SOLITARY WAVE FAMILIES IN TWO NON-INTEGRABLE MODELS USING REVERSIBLE SYSTEMS THEORY

by

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ABSTRACT

In this thesis, we apply a recently developed technique to comprehensively categorize all possible families of solitary wave solutions in two models of topical interest.

The models considered are:

a. the Generalized Pochammer-Chree Equations, which govern the propagation of longitudinal waves in elastic rods,

and

b. a generalized microstructure PDE.

Limited analytic results exist for the occurrence of one family of solitary wave solutions for each of these equations. Since, as mentioned above, solitary wave solutions often play a central role in the long-time evolution of an initial disturbance, we consider such solutions of both models here (via the normal form approach) within the framework of reversible systems theory.

Besides con rming the existence of the known family of solitary waves for each model, we nd a continuum of delocalized solitary waves (or homoclinics to small-amplitude periodic orbits). On isolated curves in the relevant parameter region, the delocalized waves reduce to genuine embedded solitons. For both models, the new family of solutions occur in regions of parameter space distinct from the known solitary wave solutions and are thus entirely new.

Directions for future work, including the dynamics of each family of solitary waves using exponential asymptotics techniques, are also mentioned.

Dedicated to my Family

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER ONE: INTRODUCTION

Solitary wave solutions of nonlinear models have become increasingly important, both as possible information carriers, as well as organizing centers for the solution dynamics in regimes where the initial conditions naturally break into stable pulses or pulse-trains. Standard techniques for investigating solitary waves of integrable nonlinear PDEs, such as the Inverse Scattering Transform, do not carry over to the non-integrable models which are of increasing relevance in modern applications. Other techniques which have been devised, such as variational ones, and exponential asymptotics methods, each yield results in certain regimes of the systems parameters.

In this thesis, we apply a recently developed technique to comprehensively categorize all possible families of solitary wave solutions in two models of topical interest.

The models considered are:

a. the Generalized Pochammer-Chree Equations, which govern the propagation of longitudinal waves in elastic rods,

$$(u \quad u_{xx})_{tt} \quad a_1 u + a_2 u^2 + a_3 u^3 \;{}_{xx} = 0 \tag{1.1}$$

and

$$(u \quad u_{xx})_{tt} \quad a_1 u + a_3 u^3 + a_5 u^5 \;{}_{xx} = 0 \tag{1.2}$$

and

b. a generalized microstructure PDE.

$$v_{tt} \quad bv_{xx} \quad \frac{\ }{2}\; v^2{}_{xx} \quad (\ v_{tt} \quad v_{xx})_{xx} = 0 \tag{1.3}$$

Limited analytic results exist for the occurrence of one family of solitary wave solutions for each of these equations. Since, as mentioned above, solitary wave solutions often play a central role in the long-time evolution of an initial disturbance, we consider such solutions of both models here (via the normal form approach) within the framework of reversible systems theory.

Besides con rming the existence of the known family of solitary waves for each model, we nd a continuum of delocalized solitary waves (or homoclinics to small-amplitude periodic orbits). On isolated curves in the relevant parameter region, the delocalized waves reduce to genuine embedded solitons. For both models, the new family of solutions occur in regions of parameter space distinct from the known solitary wave solutions and are thus entirely new.

Directions for future work, including the dynamics of each family of solitary waves using exponential asymptotics techniques, are also mentioned.

CHAPTER TWO: GENERALIZED POCHAMMER-CHREE EQUATIONS

2.1 Solitary waves: local bifurcation

Solitary waves of (1.1) and (1.2) of the form $v(x;t) = \ (x \quad ct) = \ (z)$ satisfy the fourth-order traveling wave ODE

$$_{zzzz} \quad q \ _{zz} + p \ = N_{1;2}[\ ] \tag{2.1}$$

where

$$N_1[\ ] \ = \ \frac{1}{c^2}\ 3a_3\ 2 \ \ _z^2 + \ ^2 \ _{zz} \ + 2a_2 \ \ _{zz} \ _z + \ _z^2 \tag{2.2a}$$

$$N_2[\ ] \ = \ \frac{1}{c^2}\ 3a_3\ 2 \ \ _z^2 + \ ^2 \ _{zz} \ + 5a_5\ 4 \ ^3 \ _z^2 + \ ^4 \ _{zz} \tag{2.2b}$$

$$z \qquad x \quad ct \tag{2.3a}$$

$$p \qquad 0 \tag{2.3b}$$

$$q \qquad 1 \quad \frac{a_1}{c^2} \tag{2.3c}$$

$$\tag{2.3d}$$

Equation (3.1) is invariant under the transformation z 7! z and is thus a reversible system. In this section we shall use the theory of reversible systems to characterize the homoclinic orbits to the xed point of (3.1), which correspond to pulses or solitary waves of (1.1) and (1.2) in various regions of the $(p;q)$ plane.

The linearized system corresponding to (3.1)

$$ _{zzzz} \quad q \; _{zz} + p \quad = 0 \tag{2.4}$$

has a xed point

$$ = \; _{z} = \; _{zz} = \; _{zzz} = 0 \tag{2.5}$$

Solutions $= ke^{x}$ satisfy the characteristic equation $^{4} \quad q \; ^{2} + p = 0$ from which one may deduce that the structure of the eigenvalues is distinct in two regions of $(p; q)$-space. Since $p = 0$ we have only two possible regions of eigenvalues. We denote C_0 as the positive q axis and C_1 the negative q-axis. First we shall consider the bounding curves C_0 and C_1 and their neighborhoods, then we shall discuss the possible occurrence and multiplicity of homoclinic orbits to (3.5), corresponding to pulse solitary waves of (1.1) and (1.2), in each region:

Near C_0 The eigenvalues have the structure $_{1} \; _{4} = 0; 0; \quad , (\quad 2 \; R)$ and the xed point (3.5) is a saddle-focus.

Near C_1 Here the eigenvalues have the structure $_{1} \; _{4} = 0; 0; \quad i!, (! \; 2 \; R)$. We will show by analysis of a four-dimensional normal form in Section 4 that there exists a sech^2 homoclinic orbit near C_1.

Having outlined the possible families of orbits homoclinic to the xed point (3.5) of (3.4), corresponding to pulse solitary waves of (1.1) and (1.2), we now derive normal forms near the transition curves C_0 and C_1 to con rm the existence of regular or delocalized solitary waves in the corresponding regions of $(p; q)$ parameter space.

