MATH 20800: Honors Analysis in Rn II

Problem Set 3

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Textbook: Rudin, *Principles of Mathematical Analysis*

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Problem 3.1 (8.11 done)

Suppose f Riemann integrable on $[0, A]$ for all $A < \infty$, and $f(x) \xrightarrow{x\to\infty} 1$. Prove that

$$\lim_{t\to 0} t\int_0^\infty e^{-tx} f(x)\mathrm{d}x = 1 \quad (t > 0)$$

Solution

Fix $\varepsilon > 0$. Since $f(x) \xrightarrow{x\to\infty} 1$, there exists $N > 0$ such that $x \geq N \Rightarrow |f(x) - 1| < \varepsilon/2$.

f is then Riemann integrable on $[0, N]$, therefore exists $M = \sup_{[0,N]}|f(x)| < \infty$.

Denote $D_t(x) = te^{-tx}$ then $D_t(x) \geq 0$ and simple integration yields $\int_0^\infty D_t(x) = 1\ \forall\ t > 0$, and $\int_0^N D_t(x)\mathrm{d}x = 1 - e^{-Nt}$

We can now bound:

$$\begin{aligned}
\left|t\int_0^\infty e^{tx} f(x)\mathrm{d}x - 1\right| &= \left|\int_0^\infty D_t(x)f(x)\mathrm{d}x - 1\right| \\
&= \left|\int_0^\infty D_t(x)f(x)\mathrm{d}x - \int_0^\infty D_t(x)\mathrm{d}x\right| \\
&= \left|\int_0^\infty D_t(x)(f(x) - 1)\mathrm{d}x\right| \\
&= \int_0^N D_t(x)|f(x) - 1|\mathrm{d}x + \int_N^\infty D_t(x)|f(x) - 1|\mathrm{d}x \\
&\leq \int_0^N D_t(x)(M + 1)\mathrm{d}x + 1 \times \varepsilon/2 \\
&= (M + 1)(1 - e^{-Nt}) + \varepsilon/2 \xrightarrow{t\to 0} 0
\end{aligned}$$

since $e^{-Nt} \xrightarrow{t\to 0} 1$. □

Problem 3.2 (8.12 done)

Suppose $0 < \delta < \pi$, $f(x) = 1$ if $|x| < \delta$, $f(x) = 0$ if $\delta < |x| \leq \pi$, and f is 2π-periodic.

(a) Compute Fourier coefficients of f.

(b) Conclude that

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{n-\delta}{2}$$

(c) Deduce from Parseval's theorem that

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} = \frac{n-\delta}{2}$$

(d) Let $\delta > 0$ and prove that

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 \mathrm{d}x = \frac{\pi}{2}$$

(e) Put $\delta = \pi/2$ in (c), what do you get?

Solution

(a) For $n = 0$:

$$\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\mathrm{d}x = \frac{1}{2\pi}\int_{-\delta}^{\delta} 1 \times 1\mathrm{d}x = \delta/\pi$$

and for $n \neq 0$:

$$\begin{aligned}
\hat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\delta}^{\delta} 1e^{-inx}\mathrm{d}x \\
&= \frac{1}{2\pi}\left[\frac{e^{-inx}}{-in}\right]_{-\delta}^{\delta} \\
&= \frac{-1}{2\pi in}(e^{-in\delta} - e^{in\delta}) \\
&= \frac{\sin(n\delta)}{n\pi}
\end{aligned}$$

(b) f is clearly locally Lipschitz at 0, since it is locally constant (on $(-\delta, +\delta)$ with Lipschitz constant 1). It follows that

$$\lim_{N\to\infty} s_N(f, 0) = f(0)$$

We also realize that $\hat{f}(n) = \hat{f}(-n)$, since sin is odd. It then follows that:

$$
\begin{aligned}
1 &= f(0) \\
&= \sum_{-\infty}^{\infty} \hat{f}(n) e^{0in} \\
&= \frac{\delta}{\pi} + 2\sum_{n=1}^{\infty} \hat{f}(n) \\
\Rightarrow \frac{\pi - \delta}{2\pi} &= \sum_{n=1}^{\infty} \hat{f}(n) \\
&= \sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n\pi} \\
\Rightarrow \sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} &= \frac{\pi - \delta}{2}
\end{aligned}
$$

as required.

