

MATH 20700: Honors Analysis in Rn I

Problem Set 8

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Textbook: Pugh's Real Mathematical Analysis

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Problem 8.1 (5.48 done)

Set

$$f(x, y) = \begin{cases} 1 - 1/q & \text{if } x, y \in \mathbb{Q} \cap [0, 1], y = p/q \\ 1 & \text{otherwise} \end{cases}$$

Prove that f is RI on $R = [0, 1]^2$, calculate $\underline{F}(y)$ and $\overline{F}(y)$, and prove that $\int_0^1 \underline{F}(y) dy = \int_0^1 \overline{F}(y) dy = \int_R f = 1$.

Solution

1. We first WTS f is RI on R , by showing that the set of discontinuities is a zero set in R . Recall that

$$D = \bigcup_{k \in \mathbb{N}} D_k$$

where $D_k = \{z \in R : \text{osc}_z(f) = \lim_{r \rightarrow 0} \text{diam}(B_R(z, r)) \geq \frac{1}{k}\}$.

Fix $k \in \mathbb{N}$. Let

$$\begin{aligned} N_k &= \text{set of reduced fractions less than 1 with denominators } \leq 2k \\ &= \{0 \leq p/q \leq 1 : p, q \in \mathbb{N}, \gcd(p, q) = 1, q \leq 2k\} \\ &= \{0, 1/1, 1/2, 1/3, 2/3, 1/4, 3/4, \dots, (2k-1)/2k\} \\ E_k &= \{(x, y) \in R : y \in N_k\} = [0, 1] \times N_k \end{aligned}$$

Quickly note that N_k is a finite set, since its number of elements is strictly less than $4k^2$. Therefore E_k is a zero set in R , since it is a finite union of 1-dimensional slices.

We want to show that all points in $R \setminus E_k$ has oscillation less than $1/k$.

Indeed, take any $(x_0, y_0) \in R \setminus E_k = \{(x, y) \in R : y \notin N_k\}$. Take $d = \min_{a \in N_k} \{|y_0 - a|\} > 0$.

Then for $r < d$, the ball $B_R((x_0, y_0), r) \cap E_k = \emptyset$. Take $(x, y) \in B_R((x_0, y_0), r)$, then

$$f(x, y) = \begin{cases} 1 - 1/l & \text{if } x, y \in \mathbb{Q} \cap [0, 1]; y = l'/l \text{ for some } l > 2k \text{ since } y \notin N_k \\ 1 & \text{otherwise} \end{cases}$$

Hence for any $z_1, z_2 \in B_R(z, r)$,

$$\begin{aligned} |f(z_1) - f(z_2)| &\leq |f(z_1) - 1| + |f(z_2) - 1| \\ &< |1 - 1/2k - 1| + |1 - 1/2k - 1| = 1/k \end{aligned}$$

which implies

$$\text{osc}_{(x_0, y_0)}(f) = \lim_{r \rightarrow 0} \text{diam}(B_R(z, r)) < 1/k$$

Therefore all points in $R \setminus E_k$ has oscillation less than $1/k$, which implies $D_k \subset E_k$, which implies D_k is a zero set.

D is then a countable union of zero sets, and is therefore a zero set. f is RI as required.

2. To find $\underline{F}(y)$ and $\overline{F}(y)$, we first have to define $f_y(x)$, which is:

$$f_y(x) = \begin{cases} 1 - \mathbb{1}_{\mathbb{Q}}(x)/q & \text{if } y = p/q \\ 1 & \text{otherwise} \end{cases}$$

Case 1: $y \in \mathbb{Q}, y = p/q$.

$$\begin{aligned} \underline{F}(y) &= \underline{\int_0^1} (1 - \mathbb{1}_{\mathbb{Q}}(x)/q) dx \\ &= \sup_{P \in \mathcal{P}([0,1])} L(1 - \mathbb{1}_{\mathbb{Q}}(x)/q, P) \end{aligned}$$

For any partition P of $[0, 1]$, $m_i = 1 - 1/q \forall i$, since the rationals are dense in $[0, 1]$. Therefore $L(1 - \mathbb{1}_{\mathbb{Q}}(x)/q, P) = 1 - 1/q \forall P$. Therefore

$$\underline{F}(y) = 1 - 1/q$$

On the other hand,

$$\begin{aligned} \overline{F}(y) &= \overline{\int_0^1} (1 - \mathbb{1}_{\mathbb{Q}}(x)/q) dx \\ &= \inf_{P \in \mathcal{P}([0,1])} U(1 - \mathbb{1}_{\mathbb{Q}}(x)/q, P) \end{aligned}$$

For any partition P of $[0, 1]$, $M_i = 1 \forall i$, since the irrationals are dense in $[0, 1]$. Therefore $U(1 - \mathbb{1}_{\mathbb{Q}}(x)/q, P) = 1 \forall P$. Therefore

$$\overline{F}(y) = 1$$

Case 2: $y \notin \mathbb{Q}$, so $f_y \equiv 1$. Then it's clear that

$$\overline{F}(y) = \underline{F}(y) = 1$$

In conclusion,

$$\underline{F}(y) = \begin{cases} 1 - 1/q & \text{if } y = p/q \\ 1 & \text{otherwise} \end{cases}$$

$$\overline{F}(y) \equiv 1$$

Since f is RI on R , it follows that

$$\int_0^1 \underline{F}(y) dy = \int_R f = \int_0^1 \overline{F}(y) dy = \int_0^1 1 dy = 1$$

as required. □

Problem 8.2 (5.49 done)

