Math 20250
Abstract Linear Algebra

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Lecture 1

Abelian Group, Field, Equivalence

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Goal. Vector spaces and maps between vector spaces (linear transformations)

1.1 Abelian Group

Definition 1.1 (Abelian Group).

A pair $(A, *)$ is an **Abelian group** if A is a set and $*$ is a map: $A \times A \mapsto A$ (closure is implied) with the following properties:

1. (Additive Associativity)
$$(x * y) * z = x * (y * z),\ \forall\, x, y, z \in A$$

2. (Additive Commutativity)
$$x * y = y * x,\ \forall\, x, y \in A$$

3. (Additive Identity)
$$\exists\, 0 \in A : 0 * x = x * 0 = x,\ \forall\, x \in A$$

4. (Additive Inverse)
$$\forall\, x \in A,\ \exists\, (-x) \in A : x * (-x) = (-x) * x = 0$$

Remark. ($*$ is just a symbol, soon to be $+$). Typically write as $(A, +)$ or simply A

Example.

1. $(\mathbb{Z}, +)$ is an Abelian group
2. $(\mathbb{Q}, +)$ is an Abelian group
3. $(\mathbb{Z}, \times)$ is **NOT** an Abelian group (because identity $= 1$, and 0 does not have a multiplicative inverse)
4. $(\mathbb{Q}, \times)$ is also not an Abelian group (0 does not have a multiplicative inverse)
5. $(\mathbb{Q}\backslash\{0\}, \times)$ is an Abelian group (identity is 1)
6. $(\mathbb{N}, \times)$ is NOT a group

Remark. A crucial difference between $\mathbb{Z}$ and $\mathbb{Q}\backslash\{0\}$ is that $\mathbb{Q}\backslash\{0\}$ has both $+$ and $\times$ while $\mathbb{Z}$ only has $+$. This gives us inspiration for the definition of a field!

Definition 1.2 (Field).

A **field** is a triple $(F, +, \cdot)$ such that

1. $(F, +)$ is an Abelian group with identity 0
2. (Multiplicative Associativity)
$$(x \cdot y) \cdot z = x \cdot (y \cdot z),\ \forall\, x, y, z \in F$$
3. (Multiplicative Commutativity)
$$x \cdot y = y \cdot x,\ \forall\, x, y \in F$$
4. (Distributivity) ($+$ and $\cdot$ talking in the following way)
$$x \cdot (y + z) = (x \cdot y) + (x \cdot z),\ \forall\, x, y, z \in F$$
5. (Multiplicative Identity)
$$\exists\, 1 \in F : 1 \cdot x = x,\ \forall\, x \in F$$
6. (Multiplicative Inverse)
$$\forall\, x \in F \backslash \{0\},\ \exists\, y \in F : x \cdot y = 1$$

Remark. In a field $(F, +, \cdot)$, assume that $1 \neq 0$

Example.

1. $(\mathbb{Z}, +, \cdot)$ is not a field (because property 6 failed)
2. $(\mathbb{Q}, +, \cdot)$ is a field
3. $(\mathbb{R}, +, \cdot)$ and $(\mathbb{C}, +, \cdot)$ are fields.

1.2 Finite Fields

Recall. $p \in \mathbb{Z}$ is a prime if $\forall\, m \in \mathbb{N} : m \mid p \Rightarrow m = 1 \text{ or } m = p$

Definition 1.3 ($\mathbb{F}_p$ for p prime).

$$\mathbb{F}_p = \{[0], [1], \ldots, [p-1]\}$$

Then define the operations for $[a], [b] \in \mathbb{F}_p$

$$[a] + [b] = [a + b \mod p]; [a] \cdot [b] = [a \cdot b \mod p]$$

Then $\mathbb{F}_p$ is a field, but this is not trivial.

Lemma 1.1.

1. $(\mathbb{F}_p, +)$ is an Abelian group
2. $(\mathbb{F}_p, +, \cdot)$ is a field

Example. $\mathbb{F}_5 = \{[0], [1], [2], [3], [4]\}$

$$[1] + [2] = [3], [2] + [4] = [1], [4] + [4] = [3], [2] + [3] = [0]$$

Then it is trivial that [0] is additive identity, and every element has additive inverse. [1] is multiplicative identity, and every element except [0] has multiplicative inverse. Therefore $\mathbb{F}_5$ is indeed a field.

