1

$\boxed{10/9/23}$

compactness, uniform continuity & connectedness.

More on Compactness

Thm: A countable nested intersection of nonempty cpcts is cpct:

$K_1 \supset K_2 \supset \dots$ compact, $K_i \neq \emptyset \Rightarrow$

$K = \cap K_i \neq \emptyset$ is cpct.

Pf ① compact: K closed, $K \subseteq K_1$ cpct $\Rightarrow K$ cpct.

② nonempty: choose $x_n \in K_n$ $\forall n$. Then $\exists\, x_{n_j} \to x \in K_1$.

nestedness $\Rightarrow x \in K$. ☒

2

Rmk: Gives a way to construct strange cpct sets, e.g. middle thirds Cantor set:

$K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$

Thm If $f: M \to N$ is cts bijection & M cpct then f is homeo.

Pf: To show f^{-1} cts suffices to show $f(A)$ closed $\forall A$ closed in M.

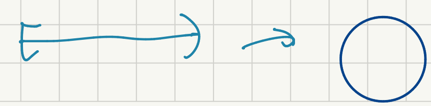
$f(A) = (f^{-1})^{pre}(A)$

But: A closed $\Rightarrow A$ cpct $\Rightarrow f(A)$ cpct

$\Rightarrow f(A)$ closed

☒

Rmk: Not true if M not cpct.



③

Thm (Extreme value Thm)
$f: M \to \mathbb{R}$ cts $X \subseteq M$ cpct $\Rightarrow$
f assumes max & min on X

Pf $f(X) \subseteq \mathbb{R}$ is compact
$\Rightarrow$ closed & bounded. Take
$M = l.u.b(f(X))$ $m = g.l.b.(f(X))$.

Thm $f: M \to N$ cts M cpct
$\Rightarrow f$ uniformly cts: $\forall \varepsilon > 0$
$\exists \delta > 0$ s.t. $d_M(x, x') < \delta \Rightarrow d_N(f(x), f(x')) < \varepsilon$.

Pf by contradiction.
Not unif cts $\Rightarrow \exists \varepsilon > 0$ & $(p_n), (q_n)$
s.t. $d(p_n, q_n) < \frac{1}{n}$ & $d(fp_n, fq_n) > \varepsilon$
compactness $\Rightarrow \exists n_k$ $p_{n_k} \to p \Rightarrow q_{n_k} \to p$
f cts $\Rightarrow f(p_{n_k}) \to f(p)$ & $f(q_{n_k}) \to f(p)$
but $d(f(p_{n_k}), f(q_{n_k})) > \varepsilon$ contradiction

④

Connectedness

$(X, \mathcal{T})$ top space.
Df $A \subseteq X$ disconnected if
$A = B \sqcup C$
$B \cap C = \emptyset$, B, C open.
A connected if not disconn.
$A \subseteq X$ path connected if $\forall x, x' \in A$
$\exists \gamma: [0,1] \to A$ cts.
s.t. $\gamma(0) = x$ $\gamma(1) = x'$

Thm Path conn $\Rightarrow$ conn, but
not conversely.

Thm (IVT) $f: X \to Y$ cts,
$A \subseteq X$ conn $\Rightarrow f(A)$ conn.

Coverings

$(X, \mathcal{T})$ top. space. An **open cover** of $A \subseteq X$ is $\mathcal{U} \subseteq \mathcal{T}$ s.t

$$A \subseteq \bigcup_{U \in \mathcal{U}} U$$

A **finite subcover** is $\{U_1, \dots, U_k\} \subseteq \mathcal{U}$ covering A.

Def $K \subseteq X$ is **compact** if every cover of K has a finite subcover.

Thm (Extreme value Thm) $f: X \to \mathbb{R}$
$A \subseteq X$ cpct $\Rightarrow$ f assumes max & min values on A.

Thm K cpct $\Rightarrow$ K sequentially cpct.