MATH 26200: Point-Set Topology

Problem Set 4

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Textbook: Munkres, *Topology*.

Problem 5.1 (30.2 done)

Show that if X has countable basis $\{B_n\}$, then every basis $\mathcal{C}$ for X contains a countable basis for X. (Hint: For every pair of indices n, m for which it is possible, choose $C_{n,m}$ such that $B_n \subset C_{n,m} \subset B_m$.)

Solution

Let $D = \{C \in \mathcal{C} : B_n \subset C \subset B_m, (n, m) \in \mathbb{N}^2\}$.

D is countable, so it remains for us to show that indeed D is a basis for X.

Take any $x \in U$, U open in X, then since $\{B_n\}$ is a basis for X, there exists some $B_{m'}$ such that $x \in B_{m'} \subset U$.

Since $\mathcal{C}$ is a basis for X and $B_{m'}$ is open, there exists some C_x such that

$$x \in C_x \subset B_{m'} \subset U$$

Since $\{B_n\}$ is a basis for X and C_x is open, there exists some $B_{n'}$ such that

$$x \in B_{n'} \subset C_x \subset B_{m'}$$

Then $C_x \in D$, since $B_{n'} \subset C_x \subset B_{m'}$.

It follows that for all U open, for all $x \in U$, there exists $C_x \in D$ such that $x \in C_x \subset U$. D is therefore a basis for X. $\square$

Problem 5.2 (31.7 done)

Let $p : X \to Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact for each $y \in Y$. (Such a map is called a *perfect map*.)

- **(a)** Show that if X is Hausdorff, then so is Y.
- **(b)** Show that if X is regular, then so is Y.
- **(c)** Show that if X is locally compact, then so is Y.
- **(d)** Show that if X is second-countable, then so is Y. (Hint: Let $\mathcal{B}$ be a countable basis for X. For each finite subset J of $\mathcal{B}$, let U_J be the union of all sets of the form $p^{-1}(W)$, for W open in Y, that are contained in the union of the elements of J.)

Solution

We use the following lemmas:

Lemma (Lemma 1)

If $p : X \to Y$ is a perfect map, $B \subset Y$ and U is an open set containing $p^{-1}(B)$ then there exists some open $B \subset W \subset Y$ such that $p^{-1}(W) \subset U$.

Proof (Lemma 1)

$p^{-1}(B) \subset U$. p is closed, so $p(X - U)$ is closed in Y. Then consider $W = Y - p(X - U)$ is open. Since $p^{-1}(B) \subset U$, $y \in W$. By construction, we also have that $p^{-1}(W) \subset U$. $\square$

Lemma (Lemma 2)

If C and K are compact subsets of Hausdorff X such that $C \cap K = \varnothing$, then there exists open, disjoint $U \supset C, V \supset K$.

Proof (Lemma 2)

Take $c \in C$, then $c \notin K$. From Lemma 26.4, we know that there exists open disjoint U_c and V_c such that $U_c \ni c, V_c \supset K$.

Since C is compact, the open cover $\{U_c\}_{c \in C}$ reduces to a subcover $\{U_{c_1}, \ldots, U_{c_n}\}$. Then $U = \bigcup_{k=1}^{n} U_{c_k}$ and $V = \bigcap_{k=1}^{n} V_{c_k}$ satisfies our requirements. $\square$

(a) Suppose that X is Hausdorff. Then $p^{-1}(\{y_1\})$ and $p^{-1}(\{y_2\})$ are disjoint, compact subsets of Hausdorff X. Using Lemma 2, It follows that there exists open, disjoint $U_1, U_2 \subset X$ such that $U_1 \supset p^{-1}(\{y_1\}), U_2 \supset p^{-1}(\{y_2\})$. Using Lemma 1, it follows that there exists open $V_1, V_2 \subset Y$ such that $y_1 \in V_1, y_2 \in V_2$ and $p^{-1}(V_1) \subset U_1, p^{-1}(V_2) \subset U_2$. Since $U_1 \cap U_2 = \varnothing \Rightarrow V_1 \cap V_2 = \varnothing$. Y is therefore Hausdorff.

(b) Perform the same proof, since B in Lemma 1 is general and not restricted to $\{y_1\}$.

(c) Suppose X is locally compact. WTS Y is also locally compact, meaning for any $y \in Y$, there exists open neighborhood V and compact K such that $y \in V \subset K$.

