SOLITARY-WAVE FAMILIES OF DISPERSIVE NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

by

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ABSTRACT

The abstract page is an essay-style summary of the purposes, methodology, findings or conclusions. It should not contain tables or figures of any kind. It is double-spaced. The heading ABSTRACT should be centered, without punctuation, at the top margin. If more than one page is necessary, continue on the following page. Do not repeat the heading or use the word “continued.”

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ACKNOWLEDGMENTS

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CHAPTER ONE: INTRODUCTION

The propagation of longitudinal derformation waves in elastic rods is governed ([1], [10], [11]) by the Generalized Pochammer-Chree Equations:

$$(u - u_{xx})_{tt} - \left(a_1 u + a_2 u^2 + a_3 u^3\right)_{xx} = 0 \tag{1.1}$$

and

$$(u - u_{xx})_{tt} - \left(a_1 u + a_3 u^3 + a_5 u^5\right)_{xx} = 0 \tag{1.2}$$

corresponding to different constitutive relations.

References [1], [10], [11] also discuss the primary references, including derivations and applications of these equations in various fields. In addition, motivated by experimental and numerical results, there are derivations of special families of solitary wave solutions by the extended $Tanh$ method [1], and other ansatzen [11]. These extend earlier solitary wave solutions given by Bogolubsky [3] and Clarkson et. al [8] for special cases of (1.1) and (1.2). In addition, [10] generalizes the existence results in [9] for solitary waves of (1.1) and (1.2).

In this paper, we initiate a fresh approach to the solitary wave solutions of the Generalized Pochammer-Chree equations (1.1) and (1.2). We invoke the theory of reversible systems and the method of normal forms to categorize the possible solitary waves of (1.1) and (1.2) much more completely than done so far. As we shall see, several families of solitary waves exist in various regions of parameter space. Our main focus here will be on delineating the possible occurence and multiplicity of solitary waves in different parameter regimes. Certain delicate questions relating to specific waves or wave families will form the basis of future work.

The remainder of this paper is organized as follows. In Section 2, we delineate the possible families of solitary waves in various parameter domains and on certain important curves using the theory of reversible systems. In Section 3 and 4, we next focus on the various transition curves and derive normal forms in their vicinity to confirm the existence of families of regular or delocalized solitary-wave solutions in their vicinity.

1.1 First-level Subheading

First-level subheadings are centered, typically underlined and occur in upper/lower case letters. In your styles menu, you will usually refer to this as Heading 2.

1.1.1 Second-level Subheading

Second-level Subheadings are usually centered in upper/lower case letters with no additional formatting. These are referred to as Heading 3.

Third-level Subheading Acts the same as command `\paragraph`. I do not see any reason to go any deeper with that.

CHAPTER TWO: LITERATURE REVIEW

Solitary waves of (1.1) and (1.2) of the form $v(x,t) = \phi(x - ct) = \phi(z)$ satisfy the fourth-order travelling wave ODE

$$\phi_{zzzz} - q\phi_{zz} + p\phi = \mathcal{N}_{1,2}[\phi] \tag{2.1}$$

where

$$\mathcal{N}_1[\phi] = -\frac{1}{c^2}\left[3a_3\left(2\phi\phi_z^2 + \phi^2\phi_{zz}\right) + 2a_2\left(\phi_{zz}\phi_z + \phi_z^2\right)\right] \tag{2.2a}$$

$$\mathcal{N}_2[\phi] = -\frac{1}{c^2}\left[3a_3\left(2\phi\phi_z^2 + \phi^2\phi_{zz}\right) + 5a_5\left(4\phi^3\phi_z^2 + \phi^4\phi_{zz}\right)\right] \tag{2.2b}$$

$$z \equiv x - ct \tag{2.3a}$$

$$p \equiv 0 \tag{2.3b}$$

$$q \equiv 1 - \frac{a_1}{c^2} \tag{2.3c}$$

$$\tag{2.3d}$$

Equation (2.1) is invariant under the transformation $z \mapsto -z$ and is thus a reversible system. In this section we shall use the theory of reversible systems to characterize the homoclinic orbits to the fixed point of (2.1), which correspond to pulses or solitary waves of the Micro-Structure PDE in various regions of the (p,q) plane.

