

Math 20250: Abstract Linear Algebra
Problem Set 1

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02 Apr 2023

Problem 1.1 (Sec 1.3. Problem 3). If

$$A = \begin{bmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$$

find all solutions of $AX = 2X$ and all solutions of $AX = 3X$. (The symbol cX denotes the matrix each entry of which is c times the corresponding entry of X)

Solution. We first solve $AX = 2X$.

$$AX + (-2)X = 0$$

which is equivalent to $A_1X = 0$ where

$$A_1 = \begin{bmatrix} 6-2 & -4-2 & 0-2 \\ 4-2 & -2-2 & 0-2 \\ -1-2 & 0-2 & 3-2 \end{bmatrix} = \begin{bmatrix} 4 & -6 & -2 \\ 2 & -4 & -2 \\ -3 & -2 & 1 \end{bmatrix}$$

Row-reducing:

$$\begin{aligned} \begin{bmatrix} 4 & -6 & -2 \\ 2 & -4 & -2 \\ -3 & -2 & 1 \end{bmatrix} &\xrightarrow{(1)=\frac{1}{4}(1)} \begin{bmatrix} 1 & \frac{-3}{2} & \frac{-1}{2} \\ 2 & -4 & -2 \\ -3 & -2 & 1 \end{bmatrix} \xrightarrow{(2)=(2)-2(1), (3)=(3)+3(1)} \begin{bmatrix} 1 & \frac{-3}{2} & \frac{-1}{2} \\ 0 & -1 & -1 \\ 0 & \frac{-13}{2} & \frac{-1}{2} \end{bmatrix} \\ &\xrightarrow{(1)=(1)-\frac{3}{2}(2), (2)=(-1)(2), (3)=(3)-\frac{13}{2}(2)} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 6 \end{bmatrix} \\ &\xrightarrow{(1)=(1)-\frac{1}{6}(3), (2)=(2)-\frac{1}{6}(3), (3)=\frac{1}{6}(3)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Therefore $x_1 = x_2 = x_3 = 0$

Solving $AX = 3X$ is equivalent to

$$\begin{bmatrix} 6-3 & -4-3 & 0-3 \\ 4-3 & -2-3 & 0-3 \\ -1-3 & 0-3 & 3-3 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -7 & -3 \\ 1 & -5 & -3 \\ -4 & -3 & 0 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Row-reducing:

$$\begin{aligned} \begin{bmatrix} 3 & -7 & -3 \\ 1 & -5 & -3 \\ -4 & -3 & 0 \end{bmatrix} &\xrightarrow{(1) \leftrightarrow (2)} \begin{bmatrix} 1 & -5 & -3 \\ 3 & -7 & -3 \\ -4 & 3 & 0 \end{bmatrix} \xrightarrow{(2)=(2)-3(1), (3)=(3)+4(1)} \\ \begin{bmatrix} 1 & -5 & -3 \\ 0 & 8 & 6 \\ 0 & -17 & -12 \end{bmatrix} &\xrightarrow{(2)=\frac{1}{8}(2)} \begin{bmatrix} 1 & -5 & -3 \\ 0 & 1 & \frac{3}{4} \\ 0 & -17 & -12 \end{bmatrix} \xrightarrow{(1)=(1)+5(2), (3)=(3)+17(2)} \begin{bmatrix} 1 & 0 & \frac{3}{4} \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & \frac{3}{4} \end{bmatrix} \\ &\xrightarrow{(1)=(1)-(3), (2)=(2)-(3), (3)=\frac{4}{3}(3)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Therefore $x_1 = x_2 = x_3 = 0$

□

Problem 1.2 (Sec 1.5. Problem 1). Let

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, C = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Compute ABC and CAB

Solution.

$$AB = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3 + (-1) \cdot 1 + 1 \cdot (-1) \\ 1 \cdot 3 + 2 \cdot 1 + 1 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$\Rightarrow ABC = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 \cdot 1 + 4 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + (-1) \cdot 1 & 1 \cdot (-1) + (-1) \cdot 2 & 1 \cdot 1 + (-1) \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \end{bmatrix}$$

$$\Rightarrow CAB = \begin{bmatrix} 1 & -3 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + (-3) \cdot 1 + 0 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

In conclusion, $ABC = CAB = [0]$

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Problem 1.3 (Sec 1.5. Problem 2). Let

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix}$$

Verify directly that $A(AB) = A^2B$

Solution.

