MATH 26200: Point-Set Topology

Take-home Midterm Exam

Hung Le Tran

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Textbook: Munkres, *Topology*

Problem 4.1 (done)

Let $\mathbb{R}$ denote the real numbers. Give $\mathbb{R}$ the topology in which the closed sets (other than all of $\mathbb{R}$) are the finite subsets. Verify that this is a topology, and prove that if $p(x)$ is a polynomial (with real coefficients), the function $x \mapsto p(x)$ is continuous in this topology.

Solution

We verify the properties of a topology through its closed sets:

1. $\mathbb{R}$ is closed by hypothesis. $\varnothing$ has $0 < \infty$ elements, so is closed.
2. Arbitrary intersections of closed $\{K_\alpha\}_{\alpha \in A}$, $\bigcap K_\alpha$, has at most $|K_\alpha|$ elements for some $\alpha \in A$, but $|K_\alpha| < \infty$ so $\bigcap K_\alpha$ is also closed.
3. Finite unions of clsoed $\{K_i\}_{i \in [N]}$ has at most $\sum_{i=1}^N |K_i| < \infty$, so is also closed.

It follows that this is indeed a topology.

Now, let $p(x)$ be any polynomial in x. Let $N = \deg(p(x)) \in \mathbb{Z}_{\geq 0}$. To show that p is continuous, want to show preimages of closed sets are closed. Let $K = \{y_1, \ldots, y_n\}$ be any closed set. Then:

$$\begin{aligned} p^{-1}(K) &= \bigcup_{i \in [n]} p^{-1}(y_i) \\ &= \bigcup_{i \in [n]} \{x : p(x) - y_i = 0\} \end{aligned}$$

And we have $\deg(p(x)) = N \Rightarrow \deg(p(x) - y_i) = N$ so it has at most N roots. Therefore:

$$|p^{-1}(K)| \leq Nn < \infty$$

so $p^{-1}(K)$ is closed, as required. □

Problem 4.2 (18.1 done)

Prove that for functions $f : \mathbb{R} \to \mathbb{R}$, the $\varepsilon - \delta$ definition of continuity implies the open set definition.

Solution

WTS if a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the $\varepsilon - \delta$ condition, then it is continuous in the open set definition.

Take any $W \subset \mathbb{R}$ open, want to prove that $f^{-1}(W)$ is open. Take $x \in f^{-1}(W)$, i.e., $f(x) \in W$. $f(x) \in W$ open, so there exists a basis $f(x) \in B(f(x'), \varepsilon') \subset W$. Define $\varepsilon := \frac{1}{2}\min\{\varepsilon' - |f(x) - f(x')|, |f(x) - f(x')|\} > 0$, then $f(x) \in B(f(x), \varepsilon) \subset B(f(x'), \varepsilon') \subset W$.

From the $\varepsilon - \delta$ condition, it follows that there exists some $\delta > 0$ such that $f(B(x, \delta)) \subset B(f(x), \varepsilon) \subset W$. It therefore follows that $B(x, \delta) \subset f^{-1}(W)$. We can do this for all $x \in f^{-1}(W)$, so it is open. f is therefore continuous in the open set definition. $\square$

Problem 4.3 (19.8 done)

Given sequences $(a_1, a_2, \dots)$ and $(b_1, b_2, \dots)$ of real numbers with $a_i > 0$ for all i, define $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by the equaiton:

$$h((x_1, x_2, \dots)) = (a_1x_1 + b_1, a_2x_2 + b_2, \dots)$$

Show that if $\mathbb{R}^\omega$ is given the product topology, h is a homeomorphism of $\mathbb{R}^\omega$ with itself. What happens if $\mathbb{R}^\omega$ is given the box topology?

Solution

To show that h is homeo, we want to show that it is bijective, continuous and that h^{-1} is also continuous.

1. We write
$$g((y_1, y_2, \dots)) = \left(\frac{y_1 - b_1}{a_1}, \frac{y_2 - b_2}{a_2}, \dots\right)$$
This map is well-defined since $a_i > 0 \ \forall\ i \in \mathbb{N}$. It's trivial that $h \circ g = g \circ h = id$. It follows that h is a bijection, with inverse $h^{-1} = g$.

2. WTS each of h's coordinate functions is continuous. Define, for each i, $f_i : \mathbb{R} \to \mathbb{R}, f_i(x) = a_ix + b_i$ then f_i is trivially continuous. Then $h_i(x) = f_i \circ \pi_i$ is a composition of 2 continuous functions, and is therefore also continuous. Since each of h's coordinate functions is continuous, h is also continuous.

3. $g = h^{-1}$ is continuous because of the same reason, each g_i is similarly continuous.

Therefore indeed h is homeo.

In the box topology, claim that h is also a homeo. First, trivially, it is still a bijection.