Using the FTC, give a direct proof of Green's Formulas

$$-\iint_R f_y dx dy = \int_{\partial R} f dx$$

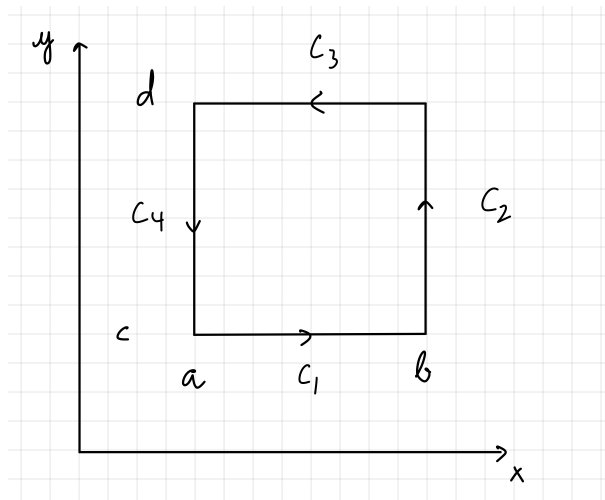
and

$$\iint_R g_x dx dy = \int_{\partial R} g dy$$

where R is a square in the plane and $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$ are smooth. (Assume that the edges of the square are parallel to the coordinate axes.)

Solution

Let $R = [a, b] \times [c, d]$.



Then

$$\begin{aligned}
-\iint_R f_y dx dy &= -\int_a^b \int_c^d f_y dy dx \\
&= -\int_a^b f(x, y=d) - f(x, y=c) dx \\
&= \int_a^b f(x, y=c) dx - \int_a^b f(x, y=d) dx \\
&= \int_{C_1} f dx + \int_{C_3} f dx \\
&= \int_{\partial R} f dx
\end{aligned}$$

since on C_2 and C_4 , x stays unchanged so $\int_{C_2} f dx = \int_{C_4} f dx = 0$. Similarly,

$$\begin{aligned}
\iint_R g_x dx dy &= \int_c^d g(x=a, y) - g(x=b, y) dy \\
&= \int_c^d g(x=b, y) dy - \int_c^d g(x=a, y) dy \\
&= \int_{\partial R} g dy
\end{aligned}$$

since $\int_{C_1} g dy = \int_{C_3} g dy = 0$. □

Problem 8.3 (5.65 done)

Show that the 1-form defined on $\mathbb{R}^2 \setminus \{(0, 0)\}$ by

$$\omega = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$$

is closed but not exact. Why do you think that this 1-form is often referred to as $d\theta$ and why is the name problematic?

Solution

1. For ω ,

$$g_1(x, y) = \frac{-y}{x^2 + y^2}, g_2(x, y) = \frac{x}{x^2 + y^2}$$

Checking for closedness:

$$\begin{aligned}
\frac{\partial g_1}{\partial y} &= y(x^2 + y^2)^{-2}(2y) - (x^2 + y^2)^{-1} \\
&= (x^2 + y^2)^{-2}(y^2 - x^2) \\
\frac{\partial g_2}{\partial x} &= -x(x^2 + y^2)^{-2}(2x) + (x^2 + y^2)^{-1} \\
&= (x^2 + y^2)^{-2}(y^2 - x^2)
\end{aligned}$$

so they're equal, so ω is indeed closed.

2. We show that ω is not exact by showing that there exists a 1-cell φ such that $\int_{\varphi} \omega \neq 0$.
Indeed, on the unit circle:

$$\varphi : [0, 1] \rightarrow \mathbb{R}^2, t \mapsto (\cos 2\pi t, \sin 2\pi t)$$

we have the pullback:

$$\begin{aligned} \varphi^* \omega &= \sum_{i=1}^2 g_i(\varphi(t)) \varphi'_i(t) dt \\ &= 2\pi [-(\sin 2\pi t)(-\sin 2\pi t) + (\cos 2\pi t)(\cos 2\pi t)] dt \\ &= 2\pi dt \end{aligned}$$

therefore

$$\int_{\varphi} \omega = \int_0^1 2\pi dt = 2\pi \neq 0$$

as required.

This 1-form is often referred to as $d\theta$ as it measures the change in the polar coordinate θ in Cartesian coordinates x, y . However, since it is not exact, the notation $d\theta$ is problematic, since it suggests that $\omega = d\theta$ is exact. \square

Problem 8.4 (5.67 done)

Show that the 2-form defined on the spherical shell by

$$\omega = \frac{x}{r^3} dy \wedge dz + \frac{y}{r^3} dz \wedge dx + \frac{z}{r^3} dx \wedge dy$$

is closed but not exact.

