MATH 20800: Honors Analysis in Rn II

Problem Set 3

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Textbook: Pugh's Real Mathematical Analysis, Rudin's Principles of Mathematical Analysis

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Problem 3.1 (Rudin 7.20 done)

If f is continuous on $[0,1]$, and $\int_0^1 f(x)x^n \mathrm{d}x = 0$ for all $n \geq 0$, prove that $f(x) = 0$ on $[0,1]$.

Solution

Let $P(x) = \sum_{k=0}^{N} c_k x^k$ be any polynomial in x, then

$$\begin{aligned}\int_0^1 f(x)P(x) &= \int_0^1 f(x)\sum_{k=0}^{N} c_k x^k \\ &= \sum_{k=0}^{n} c_k \int_0^1 f(x)x^k = 0\end{aligned}$$

From Weierstrass, we know that there exists $\{P_n\} \subset C_0([0,1],\mathbb{R})$ such that $P_n \rightrightarrows f$, i.e., that

$$d_{sup}(f, P_n) \xrightarrow{n\to\infty} 0.$$

From the above observation, it follows that:

$$\int_0^1 f(x)P_n(x)\mathrm{d}x = 0.$$

Also, since f is continuous on compact $[0,1]$, there exists $M \geq 0$ satisfying $|f| \leq M$ on $[0,1]$.

Therefore,

$$\begin{aligned}\left|\int_0^1 f(x)^2 \mathrm{d}x\right| &= \left|\int_0^1 f(x)[f(x)-P_n(x)]\mathrm{d}x\right| \\ &\le \left|\int_0^1 |f||f(x)-P_n(x)|\mathrm{d}x\right| \\ &\le 1\times M\times d_{sup}(f,P_n) = Md_{sup}(f,P_n)\end{aligned}$$

gets arbitrarily small. $\left|\int_0^1 f(x)^2\mathrm{d}x\right| \ge 0$ so $\left|\int_0^1 f(x)^2\mathrm{d}x\right| = 0$. f^2 is continuous and non-negative. It must therefore be concluded that $f^2 \equiv 0$ on $[0,1]$ (otherwise, if $f^2(x_0) = c > 0$ for some x_0, then there is a neighborhood of size δ, within which the infimum is $\ge c/2$, making the Riemann integral positive).

Therefore $f \equiv 0$. $\square$

Problem 3.2 (Rudin 7.21 done)

Let K be the unit circle in the complex plane, i.e., $\{z \in \mathbb{C} : |z| = 1\}$. Consider the algebra $\mathcal{A}$ of all functions of the form

$$f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta} \quad (\theta \in \mathbb{R})$$

Show that $\mathcal{A}$ separates points on K and $\mathcal{A}$ vanishes at no point of K, but nevertheless there are continuous functions on K which are not in the uniform closure of $\mathcal{A}$.

Solution

1. $\mathcal{A}$ separates points on K: Given $e^{i\theta_1} \ne e^{i\theta_2}$, then we have $f = id, f(e^{i\theta}) = e^{i\theta}$ that trivially separates them.

2. $\mathcal{A}$ vanishes at no point of K: Given any $e^{i\theta} \in K$, $f = id$ trivally does not vanish there.

3. WTS

$$\int_0^{2\pi} f(e^{i\theta})e^{i\theta}d\theta = 0$$

for every $f \in \mathcal{A}$ and for every g in the uniform closure of $\mathcal{A}$.

3.1. Take any $f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta} \in \mathcal{A}$. Then

$$\begin{aligned}\int_0^{2\pi} f(e^{i\theta})e^{i\theta}d\theta &= \int_0^{2\pi}\sum_{n=0}^{N} c_n e^{in\theta}e^{i\theta}d\theta \\ &= \sum_{n=0}^{N} c_n \int_0^{2\pi} e^{i(n+1)\theta}d\theta \\ &= \sum_{n=0}^{N} c_n \left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_0^{2\pi} \\ &= 0\end{aligned}$$

3.2. Then, take g in the uniform closure of $\mathcal{A}$, i.e., for every $\varepsilon > 0$, there exists $f \in A$ such that $d_{sup}(f, g) < \varepsilon$.

