MATH 20800: Honors Analysis in Rn II

Problem Set 5

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Notation

Let V, W be normed vector spaces. Then $\mathcal{B}(V,W) = \mathcal{L}(V,W)$ is the space of bounded (hence continuous) linear operators from V to W.

Problem 5.1 (Typed Problem 1 done)

Let B be a Banach space.

(a) Prove that if $T \in \mathcal{B}(B,B)$ and $\|I - T\| < 1$ where I is the identity operator, then T is invertible and in fact $\sum_{n=0}^{\infty}(I-T)^n$ converges in $\mathcal{B}(B,B)$ to T^{-1}.

(b) Prove that the set of invertible operators is open in $\mathcal{B}(B,B)$.

Solution

(a) Write the operator

$$Q \coloneqq \sum_{n=0}^{\infty}(I-T)^n : B \to B$$

A priori, Q might not be well-defined pointwise at all, not even considering if $Q \in \mathcal{B}(B,B)$. We intend to show at once that $Q \in \mathcal{B}(B,B)$.

First, we use the fact that if $T \in \mathcal{B}(B,B)$ then $\|T^n\| \leq \|T\|^n$ for any $n \in \mathbb{N}$. This is true by the submultiplicativity of operator norms. It follows that

$$\|Q\| \leq \sum_{n=0}^{\infty}\|(I-T)^n\| \leq \sum_{n=0}^{\infty}\|I-T\|^n < \infty$$

since $\|I - T\| < 1$. Therefore $Q \in \mathcal{B}(B,B)$.

We now want to show that in fact $T^{-1} = Q$, by showing that $T \circ Q = Q \circ T = I$. Apply linearity:

$$\begin{aligned} T \circ Q &= \left((I - (I-T)) \circ \sum_{n=0}^{\infty}(I-T)^n\right) \\ &= \sum_{n=0}^{\infty}(I-T)^n - \sum_{n=1}^{\infty}(I-T)^n = I \end{aligned}$$

and similarly

$$\begin{aligned} Q \circ T &= \left(\sum_{n=0}^{\infty}(I-T)^n \circ (I - (I-T))\right) \\ &= \sum_{n=0}^{\infty}(I-T)^n - \sum_{n=1}^{\infty}(I-T)^n = I \end{aligned}$$

It follows that T is invertible; its inverse is $Q \in \mathcal{B}(B,B)$.

(b) Let S be the set of invertible operators in $\mathcal{B}(B,B)$, take $T \in S$.

Consider $\|T^{-1}\|$. If $\|T^{-1}\| = 0 \Rightarrow T^{-1} = 0 \Rightarrow T^{-1}T \neq I, \Rightarrow\Leftarrow$. So $\|T^{-1}\| > 0$.

Then for all operator $Q \in B(T, 1/\|T^{-1}\|)$ we have that

$$\|T - Q\| < 1/\|T^{-1}\| \Rightarrow \|T^{-1}(T - Q)\| < 1 \Rightarrow \|I - T^{-1}Q\| < 1$$

which implies $T^{-1}Q$ is invertible. Composition of invertible maps is invertible. So $Q = T \circ T^{-1}Q$ is invertible.

So we've found ball $B(T, 1/\|T^{-1}\|)$ around T that is inside S, so S is open as required. $\square$

Problem 5.2 (Typed Problem 2 done)

Let V be a normed vector space and $W \subset V$ a proper closed subspace.

(a) Prove that $\|v + W\| := \inf_{w \in W} \|v + w\|$ is a norm on V/W.

(b) Prove that for any $\varepsilon > 0$ there exists $v \in V$ such that $\|v\| = 1$ and $\|v + W\| \geq 1 - \varepsilon$.

Hint: Let $u \in V \backslash W$. Then $\|u + W\| > 0$ and there exists $w \in W$ such that $\|u + W\| \leq \|u + w\|$ and

$$\|u + w\| \leq \|u + W\| + \varepsilon \|u + W\|.$$

Now consider $\frac{u+w}{\|u+w\|}$.

Solution

(a) We first have to check that this is a well-defined function, i.e., that for $\|v+W\| = \inf_{w \in W} \|v+w\|$, it does not matter which representative v we pick on the RHS. Indeed, for any $v_1, v_2 \in v+W \Rightarrow v_1 = v_2+w'$ for some $w' \in W$. It then follows that

$$\inf_{w \in W} \|v_1 + w\| = \inf_{w \in W} \|v_2 + w' + w\| = \inf_{w'+w \in W} \|v_2 + w' + w\|$$

since W is a subspace of V. So indeed it does not matter which representative we pick.