2.2 Normal form near C_0: solitary wave solutions

Using (3.4), the curve C_0, corresponding to $= 0; 0; \tilde{}$, is given by

$$C_0 : p = 0; q > 0 \tag{2.6}$$

Using (3.3c) implies

$$a_1 < c^2 \tag{2.7}$$

Denoting by y_1, (3.1) may be written as the two systems

$$\frac{dy_1}{dz} = y_2 \tag{2.8a}$$

$$\frac{dy_2}{dz} = y_3 \tag{2.8b}$$

$$\frac{dy_3}{dz} = y_4 \tag{2.8c}$$

$$\frac{dy_4}{dz} = qy_3 \quad py_1 \quad N_{1;2}(Y) \tag{2.8d}$$

where

$$N_1(Y) = \frac{1}{c^2} \; 3a_3 \; 2y_1y_2^2 + y_1^2y_3 \; + 2a_2 \; y_3y_2 + y_2^2 \tag{2.9a}$$

$$N_2(Y) = \frac{1}{c^2} \; 3a_3 \; 2y_1y_2^2 + y_1^2y_3 \; + 5a_5 \; 4y_1^3y_2^2 + y_1^4y_3 \tag{2.9b}$$

We wish to rewrite this as a rst order reversible system in order to invoke the relevant theory [1]. To that end, de ning $Y = \langle y_1; y_2; y_3; y_4 \rangle^T$ equation (3.8) may be written

$$\frac{dY}{dz} = L_{pq}Y \quad G_{1;2}(Y; Y) \tag{2.10}$$

where

$$
\boldsymbol{L_{pq}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \boldsymbol{q}/3 & 0 & 1 & 0 \\ 0 & \boldsymbol{q}/3 & 0 & 1 \\ \boldsymbol{q}^2 \quad \boldsymbol{p} & 0 & \boldsymbol{q}/3 & 0 \end{pmatrix} \tag{2.11}
$$

Since $\boldsymbol{p} = 0$ for (1.1) and (1.2), we have

$$
\frac{\boldsymbol{dY}}{\boldsymbol{dz}} = \boldsymbol{L}_{0\boldsymbol{q}}\boldsymbol{Y} \quad \boldsymbol{G}_{1,2}(\boldsymbol{Y},\boldsymbol{Y}) \tag{2.12}
$$

where

$$
\boldsymbol{G}_{1,2}(\boldsymbol{Y},\boldsymbol{Y}) = \langle 0, 0, 0, \quad N_{1,2}(\boldsymbol{Y})\rangle^{\boldsymbol{T}} \tag{2.13}
$$

Next we calculate the normal form of (3.11) near $\boldsymbol{C}_0$. The procedure is closely modeled on [1] and many intermediate steps may be found there.

2.2.1 Near $\boldsymbol{C}_0$

Near $\boldsymbol{C}_0$ the dynamics reduce to a two-dimensional Center Manifold

$$
\boldsymbol{Y} = \boldsymbol{A}\ {}_0 + \boldsymbol{B}\ {}_1 + \quad (\ ,\boldsymbol{A},\boldsymbol{B}) \tag{2.14}
$$

and the corresponding normal form is

$$
\frac{\boldsymbol{dA}}{\boldsymbol{dz}} = \boldsymbol{B} \tag{2.15a}
$$

$$
\frac{\boldsymbol{dB}}{\boldsymbol{dz}} = \boldsymbol{b}\ \boldsymbol{A} + \boldsymbol{c}\boldsymbol{A}^2 \tag{2.15b}
$$

Here,

$$
= \frac{\boldsymbol{q}^2}{9} \quad \boldsymbol{p} \qquad \frac{\boldsymbol{q}}{3}\ ^2 = \quad \boldsymbol{p} \tag{2.16}
$$

measures the perturbation around $\boldsymbol{C}_0$, and

$$_0 = \langle 1; 0; \quad \boldsymbol{q}/3; 0\rangle^{\boldsymbol{T}} \tag{2.17a}$$

$$_1 = \langle 0; 1; 0; \quad 2\boldsymbol{q}/3\rangle^{\boldsymbol{T}} \tag{2.17b}$$

The linear eigenvalue of (3.14) satis es

$$^2 = \boldsymbol{b} \tag{2.18}$$

The characteristic equation of the linear part of (3.11) is

$$^4 \quad \boldsymbol{q} \quad ^2 \qquad = 0 \tag{2.19}$$

Hence, the eigenvalues near zero (the Center Manifold) satisfy $^4 \qquad ^2$ and hence

$$^2 \qquad \overline{\boldsymbol{q}} \tag{2.20}$$

Matching (3.17) and (3.19) implies

$$\boldsymbol{b} = \quad \frac{1}{\boldsymbol{q}} \tag{2.21}$$

and only the nonlinear coe cient $\boldsymbol{c}$ remains to be determined in the normal form (3.14).

In order to determine $\boldsymbol{c}$ (the coe cient of $\boldsymbol{A}^2$ in (3.14)) we calculate $\frac{d\boldsymbol{Y}}{d\boldsymbol{z}}$ in two ways and match the $O(\boldsymbol{A}^2)$ terms. To this end, using the standard 'suspension' trick of treating the perturbation parameter as a variable, we expand the function in (3.13) as

$$(\; ; \boldsymbol{A}; \boldsymbol{B}) = \boldsymbol{A} \quad {}^{1}_{10} + \boldsymbol{B} \quad {}^{1}_{01} + \boldsymbol{A}^2 \quad {}^{0}_{20} + \boldsymbol{A}\boldsymbol{B} \quad {}^{0}_{11} + \boldsymbol{B}^2 \quad {}^{0}_{02} + \tag{2.22}$$

where the subscripts denote powers of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively, and the superscript denotes the power of . In the rst way of computing $\boldsymbol{dY{=}dz}$, we take the $\boldsymbol{z}$ derivative of (3.13) (using (3.14) and (3.21)). The coe cient of $\boldsymbol{A}^2$ in the resulting expression is $\boldsymbol{c}\ _1$. In the second way of computing $\boldsymbol{dY{=}dz}$, we use (3.13) and (3.21) in (3.9). The coe cient of $\boldsymbol{A}^2$ in the resulting expression is $\boldsymbol{L}_{0;\boldsymbol{q}}\ {}^{0}_{20}\quad \boldsymbol{G}_{1;2}(\ _0\boldsymbol{;}\ _0)$. Hence

$$\boldsymbol{c}\ _1 = \boldsymbol{L}_{0\boldsymbol{q}}\ {}^{0}_{20}\quad \boldsymbol{F}_2(\ _0\boldsymbol{;}\ _0) \tag{2.23}$$

