Assignment **ORD 1**

Solutions by **Hung Le Tran** `conghunglt(at)u.e.`

This document contains my solutions to the Gradescope assignment named on the top of this page. Specifically, my solutions to the following problems are included:

I did not forget

- to REFRESH my browser for the latest information about each problem
- to link problems to pages.
  This page is linked to the problems I did not solve.
- to update the items marked *** in the template (my name, email, the Gradescope title of the assignment, the list of problems solved, the `\lhead` statements (left page headers: list of (sub)problems solved on each page)

- to make sure no subproblem solution spills over to the next page (except when this is unavoidable, i.e., when the solution to a subproblem does not fit on a page)
- if a problem takes more than one page, I linked each of those pages to the problem
- I took care not to defeat the mechanisms provided by this template.

With each problem, **I stated my sources and collaborations**.
By submitting this solution *I certify* that
my statement of sources and collaborations is accurate and complete.
I understand that without this certification, my solutions will not be accepted.

(done) 01.31 Question.
Give a simple sentence in plain English expressing the negation of the statement that “the sequence S is eventually nonzero.” Do not use the word “not” and avoid mathematical usage such as “there exist(s)”. You may use the word “zero” and variants of the word “infinity”.

Sources and collaborations.
None

Proof.
Sequence S has an infinite number of zeros. $\square$

(done) 01.35 Question.
Denote the Fibonacci numbers: $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n$. Let $m \in \mathbb{N}$. Prove that the sequence ($F_n \bmod m$) is periodic with period $\leq m^2 - 1$.
Example: The ($F_n \bmod 3$) sequence is periodic with period 8; the repeating part is $0, 1, 1, 2, 0, 2, 2, 1$.

Sources and collaborations.
None.

Proof. Fix $m \in \mathbb{N}$. Define a new sequence $G_n \coloneqq F_n \bmod m$. WTS (G_n) is periodic with period $\leq m^2 - 1$. It trivially follows that $0 \leq G_n \leq m - 1$.

Investigate the tuples $\{(G_i, G_{i+1}) : 0 \leq i \leq m^2 - 1\}$.

If there exists some i such that $G_i = G_{i+1} = 0$ then we are done, since the G_n sequence would be identically zero and thus have period 1.

If there doesn't, then there remains $m^2 - 1$ possible 2-tuples of numbers from 0 to $m - 1$, but there are m^2 such tuples, so by the Pigeonhole Principle, it follows that there exists some $0 \leq i < j \leq m^2 - 1$ such that $G_i = G_j, G_{i+1} = G_{j+1}$.

Let $d = j - i \leq m^2 - 1$. WTS G_n is periodic with period d, i.e., $\forall\, n \in \mathbb{N}_0, G_{n+d} = G_n$. We shall use the following Lemma to continue.

Lemma 1.
If $G_k = G_{k+d}, G_{k+1} = G_{k+1+d}$ then $G_{k-1} = G_{k-1+d}$ and $G_{k+2} = G_{k+2+d}$.

Proof of Lemma 1.
We have $G_{k-1} = F_{k-1} \bmod m = F_{k+1} \bmod m - F_k \bmod m = G_{k+1} - G_k = G_{k+1+d} - G_{k+d} = F_{k+1+d} \bmod m - F_{k+d} \bmod m = F_{k-1+d} \bmod m = G_{k-1+d}$ and $G_{k+2} = F_{k+2} \bmod m = F_k \bmod m + F_{k+1} \bmod m = G_k + G_{k+1} = G_{k+d} + G_{k+1+d} = G_{k+2+d}$ as required. □

Back to main proof.
Use Lemma 1 to perform mathematical induction from i downwards, so that $G_n = G_{n+d}$ for all $n \leq i$, then perform mathematical induction from i upwards, so that $G_n = G_{n+d}$ for all $n \geq i$. It then follows that $G_n = G_{n+d}$ for all $n \in \mathbb{N}_0$. Hence (G_n) has period $d \leq m^2 - 1$. □

(done) 01.45 Question.
Find the quotients of all Fibonacci-type geometric progressions. *Hint.* There are two such ratios; one of them is the golden ratio.

Sources and collaborations.
None.

