9/29/23

Last time: constructed $\mathbb{R}$ with cuts & sequences

This time: **Metric Spaces**

**Def** A metric space is $(M, d)$,
$d: M \times M \longrightarrow \mathbb{R}_{\geq 0}$
satisfying: $\forall x, y, z \in M$:

- $d(x,y) \geq 0$ & $d(x,y) = 0 \Leftrightarrow x = y$ (positive definite)
- $d(x,y) = d(y,x)$ (symmetry)
- $d(x,z) \leq d(x,y) + d(y,z)$. ($\Delta \neq$)

$d$ = "distance function"

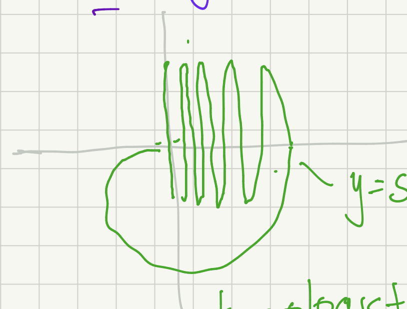
**Examples**

$(\mathbb{Q}, d), (\mathbb{R}, d)$ $\quad d(x,y) = |x-y|$

$(\mathbb{R}^k, d)$ $\quad d(x,y) = \langle x, y \rangle^{1/2}$

- $M \subseteq \mathbb{R}^n$ $\quad d$ = induced distance

$d(x,y) = \|x-y\|$



$y = \sin \frac{1}{x}$

topologist's sine circle

trefoil

- discrete metric:

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}.$$

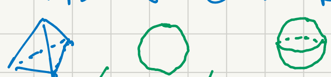
$$\underbrace{x \overset{1}{\quad} y \overset{1}{\quad} z}_{1}$$

cases for $\Delta \neq$

# Isomorphism/equivalence

**Def** An *isometry* is a bijection
$$f:(X,d_X)\longrightarrow(Y,d_Y)$$
satisfying: $\forall x,x'\in X$
$$d(f(x),f(x'))=d(x,x')$$

**Rmk:** Even isometries from $(X,d_X)$ to itself can be interesting

 etc

**Question** Consider $(\mathbb{Z},d_{\mathbb{R}})$ & $(\mathbb{Z},d_{dis})$. Are they isometric?

# Convergence & limit points

**Def** A sequence of points $(p_n)_{n\geq 1}$ in $M$ *converges* if $\exists\, p\in M$, $\forall \varepsilon>0$ $\exists N\geq 1$ $n\geq N \Rightarrow d(p_n,p)<\varepsilon$

write $p_n\to p$

**Def** Let $X\subseteq M$. Then $x\in M$ is *limit point of X* if $\exists (x_n)_{n\geq 1}$, $x_n\in X$ s.t. $x_n\to x$.

limit points of $[\longrightarrow)$ ?

# Open & closed

**Def** $K\subseteq M$ is *closed* if it contains all of its limit pts.

**Def** $O\subseteq M$ is *open* if $\forall x\in O$ $\exists r>0$ s.t. $d(x,y)<r \Rightarrow y\in O$.

**Prop** $\mathcal{O} \subseteq M$ is open iff $\mathcal{O}^c = M \setminus \mathcal{O}$ is closed.

**Pf**: ($\Rightarrow$) s'pose $\mathcal{O}$ open. wts $\mathcal{O}^c$ closed. By contradiction suppose $\exists\, p_n \in \mathcal{O}^c$ $p_n \to p$ $p \in \mathcal{O}^c$

Then $p \in \mathcal{O}$. $\Rightarrow \exists\, \varepsilon > 0$ s.t. $d(p,q) < \varepsilon \Rightarrow q \in \mathcal{O}$ ☆

But $p_n \to p \Rightarrow \exists N$ s.t. $d(p_N, p) < \varepsilon$

☆ $\Rightarrow p_N \in \mathcal{O}$, a contradiction.

($\Leftarrow$) s'pose $\mathcal{O}^c$ closed & suppose $\mathcal{O}$ is not open.

$\Rightarrow \exists\, p \in \mathcal{O}$ s.t. $\forall \varepsilon > 0$ $\exists\, q \in \mathcal{O}^c$ s.t. $d(p,q) < \varepsilon$.

Let $\varepsilon = 1/n$, obtain $\underset{\in \mathcal{O}^c}{p_n} \to \underset{\in \mathcal{O}}{p}$, a contradiction ($\mathcal{O}^c$ closed) ☒

## Continuity

$(X, d_X)$ & $(Y, d_Y)$ metric spaces.

3 definitions of $f: X \to Y$ cts.

(1) $\forall (x_n)$ in $X$, $x_n \to x$ $\Rightarrow f(x_n) \to f(x)$

(2) $\forall \varepsilon$, $\exists \delta$ s.t. $\forall x, x' \in X$ $d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$.

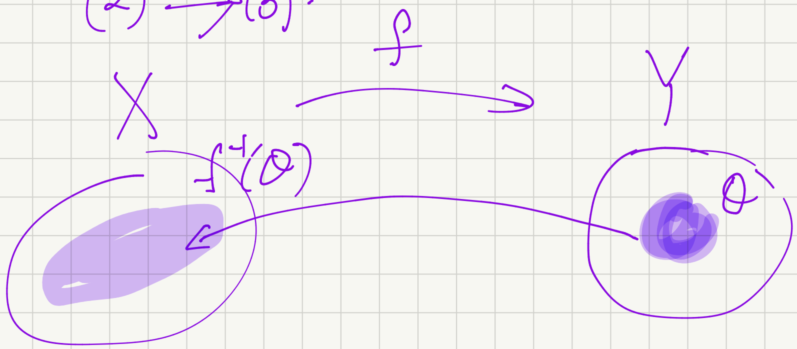
(3) $\forall$ open set $\mathcal{O} \subseteq Y$, $f^{-1}(\mathcal{O}) = \{x \in X : f(x) \in \mathcal{O}\}$ is open.

**Proposition**: All 3 conditions are equivalent

**Pf** Thm 11, p. 72

$(1) \Leftrightarrow (2)$ is thm 4, p. 65

$(2) \Rightarrow (3)$:




S'pose $\mathcal{O}$ open let $x \in f^{-1}(\mathcal{O})$
then $f(x) \in \mathcal{O}$. $\Rightarrow \exists \varepsilon > 0$ s.t.
$N_\varepsilon(f(x)) \subseteq \mathcal{O}$. Continuity $\Rightarrow$
$\exists \delta$ s.t. $f(N_\delta(x)) \subseteq \mathcal{O}$
$\Rightarrow X_\delta(x) \subseteq f^{-1}(\mathcal{O}) \Rightarrow f^{-1}(\mathcal{O})$ open
The rest is LTR $\boxtimes$

**Examples**

- $id_X$, constant maps are cts
- Lipschitz maps.

Suppose $f: X \to Y$ & $\exists L > 0$ s.t. $d_Y(f(x), f(x')) \le L \cdot d(x, x')$.

**Lemma** f is continuous.