Solution

1. We compute:

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{x}{(x^2 + y^2 + z^2)^{3/2}} \right) &= (x^2 + y^2 + z^2)^{-5/2} (y^2 + z^2 - 2x^2) = r^{-5} (y^2 + z^2 - 2x^2) \\ \frac{\partial}{\partial y} \left(\frac{x}{(x^2 + y^2 + z^2)^{3/2}} \right) &= 3xyr^{-5} \\ \frac{\partial}{\partial z} \left(\frac{x}{(x^2 + y^2 + z^2)^{3/2}} \right) &= 3x zr^{-5} \end{aligned}$$

and similarly for $\frac{y}{r^3}, \frac{z}{r^3}$. Therefore

$$\begin{aligned} d \left(\frac{x}{r^3} dy \wedge dz \right) &= r^{-5} ((y^2 + z^2 - 2x^2) dx + 3xy dy + 3xz dz) \wedge dy \wedge dz \\ &= r^{-5} (y^2 + z^2 - 2x^2) dx \wedge dy \wedge dz \\ d \left(\frac{y}{r^3} dz \wedge dx \right) &= r^{-5} (z^2 + x^2 - 2y^2) dy \wedge dz \wedge dx \\ d \left(\frac{z}{r^3} dx \wedge dy \right) &= r^{-5} (x^2 + y^2 - 2z^2) dz \wedge dx \wedge dy \end{aligned}$$

Then,

$$\begin{aligned}
dy \wedge dz \wedge dx &= -dy \wedge dz \wedge dx \\
&= dx \wedge dy \wedge dz \\
dz \wedge dx \wedge dy &= -dx \wedge dz \wedge dy \\
&= dx \wedge dy \wedge dz
\end{aligned}$$

Therefore

$$\begin{aligned}
d\omega &= d\left(\frac{x}{r^3}dy \wedge dz\right) + d\left(\frac{y}{r^3}dz \wedge dx\right) + d\left(\frac{z}{r^3}dx \wedge dy\right) \\
&= r^{-5}[(y^2 + z^2 - 2x^2) + (z^2 + x^2 - 2y^2) + (x^2 + y^2 - 2z^2)]dx \wedge dy \wedge dz \\
&= 0
\end{aligned}$$

and ω is therefore closed.

2. Assume that it is exact, then $\omega = d\alpha$ for some 1-form α .

We try to compute $\int_{S^2} \omega$, i.e. integrating ω against S^2 , with parameterization:

$$\rho(\varphi, \theta) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$$

from $[0, 2\pi] \times [0, \pi]$.

Then

$$\begin{aligned}
dx &= -\sin \theta \sin \varphi d\varphi + \cos \theta \cos \varphi d\theta \\
dy &= \sin \theta \cos \varphi d\varphi + \cos \theta \sin \varphi d\theta \\
dz &= -\sin \theta d\theta
\end{aligned}$$

which implies

$$\begin{aligned}
dx \wedge dy &= -\sin \theta \cos \theta (\sin^2 \varphi + \cos^2 \varphi) d\varphi \wedge d\theta \\
&= -\sin \theta \cos \theta d\varphi d\theta \\
dy \wedge dz &= -\sin^2 \theta \cos \varphi d\varphi \wedge d\theta \\
dz \wedge dx &= -\sin^2 \theta \sin \varphi d\varphi \wedge d\theta
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{S^2} \omega &= \int_{[0, 2\pi] \times [0, \pi]} [(\sin \theta \cos \varphi)(-\sin^2 \theta \cos \varphi) \\
&\quad + (\sin \theta \sin \varphi)(-\sin^2 \theta \sin \varphi) + \cos \theta(-\sin \theta \cos \theta)] d\varphi \wedge d\theta \\
&= \int_0^\pi \int_0^{2\pi} (-\sin^3 \theta - \sin \theta \cos^2 \theta) d\varphi d\theta \\
&= \int_0^\pi \int_0^{2\pi} -\sin \theta d\varphi d\theta \\
&= \int_0^\pi -2\pi \sin \theta d\theta = [2\pi \cos \theta]_0^\pi = -4\pi
\end{aligned}$$

However, since $\omega = d\alpha$,

$$\begin{aligned}\int_{S^2} \omega &= \int_{S^2} d\alpha \\ &= \int_{\partial S^2} \alpha\end{aligned}$$

but ∂S^2 is the 1-chain:

$$\begin{aligned}\partial\rho &= (-1)^2(\rho(2\pi, \theta) - \rho(0, \theta)) + (-1)^3(\rho(\theta, \pi) - \rho(\theta, 0)) \\ &= [(\sin \theta, 0, \cos \theta) - (\sin \theta, 0, \cos \theta)] - [(0, 0, -1) - (0, 0, -1)] \\ &= 0 \\ \Rightarrow \int_{\partial S^2} \alpha &= 0 \quad \square\end{aligned}$$

Therefore ω is not exact. \square

Problem 8.5 (III done)

This exercise explores the question (asked in class): suppose Df_p is invertible at every point of the domain: is f injective? Recall this is true for $f : (a, b) \rightarrow \mathbb{R}$ by the one dimensional inverse function theorem.

- (a) Consider the map $f : A \rightarrow \mathbb{R}^2$, where $A = \{z \in \mathbb{R}^2 : 1 < |z| < 2\}$ given by

$$f(x, y) = (x^2 - y^2, 2xy).$$

Is it C^1 in A , with invertible derivative? Is it injective?

