MATH 26200: Point-Set Topology
Problem Set 6

Hung Le Tran

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Textbook: Munkres, *Topology*

Problem 6.1 (28.1 done)

Give $[0,1]^\omega$ the uniform topology. Find an infinite subset of this space that has no limit point.

Solution

Let
$$x^{(n)} := (0, \dots, 0, 1, 0, \dots) \in [0,1]^\omega$$
where 1 is at the n-th index.

Consider set $A = \{x^{(n)} : n \in \mathbb{N}\}$. Since $[0,1]^\omega$ is endowed with the uniform metric, if A has some limit point $x \in [0,1]^\omega$ then there must necessarily be a sequence $\{x^{(n_j)}\}_{j \in \mathbb{N}}$ such that $x_{n_j} \xrightarrow{j \to \infty} x$ (note that n_j here is not required to be increasing in j).

This implies for all $\varepsilon > 0$, there exists $J \in \mathbb{N}$ such that for all $j \geq J$,
$$\bar{\rho}(x^{(n_j)}, x) < \varepsilon$$

Pick $\varepsilon = 1/2$. Then for J, we have that
$$\varepsilon > \bar{\rho}(x^{(n_J)}, x) \geq \bar{d}(1, x_{n_J}) = |x_{n_J} - 1|$$
hence $x_{n_J} > 1 - \varepsilon$. It then follows that
$$\bar{\rho}(x^{(n_{J+1})}, x) \geq \bar{d}(x^{(n_{J+1})}_{n_J}, x_{n_J}) = 1 - \varepsilon = \varepsilon, \Rightarrow\Leftarrow$$

It follows that A has no limit point. $\square$

Problem 6.2 (43.4 done)

Show that the metric space (X,d) is complete if and only if for every nested sequence $A_1 \supset A_2 \supset \cdots$ of nonempty closed set of X such that $\operatorname{diam} A_n \to 0$, the intersection of the sets A_n is nonempty.

Solution

$\boxed{\Rightarrow}$ By hypothesis, (X,d) is complete.

Take sequence $A_1 \supset A_2 \supset \cdots$ of nonempty closed set of X such that $\operatorname{diam} A_n \to 0$, and let $A = \bigcap_{n \in \mathbb{N}} A_n$. WTS $A \neq \varnothing$.

For every $n \in \mathbb{N}$, since $A_n \neq \varnothing$, take $x_n \in A_n$. Consider the sequence $(x_n)_{n \in \mathbb{N}}$. Since $A_n \subset A_{n-1} \subset \cdots \subset A_1$, it follows that for all n, $x_n \in A_m$ for all $m \leq n$. It then follows that $d(x_n, x_m) \leq \operatorname{diam} A_m$. Take $m \leq n$ to ∞, since $\operatorname{diam} A_m \xrightarrow{m \to \infty} 0$, it follows that $(x_n)_{n \in \mathbb{N}}$ is Cauchy.

Since X is complete, it follows that $x_n \xrightarrow{n \to \infty} x \in X$. WTS $x \in \bigcap_{n \in \mathbb{N}} A_n$.

Indeed, for each A_m, since it is closed in complete X, it is also complete. Then consider the cutoff-ed sequence $(y_n = x_{n+m})_{n \in \mathbb{N}}$. It is a sequence in A_m, and is Cauchy as we noted. It follows that $y_n \xrightarrow{n \to \infty} x' \in A_m$. But then (y_n) is just a subsequence of the original (x_n), hence $x' = x$. It then follows that $x \in A_m \ \forall\, m \in \mathbb{N} \Rightarrow x \in \bigcap_{m \in \mathbb{N}} A_m \Rightarrow A \neq \varnothing$ as required. $\square$

$\boxed{\Leftarrow}$ Take Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X. WTS $x_n \xrightarrow{n\to\infty} x \in X$.