1.3 Vector Spaces in brief

Intuition. The motivation for vector spaces and maps between them (linear transformations) is essentially to solve linear equations. Let $(\mathbb{K}, +, \cdot)$ be a field. We then are interested in systems of linear equations / $\mathbb{K}$; if there are solutions, and if there are how many.

We then inspect a system of linear equations of n unknowns, m relations:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\cdots &= \cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

where $a_{ij}, b_k \in \mathbb{K}$.

Example.

$$\begin{aligned}
2x_1 - x_2 + x_3 &= 0 \qquad (1) \\
x_1 + 3x_2 + 4x_3 &= 0 \qquad (2)
\end{aligned}$$

over some field $\mathbb{K}$.

Explanation. Then, $3 \times (1) + (2)$ (carrying out the operations in $\mathbb{K}$) yields

$$\begin{aligned}
7x_1 + 7x_3 &= 0 \\
7 \cdot (x_1 + x_3) &= 0
\end{aligned} \qquad (3)$$

Then, we have 2 cases.
Case 1: $7 \neq 0$ in $\mathbb{K}$, then $\exists\, 7^{-1} \in \mathbb{K} : 7^{-1} \cdot 7 = 1$.
Then (3) $\Rightarrow 7^{-1} \cdot (7 \cdot (x_1 + x_3)) = 0$

$$\begin{aligned}
((7^{-1}) \cdot 7) \cdot (x_1 + x_3) &= 0 \\
1 \cdot (x_1 + x_3) &= 0 \\
\Rightarrow x_1 + x_3 &= 0 \\
\Rightarrow x_1 &= -x_3
\end{aligned}$$

Let $x_3 = a \Rightarrow x_1 = -a \Rightarrow x_2 = 2x_1 + x_3 = -a$.
$\Rightarrow \{(-a, -a, a) \mid a \in \mathbb{K}\}$ are solutions.

Case 2: $7 = 0$ in $\mathbb{K}$ (e.g. in $\mathbb{F}_7$) then (3) is automatically true.
Let $x_1 = a, x_3 = b \Rightarrow x_2 = 2x_1 + x_3 = 2a + b$
$\Rightarrow \{(a, 2a + b, b) \mid a, b \in \mathbb{K}\}$ are solutions. □

Remark. When doing $3\times$(1)+(2), how do we know if we're gaining or losing information? e.g in $\mathbb{F}_7$ we can just multiply by 7 and get nothing new! Therefore some kind of "equivalence" concept must be introduced!

Definition 1.4 (Linear combination)**.**

Suppose $S = \{\sum a_{ij}x_j = b_i\}_{1\leq i\leq m, 1\leq j\leq n}$ is a system of linear equations over $\mathbb{K}$. $S' = \{\sum a'_{ij}x_j = b_i\}_{1\leq i\leq m, 1\leq j\leq n}$ is another system of linear equations (not too important how many equations there are in S'). Then, S' is a **linear combination** of S if every linear equations $\sum a'_{ij}x_j = b_i$ in S' can be obtained as linear combinations of equations in S, i.e. $\sum a'_{ij}x_j = b'_i$ is obtained through

$$\sum c_i \left(\sum a_{ij}x_j\right) = \sum c_i b_i, 1 \leq i \leq m, \text{ for some } c_i \in \mathbb{K}$$

Definition 1.5 (Equivalance)**.**

2 systems S, S' are **equivalent** if S' is a linear combination of S and vice versa. Denote $\mathbf{S} \sim \mathbf{S'}$

Example. In previous example, $S = \{(1), (2)\}, S' = \{(1), (3)\}, S'' = \{(2), (3)\}, S''' = \{(3)\}$. Then, $S \not\sim S'', S \sim S'$ always , $S \sim S''$ only if 3 is invertible

Explanation.
From S', (1) = (1), (2) = (3) - 3 · (1). Therefore S is a linear combination of S'. $\Rightarrow S \sim S'$.
From S'', (2) = (2), 3 · (1) = (3) - (2). If $3^{-1} \in \mathbb{K}$(i.e. $3 \neq 0$) then (1) = 3^{-1}((3) - (2)) is thus recoverable from S'', then $S \sim S''$. Otherwise, no. □

Lecture 2

Matrices

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Proposition 2.1. If 2 systems of linear equations are equivalent, $S \sim S'$ then they have the same set of solutions

Remark. Why is this important? This becomes important if we have a complicated system and want to transform into a simpler system to solve.