$$\begin{aligned} AB &= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 2 + (-1) \cdot 1 + 1 \cdot 4 & 1 \cdot (-2) + (-1) \cdot 3 + 1 \cdot 4 \\ 2 \cdot 2 + 0 \cdot 1 + 1 \cdot 4 & 2 \cdot (-2) + 0 \cdot 3 + 1 \cdot 4 \\ 3 \cdot 2 + 0 \cdot 1 + 1 \cdot 4 & 3 \cdot (-2) + 0 \cdot 3 + 1 \cdot 4 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 8 & 0 \\ 10 & -2 \end{bmatrix} \\ \Rightarrow A(AB) &= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 8 & 0 \\ 10 & -2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + (-1) \cdot 8 + 1 \cdot 10 & 1 \cdot (-1) + (-1) \cdot 0 + 1 \cdot (-2) \\ 2 \cdot 5 + 0 \cdot 8 + 1 \cdot 10 & 2 \cdot (-1) + 0 \cdot 0 + 1 \cdot (-2) \\ 3 \cdot 5 + 0 \cdot 8 + 1 \cdot 10 & 3 \cdot (-1) + 0 \cdot 0 + 1 \cdot (-2) \end{bmatrix} \\ &= \begin{bmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{bmatrix} \end{aligned}$$

Meanwhile,

$$\begin{aligned}
 A^2 &= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 \cdot 1 + (-1) \cdot 2 + 1 \cdot 3 & 1 \cdot (-1) + (-1) \cdot 0 + 1 \cdot 0 & 1 \cdot 1 + (-1) \cdot 1 + 1 \cdot 1 \\ 2 \cdot 1 + 0 \cdot 2 + 1 \cdot 3 & 2 \cdot (-1) + 0 \cdot 0 + 1 \cdot 0 & 2 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 \\ 3 \cdot 1 + 0 \cdot 2 + 1 \cdot 3 & 3 \cdot (-1) + 0 \cdot 0 + 1 \cdot 0 & 3 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 5 & -2 & 3 \\ 6 & -3 & 4 \end{bmatrix} \\
 \Rightarrow A^2 B &= \begin{bmatrix} 2 & -1 & 1 \\ 5 & -2 & 3 \\ 6 & -3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix} \\
 &= \begin{bmatrix} 2 \cdot 2 + (-1) \cdot 1 + 1 \cdot 4 & 2 \cdot (-2) + (-1) \cdot 3 + 1 \cdot 4 \\ 5 \cdot 2 + (-2) \cdot 1 + 3 \cdot 4 & 5 \cdot (-2) + (-2) \cdot 3 + 3 \cdot 4 \\ 6 \cdot 2 + (-3) \cdot 1 + 4 \cdot 4 & 6 \cdot (-2) + (-3) \cdot 3 + 4 \cdot 4 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{bmatrix} \\
 \text{Therefore, } A(AB) &= A^2(B) = \begin{bmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{bmatrix} \quad \square
 \end{aligned}$$

Problem 1.4 (Sec 2.1. Problem 3). If \mathbb{C} is the field of complex numbers, which vectors in \mathbb{C}^3 are linear combinations of $(1, 0, -1)$, $(0, 1, 1)$ and $(1, 1, 1)$?

Solution. $\mathbb{C}^3 \doteq \{(x_1, x_2, x_3) \mid x_i \in \mathbb{C}\}$. We want to prove that any vector in \mathbb{C}^3 is a linear combination of the vectors above.

$$\forall v = (x_1, x_2, x_3) \in \mathbb{C}^3,$$

$$v = (x_2 - x_3)(1, 0, -1) + (2x_2 - x_1 - x_3)(0, 1, 1) + (x_1 + x_3 - x_2)(1, 1, 1)$$

is therefore a linear combination. \square

Problem 1.5 (Sec 2.1. Problem 7). Let V be the set of pairs (x, y) of real numbers and let \mathbb{F} be the field of real numbers. Define

$$\begin{aligned}
 (x, y) + (x_1, y_1) &= (x + x_1, 0) \\
 c(x, y) &= (cx, 0)
 \end{aligned}$$

Is V , with these operations, a vector space?

Solution. No. Counterexample against $1 \cdot \alpha = \alpha \ \forall \alpha \in V$:

$$1 \cdot (1, 1) = (1, 0) \neq (1, 1)$$

Therefore V is not a vector space.

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