Let $A_n = \{x_k : k \geq n\} = \{x_n, x_{n+1}, \ldots\}$. Then since (x_n) is Cauchy, it follows that $\operatorname{diam} A_n \xrightarrow{n\to\infty} 0$. Consider $\operatorname{diam} \overline{A_n}$: then every point in the closure is well approximated by points in A_n, therefore

$$\operatorname{diam} \overline{A_n} \xrightarrow{n\to\infty} 0$$

By construction, we have that $A_1 \supset A_2 \supset \ldots \Rightarrow \overline{A_1} \supset \overline{A_2} \supset \ldots$, with $\operatorname{diam} \overline{A_n} \xrightarrow{n\to\infty} 0$. By hypothesis, therefore, $\bigcap_{n\in\mathbb{N}} \overline{A_n} \neq \varnothing$. Take $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n} \Rightarrow \ \forall\, n \in \mathbb{N}, x \in \overline{A_n}$. But $x_n \in \overline{A_n}$ and $\operatorname{diam} \overline{A_n} \xrightarrow{n\to\infty} 0 \Rightarrow x_n \xrightarrow{n\to\infty} x$. $x \in X$, so X is complete. $\square$

Problem 6.3 (43.8 done)

If X and Y are spaces, define

$$e : X \times \mathcal{C}(X,Y) \to Y$$

by the equation $e(x,f) = f(x)$; the map e is called the **evaluation map**. Show that if d is a metric for Y and $\mathcal{C}(X,Y)$ has the corresponding uniform topology, then e is continuous.

Solution

Recall that the uniform topology on $\mathcal{C}(X,Y)$ is the topology induced by the metric $\bar{\rho}$, where for $f,g \in \mathcal{C}(X,Y)$,

$$\bar{\rho}(f,g) = \sup\{\bar{d}(f(\alpha), g(\alpha)) : \alpha \in X\}$$

Take typical basis element $B(y,\varepsilon)$ of Y. WTS $e^{-1}(B(y,\varepsilon))$ is open in $X \times \mathcal{C}(X,Y)$. Take any $x \times f \in e^{-1}(B(y,\varepsilon))$. WTS there exists an open neighborhood of $x \times f$ that is contained in $e^{-1}(B(y,\varepsilon))$. Indeed, consider

$$U := f^{-1}(fx, \varepsilon/2) \times B(f, \varepsilon/2)$$

It is clear that $x \times f \in U$. Then, take any $x', f' \in U$. Then

$$d(f'x', y) \leq d(f'x', fx') + d(fx', fx) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

so it follows that $x \times f \in U \subset e^{-1}(B(y,\varepsilon))$ as required. $\square$

Problem 6.4 (45.7 done)

Let (X,d) be a metric space. If $A \subset X$ and $\varepsilon > 0$, let $U(A,\varepsilon)$ be the ε-neighborhood of A. Let $\mathcal{H}$ be the collection of all (nonempty) closed, bounded, subsets of X. If $A,B \in \mathcal{H}$, define

$$D(A,B) = \inf\{\varepsilon \mid A \subset U(B,\varepsilon) \text{ and } B \subset U(A,\varepsilon)\}$$

(a) Show that D is a metric on $\mathcal{H}$, it is called the **Hausdorff metric**.

(b) Show that if (X,d) is complete, so is $(\mathcal{H}, D)$. [Hint: Let A_n be a Cauchy sequence in $\mathcal{H}$, by passing to a subsequence, assume $D(A_n, A_{n+1}) < \frac{1}{2^n}$. Define A to be the set of all points x that are the limits of sequences $x_1, x_2, \ldots$ such that $x_i \in A_i$ for each i, and $d(x_i, x_{i+1}) < \frac{1}{2^i}$. Show $A_n \to \overline{A}$.]

(c) Show that if (X,d) is totally bounded, so is $(\mathcal{H}, D)$. [Hint: Given ε, choose $\delta < \varepsilon$ and let S be a finite subset of X such that the collection $\{B_d(x,\delta) \mid x \in S\}$ covers X. Let $\mathcal{A}$ be the collection of all nonempty subsets of S; show that $\{B_D(A,\varepsilon) \mid A \in \mathcal{A}\}$ covers $\mathcal{H}$.]

(d) Theorem: If X is compact in the metric d, then the space $\mathcal{H}$ is compact in the Hausdorff metric D.

Solution

(a) WTS D is a metric.

- $D(A,B) \geq 0$ since it is infimum of set of positive numbers.
- If $D(A,B) = 0$ then for all $\varepsilon > 0$, $A \subset U(B,\varepsilon), B \subset U(A,\varepsilon)$. $A \subset U(B,\varepsilon)$ means that for every point $a \in A, a \in U(B,\varepsilon)$, i.e., $B(a,\varepsilon) \cap B \neq \varnothing$. So every point $a \in A$ is in the closure of B. B is

closed so $\overline{B} = B$. Therefore $A \subset B$. Similarly, $B \subset A$. Hence $A = B$ as required.