(c) f is discontinuous on a zero set ($\{-\delta, \delta\}$), hence it is Riemann integrable. Therefore we can apply Parseval's Theorem:

$$
\begin{aligned}
\sum_{-\infty}^{\infty} |\hat{f}(n)|^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \mathrm{d}x \\
\left(\frac{\delta}{\pi}\right)^2 + 2\sum_{n=1}^{\infty} \left(\frac{\sin(n\delta)}{n\pi}\right)^2 &= \frac{1}{2\pi} \int_{-\delta}^{\delta} 1 \mathrm{d}x = \frac{\delta}{\pi} \\
\Rightarrow \sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2} &= \frac{\pi\delta - \delta^2}{2} \\
\Rightarrow \sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} &= \frac{\pi - \delta}{2}
\end{aligned}
$$

as required.

(d) Letting $\delta \to 0$, then

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} &= \sum_{n=1}^{\infty} \left(\frac{\sin(n\delta)}{n\delta}\right)^2 \delta \\
&\xrightarrow{\delta \to 0} \int_0^{\infty} \left(\frac{\sin(x)}{x}\right)^2
\end{aligned}
$$

since each $\sum_{n=1}^{\infty} \left(\frac{\sin(n\delta)}{n\delta}\right)^2 \delta$ is the Riemann sum of the integral.

Then $RHS \xrightarrow{\delta \to 0} \pi/2$, hence

$$
\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 = \frac{\pi}{2}
$$

(e) Putting $\delta = \pi/2$ in **(c)**. We have that $\sin(n\frac{\pi}{2})^2 = 1$ when n odd, and $= 0$ when n even, so

$$\sum_{n \text{ odd}} \frac{1}{n^2\pi/2} = \frac{\pi}{4}$$

hence

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}$$

$\square$

Problem 3.3 (8.13 done)

Put $f(x) = x$ if $0 \le x < 2\pi$, and apply Parseval's theorem to conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Solution

$f(x) = x$, hence

$$\begin{aligned}
\hat{f}(n) &= \frac{1}{2\pi}\int_0^{2\pi} xe^{-inx}\mathrm{d}x \\
&= \frac{1}{2\pi}\left[\frac{(inx+1)e^{-inx}}{n^2}\right]_0^{2\pi} \\
&= \frac{1}{2\pi}\frac{2\pi i}{n} = \frac{i}{n}
\end{aligned}$$

with the exception of $n = 0$:

$$\hat{f}(0) = \frac{1}{2\pi}\int_0^{2\pi} x\mathrm{d}x = \pi$$

Also notice that $|\hat{f}(n)|^2 = |\hat{f}(-n)|^2 = \frac{1}{n^2}$, so

$$\begin{aligned}
\pi + 2\sum_{n=1}^{\infty}\frac{1}{n^2} &= \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{2\pi} x^2\mathrm{d}x \\
&= \frac{4\pi^2}{3} \\
\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} &= \frac{\pi^2}{6}
\end{aligned}$$

as required. $\square$

Problem 3.4 (8.14 done)

If $f(x) = (\pi - |x|)^2$ on $[-\pi, \pi]$, prove that

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos(nx)$$

and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Solution

$f(x) = (\pi - |x|)^2$ on $[-\pi, \pi]$. We have that

$$\begin{aligned} |f(x+t) - f(x)| &= ||x+t| - |x|||2\pi - |x+t| - |x|| \\ &\leq |t| 2\pi \end{aligned}$$

so f is locally Lipschitz at all points with $M = 2\pi$, since locally $x + t$ and x have the same sign.

Therefore, it follows that $s_N(f, x) \xrightarrow{N\to\infty} f(x)$ on $[-\pi, \pi]$.

