MATH 20700
Honors Analysis in $\mathbb{R}^n$ I

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Disclaimer: This document will inevitably contain some mistakes, both simple typos and serious logical and mathematical errors. Take what you read with a grain of salt as it is made by an undergraduate student going through the learning process himself. If you do find any error, I would really appreciate it if you can let me know by email at conghungletran@gmail.com.

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Lecture 1

Construction of Reals

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1.1 Overview of the construction of reals in 3 easy steps

1. Use set theory (axiomatic) to construct $\mathbb{N}$ and $\mathbb{Z}$, with notions of $<, +, \cdot, |\cdot|$.
2. Construct $\mathbb{Q}$:
$$\mathbb{Q} = \{(p, q) : p \in \mathbb{Z}, q \in \mathbb{Z}_{\geq 0}\}/\sim$$
where $(p, q) \sim (r, s) \Leftrightarrow ps - qr = 0$.

   Subsequently define $<, +, \cdot, |\cdot|$ on $\mathbb{Q}$. Let us note that $\mathbb{Z}$ naturally embeds in $\mathbb{Q}$, with the correspondence $n \mapsto [(n, 1)]$.
3. Construct $\mathbb{R}$. There are 2 ways to perform this step:
   - Dedekind cuts. This is the natural, elegant way of doing it. It is a method adapted to extend the ordering notion $<$ to a bigger field ($\mathbb{R}$).
   - Cauchy sequences. This method is adapted to extend $|\cdot|$ to a bigger field. Overall, this is a more general method for other "completions".

   Both methods "complete" $\mathbb{Q}$, but in a priori different ways: Cuts make $<$ complete, and thus giving rise to the LUB property; while Cauchy sequences make $|\cdot|$ complete, and thus Cauchy sequences converge (in the field). They both produce the same isomorphic $\mathbb{R}$, and $\mathbb{Q}$ is dense in $\mathbb{R}$ in both constructions.

1.2 Dedekind cuts

The big idea of Dedekind cuts is to fill in the holes between the rationals.

Definition 1.1 (Dedekind cut)

A **Dedekind cut** is a pair $A \mid B$ with $A, B \subseteq \mathbb{Q}$ such that

1. $A \sqcup B = \mathbb{Q}$
2. $\forall\, x \in A, y \in B, x < y$
3. A has no greatest element in $\mathbb{Q}$

Example

1. $A = \{x : x < \frac{1}{2}\} \mid A^C$

   Then this cut is a rational cut, since B has a least element in $\mathbb{Q}$, namely $\frac{1}{2}$. Generalizing this, for all $z \in \mathbb{Q}$, there exists a cut:
$$z^* = \{x : x < z\} \mid rest$$
that corresponds to that rational.
2. $A = \{x : x^2 < 2\} \mid A^C$

   This is an irrational cut, since B has no least element in $\mathbb{Q}$.

Definition 1.2 (Reals from Dedekind cuts)

Let $\mathbb{R} = \mathbb{R}_{Ded}$ be the set of all Dedekind cuts.

Properties

1. $\mathbb{Q} \subset \mathbb{R}$ with the naturally embedding $z \mapsto z^*$ above.
2. $<, +, \cdot$ extend naturallly.

Note, that by "extending" we mean that the operation on $\mathbb{R}$ agrees with the notion on $\mathbb{Q}$.

3. We can also define $0, -x$, so the constructed $\mathbb{R}$ is indeed an ordered field.
4. Define $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$
5. Lastly, it is nontrivial that $\mathbb{R}$ has the LUB property, and that Cauchy sequences converge.
6. And that $\mathbb{Q}$ and $\mathbb{R}\backslash\mathbb{Q}$ are dense in $\mathbb{R}$.

Definition 1.3 (Cauchy sequences in $\mathbb{R}$)

$(a_n)_{n\geq 1} \subseteq \mathbb{R}$ is **Cauchy** if $\forall\, \varepsilon > 0 \in \mathbb{R}$, $\exists\, N \in \mathbb{N}$ such that $n, m \geq N \Rightarrow |a_n - a_m| < \varepsilon$.

1.3 Cauchy sequences

The big idea of Cauchy sequences is to "complete the voyages" in $\mathbb{Q}$, in which the $<$ is not used in the construction, and only $|\cdot|$.

Definition 1.4 (Cauchy sequences in $\mathbb{Q}$)

$(a_n)_{n\geq 1} \subseteq \mathbb{Q}$ is **Cauchy** if $\forall\, \varepsilon > 0 \in \mathbb{Q}$, $\exists\, N \in \mathbb{N}$ such that $n, m \geq N \Rightarrow |a_n - a_m| < \varepsilon$.

Definition 1.5 (Reals from Cauchy sequences)

$$\mathbb{R} = \mathbb{R}_{Cau} = \{\text{ Cauchy sequences } (a_n)\}/\sim$$

where $(a_n) \sim (b_n) \Leftrightarrow \forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $n \geq N \Rightarrow |a_n - b_n| < \varepsilon$.

