This document lists out definitions in Math that I couldn't possibly organize in my tiny brain. The definitions are expected to be built off of the previous ones.

1 Binary Relations

Definition 1.1 (Binary Relation)

A **binary relation** R over sets X, Y is a subset of $X \times Y$. $(x, y) \in R$ is equivalent to xRy. We say x *is* $R-$*related to* y.
In other words, R imposes some condition that $(x, y) \in R$ must satisfy to be included in the set.
e.g. $x \geq y$

Definition 1.2 (Homogenous Relation/Endorelation)

A **homogenous relation** is a binary relation from a set to itself, i.e. $R : X \times X \to X$.

Definition 1.3 (Reflexive, Irreflexive, Symmetric, Antisymmetric, Asymmetric, Transitive, Connected, Strongly Connected)

Let R be a homogenous relation over set X. Then the following properties are defined as:

1. R is **reflexive** if $\forall\, x \in X, xRx$ e.g. $\geq$
2. R is **irreflexive** if $\forall\, x \in X, \neg xRx$ e.g. $>$
3. R is **symmetric** if $\forall\, x, y \in X, xRy \Leftrightarrow yRx$ e.g. shares the same house
4. R is **antisymmetric** if $\forall\, x, y \in X, xRy \land yRx \Rightarrow x = y$ e.g. $\geq$
5. R is **asymmetric** if $\forall\, x, y \in X, xRy \Rightarrow \neg yRx$ e.g. $>$
6. R is **transitive** if $\forall\, x, y, z \in X, xRy \land yRz \Rightarrow xRz$ e.g. $>, \geq$
7. R is **connected** if $\forall\, x, y \in X, x \neq y \Rightarrow xRy \lor yRx$
8. R is **strongly connected** if $\forall\, x, y \in X, xRy \lor yRx$

Remark

In some sense, **asymmetry** is **antisymmetry + irreflexivity**; antisymmetry gets upgraded when there is irreflexivity.

Definition 1.4 (Partial Order)

A **partial order** is a relation that is reflexive, antisymmetric and transitive.
e.g. $\geq$

Definition 1.5 (Strict Partial Order)

A **strict partial order** is a relation that is irreflexive, asymmetric and transitive.
e.g. $>$

Definition 1.6 (Total Order)

A **total order** is a relation that is reflexive, antisymmetric, transitive and connected.

Definition 1.7 (Strict Total Order)

A **strict total order** is a relation that is irreflexive, asymmetric, transitive and connected.

Remark

In short, *total* adds on the requirement that it must be *connected*, while *strict* changes *reflexivity* into *irreflexivity* (and since everyone has *antisymmetry*, this changes into *asymmetry* too)

Definition 1.8 (Equivalence Relation)

An **equivalence relation** is a relation that is reflexive, symmetric and transitive.

2 Basic Algebraic Structures

Definition 2.1 (Group)

A non-empty set G equipped with a binary operation $\cdot : G \times G \to G$ is called a **group** if they satisfy the following *group axioms*:

1. (Associativity) $\forall\, a, b, c \in G, (a \cdot b) \cdot c = a \cdot (b \cdot c)$
2. (Identity) $\exists\, e \in G\, \text{s.t.}\ \forall\, g \in G, e \cdot g = g \cdot e = g$
3. (Inverse) $\forall\, g \in G,\ \exists\, g' \in G\, \text{s.t.}\, g'g = e$. Then $g^{-1} := g'$

Remark

The above axioms of a group imply:

1. Identity is unique
2. Inverse is unique

Even when one restricts the axioms to just simply having left identity and left inverse, the above remark still holds

Definition 2.2 (Subgroup)

$H \subseteq G$ is a **subgroup** of G if

1. $e \in H$
2. H is closed under $\cdot$ and taking inverses

Definition 2.3 (Abelian Group)

A **group** $(G, \cdot)$ is **Abelian** if the operation $\cdot$ is also commutative.
Recap: Associativity, Commutativity, Identity, Inverse

Definition 2.4 (Ring)

A **ring** is a set R equipped with two binary operations (addition, multiplication) $+, \cdot : R \times R \to R$ s.t. they satisfy the following *ring axioms*:

1. $(R, +)$ is an Abelian group (Associativity, Commutativity, Identity, Inverse)
2. (Multiplicative Associativity) $\forall\, a, b, c \in R, (a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. (Multiplicative Identity) $\forall\, r \in R,\ \exists\, 1 \in R\, \text{s.t.}\, 1 \cdot r = r \cdot 1 = r$
4. (Left, Right Distributivity) This property governs how the 2 operations interact

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

$$(b + c) \cdot a = (b \cdot a) + (c \cdot a)$$

Note that multiplicative commutativity and multiplicative inverse are not here!

