MATH 20700: Honors Analysis in Rn I

Problem Set 2

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Textbook: Pugh's Real Mathematical Analysis

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Problem 2.1 (2.30)

Consider a two-point set $M = \{a, b\}$ whose topology consists of the two sets, M and the empty set. Why does this topology not arise from a metric on M?

Solution

Suppose there exists a metric d that induces the topology $\mathcal{T} = \{\{a, b\}, \varnothing\}$ on M. Let $\varepsilon = d(a, b)/2$, then $B_M(x, \varepsilon) = \{a\}$, so $\{a\}$ is an open set. But $\{a\} \notin \mathcal{T}$, $\Rightarrow\Leftarrow$

Therefore $\mathcal{T}$ can't be a topology induced by a metric. $\square$

Problem 2.2 (2.36)

Construct a set with exactly three cluster points.

Solution

$$S = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\} \cup \{3 - \frac{1}{n} \mid n \in \mathbb{N}\} \cup \{5 - \frac{1}{n} \mid n \in \mathbb{N}\}$$

S clearly has cluster points 1, 3, 5. The set of cluster points S' is a subset of $\lim S = \{1, 3, 5\}$. So it has none other than the 3 above. $\square$

Problem 2.3 (2.44)

Consider a function $f : M \to \mathbb{R}$. Its graph is the set

$$\{(p, y) \in M \times \mathbb{R} : y = fp\}$$

- **(a)** Prove that if f is continuous then its graph is closed (as a subset of $M \times \mathbb{R}$).
- **(b)** Prove that if f is continuous and M is compact then its graph is compact
- **(c)** Prove that if the graph of f is compact then f is continuous
- **(d)** What if the graph is merely closed? Give an example of a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ whose graph is closed.

Solution

We freely use Pugh's Theorem 17 (Ch2.3) in this problem.

Let $G = \{(p, y) \in M \times \mathbb{R} : y = fp\}$ be the graph.

(a) Let $\{(p_n, y_n)\} \subset G$ be a sequence that converges to $(p, y) \in M \times \mathbb{R}$. WLOG, we use the metric d_{max} on $M \times \mathbb{R}$. WTS $(p, y) \in G$.

$(p_n, y_n) \xrightarrow{n\to\infty} (p, y) \Rightarrow p_n \xrightarrow{n\to\infty} p, y_n \xrightarrow{n\to\infty} y$. But f is continuous, so $y_n = f(p_n) \xrightarrow{n\to\infty} f(p) \Rightarrow y = f(p) \Rightarrow (p, y) \in G$. $\square$

(b) Let $\{(p_n, y_n)\} \subset G$. WTS there exists a subsequence $\{(p_{n_j}, y_{n_j})\}$ that converges in G.

Since M is compact and $\{p_n\} \subset M$, there exists a subsequence $\{p_{n_j}\}$ such that $p_{n_j} \xrightarrow{j\to\infty} p \in M$.

f is continuous, so $f(p_{n_j}) \xrightarrow{j\to\infty} f(p) \Leftrightarrow y_{n_j} \xrightarrow{j\to\infty} f(p) =: y$. This implies

$$(p_{n_j}, y_{n_j}) \xrightarrow{j\to\infty} (p, y) \in G \quad \square$$

(c) Let $\{p_n\} \subset M$ such that $p_n \xrightarrow{n\to\infty} p \in M$. WTS $f(p_n) \xrightarrow{n\to\infty} f(p)$.

Suppose not:

$$\neg(\ \forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N},\ \forall\, n \geq N, d(f(p_n), f(p)) < \varepsilon)$$
$$\Leftrightarrow\ \exists\, \varepsilon > 0,\ \forall\, N \in \mathbb{N},\ \exists\, n \geq N, d(f(p_n), f(p)) \geq \varepsilon$$

Then we construct another sequence q_n, defined as follows: Choose $N_0 = 1$ and choose $N_1 \geq N_0$ satisfying the conditions above. Assign $q_1 = p_{N_1}$. Inductively, choose $N_{k+1} \geq N_k$ satisfying the conditions above, and assign $q_{k+1} = p_{N_{k+1}}$.

