1 Week 6 Reading

1. (Page 37, 39) The idea of "reflecting the singularity" when computing Green's function in specific cases.
2. (Page 49) Duhamel's Principle and the intuition behind integrating with respect to s.
3. (Page 52 - 54) Mean-value property for heat equations. I wanna hear some remarks/motivation on why the property should be true, as the constructions now seem kinda random to me. e.g. why take such a heat ball. I can also follow the computation (applying IBP, etc.) but don't get the high-level overview of the proof.
4. Intuition for "energy method" to prove uniqueness in Laplace's and Heat Equations
5. Differences between the "fundamental solutions" of the Laplace's Equation and the Heat Equation.
   Laplace's Equation $\Delta u = 0$ has *fundamental solution*

   $$\Phi(x) = \begin{cases} \dfrac{1}{2\pi}\log|x| & (n = 2) \\ \dfrac{1}{n(n-2)\alpha(n)}\dfrac{1}{|x|^{n-2}} & (n = 3) \end{cases}$$

   which is defined on $\mathbb{R}^n\backslash\{0\}$.
   Then a solution to Poisson's equation $-\Delta u = f$ in $\mathbb{R}^n$ would be

   $$u(x) = (\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)dy \tag{1}$$

   and I get that one can think of Φ as the response to the impulse $-\Delta\Phi = \delta_0$, then "summing up throughout"
   Heat Equation $u_t - \Delta u = 0$ has *fundamental solution*

   $$\Phi(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{n/2}}e^{-\frac{|x|^2}{4t}} & (x \in \mathbb{R}^n, t > 0) \\ 0 & (x \in \mathbb{R}^n, t < 0) \end{cases}$$

   Then a solution to the initial-value problem

   $$\begin{cases} u_t - \Delta u = 0 & (x,t) \in (\mathbb{R}^n \times (0,\infty)) \\ u = g & (x,t) \in (\mathbb{R}^n \times \{t = 0\}) \end{cases}$$

   is

   $$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy$$

   Rewriting,

   $$u^t(x) = (\Phi(\cdot,t) * g)(x) \tag{2}$$

   and now one thinks of Φ encoding:

   1. Across the x-axis, the heat impulse δ_0
   2. Across the t-axis, the propagating effect of the above impulse (i.e. conforming to the heat equation)

   I'm still kinda unsettled by how the constructions of u from Φ look so similar between the 2 cases (although a lot of differences can be seen) yet yielding different behavior when "convoluting" with another function. For example, I have thought that maybe when constructing u as such in (2) we would have some "erratic" behavior like in the case of the Poisson's equation in (1), something like $u_t - \Delta u = g$ (The input shape doesn't even match here: u on space time, g on space).

   But then indeed this "erratic" behavior is achieved later when solving for the non-homogeneous heat equation with initial value

   $$\begin{cases} u_t - \Delta u = f & (\mathbb{R}^n \times (0,\infty)) \\ u = 0 & (\mathbb{R}^n \times \{t = 0\}) \end{cases}$$

and we have the solution

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s) dy ds$$

that seems more similar to the philosophy used when we were solving for Poisson's Equation. I think I'm failing to see something from the bigger picture here.