MATH 20800: Honors Analysis in Rn II

Problem Set 1

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Problem 1.1 (5.57 done)

Show that $\mathrm{d} : \Omega^k \to \Omega^{k+1}$ is a linear vector space homomorphism.

Solution

Let $\alpha, \beta \in \Omega^k; c \in \mathbb{R}$. WLOG, $\alpha = f\mathrm{d}x_I, \beta = g\mathrm{d}x_J$ where I, J are increasing $k-$tuples. The linearity of d for simple forms trivially implies linearity for general forms.f

To show that d is a linear transformation, WTS $\mathrm{d}(\alpha + c\beta) = \mathrm{d}\alpha + c\mathrm{d}\beta$.

If I, J are the same tuple, then:

$$
\begin{aligned}
\mathrm{d}(\alpha + c\beta) &= \mathrm{d}((f + cg)\mathrm{d}x_I) \\
&= \mathrm{d}(f + cg) \wedge \mathrm{d}x_I \\
&= \left(\sum_{i=1}^{n} \frac{\partial (f + cg)}{\partial x_i}(x)\mathrm{d}x_i \right) \wedge \mathrm{d}x_I \\
&= \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)\mathrm{d}x_i + \sum_{i=1}^{n} c\frac{\partial g}{\partial x_i}(x)\mathrm{d}x_i \right) \wedge \mathrm{d}x_I \\
&= \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)\mathrm{d}x_i \right) \wedge dx_I + \left(\sum_{i=1}^{n} c\frac{\partial g}{\partial x_i}(x)\mathrm{d}x_i \right) \wedge \mathrm{d}x_I \quad \text{(wedge product distributes)} \\
&= \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)\mathrm{d}x_i \right) \wedge dx_I + c\left(\sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(x)\mathrm{d}x_i \right) \wedge \mathrm{d}x_I \\
&= \mathrm{d}f \wedge \mathrm{d}x_I + c\mathrm{d}g \wedge \mathrm{d}x_I \\
&= \mathrm{d}\alpha + c\mathrm{d}\beta
\end{aligned}
$$

Otherwise,

$$\begin{aligned}
\mathrm{d}(\alpha + c\beta) &= \mathrm{d}(f\mathrm{d}x_I + cg\mathrm{d}x_J) \\
&= \mathrm{d}f \wedge \mathrm{d}x_I + \mathrm{d}(cg) \wedge \mathrm{d}x_J \\
&= \mathrm{d}\alpha + \left(\sum_{i=1}^{n} \frac{\partial(cg)}{\partial x_i}(x)\mathrm{d}x_i \right) \wedge \mathrm{d}x_J \\
&= \mathrm{d}\alpha + \left(\sum_{i=1}^{n} c\frac{\partial g}{\partial x_i}(x)\mathrm{d}x_i \right) \wedge \mathrm{d}x_J \\
&= \mathrm{d}\alpha + c\left(\sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(x)\mathrm{d}x_i \right) \wedge \mathrm{d}x_J \\
&= \mathrm{d}\alpha + c\mathrm{d}g \wedge \mathrm{d}x_J \\
&= \mathrm{d}\alpha + c\mathrm{d}\beta
\end{aligned}$$

□

Problem 1.2 (5.61 done)

Assume $\mathrm{d}^2 f = 0$ for all smooth functions f, and prove that $\mathrm{d}^2\omega = 0$ for all smooth k-forms ω.

Solution

Let ω be any smooth $k-$form. If $k = 0$, then ω is a smooth function so $\mathrm{d}^2\omega = 0$ by assumption. Otherwise, WLOG, let $\omega = f\mathrm{d}x_I$ for some increasing k-tuple I. Then

$$\begin{aligned}
\mathrm{d}^2\lambda = \mathrm{d}(\mathrm{d}\lambda) = \mathrm{d}(\mathrm{d}f \wedge \mathrm{d}x_I) \\
&= \mathrm{d}^2 f \wedge \mathrm{d}x_I + (-1)^{l+1}\mathrm{d}f \wedge \mathrm{d}(\mathrm{d}x_I) \\
&= (-1)^{l+1}\mathrm{d}f \wedge \mathrm{d}(\mathrm{d}x_I)
\end{aligned}$$

We compute

$$\mathrm{d}(\mathrm{d}x_I) = \left(\sum_{i=1}^{n} \frac{\partial(1)}{\partial x_i}(x)\mathrm{d}x_i \right) \wedge \mathrm{d}x_I = 0$$

Therefore $\mathrm{d}^2\lambda = (-1)^{l+1}\mathrm{d}f \wedge 0 = 0$ as required. □

Problem 1.3 (5.62 done)

Does there exist a continuous mapping from the circle to itself that has no fixed-point? What about the 2-torus? The 2-sphere?

