MATH 20700: Honors Analysis in Rn I

Problem Set 8

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Textbook: Pugh's Real Mathematical Analysis

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Problem 8.1 (5.48 done)

Set

$$f(x,y) = \begin{cases} 1 - 1/q & \text{if } x, y \in \mathbb{Q} \cap [0,1], y = p/q \\ 1 & \text{otherwise} \end{cases}$$

Prove that f is RI on $R = [0,1]^2$, calculate $\underline{F}(y)$ and $\overline{F}(y)$, and prove that $\int_0^1 \underline{F}(y) = \int_0^1 \overline{F}(y)dy = \int_R f = 1$.

Solution

1. We first WTS f is RI on R, by showing that the set of discontinuities is a zero set in R. Recall that

$$D = \bigcup_{k \in \mathbb{N}} D_k$$

where $D_k = \{z \in R : osc_z(f) = \lim_{r \to 0} diam(B_R(z,r)) \geq \frac{1}{k}\}$.

Fix $k \in \mathbb{N}$. Let

$$\begin{aligned} N_k &= \text{ set of reduced fractions less than 1 with denominators} \leq 2k \\ &= \{0 \leq p/q \leq 1 : p, q \in \mathbb{N}, gcd(p,q) = 1, q \leq 2k\} \\ &= \{0, 1/1, 1/2, 1/3, 2/3, 1/4, 3/4, \ldots, (2k-1)/2k\} \\ E_k &= \{(x,y) \in R : y \in N_k\} = [0,1] \times N_k \end{aligned}$$

Quickly note that N_k is a finite set, since its number of elements is strictly less than $4k^2$. Therefore E_k is a zero set in R, since it is a finite union of 1-dimensional slices.

We want to show that all points in $R \backslash E_k$ has oscillation less than $1/k$.

Indeed, take any $(x_0, y_0) \in R \backslash E_k = \{(x,y) \in R : y \notin N_k\}$. Take $d = \min_{a \in N_k}\{|y_0 - a|\} > 0$.

Then for $r < d$, the ball $B_R((x_0, y_0), r) \cap E_k = \varnothing$. Take $(x, y) \in B_R((x_0, y_0), r)$, then

$$f(x,y) = \begin{cases} 1 - 1/l & \text{if } x, y \in \mathbb{Q} \cap [0,1]; y = l'/l \text{ for some } l > 2k \text{ since } y \notin N_k \\ 1 & \text{otherwise} \end{cases}$$

Hence for any $z_1, z_2 \in B_R(z, r)$,

$$\begin{aligned} |f(z_1) - f(z_2)| &\le |f(z_1) - 1| + |f(z_2) - 1| \\ &< |1 - 1/2k - 1| + |1 - 1/2k - 1| = 1/k \end{aligned}$$

which implies

$$osc_{(x_0,y_0)}(f) = \lim_{r \to 0} diam(B_R(z, r)) < 1/k$$

Therefore all points in $R \backslash E_k$ has oscillation less than $1/k$, which implies $D_k \subset E_k$, which implies D_k is a zero set.

D is then a countable union of zero sets, and is therefore a zero set. f is RI as required.

2. To find $\underline{F}(y)$ and $\overline{F}(y)$, we first have to define $f_y(x)$, which is:

$$f_y(x) = \begin{cases} 1 - \mathbb{1}_{\mathbb{Q}}(x)/q & \text{if } y = p/q \\ 1 & \text{otherwise} \end{cases}$$

Case 1: $y \in \mathbb{Q}, y = p/q$.

$$\begin{aligned} \underline{F}(y) &= \underline{\int_0^1} (1 - \mathbb{1}_{\mathbb{Q}}(x)/q) dx \\ &= \sup_{P \in \mathcal{P}([0,1])} L(1 - \mathbb{1}_{\mathbb{Q}}(x)/q, P) \end{aligned}$$

For any partition P of $[0, 1]$, $m_i = 1 - 1/q \;\forall\; i$, since the rationals are dense in [0, 1]. Therefore $L(1 - \mathbb{1}_{\mathbb{Q}}(x)/q, P) = 1 - 1/q \;\forall\; P$. Therefore

$$\underline{F}(y) = 1 - 1/q$$

On the other hand,

$$\begin{aligned} \overline{F}(y) &= \overline{\int_0^1} (1 - \mathbb{1}_{\mathbb{Q}}(x)/q) dx \\ &= \inf_{P \in \mathcal{P}([0,1])} U(1 - \mathbb{1}_{\mathbb{Q}}(x)/q, P) \end{aligned}$$

For any partition P of $[0, 1]$, $M_i = 1 \;\forall\; i$, since the irrationals are dense in [0, 1]. Therefore $U(1 - \mathbb{1}_{\mathbb{Q}}(x)/q, P) = 1 \;\forall\; P$. Therefore

$$\overline{F}(y) = 1$$

Case 2: $y \notin \mathbb{Q}$, so $f_y \equiv 1$. Then it's clear that

$$\overline{F}(y) = \underline{F}(y) = 1$$

In conclusion,

$$\underline{F}(y) = \begin{cases} 1 - 1/q & \text{if } y = p/q \\ 1 & \text{otherwise} \end{cases}$$

$$\overline{F}(y) \equiv 1$$

Since f is RI on R, it follows that

$$\int_0^1 \underline{F}(y) = \int_R f = \int_0^1 \overline{F}(y) = \int_0^1 1 dy = 1$$

as required. □

Problem 8.2 (5.49 done)

