Some Curious Recursive Functions: Hofstadter's G and After

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Nested Recursions

From the book "Gödel,Escher,Bach" [1] :

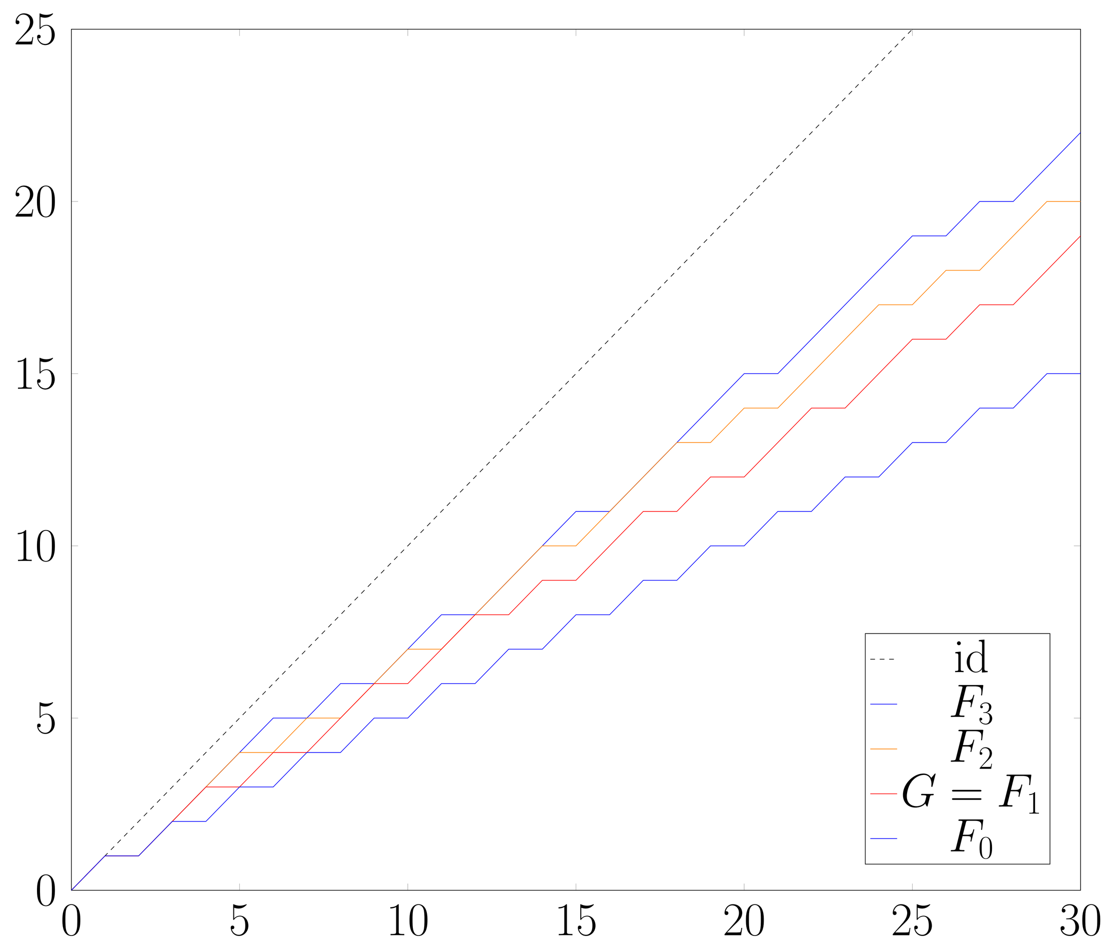
Definition: Hofstadter's G function

$$\begin{cases} G(0) = 0 \\ G(n) = n - G(G(n-1)) & \text{for all } n > 0 \end{cases}$$

More generally, with $k+1$ nested recursive calls:

Definition: the F_k functions

$$\begin{cases} F_k(0) = 0 \\ F_k(n) = n - F_k^{(k+1)}(n-1) & \text{for all } n > 0 \end{cases}$$



Theorem (with Shuo Li, nov. 2023!):

$\forall k, \forall n, F_k(n) \leq F_{k+1}(n)$

Fibonacci-like Sequences

For any $k \geq 0$:

Definition: the A_k sequences

$$\begin{cases} A^k_n = n+1 & \text{when } n \leq k \\ A^k_{n+1} = A^k_n + A^k_{n-k} & \text{when } n+1 > k \end{cases}$$

- A^0 : 1 2 4 8 16 32 64 128 256 ...
- A^1 : 1 2 3 5 8 13 21 34 55 89 ... (Fibonacci)
- A^2 : 1 2 3 4 6 9 13 19 28 41 ... (Narayana's Cows)
- A^3 : 1 2 3 4 5 7 10 14 19 26 ...

