MATH 26200: Point-Set Topology
Problem Set 5

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16 Feb 2024

Problem 5.1 (Problem 1 done)

Let X be a set. Let $\mathcal{C}$ be a nonempty family of subsets of X with the finite intersection property. Prove that $\mathcal{C}$ is contained in some filter, and (using Zorn's lemma) that every filter is contained in some maximal filter. Finally, prove that a filter is maximal if and only if it is an ultrafilter. (Please write out full proofs of these facts, just from the definitions of filter and ultrafilter).

Solution

1. WTS $\mathcal{C}$ is in some filter. $\mathcal{C}$ is nonempty family of subsets of X with fip. We use $\mathcal{C}$ to generate a filter as follows:

$$\mathcal{F} = \{B \subset X \mid \exists\, C_1, \dots, C_n \text{ such that } \bigcap_{k\in[n]} C_k \subset B\}$$

Clearly, $\mathcal{F}$ is a family of subsets of X. It remains to show that $\mathcal{F}$ satifies the conditions of a filter.

- $\varnothing \notin \mathcal{F}$, since $\mathcal{C}$ has fip, so $\bigcap_{k\in[n]} C_k \neq \varnothing \Rightarrow \not\subset \varnothing$.
- $X \in \mathcal{F}$, since $C_1 \subset X$.
- If $B_1, B_2 \in \mathcal{F}$, i.e., $B_1 \supset \bigcap_{k\in[n]} C_k^{(1)}, B_2 \supset \bigcap_{j\in[m]} C_j^{(2)}$ then

  $$B_1 \cap B_2 \supset \bigcap_{k\in[n]} C_k^{(1)} \cap \bigcap_{j\in[m]} C_j^{(2)}$$

  so $B_1 \cap B_2 \in \mathcal{F}$ too.
- If $B \in \mathcal{F}$ and $B \subset A$ then there exists

  $$\bigcap_{k\in[n]} C_k \subset B \subset A \Rightarrow A \in \mathcal{F}$$

Therefore $\mathcal{F}$ is indeed a filter.

2. WTS every filter is contained in some maximal filter.

Zorn's lemma states that a partially ordered set such that every of its totally ordered subsets has an upper bound then it has at least one maximal element.

Take some filter $\mathcal{F}_0$. Then let's look at set $P = \{ \text{filter } \mathcal{F}' \mid \mathcal{F}_0 \subset \mathcal{F}'\}$, with the order that $\mathcal{F}_1 < \mathcal{F}_2$ if $\mathcal{F}_1 \subset \mathcal{F}_2$.

Then any totally ordered subset of P, i.e., any chain in P, $Q = \{\mathcal{F}_\alpha\}_{\alpha\in A}$, has an upper bound in P, namely,

$$\mathcal{F} = \bigcup_{\alpha\in A} \mathcal{F}_\alpha$$

.

Since Q is totally ordered, let A reflect its ordering and be a totally ordered index set, i.e., $\mathcal{F}_\alpha < \mathcal{F}_\beta \Leftrightarrow \alpha < \beta$.

WTS $\mathcal{F}$ is a filter in P.

- Clearly, $\mathcal{F}_0 \subset \mathcal{F}, \varnothing \notin \mathcal{F}, X \in \mathcal{F}$.
- Take $B_1, B_2 \in \mathcal{F}$. Then $B_1 \in \mathcal{F}_{\alpha_1}, B_2 \in \mathcal{F}_{\alpha_2}$. WLOG, $\alpha_1 < \alpha_2$ so $\mathcal{F}_{\alpha_1} \subset \mathcal{F}_{\alpha_2} \Rightarrow B_1 \in \mathcal{F}_{\alpha_2}$. It follows that $B_1 \cap B_2 \in \mathcal{F}_{\alpha_2} \subset \mathcal{F}$.
- Take $B \in \mathcal{F}$ and $B \subset A$. Then $B \in \mathcal{F}_\alpha \Rightarrow A \in \mathcal{F}_\alpha \subset \mathcal{F}$.

So $\mathcal{F}$ is indeed a filter in P. And $\mathcal{F}_\alpha < \mathcal{F} \,\forall\, \alpha \in A$ so $\mathcal{F}$ is indeed an upper bound for the chain.