We check through requirements of a norm:

- $\|v + W\|$ is infimum of non negative things, so it is nonnegative
- If $\|v + W\| = 0$, fix representative $v \in v + W$, then for all $\varepsilon > 0$, there exists $w \in W$ such that $\varepsilon > \|v + w\| = \|v - (-w)\|$. Since $(-w) \in W$, it follows that $v \in \overline{W}$. $\overline{W} = W$ is W is closed, so $v \in W$.
- $\|\lambda v + W\| = \inf_{w \in W} \|\lambda v + w\| = \inf_{w \in W} \|\lambda(v + w/\lambda)\| = |\lambda| \inf_{w/\lambda \in W} \|v + w/\lambda\| = |\lambda| \|v + W\|$.
- For any $v_1, v_2 \in V; w \in W$, we have that $\|v_1+v_2+w\| \leq \|v_1+w/2\|+\|v_2+w/2\| \Rightarrow \|v_1+v_2+W\| \leq \|v_1 + W\| + \|v_2 + W\|$.

Thus $\|v + W\|$ is a norm.

(b) Fix $\varepsilon > 0$. Let $u \in V \backslash W$. Then $u + W \neq 0 \Rightarrow \|u + W\| > 0$.

Since $\|u+W\| = \inf_{w \in W} \|u+w\|$, there exists $w \in W$ such that $\|u+W\| \leq \|u+w\| \leq \|u+W\|+\varepsilon\|u+W\| = (1+\varepsilon)\|u+W\|$. It then follows that if we define $v = \frac{u+w}{\|u+w\|}$ then clearly $\|v\| = 1$ and

$$\|v + W\| = \left\| \frac{u + w}{\|u + w\|} + W \right\| = \frac{1}{\|u + w\|} \|u + W\| \geq \frac{1}{1 + \varepsilon} > \frac{1 - \varepsilon^2}{1 + \varepsilon} = 1 - \varepsilon$$

as required. $\square$

Problem 5.3 (Typed Problem 3 done)

Let V be a Banach space and $W \subset V$ a proper closed subspace. Prove that V/W with the norm defined in problem 2 is a Banach space.

Hint: Suppose that the series $\sum_n (v_n + W)$ is absolutely summable, i.e., $\sum_n \|v_n + W\|$ converges. We wish to prove that $\sum_n (v_n + W)$ converges in V/W. For each $n \in \mathbb{N}$, there exists $w_n \in W$ such that

$$\|v_n + w_n\| \le \|v_n + W\| + 2^{-n}$$

Then $\sum_n (v_n + w_n)$ is absolutely summable, and since V is a Banach space, there exists $v \in V$ such that $v = \sum_n (v_n + w_n)$. Prove that $v + W = \sum_n (v_n + W)$, i.e.,

$$\lim_{N\to\infty} v + W - \sum_{n=1}^{N} (v_n + W) = 0$$

Solution

To show that V/W is Banach, we show that every absolutely summable series is summable, i.e., suppose $\sum_{n=1}^{\infty} \|v_n + W\|$ converges, WTS $\sum_{n=1}^{\infty} (v_n + W)$ converges in V/W.

Indeed, by definition of $\|v_n + W\|$, there exists some w_n such that

$$\|v_n + W\| \le \|v_n + w_n\| \le \|v_n + W\| + 2^{-n}$$

It then follows that

$$\sum_{n=1}^{\infty} \|v_n + w_n\| \le \sum_{n=1}^{\infty} \|v_n + W\| + \sum_{n=1}^{\infty} 2^{-n} = \sum_{n=1}^{\infty} \|v_n + W\| + 1$$

so $\sum_n \|v_n + w_n\|$ is absolutely summable. They are points in Banach V, so $\sum_n (v_n + w_n)$ is summable in V. Set $v := \sum_{n=1}^{\infty} (v_n + w_n) \in V$.

We now WTS the partial sum $\sum_{n=1}^{N} (v_n + W) \xrightarrow{N\to\infty} v + w$ in V/W. Indeed,

$$\begin{aligned} \left\| v + W - \sum_{n=1}^{N} (v_n + W) \right\| &= \left\| v - \sum_{n=1}^{N} (v_n + w_n) \right\| \\ &= \left\| \sum_{n=N}^{\infty} (v_n + w_n) \right\| \le \sum_{n=N}^{\infty} \|v_n + w_n\| \xrightarrow{N\to\infty} 0 \end{aligned}$$

since $\sum_n \|v_n + w_n\|$ is absolutely summable.