Remark

One can derive that when a ring has $0 = 1$ (additive identity = multiplicative identity), then it must be a trivial *zero ring* of $\{0\}$

Definition 2.5 (Rng)

A **rng** is a ring without the requirement of a multiplicative identity.
Recap: Abelian group, multiplicative associativity, distributivity

Definition 2.6 (Commutative Ring)

A **commutative ring** is a ring, with the additional requirement of multiplication being commutative.
Recap: Abelian group; multiplicative associativity, commutativity and identity; distributivity

Definition 2.7 (Field)

A **field** is a set F equipped with 2 binary operations (addition, multiplication) $+, \cdot : F \times F \to F$ s.t.

1. Addition and multiplication are associative
2. Addition and multiplication are commutative
3. There is an additive inverse 0
4. $\forall\, a \in F \setminus 0,\ \exists\, a^{-1}$ s.t. $a^{-1}a = 1$
5. Distributivity

In short, a field is a commutative ring where non-zero elements have multiplicative inverses. The non-zero elements then form a group equipped with multiplication with 1 as their identity.
Recap: Abelian group; multiplicative associativity, commutativity, identity and inverse (inverse only for non-zero); distributivity

Definition 2.8 (Ordered Field)

[Munkres - Ch.1 p.32]

3 The -isms

Definition 3.1 (Homomorphism)

homo: Greek *homos*, meaning "same"
morphism: Greek *morphism*, meaning "shape, form"
A **homomorphism** is a map $f : A \to B$, where A, B are (very generically) algebraic structures of the same type G and are therefore equipped with the same kind(s) of operation, WLOG, namely $\cdot_A, \cdot_B : G^k \to G$ (e.g. groups, vector spaces). A homomorphism f then preserves the structures of these operations, i.e.

$$f(x) \cdot_B f(y) = f(x \cdot_A y)\ \forall\, x, y \in A$$

if $\cdot$ is binary, and the same concept applies for the general $k-$ary case.

Example

A **group homomorphism** is a homomorphism in a group, where the homomorphism preserves the $\cdot$ equipped by the group.

Definition 3.2 (Isomorphism)

iso: Greek *isos*, meaning "equal"
An **isomorphism** is a homomorphism that is bijective.

Remark

For me, it's so common to just intuitively think that a homomorphism must be bijective, but no!
e.g. $f : \mathbb{Z} \to \{0\}$, both equipped with $+$

Definition 3.3 (Endomorphism)

endo: Greek *endon*, meaning "in, within"
An **endomorphism** is a homomorphism that has the same domain and codomain, i.e.

$$f : A \to A$$

Definition 3.4 (Automorphism)

auto: Greek *autos*, meaning "self"
An **automorphism** is an endomorphism that is also an isomorphism, (or vice versa)

4 The -algebras

This section stemmed from my need to study probability and felt the urgent need to categorize what I'm working with immediately.

Definition 4.1 (Field of sets, Algebra)

A **field of sets** is a pair $(X, \mathcal{F})$, where X is a set and $\mathcal{F}$ is a *collection* of subsets of X, that satisfies:

1. $\emptyset \in \mathcal{F}$
2. (Closed under complementation)
$$\mathcal{F} \setminus F \in \mathcal{F} \;\forall\; F \in \mathcal{F}$$
3. (Closed under finite unions)
$$\bigcup_{k=1}^{n} F_k \in \mathcal{F} \;\forall\; F_1, F_2, \ldots, F_k \in \mathcal{F}$$

$\mathcal{F}$ is then called an **algebra over** X

Remark

The property that $\mathcal{F}$ is closed under finite unions also implies that it is closed under finite intersections, simply by applying De Morgan's Law.
Furthermore, one can think of $\mathcal{F}$ as consisting of *the admissible sets of* X, the *complexes of* X, the nice ones that we can handle and get a hold of. In most contexts, this $\mathcal{F}$ would be where we define a lot of things on, as not all subsets of X are nice to work with.