Using Zorn's Lemma, then there exists a maximal element in P. The partial order that we imposed on P is precisely the requirement for a maximal filter, so we've found our maximal filter.

3. WTS a filter is maximal if and only if it is an ultrafilter.

$\boxed{\Rightarrow}$ Let $\mathcal{F}$ is a maximal filter, that is, for all filter $\mathcal{F}'$ we have $\mathcal{F}' \subset \mathcal{F}$.

Suppose that $\mathcal{F}$ is not an ultrafilter, i.e., there exists $A \subset X$ such that either $A, A^C \notin \mathcal{F}$ or $A, A^C \in \mathcal{F}$.

Case 1: If $A, A^C \notin \mathcal{F}$ then let us investigate $\mathcal{G} = \mathcal{F} \cup \{A\}$. Since $A^C \notin \mathcal{F}$, for every $B \in \mathcal{F}, B \not\subset A^C \Rightarrow B \cap A \neq \varnothing$. Since $\mathcal{F}$ already satisfies the fip, so does $\mathcal{F}'$ too. $\mathcal{G}$ is then a collection of subsets of X satisfying the fip, and so as shown previously, can be extended to a filter $\mathcal{F}'$. But $\mathcal{F}' \not\subset \mathcal{F}$ so $\mathcal{F}$ is not a maximal filter, $\Rightarrow\Leftarrow$

Case 2: If $A, A^C \in \mathcal{F} \Rightarrow \varnothing \in \mathcal{F}, \Rightarrow\Leftarrow$.

Therefore $\mathcal{F}$ is indeed an ultrafilter.

$\boxed{\Leftarrow}$ Let $\mathcal{F}$ be an ultrafilter. Suppose $\mathcal{F}$ is not maximal, i.e., there exists $\mathcal{F}'$ such that it strictly extends $\mathcal{F}$. Then take $A = \mathcal{F}' - \mathcal{F}$. $A \notin \mathcal{F} \Rightarrow A^C \in \mathcal{F} \subset \mathcal{F}' \Rightarrow A^C \in \mathcal{F}'$. But then $A, A^C \in \mathcal{F}' \Rightarrow \varnothing \in \mathcal{F}', \Rightarrow\Leftarrow$ $\square$

Problem 5.2 (Problem 2)

Recall from class the following construction: Let X be a set (which can be topologized as a discrete space). Let UX be the set of ultrafilters on X. For every subset A of X let $[A]$ denote the set of ultrafilters on X that contain A (thus $[A]$ is a subset of UX).

(a) Show that the sets of the form $[A]$ form a basis for a topology on UX.

(b) Show that the map $X \to UX$ that takes each point x in X to the principal ultrafilter F_x in UX is injective, and the image of X under this map is discrete in the subspace topology on UX.

(c) Show that UX is compact and Hausdorff.

(d) (Bonus) Show that UX is the Stone-Cech compactification of X (Hint: if $f : X \to K$ is a continuous map to a compact Hausdorff space, and F is an ultrafilter on X, then $f_* F$ is an ultrafilter on K, which necessarily converges to some unique point y. Show that the map $g : UX \to K$ sending each F to y in this way is a continuous extension of f to UX.)

Solution

(a) UX is the set of ultrafilters on X. WTS $\{[A] : A \subset X\}$ forms a basis for a topology on UX.

- For each $\mathcal{F} \in UX$, take any $U \subset X$, then either $U \in \mathcal{F}$ or $U^C \in \mathcal{F}$. WLOG $U \in \mathcal{F} \Rightarrow \mathcal{F} \in [U]$.
- Take $\mathcal{F} \in [A_1] \cap [A_2]$ then it's clear that $\mathcal{F} \in [A_1 \cap A_2]$

So $\{[A] : A \subset X\}$ satisfies the conditions of a basis.

