MATH 20700: Honors Analysis in Rn I

Problem Set 4

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Textbook: Pugh's Real Mathematical Analysis

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Problem 4.1 (3.39 done)

Consider the characteristic functions $f(x)$ and $g(x)$ of the intervals $[1, 4]$ and $[2, 5]$. The derivatives f' and g' exist almost everywhere. The integration-by-parts formula says that

$$\int_0^3 f(x)g'(x)dx = f(3)g(3) - f(0)g(0) - \int_0^3 f'(x)g(x)dx$$

But both integrals are zero, while $f(3)g(3) - f(0)g(0) = 1$. Where is the error?

Solution

$f = \chi_{[1,4]}, g = \chi_{[2,5]}$ are not differentiable on $[0, 3]$ to be able to apply IBP. Specifically, f is not differentiable at 1:

$$\lim_{t\to 1+} \frac{f(t) - f(1)}{t - 1} = 0$$

$$\lim_{t\to 1-} \frac{f(t) - f(1)}{t - 1} = \lim_{t\to 1-} \frac{-1}{t - 1} = \infty$$

Similarly, g is not differentiable at 2.

Therefore IBP is not applicable. $\square$

Problem 4.2 (3.42 done)

Suppose that $\psi : [c, d] \to [a, b]$ is continuous and for every zero set $Z \subset [a, b]$, $\psi^{pre}(Z)$ is a zero set in $[c, d]$.

(a) If f is RI, prove that $f \circ \psi$ is RI.

(b) Derive Corollary 32 from (a)

Solution

$f : [a, b] \to \mathbb{R}$

(a) Let D be the set of discontinuities of f. Since f is RI, D is a zero set.

Since ψ is continuous, $\psi^{Pre}(D)$ is the set of discontinuities of $f \circ \psi$ (Pugh). D is a zero set, so it follows that $\psi^{Pre}(D)$ is a zero set too. Thus the set of discontinuities of $f \circ \psi$ is a zero set. $f \circ \psi$ is RI. $\square$

(b) Want to show Corollary 32: If $f : [a, b] \to \mathbb{R}$ is RI and $\psi : [c, d] \to [a, b]$ is a homeomorphism whose inverse satisfies a Lipschitz condition then $f \circ \psi$ is RI.

Let ψ^{-1} be $L-$Lipschitz.

We want to show that ψ satisfies the conditions above in (a), i.e., that for every set $Z \subset [a, b]$, $\psi^{Pre}(Z)$ is a zero set.

Fix $\varepsilon > 0$. Given zero set $Z \subset [a, b]$. Then Z can be covered by finitely many open intervals $\{(a_i, b_i) \mid i \leq k\}$ whose lengths add up to less than ε/L. Since ψ is a homeomorphism, $\psi^{-1}(Z)$ is covered by the finitely many open intervals $(a'_i, b'_i) = \psi^{-1}[(a_i, b_i)]$. And we have

$$\sum |b'_i - a'_i| \leq \sum L(b_i - a_i) < L\frac{\varepsilon}{L} = \varepsilon$$

so $\psi^{Pre}(Z) = \psi^{-1}(Z)$ is indeed a zero set. Now we can apply (a) to get that $f \circ \psi$ is RI. $\square$

Problem 4.3 (3.43 done)

Let $\psi(x) = x \sin(1/x)$ for $0 < x \leq 1$ and $\psi(0) = 0$.

(a) If $f : [-1, 1] \to \mathbb{R}$ is RI, prove that $f \circ \psi$ is RI.

(b) What happens for $\psi(x) = \sqrt{x} \sin 1/x$?

