MATH 26200: Point-Set Topology

Problem Set 2

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Textbook: Munkres, *Topology*

Problem 2.1 (17.5 done)

Let X be an ordered set in the order topology. Show that $\overline{(a,b)} \subset [a,b]$. Under what conditions does equality hold?

Solution

Let $x \in \overline{(a,b)}$. WTS $x \in [a,b] \Leftrightarrow x \geq a, x \leq b$.

We first prove that $x \geq a$. Suppose not, that is, $x < a$. Then there exists a neighborhood of x that does not intersect (a,b), namely, the open ray $(-\infty, a)$, which contains x, and:

$$(-\infty, a) \cap (a,b) = \varnothing$$

This is a contradiction, since $x \in \overline{(a,b)}$. Therefore $x \geq a$.

Similarly, $x \leq b$. Therefore $x \in [a,b]$, which implies $\overline{(a,b)} \subset [a,b]$.

Equality holds when both a and b are limit points of (a,b). $\square$

Problem 2.2 (17.6 done)

Let A, B and A_α denote subsets of a space X. Prove the following.

(a) If $A \subset B$, then $\overline{A} \subset \overline{B}$

(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(c) $\overline{\bigcup A_\alpha} \supset \bigcup \overline{A_\alpha}$, and give an example where equality fails.

Solution

(a) Let $x \in \overline{A}$, WTS $x \in \overline{B}$. Take any open neighborhood U of x.

Since $x \in \overline{A}$, $U \cap A \neq \varnothing$. But $A \subset B \Rightarrow U \cap B \neq \varnothing$ too.

This means that every open neighborhood U of x has non-empty intersection with B.

Therefore $x \in \overline{B}$. $\square$

(b) 1. WTS $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$.

$\overline{A}, \overline{B}$ are closed in X. It follows that $\overline{A} \cup \overline{B}$ is also closed in X.

Also, $A \subset \overline{A}, B \subset \overline{B} \Rightarrow A \cup B \subset \overline{A} \cup \overline{B}$.

$\overline{A \cup B}$ is the smallest subset that contains $A \cup B$, so $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$.

2. WTS $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$.

Take $x \in \overline{A} \cup \overline{B}$. WLOG, $x \in \overline{A}$. This implies every open neighborhood U of x satisfies:

$$U \cap A \neq \varnothing \Rightarrow U \cap (A \cup B) \neq \varnothing$$

It follows that $x \in \overline{A \cup B}$. Therefore $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$ as required.

3. From **1.**, **2.**, it follows that $\overline{A \cup B} = \overline{A} \cup \overline{B}$. $\square$

(c) Take $x \in \bigcup \overline{A_\alpha}$. WTS $x \in \overline{\bigcup A_\alpha}$.

Since $x \in \bigcup \overline{A_\alpha}$, $x \in \overline{A_\beta}$ for some β.

It follows that every open neighborhood U of x satisfies:

$$U \cap A_\beta \neq \varnothing \Rightarrow U \cap \bigcup A_\alpha \neq \varnothing$$

and therefore $x \in \overline{\bigcup A_\alpha}$. It follows that $\overline{\bigcup A_\alpha} \supset \bigcup \overline{A_\alpha}$ as required.

An example of when equality fails:

$$A_\alpha \coloneqq \{\alpha\} \;\forall\, \alpha \in \mathbb{Q}$$

Then $\overline{A_\alpha} = \{\alpha\} \Rightarrow \bigcup \overline{A_\alpha} = \mathbb{Q}$, while

$$\overline{\bigcup A_\alpha} = \overline{\mathbb{Q}} = \mathbb{R} \neq \mathbb{Q}$$

$\square$

Problem 2.3 (17.13 done)

Show that X is Hausdorff iff the *diagonal* $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$.

Solution

$\boxed{\Rightarrow}$ By hypothesis, X is Hausdorff. WTS $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$. Equivalently, WTS $\overline{\Delta} = \Delta$.

Suppose not, i.e., that there exists $y \times z \in \overline{\Delta} - \Delta$. $y \times z \notin \Delta$ so $y \neq z$. Since X is Hausdorff, there exists $U \ni y, V \ni z$ open such that $U \cap V = \varnothing$.