(b) Take $x \neq y \in X$. Let the map be $F : x \mapsto \mathcal{F}_x$. Then the principal filters $\mathcal{F}_x, \mathcal{F}_y$ are ultra filters, since for any $A \subset X$, either $x \in A$ or $x \in A^C$; similar with y. And $\mathcal{F}_x \neq \mathcal{F}_y$ since $\{x\} \in \mathcal{F}_x, \{x\} \notin \mathcal{F}_y$. The map is therefore injective.

Then consider $\mathcal{F}_x \in F(X)$. Want to show that $\{\mathcal{F}_x\}$ is open in $F(X)$ in the subspace topology on UX.

Denote $V = [\{x\}]$ is a basis element of the topology of UX and is therefore open in UX. Then clearly $\mathcal{F}_x \in V$ since it is an ultrafilter and it is a super set of $\{x\}$.

Furthermore, there doesn't exist any other $\mathcal{F}_y$ with $y \neq x$ that is in V, since $\{x\} \not\subset \mathcal{F}_y$. It follows that

$$\{\mathcal{F}_x\} = V \cap f(X)$$

and is therefore open in $f(X)$ in the subspace topology of UX.

(c) 1. WTS UX is compact. We want to show that every family of closed subsets having the finite intersection property has non-empty intersection. Every closed set of UX has the form:

$$UX - \bigcup_i [A_i] = [\bigcap_i A_i^C] = [B]$$

Therefore, let our family of closed subsets be $\{[B_i]\}_{i\in I}$ with the finite intersection property. Then we have for all $i_1, \ldots, i_n$,

$$\varnothing \neq \bigcap_{k\in[n]} [B_{i_k}] = [\bigcap_{k\in[n]} B_{i_k}]$$

so $\bigcap_{k\in[n]} B_{i_k} \neq \varnothing$. It follows that $\{B_i\}$ has the finite intersection property, and can therefore be extended to a filter, and then to an ultrafilter $\mathcal{U}$, so that $B_i \in \mathcal{U}\ \forall\, i \in I$. It then follows that

$$\mathcal{U} \in [B_i]\ \forall\, i \in I \Rightarrow \mathcal{U} \in \bigcap_{i\in I} [B_i]$$

so $\bigcap_{i\in I}[B_i] \neq \varnothing$ as required.

2. WTS UX is Hausdorff. Take $\mathcal{F}_1 \neq \mathcal{F}_2 \in UX$. Since $\mathcal{F}_1 \neq \mathcal{F}_2$ and they are both ultrafilters so neither one can be contained in the other, there exists some $A \subset X$ such that $A \in \mathcal{F}_1 - \mathcal{F}_2$. $\mathcal{F}_2$ is an ultrafilter, so $A^c \in \mathcal{F}_2$. Then $[A]$ and $[A^C]$ are open neighborhoods of $\mathcal{F}_1$ and $\mathcal{F}_2$, and they are clearly disjoint. It follows that UX is Hausdorff. $\square$

Problem 5.3 (Problem 3)

For each positive integer n give an example of a topological space that has at least n different compactifications. (Bonus: Give an example of a topological space that has infinitely many different compactifications.)

Solution

Fix n. Then take $X = \bigcup_{k=1}^{n}(2k-2, 2k-1)$ in the usual topology in $\mathbb{R}$. For example, for $n = 2, X = (0,1) \cup (2,3)$, i.e., in X, we have n translated copies of the open unit interval $(0,1)$.

Then we can compactify X into the box $[-4n, 4n]^2 \subset \mathbb{R}^2$ (overly big box), which is compact Hausdorff, by specify the embedding for each of the translated copies. For each translated copy, we can either use the 2-point compactification to map $(2k-2, 2k-1)$ to $[(2k-2, 0), (2k-1, 0)]$, or to use the 1-point compactification to map $(2k-2, 2k-1)$ to the circle $B((2k-2, 0), 1)$.

We therefore have 2^n different compactifications for X. $\square$

Problem 5.4 (38.2)

Show that the bounded continuous function $g : (0,1) \to \mathbb{R}$ defined by $g(x) = \cos(\frac{1}{x})$ cannot be extended to the compactification of Example 3. Define an embedding $h : (0,1) \to [0,1]^3$ such that the functions $x, \sin(\frac{1}{x}), \cos(\frac{1}{x})$ are all extendable to the compactification induced by h.