Using (3.16) and (3.12) and denoting ${}^{0}_{20} = h\boldsymbol{x}_1\boldsymbol{;}\boldsymbol{x}_2\boldsymbol{;}\boldsymbol{x}_3\boldsymbol{;}\boldsymbol{x}_4 i$ in (3.22) yields the equations

$$0 = \boldsymbol{x}_2 \tag{2.24a}$$

$$\boldsymbol{c} = \frac{\boldsymbol{q}}{3}\boldsymbol{x}_1 + \boldsymbol{x}_3 \tag{2.24b}$$

$$0 = \frac{\boldsymbol{q}}{3}\boldsymbol{x}_2 + \boldsymbol{x}_4 \quad =)\quad \boldsymbol{x}_4 = 0 \text{ using (3.23b)} \tag{2.24c}$$

and

$$\frac{2\boldsymbol{q}}{3}\boldsymbol{c} = \frac{\boldsymbol{q}}{3}\quad \frac{\boldsymbol{q}}{3}\boldsymbol{x}_1 + \boldsymbol{x}_3 \quad + \frac{\boldsymbol{q}}{3\boldsymbol{c}^2}(3\boldsymbol{a}_3 + 5\boldsymbol{a}_5) = \frac{\boldsymbol{q}}{3}\boldsymbol{c} + \frac{\boldsymbol{q}}{3\boldsymbol{c}^2}(3\boldsymbol{a}_3 + 5\boldsymbol{a}_5) \text{ using (3.23b)} \tag{2.25}$$

Hence we obtain

$$\boldsymbol{c} = \quad \frac{1}{3\boldsymbol{c}^2}(3\boldsymbol{a}_3 + 5\boldsymbol{a}_5) \tag{2.26}$$

Therefore, the normal form near $\boldsymbol{C}_0$ is

$$\frac{\boldsymbol{dA}}{\boldsymbol{dz}} = \boldsymbol{B} \tag{2.27a}$$

$$\frac{\boldsymbol{dB}}{\boldsymbol{dz}} = \quad \frac{-}{\boldsymbol{q}}\boldsymbol{A} \quad \frac{\boldsymbol{a}_3}{\boldsymbol{c}^2}\boldsymbol{A}^2 \tag{2.27b}$$

for (1.1) and

$$\frac{dA}{dz} = B \tag{2.28a}$$

$$\frac{dB}{dz} = \frac{-}{q}A \quad \frac{1}{3c^2}(3a_3 + 5a_5)A^2 \tag{2.28b}$$

for (1.2).

The normal form (2.27) admits a homoclinic solution (near C_0) of the form

$$A(z) = \ell \operatorname{sech}^2(kz) \tag{2.29}$$

with

$$k = \sqrt{\frac{\quad}{4q}} \tag{2.30a}$$

$$\ell = \frac{3 \quad c^2}{2qa_3} \tag{2.30b}$$

Similarly, the normal form (2.28) admits a homoclinic solution (near C_0) of the form

$$A(z) = \ell \operatorname{sech}^2(kz) \tag{2.31}$$

with

$$k = \sqrt{\frac{\quad}{4q}} \tag{2.32a}$$

$$\ell = \frac{3 \quad c^2}{2q(3a_3 + 5a_5)} \tag{2.32b}$$

Hence, since $\quad = \quad p$, and the curve C_0 corresponds to $p = 0; q > 0$, solitary waves of the form (3.27) exist in the vicinity of C_0 for

$$p > 0; q > 0 \tag{2.33}$$

which implies that $\boldsymbol{a}_1 < \boldsymbol{c}^2$ (such that $\boldsymbol{k}$ in (3.28a) is real.) As mentioned in section 2, one may show the persistence of this homoclinic solution in the original traveling wave ODE (3.4). Thus, we have demonstrated the existence of solitary waves of (1.2) for $\boldsymbol{p} = 0^+; \boldsymbol{q} > 0$.

Similarly, the curve $\boldsymbol{C}_1$ corresponds to $\boldsymbol{p} = 0; \boldsymbol{q} < 0$, solitary waves of the form (3.27) exist in the vicinity of $\boldsymbol{C}_1$ for

$$\boldsymbol{p} < 0; \boldsymbol{q} < 0 \tag{2.34}$$

which implies $\boldsymbol{a}_1 > \boldsymbol{c}^2$.

Again, one may show the persistence of this homoclinic solution in the original traveling wave ODE (3.4). Thus, we have demonstrated the existence of solitary waves of (1.2) for $\boldsymbol{p} = 0 \;\; ; \boldsymbol{q} < 0$.

2.3 Normal form near $\boldsymbol{C}_1$: possible solitary wave solutions

Using (3.4), the curve $\boldsymbol{C}_1$, corresponding to $\;\; = 0; 0 \;\; \boldsymbol{i!}$, is given by

$$\boldsymbol{C}_1 : \boldsymbol{p} = 0; \boldsymbol{q} < 0 \tag{2.35}$$

Which implies

$$\boldsymbol{a}_1 > \boldsymbol{c}^2 \tag{2.36}$$

In order to investigate the possibility of a sech^2 homoclinic orbit in the neighborhood of $\boldsymbol{C}_1$ and delocalized solitary waves, we next compute the normal form near $\boldsymbol{C}_1$ following the procedure in [1].

Near C_1 the dynamics reduce to a four-dimensional Center Manifold [1]. Since all the eigenvalues are non-hyperbolic, the Center Manifold has the form (a nonlinear coordinate change [1])

$$Y = A\ _0 + B\ _0 + C\ _+ + C\quad + \quad(\ ; A; B; C; C) \tag{2.37}$$

with a corresponding four-dimensional normal form

$$\frac{dA}{dz} = B \tag{2.38a}$$

$$\frac{dB}{dz} = \ A + b\ A^2 + c\ |C|^2 \tag{2.38b}$$

$$\frac{dC}{dz} = id_0C + i\ d_1C + id_2AC \tag{2.38c}$$

Here C is complex, C is the complex conjugate of C, $\ ;\ _0;\ _1$ are given previously and the two new complex eigenvectors co-spanning the Center Manifold are

$$= \left(1;\quad ; 2q{=}3; \frac{}{3}q\right)^T \tag{2.39}$$

Using (2.38b) and (3.14b) implies

$$= b = \frac{}{q} \tag{2.40}$$

Also from the characteristic equation (3.18), the two non-zero (imaginary) roots are

$$_2 = \frac{q + \sqrt{q^2 + 4}}{2} \quad q \text{ for } \quad \text{small} \tag{2.41}$$

