

Math 20250: Abstract Linear Algebra
Problem Set 2

Hung Le Tran

09 Apr 2023

Problem 2.1 (Sec 2.2. Problem 3)

Is the vector $(3, -1, 0, -1)$ in the subspace of \mathbb{R}^5 spanned by the vectors $(2, -1, 3, 2)$, $(-1, 1, 1, -3)$ and $(1, 1, 9, -5)$?

Solution

For $(3, -1, 0, -1)$ to be in the abovementioned subspace, it must be a linear combination of the vectors $(2, -1, 3, 2)$, $(-1, 1, 1, -3)$ and $(1, 1, 9, -5)$ in \mathbb{R}^5 , i.e.

$$\exists c_1, c_2, c_3 \in \mathbb{R} \text{ s.t. } (3, -1, 0, -1) = c_1(2, -1, 3, 2) + c_2(-1, 1, 1, -3) + c_3(1, 1, 9, -5)$$

It follows that

$$\begin{aligned} 2c_1 - c_2 + c_3 &= 3 \\ -c_1 + c_2 + c_3 &= -1 \\ 3c_1 + c_2 + 9c_3 &= 0 \\ 2c_1 - 3c_2 - 5c_3 &= -1 \end{aligned}$$

which is equivalent to:

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 1 \\ 3 & 1 & 9 \\ 2 & -3 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

which is then equivalent to the following augmented matrix:

$$\begin{bmatrix} 2 & -1 & 1 & 3 \\ -1 & 1 & 1 & -1 \\ 3 & 1 & 9 & 0 \\ 2 & -3 & -5 & -1 \end{bmatrix}$$

on which we can carry out simplifying row operations to reach the row-reduced echelon form:

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system therefore has no solution, as the third row of the augmented matrix fails ($0c_1 + 0c_2 + 0c_3 \neq 1$)

Therefore $(3, -1, 0, 1)$ is not in the subspace. \square

Problem 2.2 (Sec 2.2. Problem 5)

Let \mathbb{F} be a field and let $n \geq 2$ be a positive integer. Let V be the vector space of all $n \times n$

matrices over \mathbb{F} . Which of the following sets of matrices $A \in V$ are subspaces of V ?

1. all invertible A
2. all non-invertible A
3. all A s.t. $AB = BA$, where B is some fixed matrix in V
4. all A s.t. $A^2 = A$

Solution 1. All invertible A

No. It is not true that \forall invertible $A_1, A_2 \in V, (A_1 + A_2)$ is also invertible.

Counter example:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow A_3 = A_1 + A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which is not invertible, since it admits $X = \begin{bmatrix} a \\ -a \end{bmatrix}$ for arbitrary $a \in \mathbb{F}$ as a solution to $A_3 X = 0$

2. All non-invertible A

No. It is not true that \forall non-invertible $A_1, A_2 \in V, (A_1 + A_2)$ is also non-invertible.

Counter example:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow A_3 = A_1 + A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A_1 is non-invertible since it admits $X = \begin{bmatrix} 0 \\ a \end{bmatrix}$ for arbitrary $a \in \mathbb{F}$ as a solution to $A_1 X = 0$

A_2 is non-invertible since it admits $X = \begin{bmatrix} a \\ 0 \end{bmatrix}$ for arbitrary $a \in \mathbb{F}$ as a solution to $A_2 X = 0$

Meanwhile, $A_3 = I$ is trivially invertible.

3. All A s.t. $AB = BA$, B is fixed in V

Yes. We want to show that given A_1, A_2 s.t. $A_1 B = B A_1, A_2 B = B A_2$ then

$$\begin{aligned} (A_1 + A_2)B &= B(A_1 + A_2) \\ (cA_1)B &= B(cA_1) \quad \forall c \in \mathbb{K} \end{aligned}$$

First, we have:

$$\begin{aligned}
[(A_1 + A_2)B]_{ij} &= \sum_{k=1}^n (A_1 + A_2)_{ik} B_{kj} \\
&= \sum_{k=1}^n (A_{1,ik} B_{kj} + A_{2,ik} B_{kj}) \\
&= \sum_{k=1}^n (A_{1,ik} B_{kj}) + \sum_{k=1}^n (A_{2,ik} B_{kj})
\end{aligned}$$

However,

$$A_1 B = B A_1 \Rightarrow \sum_{k=1}^n (A_{1,ik} B_{kj}) = [A_1 B]_{ij} = [B A_1]_{ij} = \sum_{k=1}^n (B_{ik} A_{1,kj})$$

and similarly for A_2 .

It follows that:

$$\begin{aligned}
[(A_1 + A_2)B]_{ij} &= \sum_{k=1}^n (A_{1,ik} B_{kj}) + \sum_{k=1}^n (A_{2,ik} B_{kj}) \\
&= \sum_{k=1}^n (B_{ik} A_{1,kj}) + \sum_{k=1}^n (B_{ik} A_{2,kj}) \\
&= \sum_{k=1}^n B_{ik} (A_{1,kj} + A_{2,kj}) \\
&= \sum_{k=1}^n B_{ik} (A_1 + A_2)_{kj} \\
&= [B(A_1 + A_2)]_{ij}
\end{aligned}$$

It follows that indeed $(A_1 + A_2)B = B(A_1 + A_2)$.

