Generalizing some Hofstadter functions

G, H and beyond

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Some nested recursions

From the book "Gödel,Escher,Bach":

Definition (Hofstadter's G function)

$$\begin{cases} G(0) = 0 \\ G(n) = n - G(G(n-1)) & \text{for all } n \in \mathbb{N}_*. \end{cases}$$

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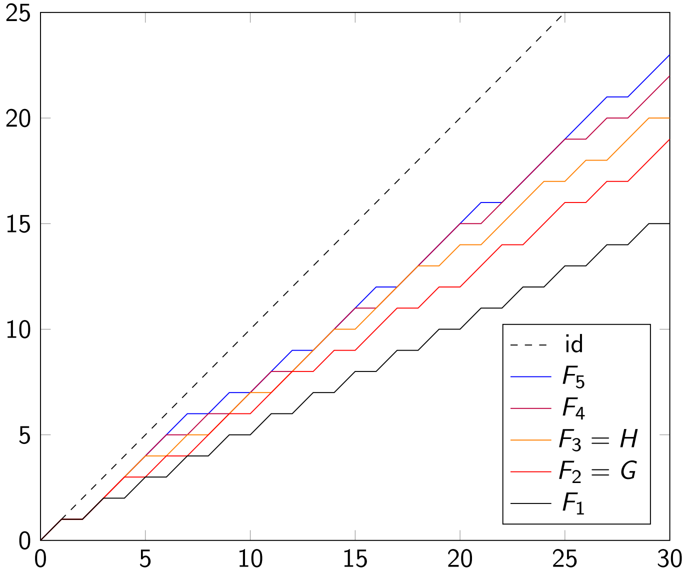
For $k \in \mathbb{N}$, we generalize to k nested recursive calls:

Definition (the F_k functions)

$$\begin{cases} F_k(0) = 0 & \\ F_k(n) = n - F_k^{(k)}(n-1) & \text{for all } n \in \mathbb{N}_* \end{cases}$$

where $F_k^{(k)}$ is the k-th iterate $F_k \circ F_k \circ \cdots \circ F_k$.

Plotting the early F_k



Outline

1. Morphic words and pointwise monotonicity
2. Numerical systems and discrepancy
3. The Coq formalization

Part I

Morphic words and pointwise monotonicity

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  - $F_1(n) = n - F_1(n-1) = 1 + F_1(n-2)$ when $n \geq 2$
  - Hence $F_1(n) = \lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$

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  - Hence $F_1(n) = \lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$
- $F_2 = G$ is already well studied, see OEIS A5206.
  In particular $F_2(n) = \lfloor (n+1)/\varphi \rfloor$ where φ is the Golden Ratio.

Basic properties

$$\begin{cases} F_k(0) = 0 & \\ F_k(n) = n - F_k^{(k)}(n-1) & \text{for all } n > 0 \end{cases}$$

- Well-defined since $0 \leq F_k(n) \leq n$
- $F_k(0) = 0$, $F_k(1) = 1$ then $n/2 \leq F_k(n) < n$
- F_k is made of a mix of flats and $+1$ steps
- Hence each F_k is increasing, onto, but not one-to-one
- Never two flats in a row
- At most k steps in a row

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F_k may be seen as an infinite word of flats and steps. For instance F_3 is $+{=}{+}{+}{+}{=}{+}{=}{+}{+}{=}{+}{+}{+}{=}\cdots$. Too coarse, no nice properties for $k > 2$.

A letter substitution and its morphic word

Let $k > 0$. We use $\mathcal{A} = \{1..k\}$ as alphabet.

Definition (substitution τ_k and morphic word x_k)

$$
\begin{aligned}
&\mathcal{A} \to \mathcal{A}^* & \\
\tau_k(n) &= (n{+}1) & \text{for } n < k \\
\tau_k(k) &= k.1 &
\end{aligned}
$$

From letter k, τ_k leads to an infinite morphic word x_k, fixed-point of τ_k.

