On differential forms

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January 17, 2024

1 Preface

This short exposition is in no way an ultra-rigorous study of differential forms, but is hopefully helpful for students who are struggling to get some grasp of what forms sort of mean. The accompanying text should be Pugh's *Real Mathematical Analysis*. I've tried reading MIT's course notes on differential forms, but ramming through the multilinear algebra was a little bit too much. Terence Tao's short notes on differential forms also give proper motivation to this subject as well.

2 Motivation

I quote Tao's notes:

> The concept of integration is of course fundamental in single-variable calculus. Actually, there are three concepts of integration which appear in the subject: the *indefinite integral* $\int_f$ (also known as the anti-derivative), the *unsigned definite integral* $\int_{[a,b]} f(x)dx$ (which one would use to find area under a curve, or the mass of a one-dimensional object of varying density), and the *signed definite integral* $\int_a^b f(x)dx$ (which one would use for instance to compute the work required to move a particle from a to b). For simplicity we shall restrict attention here to functions $f : \mathbb{R} \to \mathbb{R}$ which are continuous on the entire real line (and similarly, when we come to differential forms, we shall only discuss forms which are continuous on the entire domain). [...]
>
> These three integration concepts are of course closely related to each other in single-variable calculus; indeed, the fundamental theorem of calculus relates the signed definite integral $\int_a^b f(x)dx$ to any one of the indefinite integrals $F = \int f$ by the formula
> $$\int_a^b f(x)dx = F(b) - F(a)$$
> while the signed and unsigned integral are related by the simple identity
> $$\int_a^b f(x)dx = -\int_b^a f(x)dx = \int_{[a,b]} f(x)dx$$
> which is valid whenever $a \leq b$.
>
> When one moves from single-variable calculus to several-variable calculus, though, these three concepts begin to diverge significantly from each other. The *indefinite integral* generalises to the notion of a *solution to a differential equation*, or of an integral of a connection, vector field, or bundle. The *unsigned definite integral* generalises to the *Lebesgue integral*, or more generally to integration on a measure space. Finally, the *signed definite integral* generalises to the *integration of forms*, which will be our focus here. While these three concepts still have some relation to each other, they are not as interchangeable as they are in the single-variable setting.

For me, as for Tao, differential forms capture some sort of *orientation* in taking integrals, some sort of *oriented integral* that is not captured by the Lebesgue integral.

Our goal: The **General Stokes' Theorem**:

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega$$

That might have meant nothing to you; but you must have seen its instances in some particular cases: Gauss' divergence theorem, Green's identities, and so on.

3 Surfaces and Forms

Forms capture the integration over "lower-dimensional" things in "higher dimensional" spaces, i.e., the sphere S^2 in space $\mathbb{R}^3$. How can we capture that?

Definition 3.1 (k-surfaces $\equiv$ k-cells in $\mathbb{R}^n$)

A **k-surface** in $E \subset \mathbb{R}^n$ is a smooth *map* $\varphi : I^k \to E \subset \mathbb{R}^n$.

Remember, it is a map, not the image of the map, though when we think of "surfaces", we can also imagine its image instead. In this way, the sphere S^2 is the image of some 2-surface in $\mathbb{R}^3$, but NOT the surface itself!

Define $C_k(\mathbb{R}^n)$ to be the set of k-surfaces in $\mathbb{R}^n$.

Definition 3.2 (Functionals on surfaces)

Define $C^k(\mathbb{R}^n)$ to be the set of functionals on $C_k(\mathbb{R}^n)$, that is, on the set of k-surfaces in $\mathbb{R}^n$. That is, each $f \in C^k(\mathbb{R}^n)$ sends

$$f : C_k(\mathbb{R}^n) \to \mathbb{R}, \varphi \mapsto f(\varphi) \in \mathbb{R}$$

We jump directly into the definition of forms.