Proof (Proposition 2.1)**.** If $(x_1 = \alpha_1, x_2 = \alpha_2, \dots, x_n = \alpha_n)$ is a solution ot S then we claim that it's also a solution of S' and vice versa. This is trivial because $S \sim S'$. $\square$

Definition 2.1 (Matrix)**.**

Let $\mathbb{K}$ be a field. Then an $\mathbf{m \times n}$ **matrix** with coefficients in $\mathbb{K}$, is an ordered tuple of elements in $\mathbb{K}$, typically written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{M}_{m \times n}(\mathbb{K})$$

Definition 2.2 (Matrix Multiplication)**.**

If $T_1 \in \mathbb{M}_{m \times n}(\mathbb{K}), T_2 \in \mathbb{M}_{n \times l}(\mathbb{K})$ then $T_1 \cdot T_2 \in \mathbb{M}_{m \times l}(\mathbb{K})$ (where $m, n, l \in \mathbb{N}$). Specifically,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1l} \\ b_{21} & b_{22} & \cdots & b_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nl} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & \cdots & c_{1l} \\ c_{21} & c_{22} & \cdots & \cdots & c_{2l} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ c_{m1} & c_{m2} & \cdots & \cdots & c_{ml} \end{bmatrix}$$

where

$$\begin{aligned} c_{ij} &= \text{the “inner product” of i-th row of } T_1 \text{ and j-th row of } T_2 \\ &= \sum_{t=1}^{n} a_{it} b_{tj} \\ &\forall\, (i,j), 1 \le i \le m, 1 \le j \le l \end{aligned}$$

In particular, if $T_1, T_2 \in \mathbb{M}_n := \mathbb{M}_{n \times n}(\mathbb{K})$ then $T_1 \cdot T_2$ and $T_2 \cdot T_1$ are both valid. In general, they're often not equal.

Observe. We can write system of linear equations as

$$T \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

where

$$T \in \mathbb{M}_{m\times n}(\mathbb{K}), \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{M}_{n\times 1}(\text{indeterminants}), \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{M}_{m\times 1}(\mathbb{K})$$

Then, finding solutions to S is equivalent to finding $(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{K}$ such that

$$T \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Exercise 2.1. If $T_1, T_2, T_3 \in \mathbb{M}_n(\mathbb{K})$ then $(T_1 \cdot T_2) \cdot T_3 = T_1 \cdot (T_2 \cdot T_3)$. This is by no means obvious.

Definition 2.3 (Identity Matrix).

$$I_n = id_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \vdots & \cdots & \ddots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \in \mathbb{M}_n(\mathbb{K})$$

Observe.

$$I_n \cdot T = T \cdot I_n, \ \forall\, T \in \mathbb{M}_n(\mathbb{K})$$

Thus, $(\mathbb{M}_n(\mathbb{K}), \cdot)$ is “trying” to be a group, but it's not.

Definition 2.4 (Invertible Matrix).

A matrix $T \in \mathbb{M}_n(\mathbb{K})$ is **invertible** if $\exists\, T' \in \mathbb{M}_n(\mathbb{K})$ such that

$$T \cdot T' = I_n$$

Exercise 2.2. If $T \cdot T' = I_n \Rightarrow T' \cdot T = I_n$

Definition 2.5 (General Linear Group $GL_n(\mathbb{K})$).

$$GL_n(\mathbb{K}) = \{T \in \mathbb{M}_n(\mathbb{K}) \mid T \text{ is invertible}\}$$

Remark. Then $GL_n(\mathbb{K})$ is a group.

Definition 2.6 (Elementary Row operations).

Let S be the system of equations:

$$\sum a_{1j}x_j = b_1 \quad (1)$$
$$\sum a_{2j}x_j = b_2 \quad (2)$$
$$\vdots = \vdots$$
$$\sum a_{mj}x_j = b_m \quad (\mathrm{m})$$

then there are 3 **elementary row operations**:

1. Switching 2 of the equations
2. Replace (i) with $c \cdot$ (i) where $c \neq 0$
3. Replace (i) by (i) + d(j) where $i \neq j$

Proposition 2.2. If S' can be obtained from S via a finite sequence of elementary row operations then $S \sim S'$.

Corollary 2.1. S can also be obtained from S' via a finite sequence of elementary row operations.

Corollary 2.2. If S' can be obtained from S via a finite sequence of elementary row operations then they have the same solutions.