Solution

1. The compactification of Example 3 is the compactification induced by

$$\begin{aligned} h : (0,1) &\to [-1,1]^2 \\ x &\mapsto x \times \sin(1/x) \end{aligned}$$

$X = (0,1), Z = [-1,1]^2$.

Let Y be the compactification of X, and H be the extension of h.

Suppose for sake of contradiction, that g can be extended to a continuous map G on Y.

Consider the sequence of points $\{h(\frac{1}{n\pi})\}_{n\in\mathbb{N}} \subset Z$.

Since G is continuous and H is an embedding, on the one hand,

$$GH^{-1}(h(\frac{1}{n\pi})) = G(\frac{1}{n\pi}) \xrightarrow{n\to\infty} G(0)$$

but on the other hand,

$$GH^{-1}(h(\frac{1}{n\pi})) = g(\frac{1}{n\pi}) = (-1)^n \tag{1}$$

which does not converge.

2. Define

$$h : (0,1) \to [0,1]^3, x \mapsto x \times \sin(1/x) \times \cos(1/x)$$

□

Problem 5.5 (38.4)

Let Y be an arbitrary compactification of X, let $\beta(X)$ be the Stone-Cech compactification. Show there is a continuous surjective closed map $g : \beta(X) \to Y$ that equals the identity on X.

(This exercise makes prices what we mean by saying that $\beta(X)$ is the "maximal" compactification of X. It shows that every compactification of X is equivalent to a quotient space of $\beta(X)$.)

Solution

Y is a compactification of X, so view X as a subspace of Y with $\overline{X} = Y$ and we can inspect the continuous inclusion map

$$\iota : X \to Y$$

The universal property of the Stone Cech compactification then implies that there exists a unique extension of ι to continuous $g : \beta(X) \to Y$.

Note that $\beta(X)$ is compact, a closed subset of $\beta(X)$ is compact, so its image through g is compact, and is therefore closed in Hausdorff Y. It follows that g is a closed map.

It is a surjective map, since $\overline{i(X)} = Y \Rightarrow \overline{g(\beta X)} = Y$, but $\beta(X)$ is compact so $g(\beta X)$ is compact in Hausdorff Y, so $g(\beta X)$ is closed in Y, hence $Y = \overline{(g(\beta X))} = g(\beta X)$.

It is clear that it is exactly the identity on X, since g is an extension of ι. □

Problem 5.6 (38.6)

Let X be completely regular. Show that X is connected if and only if $\beta(X)$ is connected. [Hint: If $X = A \cup B$ is a separation of X, let $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$]

Solution

$\boxed{\Rightarrow}$ By hypothesis, X is connected. Suppose for sake of contradiction that $\beta(X)$ is not connected, i.e., there exists separation $\beta(X) = A \sqcup B$ where A, B are disjoint and clopen in $\beta(X)$.

Then consider $X \cap A, X \cap B$.

Since A, B are disjoint, they are also disjoint.

Also, $X \cap A$ is also open in X, since the conditions of the compactification is that the subspace topology of Y on X aligns with its topology. Same with $X \cap B$.

They are also nonempty, since $\overline{X} = \beta(X)$.

It then follows that $X = (X \cap A) \sqcup (X \cap B)$ is a separation of connected $X, \Rightarrow\Leftarrow$

Therefore $\beta(X)$ is connected.

$\boxed{\Leftarrow}$ By hypothesis, βX is connected. Suppose for sake of contradiction that X is not connected, i.e., there exists separation $X = A \sqcup B$ where A, B are disjoint and clopen in X. Then define $f : X \to$

$\{0,1\}, f(x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \in B \end{cases}$ then $f^{-1}(\{1\}) = A, f^{-1}(\{0\}) = B$ are both open, so f is continuous.

It follows that there is a continuous extension of f, called $g : \beta(X) \to \{0,1\}$. But then we can write $\beta(X) = g^{-1}(\{1\}) \sqcup g^{-1}(\{0\})$ which are disjoint (clearly), and open, since g is continuous. This is then a separation for connected $\beta(X), \Rightarrow\Leftarrow$. It follows that X is connected. $\square$