The linearized system corresponding to (2.1)

$$\phi_{zzzz} - q\phi_{zz} + p\phi = 0 \tag{2.4}$$

has a fixed point

$$\phi = \phi_z = \phi_{zz} = \phi_{zzz} = 0 \tag{2.5}$$

Solutions $\phi = ke^{\lambda x}$ satisfy the characteristic equation $\lambda^4 - q\lambda^2 + p = 0$ from which one may deduce that the structure of the eigenvalues is distinct in two regions of (p, q)-space. Since $p = 0$ we have only two possible regions of eigenvalues. We denote C_0 as the positive q axis and C_1 the negative q-axis. First we shall consider the bounding curves C_0 and C_1 and their neighborhoods, then we shall discusss the possible occurence and multiplicities of homoclinic orbits to (2.5), corresponding to pulse solitary waves of (1.1) and (1.2), in each region:

Near C_0 The eigenvalues have the structure $\lambda_{1-4} = 0, 0, \pm\lambda$, $(\lambda \in \mathbb{R})$ and the fixed point (2.5) is a saddle-focus.

Near C_1 Here the eigenvalues have the structure $\lambda_{1-4} = 0, 0, \pm i\omega$, $(\omega \in \mathbb{R})$. We will show by analysis of a four-dimensional normal form in Section 4 that there exists a *sech*^{2} homoclinic orbit near C_1.

Having outlined the possible families of orbits homoclinic to the fixed point (2.5) of (2.4), corresponding to pulse solitary waves of (1.1) and (1.2), we now derive normal forms near the transition curves C_0 and C_1 to confirm the existence of regular or delocalized solitary waves in the corresponding regions of (p, q) parameter space.

2.1 First-level Subheading

CHAPTER THREE: METHODOLOGY

3.1 Normal form near C_1: solitary-wave solutions

Using (2.4), the curve C_0, corresponding to $\lambda = 0, 0, \pm\tilde{\lambda}$, is given by

$$C_0 : p = 0, q > 0 \tag{3.1}$$

Using (2.3c) implies

$$a_1 < c^2 \tag{3.2}$$

Denoting ϕ by y_1, (2.1) may be written as the two systems

$$\begin{aligned}
\frac{dy_1}{dz} &= y_2 && \text{(3.3a)}\\
\frac{dy_2}{dz} &= y_3 && \text{(3.3b)}\\
\frac{dy_3}{dz} &= y_4 && \text{(3.3c)}\\
\frac{dy_4}{dz} &= qy_3 - py_1 - N_{1,2}(Y) && \text{(3.3d)}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{N}_1(Y) &= -\frac{1}{c^2}\left[3a_3\left(2y_1y_2^2 + y_1^2y_3\right) + 2a_2\left(y_3y_2 + y_2^2\right)\right] && \text{(3.4a)}\\
\mathcal{N}_2(Y) &= -\frac{1}{c^2}\left[3a_3\left(2y_1y_2^2 + y_1^2y_3\right) + 5a_5\left(4y_1^3y_2^2 + y_1^4y_3\right)\right] && \text{(3.4b)}
\end{aligned}$$

We wish to rewrite this as a first order reversible system in order to invoke the relevant theory [4]. To that end, defining $Y = \langle y_1, y_2, y_3, y_4\rangle^T$ equation (3.3) may be written

$$\frac{dY}{dz} = L_{pq}Y - G_{1,2}(Y,Y) \tag{3.5}$$

where

$$L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2 - p & 0 & q/3 & 0 \end{pmatrix} \tag{3.6}$$

Since $p = 0$ for (1.1) and (1.2), we have

$$\frac{dY}{dz} = L_{0q}Y - G_{1,2}(Y, Y) \tag{3.7}$$

where

$$G_{1,2}(Y, Y) = \langle 0, 0, 0, -\mathcal{N}_{1,2}(Y)\rangle^T \tag{3.8}$$

Next we calculate the normal form of (3.7) near C_0. The procedure is closely modeled on [4] and many intermediate steps may be found there.

