10/13/23

1

Calculus in One Variable

Differentiation

$f:(a,b) \to \mathbb{R}$ diff'ble at $x \in (a,b)$

if $\lim_{y \to x} \frac{f(y)-f(x)}{y-x}$ exists L

$L = f'(x)$

$f(y) = f(x) + f'(x)(y-x) + o(|y-x|)$

useful not'n: $o(h)$ denotes a function satisfying $\lim_{h \to 0} \frac{o(h)}{h} = 0$

linearity, chain rule, Leibniz (product) rule.

Mean Value Theorem (Rolle's thm)

$f:[a,b] \to \mathbb{R}$ cts, diff'ble on $(a,b) \Rightarrow \exists\, \theta \in (a,b)$ s.t.

2

$$\frac{f(b)-f(a)}{b-a} = f'(\theta).$$

(Pf uses fact that $f'(\alpha)=0$ if α = max or min of diff'ble fcn)

Interesting consequence: Darboux continuity of f'!

Thm If f' is diff'ble on (a,b), then f' has IV property.

(Proof uses MVT)

Theorem (Inverse function thm)

S'pose $f:(a,b) \to \mathbb{R}$ is differentiable & $f'(x) \neq 0$ $\forall x \in (a,b)$,

Then f is invertible on $f(a,b)$, f^{-1} is differentiable, &

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} \quad \forall y \in f(a,b)$$

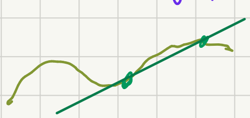
Proof. Let $C \subseteq \mathbb{R}^2$. We say C is (1D &) differentiable if $\forall z \in C$, the secant lines

$$S_{z,w}(t) = z + t(w-z)$$

converge as $w \to z$ to T_z

(this is $\Leftrightarrow$ to saying angles $\in [0, \pi)$ converge)

Facts ① If $f : (a,b) \to \mathbb{R}$, then f is diff'ble $\Leftrightarrow$ graph(f) is diff'ble & no tan. line T_z is vertical



② If C is diff'ble, then so is $\hat{C} = \{(y,x) : (x,y) \in C\}$ (flipped secant lines still converge).

- f function $\Rightarrow$ no vertical $S_{z,w}$
- f is diff'ble on (a,b)
  $\Rightarrow C =$ graph f diff'ble, no vertical T_z
- $f'(x) \neq 0 \ \forall x$
  $\Rightarrow$ no horizontal $S_{z,w}$ or T_z (follows from IV prop of f')

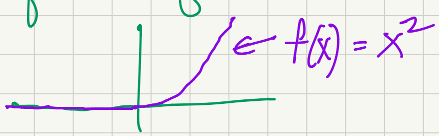
$\Rightarrow \hat{C}$ has no vertical (or horiz.) $S_{z,w}$ or T_z $\Rightarrow \hat{C} = \text{graph}(f^{-1})$ & f^{-1} is diff'ble, with $(f^{-1})' \circ f = \frac{1}{f'}$

C^r

Def Say $f:(a,b)\to\mathbb{R}$ is

- C^1 if f differentiable & f' is continuous.
- C^r if $f^{(r-1)}$ " " $f^{(r-1)}$ ... $(r\geq 2)$.
- C^∞ if C^r $\forall r\geq 1$

Example of C^1 fcn not C^2



- C^ω, or analytic if $\forall x_0\in(a,b)$ $\exists\, \varepsilon>0$ s.t. $|x-x_0|<\varepsilon \Rightarrow$

$$f(x)=\sum_{n=0}^{\infty} a_n (x-x_0)^n,$$

Where $a_n = \dfrac{f^{(n)}(x)}{n!}$