Lecture 3

Vector Spaces

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3.1 Elementary Row Operations and Systems of Linear Equations

Question: What are we doing to the matrices $A, B(Ax = B)$ (A of size $m \times n$, B of size $n \times 1$) when elementary row operations are carried out?
Answer: The row operations operate on the **rows** of A (switching rows, multiplying by scalar, adding other rows)

Example.

$$A_0 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \overset{(1')=(1)+-2(3)}{\sim} A_1 = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \sim \cdots \sim A_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b\ldots \\ b\ldots \\ b\ldots \end{bmatrix}$$

We eventually arrived $LHS = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ itself, due to the properties of I_3. By "simplying" rows this way, we can therefore solve systems of linear equations.

Definition 3.1 (Row-reduced Matrix)**.**

The **row-reduced** form of a matrix has 1 as the leading non-zero coefficient for each of its rows (0-padded on the left). Furthermore, each column which contains the leading non-zero entry of some row has all its other entries as 0. By convention, the leading coefficient of a row of higher row index also has a higher column index.

Proof (Proposition 2.2)**.** We only provide a sketch of the proof. We re-enumerate the types of operations:

1. $(i) \leftrightarrow (j)$
2. $(i) \to c(i), c \neq 0$
3. $(i) \to (i) + d(j), j \neq i$

Explanations:

1. Trivial

2. Clearly S' is obtainable from S, and trivially all other equations except for (i) of S are obtainable from S'. However, $(i) = c^{-1}(c(i)) = c^{-1}(i')$. Therefore $S \sim S'$.

3. Similarly, S' is clearly obtainable from S, while $(i) = (i') - d(j) = (i') - d(j')$. Therefore $S \sim S'$.

$\square$

3.2 Vector Spaces

Definition 3.2 (Vector Space).

Let $\mathbb{K}$ be a field. A **vector space over** $\mathbb{K}$ ("$\mathbb{K}$-vector space")("k-vs") is an Abelian group V with a map: $\mathbb{K} \times V \to V$ ($\mathbb{K}$-action on V). An element in V is called a **vector**. They have to satisfy $\forall\, a, b \in \mathbb{K};\ \forall\, v, v_1, v_2 \in V$:

1. $0 \cdot v = 0$
   $1 \cdot v = v$

2. $(a + b) \cdot v = (a \cdot v) + (b \cdot v)$
   $(a \cdot b) \cdot v = a \cdot (b \cdot v)$

3. $a \cdot (v_1 + v_2) = (a \cdot v_1) + (a \cdot v_2)$

Essentially, $\mathbb{K}, V$ with operations:

1. $+ : \mathbb{K} \times \mathbb{K} \to \mathbb{K}, \cdot : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ (Field)

2. $+ : V \times V \to V$ (Abelian group)

3. $\cdot : \mathbb{K} \times V \to V$ (Action)

Example. Field $\mathbb{K} = \mathbb{R}$, $V = \mathbb{R}^n \doteq \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\}$. Indeed, $\mathbb{R}^n$ is an Abelian group.

Definition 3.3 (Linear Combination).

Let V be a k-vs. If $v_1, v_2, \ldots, v_r \in V; r \in \mathbb{N}$ then a **linear combination** of $\{v_1, v_2, \ldots, v_r\}$ is a vector of the form

$$c_1 \cdot v_1 + c_2 \cdot v_2 + \cdots + c_r \cdot v_r \text{ where } c_i \in \mathbb{K}$$

Definition 3.4 (Linear Span).

Then the **linear span** of $v_1, v_2, \ldots, v_r$ in V is the set of all such linear combinations.

Lecture 4

Linear Transformation, Homomorphism, Kernel, Image

04 Apr 2023

4.1 Vector Subspace

Definition 4.1 (Vector Subspace).

Let V be a $\mathbb{K}$-vector space. A **subspace** (or **sub-vector space**) of V is a subset $W \subseteq V$ such that W is itself a $\mathbb{K}$-vector space under addition and scaling induced from V. A priori, we know that

$$+ : W \times W \to V, \cdot : W \times W \to V$$

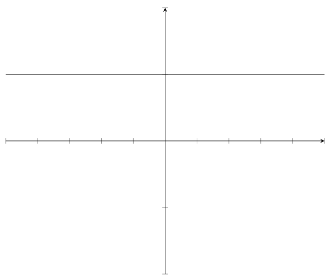
but this subspace requirement implies that

$$\forall\, x, y \in W, x + y \in W$$

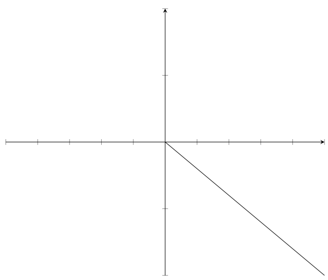
$$\forall\, \alpha \in \mathbb{K}, x \in W, \alpha \cdot x \in W$$

In other words, the subspace is closed under addition and scaling.