Solution

(a) ψ is differentiable on (0, 1):

$$\psi'(x) = \sin(1/x) + x\cos(1/x)(-1)x^{-2} = \sin(1/x) - \cos(1/x)/x.$$

Therefore, ψ has critical points satisfying:

$$\sin(1/x_c) = \cos(1/x_c)/x_c$$

Crucially, at these critical points, $\sin(1/x_c) \neq 0$ because if so then $\cos(1/x_c) = 0$ too, a contradiction.

$$\begin{aligned}
\psi''(x) &= \cos(1/x)(-x^{-2}) - \frac{x(-\sin(1/x))(-x^{-2}) - \cos(1/x)}{x^2} \\
&= -\cos(1/x)x^{-2} - \sin(1/x)x^{-3} + \cos(1/x)x^{-2} \\
&= -\sin(1/x)x^{-3} \\
\Rightarrow \psi''(x_c) &= -\sin(1/x_c)x_c^{-3} \neq 0
\end{aligned}$$

WLOG, assume $\psi''(x_c) > 0$. Since ψ'' is continuous, there exists $\delta_1 > 0$ such that $t \in (x_c, x_c + \delta_1) \Rightarrow \psi''(t) > 0$, which implies

$$\psi'(t) = \psi'(x_c) + \int_{x_c}^{x_c+t} \psi''(s)ds > \psi'(x_c) > 0$$

Similarly, there also exist $\delta_2 > 0$ such that $t \in (x_c - \delta_2, x_c) \Rightarrow \psi''(t) > 0$, which implies $\psi'(t) < 0$ on $(x_c - \delta_2, x_c)$. Therefore, it can be concluded that for each x_c, there exists an open interval $(x_c - \min\{\delta_1, \delta_2\}, x_c + \min\{\delta_1, \delta_2\})$ that has no other critical points. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $q_c \in \mathbb{Q} \cap (x_c - \min\{\delta_1, \delta_2\}, x_c + \min\{\delta_1, \delta_2\})$.

Let D be the set of critical points of ψ on $(0, 1]$. The mapping above from $x_c \mapsto q_c$ describes an injective mapping from $D \to \mathbb{Q}$. Thus D is countable.

We may now enumerate the countable critical points with the endpoints 0 and 1:

$$x_1 = 0 < x_2 < x_3 < \cdots < 1$$

Consider ψ restricted to $I_k = (x_k, x_{k+1})$. Since ψ' is continuous and can't achieve 0 in the interval (otherwise there would be another critical point in between), it has to be positive throughout or negative throughout. WLOG, suppose $\psi' > 0$ on I_k. Therefore ψ is monotone thus surjective onto $(\psi(x_{k+1}), \psi(x_k))$. Using Inverse Function Theorem, ψ is a homeomorphism. In fact, a C^1 homeomorphism, since ψ' is continuous. Using Corollary 15 (Section 3.1), ψ is a C^1 diffeomorphism.

Now, we want to show that ψ^{-1} maps zero sets to zero sets on I_k. Restrict ψ to I_k, call it $\psi_k = \psi|_k$.

$I_k = \bigcup_{n\in\mathbb{N}} [x_{k+1} + 1/n, x_k - 1/n] =: \bigcup_{n\in\mathbb{N}} J_n$. On each compact J_n, we know that ψ_k' is continuous so $\frac{1}{\psi_k'}$ is continuous. It is continuous on compact J_n so it is bounded by L. Then Inverse Function Theorem implies

$$|(\psi_k^{-1})'| = \left| \frac{1}{\psi_k' \circ \psi_k} \right| \leq L$$

Therefore ψ_k^{-1} is L-Lipschitz. As shown in 3.42(b), this implies that ψ_k^{-1} maps zero sets to zero sets.

However, this is only on J_n. But if we let Z be a zero set in I_k, then $Z = Z \cap I_k = \bigcup (Z \cap J_n) =: \bigcup Z_n$. Each Z_n is a zero set so

$$\psi_k^{-1}(Z) = \bigcup \psi_k^{-1}(Z_n)$$

is a countable union of zero sets, and is therefore a zero set.

Therefore, ψ_k^{-1} maps zero sets to zero sets.

