Math 20250: Abstract Linear Algebra

Problem Set 2

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Textbook: Linear Algebra by Hoffman and Kunze (2nd Edition)

Problem 2.1 (Sec 2.2. Problem 3)
Is the vector $(3, -1, 0, -1)$ in the subspace of $\mathbb{R}^5$ spanned by the vectors $(2, -1, 3, 2), (-1, 1, 1, -3)$ and $(1, 1, 9, -5)$?

Solution

For $(3, -1, 0, -1)$ to be in the abovementioned subspace, it must be a linear combination of the vectors $(2, -1, 3, 2), (-1, 1, 1, -3)$ and $(1, 1, 9, -5)$ in $\mathbb{R}^5$, i.e.

$$\exists\, c_1, c_2, c_3 \in \mathbb{R} \text{ such that } (3, -1, 0, -1) = c_1(2, -1, 3, 2) + c_2(-1, 1, 1, -3) + c_3(1, 1, 9, -5)$$

It follows that

$$\begin{aligned} 2c_1 - c_2 + c_3 &= 3 \\ -c_1 + c_2 + c_3 &= -1 \\ 3c_1 + c_2 + 9c_3 &= 0 \\ 2c_1 - 3c_2 - 5c_3 &= -1 \end{aligned}$$

which is equivalent to:

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 1 \\ 3 & 1 & 9 \\ 2 & -3 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

which is then equivalent to the following augmented matrix:

$$\begin{bmatrix} 2 & -1 & 1 & 3 \\ -1 & 1 & 1 & -1 \\ 3 & 1 & 9 & 0 \\ 2 & -3 & -5 & -1 \end{bmatrix}$$

on which we can carry out simplifying row operations to reach the row-reduced echelon form:

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system therefore has no solution, as the third row of the augmented matrix fails $(0c_1 + 0c_2 + 0c_3 \neq 1)$
Therefore $(3, -1, 0, 1)$ is not in the subspace. □

Problem 2.2 (Sec 2.2. Problem 5)
Let $\mathbb{F}$ be a field and let $n \geq 2$ be a positive integer. Let V be the vector space of all $n \times n$

matrices over $\mathbb{F}$. Which of the following sets of matrices $A \in V$ are subspaces of V?

1. all invertible A
2. all non-invertible A
3. all A such that $AB = BA$, where B is some fixed matrix in V
4. all A such that $A^2 = A$

Solution 1. All invertible A

No. It is not true that $\forall$ invertible $A_1, A_2 \in V, (A_1 + A_2)$ is also invertible.

Counter example:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow A_3 = A_1 + A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which is not invertible, since it admits $X = \begin{bmatrix} a \\ -a \end{bmatrix}$ for arbitrary $a \in \mathbb{F}$ as a solution to $A_3X = 0$

2. All non-invertible A

No. It is not true that $\forall$ non-invertible $A_1, A_2 \in V, (A_1 + A_2)$ is also non-invertible.

Counter example:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow A_3 = A_1 + A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A_1 is non-invertible since it admits $X = \begin{bmatrix} 0 \\ a \end{bmatrix}$ for arbitrary $a \in \mathbb{F}$ as a solution to $A_1X = 0$

A_2 is non-invertible since it admits $X = \begin{bmatrix} a \\ 0 \end{bmatrix}$ for arbitrary $a \in \mathbb{F}$ as a solution to $A_2X = 0$

Meanwhile, $A_3 = I$ is trivially invertible.

3. All A such that $AB = BA$, B is fixed in V

Yes. We want to show that given A_1, A_2 such that $A_1B = BA_1, A_2B = BA_2$ then

$$(A_1 + A_2)B = B(A_1 + A_2)$$
$$(cA_1)B = B(cA_1) \; \forall \, c \in \mathbb{K}$$

First, we have:

$$\begin{aligned}
[(A_1 + A_2)B]_{ij} &= \sum_{k=1}^{n} (A_1 + A_2)_{ik} B_{kj} \\
&= \sum_{k=1}^{n} (A_{1,ik} B_{kj} + A_{2,ik} B_{kj}) \\
&= \sum_{k=1}^{n} (A_{1,ik} B_{kj}) + \sum_{k=1}^{n} (A_{2,ik} B_{kj})
\end{aligned}$$

However,

$$A_1 B = B A_1 \Rightarrow \sum_{k=1}^{n} (A_{1,ik} B_{kj}) = [A_1 B]_{ij} = [B A_1]_{ij} = \sum_{k=1}^{n} (B_{ik} A_{1,kj})$$

and similarly for A_2.

It follows that:

$$\begin{aligned}
[(A_1 + A_2)B]_{ij} &= \sum_{k=1}^{n} (A_{1,ik} B_{kj}) + \sum_{k=1}^{n} (A_{2,ik} B_{kj}) \\
&= \sum_{k=1}^{n} (B_{ik} A_{1,kj}) + \sum_{k=1}^{n} (B_{ik} A_{2,kj}) \\
&= \sum_{k=1}^{n} B_{ik} (A_{1,kj} + A_{2,kj}) \\
&= \sum_{k=1}^{n} B_{ik} (A_1 + A_2)_{kj} \\
&= [B(A_1 + A_2)]_{ij}
\end{aligned}$$

It follows that indeed $(A_1 + A_2)B = B(A_1 + A_2)$.

