

Math 20250: Abstract Linear Algebra  
Problem Set 2

Hung Le Tran

09 Apr 2023

**Problem 2.1** (Sec 2.2. Problem 3)

Is the vector  $(3, -1, 0, -1)$  in the subspace of  $\mathbb{R}^5$  spanned by the vectors  $(2, -1, 3, 2)$ ,  $(-1, 1, 1, -3)$  and  $(1, 1, 9, -5)$

**Solution**

For  $(3, -1, 0, -1)$  to be in the abovementioned subspace, it must be a linear combination of the vectors  $(2, -1, 3, 2)$ ,  $(-1, 1, 1, -3)$  and  $(1, 1, 9, -5)$  in  $\mathbb{R}^5$ , i.e.

$$\exists c_1, c_2, c_3 \in \mathbb{R} \text{ such that } (3, -1, 0, -1) = c_1(2, -1, 3, 2) + c_2(-1, 1, 1, -3) + c_3(1, 1, 9, -5)$$

It follows that

$$\begin{aligned} 2c_1 - c_2 + c_3 &= 3 \\ -c_1 + c_2 + c_3 &= -1 \\ 3c_1 + c_2 + 9c_3 &= 0 \\ 2c_1 - 3c_2 - 5c_3 &= -1 \end{aligned}$$

which is equivalent to:

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 1 \\ 3 & 1 & 9 \\ 2 & -3 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

which is then equivalent to the following augmented matrix:

$$\begin{bmatrix} 2 & -1 & 1 & 3 \\ -1 & 1 & 1 & -1 \\ 3 & 1 & 9 & 0 \\ 2 & -3 & -5 & -1 \end{bmatrix}$$

on which we can carry out simplifying row operations to reach the row-reduced echelon form:

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system therefore has no solution, as the third row of the augmented matrix fails ( $0c_1 + 0c_2 + 0c_3 \neq 1$ )

Therefore  $(3, -1, 0, 1)$  is not in the subspace. □

**Problem 2.2** (Sec 2.2. Problem 5)

Let  $\mathbb{F}$  be a field and let  $n \geq 2$  be a positive integer. Let  $V$  be the vector space of all  $n \times n$  matrices over  $\mathbb{F}$ . Which of the following sets of matrices  $A \in V$  are subspaces of  $V$ ?

1. all invertible  $A$
2. all non-invertible  $A$
3. all  $A$  such that  $AB = BA$ , where  $B$  is some fixed matrix in  $V$
4. all  $A$  such that  $A^2 = A$

**Solution** 1. All invertible  $A$

No. It is not true that  $\forall$  invertible  $A_1, A_2 \in V, (A_1 + A_2)$  is also invertible.

Counter example:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow A_3 = A_1 + A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which is not invertible, since it admits  $X = \begin{bmatrix} a \\ -a \end{bmatrix}$  for arbitrary  $a \in \mathbb{F}$  as a solution to  $A_3X = 0$

2. All non-invertible  $A$

No. It is not true that  $\forall$  non-invertible  $A_1, A_2 \in V, (A_1 + A_2)$  is also non-invertible.

Counter example:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow A_3 = A_1 + A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A_1$  is non-invertible since it admits  $X = \begin{bmatrix} 0 \\ a \end{bmatrix}$  for arbitrary  $a \in \mathbb{F}$  as a solution to  $A_1X = 0$

$A_2$  is non-invertible since it admits  $X = \begin{bmatrix} a \\ 0 \end{bmatrix}$  for arbitrary  $a \in \mathbb{F}$  as a solution to  $A_2X = 0$

Meanwhile,  $A_3 = I$  is trivially invertible.

3. All  $A$  such that  $AB = BA$ ,  $B$  is fixed in  $V$

Yes. We want to show that given  $A_1, A_2$  such that  $A_1B = BA_1, A_2B = BA_2$  then

$$\begin{aligned} (A_1 + A_2)B &= B(A_1 + A_2) \\ (cA_1)B &= B(cA_1) \quad \forall c \in \mathbb{K} \end{aligned}$$

First, we have:

$$\begin{aligned}
[(A_1 + A_2)B]_{ij} &= \sum_{k=1}^n (A_1 + A_2)_{ik} B_{kj} \\
&= \sum_{k=1}^n (A_{1,ik} B_{kj} + A_{2,ik} B_{kj}) \\
&= \sum_{k=1}^n (A_{1,ik} B_{kj}) + \sum_{k=1}^n (A_{2,ik} B_{kj})
\end{aligned}$$

However,

$$A_1 B = B A_1 \Rightarrow \sum_{k=1}^n (A_{1,ik} B_{kj}) = [A_1 B]_{ij} = [B A_1]_{ij} = \sum_{k=1}^n (B_{ik} A_{1,kj})$$

and similarly for  $A_2$ .

It follows that:

$$\begin{aligned}
[(A_1 + A_2)B]_{ij} &= \sum_{k=1}^n (A_{1,ik} B_{kj}) + \sum_{k=1}^n (A_{2,ik} B_{kj}) \\
&= \sum_{k=1}^n (B_{ik} A_{1,kj}) + \sum_{k=1}^n (B_{ik} A_{2,kj}) \\
&= \sum_{k=1}^n B_{ik} (A_{1,kj} + A_{2,kj}) \\
&= \sum_{k=1}^n B_{ik} (A_1 + A_2)_{kj} \\
&= [B(A_1 + A_2)]_{ij}
\end{aligned}$$

It follows that indeed  $(A_1 + A_2)B = B(A_1 + A_2)$ .

