Math 20250: Abstract Linear Algebra
Problem Set 6

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Textbook: Linear Algebra by Hoffman and Kunze (2nd Edition)

Problem 6.1 (Sec 6.5. Problem 2)

Let $\mathbb{F}$ be a commuting family of 3×3 complex matrices. How many linearly independent matrices can $\mathbb{F}$ contain? What about the $n \times n$ case?
Instructor's note: Only have to consider case $n = 3$.

Solution

□

Problem 6.2 (Sec 6.5. Problem 3)

Let T be a linear operator on an $n-$dimensional space, and suppose that T has n distinct characteristic values. Prove that any linear operator which commutes with T is a polynomial in T.

Solution

Since T has n distinct eigenvalues, T is diagonalizable. Then there exists an eigenbasis $\mathcal{B}$ s.t. $A = [T]_{\mathcal{B}}$ is a diagonal matrix with entries $A_{ii} = \lambda_i$ where $\lambda_i \neq \lambda_j$ if $i \neq j$, and $A_{ij} = 0$ if $i \neq j$.
Let T' be a linear operator that commutes with T, and let $A' = [T']_{\mathcal{B}}$, then the commutativity implies:

$$AA' = A'A \Rightarrow (AA')_{ij} = (A'A)_{ij}$$

Observe that:

$$\begin{aligned}(AA')_{ij} &= A'_{ij}\lambda_j \\ (A'A)_{ij} &= A'_{ij}\lambda_i \\ \Rightarrow A'_{ij}(\lambda_i - \lambda_j) &= 0 \; \forall \, i, j\end{aligned}$$

When $i \neq j$, $\lambda_i \neq \lambda_j \Rightarrow, A'_{ij} = 0$. When otherwise, the expression is trivial. Therefore, A' is diagonal.
We claim that we can find a polynomial f in T s.t. $f(T) = A'$, as there trivially exists a polynomial g of degree less than or equal to n s.t.

$$g(A_{ii}) = A'_{ii} \; \forall \, 1 \leq i \leq n$$

We then construct f with the same coefficients as g, which then implies $f(A) = A'$, since raising powers and scaling a diagonal matrix are accomplished by raising powers and scaling the diagonal entries themselves.

□

Problem 6.3 (Sec 6.5. Problem 4)

Let A, B, C, D be $n \times n$ complex matrices which commute. Let E be the $2n \times 2n$ matrix

$$E = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Prove that $\det E = \det(AD - BC)$

Instructor's note: We can assume that A is invertible, and A, B, C, D are all diagonalizable.

Solution

Observe that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & A(-A^{-1}B) + B \\ C & -C(A^{-1}B) + D \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & -CA^{-1}B + D \end{bmatrix}$$

It follows that

$$\begin{aligned} \det E \begin{vmatrix} I & -A^{-1}B \\ 0 & I \end{vmatrix} &= \begin{vmatrix} A & 0 \\ C & -CA^{-1}B + D \end{vmatrix} \\ \Rightarrow \det E &= \det A \det(-CA^{-1}B + D) \\ &= \det(-ACA^{-1}B + AD) \end{aligned}$$

But since A, B, C, D commute:

$$\begin{aligned} \det(-ACA^{-1}B + AD) &= \det(-CAA^{-1}B + AD) \\ &= \det(-CB + AD) \\ &= \det(AD - BC) \end{aligned}$$

□