MATH 26200: Point-Set Topology

Problem Set 1

Hung Le Tran

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Textbook: Munkres, *Topology*

Problem 1.1 (13.1 done)

Let X be a topological space, let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

Solution

Take any $x \in A$, then by hypothesis there exists $x \in U_x \subset A$ open. WTS $A = \bigcup_{x \in A} U_x$.

On one hand, for all x, $U_x \subset A \Rightarrow \bigcup U_x \subset A$.

On the other hand, $\forall\, x \in A, x \in U_x \Rightarrow A \subset \bigcup_{x \in A} U_x$.

Therefore $A = \bigcup_{x \in A} U_x$, and is therefore open. $\square$

Problem 1.2 (13.4 done)

(a) If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X?

(b) Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$.

(c) If $X = \{a, b, c\}$, let

$$\mathcal{T}_1 = \{\varnothing, X, \{a\}, \{a, b\}\}, \mathcal{T}_2 = \{\varnothing, X, \{a\}, \{b, c\}\}$$

Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

Solution

(a) WTS $\mathcal{T} = \bigcap \mathcal{T}_\alpha$ is a topology on X.

1. $\forall\, \mathcal{T}_\alpha, \varnothing, X \in \mathcal{T}_\alpha \Rightarrow \varnothing, X \in \mathcal{T}$

2. Let $\{U_\beta\} \subset \mathcal{T}$. Then $\{U_\beta\} \subset \mathcal{T}_\alpha\ \forall\, \mathcal{T}_\alpha$, i.e., each U_β is open in all $\mathcal{T}_\alpha$. It follows that their union is also open in all $\mathcal{T}_\alpha$:

$$\bigcup U_\beta \in \mathcal{T}_\alpha \ \forall \ \mathcal{T}_\alpha \Rightarrow \bigcup U_\beta \in \bigcap \mathcal{T}_\alpha = \mathcal{T}$$

3. Let $U_1, U_2 \in \mathcal{T}$, which implies $U_1, U_2 \in \mathcal{T}_\alpha \ \forall \ \mathcal{T}_\alpha$. Thus

$$U_1 \cap U_2 \in \mathcal{T}_\alpha \ \forall \ \mathcal{T}_\alpha \Rightarrow U_1 \cap U_2 \in \bigcap \mathcal{T}_\alpha = \mathcal{T}$$

We have therefore checked all requirements of a topology.

$\bigcup \mathcal{T}_\alpha$ is not necessarily a topology on X. Counter example:

$$\begin{aligned} X &= \{1, 2, 3\} \\ \mathcal{T}_1 &= \{\varnothing, \{1\}, \{1, 2, 3\}\} \\ \mathcal{T}_2 &= \{\varnothing, \{2\}, \{1, 2, 3\}\} \\ \mathcal{T}_1 \cup \mathcal{T}_2 &= \{\varnothing, \{1\}, \{2\}, \{1, 2, 3\}\} \end{aligned}$$

is not a topology since $\{1, 2\} = \{1\} \cup \{2\} \notin \mathcal{T}_1 \cup \mathcal{T}_2$.

(b) Let $\mathcal{T}$ be any topology on X.

1. Unique smallest topology containing all $\mathcal{T}_\alpha$:

For $\mathcal{T}$ to contain all $\mathcal{T}_\alpha$, it must satisfy:

$$\bigcup \mathcal{T}_\alpha \subset \mathcal{T}$$

i.e., all open sets in all $\mathcal{T}_\alpha$ must be open in $\mathcal{T}$. Then we claim that the unique smallest topology that satisfies as such is the topology generated by the subbasis $\bigcup \mathcal{T}_\alpha$ ($X \in \mathcal{T}_\alpha \subset \bigcup \mathcal{T}_\alpha$ so this trivally is a subbasis).

To prove this claim, we prove the broader result that the smallest topology that has all elements of subbasis $\mathcal{S}$ as open sets is the topology generated by this subbasis $\mathcal{S}$, $\mathcal{T}_\mathcal{S}$.