Using the FTC, give a direct proof of Green's Formulas

$$-\iint_R f_y dx dy = \int_{\partial R} f dx$$

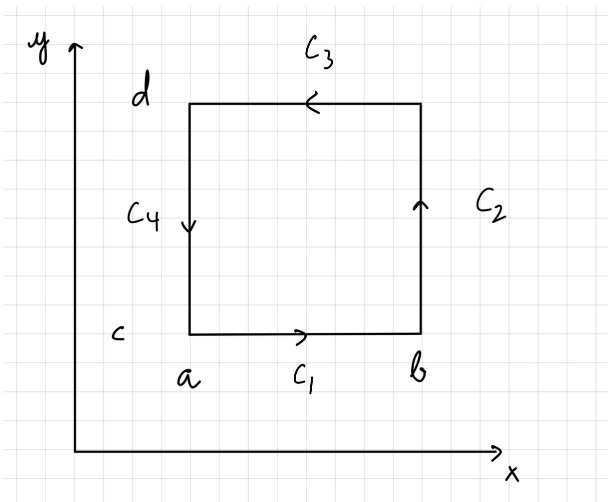
and

$$\iint_R g_x dx dy = \int_{\partial R} g dy$$

where R is a square in the plane and $f, g : \mathbb{R}^2 \to \mathbb{R}$ are smooth. (Assume that the edges of the square are parallel to the coordinate axes.)

Solution

Let $R = [a, b] \times [c, d]$.



Then

$$
\begin{aligned}
-\iint_R f_y dxdy &= -\int_a^b \int_c^d f_y dydx \\
&= -\int_a^b f(x, y=d) - f(x, y=c)dx \\
&= \int_a^b f(x, y=c)dx - \int_a^b f(x, y=d)dx \\
&= \int_{C_1} fdx + \int_{C_3} fdx \\
&= \int_{\partial R} fdx
\end{aligned}
$$

since on C_2 and C_4, x stays unchanged so $\int_{C_2} fdx = \int_{C_4} fdx = 0$. Similarly,

$$
\begin{aligned}
\iint_R g_x dxdy &= \int_c^d g(x=a, y) - g(x=b, y)dy \\
&= \int_c^d g(x=b, y)dy - \int_c^d g(x=a, y)dy \\
&= \int_{\partial R} gdy
\end{aligned}
$$

since $\int_{C_1} gdy = \int_{C_3} gdy = 0$. □

Problem 8.3 (5.65 done)

Show that the 1-form defined on $\mathbb{R}^2 \backslash \{(0,0)\}$ by

$$\omega = \frac{-y}{r^2}dx + \frac{x}{r^2}dy$$

is closed but not exact. Why do you think that this 1-form is often referred to as $d\theta$ and why is the name problematic?

Solution

1. For ω,

$$g_1(x,y) = \frac{-y}{x^2+y^2}, g_2(x,y) = \frac{x}{x^2+y^2}$$

Checking for closedness:

$$
\begin{aligned}
\frac{\partial g_1}{\partial y} &= y(x^2+y^2)^{-2}(2y) - (x^2+y^2)^{-1} \\
&= (x^2+y^2)^{-2}(y^2-x^2) \\
\frac{\partial g_2}{\partial x} &= -x(x^2+y^2)^{-2}(2x) + (x^2+y^2)^{-1} \\
&= (x^2+y^2)^{-2}(y^2-x^2)
\end{aligned}
$$

so they're equal, so ω is indeed closed.

2. We show that ω is not exact by showing that there exists a 1-cell φ such that $\int_\varphi \omega \neq 0$. Indeed, on the unit circle:

$$\varphi : [0,1] \to \mathbb{R}^2, t \mapsto (\cos 2\pi t, \sin 2\pi t)$$

we have the pullback:

$$\begin{aligned}\varphi^*\omega &= \sum_{i=1}^{2} g_i(\varphi(t))\varphi_i'(t)dt \\ &= 2\pi[-(\sin 2\pi t)(-\sin 2\pi t) + (\cos 2\pi t)(\cos 2\pi t)]dt \\ &= 2\pi dt\end{aligned}$$

therefore

$$\int_\varphi \omega = \int_0^1 2\pi dt = 2\pi \neq 0$$

as required.

This 1-form is often referred to as $d\theta$ as it measures the change in the polar coordinate θ in Cartesian coordinates x, y. However, since it is not exact, the notation $d\theta$ is problematic, since it suggests that $\omega = d\theta$ is exact. □

Problem 8.4 (5.67 done)

Show that the 2-form defined on the spherical shell by

$$\omega = \frac{x}{r^3}dy \wedge dz + \frac{y}{r^3}dz \wedge dx + \frac{z}{r^3}dx \wedge dy$$

is closed but not exact.