Accordingly,

$$\begin{aligned}\left|\int_0^{2\pi} g(e^{i\theta})e^{i\theta}d\theta\right| &= \left|\int_0^{2\pi}[g(e^{i\theta}) - f(e^{i\theta})]e^{i\theta}d\theta\right| \\ &\leq \int_0^{2\pi}\left|g(e^{i\theta}) - f(e^{i\theta})\right|\left|e^{i\theta}\right|d\theta \\ &\leq 2\pi \times 1 \times d_{sup}(f, g) = 2\pi d_{sup}(f, g)\end{aligned}$$

that gets arbitrariliy small. It follows that for any g in the uniform closure of $\mathcal{A}$,

$$\int_0^{2\pi} g(e^{i\theta})e^{i\theta}d\theta = 0.$$

However,

$$h(e^{i\theta}) = e^{-i\theta},$$

the complex conjugate function, which is trivially a continuous function on K, has

$$\int_0^{2\pi} h(e^{i\theta})e^{i\theta}d\theta = \int_0^{2\pi} 1 = 2\pi \neq 0,$$

so h is not in the uniform closure of $\mathcal{A}$. □

Problem 3.3 (Rudin 7.23 done)

Let $P_0 = 0$ and define, for $n = 0, 1, 2, \ldots$,

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}$$

Prove that $\lim_{n\to\infty} P_n(x) = |x|$ uniformly on $[-1, 1]$.

Solution

Observe that

$$\begin{aligned}|x| - P_{n+1}(x) &= |x| - P_n(x) + \frac{P_n^2(x) - x^2}{2} \\ &= (|x| - P_n(x))\left(1 - \frac{|x| + P_n(x)}{2}\right)\end{aligned}$$

We will now use induction to prove that $0 \leq P_n(x) \leq P_{n+1}(x) \leq |x|$ for $|x| \leq 1$.

It is true for $n = 0$: $P_0(x) = 0 \leq |x|, 0 \geq 0$. Suppose it is also true for $n = k$. Then

$$|x| - P_{k+1}(x) = (|x| - P_k(x))\left(1 - \frac{|x| + P_k(x)}{2}\right)$$

Then:

$$1-\frac{|x|+P_k(x)}{2}\geq 1-\frac{|x|+|x|}{2}=1-|x|\geq 0, |x|-P_k(x)\geq 0$$

and

$$1-\frac{|x|+P_k(x)}{2}\leq 1-0=1$$

so

$$|x|-P_{k+1}(x)=(|x|-P_k(x))\left(1-\frac{|x|+P_k(x)}{2}\right)\leq |x|-P_k(x), \geq 0.$$

so it follows that $0\leq P_k(x)\leq P_{k+1}(x)\leq |x|$. By induction, $0\leq P_n(x)\leq P_{n+1}(x)\leq |x|$ is true for all $n\in\mathbb{N}$ (on $[-1,1]$).

Then, we can apply

$$\begin{aligned}|x|-P_{n+1}(x)&=(|x|-P_n(x))\left(1-\frac{|x|+P_n(x)}{2}\right)\\&\leq (|x|-P_n(x))\left(1-\frac{|x|}{2}\right)\end{aligned}$$

iteratively to get

$$0\leq |x|-P_n(x)\leq (|x|-P_0(x))\left(1-\frac{|x|}{2}\right)^n=|x|\left(1-\frac{|x|}{2}\right)^n$$

Then, for all $\varepsilon>0$, for $|x|<\varepsilon$, we have that for all $n\in\mathbb{N}$ that $|x|-P_n(x)\leq |x|\times 1<\varepsilon$.

For $|x|\geq\varepsilon$, then

$$|x|-P_n(x)\leq |x|\,(1-\varepsilon/2)^n\leq (1-\varepsilon/2)^n$$

can get uniformly arbitrarily small, since $1-\frac{\varepsilon}{2}<1$.

It follows that the convergence is uniform on $[-1,1]$. □

Problem 3.4 (Pugh 4.55 done)

Let f be a real valued continuous function on the compact interval $[a,b]$. Given $\varepsilon>0$, show that there is a polynomial p such that

$$\begin{aligned}p(a)&=f(a),\\p'(a)&=0,\\|p(x)-f(x)|&<\varepsilon\end{aligned}$$

for all $x\in[a,b]$.

Solution

WLOG, $[a,b]=[0,1]$, $f(a=0)=0$ (can always scale and translate). Our goal is now to find polynomial p such that $p(0)=p'(0)=0, d_{sup}(p,f)<\varepsilon$ (d_{sup} on $[0,1]$).

