207 Homework #6 Due Wednesday, Nov 8.

I. Read Pugh, Chapter 5, Sections 2 and 3.

II. Chap 5: [9, 11, 12] 7, 8, 15, 16 (a)-(c), 17(a), 18, 20

III. For each of the examples below, draw an approximate picture of $f(B(p,\epsilon))$ for the indicated function f and point p, for ϵ very small.

(a) $f(t) = (t, t^2)$ at $p = 1$,

(b) $f(x, y) = (e^{2x^2+2x+y}, \sin(3x) - \cos(x + y))$; $p = (0, 0)$,

(c) $f(x, y) = (x^2 + y, 2xy, 3\log(y) - x)$; $p = (1, 1)$,

(d) $f(x, y, z) = (x + y + \sin(xy), y + z, x + y + z)$, $p = (0, 0, 0)$.

IV. Let $U \subset \mathbb{R}^n$, and let $f\colon U \to \mathbb{R}$ be a function. For $v, w \in \mathbb{R}^n$ (viewed as $n \times 1$ matrices), denote by $\langle v, w\rangle = v \cdot w = v^t w$ the standard Euclidean dot product. Recall that $|v| = \langle v, v\rangle^{1/2}$.

(a) Suppose that f is differentiable at $p \in U$, with derivative $Df_p \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$. Show that there is a unique vector $w \in \mathbb{R}^n$ such that, for all $v \in \mathbb{R}^n$:

$$\langle v, w\rangle = Df_p(v).$$

The vector w is called the *gradient vector of f at p* and is denoted $\mathrm{grad}_p(f)$.

(b) Show that, with respect to the standard basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$, we have

$$\mathrm{grad}_p(f) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \cdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}.$$

(c) Show that $\|Df_p\| = |\,\mathrm{grad}_p(f)|$, and that the maximum value of the function $g\colon S^{n-1} \to \mathbb{R}$ on the $(n-1)$-sphere defined by

$$g(v) = Df_p(v)$$

is attained at $v = \operatorname{grad}_p(f)/|\operatorname{grad}_p(f)|$. Thus the direction of the gradient vector field $f(p) = \operatorname{grad}_p(f)$ is the *direction of steepest ascent (increase) for the function* f, and the magnitude $|F(p)|$ is *the greatest (to first order) rate of increase of the function* f *at* p.

(d) Show that if $\gamma\colon (-\epsilon, \epsilon) \to U$ is a differentiable curve through p along which f is constant (i.e. if $\gamma(0) = p$ and $f(\gamma(t)) = f(\gamma(0))$ for all $t \in (-\epsilon, \epsilon)$), then $\gamma'(0)$ is perpendicular to $\operatorname{grad}_p(f)$; i.e.

$$\langle \operatorname{grad}_p(f), \gamma'(0) \rangle = 0.$$

In words, this means that the gradient of f is everywhere perpendicular to the level sets $f^{-1}(c), c \in \mathbb{R}$ of f.

(e) For the function $f(x, y) = \sin(2x + y) + e^{\cos(xy)}$, plot, using a computer program, the graph of the function f (in $3D$), and the level sets of f in the region $[-2, 2] \times [-2, 2]$. Then plot the gradient vector field for f. Wolfram alpha works, for example. Think about steepest ascent in terms of climbing the graph and level sets in terms of walking at a fixed elevation on the graph. The graph of the level sets is like a map of the mountain, and the gradient vector field points at the shortest, most exhausting, path up. You don't have to write down your thoughts, but do supply pics of the graphs.

(f) * Explain how to generalize this notion of gradient to any inner product on $\mathbb{R}^m$. In this way, one can use a different measure of increase of the function (if you're a budding economist or physicist, this could be useful...). The notion of gradient generalizes even more: you can let the inner product $\langle \cdot, \cdot \rangle_p$ depend on the point p! (This is the beginning of Riemannian geometry).