Theorem: F_k shifts down A^k

$\forall k, \forall n, F_k(A^k_n) = A^k_{n-1}$

Numerical Systems

Theorem (Zeckendorf):

Let $k \geq 0$. All $n \geq 0$ has a unique canonical decomposition ΣA^k_i (i.e. with indices i apart by at least $k+1$).

Theorem: F_k on decompositions

The function F_k shifts down the indices of canonical decompositions: $F_k(\Sigma A^k_i) = \Sigma A^k_{i-1}$ (with here $0 - 1 = 0$).

For instance for $k = 2$ and $n = 18$:

- $18 = A^2_0 + A^2_3 + A^2_6 = 1 + 4 + 13$
- $F_3(18) = A^2_0 + A^2_2 + A^2_5 = 1 + 3 + 9 = 13$
- $1 + 3 + 9$ no more canonical, possible renormalization

Definition: rank

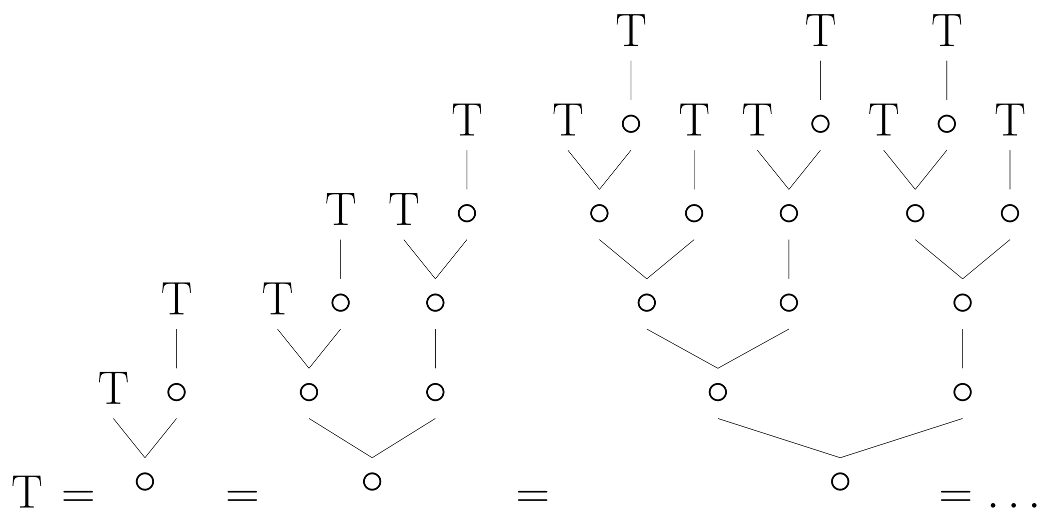
$\text{rank}_k(n)$: lowest index in the k-decomposition of n