We then calculate the Fourier coefficients:

$$\begin{aligned} \hat{f}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - |x|)^2 e^{-inx} \mathrm{d}x \\ &= \frac{1}{2\pi} \left(\int_{-\pi}^{0} (\pi + x)^2 e^{-inx} \mathrm{d}x + \int_{0}^{\pi} (\pi - x)^2 e^{-inx} \mathrm{d}x \right) \\ &= \frac{2}{n^2} - \frac{2\sin(\pi n)}{\pi n^3} = \frac{2}{n^2} \end{aligned}$$

with the exception of $n = 0$:

$$\hat{f}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - |x|)^2 \mathrm{d}x = \frac{\pi^2}{3}$$

Therefore

$$\begin{aligned} f(x) &= \sum_{-\infty}^{\infty} \hat{f}(n) e^{inx} \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left(\frac{2}{n^2} e^{inx} + \frac{2}{n^2} e^{-inx} \right) \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2} 2\cos(nx) \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos(nx) \end{aligned}$$

as required.

In particular, when $x = 0$:

$$\begin{aligned} f(0) &= (\pi - 0)^2 = \pi^2 \\ \Rightarrow \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} &= \pi^2 \\ \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} &= (\pi^2 - \pi^2/3)/4 = \frac{\pi^2}{6} \end{aligned}$$

as required.

f is continuous, so it is Riemann integrable. We can therefore apply Parseval's:

$$\begin{aligned} \sum_{-\infty}^{\infty} |\hat{f}(n)|^2 &= \frac{\pi^4}{9} + 2\sum_{n=1}^{\infty}\left(\frac{4}{n^4}\right) \\ \Rightarrow \frac{\pi^4}{9} + 2\sum_{n=1}^{\infty}\left(\frac{4}{n^4}\right) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\pi - |x|)^4 \mathrm{d}x \\ &= \frac{\pi^4}{5} \\ \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} &= (\pi^4/5 - \pi^4/9)/8 = \frac{\pi^4}{90} \end{aligned}$$

as required. □

Problem 3.5 (8.15 done)

With $D_n(x) = \sum_{k=-n}^{n} e^{inx}$, put

$$K_N(x) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(x)$$

Prove that

$$K_N(x) = \frac{1}{N+1}\frac{1 - \cos(N+1)x}{1 - \cos x}$$

and that

(a) $K_N \geq 0$

(b) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)\mathrm{d}x = 1$,

(c) $K_N(x) \leq \frac{1}{N+1}\frac{2}{1-\cos\delta}$ if $0 < \delta \leq |x| \leq \pi$

If s_N is the Nth partial sum of the Fourier series of f, consider the arithmetic means

$$\sigma_N = \frac{s_0 + \ldots + s_N}{N+1}$$

Prove that

$$\sigma_N(f; x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)\mathrm{d}t$$

and hence prove Fejer's theorem: "If f is continuous, 2π-periodic, then $\sigma_N(f;x) \to f(x)$ uniformly on $[-\pi,\pi]$."

Solution

We first rewrite

$$D_n(x) = \sum_{k=-n}^{n} e^{ikx} = \frac{e^{i(n+1)x} - e^{-inx}}{e^{ix} - 1}$$

therefore

$$\begin{aligned}
K_n(x) &= \frac{1}{N+1} \sum_{n=0}^{N} \frac{e^{i(n+1)x} - e^{-inx}}{e^{ix} - 1} \\
&= \frac{1}{N+1} \frac{1}{e^{ix} - 1} \left[\sum_{n=0}^{N} (e^{ix})^{(n+1)} - \sum_{n=0}^{N} (e^{-ix})^{n} \right] \\
&= \frac{1}{N+1} \frac{1}{e^{ix} - 1} \left(\frac{e^{i(N+2)x} - e^{ix}}{e^{ix} - 1} - \frac{e^{-i(N+1)x} - 1}{e^{-ix} - 1} \right) \\
&= \frac{1}{N+1} \frac{1}{e^{ix} - 1} \frac{1}{2 - 2\cos x} (e^{ix} - 1)(2 - e^{-i(N+1)x} - e^{i(N+1)x}) \\
&= \frac{1}{N+1} \frac{2 - 2\cos(N+1)x}{2 - 2\cos x} \\
&= \frac{1}{N+1} \frac{1 - \cos(N+1)x}{1 - \cos x}
\end{aligned}$$

as required.