Definition 4.2 ($\sigma-$field of sets, $\sigma-$algebra)

A $\sigma-$**field of sets** is a field of sets $(X, \mathcal{F})$ that also satisfies:

4. (Closed under countable unions)
$$\bigcup_{i=1}^{\infty} F_i \in \mathcal{F} \;\forall\; F_1, F_2, \cdots \in \mathcal{F}$$

$\mathcal{F}$ is then called a **$\sigma-$algebra over X**

Remark

The property that a $\sigma-$algebra is closed under countable unions also implies that it is closed under countable intersections, again, by an application of De Morgan's Law.

5 Linear Algebra

5.1 Vector Spaces

Definition 5.1 (Vector Spaces)

A **vector space** over a field F is a non-empty set V equipped with 2 binary operations (vector addition, scalar multiplication): $+ : V \times V \to V, \cdot : F \times V \to V$ that satisfies:

1. (Abelian Group) $(V, +)$ forms an abelian group
2. (Scalar and Field Multiplication)
$$(a \cdot_F b) \cdot v = a \cdot (b \cdot V)$$
3. (Field Multiplicative Identity)
$$1_F \cdot v = v \,\forall\, v \in V$$
4. (Distributivity) $\forall\, a \in F; u, v \in V$
$$a \cdot (u + v) = (a \cdot u) + (a \cdot v)$$
$$(a +_F b) \cdot v = a \cdot v + b \cdot v$$

Definition 5.2 (Linear Map)

Let V, W be vector spaces over the same field F. Then a **linear map** is a function $f : V \to W$ that is operations-preserving, i.e. satisfying:

1. (Preserving Addition)
$$f(v_1 +_V v_2) = f(v_1) +_W f(v_2) \,\forall\, v_1, v_2 \in V$$
2. (Preserving Scalar Multiplication)
$$f(c \cdot_V v) = c \cdot_W f(v) \,\forall\, c \in F, v \in V$$

More generally,

$$f(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = c_1f(v_1) + c_2f(v_2) + \cdots + c_nf(v_n)$$

In other words, a linear map is a vector space homomorphism.

Definition 5.3 (Linear Isomorphism)

A **linear isomorphism** is a linear map that is also bijective.

Definition 5.4 (Linear Operator/Linear Endomorphism)

A **linear operator** or a **linear endomorphism** is a linear map that has the same domain and codomain, i.e. a linear map $f : V \to V$.

6 Analysis

6.1 Metric Spaces

Definition 6.1 (Metric Spaces)

Definition 6.2 (Isometry)

metry: Greek *metria*, meaning "measuring, measure"
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7 Topology

Remark

On indexing:

1. $\{E_1, E_2, \ldots, E_N\}$, sometimes I like to use $\{E_k\}_{k\leq N}$, suggests a finite indexing
2. $\{E_1, E_2, \ldots\}$ or $\{E_n\}_{n\in\mathbb{N}}$ suggests a countable indexing
3. $\{E_\alpha\}$ suggests an uncountable indexing (which kinda encompasses all previous cases and alludes to the "arbitrary" nature)

Definition 7.1 (Topology, Topological Space, Open Set)

A **topology** on set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\emptyset, X \in \mathcal{T}$
2. Arbitrary union of elements in $\mathcal{T}$ is in $\mathcal{T}$, i.e.
$$\bigcup_\alpha E_\alpha \in \mathcal{T} \;\forall\; \{E_\alpha\} \subseteq \mathcal{T}$$
3. Finite intersection of elements in $\mathcal{T}$ is in $\mathcal{T}$, i.e.
$$\bigcap_{k=1}^{N} E_k \in \mathcal{T} \;\forall\; \{E_k\}_{k\leq N} \subseteq \mathcal{T}$$

A set X with a specified topology $\mathcal{T}$ is a **topological space**. $U \subseteq X$ is called an **open set** iff $U \in \mathcal{T}$, so think of $\mathcal{T}$ as a (huge) collection of open sets.

Definition 7.2 (Basis For A Topology)

Oftentimes, one is unable to specify the entire topology $\mathcal{T}$, so we can instead specify a smaller collection oof subsets of X and then define the topology using that.
Let X be a set, then a **basis** is a collection $\mathcal{B}$ of subsets of X (called **basis elements**) s.t.