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For instance:

- $x_2 = 2122121221221212212\cdots$ (Fibonacci word)
- $x_3 = 3123313123123312331\cdots$

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Spoiler: the previous $+\!=\!\cdots$ word of F_k is actually a projection of x_k where letter 1 becomes $=$ and all other letters become $+$.

Prior work

Note that τ_k is not novel, it can be seen as:

- A particular *modified Jacobi-Perron substitution* (see Pytheas Fogg)
- The substitution associated with the *Rényi expansion* of 1 in base $\beta_k = root(X^k - X^{k-1} - 1)$ (see Frougny et al.)
  - Hence the factor complexity of x_k is $n \mapsto (k{-}1)n{+}1$.

Length of substituted prefix

A useful notion relating F_k and x_k:

Definition (length L_k of substituted prefix)

$$L_k(n) := \left|\tau_k(x_k[0..n{-}1])\right|$$

where $x_k[0..n{-}1]$ is the prefix of size n of x_k.

Interestingly, the j-th iterate of L_k satisfies $L_k^j(n) = \left|\tau_k^j(x_k[0..n-1])\right|$.

Theorem

For $k, n, j > 0$, the antecedents of n by F_k^j are $L_k^j(n{-}1){+}1 \ \ldots \ L_k^j(n)$.

The proof is pretty technical (thanks Wolfgang).
Corollary: $F_k^j(L_k^j(n)) = n \leq L_k^j(F_k^j(n))$ (Galois connection).
Actually: $L_k(n) = n + F_k^{k-1}(n)$.

More relations between x_k and F_k

Consequences of the previous theorem:

- The letter $x_k[n]$ is 1 whenever $F_k(n+1) - F_k(n)$ is 0, otherwise this difference is 1.
- Counting letter 1 in $x_k[0..n{-}1]$ gives $n - F_k(n)$.
- Similarly, for $p < k$, counting letters strictly above p gives F_k^p. In particular the count of letter k is F_k^{k-1}.
- Another point of view: $x_k[n] = min(j, k)$ where j is the least value such that F_k^j is flat at n.

Letter frequencies

For $k > 0$, let α_k be the positive root of $X^k + X - 1$ and $\beta_k = 1/\alpha_k$, positive root of $X^k - X^{k-1} - 1$.

Theorem

$$\begin{aligned}
\lim_{n\to\infty} \tfrac{1}{n} F_k(n) &= \alpha_k, \\
\lim_{n\to\infty} \tfrac{1}{n} L_k(n) &= \beta_k, \\
\mathrm{frequency}_k(i) &= \alpha_k^{k+i-1} && \text{for } 1 \leq i < k, \\
\mathrm{frequency}_k(k) &= \alpha_k^{k-1} = \beta_k - 1.
\end{aligned}$$

Said otherwise, $F_k(n) = \alpha_k.n + o(n)$ when $n \to \infty$.
Could we prove it without "detour" via x_k ?

Corollary

When n is large enough, $F_k(n) < F_{k+1}(n)$.

Monotony of the F_k family

Definition

Pointwise order for functions : $f \leq h \iff \forall n, f(n) \leq h(n)$

Theorem

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- Conjectured in 2018.
- First proof by Shuo Li (Nov 2023).
- Improved version by Wolfgang Steiner.
- The key lemma proves simultaneously $L_k(n) \geq L_{k+1}(n)$ and $L_k^j(n) < L_{k+1}^{j+1}(n)$ for $k, n \geq 1$ and $j \leq k$.