Then, since $y \times z$ is a limit point of Δ, and $U \times V$ is open $X \times X$, $y \times z \in U \times V$, it follows that

$$U \times V \cap \Delta \neq \varnothing$$

Say, $x_0 \times x_o \in (U \times V) \cap \Delta$. Then $x_0 \in U, x_0 \in V \Rightarrow U \cap V \neq \varnothing, \Rightarrow\Leftarrow$.

Therefore Δ is closed in $X \times X$ as required. $\square$

$\boxed{\Leftarrow}$ By hypothesis, Δ is closed in $X \times X$. This implies $X \times X - \Delta = \{y \times z : y, z \in X; y \neq z\}$ is open.

Take $y, z \in X; y \neq z$. Then $y \times z \in X \times X - \Delta$, so there exists $U \times V$ with U, V open X such that $y \times z \in U \times V \subset X \times X - \Delta$, i.e., $U \times V \cap \Delta = \varnothing$.

WTS $U \cap V = \varnothing$. Suppose not, then there exists $w \in U \cap V$, then $w \times w \in U \times V$. But $w \times w \in \Delta$ too, so $U \times V \cap \Delta \neq \varnothing, \Rightarrow\Leftarrow$. Therefore $U \cap V = \varnothing$.

Therefore we've demonstrated U, V open with $U \ni y, V \ni z, U \cap V = \varnothing$ for all $y, z \in X; y \neq z$. X is therefore Hausdorff. $\square$

Problem 2.4 (18.12 done)

Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by the equation

$$F(x \times y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } x \times y \neq 0 \times 0 \\ 0 & \text{if } x \times y = 0 \times 0 \end{cases}$$

(a) Show that F is continuous in each variable separately.

(b) Compute the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = F(x \times x)$.

(c) Show that F is not continuous.

Solution

(a) We write F as a function in x:

$$G_y(x) = G(x; y) = F(x \times y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then for $x \neq 0$, G_y is clearly continuous at x, since it is the quotient of continuous functions in x, and the denominator $x^2 + y^2 > 0$.

It remains to show that G_y is continuous at $x = 0$. Fix $\varepsilon > 0$. Then for all x such that $|x| = |x - 0| < \delta := \varepsilon |y|$:

$$\left| \frac{xy}{x^2 + y^2} - 0 \right| \leq \left| \frac{xy}{y^2} \right| < \frac{\varepsilon |y|^2}{y^2} = \varepsilon$$

G_y is therefore continuous at $x = 0$ too. So it is continuous.

F is symmetric in x and y, so F is also continuous in y.

(b)

$$g(x) = F(x \times x) = \begin{cases} \dfrac{1}{2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(c) For sake of contradiction, suppose that F is indeed continuous. $\mathbb{R}$ is Hausdorff, so $\mathbb{R} \times \mathbb{R}$ is also Hausdorff. Then the diagonal $\Delta = \{x \times x \mid x \in \mathbb{R}\}$ is closed in $\mathbb{R} \times \mathbb{R}$, per the problem above.

F is continuous, so $F\mid_\Delta : \Delta \to \mathbb{R}$ is also continuous.

$\left\{\frac{1}{2}\right\}$ is closed in $\mathbb{R}$ so $F|_\Delta^{-1}\left(\left\{\frac{1}{2}\right\}\right)$ must also be closed in Δ. We can explicitly state, from **(b)**, that:

$$F_\Delta^{-1}\left(\left\{\frac{1}{2}\right\}\right) = (\mathbb{R} - \{0\})^2$$

$(\mathbb{R} - \{0\})^2$ is closed in Δ, and Δ is closed in $\mathbb{R}^2$, so $(\mathbb{R} - \{0\})^2$ is closed in $\mathbb{R}^2$. But it's clear that $\mathbb{R} - \{0\}$ is open in $\mathbb{R}$, so $(\mathbb{R} - \{0\})^2$ is open in $\mathbb{R}^2$, $\Rightarrow\Leftarrow$.