Therefore, it follows that $\sum_{n=1}^{\infty} (v_n + W)$ is summable, its limit is $v + W$, so V/W is Banach. $\square$

Problem 5.4 (Typed Problem 4 done)

Suppose V and W are Banach spaces, $T \in \mathcal{B}(V, W)$ and recall the following subspaces

$$\ker(T) = \{v \in V : Tv = 0\}, \quad \operatorname{range}(T) = \{Tv \in W : v \in V\}.$$

(a) Prove that $\ker(T)$ is a closed subspace of V.

(b) If V_1 and V_2 are normed linear spaces, we say a bijective linear operator $S : V_1 \to V_2$ is an isomorphism if $S \in \mathcal{B}(V_1, V_2)$ and $S^{-1} \in \mathcal{B}(V_2, V_1)$. We say V_1 and V_2 are *isomorphic* if there exists an isomorphism $S : V_1 \to V_2$.

Prove that $V/\ker(T)$ is isomorphic to $\operatorname{range}(T)$ if and only if $\operatorname{range}(T)$ is closed.

Hint: Consider the map $S : V/\ker(T) \to \operatorname{range}(T)$ given by

$$S(v + \ker(T)) = Tv,$$

and first show that S is a well-defined, bijective bounded linear operator.

Solution

(a) $T \in \mathcal{B}(V, W)$ so $\|T\| < \infty$. Take sequence $(v_n)_{n\in\mathbb{N}}$ in $\ker(T)$ such that $v_n \xrightarrow{n\to\infty} v \in V$. WTS $v \in \ker(T)$.

Indeed,

$$\begin{aligned}
\|Tv\| &= \|Tv - Tv_n\| \\
&= \|T(v - v_n)\| \\
&\leq \|T\|\|v - v_n\| \xrightarrow{n\to\infty} 0
\end{aligned}$$

so $\|Tv\| = 0 \Rightarrow Tv = 0 \Rightarrow v \in \ker(T)$. $\square$

(b)

$\boxed{\Rightarrow}$ Suppose that $V/\ker(T)$ is isomorphic to $\mathrm{range}(T)$, then there exists some isomorphism $S : V/\ker(T) \to \mathrm{range}(T)$. Then take sequence of points $(Tv_n)_{n\in\mathbb{N}}$ in $\mathrm{range}(T)$ such that $Tv_n \xrightarrow{n\to\infty} w \in W$. WTS $w \in \mathrm{range}(T)$.

Denote $w_n = Tv_n \in W$. So (w_n) is Cauchy, which implies $(S^{-1}w_n)$ is Cauchy since $\|S^{-1}\| < \infty$. $(S^{-1}w_n)$ is Cauchy in Banach $V/\ker(T)$, so converges to some $v' + \ker(T) \in V/\ker(T)$. But S is also continuous, so $S(S^{-1}w_n) = w_n \xrightarrow{n\to\infty} S(v' + \ker(T)) \in \mathrm{range}(T)$. So $w = S(v' + \ker(T)) \in \mathrm{range}(T)$, so $\mathrm{range}(T)$ is closed. $\square$

$\boxed{\Leftarrow}$ Suppose that $\mathrm{range}(T)$ is closed. $\mathrm{range}(T)$ is closed in Banach W so it is Banach. Consider the mapping

$$\begin{aligned}
S : V/\ker(T) &\to \mathrm{range}(T) \\
v + \ker(T) &\mapsto Tv
\end{aligned}$$

1. WTS this mapping is an isomorphism between $V/\ker(T)$ and $\mathrm{range}(T)$.

   First, it is well-defined, in the sense that it does not matter which representative of $v+\ker(T)$ we choose. Indeed, if $v_1, v_2 \in v + \ker(T)$, which means $v_1 = v_2 + u$ for some u such that $Tu = 0$, then $Tv_1 = T(v_2 + u) = Tv_2 + Tu = Tv_2$.

2. WTS S is linear. Linearity of S is clear from linearity of T.

   WTS it is bounded. For any v such that $\|v + \ker(T)\| = 1$, there exists some $u \in \ker(T)$ such that $\|v + u\| < 1 + 1 = 2$. Hence

   $$\|Tv\| = \|Tv + Tu\| \leq \|T\|\|v + u\| < 2\|T\|$$

   It follows that $\|S\| = \sup_{\|v+\ker(T)\|=1} \|Tv\| \leq 2\|T\| < \infty$ so it is indeed a bounded map.