3.1.1 Near C_0

Near C_0 the dynamics reduce to a 2-D Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B) \tag{3.9}$$

and the corresponding normal form is

$$\frac{dA}{dz} = B \tag{3.10a}$$

$$\frac{dB}{dz} = b\epsilon A + \tilde{c}A^2 \tag{3.10b}$$

Here,

$$\epsilon = \left(\frac{q^2}{9} - p\right) - \left(\frac{q}{3}\right)^2 = -p \tag{3.11}$$

measures the perturbation around C_0, and

$$\zeta_0 = \langle 1, 0, -q/3, 0 \rangle^T \tag{3.12a}$$

$$\zeta_1 = \langle 0, 1, 0, -2q/3 \rangle^T \tag{3.12b}$$

The linear eigenvalue of (3.10a) satisfies

$$\lambda^2 = b\epsilon \tag{3.13}$$

The characteristic equation of the linear part of (3.7) is

$$\lambda^4 - q\lambda^2 - \epsilon = 0 \tag{3.14}$$

Hence, the eigenvalues near zero (the Center Manifold) satisfy $\lambda^4 \ll \lambda^2$ and hence

$$\lambda^2 \sim -\frac{\epsilon}{q} \tag{3.15}$$

Matching (3.13) and (3.15) implies

$$b = -\frac{1}{q} \tag{3.16}$$

and only the nonlinear coefficient $\tilde{c}$ remains to be determined in the normal form (3.10a).

In order to determine $\tilde{c}$ (the coefficient of A^2 in (3.10a) we calculate $\frac{dY}{dz}$ in two ways and match the $\mathcal{O}(A^2)$ terms.

To this end, using the standard 'suspension' trick of treating the perturbation parameter ϵ as a variable, we expand the function Ψ in (3.9) as

$$\Psi(\epsilon, A, B) = \epsilon A \Psi^1_{10} + \epsilon B \Psi^1_{01} + A^2 \Psi^0_{20} + AB\Psi^0_{11} + B^2 \Psi^0_{02} + \cdots \tag{3.17}$$

where the subscripts denote powers of A and B, respectively, and the superscript denotes the power of ϵ.

In the first way of computing dY/dz, we take the z derivative of (3.9) (using (3.10a) and (3.39)). The coefficient of A^2 in the resulting expression is $\tilde{c}\zeta_1$. In the second way of computing dY/dz, use (3.9) and (3.39) in (3.5). The coefficient of A^2 in the resulting expression is $L_{0,q}\Psi_{20}^0 - G_{1,2}(\zeta_0, \zeta_0)$. Hence

$$\tilde{c}\zeta_1 = L_{0q}\Psi_{20}^0 - F_2(\zeta_0, \zeta_0) \tag{3.18}$$

Using (3.12a) and (3.8) and denoting $\Psi_{20}^0 = \langle x_1, x_2, x_3, x_4 \rangle$ in (3.18) yields the equations

$$0 = x_2 \tag{3.19a}$$

$$\tilde{c} = \frac{q}{3}x_1 + x_3 \tag{3.19b}$$

$$0 = \frac{q}{3}x_2 + x_4 \implies x_4 = 0 \text{ using (3.19b)} \tag{3.19c}$$

and

$$-\frac{2q}{3}\tilde{c} = \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) + \frac{q}{3c^2}(3a_3 + 5a_5) = \frac{q}{3}\tilde{c} + \frac{q}{3c^2}(3a_3 + 5a_5) \text{ using (3.19b)} \tag{3.20}$$

Hence we obtain

$$\tilde{c} = -\frac{1}{3c^2}(3a_3 + 5a_5) \tag{3.21}$$

Therefore, the normal form near C_0 is

$$\frac{dA}{dz} = B \tag{3.22a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{a_3}{c^2}A^2 \tag{3.22b}$$

for (1.1) and

$$\frac{dA}{dz} = B \tag{3.23a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{1}{3c^2}\left(3a_3 + 5a_5\right)A^2 \tag{3.23b}$$

for (1.2).