The other equality is trivially observed $(cA_1)B = B(cA_1)\ \forall\ c \in \mathbb{K}$, making this set a subspace of V

4. All A such that $A^2 = A$

   No. It is not true that $\forall\ A_1, A_2$ such that $A_1^2 = A_1, A_2^2 = A_2 \Rightarrow (A_1 + A_2)^2 = A_1 + A_2$

   Counterexample: $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

   It is trivially true that $A_1^2 = A^1, A_2^2 = A_2$, however

$$(A_1 + A_2)^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \neq (A_1 + A_2)$$

   Therefore this set is not a subspace of V.

$\square$

Problem 2.3 (Sec 2.3. Problem 1)

Prove that if two vectors are linearly dependent, one of them is a scalar multiple of the other.

Solution

Let v_1, v_2 be two vectors that are linearly dependent in field $\mathbb{K}$. Then there exists $a_1, a_2 \in \mathbb{K}$ not all 0 such that:

$$a_1 v_1 + a_2 v_2 = 0$$

WLOG, assume $a_1 \neq 0 \Rightarrow \; \exists\, a_1^{-1} : a_1^{-1} a_1 = 1$. It follows that

$$v_1 = a_1^{-1}(-a_2 v_2) = (-a_1^{-1} a_2) v_2$$

is a scalar multiple of v_2. □

Problem 2.4 (Sec 2.3. Problem 2)

Are the vectors

$$\begin{aligned}\alpha_1 &= (1, 1, 2, 4)\\ \alpha_2 &= (2, -1, -5, 2)\\ \alpha_3 &= (1, -1, 4, 0)\\ \alpha_4 &= (2, 1, 1, 6)\end{aligned}$$

linearly independent in $\mathbb{R}^4$?

Solution

No, because

$$\alpha_4 = (2, 1, 1, 6) = \frac{4}{3}(1, 1, 2, 4) + \frac{1}{3}(2, -1, -5, 2) = \frac{4}{3}\alpha_1 + \frac{1}{3}\alpha_2 + 0\alpha_3$$

□

Problem 2.5 (Sec 2.3. Problem 4)

Show that the vectors

$$\alpha_1 = (1, 0, -1), \alpha_2 = (1, 2, 1), \alpha_3 = (0, -3, 2)$$

form a basis for $\mathbb{R}^3$. Express each of the standard basis vectors as linear combinations of $\alpha_1, \alpha_2, \alpha_3$

Solution

We first express each of the standard basis vectors as linear combinations of $\alpha_1, \alpha_2, \alpha_3$:

$$\begin{aligned}(1, 0, 0) &= \frac{7}{10}(1, 0, -1) + \frac{3}{10}(1, 2, 1) + \frac{1}{5}(0, -3, 2)\\ (0, 1, 0) &= \frac{-1}{5}(1, 0, -1) + \frac{1}{5}(1, 2, 1) + \frac{-1}{5}(0, -3, 2)\\ (0, 0, 1) &= \frac{-3}{10}(1, 0, -1) + \frac{3}{10}(1, 2, 1) + \frac{1}{5}(0, -3, 2)\end{aligned}$$

The standard basis vectors span V, therefore each vector in V can be expressed as linear combinations of the standard basis vectors, which can then be expressed as linear combinations of $\alpha_1, \alpha_2, \alpha_3$ per the equalities above. Therefore $\alpha_1, \alpha_2, \alpha_3$ span V.
It must now be proven that they are linearly independent, which is equivalent to showing that the system of linear equations represented by the following augmented matrix has no non-trivial solutions:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 \\ -1 & 1 & 2 & 0 \end{bmatrix}$$

Indeed, its row-reduced echelon form is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, only allowing for a trivial solution of $(0,0,0)$. Thus, $\alpha_1, \alpha_2, \alpha_3$ are linearly independent.
Therefore they form a basis for V. □

Problem 2.6 ((Bonus) Sec 2.3. Problem 14)

Let V be the set of real numbers. Regard V as a vector space over the field of *rational* numbers, with the usual operations. Prove that this vector space is *not* finite-dimensional.

Solution

Suppose not; that $\dim_{\mathbb{Q}} \mathbb{R} = d \in \mathbb{N}$, meaning that there exists a basis $v_1, v_2, \ldots, v_d \in \mathbb{R}$ such that

$$\forall\, v \in \mathbb{R}\; \exists\, \alpha_1, \alpha_2, \ldots, \alpha_d \in \mathbb{Q} : v = \alpha_1 v_1 + \cdots + \alpha_d v_d$$

We first observe that for $v_1 \neq v_2 \in \mathbb{R}$, their corresponding $d-$tuple must be different, i.e. $(\alpha_{v_1,1}, \alpha_{v_2,1}, \ldots, \alpha_{v_1,d}) \neq (\alpha_{v_2,1}, \alpha_{v_2,2}, \ldots, \alpha_{v_2,d})$. This is trivial.
Therefore if we consider $\varphi : \mathbb{R} \to \mathbb{Q}^d, \varphi(v) = (\alpha_{v,1}, \alpha_{v_2}, \ldots, \alpha_{v,d})$ then φ is injective.
It follows that $|\mathbb{R}| \leq |\mathbb{Q}^d|$. However, $\mathbb{R}$ is uncountable and $\mathbb{Q}^d$ is countable. $\Rightarrow\Leftarrow$ □