- (b) Suppose that $U \subset \mathbb{R}^2$ is open, $f : U \rightarrow \mathbb{R}^n$ is C^1 , and Df_p is invertible at each $p \in U$. Prove that if $f(U)$ is closed in \mathbb{R}^n , then $f(U) = \mathbb{R}^n$.
- (c) *** (EC) Suppose that $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is C^1 , with invertible derivative. Must it be injective?
- (d) *** (EC) Characterize the open sets $U \subset \mathbb{R}^n$ with the property that if $f : U \rightarrow \mathbb{R}^n$ is C^1 , and Df_p is invertible for all $p \in U$, then f is injective.
- (e) ** (EC) Prove that if $f : \bar{A} \rightarrow \bar{A}$ is C^1 , with the property that Df_p is invertible at every $p \in \bar{A}$, then there exists an integer $k \geq 0$ such that f is k -to-one, where A is the annulus defined in (c).

Solution

- (a) 1. WTS f is C^1 . We have

$$\frac{\partial f_1}{\partial x} = 2x, \frac{\partial f_1}{\partial y} = -2y, \frac{\partial f_2}{\partial x} = -2y, \frac{\partial f_2}{\partial y} = 2x$$

which are all continuous, so the partial derivatives of f are continuous, so f is differen-

table. Writing down its Jacobian:

$$Jf_{(x,y)} = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

so

$$\begin{aligned} \|Df_{(x,y)} - Df_{(x',y')}\|_{op} &= \left\| \begin{pmatrix} 2(x-x') & -2(y-y') \\ 2(y-y') & 2(x-x') \end{pmatrix} \right\|_{op} \\ &\rightarrow 0 \text{ as } |(x,y) - (x',y')| = |(x-x', y-y')| \rightarrow 0 \end{aligned}$$

so $Df : (x,y) \mapsto Df_{(x,y)}$ is therefore continuous. f is therefore C^1 .

2. WTS the derivative is everywhere invertible. Indeed,

$$Jf^{-1}(x,y) = \frac{1}{4(x^2+y^2)} \begin{pmatrix} 2x & 2y \\ -2y & 2x \end{pmatrix}$$

after using the rule $A^{-1} = \frac{1}{\det A} \text{adj}(A)$, and realizing that $\det Jf_{(x,y)} = 4(x^2+y^2) > 0$ in the annulus.

3. WTS f is not injective. Indeed,

$$f(1,1) = (0,2) = f(-1,-1) \quad \square$$

(b) 1. WTS $f(U)$ is open.

Let $q \in f(U)$, $q = f(p)$ for some $p \in U \subset \mathbb{R}^2$. Then since f is C^1 and Df_p is invertible, by Inverse Function Theorem, f is a local C^1 diffeomorphism from a neighborhood of p to a neighborhood of q . In other words, one can find an open neighborhood around any q that is a subset of $f(U)$, therefore $f(U)$ is open.

2. Since \mathbb{R}^n is connected, and $f(U)$ is clopen and nonempty, it follows that $f(U) = \mathbb{R}^n$. \square

Problem 8.6 (IV done)

Let $g: \mathbb{R}^n \rightarrow \mathbb{R}^k$ and consider the set $S = g^{-1}(0)$. Assume that for every $p \in S$, the rank of $D_p g$ is equal to k (in other words, $D_p g: \mathbb{R}^n \rightarrow \mathbb{R}^k$ is surjective). Let $p \in S$, and define $T_p S := \ker D_p g \subset \mathbb{R}^n$.

- (a) What does the rank-nullity theorem from linear algebra tell you about the dimension of $T_p S$?
- (b) Show that if $\gamma: (-\varepsilon, \varepsilon) \rightarrow U$ is a C^1 curve through p along which g is constant (i.e. if $\gamma(0) = p$ and $g(\gamma(t)) = g(\gamma(0))$ for all $t \in (-\varepsilon, \varepsilon)$), then $\gamma'(0) \in T_p S$. This is why we call $T_p S$ the *tangent space to S at p* .
- (c) Find a basis for $T_p S$ for the following S, p

$$(i) \ S = \{(x,y,z) : -x^2 + y^2 - z^2 = -1\}, p = (\frac{1}{\sqrt{2}}, -\frac{\sqrt{3}}{\sqrt{2}}, -\sqrt{2}).$$

(ii) $S = \{(x, y, z) : -x^2 + y^2 - z^2 = -1, \text{ and } xz + 4y^2 = 5\}$, $p = (\frac{1}{\sqrt{2}}, -\frac{\sqrt{3}}{\sqrt{2}}, -\sqrt{2})$.

(d) Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ and for $p \in S$, let V_p be the orthogonal projection of $\text{grad}_p(f)$ onto the subspace $T_p S \subset \mathbb{R}^n$. Prove that if V_p is nonzero, then $\pm|V_p|$ is the maximum (resp. minimum) value of the function $G: T_p^1 S \rightarrow \mathbb{R}$, where $T_p^1 S := \{v \in T_p S : |v| = 1\}$, and

$$G(v) = D_p f(v).$$

(e) Prove that $\pm V_p$, if nonzero, points to the maximal direction of increase/decrease of f in directions tangent to the surface S .

(f) Relate (d),(e) to Lagrange multipliers.

Solution

(a)

$$\dim T_p S = \dim \ker D_p g = n - \dim \text{im } D_p g = n - k \quad \square$$

(b) WTS $\gamma'(0) \in T_p S \Leftrightarrow Dg_p(\gamma'(0)) = 0$.