Solution

1. We compute:

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) &= (x^2+y^2+z^2)^{-5/2}(y^2+z^2-2x^2) = r^{-5}(y^2+z^2-2x^2) \\ \frac{\partial}{\partial y}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) &= 3xyr^{-5} \\ \frac{\partial}{\partial y}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) &= 3xzr^{-5}\end{aligned}$$

and similarly for $\frac{y}{r^3}, \frac{z}{r^3}$. Therefore

$$\begin{aligned}d\left(\frac{x}{r^3}dy \wedge dz\right) &= r^{-5}((y^2+z^2-2x^2)dx + 3xydy + 3xzdz) \wedge dy \wedge dz \\ &= r^{-5}(y^2+z^2-2x^2)dx \wedge dy \wedge dz \\ d\left(\frac{y}{r^3}dz \wedge dx\right) &= r^{-5}(z^2+x^2-2y^2)dy \wedge dz \wedge dx \\ d\left(\frac{z}{r^3}dx \wedge dy\right) &= r^{-5}(x^2+y^2-2z^2)dz \wedge dx \wedge dy\end{aligned}$$

Then,

$$\begin{aligned}
dy \wedge dz \wedge dx &= -dy \wedge dz \wedge dx \\
&= dx \wedge dy \wedge dz \\
dz \wedge dx \wedge dy &= -dx \wedge dz \wedge dy \\
&= dx \wedge dy \wedge dz
\end{aligned}$$

Therefore

$$\begin{aligned}
d\omega &= d\left(\frac{x}{r^3} dy \wedge dz\right) + d\left(\frac{y}{r^3} dz \wedge dx\right) + d\left(\frac{z}{r^3} dx \wedge dy\right) \\
&= r^{-5}[(y^2 + z^2 - 2x^2) + (z^2 + x^2 - 2y^2) + (x^2 + y^2 - 2z^2)] dx \wedge dy \wedge dz \\
&= 0
\end{aligned}$$

and ω is therefore closed.

2. Assume that it is exact, then $\omega = d\alpha$ for some 1-form α.

We try to compute $\int_{S^2} \omega$, i.e. integrating ω against S^2, with parameterization:

$$\rho(\varphi, \theta) = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$$

from $[0, 2\pi] \times [0, \pi]$.

Then

$$\begin{aligned}
dx &= -\sin\theta \sin\varphi d\varphi + \cos\theta \cos\varphi d\theta \\
dy &= \sin\theta \cos\varphi d\varphi + \cos\theta \sin\varphi d\theta \\
dz &= -\sin\theta d\theta
\end{aligned}$$

which implies

$$\begin{aligned}
dx \wedge dy &= -\sin\theta \cos\theta (\sin^2\varphi + \cos^2\varphi) d\varphi \wedge d\theta \\
&= -\sin\theta \cos\theta d\varphi d\theta \\
dy \wedge dz &= -\sin^2\theta \cos\varphi d\varphi \wedge d\theta \\
dz \wedge dx &= -\sin^2\theta \sin\varphi d\varphi \wedge d\theta
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{S^2} \omega = &\int_{[0,2\pi]\times[0,\pi]} [(\sin\theta\cos\varphi)(-\sin^2\theta\cos\varphi) \\
&+ (\sin\theta\sin\varphi)(-\sin^2\theta\sin\varphi) + \cos\theta(-\sin\theta\cos\theta)] d\varphi \wedge d\theta \\
&= \int_0^\pi \int_0^{2\pi} (-\sin^3\theta - \sin\theta\cos^2\theta) d\varphi d\theta \\
&= \int_0^\pi \int_0^{2\pi} -\sin\theta d\varphi d\theta \\
&= \int_0^\pi -2\pi \sin\theta d\theta = [2\pi\cos\theta]_0^\pi = -4\pi
\end{aligned}$$

However, since $\omega = d\alpha$,

$$\int_{S^2} \omega = \int_{S^2} d\alpha$$
$$= \int_{\partial S^2} \alpha$$

but ∂S^2 is the $1 - chain$:

$$\begin{aligned}\partial\rho &= (-1)^2(\rho(2\pi,\theta) - \rho(0,\theta)) + (-1)^3(\rho(\theta,\pi) - \rho(\varphi,0)) \\ &= [(\sin\theta, 0, \cos\theta) - (\sin\theta, 0, \cos\theta)] - [(0,0,-1) - (0,0,-1)] \\ &= 0\end{aligned}$$

$$\Rightarrow \int_{\partial S^2} \alpha = 0 \quad \square$$

Therefore ω is not exact. $\square$

Problem 8.5 (III done)

This exercise explores the question (asked in class): suppose Df_p is invertible at every point of the domain: is f injective? Recall this is true for $f : (a,b) \to \mathbb{R}$ by the one dimensional inverse function theorem.

(a) Consider the map $f : A \to \mathbb{R}^2$, where $A = \{z \in \mathbb{R}^2 : 1 < |z| < 2\}$ given by

$$f(x,y) = (x^2 - y^2, 2xy).$$

Is it C^1 in A, with invertible derivative? Is it injective?

(b) Suppose that $U \subset \mathbb{R}^2$ is open, $f : U \to \mathbb{R}^n$ is C^1, and Df_p is invertible at each $p \in U$. Prove that if $f(U)$ is closed in $\mathbb{R}^n$, then $f(U) = \mathbb{R}^n$.

(c) *** (EC) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1, with invertible derivative. Must it be injective?

(d) *** (EC) Characterize the open sets $U \subset \mathbb{R}^n$ with the property that if $f : U \to \mathbb{R}^n$ is C^1, and Df_p is invertible for all $p \in U$, then f is injective.