Let $\mathcal{T}'$ be any topology such that $\mathcal{S} \subset \mathcal{T}'$. Then any open set in $\mathcal{T}_\mathcal{S}$ is an arbitrary union of finite intersections of sets in $\mathcal{S}$, which is also open in $\mathcal{T}'$ since $\mathcal{S} \subset \mathcal{T}'$. It follows that $\mathcal{T}_\mathcal{S} \subset \mathcal{T}'$, and $\mathcal{T}_\mathcal{S}$ is a topology, so $\mathcal{T}_\mathcal{S}$ is the smallest topology contains all sets of subbasis $\mathcal{S}$.

From this claim, our result follows naturally.

2. Unique largest topology contained in all $\mathcal{T}_\alpha$:

For $\mathcal{T}$ to be contained in all $\mathcal{T}_\alpha$, it must satisfy $\mathcal{T} \subset \bigcap \mathcal{T}_\alpha$. But since $\bigcap \mathcal{T}_\alpha$ is a topology on X itself, it must necessariy follow that $\mathcal{T} = \bigcap \mathcal{T}_\alpha$ is the unique largest topology that is contained in all $\mathcal{T}_\alpha$.

(c) The smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$ is the topology generated by the subbasis $\{\varnothing, X, \{a\}, \{a, b\}, \{b, c\}\}$:

$$\{\varnothing, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$$

The largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$ is

$$\mathcal{T}_1 \cap \mathcal{T}_2 = \{\varnothing, X, \{a\}\}$$

$\square$

Problem 1.3 (13.8 done)

(a) Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a, b) : a < b; a, b \in \mathbb{Q}\}$$

is a basis that generates the standard topology on $\mathbb{R}$.

(b) Show that the collection

$$\mathcal{C} = \{[a, b) : a < b; a, b \in \mathbb{Q}\}$$

is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

Solution

(a) Recall that the standard topology on $\mathbb{R}$ is the topology generated by the basis

$$\mathcal{B}_{standard} = \{(a, b) : a, b \in \mathbb{R}\}$$

To use Lemma 13.2, we must verify that all sets in $\mathcal{B}$ are open. Indeed they are, $\mathcal{B} \subset \mathcal{B}_{standard}$.

Thus, we now WTS for each open set $U \subset X$, and each $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.

Fix U and $x \in U$. Then there exists $B_{standard} = (p_1, p_2) \in \mathcal{B}_{standard}$ such that $x \in (p_1, p_2) \subset U$. Since the rationals are dense in $\mathbb{R}$, there exists $q_1, q_2 \in \mathbb{Q}$ such that $p_1 < q_1 < x < q_2 < p_2$, thus:

$$x \in (q_1, q_2) =: B \in \mathcal{B}, \quad B = (q_1, q_2) \subset (p_1, p_2) \subset U$$

Applying Lemma 13.2, it follows that $\mathcal{B}$ is also a basis for the standard topology.

(b) The proof that $\mathcal{C}$ is a basis on $\mathbb{R}$ is trivial.

In the lower limit topology on $\mathbb{R}$, $[\sqrt{2}, 2)$ is open. WTS $[\sqrt{2}, 2)$ is not open in the topology generated by $\mathcal{C}$.

Suppose that it is, pick $\sqrt{2} \in [\sqrt{2}, 2)$ then there must exist $C = [c, d) \in \mathcal{C}$ such that $\sqrt{2} \in C \subset [\sqrt{2}, 2)$.

This means $\sqrt{2} \in [c, d) \Rightarrow c \leq \sqrt{2} < d$, but $c \in \mathbb{Q} \Rightarrow c \neq \sqrt{2} \Rightarrow c < \sqrt{2} \Rightarrow c \notin [\sqrt{2}, 2) \Rightarrow [c, d) \not\subset [\sqrt{2}, 2), \Rightarrow\Leftarrow$.