By constructing $\{q_n\} \subset M$ (essentially a particular subsequence of $\{p_n\}$) this way,

$$\forall\, n \in \mathbb{N}, d(f(q_n), f(p)) \geq \varepsilon$$

Since G is compact, sequence $\{(q_n, f(q_n))\} \subset G$ has subsequence $\{(q_{n_j}, f(q_{n_j}))\}$ such that $(q_{n_j}, f(q_{n_j})) \xrightarrow{j\to\infty} (q, f(q)) \in G$, which implies $q_{n_j} \xrightarrow{j\to\infty} q, f(q_{n_j}) \xrightarrow{j\to\infty} f(q)$. But $\{q_{n_j}\}$ is simply a (sub) subsequence of $\{p_n\}$, so $q = p \Rightarrow f(q_{n_j}) \xrightarrow{j\to\infty} f(p)$.

But $\forall\, n \in \mathbb{N}, d(f(q_n), f(p)) \geq \varepsilon$. $\Rightarrow\Leftarrow$ $\square$

(d) Define

$$f(x) = \begin{cases} 0 & \text{for } x = 0, \\ \frac{1}{|x|} & \text{otherwise.} \end{cases}$$

then f is discontinuous at 0.

We show that G is indeed closed. Let $\{(p_n, f(p_n))\} \subset M \times \mathbb{R}$ such that $(p_n, f(p_n)) \xrightarrow{n\to\infty} (p, q) \in M \times \mathbb{R}$. WTS $(p, q) \in G \Leftrightarrow q = f(p)$.

$(p_n, f(p_n)) \xrightarrow{n\to\infty} (p, q) \Leftrightarrow p_n \xrightarrow{n\to\infty} p, f(p_n) \xrightarrow{n\to\infty} q$.

If $p \neq 0$, let $\varepsilon = |p|/2$, then $\exists\ N \in \mathbb{N}$ such that $\forall\ n \geq N, |p|/2 < |p_n/2| < 3|p|/2$. In particular, they are non-zero. Since f is trivially continuous on $\mathbb{R}\backslash\{0\}$, it is then clear that $f(p_n) \xrightarrow{n\to\infty} f(p)$, so $q = f(p)$.

If $p = 0$, with $p_n \xrightarrow{n\to\infty} 0, f(p_n)$ diverges. In particular, given any $M > 0$, let $\varepsilon = \frac{1}{2M}$ then there exists $N \in \mathbb{N}$ such that $\forall\ n \geq N, |p_n| < \varepsilon = \frac{1}{2M} \Rightarrow f(p_N) > 2M > M$. So there can't be $(p_n, f(p_n)) \xrightarrow{n\to\infty} (0, q)$ in the first place. $\square$

Problem 2.4 (2.49*)

Construct a subset $A \subset \mathbb{R}$ and a continuous bijection $f : A \to A$ that is not a homeomorphism. [Hint: By Theorem 36, A must be noncompact]

Solution

Let $A = [0, 1) \cup [2, 3) \cup [4, 5) \cup \cdots = \bigcup_{n=0}^{\infty}[2n, 2n + 1)$

And construct map f that (continuously):

1. Scales and shifts [0, 1) to [0, 1/2)
2. Scales and shifts [2, 3) to [1/2, 1)
3. Shifts [4, 5) to [2, 3), [6, 7) to [4, 5), etc., in short shifts $[2n, 2n+1)$ to $[2n-2, 2n-1)$ for all $n \geq 2$

Then f trivially maintains sequential convergence (therefore is continuous) and is a bijection. So it is a continuous bijection.

However, f sends [2, 2.5) to [1/2, 3/4), an open set in A to a nonopen set in A, so f^{-1} is not continuous. It follows that f is not homeo. $\square$

Problem 2.5 (2.56)

Prove that the 2-sphere is not homeomorphic to the plane.

Solution

We want to first show that the 2-sphere is compact, by showing that it is closed and bounded in $\mathbb{R}^3$.

It is clear that the (unit) 2-sphere is bounded by the $2 \times 2 \times 2$ box centered at the origin, so it is bounded.