(e) ** (EC) Prove that if $f : \bar{A} \to \bar{A}$ is C^1, with the property that Df_p is invertible at every $p \in \bar{A}$, then there exists an integer $k \geq 0$ such that f is $k-$to-one, where A is the annulus defined in (c).

Solution

(a) 1. WTS f is C^1. We have

$$\frac{\partial f_1}{\partial x} = 2x, \frac{\partial f_1}{\partial y} = -2y, \frac{\partial f_2}{\partial x} = -2y, \frac{\partial f_2}{\partial y} = 2x$$

which are all continuous, so the partial derivatives of f are continuous, so f is differentiable. Writing down its Jacobian:

$$Jf_{(x,y)} = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

so

$$\|Df_{(x,y)} - Df_{(x',y')}\|_{op} = \left\| \begin{pmatrix} 2(x-x') & -2(y-y') \\ 2(y-y') & 2(x-x') \end{pmatrix} \right\|_{op}$$
$$\to 0 \text{ as } |(x,y)-(x',y')| = |(x-x', y-y')| \to 0$$

so $Df : (x,y) \mapsto Df_{(x,y)}$ is therefore continuous. f is therefore C^1.

2. WTS the derivative is everywhere invertible. Indeed,

$$Jf^{-1}(x,y) = \frac{1}{4(x^2+y^2)} \begin{pmatrix} 2x & 2y \\ -2y & 2x \end{pmatrix}$$

after using the rule $A^{-1} = \frac{1}{\det A} adj(A)$, and realizing that $\det Jf_{(x,y)} = 4(x^2+y^2) > 0$ in the annulus.

3. WTS f is not injective. Indeed,

$$f(1,1) = (0,2) = f(-1,-1) \quad \square$$

(b) 1. WTS $f(U)$ is open.

Let $q \in f(U), q = f(p)$ for some $p \in U \subset \mathbb{R}^2$. Then since f is C^1 and Df_p is invertible, by Inverse Function Theorem, f is a local C^1 diffeomorphism from a neighborhood of p to a neighborhood of q. In other words, one can find an open neighborhood around any q that is a subset of $f(U)$, therefore $f(U)$ is open.

2. Since $\mathbb{R}^n$ is connected, and $f(U)$ is clopen and nonempty, it follows that $f(U) = \mathbb{R}^n$. $\square$

Problem 8.6 (IV done)

Let $g\colon \mathbb{R}^n \to \mathbb{R}^k$ and consider the set $S = g^{-1}(0)$. Assume that for every $p \in S$, the rank of $D_p g$ is equal to k (in other words, $D_p g\colon \mathbb{R}^n \to \mathbb{R}^k$ is surjective). Let $p \in S$, and define $T_p S := \ker D_p g \subset \mathbb{R}^n$.

(a) What does the rank-nullity theorem from linear algebra tell you about the dimension of $T_p S$?

(b) Show that if $\gamma\colon (-\varepsilon, \varepsilon) \to U$ is a C^1 curve through p along which g is constant (i.e. if $\gamma(0) = p$ and $g(\gamma(t)) = g(\gamma(0))$ for all $t \in (-\varepsilon, \varepsilon)$), then $\gamma'(0) \in T_p S$. This is why we call $T_p S$ the *tangent space to S at p*.

(c) Find a basis for $T_p S$ for the following S, p

(i) $S = \{(x,y,z) : -x^2 + y^2 - z^2 = -1\}$, $p = (\frac{1}{\sqrt{2}}, -\frac{\sqrt{3}}{\sqrt{2}}, -\sqrt{2})$.

(ii) $S = \{(x,y,z) : -x^2+y^2-z^2 = -1, \text{ and } xz+4y^2 = 5\}$, $p = (\frac{1}{\sqrt{2}}, -\frac{\sqrt{3}}{\sqrt{2}}, -\sqrt{2})$.

(d) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ and for $p \in S$, let V_p be the orthogonal projection of $\text{grad}_p(f)$ onto the subspace $T_pS \subset \mathbb{R}^n$. Prove that if V_p is nonzero, then $\pm|V_p|$ is the maximum (resp. minimum) value of the function $G\colon T_p^1S \to \mathbb{R}$, where $T_p^1S := \{v \in T_pS : |v| = 1\}$, and

$$G(v) = D_pf(v).$$

(e) Prove that $\pm V_p$, if nonzero, points to the maximal direction of increase/decrease of f in directions tangent to the surface S.

(f) Relate (d),(e) to Lagrange multipliers.

Solution

(a)

$$\dim T_pS = \dim\ker D_pg = n - \dim\operatorname{im} D_pg = n-k \quad \square$$

(b) WTS $\gamma'(0) \in T_pS \Leftrightarrow Dg_p(\gamma'(0)) = 0$.

Consider $g \circ \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^k$, then $g \equiv c \in \mathbb{R}^k$ since γ is a level curve of g. It follows that the derivative is 0 at $0 \in (-\varepsilon,\varepsilon)$ (0 both as the zero map and the vector 0, since domain of $g\circ\gamma$ is 1 dimensional.)

$$0 = D(g\circ\gamma)_0 = Dg(\gamma'(0))$$

as required. $\square$

(c)(i) Let $f(x,y,z) = -x^2+y^2-z^2+1$, $g(x,y,z) = xz+4y^2-5$. Recall that $p = (\frac{1}{\sqrt{2}}, -\frac{\sqrt{3}}{\sqrt{2}}, -\sqrt{2})$.