The normal form (3.22) admits a homoclinic solution (near C_0) of the form

$$A\left(z\right) = \ell sech^2\left(kz\right) \tag{3.24}$$

with

$$k = \sqrt{\frac{-\epsilon}{4q}} \tag{3.25a}$$

$$\ell = \frac{-3\epsilon c^2}{2qa_3} \tag{3.25b}$$

Similarly, the normal form (3.23) admits a homoclinic solution (near C_0) of the form

$$A\left(z\right) = \ell sech^2\left(kz\right) \tag{3.26}$$

with

$$k = \sqrt{\frac{-\epsilon}{4q}} \tag{3.27a}$$

$$\ell = \frac{-3\epsilon c^2}{2q\left(3a_3 + 5a_5\right)} \tag{3.27b}$$

Hence, since $\epsilon = -p$, and the curve C_0 corresponds to $p = 0, q > 0$, solitary waves of the form (3.24) exist in the vicinity of C_0 for

$$p > 0, q > 0 \tag{3.28}$$

which implies that $a_1 < c^2$ (such that k in (3.27a) is real.) As mentioned in section 2, one may show the persistence of this homoclinic solution in the original travelling wave ODE (2.4). Thus, we have demonstrated the existence of solitary waves of (1.2) for $p = 0^+, q > 0$.

Similarly, the curve C_1 corresponds to $p = 0, q < 0$, solitary waves of the form (3.24) exist in the vicinity of C_1 for

$$p < 0, q < 0 \tag{3.29}$$

which implies $a_1 > c^2$.

Again, one may show the persistence of this homoclinic solution in the original travelling wave ODE (2.4). Thus, we have demonstrated the existence of solitary waves of (1.2) for $p = 0^-, q < 0$.

3.2 Normal form near C_1: possible solitary-wave solutions

Using (2.4), the curve C_1, corresponding to $\lambda = 0, 0 \pm i\omega$, is given by

$$C_1 : p = 0, q < 0 \tag{3.30}$$

Which implies

$$a_1 > c^2 \tag{3.31}$$

In order to investigate the possibility of a $sech^2$ homoclinic orbit in the neighborhood of C_1 and delocalized solitary waves, we next compute the normal form near C_1 following the procedure in [4].

Near C_1 the dynamics reduce to a 4-D Center Manifold [4] Since all the eigenvalues are non-hyperbolic, the Center Manifold has the form (a nonlinear coordinate change [4])

$$Y = A\zeta_0 + B\zeta_0 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C}) \tag{3.32}$$

with a corresponding four-dimensional normal form

$$\frac{dA}{dz} = B \tag{3.33a}$$

$$\frac{dB}{dz} = \bar{\nu}A + b_* A^2 + c_* |C|^2 \tag{3.33b}$$

$$\frac{dC}{dz} = id_0 C + i\bar{\nu}d_1 C + id_2 AC \tag{3.33c}$$

Here C is complex, $\bar{C}$ is the complex conjugate of C, $\epsilon, \zeta_0, \zeta_1$ are given previously and the two new complex eigenvectors co-spanning the Center Manifold are

$$\zeta_\pm = \left\langle 1, \lambda_\pm, 2q/3, \frac{\lambda_\pm}{3}q \right\rangle^T \tag{3.34}$$

