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Henceforth we will work with metric spaces $(M,d)$ with metric topology $\mathcal{T}_d$

Recall $f: M \to N$ is a homeomorphism if bijective & $f, f^{-1}$ cts.

Examples

- translations, dilations, and rotations in $\mathbb{R}^n$ ( $f(x) = Ax + b$. $\det A \neq 0$ )
- $B_{\mathbb{R}^n}(x, r) \cong B_{\mathbb{R}^n}(y, s)$

Comparable metrics

Fix $M$, & let $d_1, d_2$ be 2 different metrics on $M$.

Say $d_1, d_2$ are comparable if $\exists\, C \geq 1$ such that $\forall x, x' \in M$

$$C^{-1} d_2(x,x') \leq d_1(x,x') \leq C \cdot d_2(x,x')$$

Prop if $d_1$ & $d_2$ are comparable, then $\mathrm{id}: (M, d_1) \to (M, d_2)$ is a homeo.

Proof Note:

$$d_2(\mathrm{id}(x), \mathrm{id}(x')) \leq C \cdot d_1(x,x')$$

$$d_1(\mathrm{id}^{-1}(x), \mathrm{id}^{-1}(x')) \leq C\, d_2(x,x').$$

To show $\mathrm{id}, \mathrm{id}^{-1}$ cts, use:

Lemma: If $f: M \to N$ is Lipschitz ($\exists L > 0$ s.t. $d_N(f(x), f(x')) \leq L \cdot d_M(x,x')$) then $f$ is cts.

Pf: $\varepsilon - \delta$ : Given $\varepsilon$, let $\delta = \frac{\varepsilon}{2L}$

## Product metrics

Let $(X, d_X)$, $(Y, d_Y)$ be metric spaces.

$M = X \times Y$.

Consider, for $p=(x,y)$ $p'(x',y')$

$$d_E(p,p') = \sqrt{d_X(x,x')^2 + d_Y(y,y')^2}$$
$$d_{\max}(p,p') = \max\{d_X(x,x'),\, d_Y(y,y')\}$$
$$d_{\text{sum}}(p,p') = d_X(x,x') + d_Y(y,y')$$

Lemma: all 3 are comparable:

$$d_{\max} \le d_E \le d_{\text{sum}} \le 2d_{\max}.$$

Cor $\mathcal{J}_{d_{\max}} = \mathcal{J}_{d_E} = \mathcal{J}_{d_{\text{sum}}}$.

convergence in any one metric $\Leftrightarrow$ in any other

closed, open.

Cor $f: M \to N$ & $g: X \to Y$ cts $\Rightarrow$ $f \times g: M \times X \to N \times Y$ cts.

## Application: Completeness of $\mathbb{R}^n$

A sequence of points $(p_n)_{n \geq 1}$ in $M$, • converges if $\exists\, p \in M$, $\forall \varepsilon > 0$ $\exists N \geq 1$ $n \geq N \Rightarrow d(p_n, p) < \varepsilon$

write $p_n \to p$

• is Cauchy if $\forall \varepsilon > 0$ $\exists N \geq 1$ $n, m \geq N \Rightarrow d(p_n, p_m) < \varepsilon$.

Lemma • $(p_n, q_n) \to (p,q)$ in $d_E, d_{\max}, d_{\text{sum}}$ $\Leftrightarrow$ $p_n \to p$ $q_n \to q$.

• $((p_n, q_n))$ Cauchy in $d_E, d_{\max}, d_{\text{sum}}$ $\Leftrightarrow$ $(p_n)$, $(q_n)$ Cauchy

Def $(X,d)$ Complete if $\forall (p_n)$ Cauchy, $\exists x \in X$: $x_n \to x$.

Recall: $\mathbb{R}$ is complete (uses LUB property)

Cor $(\mathbb{R}^n, d_E)$ complete.