Math 20250
Abstract Linear Algebra

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Disclaimer: This document will inevitably contain some mistakes, both simple typos and serious logical and mathematical errors. Take what you read with a grain of salt as it is made by an undergraduate student going through the learning process himself. If you do find any error, I would really appreciate it if you can let me know by email at conghungletran@gmail.com.

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Lecture 4

Linear Transformation, Homomorphism, Kernel, Image

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4.1 Vector Subspace

Definition 4.1 (Vector Subspace).

Let V be a $\mathbb{K}$-vector space. A **subspace** (or **sub-vector space**) of V is a subset $W \subseteq V$ such that W is itself a $\mathbb{K}$-vector space under addition and scaling induced from V. A priori, we know that

$$+ : W \times W \to V, \cdot : W \times W \to V$$

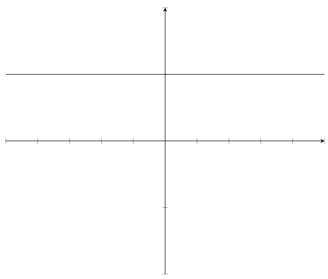
but this subspace requirement implies that

$$\forall\, x, y \in W, x + y \in W$$

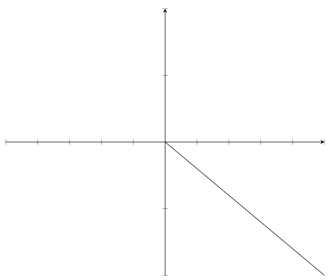
$$\forall\, \alpha \in \mathbb{K}, x \in W, \alpha \cdot x \in W$$

In other words, the subspace is closed under addition and scaling.

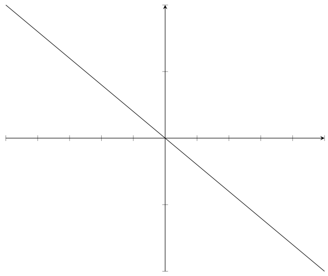
Example. Take $\mathbb{K} = \mathbb{R}, V = \mathbb{R}^2$, with ordinary addition and scaling.
Consider the subset represented by line $y = 1$.



This is not a subspace because there exists no 0 element. This kinda implies that any subspace of $\mathbb{R}^2$ must pass through the origin $(0, 0)$.
Consider another instance, this time the following ray:



This is also not a subspace, since there's no additive inverse. Therefore a subspace shall look something like this:



4.2 Mapping

Motivation. A map from sets to sets can be anything. e.g. $x : \mathbb{Z} \mapsto x^2 : \mathbb{Z}$ doesn't preserve the "group" structure $(x+y)^2 \neq x^2 + y^2$ most of the time.

Definition 4.2 (Group Homomorphism).

Let A, B be Abelian groups. Map $\psi : A \to B$ is called a **group homomorphism** if:

$$\psi(x+y) = \psi(x) + \psi(y)$$

Then $x : \mathbb{Z} \mapsto x^2 : \mathbb{Z}$ is not a group homomorphism, but $x : \mathbb{Z} \mapsto nx : \mathbb{Z}$ for fixed n is a group homomorphism.
Here, a natural question arises: If given 2 vector spaces, what maps are allowed between them? What structures do we have to preserve?

Definition 4.3 (Linear Transformation).

Let V, W be $\mathbb{K}$-vector spaces. Then a **vector space homomorphism** is also called a **linear transformation**, a map $\psi : V \to W$ such that

1. $\psi(v_1 + v_2) = \psi(v_1) + \psi(v_2)\ \forall\, v_1, v_2 \in V$
2. $\psi(\alpha \cdot v) = \alpha \cdot \psi(v)\ \forall\, \alpha \in \mathbb{K}, v \in V$

Denote $\mathbf{Hom}_{\mathbb{K}}(\mathbf{V}, \mathbf{W})$ as the set of all linear transformations $V \to W$.

