

# MATH 20800: Honors Analysis in Rn II

## Problem Set 3

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**Textbook:** Rudin, *Principles of Mathematical Analysis*

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### Problem 3.1 (8.11 done)

Suppose  $f$  Riemann integrable on  $[0, A]$  for all  $A < \infty$ , and  $f(x) \xrightarrow{x \rightarrow \infty} 1$ . Prove that

$$\lim_{t \rightarrow 0} t \int_0^{\infty} e^{-tx} f(x) dx = 1 \quad (t > 0)$$

### Solution

Fix  $\varepsilon > 0$ . Since  $f(x) \xrightarrow{x \rightarrow \infty} 1$ , there exists  $N > 0$  such that  $x \geq N \Rightarrow |f(x) - 1| < \varepsilon/2$ .

$f$  is then Riemann integrable on  $[0, N]$ , therefore exists  $M = \sup_{[0, N]} |f(x)| < \infty$ .

Denote  $D_t(x) = te^{-tx}$  then  $D_t(x) \geq 0$  and simple integration yields  $\int_0^{\infty} D_t(x) dx = 1 \forall t > 0$ , and  $\int_0^N D_t(x) dx = 1 - e^{-Nt}$

We can now bound:

$$\begin{aligned} \left| t \int_0^{\infty} e^{tx} f(x) dx - 1 \right| &= \left| \int_0^{\infty} D_t(x) f(x) dx - 1 \right| \\ &= \left| \int_0^{\infty} D_t(x) f(x) dx - \int_0^{\infty} D_t(x) dx \right| \\ &= \left| \int_0^{\infty} D_t(x) (f(x) - 1) dx \right| \\ &= \int_0^N D_t(x) |f(x) - 1| dx + \int_N^{\infty} D_t(x) |f(x) - 1| dx \\ &\leq \int_0^N D_t(x) (M + 1) dx + 1 \times \varepsilon/2 \\ &= (M + 1)(1 - e^{-Nt}) + \varepsilon/2 \xrightarrow{t \rightarrow 0} 0 \end{aligned}$$

since  $e^{-Nt} \xrightarrow{t \rightarrow 0} 1$ . □

### Problem 3.2 (8.12 done)

Suppose  $0 < \delta < \pi$ ,  $f(x) = 1$  if  $|x| < \delta$ ,  $f(x) = 0$  if  $\delta < |x| \leq \pi$ , and  $f$  is  $2\pi$ -periodic.

(a) Compute Fourier coefficients of  $f$ .

(b) Conclude that

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{n - \delta}{2}$$

(c) Deduce from Parseval's theorem that

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} = \frac{n - \delta}{2}$$

(d) Let  $\delta > 0$  and prove that

$$\int_0^{\infty} \left( \frac{\sin x}{x} \right)^2 dx = \frac{\pi}{2}$$

(e) Put  $\delta = \pi/2$  in (c), what do you get?

### Solution

(a) For  $n = 0$ :

$$\hat{f}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx = \frac{1}{2\pi} \int_{-\delta}^{\delta} 1 \times 1 dx = \delta/\pi$$

and for  $n \neq 0$ :

$$\begin{aligned} \hat{f}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\ &= \frac{1}{2\pi} \int_{-\delta}^{\delta} 1 e^{-inx} dx \\ &= \frac{1}{2\pi} \left[ \frac{e^{-inx}}{-in} \right]_{-\delta}^{\delta} \\ &= \frac{-1}{2\pi in} (e^{-in\delta} - e^{in\delta}) \\ &= \frac{\sin(n\delta)}{n\pi} \end{aligned}$$

(b)  $f$  is clearly locally Lipschitz at 0, since it is locally constant (on  $(-\delta, +\delta)$  with Lipschitz constant 1). It follows that

$$\lim_{N \rightarrow \infty} s_N(f, 0) = f(0)$$

We also realize that  $\hat{f}(n) = \hat{f}(-n)$ , since  $\sin$  is odd. It then follows that:

$$\begin{aligned}
1 &= f(0) \\
&= \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{0in} \\
&= \frac{\delta}{\pi} + 2 \sum_{n=1}^{\infty} \hat{f}(n) \\
\Rightarrow \frac{\pi - \delta}{2\pi} &= \sum_{n=1}^{\infty} \hat{f}(n) \\
&= \sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n\pi} \\
\Rightarrow \sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} &= \frac{\pi - \delta}{2}
\end{aligned}$$

as required.