Theorem: F_k flat spots

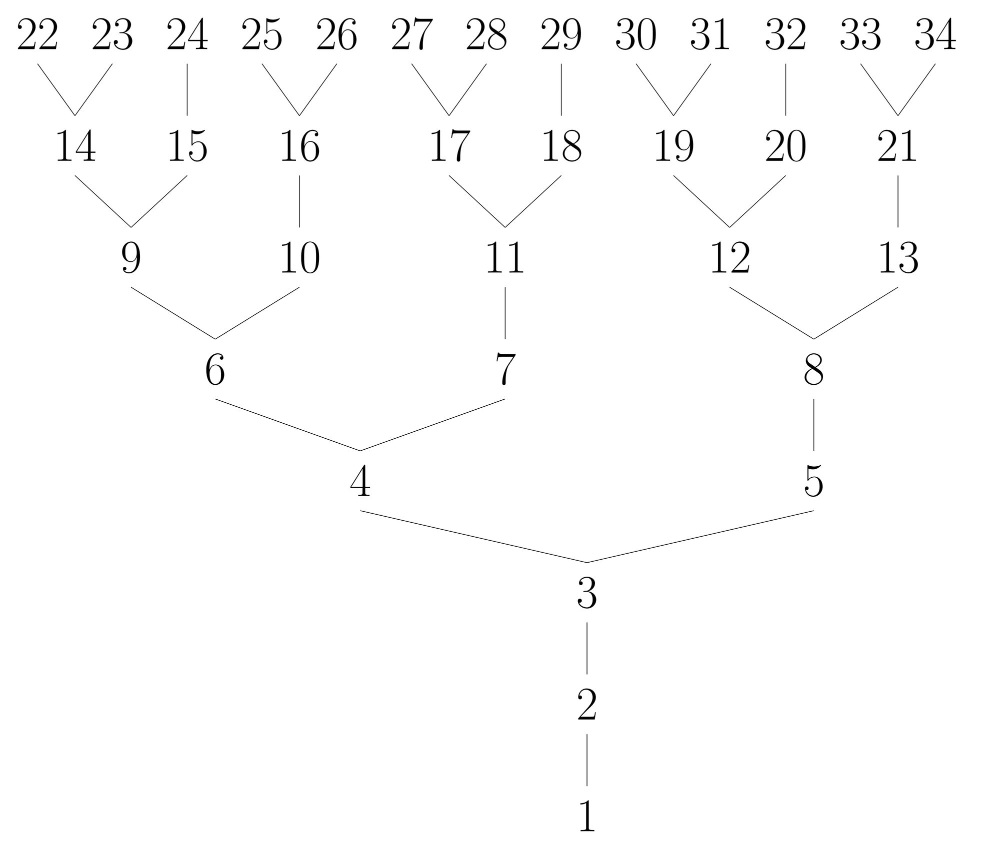
$F_k(n) = F_k(n+1)$ iff $\text{rank}_k(n) = 0$ (i.e. $n = 1 + \Sigma A^k_i$)

G as a Rational Tree

Let's repeat this branching pattern:



Now, with an ad-hoc trunk and node numbered via BFS:



Theorem:

For any node $n > 1$, its ancestor is $G(n)$.

Exercise:

Which trees correspond to functions F_k ?

Linear Equivalents

Let τ_k be the positive root of $X^{k+1} + X - 1$. It is hence algebraic, and irrational except for $k = 0$.

Theorem:

For all $k \geq 0$, when $n \to \infty$ we have $F_k(n) = \tau_k.n + o(n)$

More precisely:

- $F_0(n) = \lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$

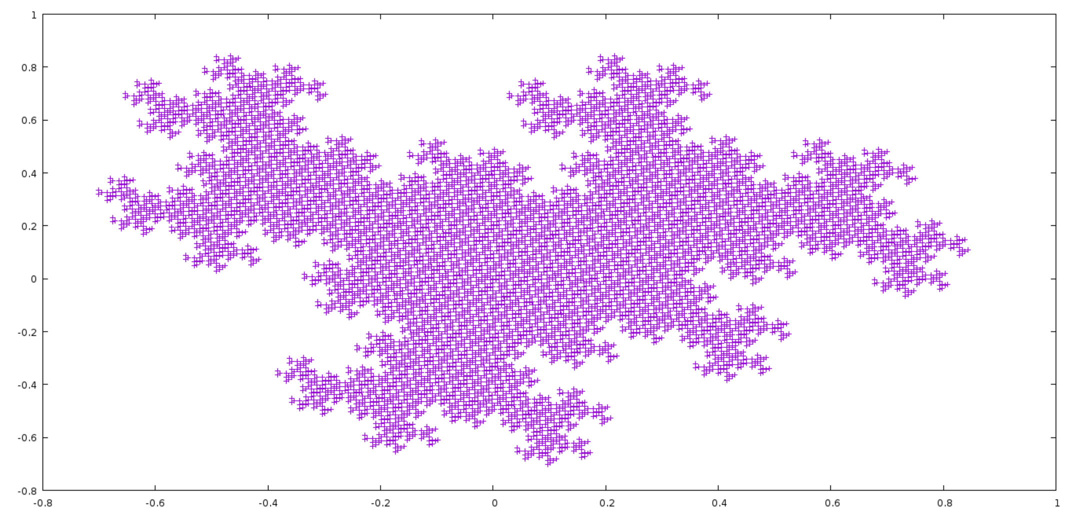
And $A^0_n = 2^n$ and we retrieve the base-2 decomposition !