Example. $\mathbb{K} = \mathbb{R}, V = W = \mathbb{R}$
$\mathrm{Hom}_{\mathbb{R}}(V, W) = \{\psi : \mathbb{R} \to \mathbb{R} \mid (1), (2) \text{ are satisfied }\}$
We claim that $\psi(1)$ uniquely determines the map ψ, because

$$\psi(\alpha) = \alpha \cdot \psi(1)$$

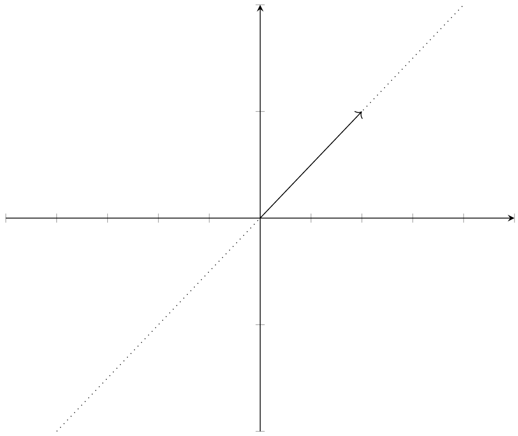
Essentially, there exists a bijection between $\mathrm{Hom}_{\mathbb{R}}(V, W)$ and $\mathbb{R}$:

$$\begin{aligned} \mathrm{Hom}_{\mathbb{R}}(V, W) &\to \mathbb{R} \\ \psi &\to \psi(1) \\ (\psi_\beta : x \mapsto x \cdot \beta) &\leftarrow \beta \end{aligned}$$

Example. $\mathbb{K} = \mathbb{R}, V = \mathbb{R}, W =$ any $\mathbb{K}$-vector space
We, similarly, claim that there is a bijection between $\mathrm{Hom}_{\mathbb{R}}(V, W)$ and W. With the same reasoning, ψ is determined by $\psi(1)$, though this time $\psi(1) \in W$.

$$\begin{aligned} \mathrm{Hom}_{\mathbb{R}}(V, W) &\to W \\ \psi &\to \psi(1) \in W \\ (\psi_\beta : x \mapsto x \cdot w) &\leftarrow w \end{aligned}$$

Example. As a sub-example of the example above, consider $W = \mathbb{R}^2$:



Then if $\psi(1) = (4, 5)$ as above (and $\psi(0) = (0, 0)$ implicit), then ψ would map the rest of $V = \mathbb{R}$ onto the dotted line above.
An interesting point to note is that if $\psi(1) = (0, 0)$, then the entire real line would get sent (and compressed) to $(0, 0)$. $\psi_{(0,0)}$ therefore contracts $\mathbb{R}$ into one point (the origin (0, 0)) while others output a subspace of $\mathbb{R}^2$.

Example. $\mathbb{K} = \mathbb{R}, V = \mathbb{R}^2, W =$ any $\mathbb{R}$-vector space
We claim that there exists a bijection between $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^2, W)$ and $W \oplus W$; as each ψ is determined by $\psi((1, 0))$ and $\psi((0, 1))$.
The notation $\oplus$ is defined as: If V, W are $\mathbb{K}$-vector spaces then

$$V \oplus W = \{(v, w) \mid v \in V, w \in W\}$$

e.g. $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$
Then $V \oplus W$ would also be a $\mathbb{K}$-vector space with operations $+, \cdot$ defined intuitively:

$$\begin{aligned} (v_1, w_1) + (v_2, w_2) &= (v_1 + v_2, w_1 + w_2) \\ \alpha \cdot (v, w) &= (\alpha \cdot v, \alpha \cdot w) \end{aligned}$$

Back to the example, $\forall\, v = (x, y) \in V, v = x(1, 0) + y(0, 1)$, therefore

$$\psi(v) = \psi((x, y)) = x \cdot \psi((1, 0)) + y \cdot \psi((0, 1))$$

ψ is therefore uniquely defined by $\psi((1, 0))$ and $\psi((0, 1))$.

Example. $\mathbb{K} = \mathbb{R}, V = \mathbb{R}^m, W =$ any $\mathbb{R}$-vector space
Think about $W = \mathbb{R}^n$ with similar reasoning.
Hint: We want to show there exists a bijection between $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R}^n)$ and $\mathbb{R}^{m \cdot n}$, but this is often rewritten as $\mathbb{M}_{m \times n}(\mathbb{R})$

4.3 Isomorphism, Kernel, Image

Every linear transformation is just a map, and we can therefore question if it is injective, surjective or bijective. In all cases, these concepts simply deal with the sets (vector spaces) as simply sets.

Definition 4.4 (Isomorphism).

A $\mathbb{K}$-linear transformation $\psi : V \to W$ is an **isomorphism** if it is bijective.

Definition 4.5 (Kernel, Image).