3. WTS it is injective. Indeed, if $v_1 + \ker(T) \neq v_2 + \ker(T)$ then $S(v_1 + \ker(T)) - S(v_2 + \ker(T)) = T(v_1 - v_2) \neq 0$.

4. WTS it is surjective. Take any $Tv \in \mathrm{range}(T)$. Then $Tv = S(v + \ker(T))$.

5. From all above, it can be concluded that S is a bijective, bounded linear operator from Banach $V/\ker(T)$ to Banach $\mathrm{range}(T)$. Applying Open Mapping Theorem, it can be concluded that S is an isomorphism between $V/\ker(T)$ and $\mathrm{range}(T)$. Thus they are isomorphic as required.

$\square$

Problem 5.5 (Typed Problem 5 done)

The following exercise shows we cannot drop certain hypotheses in the closed graph theorem and open mapping theorem. Let

$$W = \left\{ a = \{a_k\}_k : \sum_k k|a_k| < \infty \right\},$$

equipped with the ℓ^1 norm.

(a) Prove that W is a proper, dense subspace of ℓ^1 (hence, W is not complete).

Hint: Show that if $b = \{b_k\}_k \in \ell^1$ and $\varepsilon > 0$, then there exists $N \in \mathbb{N}$ such that if

$$a \coloneqq \{b_1, b_2, \ldots, b_N, 0, 0, \ldots\} \in W.$$

then $\|a - b\|_1 < \varepsilon$.

(b) Define $T : W \to \ell^1$ by $(Ta)_k = ka_k$. Prove that the graph of T is closed but T is not bounded.

(c) Let $S = T^{-1} : \ell^1 \to W$. Prove that S is bounded and surjective but is not an open mapping.

Solution

(a) Take $a = \{a_k\}_k$ where $a_k = \frac{1}{k^2}$ then $\|a\|_1 = \frac{\pi^2}{6}$ so $a \in \ell^1$, but then $\sum_k k\|a_k\| = \sum_k \frac{1}{k} = \infty$ so $a \notin W$. So W is a proper subset of ℓ^1.

To show that it is dense, pick any $b = \{b_k\}_k \in \ell^1$ and $\varepsilon > 0$. WTS there exists some $a \in W$ such that $\|a - b\|_1 < \varepsilon$.

Since $\|b\|_1 < \infty$, there exists $N \in \mathbb{N}$ such that $\sum_{k=N}^{\infty} |b_k| < \varepsilon$. Hence if we define $a \coloneqq \{b_1, b_2, \ldots, b_N, 0, 0, \ldots\}$ then

$$\sum_k k|a_k| < \infty$$

trivially so $a \in W$, while

$$\|a - b\|_1 = \sum_{k=N}^{\infty} |b_k| < \varepsilon$$

It follows that W is indeed dense in ℓ^1.

It also follows that W is not complete; for if it was complete then since every point of ℓ^1 is a limit point of W, it would also be in W, making W not proper.

(b) Take $(a^{(n)}, Ta^{(n)}) \xrightarrow{n\to\infty} (u, z) \in W \times \ell^1$, which means $a^{(n)} \xrightarrow{n\to\infty} u, Ta^{(n)} \xrightarrow{n\to\infty} z$, both with respect to ℓ^1 norm.

WTS $z = Tu$, i.e., $z_k = ku_k$.

Fix $k \in \mathbb{N}$. Then for any $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$\varepsilon/(2k) > \|a^{(N)} - u\|, \quad \varepsilon/2 > \|Ta^{(N)} - z\|$$

It then follows that

$$\begin{aligned} |z_k - ku_k| &\leq |z_k - ka_k^{(N)}| + |ka_k^{(N)} - ku_k| \\ &\leq \|Ta^{(N)} - z\| + k\|a^{(N)} - u\| \\ &< \varepsilon/2 + k\varepsilon/(2k) = \varepsilon \end{aligned}$$

This is true for all ε, so $z_k = ku_k$. It follows that $z = Tu$ and $\Gamma(T)$ is closed.

However, T is not bounded. Given any $M > 0$, construct $a = \{a_k\}_{k\in\mathbb{N}}$ where $a_k = 1$ if $k = \lceil M \rceil$ and 0 otherwise. Then $\|a\|_1 = 1$ but $\|Ta\| = \lceil M \rceil \geq M$.

