Math 20250: Abstract Linear Algebra

Problem Set 1

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Textbook: Linear Algebra by Hoffman and Kunze (2nd Edition)

Problem 1.1 (Sec 1.3. Problem 3)**.** If

$$A = \begin{bmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$$

find all solutions of $AX = 2X$ and all solutions of $AX = 3X$. (The symbol cX denotes the matrix each entry of which is c times the corresponding entry of X)

Solution. We first solve $AX = 2X$, which is equivalent to $A_1X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ where

$$A_1 = \begin{bmatrix} 6-2 & -4 & 0 \\ 4 & -2-2 & 0 \\ -1 & 0 & 3-2 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 0 \\ 4 & -4 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Row-reducing:

$$\begin{bmatrix} 4 & -4 & 0 \\ 4 & -4 & 0 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{(1)=\frac{1}{4}(1),(2)=(3)} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{(2)=(2)+(1)} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{(2)=-(2)} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Therefore $x_3 = a, x_2 = x_3 = a, x_1 = x_2 = a$ for any $a \in \mathbb{C}$.
Solving $AX = 3X$ is equivalent to

$$\begin{bmatrix} 6-3 & -4 & 0 \\ 4 & -2-3 & 0 \\ -1 & 0 & 3-3 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -4 & 0 \\ 4 & -5 & 0 \\ -1 & 0 & 0 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Row-reducing:

$$\begin{bmatrix} 3 & -4 & 0 \\ 4 & -5 & 0 \\ -1 & 0 & 0 \end{bmatrix} \xrightarrow{(1)\leftrightarrow(3)} \begin{bmatrix} -1 & 0 & 0 \\ 3 & -4 & 0 \\ 4 & -5 & 0 \end{bmatrix} \xrightarrow{(1)=-(1),(2)=(2)+3(1),(3)=(3)+4(1)}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & -5 & 0 \end{bmatrix} \xrightarrow{(2)=\frac{-1}{4}(2),(3)=\frac{-1}{5}(3)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Therefore $x_3 = a, x_2 = 0, x_1 = 0$ for any $a \in \mathbb{C}$. □

Problem 1.2 (Sec 1.5. Problem 1)**.** Let

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, C = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Compute ABC and CAB

Solution.

$$AB = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2\cdot 3 + (-1)\cdot 1 + 1\cdot(-1) \\ 1\cdot 3 + 2\cdot 1 + 1\cdot(-1) \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$\Rightarrow ABC = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} 4\cdot 1 & 4\cdot(-1) \\ 4\cdot 1 & 4\cdot(-1) \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 2 + (-1)\cdot 1 & 1\cdot(-1) + (-1)\cdot 2 & 1\cdot 1 + (-1)\cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \end{bmatrix}$$

$$\Rightarrow CAB = \begin{bmatrix} 1 & -3 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1\cdot 3 + (-3)\cdot 1 + 0\cdot(-1) \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

□

Problem 1.3 (Sec 1.5. Problem 2). Let

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix}$$

Verify directly that $A(AB) = A^2B$

Solution.

$$AB = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 2 + (-1)\cdot 1 + 1\cdot 4 & 1\cdot(-2) + (-1)\cdot 3 + 1\cdot 4 \\ 2\cdot 2 + 0\cdot 1 + 1\cdot 4 & 2\cdot(-2) + 0\cdot 3 + 1\cdot 4 \\ 3\cdot 2 + 0\cdot 1 + 1\cdot 4 & 3\cdot(-2) + 0\cdot 3 + 1\cdot 4 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 8 & 0 \\ 10 & -2 \end{bmatrix}$$

$$\Rightarrow A(AB) = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 8 & 0 \\ 10 & -2 \end{bmatrix} = \begin{bmatrix} 1\cdot 5 + (-1)\cdot 8 + 1\cdot 10 & 1\cdot(-1) + (-1)\cdot 0 + 1\cdot(-2) \\ 2\cdot 5 + 0\cdot 8 + 1\cdot 10 & 2\cdot(-1) + 0\cdot 0 + 1\cdot(-2) \\ 3\cdot 5 + 0\cdot 8 + 1\cdot 10 & 3\cdot(-1) + 0\cdot 0 + 1\cdot(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{bmatrix}$$

Meanwhile,

$$A^2 = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 1 + (-1)\cdot 2 + 1\cdot 3 & 1\cdot(-1) + (-1)\cdot 0 + 1\cdot 0 & 1\cdot 1 + (-1)\cdot 1 + 1\cdot 1 \\ 2\cdot 1 + 0\cdot 2 + 1\cdot 3 & 2\cdot(-1) + 0\cdot 0 + 1\cdot 0 & 2\cdot 1 + 0\cdot 1 + 1\cdot 1 \\ 3\cdot 1 + 0\cdot 2 + 1\cdot 3 & 3\cdot(-1) + 0\cdot 0 + 1\cdot 0 & 3\cdot 1 + 0\cdot 1 + 1\cdot 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 5 & -2 & 3 \\ 6 & -3 & 4 \end{bmatrix}$$

$$\Rightarrow A^2B = \begin{bmatrix} 2 & -1 & 1 \\ 5 & -2 & 3 \\ 6 & -3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cdot 2 + (-1)\cdot 1 + 1\cdot 4 & 2\cdot(-2) + (-1)\cdot 3 + 1\cdot 4 \\ 5\cdot 2 + (-2)\cdot 1 + 3\cdot 4 & 5\cdot(-2) + (-2)\cdot 3 + 3\cdot 4 \\ 6\cdot 2 + (-3)\cdot 1 + 4\cdot 4 & 6\cdot(-2) + (-3)\cdot 3 + 4\cdot 4 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{bmatrix}$$

Therefore, $A(AB) = A^2(B) = \begin{bmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{bmatrix}$ □

Problem 1.4 (Sec 2.1. Problem 3)**.** If $\mathbb{C}$ is the field of complex numbers, which vectors in $\mathbb{C}^3$ are linear combinations of $(1, 0, -1), (0, 1, 1)$ and $(1, 1, 1)$?

Solution. $\mathbb{C}^3 \doteq \{(x_1, x_2, x_3) \mid x_i \in \mathbb{C}\}$. We want to prove that any vector in $\mathbb{C}^3$ is a linear combination of the vectors above.
$\forall\, v = (x_1, x_2, x_3) \in \mathbb{C}^3$,

$$v = (x_2 - x_3)(1, 0, -1) + (2x_2 - x_1 - x_3)(0, 1, 1) + (x_1 + x_3 - x_2)(1, 1, 1)$$

is therefore a linear combination of $(1, 0, -1), (0, 1, 1)$ and $(1, 1, 1)$. □

Problem 1.5 (Sec 2.1. Problem 7)**.** Let V be the set of pairs (x, y) of real numbers and let $\mathbb{F}$ be the field of real numbers. Define

$$(x, y) + (x_1, y_1) = (x + x_1, 0)$$
$$c(x, y) = (cx, 0)$$

Is V, with these operations, a vector space?

Solution. No. Counterexample against the rule $1 \cdot \alpha = \alpha\ \forall\, \alpha \in V$:

$$1 \cdot (1, 1) = (1, 0) \neq (1, 1)$$

Therefore V is not a vector space. □