Some key steps in the key lemma

When proving $L_k^j(n) < L_{k+1}^{j+1}(n)$ by induction on n, simultaneously with $L_k(n) \geq L_{k+1}(n)$:

- We deal with $j = k$ via an ad-hoc equation:

$$L_{k+1}^{k+1}(n) - L_k^k(n) = L_{k+1}^k(n) - L_k^{k-1}(n)$$

- When $j < k$, consider the last letter on the right $x_{k+1}[n-1]$:
  - Either it is $k+1$, and even a k letter on the left leads to a smaller quantity: $|\tau_k^j(k)| < |\tau_{k+1}^{j+1}(k+1)|$.
  - Either it is not $k+1$, and the whole prefix $x_{k+1}[0..n-1]$ is the image by τ_{k+1} of a smaller prefix. Let m be its size. Induction hypothesis on this m (now with $j+1$ iterations): $n = L_{k+1}(m) \leq L_k(m)$ and $L_k^{j+1}(m) < L_{k+1}^{j+2}(m)$. And finally:

$$L_k^j(n) \leq L_k^j(L_k(m)) = L_k^{j+1}(m) < L_{k+1}^{j+2}(m) = L_{k+1}^{j+1}(L_{k+1}(m)) = L_{k+1}^{j+1}(n)$$

Pointwise monotonicity, summerized

Below, $f \to g$ whenever $f \leq g$ pointwise:

$$
\begin{tikzcd}
 & k=1 & k=2 & k=3 & k=4 & k=5 \\
j=1 & F_1 = \lceil \frac{n}{2} \rceil \arrow[r] & F_2 = G \arrow[r] & F_3 \arrow[r] & F_4 \arrow[r] & F_5 \\
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j=4 & F_1^4 = \lceil \frac{n}{16} \rceil \arrow[u] \arrow[r] & F_2^4 \arrow[u] \arrow[r] & F_3^4 \arrow[u] \arrow[r] & F_4^4 \arrow[u] \arrow[ul] \arrow[r] & F_5^4 \arrow[u] \arrow[ul] \\
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j=1 & \underset{0.5}{F_1 = \lceil \frac{n}{2} \rceil} \arrow[r] & \underset{0.618}{F_2 = G} \arrow[r] & \underset{0.682}{F_3} \arrow[r] & \underset{0.724}{F_4} \arrow[r] & \underset{0.754}{F_5} \\
j=2 & \underset{0.25}{F_1^2 = \lceil \frac{n}{4} \rceil} \arrow[u] \arrow[r] & \underset{0.381}{F_2^2} \arrow[u] \arrow[ul] \arrow[r] & \underset{0.465}{F_3^2} \arrow[u] \arrow[ul] \arrow[r] & \underset{0.524}{F_4^2} \arrow[u] \arrow[ul] \arrow[r] & \underset{0.569}{F_5^2} \arrow[u] \arrow[ul] \\
j=3 & \underset{0.125}{F_1^3 = \lceil \frac{n}{8} \rceil} \arrow[u] \arrow[r] & \underset{0.236}{F_2^3} \arrow[u] \arrow[ul, dotted, "?"] \arrow[r] & \underset{0.317}{F_3^3} \arrow[u] \arrow[ul] \arrow[r] & \underset{0.380}{F_4^3} \arrow[u] \arrow[ul] \arrow[r] & \underset{0.430}{F_5^3} \arrow[u] \arrow[ul] \\
j=4 & \underset{0.062}{F_1^4 = \lceil \frac{n}{16} \rceil} \arrow[u] \arrow[r] & \underset{0.145}{F_2^4} \arrow[u] \arrow[r] & \underset{0.216}{F_3^4} \arrow[u] \arrow[ul, dotted, "?"] \arrow[r] & \underset{0.275}{F_4^4} \arrow[u] \arrow[ul] \arrow[r] & \underset{0.324}{F_5^4} \arrow[u] \arrow[ul] \\
j=5 & \underset{0.031}{F_1^5 = \lceil \frac{n}{32} \rceil} \arrow[u] \arrow[r] & \underset{0.090}{F_2^5} \arrow[u] \arrow[r] & \underset{0.147}{F_3^5} \arrow[u] \arrow[r] & \underset{0.199}{F_4^5} \arrow[u] \arrow[ul, dotted, "?"] \arrow[r] & \underset{0.245}{F_5^5} \arrow[u] \arrow[ul]
\end{tikzcd}
$$

Remaining conjectures

Let $N_k := (k+1)(k+6)/2$. This appears to be the last contact:

- We proved $F_k(N_k) = F_{k+1}(N_k)$
- We conjecture $F_k(n) < F_{k+1}(n)$ for all $n > N_k$.