Now, take zero set Y in [-1, 1], then

$$\begin{aligned} \psi^{Pre}(Y) &= \bigcup_{k\in\mathbb{N}} \psi^{Pre}(Y \cap \psi(I_k)) \cup \bigcup_{k\in\mathbb{N}} \psi^{Pre}(Y \cap \{\psi(x_k)\}) \\ &= \bigcup_{k\in\mathbb{N}} \psi_k^{-1}(Y \cap \psi(I_k)) \cup \bigcup_{k\in\mathbb{N}} \psi^{Pre}(Y \cap \{\psi(x_k)\}) \end{aligned}$$

is a union of:

- a countable union of zero sets, and
- a countable union of finite sets

is therefore a zero set. □

Then, let E be the set of discontinuities of f. Since f is RI, E is zero set. Since ψ is continuous, it follows that the set of discontinuities of $f \circ \psi$ is $\psi^{Pre}(E)$, which is a zero set. $f \circ \psi$ is therefore RI.

(b) The result still holds. It only remains for us to prove that ψ'' is continuous on $(0, 1)$ and $\psi'' \neq 0$ at critical points.

$$\psi'(x) = x^{-3/2}(\sin(1/x)x - 2\cos(1/x))$$

$$\psi''(x) = -4x^{-7/2}(\sin(1/x)x^2 - 4\cos(1/x)x + 4\sin(1/x))$$

So $\psi''(x)$ is continuous. Critical points satisfy:

$$\sin(1/x_c)x_c = 2\cos(1/x_c) \Rightarrow \cos(1/x_c) \neq 0, \tan(1/x_c) = 2/x_c$$

so

$$\begin{aligned}
0 &= \psi''(x_c) \\
\Leftrightarrow 0 &= -4x_c^{-7/2}(\sin(1/x_c)x_c^2 - 4\cos(1/x_c)x_c + 4\sin(1/x_c)) \\
\Leftrightarrow 0 &= \sin(1/x_c)(x_c^2 + 4) - 4\cos(1/x_c)x_c \\
\Leftrightarrow \tan(1/x_c)(x_c^2 + 4) &= 4x_c \\
\Leftrightarrow \frac{2(x_c^2 + 4)}{x_c} &= 4x_c \\
\Leftrightarrow x_c^2 + 4 &= 2x_c^2 \\
\Leftrightarrow x_c &\in \{\pm 2\}
\end{aligned}$$

so $\psi'' \neq 0$ on $(0, 1)$. Therefore, the same proof in (a) applies. $f \circ \psi$ is therefore RI.

Problem 4.4 (4.4a done)

If $f_n : \mathbb{R} \to \mathbb{R}$ is uniformly continuous for each $n \in \mathbb{N}$ and if $f_n \rightrightarrows f$ as $n \to \infty$, prove or disprove that f is uniformly continuous.

Solution

WTS f is uniformly continuous.

Fix $\varepsilon > 0$. Since $f_n \rightrightarrows f$, there exists $N = N_\varepsilon \in \mathbb{N}$ such that

$$\forall\, n \geq N_\varepsilon, d_{sup}(f_n, f) < \varepsilon/3$$

Since f_N is uniformly continuous, there exists $\delta = \delta_{N_\varepsilon} = \delta_\varepsilon$ such that

$$|x - y| < \delta \Rightarrow |f_N(x) - f_N(y)| < \varepsilon/3$$

Then we can estimate, for $|x - y| < \delta$:

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(x)| \leq 3\varepsilon/3 = \varepsilon$$

f is therefore uniformly continuous. $\square$

Problem 4.5 (4.8 done)

Is the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \cos(n + x) + \log\left(1 + \frac{1}{\sqrt{n+2}} \sin^2(n^n x)\right)$$

equicontinuous? Prove or disprove.

Solution

(Joshua's hint on Discord)

We want to show separately that $a_n(x) = \cos(n + x)$, $b_n(x) = \log\left(1 + \frac{1}{\sqrt{n+2}} \sin^2(n^n x)\right)$ are equicontinuous.

1.

$$|a_n'(x)| = | - \sin(n + x)| \leq 1$$

Since $a_n(x)$ is continuous and differentiable everywhere on $\mathbb{R}$, MVT gives us the equicontinuity of $\{f_n\}$: given $\varepsilon > 0$, then choose $\delta = \varepsilon$, then

$$|s - t| < \delta \Rightarrow |\cos(n + s) - \cos(n + t)| = | - \sin(n + \theta)||s - t| < 1\delta < \varepsilon$$

so $\cos(n + x)$ is equicontinuous.