Let $\{p_n\} \subset S^2$ such that $p_n \xrightarrow{n\to\infty} p \in \mathbb{R}^3$. We now want to show that $p \in S^2$. Recall that $S^2 = \{z \in \mathbb{R}^3 \mid d(0, z) = 1\}$

For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall\ n \geq N, d(p_n, p) < \varepsilon$. But $d(0, p_n) = 1\ \forall\ n \in \mathbb{N}$. It follows that

$$1 - \varepsilon < d(0, p_N) - d(p_N, p) \leq d(0, p) \leq d(0, p_N) + d(p_N, p) < 1 + \varepsilon$$

which holds for all $\varepsilon > 0$. It follows that $d(0, p) = 1$, so $p \in S^2$. Therefore the 2-sphere is closed.

Since the 2-sphere is closed and bounded in $\mathbb{R}^3$, it is compact (H-B).

On the other hand, $\mathbb{R}^2$ is unbounded in $\mathbb{R}^2$, so it is not compact.

Therefore the 2-sphere (a compact set) can't be homeomorphic to the plane (a noncompact set). □

Problem 2.6 (2.66)

- **(a)** Prove that every connected open subset of $\mathbb{R}^m$ is path-connected.
- **(b)** Is the same true for open connected subsets of the circle?
- **(c)** What about connected nonopen subsets of the circle?

Solution

(a) Let $U \subset \mathbb{R}^m$ be connected and open. WTS U is path-connected.

We first want to show that any ball in $\mathbb{R}^m$ is path-connected. Let $p, q \in B(x, r) \subset U$ then we can construct

$$f : [0, 1] \to B(x, r), t \mapsto p + (q - p)t$$

that is a continuous path from p to q as required.

WTS for every $p, q \in U$, there exists a continuous path from p to q. Fix $p, q \in U$.

Then let A be the set of points $z \in U$ such that there exists a continuous path from p to z. We want to show that A is clopen. Note that $p \in A$, so A is non-empty.

1. Show that A is open.

Let $z \in A$ be arbitrary. Since U is open, we can draw $B_U(z, r) \subset U$.

We want to show that $B_U(z, r) \subset A$. Choose any $s \in B(z, r)$, then we can concatenate the continuous path from p_0 to z (since $z \in A$) and the continuous path from z to s (since the ball is path-connected) to get a continuous path from p_0 to s.

More concretely, if there exists continuous path $f_{p,z} : [0, 1] \to U$ from p to z, and $f_{z,s} : [0, 1] \to U$ from z to s then we can construct

$$f_{p,s}(t) = \begin{cases} f_{p,z}(2t) & \text{for } t < 0.5 \\ f_{z,s}(2t - 1) & \text{for } t \geq 0.5 \end{cases}$$

is a continuous path from p to s. So $s \in A$. It follows that $B(z, r) \subset A$. Therefore A is open.

2. Show that A is closed.

WTS A^c, the set of points $y \in U$ such that there doesn't exist a continuous path from p to y, is open.

Let $y \in A^c$ be arbitrary. Since U is open, we can draw $B_U(y, r) \subset U$.

WTS $B_U(y, r) \subset A^c$. Choose any $s \in B(y, r)$. Suppose not, that there exists a continuous path from p to s, then since the ball is path-connected, it follows that there also exists a continuous path from s to y. By concatenating these 2 paths, we can get a continuous path from p to y. $\Rightarrow\Leftarrow$

So there doesn't exist a continuous path from p to s for all $s \in B(y, r)$. It follows that $B_U(y, r) \subset A^c$, so A^c is open. Thus A is closed.

3. Therefore A is a nonempty clopen subset of connected U, so $A = U$. It follows that U is path-connected. $\square$

(b) Yes. WLOG, use S^1.

We now want to show that any ball $B(x, r) \subset S^1$ is path-connected. If $r \leq 2$ then $B(x, r)$ is an arc of S^1, while if $r > 2$ then $B(x, r) = S^1$. Both of which can be parameterized as $\{(\cos\theta, \sin\theta)\}$, and are therefore trivially path-connected.

Apply the same proof in (a). $\square$

(c) Yes, it is path-connected.