- By definition, $D(A, B) = D(B, A)$ trivially.
- WTS $D(A, B) + D(B, C) \geq D(A, C)$. Take $\varepsilon_{AB}, \varepsilon_{BC}$ such that $A \subset U(B, \varepsilon_{AB}), B \subset U(A, \varepsilon AB)$ and $B \subset U(C, \varepsilon_{BC}), C \subset U(B, \varepsilon_{BC})$. From $A \subset U(B, \varepsilon_{AB})$ and $B \subset U(C, \varepsilon_{BC})$ it follows that $A \subset U(C, \varepsilon_{AB} + \varepsilon_{BC})$ by triangle inequality. Vice versa, $C \subset U(A, \varepsilon_{AB} + \varepsilon_{BC})$. Taking infs, it then follows that $D(A, B) + D(B, C) \geq D(A, C)$.

Hence D is indeed a metric. $\square$

(b) (X, d) is complete.

Take $(A_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathcal{H}, D)$. WTS A_n converges to some set in $\mathcal{H}$.

Since (A_n) is Cauchy, it follows that for all $\varepsilon > 0$, there exists N_ε such that for all $n, m \geq N_\varepsilon$, we have

$$D(A_n, A_m) < \varepsilon$$

Then, for all j, set $\varepsilon = \frac{1}{2^j}$. Then there exists some N_j such that for all $n, m \geq N_j$, we have $D(A_n, A_m) < \frac{1}{2^j}$. WLOG, set the N_j be increasing j.

Define a new sequence (as a subsequence of A_n) $B_j \coloneqq A_{N_j}$, then since N_j is increasing in j, it follows that

$$D(B_n, B_{n+1}) < \frac{1}{2^n}$$

Define A to be the set of all points x that are the limits of sequences $(x_1, x_2, \ldots)$ such that for all i, $x_i \in B_i$ and $d(x_i, x_{i+1}) < \frac{1}{2^i}$. We want to show that $B_n \xrightarrow{n\to\infty} \overline{A}$.

0. We first show that A is non-empty. Take any $x_1 \in B_1$. Then since $B_1 \subset U(B_2, \frac{1}{2})$, it follows that there exists some $x_2 \in B_2$ such that $d(x_1, x_2) < \frac{1}{2}$. Perform this inductively, then we have a sequence (x_n) with $x_n \in B_n$ such that $d(x_n, x_{n+1}) < \frac{1}{2^n}$. So (x_n) is Cauchy (the sum $\sum_{k=n}^{m} \frac{1}{2^k} \xrightarrow{n\to\infty} 0$, so by triangle inequality, it is Cauchy) in complete X, so $x_n \xrightarrow{n\to\infty} x \in X$. So A is nonempty, so $\overline{A}$ is nonempty. We pay special attention to how a "valid" sequence can be "generated" from a point.

1. It is trivial that if A_n is bounded then $\overline{A}$ is also bounded.

2. Fix N. We now want to show $B_N \subset U(\overline{A}, \frac{1}{2^{N-1}})$.

Take some $b_N \in B_N$. Then by the aforementioned construction, we can form sequence $(b_n)_{n\geq N}$ such that $d(b_n, b_{n+1}) < \frac{1}{2^n}$, which is Cauchy, and since X is complete, converges to some $b \in X$. Since $B_{n+1} \subset U(B_n, \frac{1}{2^n})$, we can also pad in front of this $(b_n)_{n\geq N}$ to form $(b_n)_{n\in\mathbb{N}}$ such that $d(b_n, b_{n+1}) < \frac{1}{2^n}$ still, while the limit point stays the same. It then follows that in fact $b \in A$.

Then

$$d(b_N, b) \leq d(b_N, b_{N_\varepsilon}) + \varepsilon \leq \sum_{k=N}^{\infty} \frac{1}{2^k} + \varepsilon \leq \frac{1}{2^{N-1}} + \varepsilon$$

for all ε, hence $d(b_N, b) \leq \frac{1}{2^{N-1}}$. Therefore $B_N \subset U(A, \frac{1}{2^{N-1}}) \Rightarrow B_N \subset U(\overline{A}, \frac{1}{2^{N-1}})$.

