MATH 26200: Point-Set Topology

Problem Set 7

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Problem 7.1 (done)

Construct an explicit homeomorphism from $\{0,1\}^{\mathbb{N}}$ to $\{0,1,2\}^{\mathbb{N}}$. Here, $\{0,1\}$ and $\{0,1,2\}$ denote the 2 and 3-element sets with the discrete topology.

Solution (With Otto Reed's help)

Construct the map $f : \{0,1,2\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ as follows. Consider $g : \{0,1,2\} \to \{(0),(1,0),(1,1)\}$ such that

$$g(0) = (0), g(1) = (1,0), g(2) = (1,1)$$

then map $f(x) = (\text{concat}_{n=1}^{\infty}(g(x_n)))$.

It is easy to check that f is both injective and surjective, so it is bijective.

To show continuity, each basic open set in $\{0,1\}^{\mathbb{N}}$ is some $\prod U_i$ such that $U_i = \{0,1\}$ for all but finitely many $\{i_1 < \ldots < i_K\}$.

Its preimage is then some

$$\bigcup_j \Big(\prod_{k\in\mathbb{N}} U_k^{(j)}\Big)$$

where for all j, $U_k^{(j)} = X_k$ for all $k \geq i_K$ (a very rough bound) and each $U^{(j)}$ is the "preimage" of the sequences truncated at i_K (or $i_K + 1$ if $U_{i_K} = \{1\}$) that are contained in $\prod U_i$, is open, since $\{0,1,2\}$ has the discrete topology. Therefore $\prod_{k\in\mathbb{N}} U_k^{(j)}$ is a basic open set in $\{0,1,2\}^{\mathbb{N}}$, so the preimage of a basic open set in $\{0,1\}^{\mathbb{N}}$ is open, so f is continuous.

The same proof applies to show that f^{-1} is continuous, with the bound $k \geq 2i_K$, since g requires twice the number of indices in $\{0,1\}^{\mathbb{N}}$ than in $\{0,1,2\}^{\mathbb{N}}$ to "accommodate" restrictions from the image set.

Hence f is a homeomorphism. $\square$

Problem 7.2 (done)

Suppose X is a compact metric space, Y is Hausdorff, and $f : X \to Y$ is continuous and surjective. Show that Y is a compact metrizable space.

Solution

1. WTS Y is compact. f is surjective so $Y = f(X)$ is the continuous image of a compact set, so Y is compact.

2. WTS X is 2nd countable. X is a compact metric space. For each $x \in X$, construct and $n \in \mathbb{N}$, let $\mathcal{U}_n = \{B(x, \frac{1}{n}) : x \in X\}$ then $\mathcal{U}_n$ is an open cover of X, so it reduces to some finite subcover $\mathcal{V}_n$.

Then consider $\mathcal{V} := \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a countable set. It is basis for X, since every element of $\mathcal{V}$ is open, and if we take any $x \in U \subset X$ such that U is open in X, since $x \in U$, there exists some (WLOG) $B(x,r) \subset U \subset X$. Then there exists some N such that $\frac{1}{N} < \frac{r}{2}$. Consider the finite subcover $\mathcal{V}_N$, then there has to exist some $B(x', \frac{1}{N}) \ni x$. But $\frac{1}{N} < \frac{r}{2}$ so in fact $B(x', \frac{1}{N}) \subset B(x,r) \subset U$, and $B(x', \frac{1}{N}) \in \mathcal{V}_n \subset \mathcal{V}$. So it follows that indeed $\mathcal{V}$ is a basis for X. So X is second countable.

3. This f is also a perfect map, since if $K \subset X$ is closed then it is compact, so $f(K)$ is compact in Hausdorff Y so $f(K)$ is also closed. Also, $f^{-1}(\{y\})$ for any $y \in Y$ is the continuous inverse of a closed set so is closed, it is in compact X so it is also compact.

Hence f is a perfect map, so second countability of X implies second countability of Y (per previous HW).

$\square$

Problem 7.3 (done)

Fix a prime number p and for each integer n, let $\mathbb{Z}/p^n\mathbb{Z}$ be the abelian group consisting of integers mod p^n.