Let V be a connected, nonopen subset of the circle, and $p \neq q$ be arbitrary points in V. p and q then partition S^1 into a major closed arc A_1 and a minor closed arc A_2 that satisfies:

$$A_1 \cup A_2 = S^1, A_1 \cap A_2 = \{p, q\}$$

Case 1: If $A_1 \subset V$ or $A_2 \subset V$, then there clearly exists a continuous path in V from p to q, namely through these continuous closed arcs (that can be easily parameterized into $\{(\cos\theta, \sin\theta)\}$ on a closed interval of θ)

Case 2: There exists $k \in A_1, l \in A_2$ such that $k, l \notin V$. Then k, l similarly partitions S^1 into 2 closed arcs A_3, A_4 such that

$$A_3 \cup A_4 = S^1, A_3 \cap A_4 = \{k, l\}$$

Now consider $A_3' := A_3 \backslash \{k, l\}, A_4' := A_4 \backslash \{k, l\}$, which are open sets in S^1 that are image sets of $(\cos\theta, \sin\theta)$ on an open interval of θ. Then

$$(V \cap A_3') \cup (V \cap A_4') = V \cap (A_3' \cup A_4') = V \cap (S^1 \backslash \{k, l\}) = V$$

However, $A_3' \cap A_4' = \varnothing$ so $(V \cap A_3') \cap (V \cap A_4') = \varnothing$. Furthermore, $V \cap A_3'$ and $V \cap A_4'$ in the subspace topology of V, as they are intersections of V with an open set in the bigger topology of S^1. It follows that $(V \cap A_3') \sqcup (V \cap A_4')$ partitions V into 2 non-empty open sets, which are therefore proper clopen, making V disconnected, $\Rightarrow\Leftarrow$.

It follows that only case 1 is valid, and there exists a continuous path from any p to q. $\square$

Problem 2.7 (2.78)

$(p_1, \ldots, p_n)$ is an ε-chain in a metric space M if for each i we have $p_i \in M$ and $d(p_i, p_{i+1}) < \varepsilon$. The metric space is chain-connected if for each $\varepsilon > 0$ and each pair of points $p, q \in M$ there is an $\varepsilon-$chain from p to q.

- **(a)** Show that every connected metric space is chain-connected.
- **(b)** Show that if M is compact and chain-connected then it is connected.
- **(c)** Is $\mathbb{R} \backslash \mathbb{Z}$ chain-connected?
- **(d)** If M is complete and chain-connected, is it connected?

Solution

(a) Suppose not. Then there exists $\varepsilon > 0$ and $p_0, q_0 \in M$ such that there doesn't exist an $\varepsilon-$chain from p_0 to q_0. We emphasize that this ε is now fixed.

Let D be the set of such points in M that have no $\varepsilon-$chain from p_0. Define $C = M \backslash D$. Then $p_0 \in C, q_0 \in D$. WTS C, D are open.

We first consider C. Take arbitrary $c \in C$, then let its corresponding $\varepsilon-$chain from p_0 be $(p_1, \ldots, p_n)$.

Draw $B_M(c, \varepsilon)$. We show that C is open by showing that indeed we can draw for arbitrary c, $B_M(c, \varepsilon) \subset C$.

Indeed, $\forall\ c' \in B_M(c, \varepsilon)$, there exists an ε-chain from p_0 to c', namely $(p_1, \ldots, p_n, c')$, which is well-defined since $d(c', p_n) = d(c', c) < \varepsilon$.

Therefore $c' \in C$. So $B_M(c, \varepsilon) \subset C$.

We now consider D. The proof is very similar. Take arbitrary $d \in D$.

Draw $B_M(d, \varepsilon)$. We show that D is open by showing that we can draw for arbitrary d, $B_M(d, \varepsilon) \subset D$.

Indeed, $\forall\ d' \in B_M(d, \varepsilon)$, if there exists an $\varepsilon-$ chain from p_0 to d', for example, $(q_1, \ldots, q_n)$, then one can form an $\varepsilon-$chain from p_0 to d, which would be $(q_1, \ldots, q_n, d)$, a well-defined chain since $d(d, q_n) = d(d, d') < \varepsilon$, which would be a contradiction since $d \in D$.

Therefore there doesn't exist an $\varepsilon-$chain from p_0 to d'. So $(d' \in B_M(d, \varepsilon) \Rightarrow d' \in D)$. D is therefore open.

In conclusion, we have

$$M = C \sqcup D$$

where C and D are both open, which make them both clopen. But $p_0 \notin D, q_0 \notin C$, so they are proper clopen subsets of M, making M disconnected. $\Rightarrow\Leftarrow$

It follows that M must be chain-connected. $\square$

(b) Let M be compact and chain-connected. WTS M is connected.