Hence

$$= \quad i\sqrt{-q}; q < 0 \tag{2.42}$$

Matching this to the linear part of (2.38c) (which corresponds to the imaginary eigenvalues), $= i d_0 = i\sqrt{q}$ or

$$d_0 = \sqrt{q} \tag{2.43}$$

With a dominant balance argument after the change of variable $= \sqrt{3}$ on the characteristic equation (3.18) as $\;! \; 0$ we nd $d_1 = \frac{\sqrt{3}}{18\ ^2}$. Using $= q/3$ implies

$$d_1 = \frac{\sqrt{q}}{2q^2} \tag{2.44}$$

The remaining undetermined coe cients in the normal form are the coe cients $b\ ; c$ and d_2 which correspond to the $A^2; jCj^2$ and AC terms respectively. In order to determine them, we follow the same procedure as in Section 3 and compute dY/dz is two distinct ways. We expand the function as

$$(\ ; A; B; C; C) = \ A \ ^{1}_{1000} + \ B \ ^{1}_{0100} + A^2 \ ^{0}_{2000} + AB \ ^{0}_{1100} + AC \ ^{0}_{1010} + \ C \ ^{1}_{0010} + \tag{2.45}$$

with subscripts denoting powers of A, B, C and C, respectively, and the superscript is the power of . In the rst way, dY/dz is computed by taking the z derivative of (3.32) (using (3.33) and (3.40)) and read o the coe cients of $A^2; kCk^2; C$ and AC terms. In the second way, dY/dz is computed using (3.32) and (3.40) in (3.9) (with $p = 0$ on C_1 as given in (3.30)) and the coe cients of A, B, C and C are once again read o . Equating the coe cients of the corresponding terms in the two separate expressions for dY/dz yields the following two systems of equations:

$$O(\boldsymbol{A}^2): \quad \boldsymbol{b} \quad {}_1 \quad = \boldsymbol{L}_{0\boldsymbol{q}} \quad {}^{0}_{2000} \quad \boldsymbol{G}_{1;2}(\ {}_0; \ {}_0) \tag{2.46a}$$

$$O(j\boldsymbol{C}j^2): \quad \boldsymbol{c} \quad {}_1 \quad = \boldsymbol{L}_{0\boldsymbol{q}} \quad {}^{0}_{0011} \quad 2\boldsymbol{G}_{1;2}(\ {}_+; \quad) \tag{2.46b}$$

$$O(\ \boldsymbol{C}): \quad \frac{\boldsymbol{i}}{\boldsymbol{q}}(\boldsymbol{d}_1 \ {}_+ + \boldsymbol{d}_0 \ {}^{1}_{0010}) = \boldsymbol{L}_{0\boldsymbol{q}} \quad {}^{1}_{0010} \quad \boldsymbol{G}_{1;2}(\ {}^{1}_{0010}; \ {}^{1}_{0010}) \tag{2.46c}$$

$$O(\boldsymbol{AC}): \quad \boldsymbol{id}_2 \ {}_+ + \boldsymbol{id}_0 \ {}^{0}_{1010} = \boldsymbol{L}_{0\boldsymbol{q}} \quad {}^{0}_{1010} \quad 2\boldsymbol{G}_{1;2}(\ {}_0; \ {}_+) \tag{2.46d}$$

where we have used the fact that $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are symmetric bilinear forms. Equation (3.41b) is decoupled and yields $\boldsymbol{c} \ = \frac{8}{\boldsymbol{c}^2}(2\boldsymbol{a}_3 \quad \boldsymbol{a}_2)$ and $\boldsymbol{c} \ = \frac{1}{\boldsymbol{c}^2} \ 16\boldsymbol{a}_3 + \frac{140}{3}\boldsymbol{a}_5$ for (1.2). The only coe cient left to determine is $\boldsymbol{d}_2$ which we shall compute now.

Using ${}^{0}_{1010} = h\boldsymbol{x}_1; \boldsymbol{x}_2; \boldsymbol{x}_3; \boldsymbol{x}_4 i^{\boldsymbol{T}}$ in (3.41d) implies

$$\boldsymbol{id}_2 + \boldsymbol{id}_0\boldsymbol{x}_1 = \boldsymbol{x}_2 \tag{2.47a}$$

$$\boldsymbol{d}_0\boldsymbol{d}_2 + \boldsymbol{id}_0\boldsymbol{x}_2 = \frac{\boldsymbol{q}}{3}\boldsymbol{x}_1 + \boldsymbol{x}_3 \tag{2.47b}$$

$$\frac{2\boldsymbol{iq}}{3}\boldsymbol{d}_2 + \boldsymbol{id}_0\boldsymbol{x}_3 = \frac{\boldsymbol{q}}{3}\boldsymbol{x}_2 + \boldsymbol{x}_4 \tag{2.47c}$$

$$\frac{\boldsymbol{q}}{3}\boldsymbol{d}_0\boldsymbol{d}_2 + \boldsymbol{id}_0\boldsymbol{x}_4 = \frac{\boldsymbol{q}}{3} \quad \frac{\boldsymbol{q}}{3}\boldsymbol{x}_1 + \boldsymbol{x}_3 \qquad \frac{2\boldsymbol{q}}{\boldsymbol{c}^2} \quad \frac{7}{2}\boldsymbol{a}_3 \qquad \frac{\boldsymbol{i}}{3}\boldsymbol{d}_0\boldsymbol{a}_2 \tag{2.47d}$$

for (1.1) and

$$\boldsymbol{id}_2 + \boldsymbol{id}_0\boldsymbol{x}_1 = \boldsymbol{x}_2 \tag{2.48a}$$

$$\boldsymbol{d}_0\boldsymbol{d}_2 + \boldsymbol{id}_0\boldsymbol{x}_2 = \frac{\boldsymbol{q}}{3}\boldsymbol{x}_1 + \boldsymbol{x}_3 \tag{2.48b}$$

$$\frac{2\boldsymbol{iq}}{3}\boldsymbol{d}_2 + \boldsymbol{id}_0\boldsymbol{x}_3 = \frac{\boldsymbol{q}}{3}\boldsymbol{x}_2 + \boldsymbol{x}_4 \tag{2.48c}$$

$$\frac{\boldsymbol{q}}{3}\boldsymbol{d}_0\boldsymbol{d}_2 + \boldsymbol{id}_0\boldsymbol{x}_4 = \frac{\boldsymbol{q}}{3} \quad \frac{\boldsymbol{q}}{3}\boldsymbol{x}_1 + \boldsymbol{x}_3 \qquad \frac{2\boldsymbol{q}}{\boldsymbol{c}^2} \quad \frac{7}{2}\boldsymbol{a}_3 + \frac{32}{3}\boldsymbol{a}_5 \tag{2.48d}$$

for (1.2)

Using (2.47a) in (2.47b), (2.47b) in (2.47d) and using these in (2.47c) yields $d_2 = \frac{1}{c^2}\ \ \frac{7}{2\sqrt{q}}a_3\ \ \frac{i}{3}a_2$ for (1.1). Similarly using (2.48a) in (2.48b), (2.48b) in (2.48d) and using these in (2.48c) yields $d_2 = \frac{1}{\sqrt{q}c^2}\ \ \frac{7}{2}a_3 + \frac{32}{3}a_5$ for (1.2).