Then

$$\begin{aligned} Jf_{(x,y,z)} &= \begin{pmatrix} -2x & 2y & -2z \end{pmatrix} \\ \Rightarrow Jf_p &= \begin{pmatrix} -\sqrt{2} & -\sqrt{6} & 2\sqrt{2} \end{pmatrix} \end{aligned}$$

Now we want to find the basis for T_pS, i.e. the basis for the null space of Jf_p. Solving

$$\begin{pmatrix} -\sqrt{2} & -\sqrt{6} & 2\sqrt{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

yields

$$z = \frac{x}{2} + \frac{y\sqrt{3}}{2}$$

so a basis is $\left\{(2,0,1), (0,2,\sqrt{3})\right\}$.

(c)(ii) Similarly,

$$\begin{aligned} Jg_{(x,y,z)} &= \begin{pmatrix} z & 8y & x \end{pmatrix} \\ \Rightarrow Jg_p &= \begin{pmatrix} -\sqrt{2} & -4\sqrt{6} & \frac{1}{\sqrt{2}} \end{pmatrix} \end{aligned}$$

So now we want to find basis for the null space:

$$\begin{pmatrix} -\sqrt{2} & -\sqrt{6} & 2\sqrt{2} \\ -\sqrt{2} & -4\sqrt{6} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

has rref

$$\begin{pmatrix} 1 & 0 & \frac{-5}{2} \\ 0 & 1 & \frac{1}{2\sqrt{3}} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

so a basis is $\{(5\sqrt{3}, -1, 2\sqrt{3})\}$ $\square$

(d) We already know that $\langle grad_p(f), v\rangle = Df_p(v)$ for all v (pset 5).

Taking any $v \in T_pS$, since V_p is the orthogonal projection of $grad_p(f)$ onto T_pS, that implies $\langle grad_p(f) - V_p, v\rangle = 0$.

It follows that

$$G(v) = Df_p(v) = \langle V_p, v\rangle \; \forall \; v \in T_p^1 S$$

from which we can bound, for $v \in T_p^1 S$:

$$\begin{aligned} |G(v)| = |Df_p(v)| &= |\langle V_p, v\rangle| \\ &\leq |V_p||v| \\ &= |V_p| \end{aligned}$$

It follows that $\pm|V_p|$ are the maximum and minimum value of G on $T_p^1 S$, with equality achieved when $v \parallel V_p$.

(e) Following from (d), the equality for maximum/minimum of $Df_p(v)$ is achieved when $v \parallel V_p$, i.e., when v points in $\pm V_p$ direction. Furthermore, this direction in which $Df_p(v)$ is maximized/minimized is exactly the direction of maximal increase/decrease. Therefore $\pm V_p$ points to the direction of maximal increase/decrease.

Also, $V_p \in T_pS$, so it is tangent to surface S.

(f) Relating to Lagrange Multipliers, we WTS that if $grad_p(f)$ is not orthogonal to T_pS then p is not a local extremum.

If $grad_p(f)$ is not orthogonal to T_pS, then that implies there exists a non-zero orthogonal projection of $grad_p(f)$ onto T_pS, i.e., $V_p \neq 0$.

From (d) and (e), it follows that one can still move in the direction of $\pm V_p$ to achieve a locally higher/lower value for f, while still staying on the level curve of g, since $V_p \in T_pS = \ker Dg_p$, therefore p is not a local extremum.

Furthermore, when $grad_p(f)$ is orthogonal to T_pS, then $|G(v)| = |Df_p(v)| \leq 0 \Rightarrow Df_p \equiv 0$, so p is a local extremum!

And this condition is exactly the condition for Lagrange Multipliers. We claim that if $grad_p(f)$ is orthogonal to $T_pS = \ker Dg_p$, then $grad_p(f) \in rowspace(Jg_p)$.

This is because if $x \in \ker Dg_p$, x non trivial then $Jg_px = 0$, which implies that x is orthogonal to all rows of Jg_p, so $x \notin rowspace(Jg_p)$. Therefore $rowspace(Jg_p) \cap \ker Dg_p = \{0\}$. Their ranks add up to n, therefore

$$rowspace(Jg_p) \oplus Dg_p = \mathbb{R}^n$$

Since $grad_p(f)$ is orthogonal to $\ker Dg_p$, it must be the case that $grad_p(f) \in rowspace(Jg_p)$.

And recall that $grad_p(g) = Jg_p^T$, so $grad_p(f)$ is a linear combination of the rows of Jg_p, which are columns of $grad_p(g)$. It follows that there exists $\vec{\lambda} \in \mathbb{R}^k$ such that

$$grad_p(f) = grad_p(g)\vec{\lambda}$$

which is the requirement of Lagrange Multipliers (for k constraints). □

Problem 8.7 (V done)

(a) Compute the volume of the region $\Omega \subset \mathbb{R}^3$ bounded by $x = 0, x = 2, z = -y$ and by $z = y^2/2$.

(b) Write down a triple integral that computes the volume of the region $\Omega \subset \mathbb{R}^3$ bounded by the coordinate planes and $y = 1 - x^2$ and $y = 1 - z^2$. Don't evaluate.

(c) Compute
$$\iiint_B \left(2x + 3y^2 + 4z^3\right) dx\, dy\, dz,$$
where $B = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq 2, 0 \leq z \leq 3\}$.