The other equality is trivially observed $(cA_1)B = B(cA_1) \forall c \in \mathbb{K}$, making this set a subspace of V

4. All A s.t. $A^2 = A$

No. It is not true that $\forall A_1, A_2$ s.t. $A_1^2 = A_1, A_2^2 = A_2 \Rightarrow (A_1 + A_2)^2 = A_1 + A_2$

Counterexample: $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

It is trivially true that $A_1^2 = A_1, A_2^2 = A_2$, however

$$(A_1 + A_2)^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \neq (A_1 + A_2)$$

Therefore this set is not a subspace of V .

□

Problem 2.3 (Sec 2.3. Problem 1)

Prove that if two vectors are linearly dependent, one of them is a scalar multiple of the other.

Solution

Let v_1, v_2 be two vectors that are linearly dependent in field \mathbb{K} . Then there exists $a_1, a_2 \in \mathbb{K}$ not all 0 s.t.

$$a_1 v_1 + a_2 v_2 = 0$$

WLOG, assume $a_1 \neq 0 \Rightarrow \exists a_1^{-1} : a_1^{-1} a_1 = 1$. It follows that

$$v_1 = a_1^{-1}(-a_2 v_2) = (-a_1^{-1} a_2) v_2$$

is a scalar multiple of v_2 . □

Problem 2.4 (Sec 2.3. Problem 2)

Are the vectors

$$\alpha_1 = (1, 1, 2, 4)$$

$$\alpha_2 = (2, -1, -5, 2)$$

$$\alpha_3 = (1, -1, 4, 0)$$

$$\alpha_4 = (2, 1, 1, 6)$$

linearly independent in \mathbb{R}^4 ?

Solution

No, because

$$\alpha_4 = (2, 1, 1, 6) = \frac{4}{3}(1, 1, 2, 4) + \frac{1}{3}(2, -1, -5, 2) = \frac{4}{3}\alpha_1 + \frac{1}{3}\alpha_2 + 0\alpha_3$$

□

Problem 2.5 (Sec 2.3. Problem 4)

Show that the vectors

$$\alpha_1 = (1, 0, -1), \alpha_2 = (1, 2, 1), \alpha_3 = (0, -3, 2)$$

form a basis for \mathbb{R}^3 . Express each of the standard basis vectors as linear combinations of $\alpha_1, \alpha_2, \alpha_3$

Solution

We first express each of the standard basis vectors as linear combinations of $\alpha_1, \alpha_2, \alpha_3$:

$$(1, 0, 0) = \frac{7}{10}(1, 0, -1) + \frac{3}{10}(1, 2, 1) + \frac{1}{5}(0, -3, 2)$$

$$(0, 1, 0) = \frac{-1}{5}(1, 0, -1) + \frac{1}{5}(1, 2, 1) + \frac{-1}{5}(0, -3, 2)$$

$$(0, 0, 1) = \frac{-3}{10}(1, 0, -1) + \frac{3}{10}(1, 2, 1) + \frac{1}{5}(0, -3, 2)$$

The standard basis vectors span V , therefore each vector in V can be expressed as linear combinations of the standard basis vectors, which can then be expressed as linear combinations of $\alpha_1, \alpha_2, \alpha_3$ per the equalities above. Therefore $\alpha_1, \alpha_2, \alpha_3$ span V .

It must now be proven that they are linearly independent, which is equivalent to showing that the system of linear equations represented by the following augmented matrix has no non-trivial solutions:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 \\ -1 & 1 & 2 & 0 \end{bmatrix}$$

Indeed, its row-reduced echelon form is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, only allowing for a trivial solution of

$(0, 0, 0)$. Thus, $\alpha_1, \alpha_2, \alpha_3$ are linearly independent.

Therefore they form a basis for V . □

Problem 2.6 ((Bonus) Sec 2.3. Problem 14)

Let V be the set of real numbers. Regard V as a vector space over the field of *rational* numbers, with the usual operations. Prove that this vector space is *not* finite-dimensional.

Solution

Suppose not; that $\dim_{\mathbb{Q}} \mathbb{R} = d \in \mathbb{N}$, meaning that there exists a basis $v_1, v_2, \dots, v_d \in \mathbb{R}$ such that

$$\forall v \in \mathbb{R} \exists \alpha_1, \alpha_2, \dots, \alpha_d \in \mathbb{Q} : v = \alpha_1 v_1 + \dots + \alpha_d v_d$$

We first observe that for $v_1 \neq v_2 \in \mathbb{R}$, their corresponding d -tuple must be different, i.e. $(\alpha_{v_1,1}, \alpha_{v_1,2}, \dots, \alpha_{v_1,d}) \neq (\alpha_{v_2,1}, \alpha_{v_2,2}, \dots, \alpha_{v_2,d})$. This is trivial.

Therefore if we consider $\varphi : \mathbb{R} \rightarrow \mathbb{Q}^d, \varphi(v) = (\alpha_{v,1}, \alpha_{v,2}, \dots, \alpha_{v,d})$ then φ is injective.

It follows that $|\mathbb{R}| \leq |\mathbb{Q}^d|$. However, \mathbb{R} is uncountable and \mathbb{Q}^d is countable. $\Rightarrow \Leftarrow$ □