Suppose not. Then there exists $A, B \subset M$ such that

$$M = A \sqcup B$$

and $A, B \neq \varnothing$ and are clopen. Since $A, B \neq \varnothing$, $\exists\ a \in A, b \in B$.

We shall construct a sequence $\{t_n\} \subset A$ as follows:

Set $\varepsilon = \frac{1}{n}$, then there exists (finite) $\varepsilon-$chain $(p_{n,1}, p_{n,2}, \ldots, p_{n,N(n)})$ from a to b, i.e. $p_{n,1} = a, p_{n,N(n)=b}$. $(N(n) \geq 2$, since $a \neq b)$.

Let $T(n) = \{1 \leq k \leq N(n) \mid B_M(p_{n,k}, \frac{1}{n}) \subset A\}$, i.e., the set of indices of points in the chain whose $\varepsilon-$ neighborhoods are still within A. Clearly $1 \in T(n)$ and it has an upper bound $N(n)$, so there exists a maximum value $k(n) \in \mathbb{N}$.

Assign $t_n \coloneqq p_{n,k(n)} \in A$.

Since A is a closed subset of compact M, it follows that A is also compact. Therefore, there exists a subsequence $\{t_{n_j}\}$ such that

$$t_{n_j} \xrightarrow{j\to\infty} t \in A$$

Since A is open, we can draw a ball $B_M(t,r) \subset A$.

Since $t_{n_j} \xrightarrow{j\to\infty} t$, there exists $J_1 \in \mathbb{N}$ such that $\forall\, j > J_1, d(t_{n_j}, t) < \frac{r}{2}$.

Choose $J = \lceil \max\{J_1, \frac{4}{r}\} \rceil + 1$. Let $L = n_J$. Then

$$\frac{1}{L} \le \frac{1}{J} < \frac{r}{4}$$

Consider the original path from a to b with $\varepsilon = \frac{1}{L}$:

$$(a = p_{L,1}; p_{L,2}; \ldots; p_{L,k(L)} = t_L; p_{L,k(L)+1}; \ldots; p_{L,N(L)} = b)$$

where we pay special attention to the node after t_L.

We see that

$$\begin{aligned} d(t, p_{L,k(L)+1}) &\le d(t, t_L) + d(t_L, p_{L,k(L)+1}) \\ &< \frac{r}{2} + \frac{1}{L} \\ &< \frac{r}{2} + \frac{r}{4} = \frac{3r}{4} \end{aligned}$$

It trivially follows that $B_M(p_{L,k(L)+1}, \frac{1}{L}) \subset B_M(t,r)$.

But $B_M(t,r) \subset A$, so $B_M(p_{L,k(L)+1}) \subset A$, which contradicts with $k(L)$ being the maximum index at which the node of the path is still in A $\Rightarrow\Leftarrow$.

It follows that M must be connected. $\square$

(c) Yes.

Given $\varepsilon > 0; a, b \in \mathbb{R}\backslash\mathbb{Z}$. WLOG, assume $\varepsilon < 1$, since a larger ε'-chain can be composed through nodes consisting of $\lceil \frac{\varepsilon'}{\varepsilon} \rceil$ subnodes of the $\varepsilon-$ chain.

We will recursively define an $\varepsilon-$chain from a to b as follows:

$$p_1 \coloneqq a$$

If $b - p_n < \varepsilon, p_{n+1} \coloneqq b$, else

$$p_{n+1} = \begin{cases} p_n + \varepsilon & \text{if } p_n < \lceil p_n \rceil - \varepsilon \\ p_n + \varepsilon/2 & \text{if } \lceil p_n \rceil - \varepsilon \le p_n < \lceil p_n \rceil - \varepsilon/2 \\ p_n + \varepsilon & \text{if } \lceil p_n \rceil - \varepsilon/2 \le p_n < \lceil p_n \rceil \end{cases}$$

It is firstly trivial that p_{n+1}, as constructed, is not in $\mathbb{R}\backslash\mathbb{Z}$. It is also guaranteed that after at most $\lceil \frac{b-a}{\varepsilon/2} \rceil$ terms, p_n will come within ε of b, and therefore setting $p_{n+1} = b$, completing the $\varepsilon-$chain from a to b. $\square$

(d) No.