Take the typical basis element $\prod_{i \in \mathbb{N}} U_i$, then we have

$$h^{-1}(\prod_{i \in \mathbb{N}} U_i) = \prod_{i \in \mathbb{N}} h_i^{-1}(U_i)$$

Each h_i, as mentioned above, is continuous so $h_i^{-1}(U_i)$ is open in $\mathbb{R}$. Therefore $\prod_{i \in \mathbb{N}} h_i^{-1}(U_i)$ is open in the box topology, which implies h is continuous.

The proof for g is continuous is similar, since each g_i is similarly continuous. It follows that h is a homeo in the box topology as well. $\square$

Problem 4.4 (23.5 done)

A space is **totally disconnected** if its only connected subspaces are one-point sets. Show that if X has the discrete topology, then X is totally disconnected. Does the converse hold?

Solution

By hypothesis, X has the discrete topology. One-point sets are clearly connected, since there can't be a separation with 2 non-empty disjoint clopen subsets, which would make the number of elements in the set ≥ 2. Suppose K with $|K| \geq 2$ is also connected. Then there exists $a \in K$, and $K = \{a\} \sqcup (K - \{a\})$ is a separation of K into 2 non-empty closed sets, so K is not connected, $\Rightarrow\Leftarrow$.

It follows that the only connected subspaces are one-point sets (technically $\varnothing$ is also connected, but I figure the connected subspaces should be non-trivial), so X is totally disconnected as required.

The converse does not hold. Note that one-point sets are always connected, since its number of elements does not allow it to have a separation with 2 non-empty disjoint subsets.

Take $\mathbb{Q}$ in the subspace topology, $\mathbb{Q} \subset \mathbb{R}$. This is not the discrete topology, for each basis element $\mathbb{Q} \cap B(x, r)$ contains infinitely many points. However, we claim that $\mathbb{Q}$ is indeed totally disconnected. Suppose for sake of contradiction that there exists $U \subset \mathbb{Q}$ connected such that $|U| \geq 2$. Then we can find $q_1, q_2 \in U; q_1 < q_2$. We can then exhibit a separation of U:

$$U = (U \cap (-\infty, p) \sqcup (U \cap (p, +\infty))$$

where p is some irrational such that $q_1 < p < q_2$. They are clearly disjoint and open, and non-empty since q_1 and q_2 are respectively in them. So U is not connected. $\mathbb{Q}$ is therefore totally disconnected, but doesn't have the discrete topology. $\square$

Problem 4.5 (24.1 done)

(a) Show that no two of the spaces $(0, 1), (0, 1]$ and $[0, 1]$ are homeomorphic. [Hint: What happens if you remove a point from each of these spaces?]

(b) Suppose that there exist embeddings $f : X \to Y$ and $g : Y \to X$. Show by means of an example that X and Y need not be homeomorphic.

(c) Show $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic if $n > 1$.

Solution

(a) Removing any point α from $(0, 1)$ makes it a disconnected space: $(0, 1) = (0, \alpha) \sqcup (\alpha, 1)$, while removing 1 from $(0, 1]$ keeps the space connected ($(0, 1)$ is connected) and removing 1 from $[0, 1]$ also keeps the space connected ($[0, 1)$ is connected), so $(0, 1)$ is not homeomorphic to $(0, 1]$ nor is it homeomorphic to $[0, 1]$.

Removing any 2 distinct points from $(0, 1]$ makes it a disconnected space (removing 1 and another point $\alpha \in (0, 1)$ makes it disconnected: $(0, \alpha) \sqcup (\alpha, 1)$; and removing $\alpha, \beta \in (0, 1)$ yields $(0, \alpha) \sqcup (\alpha, 1]$), while removing 0 and 1 from $[0, 1]$ keeps the space connected ($(0, 1)$ is connected), so $(0, 1]$ is not homeomorphic to $[0, 1]$.

Why this "removing" reasoning work is that suppose there exists a homeomorphism

$f : (0, 1] \to (0, 1)$. Then $f((0, 1)) = (0, 1) - \{f(1)\}$; the LHS is connected (since f is continuous and $(0, 1)$ is connected) while the RHS is not, $\Rightarrow\Leftarrow$.

Similar for the removing-2-point case.

(b) Take $X = (0, 1), Y = [0, 1]$, then $f : X \to Y, f(x) = \frac{x}{2}$ and $g : Y \to X, g(y) = \frac{y+1}{4}$ are embeddings, but $(0, 1)$ and $[0, 1]$ are not homeomorphic as abovementioned.

(c) $n > 1$.

Removing any point α from $\mathbb{R}$ makes it a disconnected space: $(-\infty, \alpha) \sqcup (\alpha, \infty)$.