It follows that F is not continuous. $\square$

Problem 2.5 (18.13 done)

Let $A \subset X$; let $f : A \to Y$ be continuous; let Y be Hausdorff. Show that if f maybe extended to a continuous function $g : \overline{A} \to Y$, then g is uniquely determined by f.

Solution

Suppose not, i.e., that there exists $g, h : \overline{A} \to Y$ that are both continuous extension of f from A to $\overline{A}$, and $g \not\equiv h$, i.e., there exists $x \in \overline{A}$ where $g(x) \neq h(x)$. x can't be in A since $g(x) = f(x) = h(x)$, so x must be a limit point of A.

Then since $g(x) \neq h(x); g(x), h(x) \in Y$, and Y is Hausdorff, it follows that there exists U, V open in Y such that $g(x) \in U, h(x) \in V, U \cap V = \varnothing$.

Since g, h are continuous, it follows that $g^{-1}(U)$ and $h^{-1}(V)$ are also open in X.

Then $W = g^{-1}(U) \cap h^{-1}(V)$ is open in X and contains x. x is a limit point of A, so there exists $y \neq x, y \in W \subset A$.

But since $y \in A$, $g(y) = h(y)$. And $y \in W \Rightarrow g(y) \in U, h(y) \in V \Rightarrow U \cap V \neq \varnothing, \Rightarrow\Leftarrow$.

We therefore have that the continuous extension of f from A to $\overline{A}$, if possible, must be uniquely determined. $\square$

Problem 2.6 (19.4 done)

Show that $(X_1 \times \cdots \times X_{n-1}) \times X_n$ is homeomorphic with $X_1 \times \cdots \times X_n$.

Solution

Consider

$$\begin{aligned} f : (X_1 \times \cdots \times X_{n-1}) \times X_n &\to X_1 \times \cdots \times X_n \\ (x_1 \times \cdots \times x_{n-1}) \times x_n &\mapsto x_1 \times \cdots \times x_n \end{aligned}$$

1. Clearly, f is bijective.

2. WTS f is continuous. Take $U_1 \times \cdots \times U_n \in X_1 \cdots \times X_n$, a typical basis element of $X_1 \times \cdots \times X_n$, i.e., U_i is open in X_i for all $i \in [n]$.

Then its preimage $f^{-1}(U_1 \times \cdots \times U_n) = (U_1 \times \cdots \times U_{n-1}) \times U_n$.

Since U_i is open in X_i for all $i \in [n]$, particularly $i \in [n-1]$, $(U_1 \times \cdots \times U_{n-1})$ is a typical basis element in $(X_1 \times \cdots \times X_{n-1})$, so it is open. U_n is also open in X_n, so

$(U_1 \times \cdots \times U_{n-1}) \times U_n$ is a typical basis element in $(X_1 \times \cdots \times X_{n-1}) \times X_n$, so it is open. f is therefore continuous.

3. WTS f^{-1} is continuous. Take $(U_1 \times \cdots \times U_{n-1}) \times U_n \in (X_1 \times \cdots \times X_{n-1}) \times X_n$, a typical basis element of $(X_1 \times \cdots \times X_{n-1}) \times X_n$, which means that U_n is open in X_n, and $(U_1 \times \cdots \times U_{n-1})$ is open $(X_1 \times \cdots \times X_{n-1})$, which then means that U_i is open in X_i for $i \in [n-1]$. In summary, U_i is open in X_i for $i \in [n]$.

Then its preimage through f^{-1} is $f((U_1 \times \cdots U_{n-1}) \times U_n) = U_1 \times \cdots \times U_n$. It is clearly a typical basis element for $X_1 \times \cdots \times X_n$, because U_i is open in X_i for $i \in [n]$, and is therefore open. f^{-1} is therefore also continuous.

4. From **1.**, **2.**, **3.**, it follows that f is a homeomorphism. We have demonstrated a homeomorphism between $(X_1 \times \cdots \times X_{n-1}) \times X_n$ and $X_1 \times \cdots \times X_n$, so they are homeomorphic. $\square$

Problem 2.7 (19.7 done)

Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero," that is, all sequences $(x_1, x_2, \ldots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies? Justify your answer.