Since $f\in C^0([0,1],\mathbb{R})$. Fix $\varepsilon>0$. By Weierstrass, we know that there exists polynomial $g=\sum_{k=0}^N a_k x^k$ such that $d_{sup}(f,g)<\varepsilon/3$. In particular, $\varepsilon/3>d_{sup}(f,g)\geq |f(0)-g(0)|=|a_0|$.

From the previous problem, we know that there exists polynomials $P_n(x)\rightrightarrows |x|$ on

$[-1,1]$. Restrict this to $[0,1]$, then $P_n(x) \rightrightarrows x$ on $[0,1]$; and notice that in the recursive definition of $P_n(x)$, its lowest degree of x is 2.

Choose $M \in \mathbb{N}$ such that $d_{sup}(P_M, x) < \frac{\varepsilon}{3|a_1|}$. Let $P_M(x) = \sum_{k=1}^{L} b_k x^k$. Then construct

$$p(x) = a_1 P_M(x) + \sum_{k=2}^{N} a_k x^k = \sum_{k=2}^{\max\{N,L\}} c_k x^k$$

Then

$$\begin{aligned} d_{sup}(p, f) &\le d_{sup}(p, g) + d_{sup}(g, f) \\ &< \sup\{|a_0 - a_1(P_M(x) - x)|\} + \varepsilon/3 \\ &\le \sup\{|a_0|\} + \sup\{|a_1(P_m(x) - x)|\} + \varepsilon/3 \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

so this p satisfies the third condition. How about the first 2?

$$\begin{aligned} p(0) &= \sum_{k=2}^{\max\{N,L\}} c_k 0^k = 0 \\ p'(0) &= \sum_{k=2}^{\max\{N,L\}} k c_k 0^{k-1} = 0 \end{aligned}$$

And we are done. $\square$

Problem 3.5 (4.53 done)

Let f be a C^2 function on the real line. Assume that f is bounded with bounded second derivative. Let $A = \sup_x |f(x)|$ and $B = \sup_x |f''(x)|$. Prove that

$$\sup_x |f'(x)| \le 2\sqrt{AB}$$

Solution

Take any x_0. WLOG, $M = f'(x_0) > 0$. Therefore, for $t > 0$, $|f'(x_0 + t) - f'(x_0)| = |\int_{x_0}^{x_0+t} f''(s)ds| \le tB$. It follows that

$$f'(x_0 + t) \ge f'(x_0) - tB = M - tB$$

Therefore,

$$\begin{aligned} f(x_0 + M/B) - f(x_0) &= \int_{x_0}^{x_0+M/B} f'(t)\mathrm{d}t \\ &\ge \int_0^{M/B} (M - tB)\mathrm{d}t \\ &= M^2/B - B(M/B)^2/2 = \frac{M^2}{2B} \end{aligned}$$

Therefore

$$\frac{M^2}{2B} \le f(x_0 + M/B) - f(x_0) \le |f(x_0 + M/B) - f(x_0)| \le 2A \Rightarrow M \le 2\sqrt{AB}$$

Since $f'(x_0) \le 2\sqrt{AB}$ for all $x_0 \in \mathbb{R}$, it follows that $\sup f' \le 2\sqrt{AB}$. □

Problem 3.6 (4.54 done)

Let f be continuous on $\mathbb{R}$ and let

$$f_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right)$$

Prove that $f_n(x)$ converges uniformly to a limit on every finite interval $[a, b]$.

Solution

Define $g(x) = \int_0^1 f(x+t)\mathrm{d}t$. f is continuous on $\mathbb{R}$ so g is well-defined, and continuous. WTS for every $[a, b]$, $f_n \rightrightarrows g$.

Fix $[a, b]$ and $\varepsilon > 0$. f is continuous on compact interval, so is uniformly continuous. Thus there exists $\delta > 0$ such that $|u - v| < \delta \Rightarrow |fu - fv| < \varepsilon$.