(Therefore closed graph theorem doesn't work if one of the spaces is not Banach.)

(c) We write out the explicit definition for S:

$$S : \ell^1 \to W$$
$$b = \{b_k\} \mapsto a = \left\{a_k = \frac{b_k}{k}\right\}$$

1. WTS it is well-defined. If $b \in \ell^1$ then

$$\sum_k |a_k| = \sum_k |b_k/k| \le \sum_k |b_k| < \infty$$

so indeed $a \in W$.

2. WTS it is bounded. If $\|b\|_1 = 1$ then

$$\|S(b)\|_1 = \sum_k \left|\frac{b_k}{k}\right| \le \sum_k |b_k| = 1$$

so $\|S\| \le 1 < \infty$.

3. WTS it is surjective. Take $a \in W$. Then define b such that $b_k = ka_k$. Then $\sum_k |b_k| = \sum_k |ka_k| < \infty$ since $a \in W$, so $b \in \ell^1$, and clearly $S(b) = a$.

4. WTS it is not an open mapping. Consider $U = B_{\ell^1}(0,1)$. WTS $S(U)$ not open by demonstrating that we can't draw a ball (wrt ℓ^1) around $f(0) = 0$. Suppose, for sake of contradiction, that we can draw a ball $B_{\ell^1}(0, 2r) \cap W$ that is open in W. Then define $M = \lceil \frac{1}{r} \rceil$, and define a such that $a_k = r$ when $k = M$ and 0 otherwise. Clearly, $a \in B_{\ell^1}(0, 2r) \cap W$.

However if there exists some $b \in U$ such that $S(b) = a$ then that means $b_M = rM > 1 \Rightarrow \|b\|_1 > 1 \Rightarrow b \notin U, \Rightarrow\Leftarrow$.

It follows that one can't draw a ball around $S(0)$, thus $S(U)$ is not open.

$\square$

Problem 5.6 (Written Problem 1, Brezis 1.3 done)

Let $E = \{u \in C([0,1], \mathbb{R}) : u(0) = 0\}$ with the norm

$$\|u\| = \max_{t \in [0,1]} |u(t)|.$$

Consider the linear functional

$$f : u \in E \mapsto f(u) = \int_0^1 u(t)\mathrm{d}t$$

(a) Show that $f \in E^*$ and compute $\|f\|_{E^*}$.

(b) Can one find some $u \in E$ such that $\|u\| = 1$ and $f(u) = \|f\|_{E^*}$?

Solution

(a) To show that $f \in E^*$, WTS it is linear and bounded.

- Linearity of f is clear from linearity of integrals.
- It is also bounded:

$$\|f\| = \sup_{\|u\|=1} |f(u)| = \left|\int_0^1 u(t)\mathrm{d}t\right| \le \|u\| = 1 < \infty$$

Therefore $f \in E^*$.

We then compute $\|f\|_{E^*} = \sup_{\|u\|=1} |f(u)|$. We know that $\|f\| \le 1$. WTS $\|f\| = 1$, i.e., for any $1 \gg \varepsilon > 0$ there exists some $\|u\| = 1$ such that $|f(u)| = 1 - \varepsilon$. Indeed, for fixed $\varepsilon > 0$, construct u that linearly interpolates $(0,0) \to (2\varepsilon, 1) \to (1,1)$. Then $|f(u)| = 1 - \varepsilon$ as required, while $\|u\| = 1$ clearly. It follows that $\|f\|_{E^*} = 1$.

(b) Take any $u \in E$ such that $\|u\| = 1$. Fix $\varepsilon = 1/2$. Since u continuous at 0, there exists $\delta > 0$ such that $t < \delta \Rightarrow |u(t) - u(0)| < \frac{1}{2}$, that is, $|u(t)| < \frac{1}{2}$. Let $\delta' = \min\{\delta, 1\}$, then

$$\begin{aligned}\int_0^1 u(t)\mathrm{d}t &\leq \int_0^{\delta'} u(t)\mathrm{d}t + (1-\delta')\|u\| \\ &\leq \frac{1}{2}\delta' + (1-\delta') < 1 = \|f\|_{E^*}\end{aligned}$$

So there doesn't exist any u with such conditions. □

Problem 5.7 (Written Problem 2, Brezis 1.4 done)

Consider the space $E = c_0$ with its usual norm ℓ^∞ norm. For every element $u = (u_1, u_2, u_3, \ldots)$ in E define

$$f(u) = \sum_{n=1}^{\infty} \frac{1}{2^n} u_n.$$

(a) Check that f is a continuous linear functional on E and compute $\|f\|_{E^*}$.