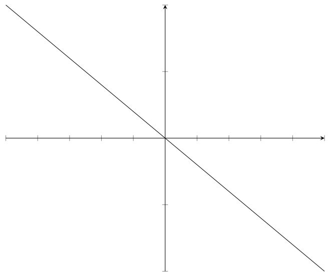
Example. Take $\mathbb{K} = \mathbb{R}, V = \mathbb{R}^2$, with ordinary addition and scaling.
Consider the subset represented by line $y = 1$.



This is not a subspace because there exists no 0 element. This kinda implies that any subspace of $\mathbb{R}^2$ must pass through the origin $(0, 0)$.
Consider another instance, this time the following ray:



This is also not a subspace, since there's no additive inverse. Therefore a subspace shall look something like this:



4.2 Mapping

Motivation. A map from sets to sets can be anything. e.g. $x : \mathbb{Z} \mapsto x^2 : \mathbb{Z}$ doesn't preserve the "group" structure $(x+y)^2 \neq x^2 + y^2$ most of the time.

Definition 4.2 (Group Homomorphism).

Let A, B be Abelian groups. Map $\psi : A \to B$ is called a **group homomorphism** if:

$$\psi(x+y) = \psi(x) + \psi(y)$$

Then $x : \mathbb{Z} \mapsto x^2 : \mathbb{Z}$ is not a group homomorphism, but $x : \mathbb{Z} \mapsto nx : \mathbb{Z}$ for fixed n is a group homomorphism.
Here, a natural question arises: If given 2 vector spaces, what maps are allowed between them? What structures do we have to preserve?

Definition 4.3 (Linear Transformation).

Let V, W be $\mathbb{K}$-vector spaces. Then a **vector space homomorphism** is also called a **linear transformation**, a map $\psi : V \to W$ such that

1. $\psi(v_1 + v_2) = \psi(v_1) + \psi(v_2) \; \forall \; v_1, v_2 \in V$
2. $\psi(\alpha \cdot v) = \alpha \cdot \psi(v) \; \forall \; \alpha \in \mathbb{K}, v \in V$

Denote $\mathbf{Hom}_{\mathbb{K}}(\mathbf{V}, \mathbf{W})$ as the set of all linear transformations $V \to W$.

Example. $\mathbb{K} = \mathbb{R}, V = W = \mathbb{R}$
$\mathrm{Hom}_{\mathbb{R}}(V, W) = \{\psi : \mathbb{R} \to \mathbb{R} \mid (1), (2) \text{ are satisfied } \}$
We claim that $\psi(1)$ uniquely determines the map ψ, because

$$\psi(\alpha) = \alpha \cdot \psi(1)$$

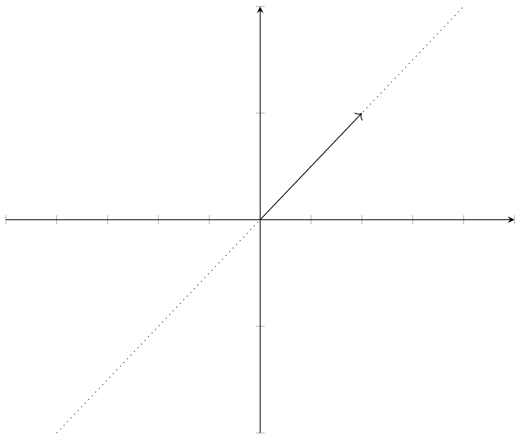
Essentially, there exists a bijection between $\mathrm{Hom}_{\mathbb{R}}(V, W)$ and $\mathbb{R}$:

$$\begin{aligned} \mathrm{Hom}_{\mathbb{R}}(V, W) &\to \mathbb{R} \\ \psi &\to \psi(1) \\ (\psi_\beta : x \mapsto x \cdot \beta) &\leftarrow \beta \end{aligned}$$

Example. $\mathbb{K} = \mathbb{R}, V = \mathbb{R}, W$ is any vector space (over $\mathbb{R}$)
We, similarly, claim that there is a bijection between $\mathrm{Hom}_{\mathbb{R}}(V, W)$ and $\mathbb{R}$. With the same reasoning, ψ is determined by $\psi(1)$, though this time $\psi(1) \in W$.

$$\begin{aligned} \mathrm{Hom}_{\mathbb{R}}(V, W) &\to W \\ \psi &\to \psi(1) \in W \\ (\psi_\beta : x \mapsto x \cdot w) &\leftarrow w \end{aligned}$$