Therefore the normal form near C_1 is

$$\frac{dA}{dz} = B \tag{2.49a}$$

$$\frac{dB}{dz} = \ \frac{\ }{q}A\ \ b\ A^2 + \frac{1}{c^2}\ \ \frac{7}{2\sqrt{q}}a_3\ \ \frac{i}{3}a_2\ \ jCj^2 \tag{2.49b}$$

$$\frac{dC}{dz} = i\sqrt{\ \ }qC\ \ i\frac{\sqrt{\ \ q}}{q^3}C\ + i\frac{1}{c^2}\ \ \frac{7}{2\sqrt{q}}a_3\ \ \frac{i}{3}a_2\ \ AC \tag{2.49c}$$

for (1.1) and

$$\frac{dA}{dz} = B \tag{2.50a}$$

$$\frac{dB}{dz} = \ \frac{\ }{q}A\ \ b\ A^2 + \frac{1}{c^2}\ \ 16a_3 + \frac{140}{3}a_5\ \ jCj^2 \tag{2.50b}$$

$$\frac{dC}{dz} = i\sqrt{\ \ }qC\ \ i\frac{\sqrt{\ \ q}}{q^3}C\ + i\frac{1}{\sqrt{\ \ }qc^2}\ \ \frac{7}{2}a_3 + \frac{32}{3}a_5\ \ AC \tag{2.50c}$$

for (1.2).

The dynamics inherent in (2.49), (2.50) may be elucidated following the discussions of [1], [2], [3] and [4]. The two rst integrals of (3.33) are

$$K = jCj^2 \tag{2.51}$$

and

$$H = B^2\ \ \frac{2}{3}b\ A^3\ \ \ A^2\ \ 2c\ KA \tag{2.52}$$

Here, the appropriate coe cients b ; and c , derived above, apply for (1.1) and (1.2). Also, c should be real, or a_2 must be zero in (1.1) for the following energy arguments to apply.

As a typical case, consider the level curve $\boldsymbol{H} = 0$ of the energy-like rst integral function $\boldsymbol{H}$. In the $(\boldsymbol{A}; \boldsymbol{B})$ phase plane, this will compromise a homoclinic orbit. The intersection of $\boldsymbol{H} = 0$ with the $\boldsymbol{A}$ axis occurs for $\frac{2}{3}\boldsymbol{b}\,\boldsymbol{A}^2 \quad \boldsymbol{A} \quad 2\boldsymbol{c}\,\boldsymbol{K} = 0$ or

$$\boldsymbol{A} = \frac{3}{4\boldsymbol{b}} \left[\quad \sqrt{\quad^2 + \frac{16\boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{K}}{3}} \right] \tag{2.53}$$

Note that $\boldsymbol{A}_+ > 0; \boldsymbol{A} < 0$ for $\boldsymbol{b}\,\boldsymbol{c} > 0$ and $\boldsymbol{b} < 0$ as relevant for us. A general homoclinic orbit, homoclinic to $\boldsymbol{A}_+$, is sketched in Figure 1 where the ow direction is deduced from (2.49a) and (2.50a) for (1.1) and (1.2), respectively. For $\boldsymbol{K} = |\boldsymbol{C}|^2 = 0$, the orbit is homoclinic to $\boldsymbol{A}_+ = 0$. For small non-zero $|\boldsymbol{K}|$, $\boldsymbol{A}_+ \quad 2\boldsymbol{c}\,\boldsymbol{K}/\;$, meaning that oscillations at in nity are then very small in this case. For $\boldsymbol{K} = 0$ this corresponds to an *orbit homoclinic to* 0 for the normal form. This is indeed valid for the normal form taken at any order. However this solution does not exist mathematically for the full original system, even though one may compute its expansion in powers of the bifurcation parameter up to any order (see [3] and [4]). This is an example of the famous challenging problem of asymptotics beyond any orders. Other solutions found on the normal form mainly persist under the perturbation from higher order terms provided by the original system [2]. These solutions are delocalized waves and their existence in Region 2 is guaranteed by the general theory for reversible systems in [3] and [4]. Also, as mentioned in Section 2, genuine solitary waves are found on isolated curves in Region 2 of Figure 1 on which the oscillation amplitudes vanish. Since these are embedded in the sea of delocalized solitary waves and in the continuous spectrum, they are referred to as embedded solitons [5]. These will further be investigated in Region 2 subsequently using a mix of exponential asymptotics and numerical shooting.

Figure 2.1: Level curves of (3.45) corresponding to various values of H.

CHAPTER THREE: SOLITARY WAVE FAMILIES OF A MICROSTRUCTURE PDE

3.1 Solitary waves; local bifurcations

Solitary waves of (1.3) of the form $v(x;t) = \quad (x \quad ct) = \quad (z)$ satisfy the fourth-order traveling wave ODE

$$_{zzzz} \quad q \; _{zz} + p \quad = N[\;] \tag{3.1}$$

where

$$N[\;] = \quad _1 \, ^2_z \quad b \; _1 \quad _{zz} \tag{3.2}$$

$$z \qquad x \quad ct \tag{3.3a}$$

$$p \qquad 0 \tag{3.3b}$$

$$q \qquad \frac{c^2 \quad b}{(\; c^2 \quad\;)} \tag{3.3c}$$

$$_1 \qquad \frac{}{(\; c^2 \quad\;)} \tag{3.3d}$$

Equation (3.1) is invariant under the transformation z 7! z and is thus a reversible system. In this section we shall use the theory of reversible systems to characterize the homoclinic orbits to the xed point of (3.1), which correspond to pulses or solitary waves of (1.3) in various regions of the $(p;q)$ plane.

The linearized system corresponding to (3.1)

$$_{zzzz} \quad q \; _{zz} + p \quad = 0 \tag{3.4}$$

has a xed point

$$= \ _z = \ _{zz} = \ _{zzz} = 0 \tag{3.5}$$

Solutions $= \boldsymbol{ke}^{\ x}$ satisfy the characteristic equation $^4 \quad \boldsymbol{q}\ ^2 + \boldsymbol{p} = 0$ from which one may deduce that the structure of the eigenvalues is distinct in two regions of $(\boldsymbol{p};\boldsymbol{q})$-space. Since $\boldsymbol{p} = 0$ we have only two possible regions of eigenvalues. We denote $\boldsymbol{C}_0$ as the positive $\boldsymbol{q}$ axis and $\boldsymbol{C}_1$ the negative $\boldsymbol{q}$-axis. First we shall consider the bounding curves $\boldsymbol{C}_0$ and $\boldsymbol{C}_1$ and their neighborhoods, then we shall discuss the possible occurrence and multiplicity of homoclinic orbits to (3.5), corresponding to pulse solitary waves of (1.3), in each region:

Near $\boldsymbol{C}_0$ The eigenvalues have the structure $_{1\ 4} = 0;0; \quad , (\ 2\ R)$ and the xed point (3.5) is a saddle-focus.

Near $\boldsymbol{C}_1$ Here the eigenvalues have the structure $_{1\ 4} = 0;0; \ \boldsymbol{i!}, (\boldsymbol{!}\ 2\ R)$. We will show by analysis of a four-dimensional normal form in Section 4 that there exists a sech^2 homoclinic orbit near $\boldsymbol{C}_1$.