(b) Can one find some $u \in E$ such that $\|u\| = 1$ and $f(u) = \|f\|_{E^*}$?

Solution

(a) Recall that

$$c_0 = \left\{ u = (u_i)_{i\in\mathbb{N}} : u_i \xrightarrow{j\to\infty} 0 \right\}$$

endowed with $\|u\|_\infty = \sup_{i\in\mathbb{N}}|u_i|$.

We first check that $f(u)$ is well-defined for each $u \in E$.

Indeed, for all $\varepsilon > 0$, since $u_i \xrightarrow{j\to\infty} 0$, there exists some $N \in \mathbb{N}$ such that $i \geq N \Rightarrow |u_i| < \varepsilon$. Then, for all $m, n \geq N$, we have

$$\begin{aligned}\left|\sum_{i=n+1}^{m} \frac{1}{2^i} u_i\right| &\leq \sum_{i=n+1}^{m} \frac{1}{2^i}\varepsilon \\ &\leq \frac{1}{2^N}\varepsilon \leq \varepsilon\end{aligned}$$

so the partial sums are Cauchy in complete $\mathbb{C}$, so the series $\sum_{i=1}^\infty \frac{1}{2^i} u_i$ indeed converges. $f(u)$ is thus well-defined.

Now WTS f is linear and bounded.

- It is linear:

$$f(u + \lambda v) = \sum_{i=1}^{\infty} \frac{1}{2^i}(u_i + \lambda v_i) = f(u) + \lambda f(v)$$

- It is bounded:

$$\begin{aligned}\|f\| &= \sup_{\|u\|=1} |f(u)| \\ &= \sup_{\|u\|=1} \left|\sum_{i=1}^{\infty} \frac{1}{2^i} u_i\right| \\ &\leq \sup_{\|u\|=1} \sum_{i=1}^{\infty} \frac{1}{2^i}\|u_i\| \\ &\leq \sup_{\|u\|=1} \|u\| = 1 < \infty\end{aligned}$$

So $f \in E^*$. We know that $\|f\| \leq 1$, WTS for any $\varepsilon > 0$, there exists some u such that $f(u) \geq 1 - \varepsilon$.

Let $N \in \mathbb{N}$ large such that $\frac{1}{2^N} < \varepsilon$. Then consider the sequence $u \in c_0$ defined by

$$u_i = \begin{cases} 1 & \text{for } i \leq N+1 \\ 0 & \text{for } i > N+1 \end{cases}$$

Then it follows that

$$f(u) = \sum_{i=1}^{\infty} \frac{u_i}{2^i} = \sum_{i=1}^{N+1} \frac{1}{2^i} = 1 - \frac{1}{2^N} \geq 1 - \varepsilon$$

while $u_i \xrightarrow{j\to\infty} 0$ clearly. Therefore $\|f\|_{E^*} = 1$.

(b) Take any $u \in c_0$ such that $\|u\|_\infty = 1$. Then there exists N such that $i \geq N \Rightarrow |u_i| < 1/3$. It follows that

$$\begin{aligned} f(u) &= \sum_{i=1}^{\infty} \frac{u_i}{2^i} \\ &= \sum_{i=1}^{N-1} \frac{u_i}{2^i} + \sum_{i=N}^{\infty} \frac{u_i}{2^i} \\ &\leq 1 - 2^{-N} + \frac{1}{3} \frac{1}{2^{N-1}} < 1 = \|f\| \end{aligned}$$

So there doesn't exist any u with such conditions. □

Problem 5.8 (Written Problem 3, Brezis 1.8 done)

Let E be a normed vector space with norm $\|\cdot\|$. Let $C \subset E$ be an open convex set such that $0 \in C$. Let p denote the gauge of C.

(a) Assume C is symmetric (i.e., $-C = C$) and C is bounded, prove that p is a norm which is equivalent to $\|\cdot\|$.

(b) Let $E = C([0,1], \mathbb{R})$ with its usual norm

$$\|u\| = \max_{t\in[0,1]} |u(t)|.$$

Let

$$C = \left\{ u \in E, \int_0^1 |u(t)|^2 \mathrm{d}t < 1 \right\}$$

Check that C is convex and symmetric and that $0 \in C$. Is C bounded in E? Compute the gauge p of C and show that p is a norm on E. Is p equivalent to $\|\cdot\|$?

