1. (a) (Hölder's inequality) Suppose that $n \in \mathbb{N}$, and let $a_k, b_k \in \mathbb{R}$, $1 \le k \le n$. Prove that if $1 < p < \infty$ and $1/p + 1/q = 1$ then

$$\sum_{k=1}^{n} |a_k b_k| \le \left[\sum_{k=1}^{n} |a_k|^p\right]^{1/p} \left[\sum_{k=1}^{n} |b_k|^q\right]^{1/q}.$$

Hint: Prove that if $A, B > 0$ and $t \in (0,1)$ then $A^t B^{1-t} \le tA + (1-t)B$ by showing the function

$$f(x) := tx + (1-t)B - x^t B^{1-t}, \quad x > 0,$$

has a minimum at $x = B$.

(b) (Minkowski's inequality) Suppose that $n \in \mathbb{N}$, and let $a_k, b_k \in \mathbb{R}$, $1 \le k \le n$. Prove that if $1 \le p < \infty$ then

$$\left[\sum_{k=1}^{n} |a_k + b_k|^p\right]^{1/p} \le \left[\sum_{k=1}^{n} |a_k|^p\right]^{1/p} + \left[\sum_{k=1}^{n} |b_k|^p\right]^{1/p}$$

Hint: By the triangle inequality

$$\sum_{k=1}^{n} |a_k + b_k|^p \le \sum_{k=1}^{n} |a_k||a_k + b_k|^{p-1} + \sum_{k=1}^{n} |b_k||a_k + b_k|^{p-1}.$$

Now apply Hölder's inequality.

2. Prove that if $1 \le p < \infty$, then ℓ^p is a Banach space (you must show it is a normed space and it is complete).

3. The set of all bounded sequences, ℓ^∞, can be identified with $\mathcal{C}_\infty(\mathbb{N})$, the set of all bounded continuous functions on the metric space $(\mathbb{N}, d_{disc})$ where d_{disc} is the discrete metric. Thus, ℓ^∞ is a Banach space. Prove that

$$c_0 := \left\{ \{a_k\}_k \in \ell^\infty \mid \lim_{k \to \infty} a_k = 0 \right\}$$

is a closed subspace of ℓ^∞ (and is thus, a Banach space).

4. Let $1 \le p \le \infty$ and

$$S := \{a = \{a_k\}_k \in \ell^p \mid \|a\|_p = 1\}.$$

(a) Prove that S is a closed subset of ℓ^p.

(b) Prove that S is not compact. Hint: Let $e_n := \{\delta_{kn}\}_k \in S$ where

$$\delta_{kn} := \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \ne n. \end{cases}$$

Show that $\{e_n\}_n$ does not have a convergent subsequence in S.

5. Let $1 \leq p < \infty$ and $1/p + 1/q = 1$.

   (a) Prove that if $a = \{a_k\}_k \in \ell^p$ and $b = \{b_k\}_k \in \ell^q$ then

   $$\sum_{k=1}^{\infty} |a_k b_k| \leq \|a\|_p \|b\|_q.$$

   (b) Let $b \in \ell^q$. Prove that $F_b : \ell^p \to \mathbb{C}$ defined via

   $$F_b(a) := \sum_{k=1}^{\infty} a_k b_k, \quad a \in \ell^p,$$

   is an element of $(\ell^p)'$, the dual space of ℓ^p, and $\|F_b\| = \|b\|_{\ell^q}$.

   (c) Prove that $F : \ell^q \mapsto (\ell^p)'$, $b \mapsto F_b$, is a bijective bounded linear operator.