(c)  $f$  is discontinuous on a zero set  $(\{-\delta, \delta\})$ , hence it is Riemann integrable. Therefore we can apply Parseval's Theorem:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx \\
\left(\frac{\delta}{\pi}\right)^2 + 2 \sum_{n=1}^{\infty} \left(\frac{\sin(n\delta)}{n\pi}\right)^2 &= \frac{1}{2\pi} \int_{-\delta}^{\delta} 1 dx = \frac{\delta}{\pi} \\
\Rightarrow \sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2} &= \frac{\pi\delta - \delta^2}{2} \\
\Rightarrow \sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} &= \frac{\pi - \delta}{2}
\end{aligned}$$

as required.

(d) Letting  $\delta \rightarrow 0$ , then

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} &= \sum_{n=1}^{\infty} \left(\frac{\sin(n\delta)}{n\delta}\right)^2 \delta \\
&\xrightarrow{\delta \rightarrow 0} \int_0^{\infty} \left(\frac{\sin(x)}{x}\right)^2
\end{aligned}$$

since each  $\sum_{n=1}^{\infty} \left(\frac{\sin(n\delta)}{n\delta}\right)^2 \delta$  is the Riemann sum of the integral.

Then  $RHS \xrightarrow{\delta \rightarrow 0} \pi/2$ , hence

$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 = \frac{\pi}{2}$$

(e) Putting  $\delta = \pi/2$  in (c). We have that  $\sin(n\frac{\pi}{2})^2 = 1$  when  $n$  odd, and  $= 0$  when  $n$  even, so

$$\sum_{n \text{ odd}} \frac{1}{n^2 \pi/2} = \frac{\pi}{4}$$

hence

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}$$

□

**Problem 3.3** (8.13 done)

Put  $f(x) = x$  if  $0 \leq x < 2\pi$ , and apply Parseval's theorem to conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

**Solution**

$f(x) = x$ , hence

$$\begin{aligned} \hat{f}(n) &= \frac{1}{2\pi} \int_0^{2\pi} x e^{-inx} dx \\ &= \frac{1}{2\pi} \left[ \frac{(inx + 1)e^{-inx}}{n^2} \right]_0^{2\pi} \\ &= \frac{1}{2\pi} \frac{2\pi i}{n} = \frac{i}{n} \end{aligned}$$

with the exception of  $n = 0$ :

$$\hat{f}(0) = \frac{1}{2\pi} \int_0^{2\pi} x dx = \pi$$

Also notice that  $|\hat{f}(n)|^2 = |\hat{f}(-n)|^2 = \frac{1}{n^2}$ , so

$$\begin{aligned} \pi + 2 \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} x^2 dx \\ &= \frac{4\pi^2}{3} \\ \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{\pi^2}{6} \end{aligned}$$

as required.

□

**Problem 3.4** (8.14 done)

If  $f(x) = (\pi - |x|)^2$  on  $[-\pi, \pi]$ , prove that

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos(nx)$$

and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

### Solution

$f(x) = (\pi - |x|)^2$  on  $[-\pi, \pi]$ . We have that

$$\begin{aligned} |f(x+t) - f(x)| &= ||x+t| - |x|| |2\pi - |x+t| - |x|| \\ &\leq |t| 2\pi \end{aligned}$$

so  $f$  is locally Lipschitz at all points with  $M = 2\pi$ , since locally  $x+t$  and  $x$  have the same sign.