Using (3.33c) and (3.10a)

$$\bar{\nu} = b\epsilon = -\frac{\epsilon}{q} \tag{3.35}$$

Also from the characteristic equation (3.14), the two non-zero (imaginary) roots are

$$\lambda^2 = \frac{q + \sqrt{q^2 + 4\epsilon}}{2} \approx q \text{ for } \epsilon \text{ small} \tag{3.36}$$

Hence

$$\lambda = \pm i\sqrt{-q}, q < 0 \tag{3.37}$$

Matching this to the linear part of (3.33c) (which corresponds to the imaginary eigenvalues), $\lambda = id_0 = i\sqrt{-q}$ or

$$d_0 = \sqrt{-q} \tag{3.38}$$

If we do a dominant balance argument after the change of variable $\epsilon = \sqrt{-3\alpha}$ on the characteristic equation as $\lambda \to 0$ then we find $d_1 = \frac{\sqrt{-3\alpha}}{18\alpha^2}$. Using $\alpha = q/3$ we find $d_1 = \frac{\sqrt{-q}}{2q^2}$.

The remaining undetermined coefficients in the normal form are the coefficients b_*, c_* and d_2 which correspond to the $A^2, |C|^2$ and AC terms respectively. In order to determine them, we follow the same procedure as in Section 3 and compute dY/dz is two distinct ways. We expand the function Ψ as

$$\Psi(\epsilon, A, B, C, \bar{C}) = \epsilon A\Psi^1_{1000} + \epsilon B\Psi^1_{0100} + A^2\Psi^0_{2000} + AB\Psi^0_{1100} + AC\Psi^0_{1010} + \epsilon C\Psi^1_{0010} + \cdots \quad (3.39)$$

with subscripts denoting powers of A, B, C and $\bar{C}$, respectively, and the superscript is the power of ϵ. In the first way, dY/dz is computed by taking the z derivative of (3.32) (using (3.33) and (3.39)) and read off the coefficients of $A^2, \|C\|^2, C\epsilon$ and AC terms.

In the second way, dY/dz is computed using (3.32) and (3.39) in (3.5) (with $p = 0$ on C_1 as given in (3.30)) and the coefficients of A, B, C and $\bar{C}$ are once again read off.

Equating the coefficients of the corresponding terms in the two separate expressions for dY/dz yields the following two systems of equations:

$$\mathcal{O}(A^2): \quad b_*\zeta_1 = L_{0q}\Psi^0_{2000} - G_{1,2}(\zeta_0, \zeta_0) \quad (3.40a)$$

$$\mathcal{O}(|C|^2): \quad c_*\zeta_1 = L_{0q}\Psi^0_{0011} - 2G_{1,2}(\zeta_+, \zeta_-) \quad (3.40b)$$

$$\mathcal{O}(\epsilon C): \quad -\tfrac{i}{q}\left(d_1\zeta_+ + d_0\Psi^1_{0010}\right) = L_{0q}\Psi^1_{0010} - G_{1,2}(\Psi^1_{0010}, \Psi^1_{0010}) \quad (3.40c)$$

$$\mathcal{O}(AC): \quad id_2\zeta_+ + id_0\Psi^0_{1010} = L_{0q}\Psi^0_{1010} - 2G_{1,2}(\zeta_0, \zeta_+) \quad (3.40d)$$

where we have used the fact that G_1 and G_2 are symmetric bilinear forms. Equation (3.40b) is decoupled and yields $c_* = \frac{8}{c^2}\left(2a_3 - a_2\right)$ for (1.1) and $c_* = \frac{1}{c^2}\left(16a_3 + \frac{140}{3}a_5\right)$ for (1.2). The only coefficient left to determine is d_2 which we shall compute now.