Solution

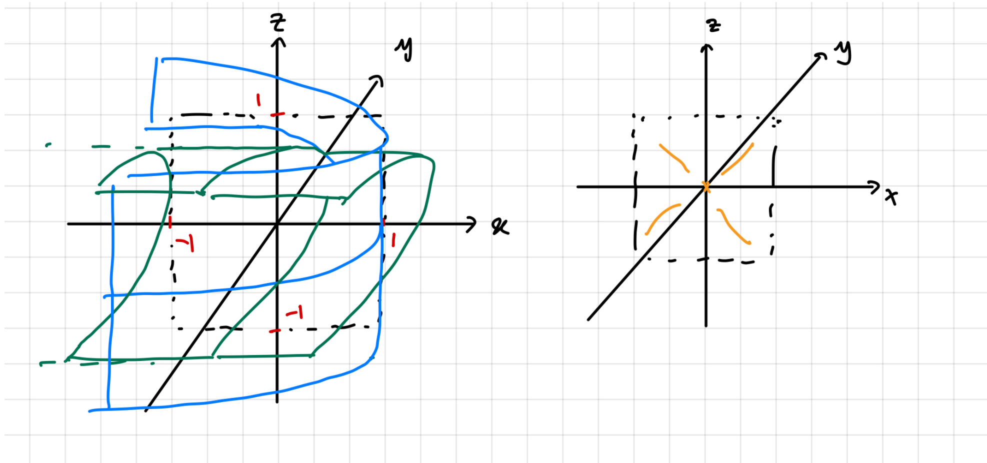
(a) Solve for the bounds for y of the bounded region:

$$-y = y^2/2 \Rightarrow y = 0, -2$$

It follows that the volume is:

$$\begin{aligned}\left|\int_0^2 \int_{-2}^0 \int_{-y}^{y^2/2} 1dzdydx\right| &= \left|\int_0^2 \int_{-2}^0 (y^2/2 + y)dydx\right| \\ &= \int_0^2 (-2/3)dx \\ &= 4/3\end{aligned}$$

(b)



The bounded volume is on the $[-1, 1] \times [-1, 1]$ square on the $xz-$plane, which consists of 4 identical volumes on each of the quadrant of the $xz-$plane. WLOG, consider the volume on the first quadrant, with $x, z \in [0, 1]$. Then for each (x, z), $y_{min} = 1 - 1^2 = 0$, while

$$y_{max} = \begin{cases} 1 - x^2 & \text{if } x \geq z \\ 1 - z^2 & \text{if } x \leq z \end{cases}$$

Taking volumes below and above the $z = x$ line, then the volume of this first-quadrant volume is

$$V_1 = \left| \int_0^1 \int_0^x \int_0^{1-x^2} 1 dydzdx \right| + \left| \int_0^1 \int_x^1 \int_0^{1-z^2} 1 dydzdx \right|$$

and the total volume throughout all 4 quadrants is just $4V_1$.

(c)

$$\begin{aligned} \int_0^3 \int_0^2 \int_0^1 (2x + 3y^2 + 4z^3) dxdydz &= \int_0^3 \int_0^2 \left[x^2 + 3y^2x + 4z^3x \right]_0^1 dydz \\ &= \int_0^3 \int_0^2 (1 + 3y^2 + 4z^3) \\ &= \int_0^3 \left[y + y^3 + 4z^3y \right]_0^2 dz \\ &= \int_0^3 (10 + 8z^3) dz \\ &= 192 \end{aligned}$$

□

Problem 8.8 (VI done)

Let $\omega = (y \cos xy + \mathrm{e}^x)\, \mathrm{d}x + (x \cos xy + 2y)\mathrm{d}y$.

(a) Evaluate $\int_\Gamma \omega$ along the segment of the parabola $y = x^2$ from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Use the parameterization $\varphi : [0, 1] \to \mathbb{R}^2$

$$\varphi(t) = (t, t^2)$$

(b) Evaluate $\int_\Gamma \omega$ for the case where Γ is the straight line joining the origin to the point $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Do the same for the case where Γ consists of the segment $0 \leqslant x \leqslant \alpha$ on the x-axis, followed by the segment $x = \alpha, 0 \leqslant y \leqslant \beta$.

(c) Find a function $f(x, y)$ such that $\omega = df$.

Solution

(a) We get the pullback:

$$\varphi^*\omega = [(t^2 \cos(t^3) + e^t) + (t \cos(t^3) + 2t^2)(2t)]dt = (3t^2 \cos(t^3) + 4t^3 + e^t)dt$$

Then

$$\begin{aligned} \int_\Gamma \omega &= \int_0^1 (3t^2 \cos(t^3) + 4t^3 + e^t)dt \\ &= \left[\sin(t^3) + t^4 + e^t\right]_0^1 \\ &= \sin(1) + 1 + e - 1 = \sin(1) + e \quad \square \end{aligned}$$