2. WTS (b_n) is uniformly equicontinuous.

2.1. WTS each b_n is uniformly continuous.

We do this by showing that b_n is Lipschitz.

$$|b_n'(x)| = \left| \frac{1}{1 + \frac{1}{\sqrt{n+2}} \sin^2(n^n x)} \left(\frac{2n^n}{\sqrt{n+2}} \sin(n^n x) \cos(n^n x) \right) \right| \leq \left| \frac{n^n}{\sqrt{n+2}} \sin(2n^n x) \right| \leq C_n$$

Similar to **1.**, b_n is C_n-Lipschitz. Given $\varepsilon > 0$, then we can choose $\delta_n = \varepsilon / C_n$, then

$$|s - t| < \delta_n \Rightarrow |b_n(s) - b_n(t)| \leq C_n |s - t| < \varepsilon$$

Therefore b_n is uniformly continuous.

2.2. WTS (b_n) is uniformly Cauchy.

Note that for fixed n,

$$0 = \log 1 \leq |b_n(x)| \leq \log\left(1 + \frac{1}{\sqrt{n+2}}\right)$$

Therefore, for all $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $\log\left(1 + \frac{1}{\sqrt{n+2}}\right) < \varepsilon/2$. Then for all $n, m \geq N$, for all $x \in \mathbb{R}$,

$$\begin{aligned}|b_n(x) - b_m(x)| &\leq |b_n(x)| + |b_m(x)| \\ &\leq 2\log\left(1 + \frac{1}{\sqrt{N+2}}\right) < \varepsilon\end{aligned}$$

Therefore for all $n, m \geq N$,

$$d_{sup}(b_n, b_m) < \varepsilon$$

(b_n) is therefore uniformly Cauchy.

2.3. Given $\varepsilon > 0$, then since (b_n) is Cauchy, there exists $N \in \mathbb{N}$ such that

$$n, m \geq N \Rightarrow d_{sup}(b_n, b_m) < \varepsilon/3$$

Since each $\{b_n\}_{1\leq n\leq N}$ is uniformly continuous, there exists $\{\delta_n\}_{1\leq n\leq N}$ such that

$$|s - t| < \delta_n \Rightarrow |b_n(s) - b_n(t)| < \varepsilon/3$$

Take $\delta = \min\{\delta_n : 1 \leq n \leq N\}$. Then, for $|s - t| < \delta$,

- If $1 \leq n \leq N$ then $|b_n(s) - b_n(t)| < \varepsilon/3 < \varepsilon$, since $\delta \leq \delta_n$.
- If $n > N$ then

$$|b_n(s) - b_n(t)| \leq |b_n(s) - b_N(s)| + |b_N(s) - b_N(t)| + |b_N(t) - b_n(t) < 3\varepsilon/3 = \varepsilon$$

It follows that

$$|s - t| < \delta \Rightarrow |b_n(s) - b_n(t)| < \varepsilon \; \forall \; n \in \mathbb{N}$$

(b_n) is therefore uniformly equicontinuous.

3. Since (a_n) and (b_n) are both uniformly equicontinuous, given $\varepsilon > 0$, there exists $\delta_a, \delta_b > 0$ such that

$$|s - t| < \delta_a \Rightarrow |a_n(s) - a_n(t)| < \varepsilon/2, |s - t| < \delta_b \Rightarrow |b_n(s) - b_n(t)| < \varepsilon/2$$

Take $\delta = \min\{\delta_a, \delta_b\}$ then

$$|s - t| < \delta \Rightarrow |f_n(s) - f_n(t)| \leq |a_n(s) - a_n(t)| + |b_n(s) - b_n(t)| < 2\varepsilon/2 = \varepsilon$$

(f_n) is therefore uniformly equicontinuous. □

Problem 4.6 (4.9 done)

If $f : \mathbb{R} \to \mathbb{R}$ is continuous and the sequence $f_n(x) = f(nx)$ is equicontinuous, what can be said about f?

Solution

WTS f is a constant function, i.e. $f(x) = C \; \forall \; x$ for some $C \in \mathbb{R}$.