Using $\Psi^0_{1010} = \langle x_1, x_2, x_3, x_4\rangle^T$ in (3.40d) implies

$$
\begin{aligned}
id_2 + id_0 x_1 &= x_2 && (3.41a)\\
-d_0 d_2 + id_0 x_2 &= \frac{q}{3}x_1 + x_3 && (3.41b)\\
\frac{2iq}{3}d_2 + id_0 x_3 &= \frac{q}{3}x_2 + x_4 && (3.41c)\\
-\frac{q}{3}d_0 d_2 + id_0 x_4 &= \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) - \frac{2q}{c^2}\left(\frac{7}{2}a_3 - \frac{i}{3}d_0 a_2\right) && (3.41d)
\end{aligned}
$$

for (1.1) and

$$
\begin{aligned}
id_2 + id_0 x_1 &= x_2 && (3.42a)\\
-d_0 d_2 + id_0 x_2 &= \frac{q}{3}x_1 + x_3 && (3.42b)\\
\frac{2iq}{3}d_2 + id_0 x_3 &= \frac{q}{3}x_2 + x_4 && (3.42c)\\
-\frac{q}{3}d_0 d_2 + id_0 x_4 &= \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) - \frac{2q}{c^2}\left(\frac{7}{2}a_3 + \frac{32}{3}a_5\right) && (3.42d)
\end{aligned}
$$

for (1.2)

Using (3.42a) in (3.42b) , (3.42b) in (3.42d) and using these in (3.42c) yields $d_2 = \frac{1}{c^2}\left(\frac{7}{2\sqrt{-q}}a_3 - \frac{i}{3}a_2\right)$ for (1.1) and $d_2 = \frac{1}{\sqrt{-q}c^2}\left(\frac{7}{2}a_3 + \frac{32}{3}a_5\right)$ for (1.2).

Therefore the normal form near C_1 is

$$
\begin{aligned}
\frac{dA}{dz} &= B && (3.43a)\\
\frac{dB}{dz} &= -\frac{\epsilon}{q}A - b_* A^2 + \frac{1}{c^2}\left(\frac{7}{2\sqrt{-q}}a_3 - \frac{i}{3}a_2\right)|C|^2 && (3.43b)\\
\frac{dC}{dz} &= i\sqrt{-q}C - i\frac{\sqrt{-q}}{q^3}C\epsilon + i\frac{1}{c^2}\left(\frac{7}{2\sqrt{-q}}a_3 - \frac{i}{3}a_2\right)AC && (3.43c)
\end{aligned}
$$

for (1.1) and

$$\frac{dA}{dz} = B \tag{3.44a}$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - b_* A^2 + \frac{1}{c^2}\left(16a_3 + \frac{140}{3}a_5\right)|C|^2 \tag{3.44b}$$

$$\frac{dC}{dz} = i\sqrt{-q}C - i\frac{\sqrt{-q}}{q^3}C\epsilon + i\frac{1}{\sqrt{-q}c^2}\left(\frac{7}{2}a_3 + \frac{32}{3}a_5\right)AC \tag{3.44c}$$

for (1.2).

The dynamics inherent in (3.43a), (3.44a) may be elucidated following the discussions of [5], [6],[7], and [4]. The two first integrals of (3.33) are

$$K = |C|^2 \tag{3.45}$$

and

$$H = B^2 - \frac{2}{3}b_* - \bar{\nu}A^2 - 2c_* K A_* \tag{3.46}$$

Here, the appropriate coefficients $b_*, \bar{\nu}$ and c_*, derived above, apply for (1.1) and (1.2). Also, c_* should be real, or a_2 must be zero in (1.1) for the following energy arguments to apply.

As a typical case, consider the level curve $H = 0$ of the energy-like first integral function H. In the (A, B) phase plane, this will compromise a homoclinic orbit. The intersection of $H = 0$ with the A axis occours for $\frac{2}{3}b_* A^2 - \bar{\nu}A - 2c_* K = 0$ or

$$A_{\mp} = \frac{3}{4b_*}\left[\bar{\nu} \pm \sqrt{\bar{\nu}^2 + \frac{16b_* c_* K}{3}}\right] \tag{3.47}$$