Therefore $[\sqrt{2}, 2)$ is not open in the topology generated by $\mathcal{C}$, so $\mathcal{C}$ must generate a topology different from the lower limit topology. $\square$

Problem 1.4 (16.4 done)

A map $f : X \to Y$ is said to be an **open** map if for every open set $U \subset X$, the set $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

Solution

To show that image of every open set in $X \times Y$ is open in the target space, we only need to show that the image of every basis element is open.

Take basis element $U \times V$ of $X \times Y$, where $U \subset X, V \subset Y$ open by definition of product topology. Then

$$\pi_1(U \times V) = U \text{ is open in } X$$
$$\pi_2(U \times V) = V \text{ is open in } Y$$

Then, since every open set is some union of basis elements, the image of every open set is the union of image of basis elements, and is therefore also open. It follows that π_1, π_2 are open maps. □

Problem 1.5 (16.5 done)

Let X and X' denote a single set in the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively. Let Y and Y' denote a single set in the topologies $\mathcal{U}$ and $\mathcal{U}'$ respectively. Assume these sets are nonempty.

(a) Show that if $\mathcal{T}' \supset \mathcal{T}$ and $\mathcal{U}' \supset \mathcal{U}$, then the product topology on $X' \times Y'$ is finer than the product topology on $X \times Y$.

(b) Does the converse hold? Justify your answer.

Solution

(a) Let $\mathcal{B}, \mathcal{B}'$ be the basis that generates the topology on $X \times Y$ and $X' \times Y'$ respectively. Then for every basis element $U \times V$ of the topology on $X \times Y$, from definition, $U \in \mathcal{T} \subset \mathcal{T}', V \in \mathcal{U} \subset \mathcal{U}'$, which implies $U \times V$ is also a basis element of $X' \times Y'$. It follows that the product topology on $X' \times Y'$ is finer than the product topology on $X \times Y$.

(b) We state the converse: If the product topology on $X' \times Y'$ is finer than the product topology on $X \times Y$, then $\mathcal{T}' \supset \mathcal{T}$ and $\mathcal{U}' \supset \mathcal{U}$. We claim that this is true.

Pick any $U \in \mathcal{T}, V \in \mathcal{U}$. We want to show that $U \in \mathcal{T}', V \in \mathcal{U}'$.

Since $U \in \mathcal{T}, V \in \mathcal{U}$, $U \times V$ is clearly open in $X \times Y$: it is a basis element in the product topology. Since the product topology of $X' \times Y'$ is finer than that of $X \times Y$, it follows that $U \times V$ is also open in $X' \times Y'$. Then, to show that $U \in \mathcal{T}'$ and $V \in \mathcal{U}'$, take any $(x, y) \in U \times V$. Since $U \times V$ is open in $X' \times V'$, there exists basis element $A \times B$ of $X' \times Y'$ such that $(x, y) \in A \times B \subset U \times V \Rightarrow A \subset U, B \subset V$.

Since $A \times B$ is a basis element, it follows that $A \in \mathcal{T}', B \in \mathcal{U}'$. Therefore, for all $x \in U, y \in V$, there exists $A \subset X', B \subset Y'$ open such that $x \in A \subset U, y \in B \subset V$. This implies $U \in \mathcal{T}', V \in \mathcal{U}'$ as required. □

Problem 1.6 (16.6 done)

Show that the countable collection

$$\{(a, b) \times (c, d) : a < b \text{ and } c < d; a, b, c, d \in \mathbb{Q}\}$$

is a basis for $\mathbb{R}^2$.

Solution

From Problem 13.8 (problem 3 in this problem set), we've shown that

$$\mathcal{B} = \{(a, b) : a < b; a, b \in \mathbb{Q}\}$$

is a basis that generates the standard topology on $\mathbb{R}$. Using Theorem 15.1, it follows that

$$\{B_1 \times B_2 : B_1, B_2 \in \mathcal{B}\} = \{(a, b) \times (c, d) : a < b \text{ and } c < d; a, b, c, d \in \mathbb{Q}\}$$

forms a basis for $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. $\square$