- **(a)** Show that there is a directed system of surjective homomorphisms $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n-1}\mathbb{Z}$ given by "reduction mod p^{n-1}".
- **(b)** Let $\mathbb{Z}_p$ denote the inverse limit of this system, with the inverse limit topology (where each $\mathbb{Z}/p^n\mathbb{Z}$ has the discrete topology). Show that $\mathbb{Z}_p$ is homeomorphic to a Cantor set.
- **(c)** Show that $\mathbb{Z}_p$ admits the natural structure of an abelian group (compatible with the group structures on all the $\mathbb{Z}/p^n\mathbb{Z}$), and with respect to this group structure, the operations of addition and inverse are continuous.

Solution

(a) Take any $n \in \mathbb{N}$. Then we can construct

$$\begin{aligned} f_n : \mathbb{Z}/p^n\mathbb{Z} &\to \mathbb{Z}/p^{n-1}\mathbb{Z} \\ a &\mapsto a \bmod p^{n-1} \end{aligned}$$

It is surjective, since for any $b \in \mathbb{Z}/p^{n-1}\mathbb{Z}$, we have that $f(b) = b$. It is also a homomorphism, since $f_n(0) = 0, f_n(a_1 + a_2) = (a_1 + a_2) \bmod p^{n-1} = a_1 \bmod p^{n-1} + a_2 \bmod p^{n-1} = f_n(a_1) + f_n(a_2)$. $\square$

(b) We have

$$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$$

Each $\mathbb{Z}/p^n\mathbb{Z}$ is a finite, discrete space, so $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ is a compact and totally disconnected space. So $\mathbb{Z}_p$ is homeomorphic to a Cantor set. $\square$

(c) Take $x = (x_n), y = (y_n) \in \mathbb{Z}_p$. Then $inv(x) = (-x_n) \in \mathbb{Z}_p$ and $x + y := (x_n + y_n) \in \mathbb{Z}_p$, and this is well-defined since each f_n is a group homomorphism. The coordinate wise group operation is exactly the group operation on all the $\mathbb{Z}/p^n\mathbb{Z}$.

Consider the addition and inverse operations:

$$+ : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p, \quad inv : \mathbb{Z}_p \to \mathbb{Z}_p$$

To see this, we know that for the inverse operation, on each $\mathbb{Z}/p^n\mathbb{Z}$, $inv_n : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ is continuous (domain has discrete topology). View another directed system of $\mathbb{Z}/p^n\mathbb{Z}$ with inverse limit $\mathbb{Z}_p$, with the same f_n, then from claim in class, there must uniquely exist some $\phi : \mathbb{Z}_p \to \mathbb{Z}_p$ such that for all n,

$$\pi_n \phi = inv_n \pi_n$$

where $\pi_n : \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$ projects onto the $\mathbb{Z}/p^n\mathbb{Z}$ coordinate. It then follows that $\phi(x)_n = -x_n \Rightarrow \phi = inv$. ϕ is continuous so inv is continuous.

For the addition operation, consider $+(x, y) = x + y$. Then take a basic open set containing $x + y$, namely, $\prod U_n$ for some $U_n \subset \mathbb{Z}/p^n\mathbb{Z}$ open such that $U_n = \mathbb{Z}/p^n\mathbb{Z}$ for all but finitely many $\{n_1, \dots, n_J\}$. Then $x_{n_j} + y_{n_j} \in U_{n_j}\ \forall\ j \in [J]$. But then on $\mathbb{Z}/p^{n_j}\mathbb{Z}$, $+$ is continuous, so it follows that $+^{-1}(U_{n_j})$ is open in $\mathbb{Z}/p^{n_j}\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}$. It is a finite set, so it is the finite union of basic open sets

$$+^{-1}(U_{n_j}) = \bigcup_{\alpha=1}^{M_j} V_{\alpha,j} \times W_{\alpha,j}$$

So $+^{-1}(x+y) = \bigcup_{j\in J}\bigcup_{\alpha=1}^{M_j} V_{\alpha,j}\times W_{\alpha,j}$ is a union of basic open sets in $\prod \mathbb{Z}/p^n\mathbb{Z}$, so is also open in the subspace topology, i.e., open in $\mathbb{Z}_p$, that contains $x\times y$. It follows that $+$ is indeed continuous.