Solution

Yes. We note that there exists a center for each of the 3 shapes. Taking each point to its reflection across that center is a continuous mapping, and there is no fixed point for that mapping. □

Problem 1.4 (5.63 done)

Show that a smooth map $T : U \to V$ induces a linear map of cohomology groups

$H^k(V) \to H^k(U)$ defined by

$$T^* : [\omega] \mapsto [T^*\omega]$$

Here, $[\omega]$ denotes the equivalence class of $\omega \in Z^k(V)$ in $H^k(V)$. The question amounts to showing that the pullback of a closed form ω is closed and that its cohomology class depends only on the cohomology class of ω.

Solution

1. WTS if $\omega \in Z^k(V)$ then $T^*\omega \in Z^k(U)$, so that the mapping T^* is indeed $H^k(V) \to H^k(U)$ and thus the cohomology class $[T^*\omega]$ is well-defined.

Let $\omega \in Z^k(V)$, which means $\mathrm{d}\omega = 0$.

Then $\mathrm{d}T^*\omega = T^*\mathrm{d}\omega = T^*0 = 0$ so $\mathrm{d}T^*\omega \in Z^k(U)$ as required.

2. WTS if $\omega_1 \in [\omega]$ then $T^*\omega_1 \in [T^*\omega]$, i.e., that the mapping between the equivalence classes is independent of representative.

Since $\omega_1 \in [\omega] \in H^k(V) = B^k(V)/Z_k(V)$, there exists $\lambda \in \Omega^{k-1}(V)$ such that

$$\omega_1 = \omega + \mathrm{d}\lambda$$

Then

$$\begin{aligned} T^*\omega_1 &= T^*(\omega + \mathrm{d}\lambda) \\ &= T^*\omega + T^*\mathrm{d}\lambda \\ &= T^*\omega + dT^*\lambda \end{aligned}$$

But $T^*\omega \in Z^k(U), \mathrm{d}T^*\lambda \in B^k(U)$ so $T^*\omega + \mathrm{d}T^*\lambda \in [T^*\omega] \in H^k(U)$. Thus $T^*\omega_1 \in [T^*\omega]$ as required.

The linearity of T^* on cohomology classes is trivial from the linearity of T^* on forms. $\square$

Problem 1.5 (Problem 1 done)

If ω and λ are k- and m-forms respectively, prove that

$$\omega \wedge \lambda = (-1)^{km} \lambda \wedge \omega$$

Solution

WLOG, let $\omega = f\mathrm{d}x_I, \lambda = g\mathrm{d}x_J$ where I and J are k- and m-tuples respectively.

Then

$$\begin{aligned} \omega \wedge \lambda &= (f\mathrm{d}x_I) \wedge (g\mathrm{d}x_J) \\ &= (fg)\mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \ldots \wedge \mathrm{d}x_{i_k} \wedge \mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \ldots \wedge \mathrm{d}x_{j_m} \\ &= (fg)(-1)^m \mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \ldots \wedge \mathrm{d}x_{i_k - 1} \wedge \mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \ldots \wedge \mathrm{d}x_{j_m} \wedge \mathrm{d}x_{i_k} \\ &\text{(perform the same “movement” } (k-1) \text{ more times)} \\ &= (fg)(-1)^{km} \mathrm{d}x_J \wedge \mathrm{d}x_I \\ &= (-1)^{km}(g\mathrm{d}x_J) \wedge (f\mathrm{d}x_I) = (-1)^{km} \lambda \wedge \omega \end{aligned}$$

The linearity of wedge product promotes the result for simple forms to general forms. $\square$

Problem 1.6 (Problem 2 done)

Consider the 1-form $\eta = \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2+y^2}$ on $\mathbb{R}^2 \backslash \{0\}$.

(a) Prove that $\mathrm{d}\eta = 0$.

(b) Let $\gamma = (r\cos t, r\sin t)$ for some $r > 0$, and let Γ be C^1-curve in $\mathbb{R}^2\backslash\{0\}$ with parameter interval $[0, 2\pi]$ and $\Gamma(0) = \Gamma(2\pi)$, such that the intervals $[\gamma(t), \Gamma(t)]$ do not contain $(0, 0)$ for any $t \in [0, 2\pi]$.

Prove that
$$\int_\Gamma \eta = 2\pi$$

(Hint: For $t \in [0, 2\pi], u \in [0, 1]$, define $\Phi(t, u) = (1-u)\Gamma(t) + u\gamma(t)$. Then Φ is a 2-surface in $\mathbb{R}^2\backslash\{0\}$ with domain $[0, 2\pi] \times [0, 1]$. Show $\partial\Phi = \Gamma - \gamma$. Deduce that $\int_\Gamma \eta = \int_\gamma \eta$ and compute $\int_\gamma \eta$.)