Suppose not. Then there exists $a \neq b \in \mathbb{R}$ such that $f(a) \neq f(b)$.

Let $\varepsilon = \frac{|f(b)-f(a)|}{2}$. Then there exists δ such that

$$|x-y| < \delta \Rightarrow |f(nx) - f(ny)| < \varepsilon \; \forall \; n \in \mathbb{N}$$

Choose N sufficiently large, such that $|\frac{b-a}{N}| < \delta$, i.e. $N > \frac{|b-a|}{\delta}$. Then that implies

$$|b/N - a/N| < \delta \Rightarrow |f(b) - f(a)| = |f(N(b/N)) - f(N(a/N))| < \varepsilon = \frac{|f(b)-f(a)|}{2} \Rightarrow\Leftarrow$$

It follows that f is a constant function. $\square$

Problem 4.7 (4.14 done)

Recall from Exercise 2.78 that a metric space M is chain connected if for each $\varepsilon > 0$ and each $p, q \in M$ there is a chain $p = p_0, \ldots, p_n = q$ in M such that

$$d(p_{k-1}, p_k) < \varepsilon \text{ for } 1 \leq k \leq n.$$

A family $\mathcal{F}$ of functions $f : M \to \mathbb{R}$ is bounded at $p \in M$ if the set $\{f(p) : f \in \mathcal{F}\}$ is bounded in $\mathbb{R}$.

Show that M is chain connected if and only if pointwise boundedness of an equicontinuous family at one point of M implies pointwise boundedness at every point of M.

Solution

$\boxed{\Rightarrow}$ Let M be chain connected. And let $\mathcal{F}$ be an equicontinuous family of functions that is bounded at some $p \in M$, i.e. $\{f(p) : f \in \mathcal{F}\}$ is bounded in $\mathbb{R}$. WTS $\mathcal{F}$ is pointwise bounded at every point $q \in M$.

$\{f(p) : f \in \mathcal{F}\}$ is bounded in $\mathbb{R}$. Thus there exists $L \in \mathbb{R}$ such that $|f(p)| < L$ for all $f \in \mathcal{F}$

Take arbitrary $q \in M$. Fix $\varepsilon = 1$. Since $\mathcal{F}$ is equicontinuous, there exists $\delta > 0$ such that

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < 1 \; \forall \; f \in \mathcal{F}$$

Since M is chain connected, there exists a chain $p = p_0, \ldots, p_N = q$ in M such that $d(p_{k-1}, p_k) < \delta \; \forall \; 1 \leq k \leq N$. Now we can make the estimate for all $f \in \mathcal{F}$,

$$|f(q) - f(p)| \leq \sum_{k=1}^{N} |f(p_k) - f(p_{k-1})| < N$$

therefore for all $f \in \mathcal{F}$

$$|f(q)| \leq N + |f(p)| < N + L$$

so $\mathcal{F}$ is pointwise bounded at $q \in M$. $\square$

$\boxed{\Leftarrow}$ We have that for all equicontinuous family $\mathcal{F}$ that is pointwise bounded at one point $p \in M$, it is pointwise boundedness at every point in M.

Let's prove by contradiction. Suppose M is not chain-connected. Then there exists $\varepsilon_0 > 0; p, q \in M$ such that there is no ε_0-chain from p to q. Let A be the set of points that can be ε_0-chain-connected from p, and let B be the set of points that cannot. Then

$p \in A, q \in B$. Also,

$$M = A \sqcup B$$

so if $c \in M$ then $c \in A$ or $c \in B$. Define family $\mathcal{F} = \{f_n : M \to \mathbb{R}\}$ as follows

$$f_n(x) = \begin{cases} 0 & \text{for } x \in A \\ n & \text{for } x \in B \end{cases}$$

then $f_n(p) = 0 \; \forall \; n \in \mathbb{N}$, since $p \in A$. Thus $\mathcal{F}$ is pointwise bounded at p.

Also note that for all $a \in A, b \in B$, we have $d(a, b) \geq \varepsilon_0$. Because otherwise, since there exists an ε_0-chain from p to a, we can concatenate this chain with b as the last node to get an ε_0-chain from p to b, making $b \in A$.