(b) 1. Straight line joining origin to the point $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is parameterized:

$$\Gamma(t) = (\alpha t, \beta t)$$

has pullback

$$\begin{aligned} \Gamma^*\omega &= [\alpha(\beta t \cos(\alpha\beta t^2) + e^{\alpha t}) + \beta(\alpha t \cos(\alpha\beta t^2) + 2\beta t)]dt \\ &= [2\alpha\beta t \cos(\alpha\beta t^2) + e^{\alpha t} + 2\beta^2 t]dt \end{aligned}$$

so

$$\begin{aligned} \int_\Gamma \omega = \int_0^1 \Gamma^*\omega &= \int_0^1 [2\alpha\beta t \cos(\alpha\beta t^2) + \alpha e^{\alpha t} + 2\beta^2 t]dt \\ &= \sin(\alpha\beta) + e^\alpha - e + \beta^2 \end{aligned}$$

2. Let ρ parameterize path from $(0, 0)$ to $(\alpha, 0)$, and σ parameterize path from $(\alpha, 0)$ to (α, β).

$$\rho(t) = (\alpha t, 0), \sigma(t) = (\alpha, \beta t).$$

They have pullback:

$$\begin{aligned}\rho^*\omega &= \alpha(e^{\alpha t})dt \\ \sigma^*\omega &= \beta(\alpha\cos(\alpha\beta t) + 2\beta t)dt\end{aligned}$$

therefore

$$\begin{aligned}\int_\Gamma \omega &= \int_\rho \omega + \int_\sigma \omega \\ &= \int_0^1 (\alpha e^{\alpha t} + \alpha\beta\cos(\alpha\beta t) + 2\beta^2 t)dt \\ &= \left[e^{\alpha t} + \sin(\alpha\beta t) + \beta^2 t^2\right]_0^1 \\ &= e^\alpha - e + \sin(\alpha\beta) + \beta^2 \quad \square\end{aligned}$$

(c)

$$f(x, y) = \sin xy + e^x + y^2$$

has

$$\begin{aligned}\frac{\partial f}{\partial x} &= y\cos xy + e^x \\ \frac{\partial f}{\partial y} &= x\cos xy + 2y\end{aligned}$$

so

$$\omega = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = df$$

as required. $\square$

Problem 8.9 (VII done)

Let $\omega = y\ \mathrm{d}x - x\ \mathrm{d}y$.

(a) Evaluate $\int_\gamma \omega$ along the semicircle γ from $\begin{pmatrix} r-1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} r+1 \\ 0 \end{pmatrix}$ defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r - \cos t \\ \sin t \end{pmatrix}$$

for $0 < t < \pi$.

(b) Show explicitly that you can obtain a different value from that in (a) by choosing a different curve joining $\begin{pmatrix} r-1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} r+1 \\ 0 \end{pmatrix}$.

Solution

(a) With parameterization $\gamma(t) = (r - \cos t, \sin t)$ on $(0, \pi)$:

$$\begin{aligned}
\int_\gamma \omega &= \int_0^\pi \sin^2 t - (r - \cos t)\cos t dt \\
&= \int_0^\pi -r\cos t + 1 dt \\
&= [-r\sin t + t]_0^\pi \\
&= \pi
\end{aligned}$$

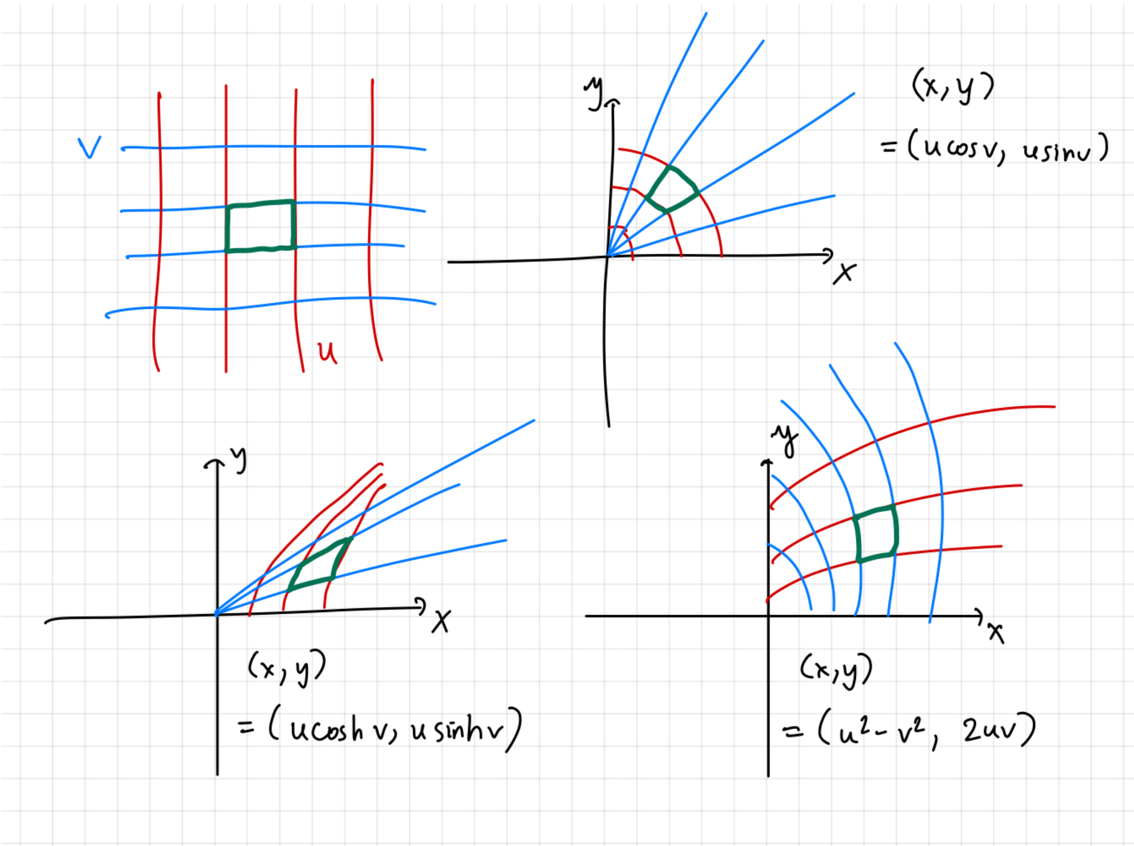
(b) Parameterize $\rho(t) = (r - 1 + 2t, 0)$. Then $y = 0$ throughout and $dy = 0$. So $\int_\rho \omega = 0$, different from (a). $\square$