3. We now want to show $\overline{A} \subset U(B_N, \frac{1}{2^{N-1}})$.

Take $a \in \overline{A}$. Then for all $\varepsilon > 0$, there exists some $a_\varepsilon \in A$ such that $d(a, a_\varepsilon) < \varepsilon$. Since $a_\varepsilon \in A$, there exists some sequence $(b_{n,\varepsilon})$ such that $b_{n,\varepsilon} \in B_n$ and $d(b_{n,\varepsilon}, b_{n+1,\varepsilon}) < \frac{1}{2^n}$ for all n. Similar to above, we have that

$$d(a_\varepsilon, b_{n,\varepsilon}) \leq \frac{1}{2^{n-1}}$$

It then follows that there for every n, ε, there exists $b_{n,\varepsilon}$ such that

$$d(a, b_{n,\varepsilon}) \leq d(a, a_\varepsilon) + d(a_\varepsilon, b_{n,\varepsilon}) < \frac{1}{2^{n-1}} + \varepsilon$$

hence $d(a, b_{n,\varepsilon}) \leq \frac{1}{2^{n-1}}$. It then follows that $\overline{A} \subset U(B_n, \frac{1}{2^{n-1}})$ for all n.

4. From all the above, it then follows $\overline{A} \in \mathcal{H}$ and so it is well-defined to conclude $D(\overline{A}, B_n) < \frac{1}{2^{n-1}}$ for all n. Hence $B_n \xrightarrow{n\to\infty} \overline{A}$. $\square$

(c) (X, d) is totally bounded. We want to show that for all $\varepsilon > 0$, we can cover $\mathcal{H}$ by finitely many ε-balls. Select $\delta = \varepsilon/2$. Then since X is totally bounded, there exists a finite cover of X using δ-balls, namely, $\{B(x_i, \delta) : x_i \in X\}_{i\leq N}$.

Let $S = \{x_i : i \leq N\}$ and take $\mathcal{A} \subset 2^S$ to be the collection of nonempty subsets of S. Since $|S| \leq N \Rightarrow |\mathcal{A}| \leq |2^S| \leq 2^N$.

We therefore WTS $\{B_D(A, \varepsilon) \mid A \in \mathcal{A}\}$ covers $\mathcal{H}$ to demonstrate a finite ε-balls cover of $\mathcal{H}$, i.e., that for any $C \in \mathcal{H}$, then there exists some $A \in \mathcal{A}$ such that $C \in B_D(A, \varepsilon)$.

Now, C is a nonempty, closed, bounded subset of X. Define $A := \{x_i : i \leq N, B(x_i, \delta) \cap C \neq \varnothing\} \in \mathcal{A}$. Then WTS $C \in B_D(A, \varepsilon) \Leftrightarrow C \subset U(A, \varepsilon), A \subset U(C, \varepsilon)$.

Since $\{B(x_i, \delta) : i \leq N\}$ cover X, and $B(x_i, \delta) \cap C = \varnothing$ if $x_i \notin A$, it then follows that $\{B(x_i, \delta) : x_i \in A\}$ covers C. But then since $\delta < \varepsilon$, it then follows that $C \subset U(A, \varepsilon)$.

Furthermore, since for all $x_i \in A$, $B(x_i, \delta) \cap C \neq \varnothing$ and $\delta < \varepsilon$, it then follows $A \subset U(C, \varepsilon)$.

From the 2 paragraphs above, we can therefore conclude that $C \in B_D(A, \varepsilon)$ as required. $\square$

(d) (X, d) compact implies (X, d) complete and totally bounded implies $(\mathcal{H}, D)$ complete and totally bounded implies $(\mathcal{H}, D)$ compact.