Now, given any $\varepsilon > 0$, we can choose $\delta = \varepsilon_0$. Then for $d(s, t) < \delta$, since if one of s and t is in A and the other in B then $d(s, t) \geq \varepsilon_0 > \delta$, so they must be both in A or both in B. Therefore $f_n(s) = f_n(t) \Rightarrow d(f_n(s), f_n(t)) = 0 < \varepsilon$.

Therefore, $\mathcal{F}$ is uniformly equicontinuous.

The problem's assumption then yields that $\mathcal{F}$ is pointwise bounded at every point in M. But $\{f_n(q)\} = \mathbb{N}$ is clearly NOT bounded! $\Rightarrow\Leftarrow$

By contradiction, M must therefore be chain-connected. $\square$

Problem 4.8 (4.23 a-c done)

Let M be a compact metric space, and let (i_n) be a sequence of isometries $i_n : M \to M$.

(a) Prove that there exists a subsequence i_{n_k} that converges to an isometry i as $k \to \infty$.

(b) Infer that the space of self-isometries of M is compact.

(c) Does the inverse isometry $i_{n_k}^{-1}$ converge to i^{-1}? (Proof or counterexample.)

Solution

(a) We want to show that the familiy of functions $i_n : X \to Y$ is uniformly equicontinuous. In fact, given $\varepsilon > 0$, we can choose $\delta = \varepsilon$. Then,

$$d(x, y) < \delta \Rightarrow d(i_n(x), i_n(y)) = d(x, y) < \delta = \varepsilon \; \forall \; n \in \mathbb{N}$$

which gives us uniformly equicontinuous (i_n).

Now we want to show that the space of self-isometries $S = \{i : M \to M\}$ is complete. As a result from above, we've seen that each self-isometry is continuous. Thus, since $C^0(M, \mathbb{R})$ is complete, if the sequence of isometries i_n is Cauchy then $i_n \xrightarrow{n\to\infty} i \in C^0$. Now we want to show $i \in S$. Indeed, since (i_n) is uniformly equicontinuous, given $\varepsilon > 0$, there exists $N = N_\varepsilon \in \mathbb{N}$ such that for all $n \geq N$, $d_{sup}(i_n, i) < \varepsilon/2$. It follows that, for all $x, y \in M$,

$$d(i(x), i(y)) \leq d(i(x), i_N(x)) + d(i_N(x), i_N(y)) + d(i_N(y), i(y)) < \varepsilon + d(x, y)$$

and

$$d(x, y) = d(i_N(x), i_N(y)) \leq d(i_N(x), i(x)) + d(i(x), i(y)) + d(i(y), i_N(y)) < \varepsilon + d(i(x), i(y))$$

We therefore have the inequalities:

$$d(x,y) - \varepsilon < d(i(x), i(y)) < d(x,y) + \varepsilon$$

that holds for all $\varepsilon > 0; x, y \in M$. It follows that $d(i(x), i(y)) = d(x,y)\ \forall\ x, y \in M$ and i is indeed an isometry.

Therefore the space of self-isometries S is complete.

For each $x \in M$, we have that $(i_n(x))_{n\in\mathbb{N}}$ lies in a compact subset of M, namely M itself. Therefore, using Theorem 37 (Section 4.8), (i_n) has a uniformly Cauchy subsequence. Since S is complete, the uniformly Cauchy subsequence is convergent. □

(b) Restating what we proved from (a): for all $\{i_n\} \subseteq S$, there exists a convergent subsequence $\{i_{n_k}\}$. S, the space of self-isometries from M to M, is therefore compact. □

(c) (With Prof Wilkinson's provided hint) We first show that the inverse i_n^{-1} is well-defined, i.e., that all self-isometries $f : M \to M$ for M compact is a bijection.

Take arbitrary $x_0 \in M$ and arbitrary isometry $f : M \to M$. Then we can construct the following sequence:

$$x_1 = f(x_0), x_2 = (f \circ f)(x_0), \ldots, x_n = f^{\circ n}(x_0), \ldots$$

Clearly, $\{x_n\} \subseteq f(M)$. Since M is compact, there exists a subsequence $(x_{n_j})_{j\in\mathbb{N}}$ that converges to some $x \in M$.