Problem 8.10 (VIII done)

In each of the following cases, u and v are functions on a plane where x and y are affine coordinates. Express $\mathrm{d}x \wedge \mathrm{d}y$ in terms of $\mathrm{d}u \wedge \mathrm{d}v$. Make a sketch showing typical curves $u =$ constant and $v =$ constant in the first quadrant $(x, y > 0)$ and try to give a geometric interpretation to the relations between $\mathrm{d}x \wedge \mathrm{d}y$ and $\mathrm{d}u \wedge \mathrm{d}v$ by applying both to a parallelogram whose sides are tangent to $u =$ constant and $v =$ constant respectively.

- **(a)** $x = u\cos v$. $\quad y = u\sin v$.
- **(b)** $x = u\cosh v$, $\quad y = u\sinh v$.
- **(c)** $x = u^2 - v^2$, $\quad y = 2uv$.

Solution



(a)

$$\begin{aligned} \mathrm{d}x &= \cos v \mathrm{d}u - u \sin v \mathrm{d}v \\ \mathrm{d}y &= \sin v \mathrm{d}u + u \cos v \mathrm{d}v \\ \Rightarrow \mathrm{d}x \wedge \mathrm{d}y &= u(\cos^2 v + \sin^2 v) \mathrm{d}u \wedge \mathrm{d}v \\ &= u \mathrm{d}u \wedge \mathrm{d}v \end{aligned}$$

(b)

$$\begin{aligned} \mathrm{d}x &= \cosh v \mathrm{d}u + u \sinh v \mathrm{d}v \\ \mathrm{d}y &= \sinh v \mathrm{d}u + u \cosh v \mathrm{d}v \\ \Rightarrow \mathrm{d}x \wedge \mathrm{d}y &= u(\cosh^2 v - \sinh^2 v) \mathrm{d}u \wedge \mathrm{d}v \\ &= u \mathrm{d}u \wedge \mathrm{d}v \end{aligned}$$

(c)

$$\begin{aligned} \mathrm{d}x &= 2u \mathrm{d}u - 2v \mathrm{d}v \\ \mathrm{d}y &= 2v \mathrm{d}u + 2u \mathrm{d}v \\ \Rightarrow \mathrm{d}x \wedge \mathrm{d}y &= (4u^2 + 4v^2) \mathrm{d}u \wedge \mathrm{d}v \end{aligned}$$

$\square$

Problem 8.11 (IX done)

(a) Show, by reversing the order of integration, that

$$\int_0^a \left(\int_0^y \mathrm{e}^{m(a-x)} f(x) \mathrm{d}x \right) \mathrm{d}y = \int_0^a (a-x) \mathrm{e}^{m(a-x)} f(x) \mathrm{d}x$$

where a and m are constants, $a > 0$.

(b) Show that $\int_0^x \left(\int_0^v \left[\int_0^u f(t) \mathrm{d}t \right] \mathrm{d}u \right) \mathrm{d}v = \frac{1}{2} \int_0^x (x-t)^2 f(t) \mathrm{d}t$. If you do this in two steps, you never actually have to consider a triple integral!

Solution

(a)

$$\begin{aligned}
\int_0^a \left(\int_0^y e^{m(a-x)} f(x) dx \right) dy &= \int_0^a \int_x^a e^{m(a-x)} f(x) dy dx \\
&= \int_0^a (a-x) e^{m(a-x)} f(x) dx
\end{aligned}$$

as required.

(b)

$$\begin{aligned}
\int_0^x \left(\int_0^v \left[\int_0^u f(t) \mathrm{d}t \right] \mathrm{d}u \right) \mathrm{d}v &= \int_0^x \int_0^v \int_t^v f(t) \mathrm{d}u \mathrm{d}t \mathrm{d}v \\
&= \int_0^x \int_0^v \left[u f(t) \right]_{u=t}^{u=v} \mathrm{d}t \mathrm{d}v \\
&= \int_0^x \int_0^v (v-t) f(t) \mathrm{d}t \mathrm{d}v \\
&= \int_0^x \int_t^x (v-t) f(t) \mathrm{d}v \mathrm{d}t \\
&= \int_0^x \left[\frac{(v-t)^2}{2} \right]_{v=t}^{v=x} \mathrm{d}t \\
&= \frac{1}{2} \int_0^x (x-t)^2 f(t) \mathrm{d}t
\end{aligned}$$

as required. □