For all $x \in p^{-1}(\{y\})$, since X is locally compact, there exists open U_x and compact C_x such that $x \in U_x \subset C_x$. $\{U_x\}$ is then an open cover of compact $p^{-1}(y)$, hence reduces to finite subcover $\{U_{x_1}, \ldots, U_{x_n}\}$. Then let $U = \bigcup_{k=1}^{n} U_{x_k}, C = \bigcup_{k=1}^{n} C_{x_k}$ then $p^{-1}(y) \subset U$ and C is a finite union of compacts and is therefore compact.

Using Lemma 1, then there exists V open in Y such that $y \in V$ and $p^{-1}(V) \subset U \subset C \Rightarrow V \subset p(C)$. p is continuous and C is compact so $p(C)$ is also compact.

(d) Suppose X is second-countable. Let $\mathcal{B} = \{B_j\}_{j \in \mathbb{N}}$ be a countable basis for X. For each finite subset J of $\mathcal{B}$, let U_J be the union of all sets $p^{-1}(W)$ for some W open in Y such that $p^{-1}(W) \subset \bigcup_{j \in J} B_j$. The number of finite subsets of a countable set is countable, so $\{U_J\}$ is countable, and also $\{p(U_J)\}$.

WTS $\{p(U_J)\}$ is a basis for Y. Take $W \subset Y$ open. Then $p^{-1}(W) = \bigcup_{y \in W} p^{-1}(\{y\})$ is a union of compacts. p is continuous and W is open, so $p^{-1}(W)$ is also open, and is therefore a union of basis elemenets, i.e., $\{B_j\}_{j \in J_W}$. Each $p^{-1}(y)$ is compact, and can therefore be covered by finitely many $\{B_j\}_{j \in J_y \subset J_W}$. Using Lemma 1, it then follows that there exists open $V_y \subset Y$ such that $y \in V_y, p^{-1}(V_y) \subset \bigcup_{j \in J_y} B_j$, so $p^{-1}(V_y) \subset U_{J_y} \subset \bigcup_{j \in J_W} \subset p^{-1}(W)$, which implies that $W = \bigcup_{y \in W} U_{J_y}$ as required. $\square$

Problem 5.3 (32.5 done)

Is $\mathbb{R}^\omega$ normal in the product topology? In the uniform topology?

Solution

$\mathbb{R}^\omega$ is metrizable in both topologies, and is therefore normal in both. $\square$

Problem 5.4 (32.6 done)

A space X is said to be *completely normal* if every subspace of X is normal. Show that X is completely normal iff for every pair A, B of separated sets in X (that is, sets such that $\overline{A} \cap B = \varnothing = A \cap \overline{B}$), there exist disjoint open sets containing them. (Hint: If X is completely normal, consider $X - (\overline{A} \cap \overline{B})$).

Solution

$\boxed{\Rightarrow}$ Suppose X is completely normal. Let A, B be a pair of separated sets in X, meaning $\overline{A} \cap B = \varnothing = A \cap \overline{B}$.

Consider $M = X - (\overline{A} \cap \overline{B}) \subset X$, then M is normal. $\overline{A} \cap \overline{B}$ is closed, so M is open in X.

$M = (X - \overline{A}) \cup (X - \overline{B})$. Since $A \cap \overline{B} = \varnothing \Rightarrow A \subset X - \overline{B} \Rightarrow A \subset M$. Similarly, $B \subset M$.

The closure of A in M is then $cl_M(A) = \overline{A} \cap M = \overline{A} \cup (X - \overline{B}) = \overline{A} - \overline{B}$, and similarly closure of B in M is $cl_M(B) = \overline{B} \cap M = \overline{B} - \overline{A}$. They are disjoint closed sets in normal M and therefore there are disjoint open sets $U, V \subset M$ such that $U \supset cl_M(A), V \supset cl_M(B)$. It then also follows that $U \supset A, V \supset B$. U, V are open in M open in X, so U, V are open in X. $\square$

$\boxed{\Leftarrow}$ WTS X is completely normal. Take subspace M of X and disjoint closed subsets A, B of M. Then $cl_M(A) = A, cl_M(B) = B$. Then since $A, B \subset M$,

$$\overline{A} \cap B = (\overline{A} \cap M) \cap B = cl_M(A) \cap B = A \cap B = \varnothing$$

Similarly, $\overline{B} \cap A = \varnothing$. Hence A, B are separated sets in X, so there exists disjoint open sets containing them. These open sets, when intersecting with M, are open in the subspace topology and of course still disjoint. M is therefore normal, making X completely normal. $\square$

Problem 5.5 (33.4 done)

Recall that A is a G_δ set in X if A is the intersection of a countable collection of open sets in X.