Let $\psi : V \to W$ be a linear transformation over $\mathbb{K}$. Then:

1. **Kernel**: $\ker(\psi) \coloneqq \{v \in V \mid \psi(v) = 0\} \subseteq V$
2. **Image**: $\mathrm{im}(\psi) \coloneqq \{w \in W \mid \exists\, v \in V \text{ such that } \psi(v) = w\}$

Lemma 4.1.

1. $\ker(\psi)$ is a $\mathbb{K}$-vector subspace of V
2. $\mathrm{im}(\psi)$ is a $\mathbb{K}$-vector subspace of W

Proof (Lemma)**.** We want to show that if $x, y \in \ker(\psi)$ then $x + y \in \ker(\psi)$.

$$\begin{aligned} \psi(x+y) &= \psi(x) + \psi(y) (\text{ since } \psi \text{ is a linear transformation }) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Therefore $x + y \in \ker(\psi)$
Furthermore, $\forall\, \alpha \in \mathbb{K}, x \in \ker(\psi)$ then

$$\psi(\alpha, x) = \alpha \cdot \psi(x) = \alpha \cdot 0 = 0 \Rightarrow \alpha \cdot x \in \ker(\psi)$$

Therefore $\ker(\psi)$ is a subspace.
Similarly, $\mathrm{im}(\psi)$ is a subspace. □

Definition 4.6 (Finite Dimensional, Dimension).

1. Let V be a $\mathbb{K}$-vector space. V is called **finite dimensional** if there exists a surjective linear transformation $\mathbb{K}^r \to V$ where $r \in \mathbb{Z}_{\geq 0}$. As a consequence, $\mathbb{K}^r$ is also finite dimensional, with an identity mapping.

2. If V is finite dimensional then **dimension** of V is defined as

$$\dim V := \min\{k \in \mathbb{Z}_{\geq 0} \mid \exists \text{ linear transformation } \mathbb{K}^r \to V\}$$

Lecture 5

Span, Linear Independence, Basis

06 Apr 2023

Recall. Linear Combination: Let $V = \mathbb{K}$-vector space with $v_1, v_2, \ldots, v_r \in V$ then

$$\mathbb{K}\langle v_1, v_2, \ldots, v_r\rangle := \{w \in W \mid = w = a_1 v_1 + \cdots + a_r v_r; a_i \in \mathbb{K}\} \subseteq V(\text{ is a subspace of } V)$$

Definition 5.1 (Span).

$\{v_1, v_2, \ldots, v_r\}$ span V if

$$\mathbb{K}\langle v_1, v_2, \ldots, v_r\rangle = V$$

i.e. equality is achieved: every vector in V can be written as linear combinations of $\{v_1, v_2, \ldots, v_r\}$

Connecting to the previous lecture, let $\psi : \mathbb{K}^r \to V$ then $\psi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^r, V) \xrightarrow{\sim} V^{\oplus r}$, i.e. ψ corresponds to $(v_1, v_2, \ldots, v_r)$ in V.
In particular, $(v_1, v_2, \ldots, v_r) \in V^{\oplus r}$ determines the map:

$$\begin{aligned}
\psi : (1, 0, \ldots, 0) \in \mathbb{K}^r &\to v_1 \\
(0, 1, \ldots, 0) \in \mathbb{K}^r &\to v_2 \\
&\vdots\vdots \\
(0, 0, \ldots, 1) \in \mathbb{K}^r &\to v_r \\
(\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{K}^r &\to \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_r v_r
\end{aligned}$$

Lemma 5.1.

1. Let $\psi : \mathbb{K}^r \to V$ be a linear transformation determined by $v_1, v_2, \ldots, v_r \in V$, i.e. $\psi(\alpha_1, \alpha_2, \ldots, \alpha_r) := \sum_{i=1}^r \alpha_i v_i$, then

$$\mathrm{im}(\psi) = \mathbb{K}\langle v_1, v_2, \ldots, v_r\rangle$$

   is a subspace of V

2. $\{v_1, v_2, \ldots, v_r\}$ span $V \Leftrightarrow \psi$ is surjective

   i.e. a surjection $\mathbb{K}^r \to V$ corresponds to r vectors $v_1, v_2, \ldots, v_r \in V$ that span V

Remark. V is finite dimensional when $\exists$ surjection $\mathbb{K}^d \to V$