Having outlined the possible families of orbits homoclinic to the xed point (3.5) of (3.4), corresponding to pulse solitary waves of (1.3), we now derive normal forms near the transition curves $\boldsymbol{C}_0$ and $\boldsymbol{C}_1$ to con rm the existence of regular or delocalized solitary waves in the corresponding regions of $(\boldsymbol{p};\boldsymbol{q})$ parameter space.

3.2 Normal form near $\boldsymbol{C}_0$: solitary wave solutions

Using (3.4), the curve $\boldsymbol{C}_0$, corresponding to $= 0;0; \ \tilde{}$, is given by

$$\boldsymbol{C}_0 : \boldsymbol{p} = 0; \boldsymbol{q} > 0 \tag{3.6}$$

Using (3.3c) implies

$$\frac{c^2 \quad b}{(\ c^2 \quad)} > 0 \tag{3.7}$$

Denoting   by y_1, equation (3.1) may be written as the system

$$\frac{dy_1}{dz} = y_2 \tag{3.8a}$$

$$\frac{dy_2}{dz} = y_3 \tag{3.8b}$$

$$\frac{dy_3}{dz} = y_4 \tag{3.8c}$$

$$\frac{dy_4}{dz} = qy_3 \quad py_1 \qquad {}_1 y_2^2 + b \quad {}_1 y_1 y_3 \tag{3.8d}$$

We wish to rewrite this as a rst order reversible system in order to invoke the relevant theory [1]. To that end, de ning $Y = \langle y_1, y_2, y_3, y_4 \rangle^T$, equation (3.8) may be written

$$\frac{dY}{dz} = L_{pq} Y \quad F_2(Y, Y) \tag{3.9}$$

where

$$L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2 \quad p & 0 & q/3 & 0 \end{pmatrix} \tag{3.10}$$

Since $p = 0$ for (1.3), we have

$$\frac{dY}{dz} = L_{0q} Y \quad F_2(Y, Y) \tag{3.11}$$

where

$$F_2(Y, Y) = \quad 0, 0, 0, \quad {}_1 y_2^2 + b \quad {}_1 y_1 y_3 \quad {}^T \tag{3.12}$$

Next we calculate the normal form of (3.11) near $\boldsymbol{C}_0$. The procedure is closely modeled on [1] and many intermediate steps may be found there.

3.2.1 Near $\boldsymbol{C}_0$

Near $\boldsymbol{C}_0$ the dynamics reduce to a two-dimensional Center Manifold

$$\boldsymbol{Y} = \boldsymbol{A}\ _0 + \boldsymbol{B}\ _1 + \quad (\ ;\boldsymbol{A};\boldsymbol{B}) \tag{3.13}$$

and the corresponding normal form is

$$\frac{d\boldsymbol{A}}{d\boldsymbol{z}} = \boldsymbol{B} \tag{3.14a}$$

$$\frac{d\boldsymbol{B}}{d\boldsymbol{z}} = \boldsymbol{b}\ \boldsymbol{A} + \boldsymbol{c}\boldsymbol{A}^2 \tag{3.14b}$$

Here,

$$= \frac{\boldsymbol{q}^2}{9} \quad \boldsymbol{p} \qquad \frac{\boldsymbol{q}}{3}\ ^2 = \quad \boldsymbol{p} \tag{3.15}$$

measures the perturbation around $\boldsymbol{C}_0$, and

$$_0 = h1; 0; \quad \boldsymbol{q}{=}3; 0i^{\boldsymbol{T}} \tag{3.16a}$$

$$_1 = h0; 1; 0; \quad 2\boldsymbol{q}{=}3i^{\boldsymbol{T}} \tag{3.16b}$$

The linear eigenvalue of (3.14) satis es

$$^2 = \boldsymbol{b} \tag{3.17}$$

The characteristic equation of the linear part of (3.11) is

$$^4 \quad \boldsymbol{q}\ ^2 \qquad = 0 \tag{3.18}$$

Hence, the eigenvalues near zero (the Center Manifold) satisfy ${}^{4} \quad {}^{2}$ and hence

$$ {}^{2} \qquad \frac{}{\boldsymbol{q}} \tag{3.19}$$

Matching (3.17) and (3.19)

$$\boldsymbol{b} = \frac{1}{\boldsymbol{q}} \tag{3.20}$$

and only the nonlinear coe cient $\boldsymbol{c}$ remains to be determined in the normal form (3.14).

In order to determine $\boldsymbol{c}$ (the coe cient of $\boldsymbol{A}^2$ in (3.14)) we calculate $\frac{\boldsymbol{dY}}{\boldsymbol{dz}}$ in two ways and match the $O(\boldsymbol{A}^2)$ terms. To this end, using the standard 'suspension' trick of treating the perturbation parameter as a variable, we expand the function in (3.13) as

$$(\,;\boldsymbol{A};\boldsymbol{B}) = \boldsymbol{A} \; {}^{1}_{10} + \boldsymbol{B} \; {}^{1}_{01} + \boldsymbol{A}^2 \; {}^{0}_{20} + \boldsymbol{AB} \; {}^{0}_{11} + \boldsymbol{B}^2 \; {}^{0}_{02} + \tag{3.21}$$

where the subscripts denote powers of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively, and the superscript denotes the power of .

In the rst way of computing $\boldsymbol{dY=dz}$, we take the $\boldsymbol{z}$ derivative of (3.13) (using (3.14) and (3.21)). The coe cient of $\boldsymbol{A}^2$ in the resulting expression is $\boldsymbol{c} \; {}_{1}$. In the second way of computing $\boldsymbol{dY=dz}$, we use (3.13) and (3.21) in (3.9). The coe cient of $\boldsymbol{A}^2$ in the resulting expression is $\boldsymbol{L}_{0;\boldsymbol{q}} \; {}^{0}_{20} \quad \boldsymbol{F}_2(\; {}_0; \; {}_0)$. Hence

$$\boldsymbol{c} \; {}_{1} = \boldsymbol{L}_{0\boldsymbol{q}} \; {}^{0}_{20} \quad \boldsymbol{F}_2(\; {}_0; \; {}_0) \tag{3.22}$$

Using (3.16) and (3.12) and denoting $^{0}_{20} = h\boldsymbol{x}_1; \boldsymbol{x}_2; \boldsymbol{x}_3; \boldsymbol{x}_4 i$ in (3.22) yields the equations

$$0 = \boldsymbol{x}_2 \tag{3.23a}$$

$$\boldsymbol{c} = \frac{\boldsymbol{q}}{3}\boldsymbol{x}_1 + \boldsymbol{x}_3 \tag{3.23b}$$

$$0 = \frac{\boldsymbol{q}}{3}\boldsymbol{x}_2 + \boldsymbol{x}_4 \;\; =) \quad \boldsymbol{x}_4 = 0 \text{ using (3.23b)} \tag{3.23c}$$

and

$$\frac{2\boldsymbol{q}}{3}\boldsymbol{c} = \frac{\boldsymbol{q}}{3} \quad \frac{\boldsymbol{q}}{3}\boldsymbol{x}_1 + \boldsymbol{x}_3 \quad + \frac{2\boldsymbol{q}}{3} = \frac{\boldsymbol{q}}{3}\boldsymbol{c} + \frac{\boldsymbol{b} \;\; _1}{3} \text{ using (3.23b)} \tag{3.24}$$