$\square$

Problem 6.5 (46.6 done)

Show that in the compact-open topology, $\mathcal{C}(X, Y)$ is Hausdorff if Y is Hausdorff, and regular if Y is regular. [Hint: If $\overline{U} \subset V$ then $\overline{S(C, U)} \subset S(C, V)$]

Solution

1. Y is Hausdorff.

Take $f \neq g \in \mathcal{C}(X, Y)$. Since $f \neq g$, it implies that there exists $a \in X$ such that $f(a) \neq g(a)$. Since Y is Hausdorff, there exists U_f, U_g disjoint and open in Y such that $f(a) \in U_f, g(a) \in U_g$. Then $S(\{a\}, U_f) \ni f, S(\{a\}, U_g) \ni g$ and they are disjoint, hence $\mathcal{C}(X, Y)$ is also Hausdorff.

2. Y is regular.

Recall regularity is equivalent to saying that given an open neighborhood of a point, one can always find a smaller neighborhood around that point whose closure is contained in the first open neighborhood.

To prove this, we shall only do this for the subbasis that generates the compact-open topology, since closure of the intersection is a subset of the intersection of closures.

Take $f \in S(C, U)$ where $f \in \mathcal{C}(X, Y)$, $C \subset X$ is compact and $U \subset Y$ open. Since $f \in S(C, U)$, it follows that $f(C) \subset U$.

Since C is compact and f is continuous, $f(C)$ is compact. For each $y \in f(C) \subset U$, then, since Y is regular, there exists some V_y open such that $y \in V_y \subset \overline{V_y} \subset U$. Then $\{V_y : y \in f(C)\}$ is an open cover of $f(C)$ and hence reduces to some finite subcover $\{V_{y_i}\}_{i\leq N}$. Let $V = \cup_{i\leq N} V_{y_i}$, then V is open and finiteness gives us $\overline{V} = \cup_{i\leq N} \overline{V_{y_i}} \subset Y$, and also that $f(C) \subset V$, i.e., $f \in S(C, V)$.

WTS $\overline{S(C, V)} \subset S(C, U)$ by showing that $\overline{S(C, V)} \subset \{f \in \mathcal{C}(X, Y) \mid f(C) \subset \overline{V}\}$. Suppose $f \in \mathcal{C}(X, Y) \backslash \{f \in \mathcal{C}(X, Y) \mid f(C) \subset \overline{V}\}$. This implies there exists some $x_0 \in C$ such that $f(x_0) \notin \overline{V}$. Then there exists some $W \ni f(x_0)$ open such that $W \cap \overline{V} = \varnothing$. It then follows that $f \in S(\{x_0\}, W)$ and $S(\{x_0\}, W) \cap \{f \in \mathcal{C}(X, Y) \mid f(C) \subset \overline{V}\} = \varnothing$. It follows that $\{f \in \mathcal{C}(X, Y) \mid f(C) \subset \overline{V}\}$ is closed.

By definition, $S(C, V) \subset \{f \in \mathcal{C}(X, Y) \mid f(C) \subset \overline{V}\} \Rightarrow \overline{S(C, V)} \subset \overline{\{f \in \mathcal{C}(X, Y) \mid f(C) \subset \overline{V}\}} = \{f \in \mathcal{C}(X, Y) \mid f(C) \subset \overline{V}\} \subset S(C, U)$ as required. $\square$

Problem 6.6 (47.5)

Let (Y, d) be a metric space; let $f_n : X \to Y$ be a sequence of continuous functions; let $f : X \to Y$ be a function (not necessarily continuous). Suppose f_n converges to f is the topology of pointwise convergence. Show that if $\{f_n\}$ is equicontinuous, then f is continuous and f_n converges to f in the topology of compact convergence.

Solution

Consider $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ is a family of functions in $\mathcal{C}(X, Y)$. WTS $\mathcal{F}_a \coloneqq \{f_n(a) : n \in \mathbb{N}\}$ has compact closure for each $a \in X$.

Since $f_n \xrightarrow{n\to\infty} f$ in the topology of pointwise convergence, $f_n(a) \xrightarrow{n\to\infty} f(a)$. It then follows that $\overline{\mathcal{F}_a}$ is sequentially compact. It is sequentially compact in the metric space (Y, d), so it is compact.

Apply Ascoli's Theorem, it then follows that $\overline{\mathcal{F}}$ is compact in $\mathcal{C}(X, Y)$ in the topology of compact convergence, which implies that $f_n \xrightarrow{n\to\infty} f$ in this topology. $\mathcal{F}$ is compact in Hausdorff $\mathcal{C}(X, Y)$, hence is closed, so $f \in C(X, Y)$ as required. $\square$