(a) $\cos(N+1)x, \cos x \le 1 \Rightarrow K_n(x) \ge 0$.

(b) We know that $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x)\mathrm{d}x = 1$ hence

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x)\mathrm{d}x = \frac{1}{N+1} \sum_{n=0}^{N} \int_{-\pi}^{\pi} D_n(x)\mathrm{d}x = \frac{1}{N+1}(N+1) = 1$$

(c) If $0 < \delta \le |x| \le \pi$ then

$$K_N(x) = \frac{1}{N+1} \frac{1 - \cos(N+1)x}{1 - \cos x} \le \frac{1}{N+1} \frac{1 - (-1)}{1 - \cos\delta} = \frac{1}{N+1} \frac{2}{1 - \cos\delta}$$

We've therefore proven all properties of $K_N(x)$.

Now,

$$\begin{aligned}
\sigma_N(x) &= \frac{1}{N+1}\sum_{n=0}^{N} \\
&= \frac{1}{N+1}\sum_{n=0}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_n(t)\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\sum_{n=0}^{N} D_n(t)\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)\mathrm{d}t
\end{aligned}$$

We are now ready to prove Fejer's Theorem: Suppose that f is continuous with period 2π, then $\sigma_N(f,x) \xrightarrow{N\to\infty} f(x)$ uniformly on $[-\pi,\pi]$.

f is continuous on $[-\pi,\pi]$ and is 2π-periodic, hence there exists $M \geq \|f\|$.

For all $\varepsilon > 0$, since f is continuous on $[-\pi,\pi]$ and 2π-periodic, it is uniformly continuous, hence there exists $\delta > 0$ such that

$$|u-v| < \delta \Rightarrow |fu - fv| < \varepsilon/2$$

Then, we have that

$$\begin{aligned}
|\sigma_N(f;x) - f(x)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)\mathrm{d}t - f(x)\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)\mathrm{d}t\right| \\
&= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} K_N(t)(f(x-t)-f(x))\mathrm{d}t\right| \\
&\leq \frac{1}{2\pi}\left|\int_{\delta\leq|t|} K_N(t)(f(x-t)-f(x))\mathrm{d}t\right| \\
&+ \frac{1}{2\pi}\left|\int_{-\delta}^{\delta} K_N(t)(f(x-t)-f(x))\mathrm{d}t\right| \\
&\leq \frac{1}{2\pi}2M\int_{\delta\leq|t|} K_N(t)\mathrm{d}t + \frac{1}{2\pi}\int_{-\delta}^{\delta} K_N(t)\varepsilon/2\mathrm{d}t \\
&\leq \frac{1}{2\pi}2M\frac{1}{N+1}\frac{2}{1-\cos\delta} + \varepsilon/2
\end{aligned}$$

There then exists N_1 big enough such that $\frac{1}{2\pi}2M\frac{1}{N_1+1}\frac{2}{1-\cos\delta} < \varepsilon/2$, then for $N \geq N_1$, $\|\sigma_N - f\| < \varepsilon$, hence the convergence is uniform. □

Problem 3.6 (8.16 done)

Prove a pointwise version of Fejer's theorem: If f Riemann integrable and $f(x+), f(x-)$ exist for some x, then

$$\lim_{N\to\infty}\sigma_N(f;x) = \frac{1}{2}[f(x+)+f(x-)]$$

Solution

Note that $K_n(x) = \frac{1}{N+1}\frac{1-\cos(N+1)x}{1-\cos x}$ is even. Therefore $\int_{-\pi}^{0} K_N(t)\mathrm{d}t = \int_0^{\pi} K_N(t)\mathrm{d}t = \frac{1}{2}$.