N_k also appears to be the last contact between L_{k+1} and L_{k+2}.

Part II

Numerical systems and discrepancy

Quiz !

Let $k > 0$. We say that a set of integers S is *k-sparse* if two distinct elements of S are always separated by at least k. How many k-sparse subsets of $\{1..n\}$ could you form ?

A Fibonacci-like family of sequences

For $k > 0$:

$$\begin{cases} A_{k,n} &= n{+}1 \qquad\qquad\qquad\qquad \text{when } n \leq k \\ A_{k,n} &= A_{k,n-1} + A_{k,n-k} \qquad \text{when } n \geq k \end{cases}$$

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- $A_{1,n}$: 1 2 4 8 16 32 64 128 256 512 ... (Powers of 2)
- $A_{2,n}$: 1 2 3 5 8 13 21 34 55 89 ... (Fibonacci)
- $A_{3,n}$: 1 2 3 4 6 9 13 19 28 41 ... (Narayana's Cows)
- $A_{4,n}$: 1 2 3 4 5 7 10 14 19 26 ...

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Actually: $A_{k,n} = L_k^n(1)$

Zeckendorf decomposition

Let $k > 0$.

Theorem (Zeckendorf)

All natural number can be written as a sum of $A_{k,i}$ numbers. This decomposition is unique when its indices i form a k-sparse set.

F_k is a bitwise right shift

Theorem

F_k is a right shift for such a decomposition : $F_k(\Sigma A_{k,i}) = \Sigma A_{k,i-1}$ (with the convention $A_{k,0-1} = A_{k,0} = 1$)

- Beware, this shifted decomposition might not be k-sparse anymore
- Not so new : a variant of F_k is already known to be a right shift on these decompositions (Meek & van Rees, 1981).
- Key property : F_k is flat at n iff the decomposition of n contains $A_{k,0} = 1$.
- More generally, $F_k^{(j)}$ is "flat" at n iff $j > rank(n)$ where the rank of n is the smallest index in the decomposition of n.

Discrepancy

Definition (Discrepancy)

$\Delta_k := Sup_n \, |F_k(n) - \alpha_k n|$

- $F_1(n) = \lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$ hence $\Delta_1 = 0.5$
- $F_2(n) = \lfloor \alpha_2.(n+1) \rfloor$ with $\alpha_2 = \varphi - 1 \approx 0.618...$ hence $\Delta_2 = \varphi - 1$

New results:

- $\Delta_3 < 1$
- $\Delta_4 < 2$
- For $k \geq 5$, $\Delta_k = \infty$

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- $\Delta_4 < 2$
- For $k \geq 5$, $\Delta_k = \infty$

This proves two conjectures of OEIS:

- $F_3(n) \in \lfloor \alpha_3.n \rfloor + \{0, 1\}$
- $F_4(n) \in \lfloor \alpha_4.n \rfloor + \{-1, 0, 1, 2\}$

Solving the Fibonacci-like recurrences

Let $r_{k,i}$ be the k roots of $X^k - X^{k-1} - 1$ and $c_{k,i}$ be $r_{k,i}^k\,(kr_{k,i}-(k-1))^{-1}$.

Theorem

For all n:

$$A_{k,n} = \sum_{i=0}^{k-1} c_{k,i}\, r_{k,i}^n$$

- We obtain these coefficients by inversing a Vandermonde matrix.
- Trick: temporary consider $\tilde{A}_{k,n}$ with the same recursion but initial values $0\ 0\ \cdots\ 0\ 1$.

Computing discrepancies

Let $d_{k,i} = c_{k,i}\,(r_{k,i}^{-1} - \alpha_k)$ and $D_k(n)$ be the Zeckendorf k-decomposition of n.