Proof.
Let $(a_n = aq^n)_{n\in\mathbb{N}_0}$ be a geometric progression for some $a \in \mathbb{R}$. If it is Fibonacci-type then for all $n \in \mathbb{N}_0$ we have

$$a_{n+2} = a_n + a_{n+1}$$

This implies

$$aq^{n+2} = aq^n + aq^{n+1}$$

The trivial sequence of all zeros would admit any number as a quotient. Let us, then, only consider non-trivial sequences, hence with $a, q \neq 0$. Then the above equation implies

$$q^2 - q - 1 = 0$$

which has roots $\frac{1+\sqrt{5}}{2}$ (golden ratio) and $\frac{1-\sqrt{5}}{2}$.

In conclusion, the valid quotients are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$. $\square$

(done) 01.54 Question.
Find two bounded sequences, (a_n) and (b_n) such that $\limsup(a_n+b_n) < \limsup a_n + \limsup b_n$.

Sources and collaborations.
None.

Proof.
Consider $(a_n) = (1,0,1,0,\ldots)$ the $\{0,1\}$-alternating sequence that starts with 1, and $(b_n) = (0,1,0,1,\ldots)$ the $\{0,1\}$-alternating sequence that starts with 0. Then $\limsup a_n = \limsup b_n = 1$ and $\limsup a_n + b_n = 1$. $\square$

(done) 01.62 Question.
Prove that there exist real numbers a, b such that $\binom{n}{5} \sim a \cdot n^b$. Find a, b. Make your proof elegant.

Sources and collaborations.
None.

Proof.

$$\binom{n}{5} = \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}$$

is a degree 5 polynomial in n with leading term $\frac{n^5}{5!} = \frac{n^5}{120}$ so it is asymptotically equal to $a \cdot n^b$ where $a = \frac{1}{120}$ and $b = 5$. $\square$

(done) 01.68 Question.
Prove $\sqrt{n^2+1} - n \sim 1/(2n)$.

Sources and collaborations.
None.

Proof.
We have

$$\begin{aligned}\sqrt{n^2+1} - n &= \frac{n^2+1-n^2}{\sqrt{n^2+1}+n} \\ &= \frac{1}{\sqrt{n^2+1}+n}\end{aligned}$$

and

$$\lim_{n\to\infty} \frac{1/(\sqrt{n^2+1}+n)}{1/(2n)} = \lim_{n\to\infty} \frac{2n}{\sqrt{n^2+1}+n} = 1$$

so $\sqrt{n^2+1} - n \sim 1/(2n)$ as required. $\square$

(done) 01.71 Question.
If f is a function, differentiable at zero, $f(0) = 0$, and $f'(0) \neq 0$, then $f(1/n) \sim f'(0)/n$.

Sources and collaborations.
None.

Proof.
f is differentiable at 0 so we can expand:

$$f(x) = f(0) + f'(0)x + o(|x|) = f'(0)x + R(x)$$

where $\lim_{x \to 0} \frac{R(x)}{|x|} = 0$.

It then follows that

$$\lim_{n\to\infty} \frac{f(1/n)}{f'(0)/n} = \lim_{n\to\infty} \frac{f'(0)/n + R(1/n)}{f'(0)/n} = 1 + \lim_{n\to\infty} \frac{R(1/n)}{f'(0)/n} = 1$$

□

(done) 01.77 Question.
Prove that there exist numbers a, b, c such that $\binom{2n}{n} \sim an^b c^n$. Find a, b, c.

Sources and collaborations.
None

Proof.
Using Stirling's Formula, we have

$$\begin{aligned}\binom{2n}{n} &= \frac{(2n)!}{n!n!} \\ &\sim \sqrt{2\pi(2n)}\left(\frac{2n}{e}\right)^{2n}\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}\right)^{-2} \\ &= \frac{1}{\sqrt{n}}n^{-1/2}4^n\end{aligned}$$

so $a = \frac{1}{\sqrt{\pi}}, b = -1/2, c = 4$. $\square$

(done) 01.87(a) Question.
Assume $a_n, b_n > 1$. Consider the following statements:
(A) $a_n \sim b_n$
(B) $\ln a_n \sim \ln b_n$
Prove (A) does not imply (B).

Sources and collaborations.
None.

Proof.
Consider $(a_n = e^{2/n})$ and $(b_n = e^{1/n})$ then we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{e^{2/n}}{e^{1/n}} = \lim_{n\to\infty} e^{1/n} = 1$$

so $a_n \sim b_n$ but

$$\lim_{n\to\infty} \frac{\ln a_n}{\ln b_n} = \lim_{n\to\infty} \frac{2/n}{1/n} = 2$$

so $\ln a_n \not\sim \ln b_n$.