Since there exists $f(x+), f(x-)$, there exists $\delta_1, \delta_2 > 0$ such that $0 < t < \delta_1 \Rightarrow |f(x+t) - f(x+)| < \varepsilon/2$ and $0 < t < \delta_2 \Rightarrow |f(x-t) - f(x-)| < \varepsilon/2$. Take $\delta = \min\{\delta_1, \delta_2\}$.

Then,

$$
\begin{aligned}
&\left|\sigma_N(f;x) - \frac{1}{2}(f(x+) + f(x-))\right| \\
&= \frac{1}{2\pi}\left|\int_{-\pi}^{0} f(x-t)K_N(t)\mathrm{d}t - f(x+)\int_{-\pi}^{0} K_N(t)\mathrm{d}t + \int_0^{\pi} f(x-t)K_N(t)\mathrm{d}t - f(x-)\int_0^{\pi} K_N(t)\mathrm{d}t\right| \\
&= \frac{1}{2\pi}\left|\int_{-\pi}^{0} (f(x-t) - f(x+))K_N(t)\mathrm{d}t + \int_0^{\pi} (f(x-t) - f(x-))K_N(t)\mathrm{d}t\right| \\
&\le \frac{1}{2\pi}\left|\int_{-\pi}^{-\delta} (f(x-t) - f(x+))K_N(t)\mathrm{d}t\right| + \frac{1}{2\pi}\left|\int_{-\delta}^{0} (f(x-t) - f(x+))K_N(t)\mathrm{d}t\right| \\
&+ \frac{1}{2\pi}\left|\int_0^{\delta} (f(x-t) - f(x+))K_N(t)\mathrm{d}t\right| + \frac{1}{2\pi}\left|\int_{\delta}^{\pi} (f(x-t) - f(x+))K_N(t)\mathrm{d}t\right| \\
&\le \frac{1}{2\pi}\left(2M\pi\frac{1}{N+1}\frac{2}{1-\cos\delta} + \pi\varepsilon/2 + \pi\varepsilon/2 + 2M\pi\frac{1}{N+1}\frac{2}{1-\cos\delta}\right) < \varepsilon
\end{aligned}
$$

for N sufficiently large. The pointwise convergence is thus proven. □

Problem 3.7 (8.17)

Assume f is bounded and monotonic on $[-\pi, \pi)$, with Fourier coefficients c_n.

(a) Use Exercise 6.17 to prove that $\{nc_n\}$ is bounded.

(b) Combine (a) with Exercise 3.14(e) to conclude that

$$\lim_{N\to\infty} s_N(f;x) = \frac{1}{2}[f(x+) + f(x-)]$$

for every x.

(c) Assume only that f Riemann integrable on $[-\pi, \pi]$ and that f is monotonic in some segment $(\alpha, \beta) \subset [-\pi, \pi]$. Prove that the conclusion of (b) holds for every $x \in (\alpha, \beta)$. (This is an application of the localization theorem.)

Solution

(a) Suppose $\|f\| < M$. Then we have

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\mathrm{d}x \\
&= \frac{1}{2\pi}\left(e^{-in\pi}/(-in) - e^{-in(-\pi)}/(-in) - \int_{-\pi}^{\pi} e^{-inx}/(-in)df\right) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}/(-in)df
\end{aligned}
$$

so

$$|nc_n| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} e^{-inx} df\right| = \frac{1}{2\pi}\left|f(\pi)e^{-in\pi} - f(-\pi)e^{in\pi}\right| \leq M/\pi$$

is therefore bounded.