The other equality is trivially observed  $(cA_1)B = B(cA_1) \forall c \in \mathbb{K}$ , making this set a subspace of  $V$

4. All  $A$  such that  $A^2 = A$

No. It is not true that  $\forall A_1, A_2$  such that  $A_1^2 = A_1, A_2^2 = A_2 \Rightarrow (A_1 + A_2)^2 = A_1 + A_2$

Counterexample:  $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

It is trivially true that  $A_1^2 = A_1, A_2^2 = A_2$ , however

$$(A_1 + A_2)^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \neq (A_1 + A_2)$$

Therefore this set is not a subspace of  $V$ .

□

**Problem 2.3** (Sec 2.3. Problem 1)

Prove that if two vectors are linearly dependent, one of them is a scalar multiple of the other.

**Solution**

Let  $v_1, v_2$  be two vectors that are linearly dependent in field  $\mathbb{K}$ . Then there exists  $a_1, a_2 \in \mathbb{K}$  not all 0 such that :

$$a_1 v_1 + a_2 v_2 = 0$$

WLOG, assume  $a_1 \neq 0 \Rightarrow \exists a_1^{-1} : a_1^{-1} a_1 = 1$ . It follows that

$$v_1 = a_1^{-1}(-a_2 v_2) = (-a_1^{-1} a_2) v_2$$

is a scalar multiple of  $v_2$ . □

**Problem 2.4** (Sec 2.3. Problem 2)

Are the vectors

$$\alpha_1 = (1, 1, 2, 4)$$

$$\alpha_2 = (2, -1, -5, 2)$$

$$\alpha_3 = (1, -1, 4, 0)$$

$$\alpha_4 = (2, 1, 1, 6)$$

linearly independent in  $\mathbb{R}^4$ ?

**Solution**

No, because

$$\alpha_4 = (2, 1, 1, 6) = \frac{4}{3}(1, 1, 2, 4) + \frac{1}{3}(2, -1, -5, 2) = \frac{4}{3}\alpha_1 + \frac{1}{3}\alpha_2 + 0\alpha_3$$

□

**Problem 2.5** (Sec 2.3. Problem 4)

Show that the vectors

$$\alpha_1 = (1, 0, -1), \alpha_2 = (1, 2, 1), \alpha_3 = (0, -3, 2)$$

form a basis for  $\mathbb{R}^3$ . Express each of the standard basis vectors as linear combinations of  $\alpha_1, \alpha_2, \alpha_3$

**Solution**

We first express each of the standard basis vectors as linear combinations of  $\alpha_1, \alpha_2, \alpha_3$ :

$$\begin{aligned} (1, 0, 0) &= \frac{7}{10}(1, 0, -1) + \frac{3}{10}(1, 2, 1) + \frac{1}{5}(0, -3, 2) \\ (0, 1, 0) &= \frac{-1}{5}(1, 0, -1) + \frac{1}{5}(1, 2, 1) + \frac{-1}{5}(0, -3, 2) \\ (0, 0, 1) &= \frac{-3}{10}(1, 0, -1) + \frac{3}{10}(1, 2, 1) + \frac{1}{5}(0, -3, 2) \end{aligned}$$

The standard basis vectors span  $V$ , therefore each vector in  $V$  can be expressed as linear combinations of the standard basis vectors, which can then be expressed as linear combinations of  $\alpha_1, \alpha_2, \alpha_3$  per the equalities above. Therefore  $\alpha_1, \alpha_2, \alpha_3$  span  $V$ .

It must now be proven that they are linearly independent, which is equivalent to showing that the system of linear equations represented by the following augmented matrix has no non-trivial solutions:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 \\ -1 & 1 & 2 & 0 \end{bmatrix}$$

Indeed, its row-reduced echelon form is  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ , only allowing for a trivial solution of

$(0, 0, 0)$ . Thus,  $\alpha_1, \alpha_2, \alpha_3$  are linearly independent.

Therefore they form a basis for  $V$ . □

### Problem 2.6 ((Bonus) Sec 2.3. Problem 14)

Let  $V$  be the set of real numbers. Regard  $V$  as a vector space over the field of *rational* numbers, with the usual operations. Prove that this vector space is *not* finite-dimensional.

### Solution

Suppose not; that  $\dim_{\mathbb{Q}} \mathbb{R} = d \in \mathbb{N}$ , meaning that there exists a basis  $v_1, v_2, \dots, v_d \in \mathbb{R}$  such that

$$\forall v \in \mathbb{R} \exists \alpha_1, \alpha_2, \dots, \alpha_d \in \mathbb{Q} : v = \alpha_1 v_1 + \dots + \alpha_d v_d$$

We first observe that for  $v_1 \neq v_2 \in \mathbb{R}$ , their corresponding  $d$ -tuple must be different, i.e.  $(\alpha_{v_1,1}, \alpha_{v_1,2}, \dots, \alpha_{v_1,d}) \neq (\alpha_{v_2,1}, \alpha_{v_2,2}, \dots, \alpha_{v_2,d})$ . This is trivial.

Therefore if we consider  $\varphi : \mathbb{R} \rightarrow \mathbb{Q}^d, \varphi(v) = (\alpha_{v,1}, \alpha_{v,2}, \dots, \alpha_{v,d})$  then  $\varphi$  is injective.

It follows that  $|\mathbb{R}| \leq |\mathbb{Q}^d|$ . However,  $\mathbb{R}$  is uncountable and  $\mathbb{Q}^d$  is countable.  $\Rightarrow \Leftarrow$  □