Fix $\varepsilon > 0$. Then there exists $J \in \mathbb{N}$ such that for all $j \geq J$,

$$d(x_{n_j}, x) < \varepsilon/2$$

Then choose $J_1 > J_2 \geq J$. Let $N_1 = n_{J_1} > N_2 = n_{J_2}$ then we have

$$d(x_{N_1}, x), d(x_{N_2}, x) < \varepsilon/2 \Rightarrow d(x_{N_1}, x_{N_2}) \leq d(x_{N_1}, x) + d(x_{N_2}, x) < \varepsilon$$

But by construction of (x_n),

$$\varepsilon > d(x_{N_1}, x_{N_2}) = d(f^{\circ N_1}(x_0), f^{\circ N_2}(x_0)) = d(f^{\circ(N_1-N_2)}(x_0), x_0) = d(x_{N_2-N_1}, x_0)$$

It follows that for all $\varepsilon > 0$, there always exists $K = K_\varepsilon, x_K \in M$ such that $d(x_K, x_0) < \varepsilon$. Therefore x_0 is a cluster point of $f(M)$.

From above, every isometry is continuous. f is therefore continuous and M is compact so $f(M)$ is compact, and therefore closed. x_0 is a cluster point of a closed set and is therefore in the set.

It follows that every self-isometry of M compact is surjective. It is clear that every isometry is also injective, because

$$x \neq y \Rightarrow d(x,y) > 0 \Rightarrow d(f(x), f(y)) = d(x,y) > 0 \Rightarrow f(x) \neq f(y)$$

Thus, every self-isometry of M compact is bijective and f^{-1} is well-defined. It is also clear that f^{-1} is an isometry:

$$d(f^{-1}(x), f^{-1}(y)) = d(f(f^{-1}(x)), f(f^{-1})(y)) = d(x, y)$$

Now, we want to show that yes, if self-isometries (i_{n_k}) converge to i then $(i_{n_k}^{-1})$ also converge to i^{-1}.

For simplicity, relabel i_{n_k} as g_k. Then since g_k converges to i, given $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that for all $k \geq K$,

$$\sup_{x \in M} \{d_M(g_k(x), i(x))\} < \varepsilon$$

Then for all $k \geq K$,

$$\begin{aligned}
\sup_{x \in M} \{d(g_k^{-1}(x), i^{-1}(x))\} &= \sup_{x \in M} \{d(i(g_k^{-1}(x)), x)\} \\
&= \sup_{x \in M} \{d(i(g_k^{-1}(x)), g_k(g_k^{-1}(x)))\} \\
&\leq \sup_{x \in M} \{d(i(x), g_k(x))\} < \varepsilon
\end{aligned}$$

Therefore g_k^{-1} does converge to i^{-1}. $\square$

Problem 4.9 (Prelim 4.8 done)

Let $h : [0, 1) \to \mathbb{R}$ be a uniformly continuous function where $[0,\ 1)$ is the half-open interval. Prove that there is a unique continuous map $g : [0, 1] \to \mathbb{R}$ such that $g(x) = h(x)$ for all $x \in [0, 1)$.

Solution

Let $a_n := h\left(1 - \frac{1}{n}\right)$. We want to show that (a_n) is Cauchy. Given $\varepsilon > 0$, then since h is uniformly continuous, there exists $\delta_1 > 0$ such that

$$|x - y| < \delta_1 \Rightarrow |h(x) - h(y)| < \varepsilon$$

Choose $N_1 \in \mathbb{N}$ such that $2/N_1 < \delta_1$. It then follows that for $m, n \geq N_1$,

$$\left|\left(1 - \frac{1}{m}\right) - \left(1 - \frac{1}{n}\right)\right| \leq \frac{2}{\min\{m, n\}} \leq \frac{2}{N_1} < \delta_1$$

which implies

$$|a_m - a_n| = |h(1 - 1/m) - h(1 - 1/n)| < \varepsilon$$

(a_n) is therefore Cauchy and converges to unique $L \in \mathbb{R}$.

