This document lists out definitions in Math that I couldn't possibly organize in my tiny brain. The definitions are expected to be built off of the previous ones.

1 Binary Relations

Definition 1.1 (Binary Relation)

A **binary relation** R over sets X, Y is a subset of $X \times Y$. $(x, y) \in R$ is equivalent to xRy. We say x *is* $R-$*related to* y.
In other words, R imposes some condition that $(x, y) \in R$ must satisfy to be included in the set.
e.g. $x \geq y$

Definition 1.2 (Homogenous Relation/Endorelation)

A **homogenous relation** is a binary relation from a set to itself, i.e. $R : X \times X \to X$.

Definition 1.3 (Reflexive, Irreflexive, Symmetric, Antisymmetric, Asymmetric, Transitive, Connected, Strongly Connected)

Let R be a homogenous relation over set X. Then the following properties are defined as:

1. R is **reflexive** if $\forall\, x \in X, xRx$ e.g. $\geq$
2. R is **irreflexive** if $\forall\, x \in X, \neg xRx$ e.g. $>$
3. R is **symmetric** if $\forall\, x, y \in X, xRy \Leftrightarrow yRx$ e.g. shares the same house
4. R is **antisymmetric** if $\forall\, x, y \in X, xRy \wedge yRx \Rightarrow x = y$ e.g. $\geq$
5. R is **asymmetric** if $\forall\, x, y \in X, xRy \Rightarrow \neg yRx$ e.g. $>$
6. R is **transitive** if $\forall\, x, y, z \in X, xRy \wedge yRz \Rightarrow xRz$ e.g. $>, \geq$
7. R is **connected** if $\forall\, x, y \in X, x \neq y \Rightarrow xRy \vee yRx$
8. R is **strongly connected** if $\forall\, x, y \in X, xRy \vee yRx$

Remark

In some sense, **asymmetry** is **antisymmetry + irreflexivity**; antisymmetry gets upgraded when there is irreflexivity.

Definition 1.4 (Partial Order)

A **partial order** is a relation that is reflexive, antisymmetric and transitive.
e.g. $\geq$

Definition 1.5 (Strict Partial Order)

A **strict partial order** is a relation that is irreflexive, asymmetric and transitive.
e.g. $>$

Definition 1.6 (Total Order)

A **total order** is a relation that is reflexive, antisymmetric, transitive and connected.

Definition 1.7 (Strict Total Order)

A **strict total order** is a relation that is irreflexive, asymmetric, transitive and connected.

Remark

In short, *total* adds on the requirement that it must be *connected*, while *strict* changes *reflexivity* into *irreflexivity* (and since everyone has *antisymmetry*, this changes into *asymmetry* too)

Definition 1.8 (Equivalence Relation)
An **equivalence relation** is a relation that is reflexive, symmetric and transitive.

2 Basic Algebraic Structures

Definition 2.1 (Group)
A non-empty set G equipped with a binary operation $\cdot : G \times G \to G$ is called a **group** if they satisfy the following *group axioms*:

1. (Associativity) $\forall\, a, b, c \in G, (a \cdot b) \cdot c = a \cdot (b \cdot c)$
2. (Identity) $\exists\, e \in G$ s.t. $\forall\, g \in G, e \cdot g = g \cdot e = g$
3. (Inverse) $\forall\, g \in G,\ \exists\, g' \in G$ s.t. $g'g = e$. Then $g^{-1} := g'$

Remark
The above axioms of a group imply:

1. Identity is unique
2. Inverse is unique

Even when one restricts the axioms to just simply having left identity and left inverse, the above remark still holds

Definition 2.2 (Subgroup)
$H \subseteq G$ is a **subgroup** of G if

1. $e \in H$
2. H is closed under $\cdot$ and taking inverses

Definition 2.3 (Abelian Group)
A **group** $(G, \cdot)$ is **Abelian** if the operation $\cdot$ is also commutative.
Recap: Associativity, Commutativity, Identity, Inverse

Definition 2.4 (Ring)
A **ring** is a set R equipped with two binary operations (addition, multiplication) $+, \cdot : R \times R \to R$ s.t. they satisfy the following *ring axioms*:

1. $(R, +)$ is an Abelian group (Associativity, Commutativity, Identity, Inverse)
2. (Multiplicative Associativity) $\forall\, a, b, c \in R, (a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. (Multiplicative Identity) $\forall\, r \in R,\ \exists\, 1 \in R$ s.t. $1 \cdot r = r \cdot 1 = r$
4. (Left, Right Distributivity) This property governs how the 2 operations interact

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

$$(b + c) \cdot a = (b \cdot a) + (c \cdot a)$$

Note that multiplicative commutativity and multiplicative inverse are not here!