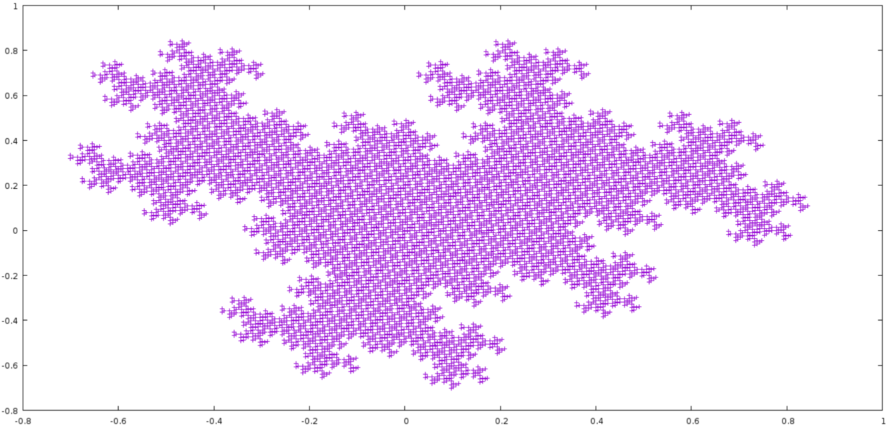
Theorem

$$F_k(n) - \alpha_k\, n = \sum_{q \in D_k(n)} \sum_{i=0}^{k-1} d_{k,i}\, r_{k,i}^{q} = \sum_{i=0}^{k-1} \left(d_{k,i} \sum_{q \in D_k(n)} r_{k,i}^{q} \right)$$

- One coefficient $d_{k,i}$ is null (the one for the positive root)
- For $k < 5$, all other roots have modulus strictly less than 1, leading to a finite discrepancy.
- For proving $\Delta_3 < 1$ and $\Delta_4 < 2$ we follow Rauzy and regroup some root powers together (up to 3 terms together for $k = 3$, i.e. q modulo 9, and up to 4 terms together for $k = 4$, i.e. q modulo 16). In these groups of root powers, a lot of cancellation happens.
- For $k \geq 5$, at least one non-real root has modulus 1 or more, leading to infinite discrepancy.

Serendipity : a Rauzy fractal

Let $\delta(n) := F_3(n) - \alpha_3.n$, then plot $(\delta(i), \delta(F_3(i)))$ for many i:



Summary

	F_1	F_2	F_3	F_4	F_5	F_k $k\geq 6$
Hofstadter's name		G	H			
Mean slope α_k = $root(X^k+X-1)$	0.5	$\varphi-1$	$\approx$ 0.682	$\approx$ 0.724	$\approx$ 0.754	α_k
Discrepancy $Sup\ \lvert F_k(n) - \alpha_k n\rvert$	0.5	$\varphi-1$	< 1	< 2	$O(ln(n))$	$O(n^a)$, $0<a<1$
Exact expression	$\lfloor \frac{n+1}{2} \rfloor$	$\lfloor \frac{n+1}{\varphi} \rfloor$	$\times$	$\times$	$\times$	$\times$
Almost expression			$\lfloor \alpha_3 n \rfloor + \{0,1\}$	$\lfloor \alpha_4 n \rfloor + \{-1,0,1,2\}$	$\times$	$\times$
Quasi-additivity	✓	✓	✓	✓	$\times$	$\times$
$\beta_k = \frac{1}{\alpha_k}$ is Pisot	✓	✓	✓	✓	✓!	$\times$

Here $\beta_2 = \varphi \approx 1.618$ is the Golden Ratio.
And $\beta_5 \approx 1.324$ is the Plastic Ratio, root of $X^3 - X - 1$, smallest Pisot number

Part III

The Coq formalization

Current status of this Coq formalization

- Freely accessible : `https://github.com/letouzey/hofstadter_g`
- Covers all results of part I and part II. And more.
- One remaining axiom for proving $\Delta_k = \infty$ for $k \geq 6$ (axiom equivalent to "the minimal Pisot is the Plastic Ratio").
- Quite large, about 20 kloc. Lots of cruft, experiments, etc.
- Coq rechecks the whole in about 2 min.
- The discrete part is self-contained (`nat`, `List`, ...)
- The parts involving $\mathbb{R}$ and $\mathbb{C}$ use two external libraries (Coquelicot, QuantumLib), with personal contributions and extensions.

