}\left[3a_3\left(2y_1y_2^2 + y_1^2y_2\right) + 5a_5\left(4y_1^3y_2^2 + y_1^4y_3\right)\right] \right\rangle^T \tag{3.13b}$$

3.2. **Near C_0.** Near C_0 the dynamics reduce to a 2-D Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B) \tag{3.14}$$

with

$$\frac{dA}{dz} = B \tag{3.15a}$$

$$\frac{dB}{dz} = b\epsilon A + \tilde{c}A^2 \tag{3.15b}$$

and the eigenvectors are $\zeta_0 = \langle 1, 0, -q/3, 0 \rangle^T$, $\zeta_1 = \langle 0, 1, 0, -2q/3 \rangle^T$, ϵ determines the distance from C_0 . A simple dominant balance argument on the linearized equation yields $b = -\frac{1}{q}$.

To determine $\tilde{c}$ we calculate $\frac{dY}{dz}$ in two ways and match the $\mathcal{O}(A^2)$ terms. Expanding $\Psi(\epsilon, A, B)$ in series

$$\Psi(\epsilon, A, B) = \epsilon A\Psi_{10}^1 + \epsilon B\Psi_{01}^1 + A^2\Psi_{20}^0 + AB\Psi_{11}^0 + B^2\Psi_{02}^0 + \cdots \tag{3.16}$$

we find

$$\tilde{c}\zeta_1 = L_{0q}\Psi_{20}^0 - F_{1,2}(\zeta_0, \zeta_0) \tag{3.17}$$

We find that $F_1(\zeta_0,\zeta_0) = \left\langle 0,0,0,-\frac{q}{c^2}a_3\right\rangle^T$ and $F_2(\zeta_0,\zeta_0) = \left\langle 0,0,0,-\frac{q}{c^2}\left(a_3+\frac{5a_5}{3}\right)\right\rangle^T$ which implies $\tilde{c} = -\frac{a_3}{c^2}$ for the GPC 1 and $\tilde{c} = -\frac{1}{c^2}\left(a_3+\frac{5}{3}a_5\right)$ for the GPC 2.

Therefore, the normal form for the GPC 1 PDE near C_0 is

$$\frac{dA}{dz} = B \tag{3.18a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{a_3}{c^2}A^2 \tag{3.18b}$$

and the normal form for the GPC 2 PDE near C_0 is

$$\frac{dA}{dz} = B \tag{3.19a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{1}{c^2}\left(a_3+\frac{5}{3}a_5\right)A^2 \tag{3.19b}$$

3.3. **Near C_1.** Near C_1 the dynamics reduce to a 4-D Center Manifold

$$Y = A\zeta_0 + B\zeta_0 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C}) \tag{3.20}$$

with

$$\frac{dA}{dz} = B \tag{3.21a}$$

$$\frac{dB}{dz} = \bar{\nu}A + b_*A^2 + c_*|C|^2 \tag{3.21b}$$

$$\frac{dC}{dz} = id_0C + i\bar{\nu}d_1C + id_2AC \tag{3.21c}$$

where $\zeta_\pm = \left\langle 1, \lambda_\pm, 2q/3, \frac{\lambda_\pm}{3}q\right\rangle^T$ where $\bar{\nu} = -\frac{\epsilon}{q}$, $b_* = -\frac{b\Delta 1}{3}$ and $\lambda_\pm = \pm i\sqrt{-q}$. Comparing to the linearized equations gives $d_0 = \sqrt{-q}$.

DONE

ACTUALLY DO THIS: If we do a dominant balance argument after the change of variable $\tilde{\epsilon} = \sqrt{-3\mu}$ on the characteristic equation as $\lambda \to 0$ then we find $d_1 = \frac{\sqrt{-3\mu}}{18\mu^2}$. Using $\mu = q/3$ we find $d_1 = \frac{\sqrt{-q}}{2q^2}$

To determine b_*, c_* and d_2 we expand

$$\Psi(\epsilon, A, B, C, \bar{C}) = \epsilon A\Psi^1_{1000} + \epsilon B\Psi^1_{0100} + A^2\Psi^0_{2000} + AB\Psi^0_{1100} + AC\Psi^0_{1010} + \epsilon C\Psi^1_{0010} + \cdots \tag{3.22}$$

Now we use (3.21b), (3.21c), (3.21c) in (3.22) when we calculate $\frac{dY}{dz}$. Matching coefficients yields

$$\mathcal{O}(A^2): \quad b_*\zeta_1 = L_{0q}\Psi^0_{2000} - F_2(\zeta_0,\zeta_0) \tag{3.23a}$$

$$\mathcal{O}(|C|^2): \quad c_*\zeta_1 = L_{0q}\Psi^0_{0011} - 2F_2(\zeta_+,\zeta_-) \tag{3.23b}$$

$$\mathcal{O}(\epsilon C): \quad -\frac{i}{q}\left(d_1\zeta_+ + d_0\Psi^1_{0010}\right) = L_{0q}\Psi^1_{0010} - F_2(\Psi^1_{0010},\Psi^1_{0010}) \tag{3.23c}$$

$$\mathcal{O}(AC): \quad id_2\zeta_+ + id_0\Psi^0_{1010} = L_{0q}\Psi^0_{1010} - 2F_2(\zeta_0,\zeta_+) \tag{3.23d}$$

(3.23e)

where we have used the fact that F_2 is a symmetric bilinear form. Equation (3.23b) is decoupled and yields $c_* = 2\Delta_1\left(\frac{1}{q}+\frac{2b}{3}\right)$ The only coefficient left to determine is d_2 which we shall compute now.

Put $\Psi^0_{1010} = \langle x_1, x_2, x_3, x_4\rangle^T$ into (3.23e) implies

$$
\begin{aligned}
&(3.24a) & id_2 + id_0 x_1 &= x_2 \\
&(3.24b) & -d_0 d_2 + id_0 x_2 &= \frac{q}{3} x_1 + x_3 \\
&(3.24c) & \frac{2iq}{3} d_2 + id_0 x_3 &= \frac{q}{3} x_2 + x_4 \\
&(3.24d) & -\frac{q}{3} d_0 d_2 + id_0 x_4 &= \frac{q}{3}\left(\frac{q}{3} x_1 + x_3\right) - \frac{2bq\Delta_1}{3}
\end{aligned}
$$

Using (3.24a) in (3.24b) , (3.24b) in (3.24d) and using these in (3.24c) yields $d_2 = \frac{b\Delta_1}{3\sqrt{-q}}$. Therefore the normal form for the MS PDE near C_1 is

$$
\begin{aligned}
&(3.25a) & \frac{dA}{dz} &= B \\
&(3.25b) & \frac{dB}{dz} &= -\frac{\epsilon}{q} A - \frac{b\Delta_1}{3} A^2 + 2\Delta_1 \left(\frac{1}{q} + \frac{2b}{3}\right) |C|^2 \\
&(3.25c) & \frac{dC}{dz} &= i\sqrt{-q} C - i\frac{\epsilon\sqrt{-q}}{q^3} C + i\frac{b\Delta_1}{3\sqrt{-q}} AC
\end{aligned}
$$

4. Analysis of Normal Forms

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References

[1] Iooss G., Adelmeyer, M., *Topics in Bifurcation Theory and Applications*, World Scientific, Singapore, 1998.

[2] S. Roy Choudhury, *Solitary-wave families of the Ostrovsky equation: An approach via reversible systems theory and normal forms* , Elsevier, (2007), pp. 1468–1479.