1 Week 2 Reading

1. (Page 21) How does the equality (given $S_0 = x$)

$$F(x) = [1 - \mathbb{P}^x(S_T = N)]F(0) + \mathbb{P}^x(S_T = N)F(N)$$

result in $\mathbb{P}^x(S_T = N) = \mathbb{P}(S_T = N \mid S_0 = x) = \frac{x}{N}$

(Added later: Derivation in continuous setting, page 73)

(NTS: Have to use the mean condition?)

2. (Page 27) (Clarification) The first expression, given the initial conditions, that

$$p_n(y) = \mathbb{P}(S_{n\wedge T_A} = y \mid S_0 = x)$$

From which expression is this equality derived from? I assume that this is not derived from the equation from the last section, where conditions are placed on the boundary ∂A and values are then projected into the interior A. In this case there are just initial conditions placed on points in A.

I want to write

$$p_n(y) = \sum_{z\in A} \mathbb{P}(S_{n\wedge T_A} = y \mid S_0 = z)p_0(z) = \mathbb{P}(S_{n\wedge T_A} = y \mid S_0 = x)p_0(x) = \mathbb{P}(S_{n\wedge T_A} = y \mid S_0 = x)$$

(the sum simplifies because $p_0(z) = 0 \ \forall \ z \neq x$), but that seems unrigorous to me, though it does make intuitive sense (The heat after n steps only came from within the system and therefore from the initial conditions).

In fact, I think the above equation comes readily from the following paragraph, about how $\{p_n(x) : x \in A\}$ is the vector $\mathbf{Q}^n f$.

Perhaps I'm having trouble because the only expression that I have right now is that

$$\begin{aligned}
\partial_n p_n(x) &= \mathcal{L} p_n(x) \\
\Leftrightarrow p_{n+1}(x) &= \mathbf{Q} p_n(x) \\
&= \mathbb{E}[p_n(S_1) \mid S_0 = x] \\
&= \sum_{|y-x|=1} p_n(y)\mathbb{P}(S_1 = y \mid S_0 = x) \\
&= \sum_{y\in A} p_n(y)\mathbb{P}(S_1 = y \mid S_0 = x) \\
&\text{(more generically this way, since the other probabilities = 0)}
\end{aligned}$$

But perhaps this can be done repeatedly to get back down to p_0, which would turn out to be

$$\begin{aligned}
&= \sum_{y\in A} p_0(y)\mathbb{P}(S_{n\wedge T_A} = y \mid S_0 = x) \\
&= \sum_{y\in A} p_0(y)\mathbb{P}(S_{n\wedge T_A} = x \mid S_0 = y) \text{ (can switch around...?!)}
\end{aligned}$$

3. (page 35) (Clarification, not sure if my understanding is correct)

Since if

$$f(x) = |x|^2 = x_1^2 + x_2^2 + \cdots + x_d^2$$

then $\mathcal{L}f(x) = 1$; for $n < T_A (\Rightarrow n \wedge T_A = n, (n+1) \wedge T_A = n+1)$ we have

$$\begin{aligned}
\mathbb{E}[f(S_{(n+1)\wedge T_A}) \mid S_0, S_1, \ldots, S_n] - f(S_{n\wedge T_A}) &= 1 \\
\Rightarrow \mathbb{E}[|S_{(n+1)\wedge T_A}|^2 \mid S_0, S_1, \ldots, S_n] &= |S_{n\wedge T_A}|^2 + 1 \\
\Rightarrow \mathbb{E}[|S_{(n+1)\wedge T_A}|^2 \mid S_0, S_1, \ldots, S_n] - (n+1)\wedge T_A &= |S_{n\wedge T_A}|^2 - n\wedge T_A \\
\Rightarrow \mathbb{E}(M_{n+1} \mid S_0, S_1, \ldots, S_n) &= M_n \quad \square
\end{aligned}$$

else if $n \geq T_A \Rightarrow n \wedge T_A = (n+1) \wedge T_A = T_A$ it's trivial that the above equality holds.

4. (Page 36) (Clarification, not sure if my understanding is correct)

   We want to show $\mathcal{L}f(y) = -1$:

$$\begin{aligned}
\mathcal{L}f(y) &= \mathbb{E}[f(S_1) \mid S_0 = y] - f(y) \\
&= \mathbb{E}[G_A(S_1, y) \mid S_0 = y] - G_A(y, y) \\
&= \mathbb{E}[G_A(S_1, y) \mid S_0 = y] - \mathbb{E}(V_y \mid S_0 = y) \\
&= \mathbb{E}[G_A(S_1, y) \mid S_0 = y] - (1 + \mathbb{E}[G_A(S_1, y) \mid S_0 = y]) \\
&= -1
\end{aligned}$$

5. (Page 36 - 38, Green's Function) I understand the construction and derivations but would like to get a more intuitive explanation of what the Green's Function represents.

6. (Page 40, 41) I understood the motivation and computation until the end of page 39, yet I find the following section in page 40 and 41 unmotivated. I guess one can say that they're made to lead up to Theorem 1.14 and 1.15, but I don't really understand the bigger picture.

   (skipped 1.5.1. Exterior Dirichlet Problem and 1.6. Exercises)

7. (Page 61) I don't fully understand the implications of Proposition 2.4.

1.1 Additional Questions

1. (Page 14) Note that this also implies that if the random walker starts at $x \neq 0$, then the probability that it will get to the origin is one.

   Answer: Can just take expected value of visits of y from x, and realize that the probability is one.

2. (Page 18, 25) "It is not hard to see that with probability one $T < \infty$ i.e. eventually the walker will reach 0 or N and then stop." "We have shown in the previous section that with probability one $T < \infty$". (Apparently, the statement works for both finite and infinite discrete state space with interior and boundary $A, \partial A \subset \mathbb{Z}^d$). Continuous state space also implies $T_U < \infty$ too, page 72. Lawler: If there's a positive probability, if keep trying then will get there.

   Answer: $\mathbb{E}[T] < \infty$

3. (Page 32) Parity issues: Shouldn't this have been accounted for in the transition matrix already?

   Answer: Apparently nonsense

4. (Page 66 - 69) **Theorem 2.8**

   Answer: Similar to the discrete analogue!

5. (Page 73) Show that construction of Brownian motion does satisfy: *continuity* and *MVP*

   Answer: Direct computation, which shouldn't be too hard.

6. (Page 71-75) Why do we take the limits as $r \to 0, R \to \infty$ to check point recurrence/neighborhood recurrence?