Fibonacci-like recurrence

An example of straightforward definition (NB: "S" is Coq jargon for +1):

```
Fixpoint A (q n : nat) :=
  match n with
  | 0 ⇒ 1
  | S m ⇒ A q m + A q (m−q)
  end.
```

```
Compute A 0 8. (* 256 *)
Compute A 1 8. (* 55 *)
```

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Compute A 0 8. (* 256 *)
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```

- But we parameterized by $q = k - 1$ instead of k. Otherwise, dealing with $k = 0$ is a pain.
- Already some magic around `m-q`

Fibonacci-like recurrence

A corresponding proof (NB: “S” is Coq jargon for $+1$):

```
Lemma A_base q n : n <= S q → A q n = S n.
Proof.
 induction n; auto.
 simpl. intros.
 replace (n−q) with 0 by lia. simpl.
 rewrite IHn; lia.
Qed.
```

Defining F_k

```
Notation "f ^^ n" := (Nat.iter n f) (at level 30, right associativity).

Fixpoint recf q p n :=
 match p, n with
 | S p, S n => S n - ((recf q p)^^(S q)) n
 | _, _ => 0
 end.

Definition f q n := recf q n n.

Compute f 0 256. (* 128 *)
Compute f 1 89.  (* 55 *)
```

- Same trick with $q = k - 1$.
- To overcome the lack of structural decrease : an extra parameter p.
- First task : prove base and step equations for this f.
- Alternative : predicative or inductive definitions, much more flexible but no computation.

A main result

```
Theorem f_grows q n : f q n <= f (S q) n.
Proof.
 ...
Qed.

Print Assumptions f_grows.
(* Closed under the global context *)
```

Alternative definition of F_k

More involved but less inefficient. Can be proved equivalent to f.

```
Fixpoint fdescend stq p n :=
  match p with
  | 0 ⇒ n
  | S p ⇒
    match stq with
    | []  ⇒ 0 (* normally won't occur *)
    | a:: _ ⇒ fdescend (skipn (n−a) stq) p a
    end
  end.

Fixpoint ftabulate q n :=
 match n with
 | 0 ⇒ [0]
 | S n ⇒
   let stq := ftabulate q n in
   (S n − fdescend stq (S q) n)::stq
 end.

Definition fopt q n := hd 0 (ftabulate q n).
```

Finite and infinite words

Relatively generic definitions:

```
Notation letter := nat (only parsing).
Definition word := list letter. (* finite word *)
Definition sequence := nat → letter. (* infinite word *)
Definition subst := letter → word.

Definition apply : subst → word → word := ·flat_map _ _.
Definition napply (s:subst) n w := (apply s ^^n) w.

Definition subst2seq s a :=
 fun n ⇒ nth n (napply s n [a]) a.
```

Examples τ_k and x_k:

```
Definition qsubst q (n:letter) :=
 if n =? q then [q; 0] else [S n].
Definition qseq q := subst2seq (qsubst q) q.
```