Therefore, it follows that  $s_N(f, x) \xrightarrow{N \rightarrow \infty} f(x)$  on  $[-\pi, \pi]$ .

We then calculate the Fourier coefficients:

$$\begin{aligned} \hat{f}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - |x|)^2 e^{-inx} dx \\ &= \frac{1}{2\pi} \left( \int_{-\pi}^0 (\pi + x)^2 e^{-inx} dx + \int_0^{\pi} (\pi - x)^2 e^{-inx} dx \right) \\ &= \frac{2}{n^2} - \frac{2 \sin(\pi n)}{\pi n^3} = \frac{2}{n^2} \end{aligned}$$

with the exception of  $n = 0$ :

$$\hat{f}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - |x|)^2 dx = \frac{\pi^2}{3}$$

Therefore

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx} \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left( \frac{2}{n^2} e^{inx} + \frac{2}{n^2} e^{-inx} \right) \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2} 2 \cos(nx) \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos(nx) \end{aligned}$$

as required.

In particular, when  $x = 0$ :

$$\begin{aligned} f(0) &= (\pi - 0)^2 = \pi^2 \\ \Rightarrow \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} &= \pi^2 \\ \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} &= (\pi^2 - \pi^2/3)/4 = \frac{\pi^2}{6} \end{aligned}$$

as required.

$f$  is continuous, so it is Riemann integrable. We can therefore apply Parseval's:

$$\begin{aligned} \sum_{-\infty}^{\infty} |\hat{f}(n)|^2 &= \frac{\pi^4}{9} + 2 \sum_{n=1}^{\infty} \left( \frac{4}{n^4} \right) \\ \Rightarrow \frac{\pi^4}{9} + 2 \sum_{n=1}^{\infty} \left( \frac{4}{n^4} \right) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - |x|)^4 dx \\ &= \frac{\pi^4}{5} \\ \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} &= (\pi^4/5 - \pi^4/9)/8 = \frac{\pi^4}{90} \end{aligned}$$

as required. □

**Problem 3.5** (8.15 done)

With  $D_n(x) = \sum_{k=-n}^n e^{ikx}$ , put

$$K_N(x) = \frac{1}{N+1} \sum_{n=0}^N D_n(x)$$

Prove that

$$K_N(x) = \frac{1}{N+1} \frac{1 - \cos(N+1)x}{1 - \cos x}$$

and that

- (a)  $K_N \geq 0$
- (b)  $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x) dx = 1$ ,
- (c)  $K_N(x) \leq \frac{1}{N+1} \frac{2}{1 - \cos \delta}$  if  $0 < \delta \leq |x| \leq \pi$

If  $s_N$  is the  $N$ th partial sum of the Fourier series of  $f$ , consider the arithmetic means

$$\sigma_N = \frac{s_0 + \dots + s_N}{N+1}$$

Prove that

$$\sigma_N(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) K_N(t) dt$$

and hence prove Fejer's theorem: "If  $f$  is continuous,  $2\pi$ -periodic, then  $\sigma_N(f; x) \rightarrow f(x)$  uniformly on  $[-\pi, \pi]$ ."

### Solution

We first rewrite

$$D_n(x) = \sum_{k=-n}^n e^{ikx} = \frac{e^{i(n+1)x} - e^{-inx}}{e^{ix} - 1}$$

therefore

$$\begin{aligned} K_n(x) &= \frac{1}{N+1} \sum_{n=0}^N \frac{e^{i(n+1)x} - e^{-inx}}{e^{ix} - 1} \\ &= \frac{1}{N+1} \frac{1}{e^{ix} - 1} \left[ \sum_{n=0}^N (e^{ix})^{(n+1)} - \sum_{n=0}^N (e^{-ix})^n \right] \\ &= \frac{1}{N+1} \frac{1}{e^{ix} - 1} \left( \frac{e^{i(N+2)x} - e^{ix}}{e^{ix} - 1} - \frac{e^{-i(N+1)x} - 1}{e^{-ix} - 1} \right) \\ &= \frac{1}{N+1} \frac{1}{e^{ix} - 1} \frac{1}{2 - 2\cos x} (e^{ix} - 1)(2 - e^{-i(N+1)x} - e^{i(N+1)x}) \\ &= \frac{1}{N+1} \frac{2 - 2\cos(N+1)x}{2 - 2\cos x} \\ &= \frac{1}{N+1} \frac{1 - \cos(N+1)x}{1 - \cos x} \end{aligned}$$

as required.