Example. As a sub-example of the example above, consider $W = \mathbb{R}^2$:



Then if $\psi(1) = (4, 5)$ as above (and $\psi(0) = (0, 0)$ implicit), then ψ would map the rest of $V = \mathbb{R}$ onto the dotted line above.
An interesting point to note is that if $\psi(1) = (0, 0)$, then the entire real line would get sent (and compressed) to $(0, 0)$. $\psi_{(0,0)}$ therefore contracts $\mathbb{R}$ into one point (the origin (0, 0)) while others output a subspace of $\mathbb{R}^2$.

Example. $\mathbb{K} = \mathbb{R}, V = \mathbb{R}^2, W =$ any $\mathbb{R}$-vector space
We claim that there exists a bijection between $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^2, W)$ and $W \oplus W$; as each ψ is determined by $\psi((1, 0))$ and $\psi((0, 1))$.
The notation $\oplus$ is defined as: If V, W are $\mathbb{K}$-vector spaces then

$$V \oplus W = \{(v, w) \mid v \in V, w \in W\}$$

e.g. $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$
Then $V \oplus W$ would also be a $\mathbb{K}$-vector space with operations $+, \cdot$ defined intuitively:

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$$
$$\alpha \cdot (v, w) = (\alpha \cdot v, \alpha \cdot w)$$

Back to the example, $\forall\, v = (x, y) \in V, v = x(1, 0) + y(0, 1)$, therefore

$$\psi(v) = \psi((x, y)) = x \cdot \psi((1, 0)) + y \cdot \psi((0, 1))$$

ψ is therefore uniquely defined by $\psi((1, 0))$ and $\psi((0, 1))$.

Example. $\mathbb{K} = \mathbb{R}, V = \mathbb{R}^m, W =$ any $\mathbb{R}$-vector space
Think about $W = \mathbb{R}^n$ with similar reasoning.
Hint: We want to show there exists a bijection between $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R}^n)$ and $\mathbb{R}^{m \cdot n}$, but this is often rewritten as $\mathbb{M}_{m \times n}(\mathbb{R})$

4.3 Isomorphism, Kernel, Image

Every linear transformation is just a map, and we can therefore question if it is injective, surjective or bijective. In all cases, these concepts simply deal with the sets (vector spaces) as simply sets.

Definition 4.4 (Isomorphism).

A $\mathbb{K}$-linear transformation $\psi : V \to W$ is an **isomorphism** if it is bijective.

Definition 4.5 (Kernel, Image).

Let $\psi : V \to W$ be a linear transformation over $\mathbb{K}$. Then:

1. **Kernel**: $\ker(\psi) := \{v \in V \mid \psi(v) = 0\} \subseteq V$
2. **Image**: $\mathrm{im}(\psi) := \{w \in W \mid \exists\, v \in V \text{ such that } \psi(v) = w\}$

Lemma 4.1.

1. $\ker(\psi)$ is a $\mathbb{K}$-vector subspace of V
2. $\mathrm{im}(\psi)$ is a $\mathbb{K}$-vector subspace of W

Proof (Lemma)**.** We want to show that if $x, y \in \ker(\psi)$ then $x + y \in \ker(\psi)$.

$$\begin{aligned} \psi(x+y) &= \psi(x) + \psi(y) (\text{ since } \psi \text{ is a linear transformation }) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Therefore $x + y \in \ker(\psi)$
Furthermore, $\forall\, \alpha \in \mathbb{K}, x \in \ker(\psi)$ then

$$\psi(\alpha, x) = \alpha \cdot \psi(x) = \alpha \cdot 0 = 0 \Rightarrow \alpha \cdot x \in \ker(\psi)$$

Therefore $\ker(\psi)$ is a subspace.
Similarly, $\mathrm{im}(\psi)$ is a subspace. □

Definition 4.6 (Finite Dimensional, Dimension).

1. Let V be a $\mathbb{K}$-vector space. V is called **finite dimensional** if there exists a surjective linear transformation $\mathbb{K}^r \to V$ where $r \in \mathbb{Z}_{\geq 0}$. As a consequence, $\mathbb{K}^r$ is also finite dimensional, with an identity mapping.

2. If V is finite dimensional then **dimension** of V is defined as

$$\dim V \coloneqq \min\{k \in \mathbb{Z}_{\geq 0} \mid \exists \text{ linear transformation } \mathbb{K}^r \to V\}$$