Note that $A_+ > 0, A_- < 0$ for $b_* c_* > 0$ and $b_* < 0$ as relevant for us. A general homoclinic orbit, homoclinic to A_+, is sketched in FIGURE where the flow direction is deduced from

(3.43b) and (3.44b) for (1.1) and (1.2), respectively. For $K = |C|^2 = 0$, the orbit is homoclinic to $A_+ = 0$. For small non-zero $|K|$, $A_+ \sim -2c_* K/\bar{\nu}$, meaming that oscillations at infinity are then very small in this case. For $K = 0$ this corresponds to an orbit to 0 for the normal form. This is indeed valid for the normal form taken at any order. However this solution does not exist mathematically for the full original system, even though one may compute its expansion in powers of the bifurcation parameter up to any order (see [6] and [7]). This is an example of the famous challenging problem of asymptotics beyond any orders. Other solutions found on the normal form mainly persist under the perturbation from higher order terms provided by the original system [5]. These solutions are delocalized waves and their existence in Region 2 is guaranteed by the general theory for reversible systems in [6] and [7]. Also, as mentioned in Section 2, genuine solitary waves are found on isolated curves in Region 2 of FIGURE on which the oscillation amplitudes vanish. Since these are embedded in the sea of delocalized solitary waves and in the continuous spectrum, they are referred to as embedded solitons [2]. These will further be investigated in Region 2 subsequently using a mix of exponential asymptotics and numerical shooting.

CHAPTER FOUR: FINDINGS

Chapter Four, also called Results or Data Analysis, usually provides detailed findings of the research. This chapter is where tables and figures most often appear, so make sure formatting is consistent.

4.1 Sample Table

The following sample table is an example of acceptable table formatting. Descriptive titles appear above tables and may appear either on one line or stacked and single-spaced. The table itself may also be single-spaced as necessary. If possible, try to keep tables and/or figures all on one page. If necessary, start the table or figure on a new page, even if this means leaving blank space on the preceding page. If you must split a table over multiple pages, repeat the table headings and continue. It is not necessary to repeat the table title.

Table 4.1: Classroom Checklist for Physical Organization (a sample table)

	Classrooms					
Physical Components	A	B	C	D	E	F
Desk Groupings for Student Interaction	5	3	3	5	3	2
Learning and Resource Centers	3	2	2	3	1	1
Flexibility of Furniture Use	3	4	3	3	2	1
Specific M/G Displays	1	1	3	2	2	2
Total out of 30 points	12	10	11	13	8	6

Degree of Application: 5=High; 4=Medium-High; 3=Medium; 2=Medium-Low; 1=Low
M/G=Multicultural/Global
A, B, C, D, E, and F are the classrooms of Alice, Betty, Carol, Donna, Elaine, and Fran respectively.

4.2 Sample Figure

The following is a sample figure with acceptable figure formatting. For figures, be sure you format both the figure and the figure title consistently. This includes placement (centered or left-justified), spacing before and after, line spacing, point size and font.