Definition 3.3 (k-forms)

$\omega \in C^k(\mathbb{R}^n)$ is called a k-form on $E \subset \mathbb{R}^n$ if there exists $\{a_{i_1}, \ldots, a_{i_k} : E \to \mathbb{R}\}_{i_1,\ldots,i_k \in \{1,\ldots,n\}}$ (permutating through all k-tuples in $\{1, \ldots, n\}$) such that

$$\int_\varphi \omega := \omega(\varphi) = \int_{I^k} \sum_{\text{all } k\text{-tuples}} a_{i_1,\ldots,i_k}(\varphi(u)) \frac{\partial(\varphi_{i_1}, \ldots, \varphi_{i_k})}{\partial(u_1, \ldots, u_k)} \mathrm{d}u$$

where the *Jacobian*

$$\frac{\partial(\varphi_{i_1}, \ldots, \varphi_{i_k})}{\partial(u_1, \ldots, u_k)}(u) := \det \begin{bmatrix} \frac{\partial \varphi_{i_1}}{\partial u_1} & \frac{\partial \varphi_{i_1}}{\partial u_2} & \cdots & \frac{\partial \varphi_{i_1}}{\partial u_k} \\ \frac{\partial \varphi_{i_2}}{\partial u_1} & \frac{\partial \varphi_{i_2}}{\partial u_2} & \cdots & \frac{\partial \varphi_{i_2}}{\partial u_k} \\ \vdots & & & \\ \frac{\partial \varphi_{i_k}}{\partial u_1} & \cdots & \cdots & \frac{\partial \varphi_{i_k}}{\partial u_k} \end{bmatrix} (u)$$

And we denote

$$\omega = \sum_{\text{all } k\text{-tuples}} a_{i_1,\ldots,i_k}(x) \mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \ldots \wedge \mathrm{d}x_{i_k}$$

This "wedge" $\wedge$ thing is now meaningless to us. It's just a convenient way of encoding the way the values that ω sends surfaces to by tracking the k-indices and their corresponding coefficients $a_{i_1,\ldots,i_k}$.

We'll often let $I = (i_1, \ldots, i_k)$ be a k-tuple with numbers chosen from $\{1, \ldots, n\}$, and $\mathrm{d}x_I := \mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \ldots \wedge \mathrm{d}x_{i_k}$.

Remark

This definition seems out of place. However, what we're really doing here is doing a grand change of variables from $\varphi(u)$ back to u, so that we can integrate over the cube I^k.

Properties

- Say $\omega = a(x)\mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \ldots \wedge \mathrm{d}x_{i_k}$. Then if $\bar{\omega} = a(x)\mathrm{d}x_{\pi I}$ for some permutation π then $\omega = \operatorname{sign}(\pi)\,\bar{\omega}$. This comes natural, through our usage of the Jacobian in the definition of forms.

Definition 3.4 (Basic k-form)

$\mathrm{d}x_I$ where I is increasing is a **basic k-form**.

Proposition 3.5

If I has a repeating index then $\mathrm{d}x_I = 0$

Proof

This is because we can perform the permutation π_0 that switch the repeating indices and still get the same I, therefore

$$\mathrm{d}x_I = \operatorname{sign}(\pi)\,\mathrm{d}x_I = -\mathrm{d}x_I \Rightarrow \mathrm{d}x_I = 0$$

□

Corollary 3.6

Every k-form can be written in terms of basic k-forms:

$$\omega = \sum_{\text{increasing } I} b_I(x)\mathrm{d}x_I$$

Warning: The a and b coefficient functions are not the same!

Proposition 3.7

$\omega = 0 \Rightarrow b_I = 0 \; \forall \; I$

Proof

Suppose not. That there exists J, v such that $b_J(v) > 0$ for some increasing J, and $v \in I^k$. Then what does it mean for $\omega = 0$? It means that $\omega(\varphi) = 0$ for all k-surfaces φ. We shall prove by contradiction, by constructing a surface where $\int_\varphi \omega$ can't be 0.

etc. □

4 Wedge Product

Let I, J be increasing p, q-tuples respectively. So $\mathrm{d}x_I, \mathrm{d}x_J$ are basic p, q-forms. Then we can define the new form

$$\mathrm{d}x_I \wedge \mathrm{d}x_J = \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_p} \wedge \mathrm{d}x_{j_1} \wedge \ldots \wedge \mathrm{d}x_{j_q}$$