Take N large enough so that $N\delta > 1$. Then for any $n \ge N$ (and thus $1/n \le 1/N < \delta$), we have for any x,

$$\begin{aligned}
|f_n(x) - g(x)| &= \left|\frac{1}{n}\sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right) - \int_0^1 f(x+t)\mathrm{d}t\right| \\
&\le \sum_{k=0}^{n-1}\left|\frac{1}{n} f\left(x + \frac{k}{n}\right) - \int_{k/n}^{(k+1)/n} f(x+t)\mathrm{d}t\right| \\
&\le \sum_{k=0}^{n-1}\left|\int_{k/n}^{(k+1)/n}\left(f\left(x + \frac{k}{n}\right) - f(x+t)\right)\mathrm{d}t\right| \\
&< \sum_{k=0}^{n-1} \varepsilon/n = \varepsilon
\end{aligned}$$

and therefore $f_n \rightrightarrows g$ on $[a, b]$. □

Problem 3.7 (Pugh 4.57 done)

Let f and f_n be functions from $\mathbb{R}$ to $\mathbb{R}$. Assume that $f_n(x_n) \to f(x)$ as $n \to \infty$ whenever $x_n \xrightarrow{n\to\infty} x$. Prove that f is continuous. (Note: the functions f_n are not assumed to be continuous.)

Solution

Suppose not. Then there exists $x_n \xrightarrow{n\to\infty} x$ such that $f(x_n) \not\to f(x)$, i.e., that there exists some $\varepsilon > 0$ such that for all $N \in \mathbb{N}$, there exists some $m \ge N$ such that $|f(x_m) - f(x)| \ge \varepsilon$.

For each x_n, take the sequence $(y_k)_{k\in\mathbb{N}} := (y_k = x_n)_{k\in\mathbb{N}}$. Trivially, $y_k \xrightarrow{k\to\infty} x_n$. Therefore

$f_k(x_n) = f_k(y_k) \xrightarrow{k\to\infty} f(x_n)$. In short, we have pointwise convergence of $\{f_k\}$ on each x_n. This implies there exists M_n such that $k \geq M_n \Rightarrow |f_k(x_n) - f(x_n)| < \varepsilon/2$. We iteratively choose $M_1, M_2, \ldots$ such that they are in strict increasing order (can always take $M_{n+1} > \max\{M_1, \ldots, M_n\}$).

Then, define a new sequence $(z_l)_{l\in\mathbb{N}}$ as follows:

$$
\begin{aligned}
z_0 = \ldots = z_{M_1-1} &= 0 \\
z_{M_1} = z_{M_1+1} = \ldots = z_{M_2-1} &= x_1 \\
z_{M_2} = z_{M_2+1} = \ldots = z_{M_3-1} &= x_2 \\
&\ldots
\end{aligned}
$$

where $z_l = x_j$ iff $M_j \leq l < M_{j+1}$.

From definition, notice that for $l \geq M_1$, $|f_l(z_l) - f(z_l)| < \varepsilon/2$, since their index, l, satisfies the pointwise condition above.

Furthermore, $z_l \to x$. So $f_l(z_l) \to f(x)$ by hypothesis, which means there exists some L' such that $\forall\, l \geq L'$, $|f_l(z_l) - f(x)| < \varepsilon/2$. Choose $L = \max\{L', M_1\}$. Then $L \leq M_N$ for some N. Then, for all $m \geq N$ ($\Rightarrow M_m \geq M_N \geq L$), we can pick some z_l such that $z_l = x_m$, which implies, $l \geq M_m \geq L$, and can bound

$$
\begin{aligned}
|f(x_m) - f(x)| &\leq |f(z_l) - f_l(z_l)| + |f_l(z_l) - f(x)| \\
&< \varepsilon/2 \text{ (by design of sequence, } l \geq M_1) \ + \varepsilon/2 \ (l \geq L) \\
&= \varepsilon
\end{aligned}
$$

We have therefore found N that was supposed to be impossible to find from the start, $\Rightarrow\Leftarrow$.

It follows that f must be continuous. $\square$

Problem 3.8 (Pugh 4.58 done)

Let $f(x), 0 \leq x \leq 1$, be a continuous real function with continuous derivative $f'(x)$. Let $M = \sup_{x\in[0,1]}|f'(x)|$. Prove, for $n = 1, 2, \ldots$,

$$
\left| \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) - \int_0^1 f(x)\mathrm{d}x \right| \leq \frac{M}{2n}
$$

