Assignment **BON 1**

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This document contains my solutions to the Gradescope assignment named on the top of this page. Specifically, my solutions to the following problems are included:

- 01.104 (pages 2-3)
- 01.159 (page 4)

I did not forget

- to REFRESH my browser for the latest information about each problem
- to link problems to pages.
  This page is linked to the problems I did not solve.
- to update the items marked *** in the template (my name, email, the Gradescope title of the assignment, the list of problems solved, the `\lhead` statements (left page headers: list of (sub)problems solved on each page)
- to make sure no subproblem solution spills over to the next page (except when this is unavoidable, i.e., when the solution to a subproblem does not fit on a page)
- if a problem takes more than one page, I linked each of those pages to the problem
- I took care not to defeat the mechanisms provided by this template.

With each problem, **I stated my sources and collaborations**.
By submitting this solution *I certify* that
my statement of sources and collaborations is accurate and complete.
I understand that without this certification, my solutions will not be accepted.

(done) 01.104 Question.
Let p_n be the n-th prime number. Consider the statement:

$$p_n \sim n \ln n$$

Prove that this statement is equivalent to the Prime Number Theorem.

Sources and collaborations.
None

Proof.
Let us restate the Prime Number Theorem (PNT).

Theorem. Let $\pi(x)$ be the numbers of prime numbers $\leq x$. Then

$$\pi(x) \sim \frac{x}{\ln x}$$

$\boxed{\Rightarrow}$ We first show that $p_n \sim n \ln n$ implies PNT.

Let $0 < \varepsilon \ll 1$ be arbitrary. Take $\delta = \varepsilon/2$.

Since $p_n \sim n \ln n$, there exists some $N = N_\delta = N_\varepsilon \in \mathbb{N}$ such that $n \geq N \Rightarrow (1-\delta) n \ln n \leq p_n \leq (1+\delta) n \ln n$.

Take $X = p_N = X_\varepsilon$. For all $x \geq X$, let $m = \pi(x) \geq \pi(X) = N$ then we have

$$p_m \leq x \leq p_{m+1}$$

but $m \geq N$ so

$$(1-\delta) m \ln m \leq x \leq (1+\delta)(m+1)\ln(m+1)$$

ln is monotonic, so:

$$\ln(1-\delta) + \ln m + \ln\ln m \leq \ln x \leq \ln(1+\delta) + \ln(m+1)\ln\ln(m+1)$$

It then follows that for all $x \geq X$,

$$\begin{aligned} \frac{(1-\delta) m \ln m}{m[\ln(1+\delta) + \ln(m+1) + \ln\ln(m+1)]} &\leq \frac{x}{\pi(x)\ln x} \\ &\leq \frac{(1+\delta)(m+1)\ln(m+1)}{m[\ln(1-\delta) + \ln m + \ln\ln m]} \end{aligned}$$

Investigating the limit as $m \to \infty$, we have:

$$\begin{aligned} \lim_{m\to\infty} \text{LHS} &= \lim_{m\to\infty} (1-\delta)\frac{\ln m}{\ln(1+\delta) + \ln(m+1) + \ln\ln(m+1)} \\ &= 1 - \delta, \end{aligned}$$

using the ln asymptotic result in 01.87(b) to get $\ln(m+1) \sim \ln m$, and that $\ln m$ grows exponentially faster than $\ln\ln(m+1)$.

Similarly, $\lim_{m\to\infty} \mathrm{RHS} = 1+\delta$.

What the 2 limits above imply is that there exists some $M = M_\varepsilon$ such that $\pi(x) \geq M_\varepsilon \Rightarrow |\mathrm{LHS} - (1-\delta)|, |\mathrm{RHS} - (1+\delta)| < \frac{\varepsilon}{2}$, implying $|\mathrm{LHS} - 1|, |\mathrm{RHS} - 1| < \varepsilon$, so

$$\left|\frac{\pi(x)\ln x}{x} - 1\right| < \varepsilon$$

Therefore, for all $x \geq p_{\lceil M_\varepsilon \rceil}$, the above inequality is achieved.

ε was arbitrary, so $\lim_{x\to\infty} \frac{\pi(x)\ln x}{x} = 1$, so $\pi(x) \sim \frac{x}{\ln x}$ as required. □

$\boxed{\Leftarrow}$ WTS PNT implies $p_n \sim n \ln n$.

Let $0 < \varepsilon \ll 1$ be arbitrary. Since $\pi(x) \sim \frac{x}{\ln x}$, there exists some X_ε such that

$$x \geq X_\varepsilon \Rightarrow (1-\varepsilon)x \leq \pi(x)\ln x \leq (1+\varepsilon)x \tag{1}$$

Take $N = N_\varepsilon = \pi(X_\varepsilon) + 1$, then for all $n \geq N$, we have that $p_n \geq p_{N+1} \geq X_\varepsilon$ so (1) holds:

$$(1-\varepsilon)p_n \leq n \ln n \leq (1+\varepsilon)p_n$$

which implies

$$1 - \varepsilon \leq \frac{n \ln n}{p_n} \leq 1 + \varepsilon$$

ε was arbitrary, so $\lim_{n\to\infty} \frac{n\ln n}{p_n} = 1 \Rightarrow p_n \sim n \ln n$ as required. □

(done) 01.159 Question.
Let Ω be a set of n elements. Let $C_1, \dots, C_m$ be an Oddtown system in Ω and let $\mathbf{v}_1, \dots, \mathbf{v}_m$ be the corresponding incidence vectors. Prove that the $\mathbf{v}_i$ are linearly independent over $\mathbb{Q}$. (Note that the Oddtown Theorem follows from this)

Sources and collaborations.
None.

Proof.
WTS Oddtown incidence vectors $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathbb{Q}^n$ are linearly independent.

Suppose there exists $(p_i/q_i)_{i\in[m]} \in \mathbb{Q}$ such that

$$\sum_{i=1}^{m} \alpha_i \mathbf{v}_i = \mathbf{0}$$

Let $a_i = (p_i/q_i)\operatorname{lcm}(q_1, \dots, q_m) \in \mathbb{Z}$ then

$$\sum_{i=1}^{m} a_i \mathbf{v}_i = \mathbf{0} \tag{2}$$

WLOG, $\gcd(a_1, \dots, a_m) = 1$ (If not, divide them all by $\gcd(a_1, \dots, a_m)$).

Define the symmetric bilinear function $\langle \cdot, \cdot \rangle : \mathbb{Q}^n \times \mathbb{Q}^n \to \mathbb{Z}, \mathbf{u} \times \mathbf{v} \mapsto \sum_{i=1}^{m} \mathbf{u}^{(i)} \mathbf{v}^{(i)}$ where $\mathbf{u}^{(i)}$ denotes the i-th index of $\mathbf{u}$. Then the Oddtown conditions mean that $\langle \mathbf{v}_i, \mathbf{v}_j \rangle$ is even if $i \neq j$ and odd if $i = j$.

For any $i \in [m]$, from (2), we get $\langle \text{LHS}, \mathbf{v}_i \rangle = \langle \text{RHS}, \mathbf{v}_i \rangle$ which implies:

$$a_i \cdot (odd) + \sum (even) = 0$$

where $(odd), (even)$ denote some odd and even integers respectively. It then follows that a_i is even.

This holds for all $i \in [m]$, so $(a_i)_{i\in[m]}$ have 2 as a common divisor, $\Rightarrow\Leftarrow$ $\square$