It then follows that (A) does not imply (B). $\square$

(done) 01.87(b) Question.
Prove (A) does imply (B) under the stronger assumption that $a_n \geq 1.01$.

Sources and collaborations.
None.

Proof.
We have $a_n \geq 1.01 \Rightarrow \ln a_n \geq \ln 1.01 =: C > 0$.

Let us investigate if the following limit exists:

$$\lim_{n\to\infty} \frac{\ln(b_n/a_n)}{\ln a_n}$$

Indeed,

$$\left|\frac{\ln(b_n/a_n)}{\ln(a_n)}\right| = \frac{|\ln(b_n/a_n)|}{\ln a_n} \leq \frac{|\ln(b_n/a_n)|}{C} \xrightarrow{n\to\infty} 0$$

since $\lim_{n\to\infty} \frac{b_n}{a_n} = 1$.

It then follows by Squeeze Theorem that

$$\lim_{n\to\infty} \frac{\ln(b_n/a_n)}{\ln a_n} = 0$$

so

$$\lim_{n\to\infty} \frac{\ln b_n}{\ln a_n} = \lim_{n\to\infty} \frac{\ln a_n + \ln(b_n/a_n)}{\ln a_n} = 1 + \lim_{n\to\infty} \frac{\ln(b_n/a_n)}{\ln a_n} = 1$$

hence $\ln b_n \sim \ln a_n$ as required. □

(done) 01.92 Question.
Prove that for all $n \geq 1$, we have $n! > \left(\frac{n}{e}\right)^n$. Use the power series expansion of e^x.

Sources and collaborations.
None.

Proof.
We have for $x \geq 1$,

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n > \frac{x^n}{n!}$$

In particular, this holds for $x = n$, which implies

$$e^n > \frac{n^n}{n!} \Rightarrow n! > \left(\frac{n}{e}\right)^n$$

$\square$

(done) 01.101 Question.
Prove $\ln(n!) \sim n \ln n$ without using Stirling's formula, using ASY 6.8 instead.

Sources and collaborations.
None.

Proof.
Using ASY 6.8, we have

$$n^n > n! > \left(\frac{n}{e}\right)^n$$

which implies

$$n \ln n > \ln n! > n(\ln n - 1)$$

which implies

$$1 + \frac{n}{\ln n!} > \frac{n \ln n}{\ln n!} > 1$$

For n large we have $1 + \frac{n}{\ln n!} = 1 + \frac{1}{\ln(n-1)!}$ so it converges to 1 as n tends to ∞. By Squeeze Theorem, it follows that $\lim_{n\to\infty} \frac{n \ln n}{\ln n!} = 1$ so $n \ln n \sim \ln n!$ as required. $\square$

(done) 01.151 Question.
Prove that the bound in the Eventown Theorem is tight, i.e., $m = 2^{\lfloor n/2 \rfloor}$ is achievable for all n.

Sources and collaborations.
None.

Proof.
Denote the members as $\Omega = \{0, \ldots, n-1\}$. Let us pair up members $(2i, 2i+1)$ for $0 \leq i \leq \lfloor n/2 \rfloor - 1$ so we have $\lfloor n/2 \rfloor$ pairs.

We claim that the system that consist of all subsets of the set of pairs (which contains $2^{\lfloor n/2 \rfloor}$ clubs) is an Eventown system.

Indeed, the intersection between each pair of clubs is even since they intersect at some whole number of pairs, which pack an even number of members.

Hence $m = 2^{\lfloor n/2 \rfloor}$ is achievable for all n. □

(done) 01.164 Question.
For every $n \in \mathbb{N}$ determine the minimum number of clubs in a maximal Oddtown system.

Sources and collaborations.
None.

Proof.
If n is odd, then the system that only has 1 club, namely the club that contains everyone, is a maximal system. Because if any other club is added, it would intersect with this club in its entirety, i.e., with an odd number of members, so not even.

If n is even, then the system that has 2 clubs, club 1 consisting of everyone except for person 1 and club 2 consisting only of person 1, is a maximal system. Because if a new club wants to be added, then if it contains person 1, its intersection with club 2 would be odd, hence invalid. It then has to exclude person 1 and is therefore contained in club 1 and we run into the same problem as above.

So the answer is 1 for odd n and 2 for even n. □