Remark

One can derive that when a ring has $0 = 1$ (additive identity = multiplicative identity), then it must be a trivial *zero ring* of $\{0\}$

Definition 2.5 (Rng)

A **rng** is a ring without the requirement of a multiplicative identity.
Recap: Abelian group, multiplicative associativity, distributivity

Definition 2.6 (Commutative Ring)

A **commutative ring** is a ring, with the additional requirement of multiplication being commutative.
Recap: Abelian group; multiplicative associativity, commutativity and identity; distributivity

Definition 2.7 (Field)

A **field** is a set F equipped with 2 binary operations (addition, multiplication) $+, \cdot : F \times F \to F$ s.t.

1. Addition and multiplication are associative
2. Addition and multiplication are commutative
3. There is an additive inverse 0
4. $\forall\, a \in F \setminus 0,\ \exists\, a^{-1}$ s.t. $a^{-1}a = 1$
5. Distributivity

In short, a field is a commutative ring where non-zero elements have multiplicative inverses. The non-zero elements then form a group equipped with multiplication with 1 as their identity.
Recap: Abelian group; multiplicative associativity, commutativity, identity and inverse (inverse only for non-zero); distributivity

Definition 2.8 (Ordered Field)

[Munkres - Ch.1 p.32]

3 The -isms

Definition 3.1 (Homomorphism)

homo: Greek *homos*, meaning "same"
morphism: Greek *morphism*, meaning "shape, form"
A **homomorphism** is a map $f : A \to B$, where A, B are (very generically) algebraic structures of the same type G and are therefore equipped with the same kind(s) of operation, WLOG, namely $\cdot_A, \cdot_B : G^k \to G$ (e.g. groups, vector spaces). A homomorphism f then preserves the structures of these operations, i.e.

$$f(x) \cdot_B f(y) = f(x \cdot_A y)\ \forall\, x, y \in A$$

if $\cdot$ is binary, and the same concept applies for the general $k-$ary case.

Example

A **group homomorphism** is a homomorphism in a group, where the homomorphism preserves the $\cdot$ equipped by the group.

Definition 3.2 (Isomorphism)
iso: Greek *isos*, meaning "equal"
An **isomorphism** is a homomorphism that is bijective.

Remark
For me, it's so common to just intuitively think that a homomorphism must be bijective, but no!
e.g. $f : \mathbb{Z} \to \{0\}$, both equipped with $+$

Definition 3.3 (Endomorphism)
endo: Greek *endon*, meaning "in, within"
An **endomorphism** is a homomorphism that has the same domain and codomain, i.e.

$$f : A \to A$$

Definition 3.4 (Automorphism)
auto: Greek *autos*, meaning "self"
An **automorphism** is an endomorphism that is also an isomorphism, (or vice versa)

4 The -algebras

This section stemmed from my need to study probability and felt the urgent need to categorize what I'm working with immediately.

Definition 4.1 (Field of sets, Algebra)
A **field of sets** is a pair $(X, \mathcal{F})$, where X is a set and $\mathcal{F}$ is a *collection* of subsets of X, that satisfies:

1. $\emptyset \in \mathcal{F}$
2. (Closed under complementation)
$$\mathcal{F} \setminus F \in \mathcal{F} \;\forall\; F \in \mathcal{F}$$
3. (Closed under finite unions)
$$\bigcup_{k=1}^{n} F_k \in \mathcal{F} \;\forall\; F_1, F_2, \ldots, F_k \in \mathcal{F}$$

$\mathcal{F}$ is then called an **algebra over** X

Remark
The property that $\mathcal{F}$ is closed under finite unions also implies that it is closed under finite intersections, simply by applying De Morgan's Law.
Furthermore, one can think of $\mathcal{F}$ as consisting of *the admissible sets of* X, the *complexes of* X, the nice ones that we can handle and get a hold of. In most contexts, this $\mathcal{F}$ would be where we define a lot of things on, as not all subsets of X are nice to work with.

Definition 4.2 ($\sigma-$field of sets, $\sigma-$algebra)
A $\sigma-$**field of sets** is a field of sets $(X, \mathcal{F})$ that also satisfies:

4. (Closed under countable unions)
$$\bigcup_{i=1}^{\infty} F_i \in \mathcal{F} \;\forall\; F_1, F_2, \cdots \in \mathcal{F}$$

$\mathcal{F}$ is then called a **$\sigma-$algebra over X**

Remark

The property that a $\sigma-$algebra is closed under countable unions also implies that it is closed under countable intersections, again, by an application of De Morgan's Law.