Consider $g \circ \gamma : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^k$, then $g \equiv c \in \mathbb{R}^k$ since γ is a level curve of g . It follows that the derivative is 0 at $0 \in (-\varepsilon, \varepsilon)$ (0 both as the zero map and the vector 0, since domain of $g \circ \gamma$ is 1 dimensional.)

$$0 = D(g \circ \gamma)_0 = Dg(\gamma'(0))$$

as required. \square

(c)(i) Let $f(x, y, z) = -x^2 + y^2 - z^2 + 1$, $g(x, y, z) = xz + 4y^2 - 5$. Recall that $p = (\frac{1}{\sqrt{2}}, -\frac{\sqrt{3}}{\sqrt{2}}, -\sqrt{2})$.

Then

$$\begin{aligned} Jf_{(x,y,z)} &= \begin{pmatrix} -2x & 2y & -2z \end{pmatrix} \\ \Rightarrow Jf_p &= \begin{pmatrix} -\sqrt{2} & -\sqrt{6} & 2\sqrt{2} \end{pmatrix} \end{aligned}$$

Now we want to find the basis for $T_p S$, i.e. the basis for the null space of Jf_p . Solving

$$\begin{pmatrix} -\sqrt{2} & -\sqrt{6} & 2\sqrt{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

yields

$$z = \frac{x}{2} + \frac{y\sqrt{3}}{2}$$

so a basis is $\{(2, 0, 1), (0, 2, \sqrt{3})\}$.

(c)(ii) Similarly,

$$\begin{aligned} Jg_{(x,y,z)} &= \begin{pmatrix} z & 8y & x \end{pmatrix} \\ \Rightarrow Jg_p &= \begin{pmatrix} -\sqrt{2} & -4\sqrt{6} & \frac{1}{\sqrt{2}} \end{pmatrix} \end{aligned}$$

So now we want to find basis for the null space:

$$\begin{pmatrix} -\sqrt{2} & -\sqrt{6} & 2\sqrt{2} \\ -\sqrt{2} & -4\sqrt{6} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

has rref

$$\begin{pmatrix} 1 & 0 & \frac{-5}{2} \\ 0 & 1 & \frac{1}{2\sqrt{3}} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

so a basis is $\{(5\sqrt{3}, -1, 2\sqrt{3})\}$ □

(d) We already know that $\langle \text{grad}_p(f), v \rangle = Df_p(v)$ for all v (pset 5).

Taking any $v \in T_p S$, since V_p is the orthogonal projection of $\text{grad}_p(f)$ onto $T_p S$, that implies $\langle \text{grad}_p(f) - V_p, v \rangle = 0$.

It follows that

$$G(v) = Df_p(v) = \langle V_p, v \rangle \quad \forall v \in T_p^1 S$$

from which we can bound, for $v \in T_p^1 S$:

$$\begin{aligned} |G(v)| &= |Df_p(v)| = |\langle V_p, v \rangle| \\ &\leq |V_p| |v| \\ &= |V_p| \end{aligned}$$

It follows that $\pm|V_p|$ are the maximum and minimum value of G on $T_p^1 S$, with equality achieved when $v \parallel V_p$.

(e) Following from (d), the equality for maximum/minimum of $Df_p(v)$ is achieved when $v \parallel V_p$, i.e., when v points in $\pm V_p$ direction. Furthermore, this direction in which $Df_p(v)$ is maximized/minimized is exactly the direction of maximal increase/decrease. Therefore $\pm V_p$ points to the direction of maximal increase/decrease.

Also, $V_p \in T_p S$, so it is tangent to surface S .

(f) Relating to Lagrange Multipliers, we WTS that if $\text{grad}_p(f)$ is not orthogonal to $T_p S$ then p is not a local extremum.

If $\text{grad}_p(f)$ is not orthogonal to $T_p S$, then that implies there exists a non-zero orthogonal projection of $\text{grad}_p(f)$ onto $T_p S$, i.e., $V_p \neq 0$.

From (d) and (e), it follows that one can still move in the direction of $\pm V_p$ to achieve a locally higher/lower value for f , while still staying on the level curve of g , since $V_p \in T_p S = \ker Dg_p$, therefore p is not a local extremum.

Furthermore, when $\text{grad}_p(f)$ is orthogonal to $T_p S$, then $|G(v)| = |Df_p(v)| \leq 0 \Rightarrow Df_p \equiv 0$, so p is a local extremum!

And this condition is exactly the condition for Lagrange Multipliers. We claim that if $\text{grad}_p(f)$ is orthogonal to $T_p S = \ker Dg_p$, then $\text{grad}_p(f) \in \text{rowspace}(Jg_p)$.

This is because if $x \in \ker Dg_p$, x non trivial then $Jg_p x = 0$, which implies that x is orthogonal to all rows of Jg_p , so $x \notin \text{rowspace}(Jg_p)$. Therefore $\text{rowspace}(Jg_p) \cap \ker Dg_p = \{0\}$. Their ranks add up to n , therefore

$$\text{rowspace}(Jg_p) \oplus Dg_p = \mathbb{R}^n$$

Since $\text{grad}_p(f)$ is orthogonal to $\ker Dg_p$, it must be the case that $\text{grad}_p(f) \in \text{rowspace}(Jg_p)$.