Hence we obtain

$$\boldsymbol{c} = \frac{\boldsymbol{b} \;\; _1}{3} \tag{3.25}$$

Therefore, the normal form for (1.3) near $\boldsymbol{C}_0$ is

$$\frac{\boldsymbol{dA}}{\boldsymbol{dz}} = \boldsymbol{B} \tag{3.26a}$$

$$\frac{\boldsymbol{dB}}{\boldsymbol{dz}} = \frac{-}{\boldsymbol{q}}\boldsymbol{A} \quad \frac{\boldsymbol{b} \;\; _1}{3}\boldsymbol{A}^2 \tag{3.26b}$$

The normal form (3.14) admits a homoclinic solution (near $\boldsymbol{C}_0$) of the form

$$\boldsymbol{A}(\boldsymbol{z}) = \text{‘}\mathrm{sech}^2(\boldsymbol{kz}) \tag{3.27}$$

with

$$\boldsymbol{k} = \text{r}\,\frac{\overline{}}{4\boldsymbol{q}} \tag{3.28a}$$

$$\text{‘} = \frac{6\boldsymbol{k}^2}{\boldsymbol{b} \;\; _1} \tag{3.28b}$$

Hence, since $\ =\ \ p$, and the curve C_0 corresponds to $p = 0; q > 0$, solitary waves of the form (3.27) exist in the vicinity of C_0 for

$$p > 0; q > 0 \tag{3.29}$$

which implies that $\frac{c^2\ \ b}{(\ c^2\ \ )} > 0$ (such that k in (3.28a) is real.) As mentioned in section 2, one may show the persistence of this homoclinic solution in the original traveling wave ODE (3.4). Thus, we have demonstrated the existence of solitary waves of (1.3) for $p = 0^+; q > 0$.

3.3 Normal form near C_1: possible solitary wave solutions

Using (3.4), the curve C_1, corresponding to $\ = 0; 0\ \ i\,!$, is given by

$$C_1 : p = 0; q < 0 \tag{3.30}$$

Which implies

$$\frac{c^2\ \ \ b}{(\ c^2\ \ \ )} < 0 \tag{3.31}$$

In order to investigate the possibility of a sech^2 homoclinic orbit in the neighborhood of C_1 and delocalized solitary waves, we next compute the normal form near C_1 following the procedure in [1].

Near C_1 the dynamics reduce to a four-dimensional Center Manifold [1]. Since all the eigenvalues are non-hyperbolic, the Center Manifold has the form (a nonlinear coordinate change [1])

$$Y = A\ _0 + B\ _0 + C\ _+ + C\ \ \ + \ \ (\ ; A; B; C; C) \tag{3.32}$$

with a corresponding four-dimensional normal form

$$\frac{dA}{dz} = B \tag{3.33a}$$

$$\frac{dB}{dz} = A + b\,A^2 + c\,|C|^2 \tag{3.33b}$$

$$\frac{dC}{dz} = id_0C + i\;d_1C + id_2AC \tag{3.33c}$$

Here C is complex, C is the complex conjugate of C, $\;;\;_0;\;_1$ are given previously and the two new complex eigenvectors co-spanning the Center Manifold are

$$= \left(1;\quad;2q=3;\frac{}{3}q\right)^T \tag{3.34}$$

Using (3.33b) and (3.14b)

$$= b = \frac{}{q} \tag{3.35}$$

Also from the characteristic equation (3.18), the two non-zero (imaginary) roots are

$$^2 = \frac{q + \sqrt{q^2+4}}{2} \quad q \text{ for } \quad \text{small} \tag{3.36}$$

Hence

$$= i\sqrt{\;q};q < 0 \tag{3.37}$$

Matching this to the linear part of (3.33c) (which corresponds to the imaginary eigenvalues), $= id_0 = i\sqrt{\;q}$ or

$$d_0 = \sqrt{\;q} \tag{3.38}$$

With a dominant balance argument after the change of variable $= \sqrt{\;3}$ on the characteristic equation (3.18) as $\to 0$ we nd $d_1 = \frac{\sqrt{\;3}}{18\;^2}$. Using $= q=3$ implies

$$d_1 = \frac{\sqrt{\;q}}{2q^2} \tag{3.39}$$

The remaining undetermined coefficients in the normal form are the coefficients $\boldsymbol{b}\ ;\boldsymbol{c}$ and $\boldsymbol{d}_2$ which correspond to the $\boldsymbol{A}^2; j\boldsymbol{C}j^2$ and $\boldsymbol{AC}$ terms respectively. In order to determine them, we follow the same procedure as in Section 3 and compute $\boldsymbol{dY{=}dz}$ is two distinct ways. We expand the function as

$$(\ ;\boldsymbol{A};\boldsymbol{B};\boldsymbol{C};\boldsymbol{C}) = \ \boldsymbol{A}\ {}^{1}_{1000} + \ \boldsymbol{B}\ {}^{1}_{0100} + \boldsymbol{A}^2\ {}^{0}_{2000} + \boldsymbol{AB}\ {}^{0}_{1100} + \boldsymbol{AC}\ {}^{0}_{1010} + \ \boldsymbol{C}\ {}^{1}_{0010} + \tag{3.40}$$

with subscripts denoting powers of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{C}$, respectively, and the superscript is the power of . In the first way, $\boldsymbol{dY{=}dz}$ is computed by taking the $\boldsymbol{z}$ derivative of (3.32) (using (3.33) and (3.40)) and read off the coefficients of $\boldsymbol{A}^2; k\boldsymbol{C}k^2; \boldsymbol{C}$ and $\boldsymbol{AC}$ terms. In the second way, $\boldsymbol{dY{=}dz}$ is computed using (3.32) and (3.40) in (3.9) (with $\boldsymbol{p} = 0$ on $\boldsymbol{C}_1$ as given in (3.30)) and the coefficients of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{C}$ are once again read off. Equating the coefficients of the corresponding terms in the two separate expressions for $\boldsymbol{dY{=}dz}$ yields the following equations:

$$O(\boldsymbol{A}^2): \quad \boldsymbol{b}\ \ _1 = \boldsymbol{L}_{0\boldsymbol{q}}\ {}^{0}_{2000} \quad \boldsymbol{F}_2(\ _0;\ _0) \tag{3.41a}$$

$$O(j\boldsymbol{C}j^2): \quad \boldsymbol{c}\ \ _1 = \boldsymbol{L}_{0\boldsymbol{q}}\ {}^{0}_{0011} \quad 2\boldsymbol{F}_2(\ _+;\ \ ) \tag{3.41b}$$