(a)  $\cos(N+1)x, \cos x \leq 1 \Rightarrow K_n(x) \geq 0$ .

(b) We know that  $\frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(x) dx = 1$  hence

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x) dx = \frac{1}{N+1} \sum_{n=0}^N \int_{-\pi}^{\pi} D_n(x) dx = \frac{1}{N+1} (N+1) = 1$$

(c) If  $0 < \delta \leq |x| \leq \pi$  then

$$K_N(x) = \frac{1}{N+1} \frac{1 - \cos(N+1)x}{1 - \cos x} \leq \frac{1}{N+1} \frac{1 - (-1)}{1 - \cos \delta} = \frac{1}{N+1} \frac{2}{1 - \cos \delta}$$

We've therefore proven all properties of  $K_N(x)$ .

Now,

$$\begin{aligned}
\sigma_N(x) &= \frac{1}{N+1} \sum_{n=0}^N \\
&= \frac{1}{N+1} \sum_{n=0}^N \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_n(t) dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) \sum_{n=0}^N D_n(t) dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) K_N(t) dt
\end{aligned}$$

We are now ready to prove Fejer's Theorem: Suppose that  $f$  is continuous with period  $2\pi$ , then  $\sigma_N(f, x) \xrightarrow{N \rightarrow \infty} f(x)$  uniformly on  $[-\pi, \pi]$ .

$f$  is continuous on  $[-\pi, \pi]$  and is  $2\pi$ -periodic, hence there exists  $M \geq \|f\|$ .

For all  $\varepsilon > 0$ , since  $f$  is continuous on  $[-\pi, \pi]$  and  $2\pi$ -periodic, it is uniformly continuous, hence there exists  $\delta > 0$  such that

$$|u - v| < \delta \Rightarrow |fu - fv| < \varepsilon/2$$

Then, we have that

$$\begin{aligned}
|\sigma_N(f; x) - f(x)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) K_N(t) dt - f(x) \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(t) dt \right| \\
&= \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} K_N(t) (f(x-t) - f(x)) dt \right| \\
&\leq \frac{1}{2\pi} \left| \int_{\delta \leq |t|} K_N(t) (f(x-t) - f(x)) dt \right| \\
&\quad + \frac{1}{2\pi} \left| \int_{-\delta}^{\delta} K_N(t) (f(x-t) - f(x)) dt \right| \\
&\leq \frac{1}{2\pi} 2M \int_{\delta \leq |t|} K_N(t) dt + \frac{1}{2\pi} \int_{-\delta}^{\delta} K_N(t) \varepsilon/2 dt \\
&\leq \frac{1}{2\pi} 2M \frac{1}{N+1} \frac{2}{1 - \cos \delta} + \varepsilon/2
\end{aligned}$$

There then exists  $N_1$  big enough such that  $\frac{1}{2\pi} 2M \frac{1}{N_1+1} \frac{2}{1 - \cos \delta} < \varepsilon/2$ , then for  $N \geq N_1$ ,  $\|\sigma_N - f\| < \varepsilon$ , hence the convergence is uniform.  $\square$

### Problem 3.6 (8.16 done)

Prove a pointwise version of Fejer's theorem: If  $f$  Riemann integrable and  $f(x+), f(x-)$  exist for some  $x$ , then

$$\lim_{N \rightarrow \infty} \sigma_N(f; x) = \frac{1}{2} [f(x+) + f(x-)]$$



### Solution

Note that  $K_n(x) = \frac{1}{N+1} \frac{1-\cos((N+1)x)}{1-\cos x}$  is even. Therefore  $\int_{-\pi}^0 K_N(t)dt = \int_0^{\pi} K_N(t)dt = \frac{1}{2}$ .