And recall that $\text{grad}_p(g) = Jg_p^T$, so $\text{grad}_p(f)$ is a linear combination of the rows of Jg_p , which are columns of $\text{grad}_p(g)$. It follows that there exists $\vec{\lambda} \in \mathbb{R}^k$ such that

$$\text{grad}_p(f) = \text{grad}_p(g)\vec{\lambda}$$

which is the requirement of Lagrange Multipliers (for k constraints). □

Problem 8.7 (V done)

- (a) Compute the volume of the region $\Omega \subset \mathbb{R}^3$ bounded by $x = 0, x = 2, z = -y$ and by $z = y^2/2$.
- (b) Write down a triple integral that computes the volume of the region $\Omega \subset \mathbb{R}^3$ bounded by the coordinate planes and $y = 1 - x^2$ and $y = 1 - z^2$. Don't evaluate.
- (c) Compute

$$\iiint_B (2x + 3y^2 + 4z^3) dx dy dz,$$

where $B = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq 2, 0 \leq z \leq 3\}$.

Solution

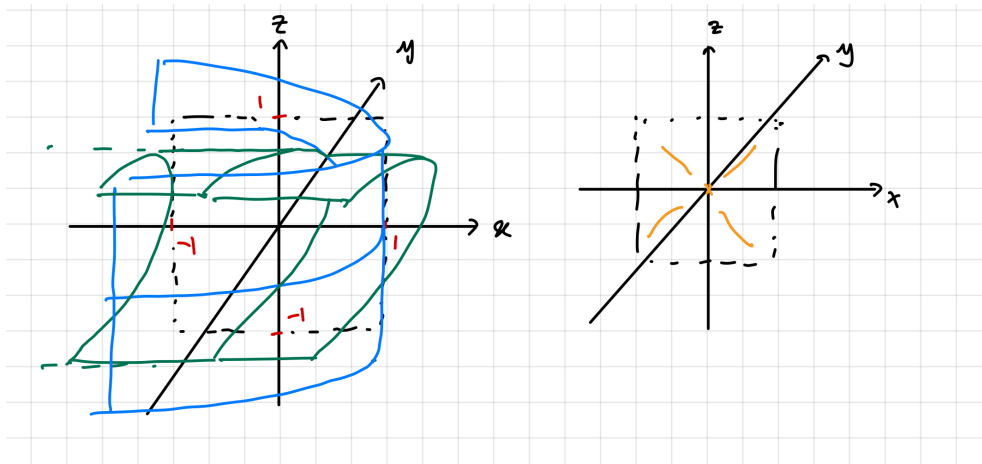
- (a) Solve for the bounds for y of the bounded region:

$$-y = y^2/2 \Rightarrow y = 0, -2$$

It follows that the volume is:

$$\begin{aligned} \left| \int_0^2 \int_{-2}^0 \int_{-y}^{y^2/2} 1 dz dy dx \right| &= \left| \int_0^2 \int_{-2}^0 (y^2/2 + y) dy dx \right| \\ &= \int_0^2 (-2/3) dx \\ &= 4/3 \end{aligned}$$

(b)



The bounded volume is on the $[-1, 1] \times [-1, 1]$ square on the xz -plane, which consists of 4 identical volumes on each of the quadrant of the xz -plane. WLOG, consider the volume on the first quadrant, with $x, z \in [0, 1]$. Then for each (x, z) , $y_{\min} = 1 - 1^2 = 0$, while

$$y_{\max} = \begin{cases} 1 - x^2 & \text{if } x \geq z \\ 1 - z^2 & \text{if } x \leq z \end{cases}$$

Taking volumes below and above the $z = x$ line, then the volume of this first-quadrant volume is

$$V_1 = \left| \int_0^1 \int_0^x \int_0^{1-x^2} 1 dy dz dx \right| + \left| \int_0^1 \int_x^1 \int_0^{1-z^2} 1 dy dz dx \right|$$

and the total volume throughout all 4 quadrants is just $4V_1$.

(c)

$$\begin{aligned} \int_0^3 \int_0^2 \int_0^1 (2x + 3y^2 + 4z^3) dx dy dz &= \int_0^3 \int_0^2 [x^2 + 3y^2x + 4z^3x]_0^1 dy dz \\ &= \int_0^3 \int_0^2 (1 + 3y^2 + 4z^3) dy dz \\ &= \int_0^3 [y + y^3 + 4z^3y]_0^2 dz \\ &= \int_0^3 (10 + 8z^3) dz \\ &= 192 \end{aligned}$$

□

Problem 8.8 (VI done)

Let $\omega = (y \cos xy + e^x) dx + (x \cos xy + 2y) dy$.

(a) Evaluate $\int_{\Gamma} \omega$ along the segment of the parabola $y = x^2$ from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Use

the parameterization $\varphi : [0, 1] \rightarrow \mathbb{R}^2$

$$\varphi(t) = (t, t^2)$$

(b) Evaluate $\int_{\Gamma} \omega$ for the case where Γ is the straight line joining the origin to the point $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Do the same for the case where Γ consists of the segment $0 \leq x \leq \alpha$ on the x -axis, followed by the segment $x = \alpha, 0 \leq y \leq \beta$.