$$O(\ \boldsymbol{C}): \quad \frac{\boldsymbol{i}}{\boldsymbol{q}}(\boldsymbol{d}_1\ \ _+ + \boldsymbol{d}_0\ {}^{1}_{0010}) = \boldsymbol{L}_{0\boldsymbol{q}}\ {}^{1}_{0010} \quad \boldsymbol{F}_2(\ {}^{1}_{0010};\ {}^{1}_{0010}) \tag{3.41c}$$

$$O(\boldsymbol{AC}): \quad \boldsymbol{id}_2\ \ _+ + \boldsymbol{id}_0\ {}^{0}_{1010} = \boldsymbol{L}_{0\boldsymbol{q}}\ {}^{0}_{1010} \quad 2\boldsymbol{F}_2(\ _0;\ _+) \tag{3.41d}$$

where we have used the fact that $\boldsymbol{F}_2$ is a symmetric bilinear form. Equation (3.41b) is decoupled and yields $\boldsymbol{c}\ = 2\ \ _1\ \frac{2\boldsymbol{b}}{3}\ \ 1$. The only coefficient left to determine is $\boldsymbol{d}_2$ which we shall compute now.

Using $\ _{1010}^{0} = \langle \boldsymbol{x}_1; \boldsymbol{x}_2; \boldsymbol{x}_3; \boldsymbol{x}_4\rangle^{\boldsymbol{T}}$ in (3.41d) implies

$$\boldsymbol{i}\boldsymbol{d}_2 + \boldsymbol{i}\boldsymbol{d}_0\boldsymbol{x}_1 = \boldsymbol{x}_2 \tag{3.42a}$$

$$\boldsymbol{d}_0\boldsymbol{d}_2 + \boldsymbol{i}\boldsymbol{d}_0\boldsymbol{x}_2 = \frac{\boldsymbol{q}}{3}\boldsymbol{x}_1 + \boldsymbol{x}_3 \tag{3.42b}$$

$$\frac{2\boldsymbol{iq}}{3}\boldsymbol{d}_2 + \boldsymbol{i}\boldsymbol{d}_0\boldsymbol{x}_3 = \frac{\boldsymbol{q}}{3}\boldsymbol{x}_2 + \boldsymbol{x}_4 \tag{3.42c}$$

$$\frac{\boldsymbol{q}}{3}\boldsymbol{d}_0\boldsymbol{d}_2 + \boldsymbol{i}\boldsymbol{d}_0\boldsymbol{x}_4 = \frac{\boldsymbol{q}}{3}\quad \frac{\boldsymbol{q}}{3}\boldsymbol{x}_1 + \boldsymbol{x}_3 \qquad \frac{2\boldsymbol{bq}\ \ _1}{3} \tag{3.42d}$$

Using (3.42a) in (3.42b) , (3.42b) in (3.42d) and using these in (3.42c) yields $\boldsymbol{d}_2 = \frac{\boldsymbol{b}\ \ _1}{3\sqrt{\boldsymbol{q}}}$.

Therefore the normal form for (1.3) near $\boldsymbol{C}_1$ is

$$\frac{\boldsymbol{dA}}{\boldsymbol{dz}} = \boldsymbol{B} \tag{3.43a}$$

$$\frac{\boldsymbol{dB}}{\boldsymbol{dz}} = \frac{-}{\boldsymbol{q}}\boldsymbol{A} \quad \frac{\boldsymbol{b}\ \ _1}{3}\boldsymbol{A}^2 + 2\ \ _1\ \ \frac{2\boldsymbol{b}}{3}\quad 1\ \ |\boldsymbol{C}|^2 \tag{3.43b}$$

$$\frac{\boldsymbol{dC}}{\boldsymbol{dz}} = \boldsymbol{i}\sqrt{\ \ }\boldsymbol{q}\boldsymbol{C} \quad \boldsymbol{i}\frac{\sqrt{\ \ \boldsymbol{q}}}{\boldsymbol{q}^3}\boldsymbol{C} + \boldsymbol{i}\frac{\boldsymbol{b}\ \ _1}{3\sqrt{\boldsymbol{q}}}\boldsymbol{AC} \tag{3.43c}$$

The dynamics inherent in (3.43) may be elucidated following the discussions of [1], [2], [3] and [4]. The two first integrals of (3.33) are

$$\boldsymbol{K} = |\boldsymbol{C}|^2 \tag{3.44}$$

and

$$\boldsymbol{H} = \boldsymbol{B}^2 \quad \frac{2}{3}\boldsymbol{b}\ \boldsymbol{A}^3 \qquad \boldsymbol{A}^2 \quad 2\boldsymbol{c}\ \boldsymbol{KA} \tag{3.45}$$

Also, $\boldsymbol{c}$ should be real for the following energy arguments to apply. As a typical case, consider the level curve $\boldsymbol{H} = 0$ of the energy-like first integral function $\boldsymbol{H}$. In the $(\boldsymbol{A}; \boldsymbol{B})$

phase plane, this will compromise a homoclinic orbit. The intersection of $H = 0$ with the A axis occurs for $\frac{2}{3}b\,A^2 \quad A \quad 2c\,K = 0$ or

$$A \quad = \frac{3}{4b}\left[\qquad \sqrt{\;^2 + \frac{16 b\,c\,K}{3}}\right] \tag{3.46}$$

Note that $A_+ > 0; A \; < 0$ for $b\,c \; > 0$ and $b \; < 0$ as relevant for us. A general homoclinic orbit, homoclinic to A_+, is sketched in Figure 1 where the ow direction is deduced from (3.43a). For $K = |C|^2 = 0$, the orbit is homoclinic to $A_+ = 0$. For small non-zero $|K|$, A_+ $2c\,K/$, meaning that oscillations at in nity are then very small in this case. For $K = 0$ this corresponds to an *orbit homoclinic to* 0 for the normal form. This is indeed valid for the normal form taken at any order. However this solution does not exist mathematically for the full original system, even though one may compute its expansion in powers of the bifurcation parameter up to any order (see [3] and [4]). This is an example of the famous challenging problem of asymptotics beyond any orders. Other solutions found on the normal form mainly persist under the perturbation from higher order terms provided by the original system [2]. These solutions are delocalized waves and their existence in Region 2 is guaranteed by the general theory for reversible systems in [3] and [4]. Also, as mentioned in Section 2, genuine solitary waves are found on isolated curves in Region 2 of Figure 1 on which the oscillation amplitudes vanish. Since these are embedded in the sea of delocalized solitary waves and in the continuous spectrum, they are referred to as embedded solitons [5]. These will further be investigated in Region 2 subsequently using a mix of exponential asymptotics and numerical shooting.

Figure 3.1: Level curves of (3.45) corresponding to various values of H.

CHAPTER FOUR: RESULTS

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