Solution

(a) Let $f(x, y) = \frac{-y}{x^2+y^2}, g(x, y) = \frac{x}{x^2+y^2}$ then $\eta = f\mathrm{d}x + g\mathrm{d}y$.

Then $f_y = g_x = \frac{y^2-x^2}{(x^2+y^2)^2}$, so

$$\begin{aligned}
\mathrm{d}\eta &= \mathrm{d}(f\mathrm{d}x + g\mathrm{d}y) \\
&= \mathrm{d}f \wedge \mathrm{d}x + \mathrm{d}g \wedge \mathrm{d}y \\
&= (f_x\mathrm{d}x + f_y\mathrm{d}y) \wedge \mathrm{d}x + (g_x\mathrm{d}x + g_y\mathrm{d}y) \wedge \mathrm{d}y \\
&= f_y\mathrm{d}y \wedge \mathrm{d}x + g_x\mathrm{d}x \wedge \mathrm{d}y \\
&= (g_x - f_y)\mathrm{d}x \wedge \mathrm{d}y = 0 \quad \square
\end{aligned}$$

(b) $\partial\Phi$ is the 1-surface:

$$\begin{aligned}
\partial\Phi(x) &= \delta^1\Phi(x) - \delta^2\Phi(x) \\
&= ((1-x)\Gamma(2\pi) + x\gamma(2\pi)) - ((1-x)\Gamma(0) + x\gamma(0)) \\
&- (\gamma(x) - \Gamma(x)) \\
&= (1-x)(\Gamma(2\pi) - \Gamma(0)) + x(\gamma(2\pi) - \gamma(0)) + \Gamma(x) - \gamma(x) \\
&= \Gamma(x) - \gamma(x)
\end{aligned}$$

By Stokes' Theorem, since $\mathrm{d}\eta = 0$, we therefore get:

$$0 = \int_\Phi \mathrm{d}\eta = \int_{\partial\Phi} \eta = \int_{\Gamma-\gamma} \eta = \int_\Gamma \eta - \int_\gamma \eta$$

which implies

$$\int_\Gamma \eta = \int_\gamma \eta$$

We can compute:

$$\begin{aligned}\int_\gamma \eta &= \int_0^{2\pi} \frac{-r\cos t}{r^2}(-r\sin t) + \frac{r\sin t}{r^2}(r\cos t)dt \\ &= \int_0^{2\pi} 2\sin t\cos t dt \\ &= \int_0^{2\pi} \sin(2t)dt = \left[\frac{-\cos(2t)}{2}\right]_0^{2\pi} = 0\end{aligned}$$

Therefore

$$\int_\Gamma \eta = \int_\gamma \eta = 0$$

as required. □

Problem 1.7 (Problem 3)

Define ζ on $\mathbb{R}^3\backslash\{0\}$ by

$$\zeta = \frac{x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y}{r^3} \quad (r = (x^2+y^2+z^2)^{1/2}),$$

Let $D = [0,\pi] \times [0,2\pi]$ and let Σ be the 2-surface in $\mathbb{R}^3$ defined on D given by

$$x = \sin u\cos v, y = \sin u\sin v, z = \cos u \quad (u \in [0,\pi], v \in [0,2\pi])$$

(a) Prove $\mathrm{d}\zeta = 0$ in $\mathbb{R}^3\backslash\{0\}$

(b) Let S denote the restriction of Σ to $E \subset D$. Prove

$$\int_S \zeta = \int_E \sin u du dv$$

(c) Suppose g, h_1, h_2, h_3 are C^2-functions on $[0,1]$ and let $(x,y,z) = \Phi(s,t)$ be the 2-surface $x = g(t)h_1(s), y = g(t)h_2(s), z = g(t)h_3(s)$. Prove, using the definition of forms, that

$$\int_\Phi \zeta = 0$$

Solution

(a) We note that all basic 3-forms with overlapping indices are 0, so the only relevant basic 3-forms in this computation are those in which all 3 indices appear. Preliminarily,

$$\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{y^2+z^2-2x^2}{r^2}$$

and similarly for y, z. Then,

$$
\begin{aligned}
\mathrm{d}\zeta &= \frac{y^2+z^2-2x^2}{r^2}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + \frac{z^2+x^2-2y^2}{r^2}\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}x \\
&+ \frac{x^2+y^2-2z^2}{r^2}\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= \left(\frac{y^2+z^2-2x^2}{r^2} + \frac{z^2+x^2-2y^2}{r^2} + \frac{x^2+y^2-2z^2}{r^2}\right)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&= 0 \quad \square
\end{aligned}
$$