(c) Find a function $f(x, y)$ such that $\omega = df$.

Solution

(a) We get the pullback:

$$\varphi^*\omega = [(t^2 \cos(t^3) + e^t) + (t \cos(t^3) + 2t^2)(2t)]dt = (3t^2 \cos(t^3) + 4t^3 + e^t)dt$$

Then

$$\begin{aligned} \int_{\Gamma} \omega &= \int_0^1 (3t^2 \cos(t^3) + 4t^3 + e^t)dt \\ &= [\sin(t^3) + t^4 + e^t]_0^1 \\ &= \sin(1) + 1 + e - 1 = \sin(1) + e \quad \square \end{aligned}$$

(b) 1. Straight line joining origin to the point $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is parameterized:

$$\Gamma(t) = (\alpha t, \beta t)$$

has pullback

$$\begin{aligned} \Gamma^*\omega &= [\alpha(\beta t \cos(\alpha\beta t^2) + e^{\alpha t}) + \beta(\alpha t \cos(\alpha\beta t^2) + 2\beta t)]dt \\ &= [2\alpha\beta t \cos(\alpha\beta t^2) + e^{\alpha t} + 2\beta^2 t]dt \end{aligned}$$

so

$$\begin{aligned} \int_{\Gamma} \omega &= \int_0^1 \Gamma^*\omega = \int_0^1 [2\alpha\beta t \cos(\alpha\beta t^2) + \alpha e^{\alpha t} + 2\beta^2 t]dt \\ &= \sin(\alpha\beta) + e^{\alpha} - e + \beta^2 \end{aligned}$$

2. Let ρ parameterize path from $(0, 0)$ to $(\alpha, 0)$, and σ parameterize path from $(\alpha, 0)$ to (α, β) .

$$\rho(t) = (\alpha t, 0), \sigma(t) = (\alpha, \beta t).$$

They have pullback:

$$\begin{aligned}\rho^*\omega &= \alpha(e^{\alpha t})dt \\ \sigma^*\omega &= \beta(\alpha \cos(\alpha\beta t) + 2\beta t)dt\end{aligned}$$

therefore

$$\begin{aligned}\int_{\Gamma} \omega &= \int_{\rho} \omega + \int_{\sigma} \omega \\ &= \int_0^1 (\alpha e^{\alpha t} + \alpha\beta \cos(\alpha\beta t) + 2\beta^2 t) dt \\ &= [e^{\alpha t} + \sin(\alpha\beta t) + \beta^2 t^2]_0^1 \\ &= e^{\alpha} - e + \sin(\alpha\beta) + \beta^2 \quad \square\end{aligned}$$

(c)

$$f(x, y) = \sin xy + e^x + y^2$$

has

$$\begin{aligned}\frac{\partial f}{\partial x} &= y \cos xy + e^x \\ \frac{\partial f}{\partial y} &= x \cos xy + 2y\end{aligned}$$

so

$$\omega = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = df$$

as required. □

Problem 8.9 (VII done)

Let $\omega = y \, dx - x \, dy$.

(a) Evaluate $\int_{\gamma} \omega$ along the semicircle γ from $\begin{pmatrix} r-1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} r+1 \\ 0 \end{pmatrix}$ defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r - \cos t \\ \sin t \end{pmatrix}$$

for $0 < t < \pi$.

(b) Show explicitly that you can obtain a different value from that in (a) by choosing a different curve joining $\begin{pmatrix} r-1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} r+1 \\ 0 \end{pmatrix}$.

Solution

(a) With parameterization $\gamma(t) = (r - \cos t, \sin t)$ on $(0, \pi)$:

$$\begin{aligned}\int_{\gamma} \omega &= \int_0^{\pi} \sin^2 t - (r - \cos t) \cos t dt \\ &= \int_0^{\pi} -r \cos t + 1 dt \\ &= [-r \sin t + t]_0^{\pi} \\ &= \pi\end{aligned}$$

(b) Parameterize $\rho(t) = (r - 1 + 2t, 0)$. Then $y = 0$ throughout and $dy = 0$. So $\int_{\rho} \omega = 0$, different from (a). \square

Problem 8.10 (VIII done)