Solution

Define $A := \mathbb{R}^\omega - \mathbb{R}^\infty$, i.e., the set of sequences such that $x_i \neq 0$ for infinitely many values of i, i.e., $x_i = 0$ for finitely many values of i.

1. In the box topology. WTS $\overline{\mathbb{R}^\infty} = \mathbb{R}^\infty \Leftrightarrow \mathbb{R}^\infty$ is closed $\Leftrightarrow A$ is open.

Take $(y_i)_{i \in \mathbb{N}} = (y_1, \ldots) \in A$, i.e., $y_i = 0$ for finitely many values of i. WLOG, $y_1 = \ldots = y_n = 0; y_i > 0\ \forall\ i \geq n+1$ for some $n \in \mathbb{N}$ (negative/mixed cases are handled similarly and trivially).

Then there exists $U := (-1,1)^n \times (\frac{y_{n+1}}{2}, \frac{3y_{n+1}}{2}) \times (\frac{y_{n+2}}{2}, \frac{3y_{n+2}}{2}) \times \ldots$ is a basis element in the box topology, and is therefore open.

Then it's clear that by construction, $(y_i) \in U$.

Furthermore, we claim that $U \subset A$. Take any $(z_i) \in U$. Since, for all $i \geq n+1$, we have that $y_i > 0$ and $z_i \in (\frac{y_{n+1}}{2}, \frac{3y_{n+1}}{2})$, it follows that $z_i \neq 0$ for all $i \geq n+1$, i.e., infinitely many values of i. It follows that $(z_i) \notin \mathbb{R}^\infty \Rightarrow U \subset A$.

We have therefore demonstrated, for all $(y_i) \in A$, there exists open U such that $(y_i) \in U \subset A$. A is therefore open in the box topology.

It follows that $\mathbb{R}^\omega - A = \mathbb{R}^\infty$ is closed, so $\overline{\mathbb{R}^\infty} = \mathbb{R}^\infty$.

2. In the product topology. WTS $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$, by showing that every point in $\mathbb{R}^\omega$ is a limit point of $\mathbb{R}^\infty$.

Take $(y_i)_{i \in \mathbb{N}} \in \mathbb{R}^\omega$. Take a typical basis element in the product topology that contains (y_i), that is:

$$B = \mathbb{R} \times \ldots \times \mathbb{R} \times U_{i_1} \times \mathbb{R} \times \ldots \mathbb{R} \times U_{i_2} \times \mathbb{R} \times \ldots \times U_{i_n} \times \mathbb{R} \times \ldots$$

where only $U_{i_1}, U_{i_2}, \ldots, U_{i_n} \neq \mathbb{R}$ at the $i_1, i_2, \ldots, i_n$-th coordinate.

For this basis element to contain (y_i), it requires:

$$y_{i_k} \in U_{i_k} \ \forall\, k \in [n],$$

with all other coordinates being trivially true (simply being in $\mathbb{R}$).

We can then immediately show that $B \cap \mathbb{R}^\infty \neq \varnothing$, by demonstrating a point of intersection (z_i) defined by

$$z_{i_k} = y_{i_k} \ \forall\, k \in [n], z_i = 0 \text{ otherwise}$$

Then $(z_i) \in \mathbb{R}^\infty$ since $z_i \neq 0$ for only finitely many values of i. At the same time, since $z_{i_k} = y_{i_k} \in U_{i_k}$ and $0 \in \mathbb{R}$, it follows that $(z_i) \in B$ too. It follows that $B \cap \mathbb{R}^\infty \neq \varnothing$.

Therefore, $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$. □

Problem 2.8 (17.21* (Bonus))

(Kuratowski) Consider the collection of all subsets A of the topological space X. The operations of closure $A \to \overline{A}$ and complementation $A \to X - A$ are functions from this collection to itself.

(a) Show that starting with a given set A, one can form no more than 14 distinct sets by applying these two operations successively.

(b) Find a subset of A of $\mathbb{R}$ (in its usual topology) for which the maximum of 14 is obtained.

Solution

□