(b) The pointwise version of Fejer's Theorem implies that for all x where the right and left limits exist,

$$\lim_{N\to\infty} \sigma_N(f;x) = \frac{1}{2}[f(x+) + f(x-)]$$

Using Exercise 3.14(a), since $|nc_n|$ is bounded, it follows that $\lim_{N\to\infty} s_N(f;x) = \lim_{N\to\infty} \sigma_N(f;x) = \frac{1}{2}[f(x+) + f(x-)]$

(c) Define g such that it agrees with f on (a,b): $g(x) = f(x)$ on (α, β), and $g(x) = f(\alpha)$ for $x \leq \alpha$ and $g(x) = f(\beta)$ for $x \geq \beta$. Then, g is clearly monotonic in $[-\pi, \pi]$, so

$$\lim_{N\to\infty} S_N(g;x) = \frac{1}{2}[g(x+) + g(x-)]$$

But on (α, β), $g(x+) = f(x+), g(x-) = f(x-)$ so on (α, β):

$$\lim_{N\to\infty} S_N(g;x) = \frac{1}{2}[f(x+) + f(x-)]$$

f and g agree on (α, β), so by the localization theorem, it implies that

$$\lim_{N\to\infty} S_N(f;x) - S_N(g;x) = 0$$

Therefore on $(\alpha, \beta), \lim_{N\to\infty} S_N(f;x) = \frac{1}{2}[f(x+) + f(x-)]$. □

Problem 3.8 (8.19 done)

Suppose that f is a continuous, 2π-periodic, real-valued function and some α such that α/π is irrational. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x + n\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\mathrm{d}t$$

Solution

We first prove that the proposition is true for e^{ikx} for any $k \in \mathbb{Z}$.

If $k = 0$ then

$$\mathrm{RHS} = 1$$
$$\mathrm{LHS} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} 1 = 1$$

else

$$
\begin{aligned}
\text{RHS} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx}\mathrm{d}x \\
&= \left[\frac{e^{ikx}}{ik}\right]_{-\pi}^{\pi} = 0
\end{aligned}
$$

and

$$
\begin{aligned}
\text{LHS} &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{ik(x+n\alpha)} \\
&= e^{ikx+ik\alpha}\lim_{N\to\infty}\frac{e^{ikN\alpha}-1}{N(e^{ik\alpha}-1)} = 0
\end{aligned}
$$

Note that this computation does not into problem for all $k \in \mathbb{Z}\backslash\{0\}$ (denominator being 0), since α/π is irrational so $e^{ik\alpha} \neq 1$ for all $k \in \mathbb{Z}\backslash\{0\}$.

Thus, indeed the proposition is true for e^{ikx} for all $k \in \mathbb{Z}$. A trigonometric polynomial is a linear combination of e^{ikx} terms, and the proposition is linear in f, so the proposition is true for all trigonometric polynomial too.

Since f is continuous and 2π-periodic, given $\varepsilon > 0$, by Stone-Weierstrass, there exists some trigonometric polynomial P such that

$$
\|f - P\|_\infty < \varepsilon/2
$$

Note that per our remarks above,

$$
\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} P(x+n\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(t)\mathrm{d}t
$$

It then follows that

$$
\begin{aligned}
&\left|\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x+n\alpha) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\mathrm{d}t\right| \leq \\
&\left|\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x+n\alpha) - \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} P(x+n\alpha)\right| + \left|\frac{1}{2\pi} - \int_{-\pi}^{\pi} P(t)\mathrm{d}t\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\mathrm{d}t\right| \\
&\qquad \leq \left(\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|f(x+n\alpha) - P(x+n\alpha)|\right) + \frac{1}{2\pi}\int_{-\pi}^{\pi}|P(t) - f(t)|\mathrm{d}t \\
&\qquad < \left(\lim_{N\to\infty}\sum_{n=1}^{N} N\varepsilon/2\right) + \frac{1}{2\pi}2\pi\varepsilon/2 \\
&\qquad = \varepsilon
\end{aligned}
$$

This is true for all ε, hence the proposition holds for any continuous, 2π-periodic f. $\square$

Problem 3.9 (8.21 done)

Let

$$L_n = \frac{1}{2\pi}|D_n(t)|\mathrm{d}t$$

Prove that there exists $C > 0$ such that

$$L_N > C \log n$$

or, more precisely, that the sequence

$$\left\{L_n - \frac{4}{\pi^2}\log n\right\}$$

is bounded.