1. $\forall\, x \in X,\ \exists\, B \in \mathcal{B}$ s.t. $x \in B$ (wherever you are, I got you)
2. If $x \in B_1 \cap B_2; B_1, B_2 \in \mathcal{B}$ then $\exists\, B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subseteq B_1 \cap B_2$

Definition 7.3 (Topology Generated By Basis)

Call this entity that we want to generate $\mathcal{T}_\mathcal{B}$ (my own notation).
Then for subset $U \subseteq X$,

$$U \in \mathcal{T}_\mathcal{B} \text{ if } \forall\, x \in U,\ \exists\, B \in \mathcal{B} \text{ s.t. } x \in B \subseteq U$$

7.1 Others

Definition 7.4 (Linear Continuum)

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Definition 8.1 (Harmonic Function)
Let U be an open subset of $\mathbb{R}^d$, then f is **harmonic** in U if and only if it is continuous and satisfies the mean value property: for every $x \in U$, $\forall\, 0 < \varepsilon < dist(x, \partial U)$,

$$f(x) = MV(f; x, \varepsilon) = \int_{|y-x|=\varepsilon} f(y)ds(y)$$

9 Wilansky's Topology for Analysis

9.1 Chapter 2: Topological Space

Recall the definition of a **topology** from above; essentially when we define a topology, just need to define what "open sets" are. Finite intersections and arbitrary unions of those open sets would automatically be in the topology. Note that $\varnothing$ and Ω are axiomatically in the topology.

Definition 9.1 (Semimetric, Metric)
Set X. Let $d : X \times X \to \mathbb{R}$ s.t.

1. $d(x, y) = d(y, x) \geq 0$
2. $d(x, x) = 0$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Then d is a **semimetric**. A **metric** is a semimetric d s.t. $d(x, y) > 0$ when $x \neq y$

Definition 9.2 (Semimetric Space, Metric Space)
A **(semi)metric space** is a set that is equipped with a (semi)metric.

Definition 9.3 (Open ball, Closed ball, Disc)
Let X be a semimetric space. Define

1. **Open ball**: $N(a, r) = \{x \mid d(x, a) < r\}$
2. **Closed ball**: $D(a, r) = \{x \mid d(x, a) \leq r\}$
3. **Sphere**: $C(a, r) = \{x \mid d(x, a) = r\}$

Definition 9.4 (Topology on semimetric space)
The **standard topology on semimetric spaces** (from now on, metric spaces should be interpreted similarly) is that

1. $\varnothing$ is *open*
2. $G \subset X$ is *open* if $\forall\, x \in G$, there exists open ball $N(x, r) \subset G$

Indeed, open balls are open.

Definition 9.5 (Neighborhood)

N is a **neighborhood** of x if there exists open set G s.t. $x \in G \subset N$
Equivalently, there exists open ball $N(x,r) \subset G \subset N$
It is here trivial that an open ball $N(x,r)$ is also a neighborhood.

Definition 9.6 (Interior, Closure)

The **interior** of a set S is the set of all points x s.t. S is a neighborhood of x.
The **closure** of a set S is the intersection of all closed sets F s.t. $S \subset F$, i.e. the smallest closed set $\bar{S}$ s.t. $S \subset \bar{S}$

Definition 9.7 (Local Base, Base)

A collection $\mathcal{B}$ of subsets of a topological space is called a **local base at** x if

1. Every member of $\mathcal{B}$ is a neighborhood of x
2. For every neighborhood N of x, there exists a set $S \in \mathcal{B}$ with $S \subset N$

A collection $\mathcal{B}$ of subsets of a topological space is called a **base** for the topology if every member of $\mathcal{B}$ is open and $\mathcal{B}$ includes a local base at each point.

9.2 Chapter 3: Convergence

Definition 9.8 (Convergence)

A sequence $\{y_n\}$ in topological space X. y_n converges to x ($y_n \to x$) if for every neighborhood N of $x, y_n \in N$ eventually, i.e. $\forall$ neighborhood N of x, $\exists\, N \in \mathbb{N}$ s.t. $\forall\, n > N, y_n \in N$

Definition 9.9 (First Countable)

A topological space is called **first countable at** x if there is a countable local base at x, and **first countable** if it is first countable at each of its points.

Definition 9.10 (Filters)

A collection $\mathcal{F}$ of subsets of a set X is a **filter** in X if

1. $X \in \mathcal{F}$
2. $\varnothing \notin \mathcal{F}$
3. $A \in \mathcal{F}, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$
4. $A \in \mathcal{F}, B \supset A \Rightarrow B \in \mathcal{F}$