Solution

We have:

$$\begin{aligned}\left|\frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) - \int_0^1 f(x)\mathrm{d}x\right| &= \left|\sum_{k=0}^{n-1} \frac{1}{n} f\left(\frac{k}{n}\right) - \sum_{k=0}^{n-1}\int_{k/n}^{(k+1)/n} f(x)\mathrm{d}x\right| \\ &\le \sum_{k=0}^{n-1}\int_{k/n}^{(k+1)/n}\left|f(x) - f\left(\frac{k}{n}\right)\right|\mathrm{d}x \\ &\le \sum_{k=0}^{n-1}\int_{k/n}^{(k+1)/n} M\left(x-\frac{k}{n}\right)\mathrm{d}x \\ &= \sum_{k=0}^{n-1}\int_0^{1/n} Mt\mathrm{d}t \\ &= \sum_{k=0}^{n-1}\frac{M}{2n^2} = \frac{M}{2n}\end{aligned}$$

as required. □

Problem 3.9 (Pugh 4.60 done)

Let f be a continuous real-valued function on $[0,\infty)$ such that

$$\lim_{x\to\infty}\left(f(x) + \int_0^x f(t)\mathrm{d}t\right)$$

exists (and is finite). Prove that $\lim_{x\to\infty} f(x) = 0$.

Solution

Notice that:

$$f(x) + \int_0^x f(t)\mathrm{d}t = \frac{1}{e^x}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\int_0^x f(t)\mathrm{d}t\right) = \frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\int_0^x f(t)\mathrm{d}t\right)}{\frac{\mathrm{d}}{\mathrm{d}x}e^x}$$

Let $g = e^x\int_0^x f(t)\mathrm{d}t, h = e^x$ then we have that

$$\lim_{x\to\infty}\frac{g'}{h'} = \lim_{x\to\infty} f(x) + \int_0^x f(t)\mathrm{d}t = L < \infty$$

We want to show that $\lim_{x\to\infty}\frac{g}{h} = L$ too. (Technically we're simply proving L'Hopital rule, but we have to be explicitly clear here, since it is not trivially clear that $g \xrightarrow{x\to\infty} \pm\infty$.)

Take any $\varepsilon > 0$. Since $\lim_{x\to\infty}\frac{g'}{h'} = L$, there exists X_1 such that $x \ge X_1 \Rightarrow \left|\frac{g'}{h'} - L\right| < \frac{\varepsilon}{2}$.

By construction, it follows that for all $x \ge X_1$,

$$\left|\frac{g(x) - g(X_1)}{h(x) - h(X_1)} - L\right| = \left|\frac{g'(\theta)}{h'(\theta)} - L\right| < \varepsilon/2$$

since $\theta \in (X_1, x) \Rightarrow \theta > X_1 \Rightarrow \left|\frac{g'(\theta)}{h'(\theta)} - L\right| < \varepsilon/2$.

Then we can estimate for all $x \geq X_1$:

$$\begin{aligned}\left|\frac{g(x)}{h(x)} - L\right| &= \left|\frac{g(x) - g(X_1)}{h(x) - h(X_1)} \frac{h(x) - h(X_1)}{h(x)} + \frac{g(X_1)}{h(x)} - L\right| \\ &\leq \left|\frac{g(x) - g(X_1)}{h(x) - h(X_1)} - L\right| + \left|\frac{g(x) - g(X_1)}{h(x) - h(X_1)} \frac{h(X_1)}{h(x)}\right| + \left|\frac{g(X_1)}{h(x)}\right| \\ &\leq \frac{\varepsilon}{2} + \left(|L| + \frac{\varepsilon}{2}\right)\left|\frac{h(X_1)}{h(x)}\right| + \left|\frac{g(X_1)}{h(x)}\right|\end{aligned}$$

goes arbitrarily small as $h(X_1), g(X_1)$ are fixed, and $h(x) \xrightarrow{x \to \infty} +\infty$.

All in all, it follows that

$$\lim_{x \to \infty} \frac{g(x)}{h(x)} = L$$

Therefore

$$\begin{aligned} L &= \lim_{x \to \infty} \int_0^x f(t)\mathrm{d}t \\ \Rightarrow \lim_{x \to \infty} f(x) &= \lim_{x \to \infty} (f(x) + \int_0^x f(t)\mathrm{d}t) - \lim_{x \to \infty} \int_0^x f(t)\mathrm{d}t \\ &= L - L = 0\end{aligned}$$

as required. □

Problem 3.10 (Pugh 4.65 done)

Let f be a continuous, strictly increasing function from $[0, \infty)$ onto $[0, \infty)$ and let $g = f^{-1}$ (the inverse, not the reciprocal). Prove that

$$\int_0^a f(x)\mathrm{d}x + \int_0^b g(y)\mathrm{d}y \geq ab$$

Solution

Fix any $c \geq 0$. Then g is strictly increasing on $[0, c]$, so it is integrable on $[0, c]$. $\int_0^b g(y)\mathrm{d}y$ is then well-defined.