Solution

We first show the lower bound:

$$\begin{aligned}
L_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|D_n(t)|\mathrm{d}t \\
&= \frac{1}{\pi}\int_0^{\pi}\left|\frac{\sin(n+\frac{1}{2})t}{\sin t/2}\right|\mathrm{d}t \\
&= \frac{2}{\pi}\int_0^{\pi/2}\frac{\left|\sin(n+\frac{1}{2})(2t)\right|}{\sin t}\mathrm{d}t \\
&> \frac{2}{\pi}\int_0^{\pi/2}\frac{\left|\sin(n+\frac{1}{2})(2t)\right|}{t}\mathrm{d}t \\
&= \frac{2}{\pi}\int_0^{(n+\frac{1}{2})\pi}\frac{|\sin u|}{u}\mathrm{d}u \\
&> \frac{2}{\pi}\int_0^{n\pi}\frac{|\sin u|}{u}\mathrm{d}u \\
&> \frac{2}{\pi}\sum_{k=0}^{n-1}\int_0^{\pi}\frac{\sin u}{(k+1)\pi}\mathrm{d}u \\
&= \frac{2}{\pi^2}\left(\sum_{k=1}^{n}\frac{1}{k}\right)[-\cos u]_0^{\pi} \\
&= \frac{4}{\pi^2}\left(\sum_{k=1}^{n}\frac{1}{k}\right) \geq \frac{4}{\pi^2}\log n
\end{aligned}$$

then the upper bound, where we first use a preliminary bound:

$$\begin{aligned}
\left|\frac{\sin(2n+1)t}{\sin t}\right| &= \left|\frac{\sin(2nt)\cos t + \cos(2nt)\sin t}{\sin t}\right| \\
&= \left|\frac{\sin(2nt)}{\tan t} + \cos(2nt)\right| \\
&\leq \left|\frac{\sin(2nt)}{\tan t}\right| + 1
\end{aligned}$$

so

$$
\begin{aligned}
L_n &= \frac{2}{\pi}\int_0^{\pi/2}\left|\frac{\sin(2n+1)t}{\sin t}\right|\mathrm{d}t \\
&\leq \frac{2}{\pi}\int_0^{\pi/2}\left|\frac{\sin(2nt)}{\tan t}\right| + 1\mathrm{d}t \\
&= 1 + \frac{2}{\pi}\int_0^{\pi/2}\left|\frac{\sin(2nt)}{\tan t}\right|\mathrm{d}t \\
&= 1 + \frac{2}{\pi}\int_0^{n\pi}\frac{|\sin u|}{u}\mathrm{d}u \\
&= 1 + \frac{2}{\pi}\int_0^{\pi}\sum_{k=0}^{n-1}\frac{\sin u}{u+k\pi}\mathrm{d}u \\
&= 1 + \frac{2}{\pi}\int_0^{\pi}\frac{\sin u}{u} + \frac{2}{\pi}\int_0^{\pi}\sum_{k=1}^{n-1}\frac{\sin u}{u+k\pi}\mathrm{d}u \\
&< C + \frac{2}{\pi}\int_0^{\pi}\sum_{k=1}^{n-1}\frac{\sin u}{u+k\pi}\mathrm{d}u \\
&= C + \frac{2}{\pi}\int_0^{\pi}\sum_{k=1}^{n-1}\frac{\sin u}{k\pi}\mathrm{d}u \\
&= C + \frac{2}{\pi}\left(\sum_{k=1}^{n-1}\frac{1}{k}\right)[-\cos u]_0^{\pi} \\
&< C + \frac{4}{\pi}(\log n + \gamma) = C + \frac{4}{\pi}\log n
\end{aligned}
$$

It follows that $\{L_n - \frac{4}{\pi^2}\log n\}$ is bounded as required. $\square$