- $G(n) = F_1(n) = \lfloor \tau_1.(n+1) \rfloor$

Here $\tau_1 = 1/\phi = \phi - 1 \approx 0.618...$

- $F_2(n) - \lfloor \tau_2.n \rfloor \in \{0, 1\}$

Here $\tau_2 \approx 0.6823...$, inverse of Pisot number P_3.
Let $\delta(n) = F_2(n) - \tau_2.n$. Then plotting $(\delta(i), \delta(F_2(i)))$ leads to this Rauzy fractal [4, 2]!



- $F_3(n) - \lfloor \tau_3.n \rfloor \in \{-1, 0, 1, 2\}$

Here $\tau_3 \approx 0.7244...$, inverse of Pisot number Q_3.

- After $k \geq 4$, $F_k(n) - \tau_k.n = o(n)$ but not bounded.

Note: τ_4 is the inverse of the Plastic number (smallest Pisot), then τ_k for $k \geq 5$ is above any Pisot inverse.

Morphic Words

We take $\mathcal{A} = \{0..k\}$ as alphabet.

Definition: the substitutions σ_k

$$\begin{cases} \mathcal{A} \to \mathcal{A}^* \\ \sigma_k(n) = n+1 & \text{for } n < k \\ \sigma_k(k) = k.0 \end{cases}$$

Definition: the morphic words m_k

The substitution σ_k is prolongeable at k. It hence admits an infinite word m_k (called *morphic*) as fixed point:

$$m_k = \lim_{n \to \infty} \sigma_k^n(k) = \sigma_k(m_k)$$

For example:

- m_1 is the Fibonacci word (with opposite letters)
- And $m_2 = 20122020120122012202...$

Theorem: alternative description of m_k

m_k is also the limit of its finite prefixes M^k_n defined as:

$$\begin{cases} M^k_n = k.0...(n-1) & \text{for } n \leq k \\ M^k_{n+1} = M^k_n.M^k_{n-k} & \text{for } n+1 > k \end{cases}$$

Also note that $|M_{k,n}| = A^k_n$

Theorem: linear complexity

The subword complexity of m_k (i.e. its number of distinct factors of size p) is $p \mapsto k.p + 1$.

In particular, m_1 is Sturmian (as expected).

Theorem: relating m_k and rank_k and F_k

- At position $n > 1$, $m_k[n] = min(k, \text{rank}_k(n))$.
- In particular this letter is 0 iff F_k is flat there.
- Hence the number of 0 in the n first letters of m_k is $n - F_k(n)$.
- For any $p \leq k$, counting the letters $\geq p$ gives $F_k^{(p)}$.
- All letters in m_k have (infinite) frequencies, for instance the frequency of 0 is $1 - \tau_k$ (see Saari [3]).

Coq formalization

- Already 90% of this poster certified in Coq: `https://github.com/letouzey/hofstadter_g`
- Nearly 20 000 lines of Coq formalization
- Several proved facts were just conjectures on OEIS.
- At first, delicate (non-structural) function definitions over `nat`, and many tedious recursions (multiple cases).
- More recently, use of real and complex numbers, polynomial, matrix (e.g. Vandermonde and its determinant), some interval arithmetic for real approximation, etc.
- Use the `QuantumLib` library for its linear algebra part!

Thanks!

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References

[1] Hofstadter, Douglas R., *Gödel, Escher, Bach: An Eternal Golden Braid*, 1979, Basic Books, Inc, NY.

[2] Pytheas Fogg, N., *Substitutions in Dynamics, Arithmetics and Combinatorics*, 2002, LNCS 1794.

[3] Saari, K., *On the Frequency of Letters in Morphic Sequences*, CSR 2006, LNCS 3967.

[4] Rauzy, G., *Nombres algébriques et substitutions.* Bulletin de la SMF, Vol 110 (1982).