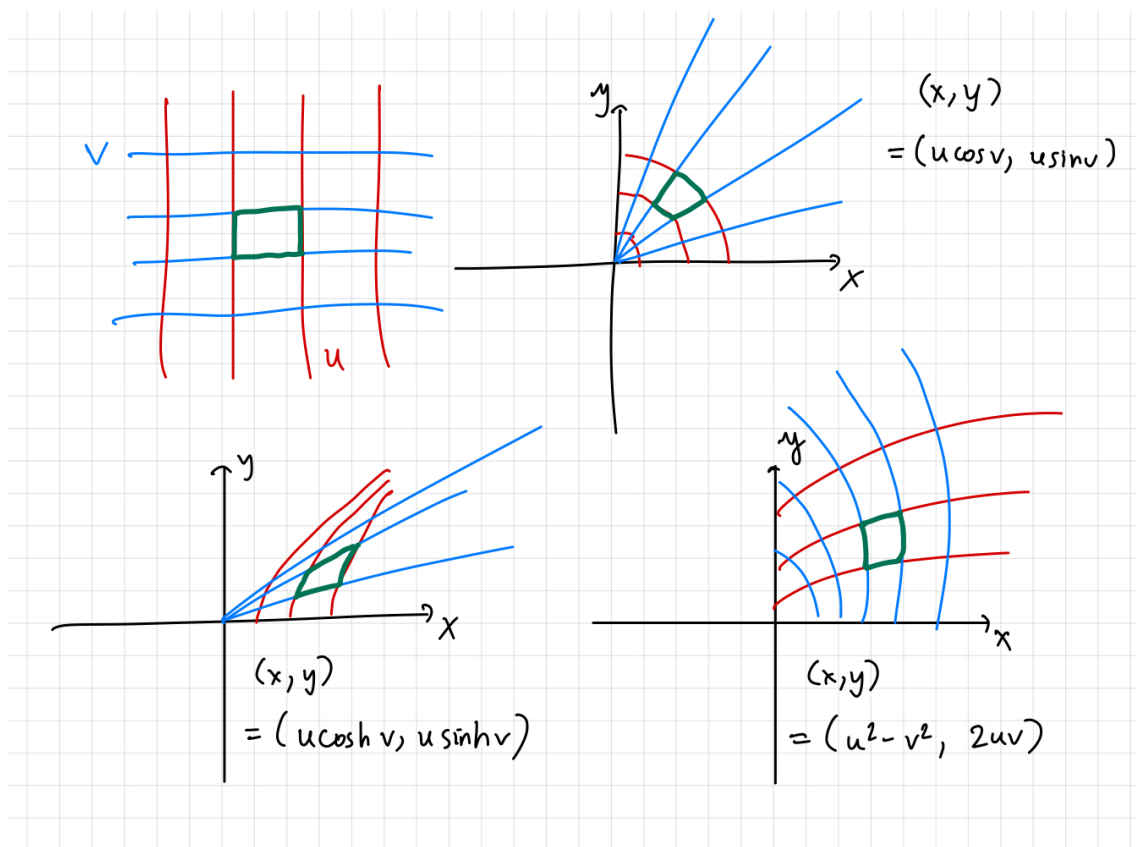
In each of the following cases, u and v are functions on a plane where x and y are affine coordinates. Express $dx \wedge dy$ in terms of $du \wedge dv$. Make a sketch showing typical curves $u = \text{constant}$ and $v = \text{constant}$ in the first quadrant ($x, y > 0$) and try to give a geometric interpretation to the relations between $dx \wedge dy$ and $du \wedge dv$ by applying both to a parallelogram whose sides are tangent to $u = \text{constant}$ and $v = \text{constant}$ respectively.

(a) $x = u \cos v, \quad y = u \sin v.$

(b) $x = u \cosh v, \quad y = u \sinh v.$

(c) $x = u^2 - v^2, \quad y = 2uv.$

Solution



(a)

$$\begin{aligned}
 dx &= \cos v du - u \sin v dv \\
 dy &= \sin v du + u \cos v dv \\
 \Rightarrow dx \wedge dy &= u(\cos^2 v + \sin^2 v) du \wedge dv \\
 &= u du \wedge dv
 \end{aligned}$$

(b)

$$\begin{aligned}
 dx &= \cosh v du + u \sinh v dv \\
 dy &= \sinh v du + u \cosh v dv \\
 \Rightarrow dx \wedge dy &= u(\cosh^2 v - \sinh^2 v) du \wedge dv \\
 &= u du \wedge dv
 \end{aligned}$$

(c)

$$\begin{aligned}
 dx &= 2u du - 2v dv \\
 dy &= 2v du + 2u dv \\
 \Rightarrow dx \wedge dy &= (4u^2 + 4v^2) du \wedge dv
 \end{aligned}$$

□

Problem 8.11 (IX done)

(a) Show, by reversing the order of integration, that

$$\int_0^a \left(\int_0^y e^{m(a-x)} f(x) dx \right) dy = \int_0^a (a-x) e^{m(a-x)} f(x) dx$$

where a and m are constants, $a > 0$.

(b) Show that $\int_0^x \left(\int_0^v \left[\int_0^u f(t) dt \right] du \right) dv = \frac{1}{2} \int_0^x (x-t)^2 f(t) dt$. If you do this in two steps, you never actually have to consider a triple integral!

Solution

(a)

$$\begin{aligned} \int_0^a \left(\int_0^y e^{m(a-x)} f(x) dx \right) dy &= \int_0^a \int_x^a e^{m(a-x)} f(x) dy dx \\ &= \int_0^a (a-x) e^{m(a-x)} f(x) dx \end{aligned}$$

as required.

(b)

$$\begin{aligned} \int_0^x \left(\int_0^v \left[\int_0^u f(t) dt \right] du \right) dv &= \int_0^x \int_0^v \int_t^v f(t) du dt dv \\ &= \int_0^x \int_0^v [uf(t)]_{u=t}^{u=v} dt dv \\ &= \int_0^x \int_0^v (v-t) f(t) dt dv \\ &= \int_0^x \int_t^x (v-t) f(t) dv dt \\ &= \int_0^x \left[\frac{(v-t)^2}{2} \right]_{v=t}^{v=x} dt \\ &= \frac{1}{2} \int_0^x (x-t)^2 f(t) dt \end{aligned}$$

as required. □