Counter-example:

Define

$$A = \{(x, 1/x) \mid x > 0\}, B = \{(x, -1/x) \mid x < 0\}$$

and $M = A \sqcup B$. We show that M is complete and chain-connected, but not connected.

First, A and B are closed in the ambient metric space $\mathbb{R}^2$ (similar to problem 2.44d). So $M = A \sqcup B$ is also closed in $\mathbb{R}^2$. $\mathbb{R}^2$ is complete so M is complete.

Second, M is chain-connected. Suppose we are given $\varepsilon > 0; p, q \in M$. If p and q are both in A or are both in B, then the chain is trivial. If $p \in A$ and $q \in B$ then we can define $x_1 = -\varepsilon/3, x_2 = \varepsilon/3$ then

$$d_E((x_1, 1/|x_1|), (x_2, 1/|x_2|)) = 2\varepsilon/3 < \varepsilon$$

so we can make and concatenate $\varepsilon-$chains from p to $(x_1, 1/x_1)$, from $(x_1, 1/x_1)$ to $(x_2, -1/x_2)$, and from $(x_2, -1/x_2)$ to q.

Lastly, it is not connected. $A = A \cap M$, A is closed in $\mathbb{R}^2$ so A is closed in M. It is also open in M because for each $p = (a, 1/a) \in A$, we can draw $B_M((a, 1/a), 1/4a) \subset A$ trivially. So A is clopen and proper. $\square$

Problem 2.8 (2.117 f)

Fold a piece of paper in half.

(a) Is this a continuous transformation of one rectangle into another?

(b) Is it injective?

(c) Draw an open set in the target rectangle, and find its preimage in the original rectangle. Is it open?

(d) What if the open set meets the crease?

The **baker's transformation** is a similar mapping. A rectangle of dough is stretched to twice its length and then folded back on itself. Is the transformation continuous? A formula for the baker's transformation in one variable is

$$f : [0, 1] \to [0, 1], f(x) = 1 - |1 - 2x|$$

The $\mathbf{n^{th}}$ iterate of f is $f^n = f \circ f \circ \cdots \circ f, n$ times. The **orbit** of a point x is

$$\{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}$$

$f^{\circ n}$ is f^n

(e) If x is rational prove that the orbit of x is a finite set.

(f) If x is irrational what is the orbit?

Solution

(a) WLOG, Let the rectangle be $[0,2] \times [0,1]$, and let the crease be at $x = 1$. Then the transformation is

$$f(x,y) = \begin{cases} (x,y) & \text{for } x < 1 \\ (2-x,y) & \text{for } x \geq 1 \end{cases}$$

which folds the rectangle $[0,2] \times [0,1]$ into the rectangle $[0,1] \times [0,1]$. The transformation is continuous, since the stepwise function on the $x-$coordinate is continuous and y is trivially continuous. □

(b) No, it is not injective.

$$f(0.5,0) = f(1.5,0) = (0.5,0) \quad \square$$

(c) Yes.

Let the open set be $U = \{(x,y)\} \subset [0,1] \times [0,1]$. Define $g(x,y) = (2-x,y)$ then

$$f^{Pre}(U) = U \cup g(U)$$

Note that g is a homeomorphism, so $g(U)$ is also open. $f^{Pre}(U)$ is a union of open sets and is therefore open. □

(d) It doesn't matter if the open set meets the crease. The above proof still holds. □

(e) We freely use the fact that if $x \in [0,1]$, x can be decomposed into the unique following binary representation:

$$x = \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_n}{2^n} + \cdots$$

i.e., each $x \in [0,1]$ corresponds to a unique sequence $\{a_{x,n}\}$ with $a_{x,n} \in \{0,1\}$.

Note that 0 corresponds to the zero-sequence, while 1 corresponds to the one-sequence.

Let $x \in [0,1]$. To be concise, we label $\{a_{x,n}\}$ as $\{a_n\}$. We pay attention to how the sequence changes as f is applied continually to x.