$\square$

Problem 7.4 (done)

A space X is *zero-dimensional* if for every point x and any open neighborhood U of x, there is a clopen set V with x in V and V in U.

(a) Show that any zero dimensional Hausdorff space is totally disconnected.

(b) Suppose X is Hausdorff, locally compact and totally disconnected. Show that it is zero dimensional.

Solution

(a) Let X be a zero dimensional Hausdorff space. Suppose X is not totally disconnected, i.e., there exists some $K\subset X$ connected with more than 1 point, say, $a\neq b\in K$. Since X is Hausdorff, there exists some open $U_a\ni a, U_b\ni b$ such that $U_a\cap U_b=\varnothing$. Since X is zero dimensional, there exists some clopen V_a such that $a\in V_a\subset U_a$. Consequently, $b\notin V_a$.

Then $V_a\cap K$ is a proper clopen subset of K, so K is not connected. $\Rightarrow\Leftarrow$ $\square$

(b) Let X be Hausdorff, locally compact and totally disconnected. WTS it is zero-dimensional.

X is Hausdorff and locally compact, so it is regular. So there exists some open V such that $x\in\overline{V}\subset U$. $\overline{V}$ is closed in Hausdorff X, so it is compact. Let us inspect $\overline{V}\ni x$. Since X is totally disconnected, $\overline{V}$ is also disconnected (with the subspace topology). $\overline{V}$ is therefore a compact and Hausdorff space. Following a Corollary from class, we have that $\{x\}$ is a component and $\{x\}\subset U$, so there exists some W clopen such that $x\in W\subset U$. We've thus found W. $\square$

Problem 7.5 (done)

Let $\mathcal{F}$ be an equicontinuous family of functions from $[0,1]$ to $[0,1]$. Show that there is a continuous function $g:[0,1]\times[0,1]\to[0,1]$ so that for every f in $\mathcal{F}$ there is some $t\in[0,1]$ so that g restricted to the horizontal interval $[0,1]\times t$ agrees with f. (Hint: using Ascoli, show $\mathcal{F}$ is contained in some compact subset G of the space of continuous functions from $[0,1]$ to $[0,1]$ with some suitable topology. Show this compact space is metrizable. Deduce that there is a surjective map from the Cantor set to G. Use this surjective map to construct the function g.)

Solution

For any $a\in[0,1]$, consider $\mathcal{F}_a=\{f(a): f\in\mathcal{F}\}$, then its closure is closed in compact $[0,1]$, so is compact. Thus, using Ascoli's Theorem, we have that $\mathcal{F}$ is contained in a compact subspace G of $\mathcal{C}([0,1],[0,1])$ in the topology of compact convergence.

Since $[0,1]$ is compact and $[0,1]$ is metric, the sets $\{B([0,1],f,\varepsilon)\}=\{f'\in Y^X \mid \sup_{x\in[0,1]}\{d(fx,f'x)<\varepsilon\}\}$ forms the basis for $\mathcal{C}([0,1],[0,1])$. But this is exactly the uniform topology induced by the uniform metric. So it is metrizable!

Therefore G is a compact metric space. So there is a continuous surjective map $h:\mathcal{C}\to G$, from the middle thirds Cantor set into G. Then for any $f\in\mathcal{F}\subset G$, there exists some $\alpha\in\mathcal{C}\subset[0,1]$ such that $h(\alpha)=f$. Then define $g\mid_{[0,1]\times\alpha}=h(\alpha)=f$.

So we've define g on $[0,1]\times\mathcal{C}\to[0,1]$, it is clearly continuous since f is continuous. Also, $\mathcal{C}$ is compact, so $[0,1]\times\mathcal{C}$ is compact in Hausdorff $[0,1]\times[0,1]$ so it's closed. Using Tietze extension theorem, we can extend g to $[0,1]\times[0,1]\to[0,1]$. And we've found our g. $\square$