5 Linear Algebra

5.1 Vector Spaces

Definition 5.1 (Vector Spaces)

A **vector space** over a field F is a non-empty set V equipped with 2 binary operations (vector addition, scalar multiplication): $+: V \times V \to V, \cdot : F \times V \to V$ that satisfies:

1. (Abelian Group) $(V, +)$ forms an abelian group
2. (Scalar and Field Multiplication)
$$(a \cdot_F b) \cdot v = a \cdot (b \cdot V)$$
3. (Field Multiplicative Identity)
$$1_F \cdot v = v \; \forall \; v \in V$$
4. (Distributivity) $\forall \; a \in F; u, v \in V$
$$a \cdot (u + v) = (a \cdot u) + (a \cdot v)$$
$$(a +_F b) \cdot v = a \cdot v + b \cdot v$$

Definition 5.2 (Linear Map)

Let V, W be vector spaces over the same field F. Then a **linear map** is a function $f : V \to W$ that is operations-preserving, i.e. satisfying:

1. (Preserving Addition)
$$f(v_1 +_V v_2) = f(v_1) +_W f(v_2) \; \forall \; v_1, v_2 \in V$$
2. (Preserving Scalar Multiplication)
$$f(c \cdot_V v) = c \cdot_W f(v) \; \forall \; c \in F, v \in V$$

More generally,

$$f(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = c_1f(v_1) + c_2f(v_2) + \cdots + c_nf(v_n)$$

In other words, a linear map is a vector space homomorphism.

Definition 5.3 (Linear Isomorphism)

A **linear isomorphism** is a linear map that is also bijective.

Definition 5.4 (Linear Operator/Linear Endomorphism)

A **linear operator** or a **linear endomorphism** is a linear map that has the same domain and codomain, i.e. a linear map $f : V \to V$.

6 Analysis

6.1 Metric Spaces

Definition 6.1 (Metric Spaces)

Definition 6.2 (Isometry)

metry: Greek *metria*, meaning "measuring, measure"
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7 Topology

Remark

On indexing:

1. $\{E_1, E_2, \ldots, E_N\}$, sometimes I like to use $\{E_k\}_{k \leq N}$, suggests a finite indexing
2. $\{E_1, E_2, \ldots\}$ or $\{E_n\}_{n \in \mathbb{N}}$ suggests a countable indexing
3. $\{E_\alpha\}$ suggests an uncountable indexing (which kinda encompasses all previous cases and alludes to the "arbitrary" nature)

Definition 7.1 (Topology, Topological Space, Open Set)

A **topology** on set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\emptyset, X \in \mathcal{T}$
2. Arbitrary union of elements in $\mathcal{T}$ is in $\mathcal{T}$, i.e.
$$\bigcup_\alpha E_\alpha \in \mathcal{T} \; \forall \; \{E_\alpha\} \subseteq \mathcal{T}$$
3. Finite intersection of elements in $\mathcal{T}$ is in $\mathcal{T}$, i.e.
$$\bigcap_{k=1}^{N} E_k \in \mathcal{T} \; \forall \; \{E_k\}_{k \leq N} \subseteq \mathcal{T}$$

A set X with a specified topology $\mathcal{T}$ is a **topological space**. $U \subseteq X$ is called an **open set** iff $U \in \mathcal{T}$, so think of $\mathcal{T}$ as a (huge) collection of open sets.

Definition 7.2 (Basis For A Topology)

Oftentimes, one is unable to specify the entire topology $\mathcal{T}$, so we can instead specify a smaller collection oof subsets of X and then define the topology using that.
Let X be a set, then a **basis** is a collection $\mathcal{B}$ of subsets of X (called **basis elements**) s.t.

1. $\forall \, x \in X, \; \exists \, B \in \mathcal{B}$ s.t. $x \in B$ (wherever you are, I got you)
2. If $x \in B_1 \cap B_2; B_1, B_2 \in \mathcal{B}$ then $\exists \, B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subseteq B_1 \cap B_2$

Definition 7.3 (Topology Generated By Basis)

Call this entity that we want to generate $\mathcal{T}_\mathcal{B}$ (my own notation).
Then for subset $U \subseteq X$,

$$U \in \mathcal{T}_\mathcal{B} \text{ if } \forall \, x \in U, \; \exists \, B \in \mathcal{B} \text{ s.t. } x \in B \subseteq U$$

7.1 Others

Definition 7.4 (Linear Continuum)

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Definition 8.1 (Harmonic Function)

Let U be an open subset of $\mathbb{R}^d$, then f is **harmonic** in U if and only if it is continuous and satisfies the mean value property: for every $x \in U$, $\forall\, 0 < \varepsilon < dist(x, \partial U)$,

$$f(x) = MV(f; x, \varepsilon) = \int_{|y-x|=\varepsilon} f(y) ds(y)$$