Math 20250: Abstract Linear Algebra Problem Set 1

Hung Le Tran

 $02~\mathrm{Apr}~2023$ 

## Textbook: Linear Algebra by Hoffman and Kunze (2nd Edition)

**Problem 1.1** (Sec 1.3. Problem 3). If

$$A = \left[ \begin{array}{rrr} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{array} \right]$$

find all solutions of AX = 2X and all solutions of AX = 3X. (The symbol cX denotes the matrix each entry of which is c times the corresponding entry of X)

**Solution.** We first solve AX = 2X, which is equivalent to  $A_1X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$  where

$$A_1 = \begin{bmatrix} 6-2 & -4 & 0 \\ 4 & -2-2 & 0 \\ -1 & 0 & 3-2 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 0 \\ 4 & -4 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Row-reducing:

$$\begin{bmatrix} 4 & -4 & 0 \\ 4 & -4 & 0 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{(1) = \frac{1}{4}(1), (2) = (3)} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{(2) = (2) + (1)} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{(2) = -(2)} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Therefore  $x_3 = a, x_2 = x_3 = a, x_1 = x_2 = a$  for any  $a \in \mathbb{C}$ . Solving AX = 3X is equivalent to

$$\begin{bmatrix} 6-3 & -4 & 0 \\ 4 & -2-3 & 0 \\ -1 & 0 & 3-3 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -4 & 0 \\ 4 & -5 & 0 \\ -1 & 0 & 0 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Row-reducing:

$$\begin{bmatrix} 3 & -4 & 0 \\ 4 & -5 & 0 \\ -1 & 0 & 0 \end{bmatrix} \xrightarrow{(1) \leftrightarrow (3)} \begin{bmatrix} -1 & 0 & 0 \\ 3 & -4 & 0 \\ 4 & -5 & 0 \end{bmatrix} \xrightarrow{(1) = -(1), (2) = (2) + 3(1), (3) = (3) + 4(1)}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & -5 & 0 \end{bmatrix} \xrightarrow{(2) = \frac{-1}{4}(2), (3) = \frac{-1}{5}(3)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Therefore  $x_3 = a, x_2 = 0, x_1 = 0$  for any  $a \in \mathbb{C}$ .

## Problem 1.2 (Sec 1.5. Problem 1). Let

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, C = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Compute ABC and CAB

Solution.

$$AB = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3 + (-1) \cdot 1 + 1 \cdot (-1) \\ 1 \cdot 3 + 2 \cdot 1 + 1 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$\Rightarrow ABC = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 \cdot 1 & 4 \cdot (-1) \\ 4 \cdot 1 & 4 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + (-1) \cdot 1 & 1 \cdot (-1) + (-1) \cdot 2 & 1 \cdot 1 + (-1) \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \end{bmatrix}$$

$$\Rightarrow CAB = \begin{bmatrix} 1 & -3 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + (-3) \cdot 1 + 0 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

## Problem 1.3 (Sec 1.5. Problem 2). Let

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix}$$

Verify directly that  $A(AB) = A^2B$ 

## Solution.

$$AB = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \cdot 2 + (-1) \cdot 1 + 1 \cdot 4 & 1 \cdot (-2) + (-1) \cdot 3 + 1 \cdot 4 \\ 2 \cdot 2 + 0 \cdot 1 + 1 \cdot 4 & 2 \cdot (-2) + 0 \cdot 3 + 1 \cdot 4 \\ 3 \cdot 2 + 0 \cdot 1 + 1 \cdot 4 & 3 \cdot (-2) + 0 \cdot 3 + 1 \cdot 4 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 8 & 0 \\ 10 & -2 \end{bmatrix}$$

$$\Rightarrow A(AB) = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 8 & 0 \\ 10 & -2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + (-1) \cdot 8 + 1 \cdot 10 & 1 \cdot (-1) + (-1) \cdot 0 + 1 \cdot (-2) \\ 2 \cdot 5 + 0 \cdot 8 + 1 \cdot 10 & 2 \cdot (-1) + 0 \cdot 0 + 1 \cdot (-2) \\ 3 \cdot 5 + 0 \cdot 8 + 1 \cdot 10 & 3 \cdot (-1) + 0 \cdot 0 + 1 \cdot (-2) \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{bmatrix}$$

Meanwhile,

$$A^{2} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 1 + (-1) \cdot 2 + 1 \cdot 3 & 1 \cdot (-1) + (-1) \cdot 0 + 1 \cdot 0 & 1 \cdot 1 + (-1) \cdot 1 + 1 \cdot 1 \\ 2 \cdot 1 + 0 \cdot 2 + 1 \cdot 3 & 2 \cdot (-1) + 0 \cdot 0 + 1 \cdot 0 & 2 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 \\ 3 \cdot 1 + 0 \cdot 2 + 1 \cdot 3 & 3 \cdot (-1) + 0 \cdot 0 + 1 \cdot 0 & 3 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 5 & -2 & 3 \\ 6 & -3 & 4 \end{bmatrix}$$

$$\Rightarrow A^{2}B = \begin{bmatrix} 2 & -1 & 1 \\ 5 & -2 & 3 \\ 6 & -3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 2 + (-1) \cdot 1 + 1 \cdot 4 & 2 \cdot (-2) + (-1) \cdot 3 + 1 \cdot 4 \\ 5 \cdot 2 + (-2) \cdot 1 + 3 \cdot 4 & 5 \cdot (-2) + (-2) \cdot 3 + 3 \cdot 4 \\ 6 \cdot 2 + (-3) \cdot 1 + 4 \cdot 4 & 6 \cdot (-2) + (-3) \cdot 3 + 4 \cdot 4 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{bmatrix}$$

Therefore, 
$$A(AB) = A^2(B) = \begin{bmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{bmatrix}$$

**Problem 1.4** (Sec 2.1. Problem 3). If  $\mathbb{C}$  is the field of complex numbers, which vectors in  $\mathbb{C}^3$  are linear combinations of (1,0,-1), (0,1,1) and (1,1,1)?

**Solution.**  $\mathbb{C}^3 \doteq \{(x_1, x_2, x_3) \mid x_i \in \mathbb{C}\}$ . We want to prove that any vector in  $\mathbb{C}^3$  is a linear combination of the vectors above.

$$\forall v = (x_1, x_2, x_3) \in \mathbb{C}^3,$$

$$v = (x_2 - x_3)(1, 0, -1) + (2x_2 - x_1 - x_3)(0, 1, 1) + (x_1 + x_3 - x_2)(1, 1, 1)$$

is therefore a linear combination of (1,0,-1), (0,1,1) and (1,1,1).

**Problem 1.5** (Sec 2.1. Problem 7). Let V be the set of pairs (x,y) of real numbers and let  $\mathbb{F}$  be the field of real numbers. Define

$$(x,y) + (x_1, y_1) = (x + x_1, 0)$$
  
 $c(x,y) = (cx, 0)$ 

Is V, with these operations, a vector space?

**Solution.** No. Counterexample against the rule  $1 \cdot \alpha = \alpha \ \forall \ \alpha \in V$ :

$$1 \cdot (1,1) = (1,0) \neq (1,1)$$

Therefore V is not a vector space.