However, removing $0 \in \mathbb{R}^n$ keeps it connected. So $\mathbb{R}^n - \{0\}$ is not homeomorphic to $\mathbb{R}$. □

Problem 4.6 (24.10 done)

Show that if U is an open connected subspace of $\mathbb{R}^2$ then U is path connected. [Hint: Show that given $x_0 \in U$, the set of points that can be joined to x_0 by a path in U is both open and closed in U.]

Solution

Take $x_0 \in U$. Let $A = \{x \in U : \text{ there exists a path } x_0 \to x\}$. Then $x_0 \in A$, so A is non-empty. WTS A is clopen in U.

Take any $x \in A$, then since U is open, there exists some $B(x, \varepsilon)$ (can always center the ball) such that $x \in B(x, \varepsilon) \subset U$. Take any $y \in B(x, \varepsilon)$, then since the ball is convex, the segment $x \to y$ is contained in $B(x, \varepsilon)$, and therefore in U. Therefore $f(t) = x+t(y-x)$ is a path from $x \to y$. $x \in A$, so there exists a path from $x_0 \to x$. Concatenate these 2 paths and we get a path from $x_0 \to y$, so $y \in A$. This works for any $y \in B(x, \varepsilon)$, so $B(x, \varepsilon) \subset A$. This works for any $x \in A$, so A is open.

A is also closed, take $X - A = \{x \in U : \text{ there is no path } x_0 \to x\}$. Take any $x \in X - A$, then since U is open, there exists some $B(x, \varepsilon)$ (can always center the ball) such that $x \in B(x, \varepsilon) \subset U$. Take any $y \in B(x, \varepsilon)$, then similarly, $f(t) = x + t(y - x)$ is a path from $x \to y$. We can conclude that $y \in X - A$, because if $y \in A$, meaning there's a path from $x_0 \to y$, then one can concatenate that path with the path from $x \to y$ in reverse to get a path from $x_0 \to x$, but $x \notin A$, so it would be a contradiction. Therefore for any $y \in B(x, \varepsilon)$, $y \in X - A \Rightarrow B(x, \varepsilon) \subset X - A$. This works for any $x \in X - A$, so $X - A$ is open, so A is closed.

A is clopen and non-empty, $A \subset U$ connected, so $A = U$. There is a path from x_0 to all $x \in U$. Then for all $x, y \in U$, concatenate path $x_0 \to x$ in reverse to path $x_0 \to y$, and we get path $x \to y$. So U is path-connected. □

Problem 4.7 (26.11 done)

Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a collection of closed connected subsets of X that is simply ordered by proper inclusion. Then

$$Y = \bigcap_{A\in\mathcal{A}} A$$

is connected. [Hint: if $C \cup D$ is a separation of Y, choose disjoint open sets U and V of X containing C and D, respectively, and show that $\bigcap_{A\in\mathcal{A}}(A - (U \cup V))$ is not empty.]

Solution

Each A is closed so $Y = \bigcap_{A \in \mathcal{A}} A$ is also closed in X.

Suppose, for the sake of contradiction, that Y is not connected, i.e., that there exists separation $Y = C \sqcup D$ of C, D non-empty, disjoint, clopen sets in Y. C, D are closed in Y, Y closed in X so they are closed in X.

We use a fact, shown in class, that compact Hausdorff spaces are normal. So X is normal. Therefore, since C, D are closed and disjoint in X, there exists open $U, V \subset X$ such that $C \subset U, D \subset V$ and $U \cap V = \varnothing$.

Then, for each $A \in \mathcal{A}$, claim that $(A \cap U) \cup (A \cap V) \neq A$. Suppose not, that $(A \cap U) \cup (A \cap V) = A$, then $U \cap V = \varnothing$ implies that $(A \cap U) \cap (A \cap V) = \varnothing$, with both $(A \cap U), (A \cap V)$ open in A, and $A \cap U \supset C$, $A \cap V \supset D$ so they are both non-empty. Hence we get a separation of A, but A is connected, $\Rightarrow\Leftarrow$.

It follows that $(A \cap U) \cup (A \cap V) \neq A \Rightarrow A - (A \cap U) \cup (A \cap V) = A - U \cup V \neq \varnothing$ for all $A \in \mathcal{A}$.

$U \cup V$ is open in X, so $A - U \cup V$ is closed in X. X is compact, so $A - U \cup V$ is compact. $\mathcal{A}$ is a collection of A ordered by proper inclusion, so $\{A - U \cup V\}_{A \in \mathcal{A}}$ is a collection of non-empty, compact, subsets ordered by proper inclusion.

It then follows that

$$\bigcap_{A \in \mathcal{A}} (A - U \cup V) \neq \varnothing$$

However,

$$\bigcap_{A \in \mathcal{A}} (A - U \cup V) = \left(\bigcap_{A \in \mathcal{A}} A \right) - U \cup V = Y - U \cup V = C \cup D - U \cup V = \varnothing, \Rightarrow\Leftarrow$$

It follows that Y is connected. $\square$