Rewriting f, we have:

$$f(x) = \begin{cases} 2x & \text{for } x < 0.5 \\ 2-2x & \text{for } x \geq 0.5 \end{cases}$$

When $x < 0.5$, it is clear that $a_1 = 0$. Then we have

$$2x = 2\left(\frac{a_2}{2^2} + \cdots\right) = \frac{a_2}{2} + \frac{a_3}{2^2} + \cdots$$

which corresponds to sequence $(a_2, a_3, \cdots)$ (sequence shifts left)

$(a_1, a_2, \ldots) \mapsto (a_2, a_3, \ldots)$.

When $x \geq 0.5$, it is clear that $a_1 = 1$. We have

$$2 - 2x = \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right) - \left(1 + \frac{a_2}{2} + \frac{a_3}{2^2} + \cdots\right) = \frac{1-a_2}{2} + \frac{1-a_3}{2} + \cdots$$

which corresponds to sequence $(1 - a_2, 1 - a_3, \cdots)$ (sequence shifts left then subtracted from one-sequence).

Now, we also freely use the fact that if x is rational then the sequence eventually repeats itself, i.e. there exists $N, k \in \mathbb{N}$ such that $\forall\ n \geq N, a_{x,n} = a_{x,n+k}$. Otherwise, the sequence doesn't.

Let x be such a rational, then there exists fixed $N, k \in \mathbb{N}$ satisfying the conditions above.

Then, after $N-1$ applications of f, from the only 2 possible transitions above, we either get the sequence

$$A_1 = (a_N, a_{N+1}, \cdots)$$

or

$$A_2 = (1 - a_N, 1 - a_{N+1}, \cdots)$$

WLOG, suppose we get A_1. k steps later, we either get A_1 or A_2.

If we get A_1, that means we have entered a loop, and so values $f^{N-1+kj}(x)$ for all $j \in \mathbb{N}$ have already appeared as $f^{N-1+k}(x)$ in the orbit of x. Thus the orbit is finite $(\leq N - 1 + k)$.

If we get A_2, k steps later, we either get A_1 or A_2. It's trivial now that there is now a loop of either length k or length $2k$. So the orbit is finite. $\square$

(f) It is clear that the orbit of x is countable. If x is irrational, then we claim that its orbit of x is a denumerable (infinite countable) subset of $[0, 1]$.

Suppose not. Then the orbit is finite. Let N be the maximum index such that $f^N(x)$ is in the orbit of x. This implies that

$$f^{N+1}(x) = f^k(x)$$

for some $k \leq N$. Then $f^{N+2}(x) = f^{k+1}(x), \ldots, f^{2N+1-k}(x) = f^N(x), \ldots$

In short,

$$f^{k+(N+1-k)t}(x) = f^k(x) \tag{1}$$

for all $t \in \mathbb{N}$.

Let $f^k(x) = (b_{k+1}, b_{k+2}, \ldots)$ (b_i can either all be a_i or all be $1 - a_i$, this doesn't matter). Then by referring to the 2 possible transitions we have as represented above in part (e), after $2(N + 1 - k)t, t \in \mathbb{N}$ transitions (an even number), we must reach

$$f^{k+2(N+1-k)t}(x) = (b_{k+2(N+1-k)t+1}, b_{k+2(N+1-k)t+2}, \ldots)$$

or more specifically, not $(1 - b_{k+2(N+1-k)t+1}, 1 - b_{k+2(N+1-k)t+2}, \ldots)$, since the number of transitions is even.

Then (1) implies that

$$b_{k+1} = b_{k+2(N+1-k)t+1}, b_{k+1} = b_{k+2(N+1-k)t+1}, \cdots$$

which is to suggest that $\{b_n\}$ eventually repeats, with maximum period of $2(N + 1 - k)$.

This implies that x is rational. $\Rightarrow\Leftarrow$

So the orbit of x must be an infinite countable set (subset of $[0, 1]$) $\square$

Problem 2.9 (2.81* (EC))

The topologist's sine curve is the set

$$\{(x, y) \mid x = 0 \text{ and } |y| \leq 1 \text{ or } 0 < x < 1 \text{ and } y = \sin(\frac{1}{x})\}$$

(It is the union of a circular arc and the topologist's sine curve.) Prove that it is path-connected but not locally path-connected. (M is **locally path-connected** if for each $p \in M$ and each neighborhood U of p there is a path-connected subneighborhood V of p.)

Solution

$\square$