(b) Since $x = \sin u \cos v, y = \sin u \sin v, z = \cos u$, we can preliminarily compute:

$$
\begin{aligned}
\frac{\partial \Sigma_{(2,3)}}{\partial(u,v)} &= \det \begin{bmatrix} \cos u \sin v & \sin u \cos v \\ -\sin u & 0 \end{bmatrix} = \sin^2 u \cos v \\
\frac{\partial \Sigma_{(3,1)}}{\partial(u,v)} &= \det \begin{bmatrix} -\sin u & 0 \\ \cos u \cos v & -\sin u \sin v \end{bmatrix} = \sin^2 u \sin v \\
\frac{\partial \Sigma_{(1,2)}}{\partial(u,v)} &= \det \begin{bmatrix} \cos u \cos v & -\sin u \sin v \\ \cos u \sin v & \sin u \cos v \end{bmatrix} = \sin u \cos u (\cos^2 v + \sin^2 v) = \sin u \cos u
\end{aligned}
$$

therefore

$$
\begin{aligned}
\int_S \zeta &= \int_S \frac{x}{r^3}\mathrm{d}y \wedge \mathrm{d}z + \frac{y}{r^3}\mathrm{d}z \wedge \mathrm{d}x + \frac{z}{r^3}\mathrm{d}x \wedge \mathrm{d}y \\
&= \int_E \left(\sin u \cos v \frac{\partial \Sigma_{(2,3)}}{\partial(u,v)} + \sin u \sin v \frac{\partial \Sigma_{(3,1)}}{\partial(u,v)} + \cos u \frac{\partial \Sigma_{(1,2)}}{\partial(u,v)}\right)\mathrm{d}u\mathrm{d}v \\
&= \int_E \big(\sin u \cos v \sin^2 u \cos v + \sin u \sin v \sin^2 u \sin v + \cos u \sin u \cos v\big)\,\mathrm{d}u\mathrm{d}v \\
&= \int_E (\sin^3 u(\cos^2 v + \sin^2 v) + \cos^2 u \sin u)\mathrm{d}u\mathrm{d}v \\
&= \int_E (\sin u(\sin^2 u + \cos^2 u))\mathrm{d}u\mathrm{d}v \\
&= \int_E \sin u \mathrm{d}u\mathrm{d}v
\end{aligned}
$$

as required. $\square$

(c) Restate that Φ is the 2-surface in $\mathbb{R}^3$, mapping $(s,t) \mapsto (g(t)h_1(s), g(t)h_2(s), g(t)h_3(s))$. First computing the Jacobians (abuse of notation: suppressing the arguments for conciseness):

$$\frac{\partial\Phi_{(2,3)}}{\partial(u,v)} = \det\begin{bmatrix} gh_2' & g'h_2 \\ gh_3' & g'h_3 \end{bmatrix} = gg'(h_3h_2' - h_2h_3')$$

$$\frac{\partial\Phi_{(3,1)}}{\partial(u,v)} = \det\begin{bmatrix} gh_3' & g'h_3 \\ gh_1' & g'h_1 \end{bmatrix} = gg'(h_1h_3' - h_3h_1')$$

$$\frac{\partial\Phi_{(1,2)}}{\partial(u,v)} = \det\begin{bmatrix} gh_1' & g'h_1 \\ gh_2' & g'h_2 \end{bmatrix} = gg'(h_2h_1' - h_1h_2')$$

$$r^3 = (g^2(h_1^2 + h_2^2 + h_3^2))^{3/2}$$

therefore

$$\begin{aligned}
\int_\Phi \zeta &= \int_{I^2} \frac{1}{(g^2(h_1^2 + h_2^2 + h_3^2))^{3/2}} \left(gh_1 \frac{\partial\Phi_{(2,3)}}{\partial(u,v)} + gh_2 \frac{\partial\Phi_{(3,1)}}{\partial(u,v)} + gh_3 \frac{\partial\Phi_{(1,2)}}{\partial(u,v)} \right) \mathrm{d}u\mathrm{d}v \\
&= \int_{I^2} \frac{gh_1gg'(h_3h_2' - h_2h_3') + gh_2gg'(h_1h_3' - h_3h_1') + gh_3gg'(h_2h_1' - h_1h_2')}{(g^2(h_1^2 + h_2^2 + h_3^2))^{3/2}} \mathrm{d}u\mathrm{d}v \\
&= \int_{I^2} \frac{g^2g'(h_1h_3h_2' - h_1h_2h_3' + h_2h_1h_3' - h_2h_3h_1' + h_3h_2h_1' - h_3h_1h_2')}{(g^2(h_1^2 + h_2^2 + h_3^2))^{3/2}} \mathrm{d}u\mathrm{d}v \\
&= \int_{I^2} 0 \mathrm{d}u\mathrm{d}v = 0
\end{aligned}$$

□