We now want to show that $\lim_{t \to 1-} g(t) = \lim_{t \to 1-} h(t) = L$.

Indeed, given $\varepsilon > 0$, there exists $\delta_2 > 0$ such that

$$|x - y| < \delta_2 \Rightarrow |h(x) - h(y)| < \varepsilon/2$$

Furthermore, since (a_n) converges to L, there exists $N_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$|a_n - L| < \varepsilon/2$$

Therefore, for $0 < 1 - t < \frac{\delta_2}{2}$, we can choose $N_3 \geq N_2$ such that $\frac{1}{N_3} < \frac{\delta_2}{2}$. Then

$$|t - (1 - 1/N_3)| \leq |1 - t| + |1/N_3| < 2\delta_2/2 = \delta_2$$

which implies

$$|h(t) - L| \leq |h(t) - h(1 - 1/N_3)| + |h(1 - 1/N_3) - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

In short, there exists $\delta = \delta_2/2$ such that

$$1 - t < \delta \Rightarrow |h(t) - L| < \varepsilon$$

Therefore $\lim_{h \to 1-} h(t) = L$.

Since g is continuous, it follows that $g(1) = \lim_{t \to 1-} g(t) = L$ is unique. g is therefore unique, since it is already defined on $[0, 1)$. □

Problem 4.10 (Prelim 4.12 done)

Let $f : [0, 1] \to \mathbb{R}$ be continuously differentiable, with $f(0) = 0$. Prove that

$$\|f\|^2 \leq \int_0^1 (f'(x))^2 dx$$

where $\|f\| = \sup\{|f(t)| : 0 \leq t \leq 1\}$.

Solution

Since f is continuously differentiable, $|f|$ is continuous on $[0, 1]$. Therefore it achieves its maximum at $x_0 \in [0, 1]$, i.e.

$$\|f\| = \sup\{|f(t)| : 0 \leq t \leq 1\} = |f(x_0)|$$

Therefore, using 2nd FTC and Cauchy-Schwarz,

$$\begin{aligned}
\|f\|^2 &= |f(x_0)|^2 \\
&= \left| f(0) + \int_0^{x_0} f'(t)dt \right|^2 \\
&= \left| \int_0^{x_0} f'(t)dt \right|^2 \\
&\leq \left(\int_0^{x_0} (f'(t))^2 dt \right) \int_0^{x_0} 1^2 dt \\
&\leq \left(\int_0^1 (f'(t))^2 dt \right) x_0 \\
&\leq \int_0^1 (f'(t))^2 dt
\end{aligned}$$

since $x_0 \in [0, 1]$. □

Problem 4.11 (Prelim 4.20 done)

Let (g_n) be a sequence of RI functions from $[0,\ 1]$ into $\mathbb{R}$ such that $|g_n(x)| \leq 1$ for all n, x. Define

$$G_n(x) = \int_0^x g_n(t)dt.$$

Prove that a subsequence of (G_n) converges uniformly.

Solution

Note that G_n is only defined on $[0, 1]$.

We first want to show that G_n is uniformly equicontinuous. Indeed,

$$\begin{aligned} |G_n(x) - G_n(y)| &= \left| \int_0^x g_n(t)dt - \int_0^y g_n(t)dt \right| \\ &= \left| \int_x^y g_n(t)dt \right| \\ &\leq \left| \int_x^y |g_n(t)|dt \right| \\ &\leq |x - y| \end{aligned}$$

Thus, given $\varepsilon > 0$, we can choose $\delta = \varepsilon$, then for all G_n,

$$|G_n(x) - G_n(y)| \leq |x - y| < \delta = \varepsilon$$

Secondly, we want to show that G_n is bounded. Indeed,

$$|G_n(x)| = \left| \int_0^x g_n(t)dt \right| \leq \left| \int_0^x |g_n(t)|dt \right| \leq |x| \leq 1$$

Therefore, combining the 2 facts above, using Arzela-Ascoli, it follows that (G_n) has a convergent subsequence. □