Solution

(a) Recall that the gauge of C is

$$p(x) = \inf\{\alpha > 0 : \alpha^{-1} x \in C\}$$

1. We first show that it is indeed a norm.

- p is the infimum of positive numbers so it is nonnegative.
- When $\lambda \geq 0, p(\lambda x) = \inf\{\alpha > 0 : \alpha^{-1}(\lambda x) \in C\} = \lambda \inf\{\alpha > 0 : \alpha^{-1}(x) \in C\} = \lambda p(x) = |\lambda| p(x)$.
- When $\lambda < 0, p(\lambda x) = \inf\{\alpha > 0 : \alpha^{-1}(\lambda x) \in C\} = \inf\{\alpha > 0 : |\lambda|\alpha^{-1}(-x) \in C\} = |\lambda| \inf\{\alpha > 0 : |\lambda|\alpha^{-1}(x) \in C\} = |\lambda| p(x)$, since $C = -C$.
- Therefore $p(\lambda x) = |\lambda| p(x)\ \forall\ \lambda \in \mathbb{R}$.
- We've proven triangle inequality for p in Brezis.
- It remains to show that if $p(x) = 0$ then $x = 0$. Suppose we have that $p(a) = 0$ and $a \neq$

$0 \Rightarrow \|a\| > 0$. Then since C is bounded, there exists some M such that $C \subset B(0, M)$. Since $0 = p(a) = \inf\{\alpha > 0 : \alpha^{-1}a \in C\}$, there exists some $\alpha < \frac{\|a\|}{M}$ such that $\alpha^{-1}a \in C$. But then

$$\|\alpha^{-1}a\| > \left\| \frac{M}{\|a\|}a \right\| = M \Rightarrow \alpha^{-1}a \notin C$$

It follows that $a = 0$ necessarily.

So p is a norm.

2. We now need to show that it is comparable to $\|\cdot\|$.

Since $0 \in C$ and C is open, there exists $r > 0$ such that $B(0, r) \subset C$. It then follows that $p(x) \leq \frac{1}{r}\|x\|$.

Since C is bounded, there exists $M > 0$ such that $C \subset B(0, M)$. Then

$$p(x) = \inf\{\alpha > 0 : \alpha^{-1}x \in C\} \geq \inf\{\alpha > 0 : \alpha^{-1}x \in B(0, M)\} = \frac{1}{M}\|x\|$$

Hence p and $\|\cdot\|$ are comparable.

(b) 1. WTS C is convex, symmetric and $0 \in C$.

- $0 \in C$ trivially.
- Take $u, v \in C$ and some $\lambda \in [0, 1]$. Then

$$\begin{aligned}
\int_0^1 |\lambda u(t) + (1-\lambda)v(t)|^2 \mathrm{d}t &= \lambda^2 \int_0^1 |u(t)|^2 \mathrm{d}t + (1-\lambda)^2 \int_0^1 |v(t)|^2 \mathrm{d}t + 2\lambda(1-\lambda) \int_0^1 u(t)v(t)\mathrm{d}t \\
&< \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda) \left(\int_0^1 |u(t)|^2 \mathrm{d}t\right) \left(\int_0^1 |v(t)|^2 \mathrm{d}t\right) \\
&< \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda) = 1 \\
\Rightarrow \lambda u + (1-\lambda)v \in C
\end{aligned}$$

  So C is convex.
- If $u \in C$ then $-u \in C$ since $|u(t)| = |(|(-u)(t))$. So C is symmetric.

2. WTS C is NOT bounded in E.

For any $M > 0$, construct $u_M(t) = \begin{cases} \sqrt{M - M^2 t} & \text{if } t \in [0, \frac{1}{M}] \\ 0 & \text{if } t \in [\frac{1}{M}, 1] \end{cases}$, which is the square root of the function that linearly interpolates $(0, M) \to (\frac{1}{M}, 0) \to (1, 0)$. Then $\int_0^1 |u_M(t)|^2 \mathrm{d}t = \frac{1}{2} < 1$ so $u_M \in C$. But $\|u_M\| = M$ is unbounded. Hence C is unbounded in E.

3. We now compute the gauge p of C. WTS it is also a norm on E, but this norm is not equivalent to $\|\cdot\|$.

Recall that

$$p(u) = \inf\{\alpha > 0 : \alpha^{-1}u \in C\} = \inf\{\alpha > 0 : \alpha^{-1} \int_0^1 |u(t)|^2 \mathrm{d}t < 1\}$$

Then it's clear that $p(u) = \int_0^1 |u(t)|^2 \mathrm{d}t = \|u\|_{L^2([0,1])}$. It is the $L^2([0, 1])$ norm so it is a norm.