In short, $\mathbb{R} = \{[(a_n)] : a_n \text{ Cauchy }\}$

Properties

We then check the operations:

1. $+, \cdot : [(a_n)] + [(b_n)] = [(a_n + b_n)]$
2. $|\cdot| : |[(a_n)]| = [(|a_n|)]$
3. $<$ takes work: $[(a_n)] < [(b_n)]$ if $a_n < b_n$ for infinitely many n.

Lecture 2

Metric Spaces

29 Sep 2023

2.1 Metric spaces

The goal of metric spaces is to generalize the notion of distance, which can just be a function that takes in 2 arguments and returns the distance between them, such that this distance satisfies certain reasonable properties.

Definition 2.1 (Metric spaces)

A **metric space** is a pair (M, d) where M is a set and $d : M \times M \to \mathbb{R}_{\geq 0}$ such that for all $x, x', y, z \in M$:

1. (Positive definite) $d(x, x') \geq 0$, equality holds iff $x = x'$.
2. (Symmetry) $d(x, x') = d(x', x)$.
3. (Triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$.

Example

1. $M = \mathbb{R}, d(x, y) = |x - y|$.
2. $M = \mathbb{R}^n, d(x, y) = \|x - y\|$ where $\|v\| = (v \cdot v)^{1/2}$. This d is the usual Euclidean distance.
3. (Induced metric) $X \subseteq (M, d)$, and define $d_X(x, x') = d_M(x, x')$. The metric on X is induced by the metric on M.
4. Using $M = \mathbb{R}^n$ (with appropriate choice of n in the examples below, the set M can really be anything): insert figure
5. (Discrete metric)

$$d(x, x') = \begin{cases} 1 & \text{if } x \neq x' \\ 0 & \text{otherwise} \end{cases}$$

2.2 Isometry and equivalence

When are (X, d_X) and (Y, d_Y) the same?

Definition 2.2 (Isometry)

$f : X \to Y$ is an **isometry** if f is bijective and

$$d_X(x, x') = d_Y(fx, fx')$$

We say that (X, d_X) and (Y, d_Y) are **isometric** if there exists such an isometry.

This is an equivalence relation!

Remark

Fix a metric space (X, d_X), the isometries $f : X \to X$ are (sometimes) interesting! They form a group! For exapmle, on the circle $S^1 \subset \mathbb{R}^2$, its isometries are rotations and line reflections.

Remark

Consider $\mathbb{Z} \subset \mathbb{R}$. Are $(\mathbb{Z}, d_{discrete})$ and $(\mathbb{Z}, d_{\mathbb{R}})$ isometric? Clearly no. Because if there exists $f : (\mathbb{Z}, d_{discrete}) \to (\mathbb{Z}, d_{\mathbb{R}})$ then $d_{discrete}(f^{-1}(0), f^{-1}(2)) = d_{\mathbb{R}}(0, 2) = 2, \Rightarrow\Leftarrow$

2.3 Convergence and limit points

An important point (pun intended) of consideration, perhaps the most as I recognized it so far, for metric spaces is *convergence*. This consideration takes place in many shapes and forms. Does a sequence converge in the metric space at all? If it does, and the points of the sequence are from a certain subset of the metric space, is this point of convergence in the subset? If a sequence doesn't converge, does then exist a convergent subsequence? And many, many more.

Definition 2.3 (Convergence)

A sequence $(x_n)_{n\geq 1}$ in (X,d) **converges** if $\exists\, x \in X$ such that

$$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow d(x_n, x) < \varepsilon$$

We write $x_n \xrightarrow{n\to\infty} x$.

Definition 2.4 (Limit point)

Given $Y \subseteq X$. Say $x \in X$ is a **limit point** of Y if there exists a sequence $(y_n) \subseteq Y$ such that $y_n \xrightarrow{n\to\infty} x$.

A word of caution: The limit point might not be in Y itself!

Example

The set of limit points of S^1 in $\mathbb{R}^2$ S^1 itself. The set of limit points of $(0,1)$ is $[0,1]$.

Definition 2.5 (Closed set)

$K \subseteq X$ is **closed** if it contains (and therefore equals to) all of its limit points.

Definition 2.6 (Open set)

$U \subseteq X$ is **open** if $\forall\, x \in U,\ \exists\, r > 0$ such that $\forall\, x' \in X, d(x,x') < r \Rightarrow x' \in U$.

In words, it is open if we can draw a positive-radius open ball around every point of the set, so that this ball is wholly contained in the set U.

Notation

In (X,d), $x \in X, r > 0$, denote:

$$B_X(x,r) = \{x' : d(x,x') < r\}$$

Then as mentioned, U open if $\forall\, x \in U,\ \exists\, r > 0$ such that $B_X(x,r) \subseteq U$.

Here comes the first non-trivial statement:

Proposition 2.7

fdfsd

Proof

fdsfds $\square$

Proof (Name)

fdsfds $\square$