Since there exists  $f(x+), f(x-)$ , there exists  $\delta_1, \delta_2 > 0$  such that  $0 < t < \delta_1 \Rightarrow |f(x+t) - f(x+)| < \varepsilon/2$  and  $0 < t < \delta_2 \Rightarrow |f(x-t) - f(x-)| < \varepsilon/2$ . Take  $\delta = \min\{\delta_1, \delta_2\}$ .

Then,

$$\begin{aligned} & \left| \sigma_N(f; x) - \frac{1}{2}(f(x+) + f(x-)) \right| \\ &= \frac{1}{2\pi} \left| \int_{-\pi}^0 f(x-t)K_N(t)dt - f(x+) \int_{-\pi}^0 K_N(t)dt + \int_0^{\pi} f(x-t)K_N(t)dt - f(x-) \int_0^{\pi} K_N(t)dt \right| \\ &= \frac{1}{2\pi} \left| \int_{-\pi}^0 (f(x-t) - f(x+))K_N(t)dt + \int_0^{\pi} (f(x-t) - f(x-))K_N(t)dt \right| \\ &\leq \frac{1}{2\pi} \left| \int_{-\pi}^{-\delta} (f(x-t) - f(x+))K_N(t)dt \right| + \frac{1}{2\pi} \left| \int_{-\delta}^0 (f(x-t) - f(x+))K_N(t)dt \right| \\ &\quad + \frac{1}{2\pi} \left| \int_0^{\delta} (f(x-t) - f(x-))K_N(t)dt \right| + \frac{1}{2\pi} \left| \int_{\delta}^{\pi} (f(x-t) - f(x-))K_N(t)dt \right| \\ &\leq \frac{1}{2\pi} \left( 2M\pi \frac{1}{N+1} \frac{2}{1-\cos \delta} + \pi\varepsilon/2 + \pi\varepsilon/2 + 2M\pi \frac{1}{N+1} \frac{2}{1-\cos \delta} \right) < \varepsilon \end{aligned}$$

for  $N$  sufficiently large. The pointwise convergence is thus proven.  $\square$

### Problem 3.7 (8.17)

Assume  $f$  is bounded and monotonic on  $[-\pi, \pi)$ , with Fourier coefficients  $c_n$ .

- (a) Use Exercise 6.17 to prove that  $\{nc_n\}$  is bounded.
- (b) Combine (a) with Exercise 3.14(e) to conclude that

$$\lim_{N \rightarrow \infty} s_N(f; x) = \frac{1}{2}[f(x+) + f(x-)]$$

for every  $x$ .

- (c) Assume only that  $f$  Riemann integrable on  $[-\pi, \pi]$  and that  $f$  is monotonic in some segment  $(\alpha, \beta) \subset [-\pi, \pi]$ . Prove that the conclusion of (b) holds for every  $x \in (\alpha, \beta)$ . (This is an application of the localization theorem.)

### Solution

(a) Suppose  $\|f\| < M$ . Then we have

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx}dx \\ &= \frac{1}{2\pi} \left( e^{-in\pi}/(-in) - e^{-in(-\pi)}/(-in) - \int_{-\pi}^{\pi} e^{-inx}/(-in)df \right) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx}/(-in)df \end{aligned}$$

so

$$|nc_n| = \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} e^{-inx} df \right| = \frac{1}{2\pi} |f(\pi)e^{-in\pi} - f(-\pi)e^{in\pi}| \leq M/\pi$$

is therefore bounded.

