207 Homework #8 Due Friday, Dec 1.

I. Read Pugh, Chapter 5, Section 7. Section 8. Read Bamberg and Sternberg Chapters 7 and 8.

II. Chap 5: [40, 44, 55, 68], 48, 49, 65, 67.

III. This exercise explores the question (asked in class): suppose Df_p is invertible at every point of the domain: is f injective? Recall this is true for $f\colon (a,b) \to \mathbb{R}$ by the one dimensional inverse function theorem.

(a) Consider the map $f\colon A \to \mathbb{R}^2$, where $A = \{z \in \mathbb{R}^2 : 1 < |z| < 2\}$ given by

$$f(x,y) = (x^2 - y^2, 2xy).$$

Is it C^1 in A, with invertible derivative? Is it injective?

(b) Suppose that $U \subset \mathbb{R}^n$ is open, $f\colon U \to \mathbb{R}^n$ is C^1, and Df_p is invertible at each $p \in U$. Prove that if $f(U)$ is closed in $\mathbb{R}^n$, then $f(U) = \mathbb{R}^n$.

(c) ** (Extra credit) Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is C^1, with invertible derivative. Must it be injective?

(d) ** (Extra credit) Characterize the open sets $U \subset \mathbb{R}^n$ with the property that if $f\colon U \to \mathbb{R}^n$ is C^1, and f is invertible at every point in U, then f is injective.

(e) ** (Extra credit) Prove that if $f\colon A \to \mathbb{R}^2$ is C^1, with the property that Df_p is invertible at every $p \in A$, then there exists an integer $k \geq 0$ such that f is k-to-one, where A is the annulus defined in (c).

IV. Let $g\colon \mathbb{R}^n \to \mathbb{R}^k$ and consider the set $S = g^{-1}(0)$. Assume that for every $p \in S$, the rank of $D_p g$ is equal to k (in other words, $D_p g\colon \mathbb{R}^n \to \mathbb{R}^k$ is surjective). Let $p \in S$, and define $T_pS := \ker D_p f \subset \mathbb{R}^n$.

(a) What does the rank-nullity theorem from linear algebra tell you about the dimension of T_pS?

(b) Show that if $\gamma\colon (-\epsilon, \epsilon) \to U$ is a C^1 curve through p along which g is constant (i.e. if $\gamma(0) = p$ and $g(\gamma(t)) = g(\gamma(0))$ for all $t \in (-\epsilon, \epsilon)$), then $\gamma'(0) \in T_pS$. This is why we call T_pS the *tangent space to p at S*.

(c) Find a basis for T_pS for the following S, p

  (i) $S = \{(x,y,z) : -x^2 + y^2 - z^2 = -1\}$, $p = (\frac{1}{\sqrt{2}}, -\frac{\sqrt{3}}{\sqrt{2}}, -\sqrt{2})$.

  (ii) $S = \{(x,y,z) : -x^2 + y^2 - z^2 = -1, \text{ and } xz + 4y^2 = 5\}$, $p = (\frac{1}{\sqrt{2}}, -\frac{\sqrt{3}}{\sqrt{2}}, -\sqrt{2})$.

(d) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ and for $p \in S$, let V_p be the orthogonal projection of $\mathrm{grad}_p(f)$ onto the subspace $T_pS \subset \mathbb{R}^n$. Prove that if V_p is nonzero, then $\pm|V_p|$ is the maximum (resp. minimum) value of the function $G\colon T_p^1S \to \mathbb{R}$, where $T_p^1S := \{v \in T_pS : |v| = 1\}$, and

$$G(v) = D_pf(v).$$

(e) Prove that $\pm V_p$, if nonzero, points to the maximal direction of increase/decrease of f in directions tangent to the surface S.

(f) Relate (d),(e) to Lagrange multlipliers.

V. (a) Compute the volume of the region $\Omega \subset \mathbb{R}^3$ bounded by $x = 0, x = 2, z = -y$ and by $z = y^2/2$.

(b) Write down a triple integral that computes the volume of the region $\Omega \subset \mathbb{R}^3$ bounded by the coordinate planes and $y = 1 - x^2$ and $y = 1 - z^2$. Don't evaluate.

(c) Compute

$$\iiint_B \left(2x + 3y^2 + 4z^3\right) dx\, dy\, dz,$$

where $B = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le 2, 0 \le z \le 3\}$.

VI. Let $\omega = (y \cos xy + \mathrm{e}^x)\, \mathrm{d}x + (x \cos xy + 2y)\mathrm{d}y$.

(a) Evaluate $\int_\Gamma \omega$ along the segment of the parabola $y = x^2$ from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Use the parameterization ϕ described by the pullback

$$\begin{pmatrix} \phi^* x \\ \phi^* y \end{pmatrix} = \begin{pmatrix} t \\ t^2 \end{pmatrix}.$$

(b) Evaluate $\int_\Gamma \omega$ for the case where Γ is the straight line joining the origin to the point $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$. Do the same for the case where Γ consists of the segment $0 \leqslant x \leqslant \alpha$ on the x-axis, followed by the segment $x = \alpha, 0 \leqslant y \leqslant \beta$.

(c) Find a function $f(x, y)$ such that $\omega = df$.

VII. Let $\omega = y\ \mathrm{d}x - x\ \mathrm{d}y$.

(a) Evaluate $\int_\gamma \omega$ along the semicircle γ from $\begin{pmatrix}r-1\\ 0\end{pmatrix}$ to $\begin{pmatrix}r+1\\ 0\end{pmatrix}$ defined by

$$\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}r - \cos t\\ \sin t\end{pmatrix}$$

for $0 < t < \pi$.

(b) Show explicitly that you can obtain a different value from that in (a) by choosing a different curve joining $\begin{pmatrix}r-1\\ 0\end{pmatrix}$ to $\begin{pmatrix}r+1\\ 0\end{pmatrix}$.

VIII. In each of the following cases, u and v are functions on a plane where x and y are affine coordinates. Express $\mathrm{d}x \wedge \mathrm{d}y$ in terms of $\mathrm{d}u \wedge \mathrm{d}v$. Make a sketch showing typical curves $u =$ constant and $v =$ constant in the first quadrant $(x, y > 0)$ and try to give a geometric interpretation to the relations between $\mathrm{d}x \wedge \mathrm{d}y$ and $\mathrm{d}u \wedge \mathrm{d}v$ by applying both to a parallelogram whose sides are tangent to $u =$ constant and $v =$ constant respectively.

(a) $x = u\cos v.\quad y = u\sin v.$

(b) $x = u\cosh v,\quad y = u\sinh v.$

(c) $x = u^2 - v^2,\quad y = 2uv.$

IX. (a) Show, by reversing the order of integration, that

$$\int_0^a \left(\int_0^y \mathrm{e}^{m(a-x)} f(x)\mathrm{d}x\right)\mathrm{d}y = \int_0^a (a-x)\mathrm{e}^{m(a-x)} f(x)\mathrm{d}x$$

where a and m are constants, $a > 0$.

(b) Show that $\int_0^x \left(\int_0^v \left[\int_0^u f(t)\mathrm{d}t\right]\mathrm{d}u\right)\mathrm{d}v = \frac{1}{2}\int_0^x (x-t)^2 f(t)\mathrm{d}t$. If you do this in two steps, you never actually have to consider a triple integral!