We use the following Lemma:

Lemma

For $a \geq 0$,

$$\int_0^a f(x)\mathrm{d}x + \int_0^{f(a)} g(y)\mathrm{d}y = af(a)$$

Proof (Lemma)

Take any partition $P = \{x_0 = 0, x_1, \dots, x_n = a\}$ of $[0, a]$. Then, since f is a bijection and strictly increasing, $Q_P \coloneqq \{f(x_0) = f(0) = 0, f(x_1), \dots, f(x_n) = f(a)\}$ is a partition of $[0, f(a)]$. In fact, it is clear that $P \mapsto Q_P$ is a bijective map between the set of partitions on $[0, a]$ and $[0, f(a)]$.

Then, we have that:

$$\begin{aligned}
L(f, P) + U(g, Q_P) &= \sum_{i=1}^{n} \inf_{t \in [x_{i-1}, x_i]} f(t)\Delta x_i + \sup_{s \in [f(x_{i-1}), f(x_i)]} g(s)\Delta f(x_i) \\
&= \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}) + x_i(f(x_i) - f(x_{i-1})) \\
&= \sum_{i=1}^{n} x_i f(x_i) - x_{i-1} f(x_{i-1}) \\
&= x_n f(x_n) - x_0 f(x_0) = af(a)
\end{aligned}$$

Similarly,

$$U(f, P) + L(g, Q_P) = af(a)$$

It follows that

$$L(f, P) + U(g, Q_P) + U(f, P) + L(g, Q_P) = 2af(a)$$

The equality holds for all P and corresponding Q_P. Fix some $\varepsilon > 0$. Since f, g are integrable, there exists some P, Q such that

$$\begin{aligned}
2\int_0^a f(x)\mathrm{d}x - \varepsilon \leq L(f, P) + U(f, P) &\leq 2\int_0^a f(x)\mathrm{d}x + \varepsilon \\
2\int_0^{f(a)} g(y)\mathrm{d}y - \varepsilon \leq L(g, Q) + U(g, Q) &\leq 2\int_0^{f(a)} g(y)\mathrm{d}y + \varepsilon
\end{aligned}$$

Then we can define P' as the refinement of P and $f^{-1}(Q) = g(Q) = \{g(y_i) = f^{-1}(y_i) : y_i \in Q\}$ on $[0, a]$, then the bound remains the same, with P replaced by P' and Q replaced by $Q_{P'}$. It then follows that

$$2\left(\int_0^a f(x)\mathrm{d}x + \int_0^{f(a)} g(y)\mathrm{d}y\right) - 2\varepsilon \leq 2af(a) \leq 2\left(\int_0^a f(x)\mathrm{d}x + \int_0^{f(a)} g(y)\mathrm{d}y\right) + 2\varepsilon$$

And this holds for all ε so it must be the case that

$$\int_0^a f(x)\mathrm{d}x + \int_0^{f(a)} g(y)\mathrm{d}y = af(a)$$

as required. $\square$

Now that the lemma is proven, we use it for our problem.

Case 1: $b \le f(a)$. Let $a' = g(b) \le a$. Then

$$\begin{aligned}\int_0^a f(x)\mathrm{d}x + \int_0^b g(y)\mathrm{d}y &= \int_{a'}^a f(x)\mathrm{d}x + \left(\int_0^{a'} f(x)\mathrm{d}x + \int_0^{f(a')} g(y)\mathrm{d}y\right) \\ &= \int_{a'}^a f(x)\mathrm{d}x + a'b \ge (a-a')f(a') + a'b = (a-a')b - a'b = ab\end{aligned}$$

Case 2: $b \ge f(a)$. Similarly, let $a' = g(b) \ge a$. Then

$$\begin{aligned}\int_0^a f(x)\mathrm{d}x + \int_0^b g(y)\mathrm{d}y &= \left(\int_0^a f(x)\mathrm{d}x + \int_0^{f(a)} g(y)\mathrm{d}y\right) + \int_{f(a)}^b g(y)\mathrm{d}y \\ &\ge af(a) + (b - f(a))a = ab\end{aligned}$$

From 2 cases, we are done. □