(b) The pointwise version of Fejer's Theorem implies that for all  $x$  where the right and left limits exist,

$$\lim_{N \rightarrow \infty} \sigma_N(f; x) = \frac{1}{2}[f(x+) + f(x-)]$$

Using Exercise 3.14(a), since  $|nc_n|$  is bounded, it follows that  $\lim_{N \rightarrow \infty} s_N(f; x) = \lim_{N \rightarrow \infty} \sigma_N(f; x) = \frac{1}{2}[f(x+) + f(x-)]$

(c) Define  $g$  such that it agrees with  $f$  on  $(a, b)$ :  $g(x) = f(x)$  on  $(\alpha, \beta)$ , and  $g(x) = f(\alpha)$  for  $x \leq \alpha$  and  $g(x) = f(\beta)$  for  $x \geq \beta$ . Then,  $g$  is clearly monotonic in  $[-\pi, \pi]$ , so

$$\lim_{N \rightarrow \infty} S_N(g; x) = \frac{1}{2}[g(x+) + g(x-)]$$

But on  $(\alpha, \beta)$ ,  $g(x+) = f(x+)$ ,  $g(x-) = f(x-)$  so on  $(\alpha, \beta)$ :

$$\lim_{N \rightarrow \infty} S_N(g; x) = \frac{1}{2}[f(x+) + f(x-)]$$

$f$  and  $g$  agree on  $(\alpha, \beta)$ , so by the localization theorem, it implies that

$$\lim_{N \rightarrow \infty} S_N(f; x) - S_N(g; x) = 0$$

Therefore on  $(\alpha, \beta)$ ,  $\lim_{N \rightarrow \infty} S_N(f; x) = \frac{1}{2}[f(x+) + f(x-)]$ . □

### Problem 3.8 (8.19 done)

Suppose that  $f$  is a continuous,  $2\pi$ -periodic, real-valued function and some  $\alpha$  such that  $\alpha/\pi$  is irrational. Prove that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(x + n\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$$

### Solution

We first prove that the proposition is true for  $e^{ikx}$  for any  $k \in \mathbb{Z}$ .

If  $k = 0$  then

$$\text{RHS} = 1$$

$$\text{LHS} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N 1 = 1$$

else

$$\begin{aligned}\text{RHS} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} dx \\ &= \left[ \frac{e^{ikx}}{ik} \right]_{-\pi}^{\pi} = 0\end{aligned}$$

and

$$\begin{aligned}\text{LHS} &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{ik(x+n\alpha)} \\ &= e^{ikx+ik\alpha} \lim_{N \rightarrow \infty} \frac{e^{ikN\alpha} - 1}{N(e^{ik\alpha} - 1)} = 0\end{aligned}$$

Note that this computation does not into problem for all  $k \in \mathbb{Z} \setminus \{0\}$  (denominator being 0), since  $\alpha/\pi$  is irrational so  $e^{ik\alpha} \neq 1$  for all  $k \in \mathbb{Z} \setminus \{0\}$ .

Thus, indeed the proposition is true for  $e^{ikx}$  for all  $k \in \mathbb{Z}$ . A trigonometric polynomial is a linear combination of  $e^{ikx}$  terms, and the proposition is linear in  $f$ , so the proposition is true for all trigonometric polynomial too.

Since  $f$  is continuous and  $2\pi$ -periodic, given  $\varepsilon > 0$ , by Stone-Weierstrass, there exists some trigonometric polynomial  $P$  such that

$$\|f - P\|_{\infty} < \varepsilon/2$$

Note that per our remarks above,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N P(x + n\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(t) dt$$

It then follows that

$$\begin{aligned}& \left| \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(x + n\alpha) - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt \right| \leq \\ & \left| \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(x + n\alpha) - \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N P(x + n\alpha) \right| + \left| \frac{1}{2\pi} - \int_{-\pi}^{\pi} P(t) dt \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt \right| \\ & \leq \left( \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N |f(x + n\alpha) - P(x + n\alpha)| \right) + \frac{1}{2\pi} \int_{-\pi}^{\pi} |P(t) - f(t)| dt \\ & < \left( \lim_{N \rightarrow \infty} \sum_{n=1}^N N\varepsilon/2 \right) + \frac{1}{2\pi} 2\pi\varepsilon/2 \\ & = \varepsilon\end{aligned}$$

This is true for all  $\varepsilon$ , hence the proposition holds for any continuous,  $2\pi$ -periodic  $f$ .  $\square$

**Problem 3.9** (8.21 done)

Let

$$L_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| dt$$

Prove that there exists  $C > 0$  such that

$$L_N > C \log n$$

or, more precisely, that the sequence

$$\left\{ L_n - \frac{4}{\pi^2} \log n \right\}$$

is bounded.