Let X be normal. Then there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ for $x \in A$, and $f(x) > 0$ for $x \notin A$ **if and only if** A is a closed G_δ set in X.

A function satisfying the requirements of this theorem is said to vanish precisely on A.

Solution

$\boxed{\Rightarrow}$ Suppose there exists continuous $f : X \to [0,1]$ such that $f(x) = 0$ for $x \in A$, $f(x) > 0$ for $x \notin A$.

It follows that

$$A = f^{-1}(\{0\}) = \bigcap_{n \in \mathbb{N}} f^{-1}\left([0, \frac{1}{n})\right)$$

Since f is continuous and $[0, 1/n)$ is open in $[0,1]$ for all $n \in \mathbb{N}$, A is thus a G_δ set.

$\boxed{\Leftarrow}$ Suppose A is a closed G_δ set in X.

$$A = \bigcap_{n \in \mathbb{N}} U_n$$

where each U_n is open in X. Then, for each n, $A \subset U_n \Rightarrow A \cap (X - U_n) = \varnothing$. Thus A and $X - U_n$ are disjoint closed subsets of X, so by Urysohn's Lemma, there exists continuous $f_n : X \to [0,1]$ such that $f_n(A) = \{0\}$ and $f_n(X - U_n) = \{1\}$.

Then define

$$s_n(x) = \sum_{k=1}^{n} \frac{1}{2^k} f_k(x)$$

then s_n is a finite linear combination of continuous functions and is therefore continuous for all $n \in \mathbb{N}$.

And define

$$f(x) = \lim_{n \to \infty} s_n(x)$$

as the pointwise convergence of s_n. The pointwise convergence exists because for fixed x, $s_n(x)$ is monotonically increasing and bounded above by 1.

We want to show that this convergence is uniform. Indeed,

$$|s_n(x) - f(x)| = \left| \sum_{k=n+1}^{\infty} \frac{1}{2^{k+1}} f_k(x) \right| \leq \left| \sum_{k=n+1}^{\infty} \frac{1}{2^{k+1}} \right| \xrightarrow{n \to \infty} 0 \text{ uniformly}$$

f is therefore the uniform limit of continuous functions, and is therefore continuous itself.

For any $x \in A$, $s_n(x) = 0 \Rightarrow f(x) = 0$. Then for $x \notin A \Rightarrow x \in X - U_m$ for some m, then $f(x) \geq s_m(x) = \frac{1}{2^m} > 0$. □

Problem 5.6 (34.1 done)

Give an example showing that a Hausdorff space with a countable basis need not be metrizable.

Solution

$\mathbb{R}$ with K-topology, i.e., the open sets are the open sets in the usual topology, and sets of the form $(a, b) - K$ where $K = \{\frac{1}{n}\}_{n \in \mathbb{N}}$.

$\mathbb{R}_K$ is finer than $\mathbb{R}$ with its usual topology, so it is Hausdorff. It has a countable basis: $\mathcal{B} = \{(a, b), (a, b) - K : a, b \in \mathbb{Q}\}$.

But $\mathbb{R}_K$ is not regular, hence it can't be metrizable since metrizable implies normal implies regular. □

Problem 5.7 (35.9)

Let $X_1 \subset X_2 \subset \cdots$ be a sequence of spaces, where X_i is a closed subspace of X_{i+1} for each i. Let X be the union of the X_i; let us topologize X by declaring a set U to be open in X if $U \cap X_i$ is open in X for each i.

(a) Show that this is a topology on X and that each space X_i is a subspace (in fact, a closed subspace) of X in this topology. This topology is called the topology *coherent* with the subspaces X_i.

(b) Show that $f : X \to Y$ is continuous if $f|_{X_i}$ is continuous for each i.

(c) Show that if each space X_i is normal, then X is normal. (Hint: Given disjoint closed sets A and B in X, set f equal to 0 on A and 1 on B, and extend f successively to $A \cup B \cup X_i$ for $i = 1, 2, \ldots$)

Solution

□