This norm is not equivalent to $\|\cdot\|$, since suppose not, that there exists some M such that

$$\|u\| \leq M\|u\|_{L^2([0,1])} \ \forall\, u \in C([0, 1], \mathbb{R})$$

Then one can construct function

$$u_M \equiv \frac{1}{2M}$$

Then $\|u\| = \frac{1}{2M}$ and $M\|u\|_{L^2([0,1])} = \frac{M}{4M^2} = \frac{1}{4M}$. So $\|u\| > M\|u\|_{L^2([0,1])}, \Rightarrow\Leftarrow$

Hence they are not equivalent. $\square$

Problem 5.9 (Written Problem 4, Brezis 2.3 done)

Let E, F be Banach spaces and $(T_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{L}(E,F)$. Assume that for every $x \in E$, $T_n x$ converges as $n \to \infty$ to a limit denoted by Tx. Show that if $x_n \xrightarrow{n\to\infty} x$ in E, then $T_n x_n \xrightarrow{n\to\infty} Tx$ in F.

Solution

From Banach-Steinhaus and its Corollary 2.3, we find that there exists some C such that $\|T_n\| \leq C \;\forall\, n$ and $T \in \mathcal{L}(E,F)$.

By hypothesis, $x_n \xrightarrow{n\to\infty} x$ in E.

Fix $\varepsilon > 0$. Since $x_n \xrightarrow{n\to\infty} x$, there exists some N_1 such that $n \geq N_1 \Rightarrow \|x_n - x\| < \varepsilon/(2C)$.

Furthermore, since $T_n x \xrightarrow{n\to\infty} Tx$, there exists some N_2 such that $n \geq N_2 \Rightarrow \|T_n x - Tx\| < \varepsilon/2$.

It then follows that for $n \geq N_1 + N_2$, we have

$$\|Tx - T_n x_n\| \leq \|Tx - T_n x\| + \|T_n x - T_n x_n\| < \varepsilon/2 + C\varepsilon/(2C) = \varepsilon$$

Hence $T_n x_n \xrightarrow{n\to\infty} Tx$. □

Problem 5.10 (Written Problem 5, Brezis 2.17 done)

Let $E = C([0,1])$ with its usual norm. Consider the operator $A : D(A) \subset E \to E$ defined by

$$D(A) = C^1([0,1]) \quad \text{and} \quad Au = u' = \frac{\mathrm{d}u}{\mathrm{d}t}$$

(a) Check that $\overline{D(A)} = E$.

(b) Is A closed?

(c) Consider the operator $B : D(B) \subset E \to E$ defined by

$$D(B) = C^2([0,1]) \quad \text{and} \quad Bu = u' = \frac{\mathrm{d}u}{\mathrm{d}t}$$

Is B closed?

Solution

(a) Clearly, $D(A) \subset E$. It remains for us to show that every $u \in E$ is the limit of some sequence in $D(A) = C^1([0,1])$.

Fix $u \in E = C([0,1])$. Then the standard mollification gives that u is the (uniform) limit of $\{u * \eta_{1/n}\}_{n\in\mathbb{N}}$. Each $u * \eta_{1/n} \in C^\infty \subset C^1([0,1])$, so indeed u is a limit point of $C^1([0,1])$.

Hence $\overline{D(A)} = E$ as required.

(b) A is closed iff it maps closed sets to closed sets.

Consider $K = \{u_n(t) \in E\}_{n\in\mathbb{N}}$ where $u_n(t) = n + \frac{t}{n}$. Then clearly $K \subset D(A)$, with $u_n' \equiv \frac{1}{n}$.

We show that K is closed by showing that it is not Cauchy in any rearrangement, hence does not have a limit point. Indeed, take any $m \neq n$, WLOG, $m > n$, then

$$\|u_m - u_n\| = \sup_{t\in[0,1]} |(m-n) + t\left(\frac{1}{m} - \frac{1}{n}\right)| \geq |(m-n)| \geq 1$$

So K is closed.

But it is clear that $u_n' \equiv \frac{1}{n} \xrightarrow{n\to\infty} u_0' \equiv 0$ (uniformly), but $u_0' \notin A(K)$. So $A(K)$ is not closed.

Hence A is not a closed map.

(c) The same example demonstrates the same point. So B is not closed.

□