**Solution**

We first show the lower bound:

$$\begin{aligned} L_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| dt \\ &= \frac{1}{\pi} \int_0^{\pi} \left| \frac{\sin(n + \frac{1}{2})t}{\sin t/2} \right| dt \\ &= \frac{2}{\pi} \int_0^{\pi/2} \frac{|\sin(n + \frac{1}{2})(2t)|}{\sin t} dt \\ &> \frac{2}{\pi} \int_0^{\pi/2} \frac{|\sin(n + \frac{1}{2})(2t)|}{t} dt \\ &= \frac{2}{\pi} \int_0^{(n+\frac{1}{2})\pi} \frac{|\sin u|}{u} du \\ &> \frac{2}{\pi} \int_0^{n\pi} \frac{|\sin u|}{u} du \\ &> \frac{2}{\pi} \sum_{k=0}^{n-1} \int_0^{\pi} \frac{\sin u}{(k+1)\pi} du \\ &= \frac{2}{\pi^2} \left( \sum_{k=1}^n \frac{1}{k} \right) [-\cos u]_0^{\pi} \\ &= \frac{4}{\pi^2} \left( \sum_{k=1}^n \frac{1}{k} \right) \geq \frac{4}{\pi^2} \log n \end{aligned}$$

then the upper bound, where we first use a preliminary bound:

$$\begin{aligned} \left| \frac{\sin(2n+1)t}{\sin t} \right| &= \left| \frac{\sin(2nt) \cos t + \cos(2nt) \sin t}{\sin t} \right| \\ &= \left| \frac{\sin(2nt)}{\tan t} + \cos(2nt) \right| \\ &\leq \left| \frac{\sin(2nt)}{\tan t} \right| + 1 \end{aligned}$$

so

$$\begin{aligned}
L_n &= \frac{2}{\pi} \int_0^{\pi/2} \left| \frac{\sin(2n+1)t}{\sin t} \right| dt \\
&\leq \frac{2}{\pi} \int_0^{\pi/2} \left| \frac{\sin(2nt)}{\tan t} \right| + 1 dt \\
&= 1 + \frac{2}{\pi} \int_0^{\pi/2} \left| \frac{\sin(2nt)}{\tan t} \right| dt \\
&= 1 + \frac{2}{\pi} \int_0^{n\pi} \frac{|\sin u|}{u} du \\
&= 1 + \frac{2}{\pi} \int_0^\pi \sum_{k=0}^{n-1} \frac{\sin u}{u + k\pi} du \\
&= 1 + \frac{2}{\pi} \int_0^\pi \frac{\sin u}{u} + \frac{2}{\pi} \int_0^\pi \sum_{k=1}^{n-1} \frac{\sin u}{u + k\pi} du \\
&< C + \frac{2}{\pi} \int_0^\pi \sum_{k=1}^{n-1} \frac{\sin u}{u + k\pi} du \\
&= C + \frac{2}{\pi} \int_0^\pi \sum_{k=1}^{n-1} \frac{\sin u}{k\pi} du \\
&= C + \frac{2}{\pi} \left( \sum_{k=1}^{n-1} \frac{1}{k} \right) [-\cos u]_0^\pi \\
&< C + \frac{4}{\pi} (\log n + \gamma) = C + \frac{4}{\pi} \log n
\end{aligned}$$

It follows that  $\{L_n - \frac{4}{\pi^2} \log n\}$  is bounded as required. □