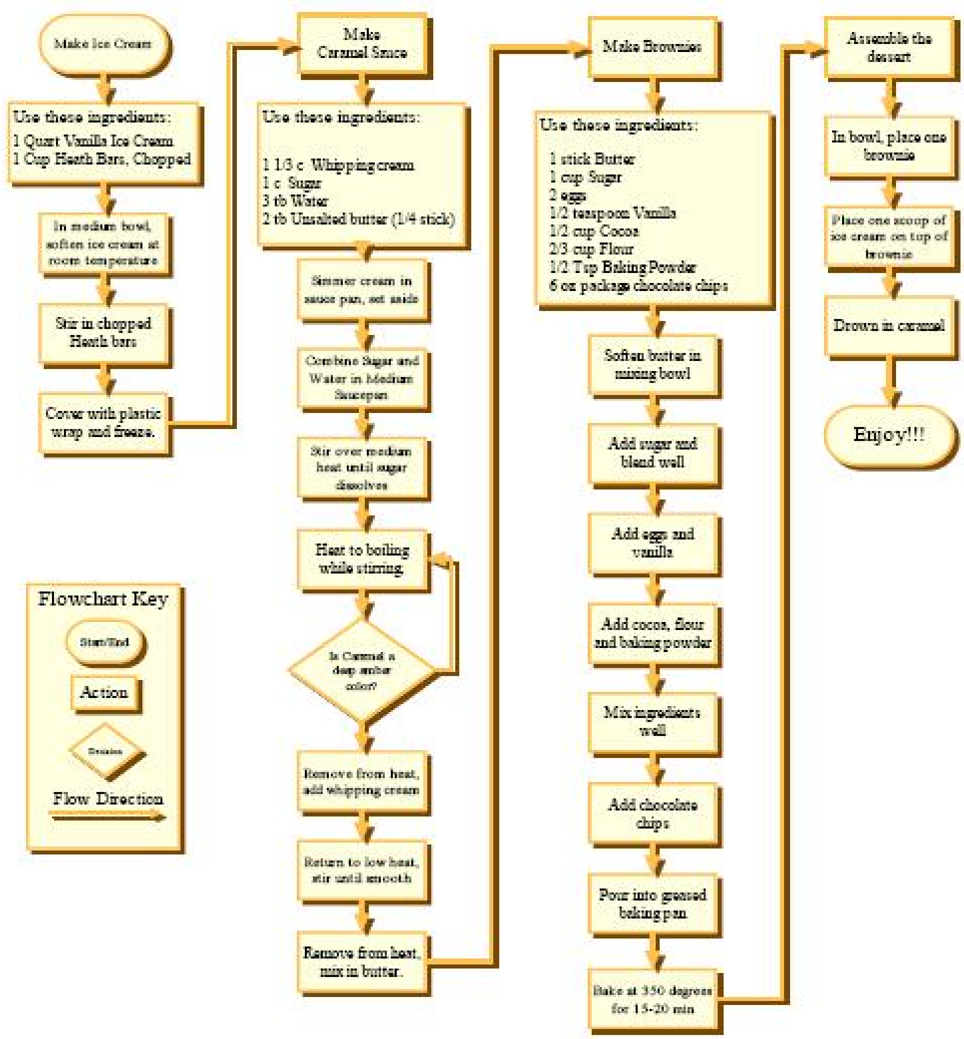

Figure 4.1: Heath Bar Caramel Brownie Sundae

CHAPTER FIVE: CONCLUSION

Chapter Five, also called “Summary”, “Conclusion” or “Recommendations”, usually presents a conclusion to the research, offers recommendations to the problem investigated, or discusses implications for future studies.

5.1 Bookmarks

A few words about bookmarks. Frontmatter entries, like the Abstract, Acknowledgments and the Table of Contents should appear in the bookmarks but not in the Table of Contents. The TOC contains only pages that appear after the Table of Contents in the document, usually beginning with the List of Figures. So, bookmark and Table of Contents entries do vary. However, bookmarks should include all major and chapter headings and at least first-level subheadings EXACTLY as they appear in the document (and the TOC). And readers should be able to link to pages within the ETD from all of the bookmarks, the TOC entries, as well as the Lists of Figures and Tables.

APPENDIX: TITLE OF APPENDIX

- Begin appendix text on the page following the buffer page
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- Use the same style and format for buffer page headings as you do for other body chapter headings.
- Letter, dont number, appendixes (e.g., APPENDIX A, APPENDIX B)
- If you have only one appendix, do not letter it at all
- Appendixes should follow the margin and other formatting requirements from the rest of the document

A.1 About References

References appear in the style of your particular reference system. While a hanging indent is preferred, some style guides use alternate formatting. References should be either double-spaced throughout, with no space between entries, or single-spaced within entries with a double space between. Below find a sample reference with a hanging indent, formatting along APA style. Here is how to cite: [**?**].

APPENDIX B: TITLE OF APPENDIX

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B.1 About References

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