Example of a larger proof I

```
Lemma Lq_LSq q n :
 L (S q) 1 n <= L q 1 n
 ∧ (0<n → forall j, j<=S q → L q j n < L (S q) (S j) n).
Proof.
 induction n as [n IH] using lt_wf_ind.
 destruct (Nat.eq_dec n 0) as [→ |N0]; [easy|].
 destruct (Nat.eq_dec n 1) as [→ |N1].
 { clear N0 IH. split; intros;
   rewrite !L_S, !L_0, !qseq_q_0, !qnsub_qword, !qword_len, !A_base; lia. }
 split.
 - rewrite !Lq1_Cqq, ← !fs_count_q, ← Nat.add_le_mono_l.
   set (c := fs q q n).
   set (c' := fs (S q) (S q) n).
   destruct (Nat.eq_dec c' 0); try lia.
   replace c' with (S (c'-1)) by lia. change (c'-1 < c).
   apply (incr_strmono_iff _ (L_incr (S q) (S q))).
   apply Nat.lt_le_trans with n; [apply steiner_thm; lia|].
   transitivity (L q q c); [apply steiner_thm; lia|].
   destruct (Nat.eq_dec q 0) as [→ |Q].
   + rewrite L_q_0. apply L_ge_n.
   + apply Nat.lt_le_incl, IH; try apply fs_lt; try apply fs_nonzero; lia.
 - intros _. destruct n; try easy.
   destruct (Nat.eq_dec (qseq (S q) n) (S q)) as [E|N].
   + intros j Hj. rewrite !L_S, E.
     rewrite qnsub_qword, qword_len.
     assert (Hx := qseq_letters q n).
     set (x := qseq q n) in *.
     generalize (qnsub_len_le q j x Hx). rewrite !A_base by lia.
     destruct (IH n lia) as (_,IH').
     specialize (IH' lia j Hj).
     lia.
```

Example of a larger proof II

```
+ destruct (qsubst_prefix_inv (S q) (qprefix (S q) (S n)))
    as (v & w & Hv & E & Hw); try apply qprefix_ok.
  destruct Hw as [→ | → ].
  2:{ rewrite take_S in Hv; apply app_inv' in Hv; trivial;
      destruct Hv as (_,[= E']);  lia. }
  rewrite app_nil_r in Hv.
  red in E.
  set (l := length v) in *.
  assert (E' : L (S q) 1 l = S n).
  { now rewrite ← (qprefix_length (S q) (S n)), Hv, E. }
  assert (Hl0 : l <> 0). { intros → . now rewrite L_0 in E'. }
  assert (Hl : l < S n).
  { rewrite ← E'. rewrite Lq1_Cqq.
    generalize (Cqq_nz (S q) l). lia. }
  destruct (IH l Hl) as (IH5,IH6). clear IH. rewrite E' in IH5.
  specialize (IH6 lia).
  assert (LT : forall j, j <= q → L q j (S n) < L (S q) (S j) (S n)).
  { intros j Hj. specialize (IH6 (S j) lia).
    rewrite ← E' at 2. rewrite L_add, Nat.add_1_r.
    eapply Nat.le_lt_trans; [|apply IH6].
    rewrite ← (Nat.add_1_r j), ← L_add. apply incr_mono; trivial.
    apply L_incr. }
  intros j Hj. destruct (Nat.eq_dec j (S q)) as [→ |Hj'].
  * generalize (steiner_trick q (S n)).
    specialize (LT q (Nat.le_refl _)). lia.
  * apply LT. lia.
Qed.
```

Reals, Matrix, Polynomial, etc

- Due to the Coq standard definition of reals, this part uses 4 logical axioms (incl. excluded middle and functional extensionality).
- ```
  Lemma Vandermonde_det n (l : list C) :
   length l = n →
   Determinant (Vandermonde n l) = multdiffs l.
  ```
- A bit of interval arithmetic for computing bounds reliably. For instance $\Delta_3 \leq 0.9959 < 1$.
- The proofs about Δ_3 automatically enumerate "sparse" subsets and bounds the corresponding 13 cases. Same for Δ_4 (69 cases).
  ```

Conclusion

About the F_k:

- Some remaining conjectures, seem to require new ideas.
- Is this specific to this particular recursion $X^k + X - 1$? What about similar functions e.g. for Rauzy's Tribonacci ? Unclear.

About Coq:

- Using Coq for this kind of study is doable, but still tricky and time consuming.
- Remaining critical parts to manually review : early definitions, final theorem statements, used axioms.
- Difficulty: lack of results in libraries (e.g. on complex, matrices, power series).
- Tempting: start proving that Plastic Ratio is the smallest Pisot.

Thank you for your attention

Preprint about part I: `https://hal.science/hal-04715451`

Coq Development: `https://github.com/letouzey/hofstadter_g`