MATH 20700: Honors Analysis in Rn I

Problem Set 5

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Textbook: Pugh's Real Mathematical Analysis

Problem 5.1 (2.13 done)

Assume $f : M \to N$ is a function from one metric space to another, satisfying the following condition: If a sequence $(p_n) \subseteq M$ converges then the sequence $(f(p_n)) \subseteq N$ converges. Prove that f is continuous.

Solution

Let $(p_n) \subseteq M$ and $p_n \xrightarrow{n\to\infty} p \in M$. Then WTS $(f(p_n)) \xrightarrow{n\to\infty} f(p)$. Consider $(q_n) \subseteq M$ defined as follows:

$$q_{2i-1} = p_i, q_{2i} = p \; \forall \, i \in \mathbb{N}$$

then (q_n) is a well-defined sequence in M. Does it converge? Yes. Given $\varepsilon > 0$, since $p_n \xrightarrow{n\to\infty} p$, there exists $N_1 \in \mathbb{N}$ such that

$$\forall \, n \geq N_1, d(p_n, p) < \varepsilon$$

Then choose $N_2 = 2N_1$, then for all $n \geq N_2$, we have if n is even then

$$d(q_n, p) = d(p, p) = 0 < \varepsilon$$

and if n is odd

$$\frac{n+1}{2} > N_1 \Rightarrow d(q_n, p) = d(p_{\frac{n+1}{2}}, p) < \varepsilon$$

Therefore $q_n \xrightarrow{n\to\infty} p$. It follows that $f(q_n) \xrightarrow{n\to\infty} y \in N$. WTS $y = f(p)$.

Suppose not, then since $f(q_n) \xrightarrow{n\to\infty} y$, for $\varepsilon' = d(y, f(p))/2$, there exists N_3 such that for all $n \geq N_3$,

$$d(f(q_n), y) < \varepsilon'/2$$

Take $N_4 = 2N_3 - 1 \geq N_3$ then

$$d(y, f(p))/2 > d(f(q_{N_4}), y) = d(f(q_{(2N_3-1)}), y) = d(f(p), y) \Rightarrow\Leftarrow$$

It follows that $y = f(p)$. Therefore $f(q_n) \xrightarrow{n\to\infty} f(p)$.

But then $(f(p_n))$ is a convergent subsequence of $(f(q_n))$, so it has to converge to the same limit. Thus $f(p_n) \xrightarrow{n\to\infty} f(p)$. f is therefore continuous. $\square$

Problem 5.2 (2.27 done)

If $S, T \subseteq M$, a metric space, and $S \subseteq T$, prove that

(a) $\overline{S} \subseteq \overline{T}$

(b) $int(S) \subseteq int(T)$

Solution

(a) Let $s \in \overline{S}$. Then there exists a sequence $(p_n) \subseteq S$ such that $p_n \xrightarrow{n\to\infty} s$. But $S \subseteq T \Rightarrow (p_n) \subseteq T$ too. And $p_n \xrightarrow{n\to\infty} s \in M$ so s is a limit point of T. In other words, $s \in \overline{T}$. Therefore $\overline{S} \subseteq \overline{T}$. $\square$

(b) Recall that $int(S)$ is the largest open set in M that is a subset of S. Let $s \in int(S)$. Then there exists $r > 0$ such that

$$B_M(s, r) \subseteq S$$

But $S \subseteq T \subseteq M$ so $B_M(s, r) \subseteq T$.

Suppose that $s \notin int(T)$ then we can construct

$$I = int(T) \cup B_M(s, r)$$

is a union of open sets in M and is therefore open. Also, $int(T), B_M(s, r) \subseteq T \Rightarrow I \subseteq T$.

However, $int(T) \subseteq I, s \in I, s \notin int(T)$ so $int(T)$ is not the largest open set in M that is a subset of T. $\Rightarrow\Leftarrow$

It follows that $s \in int(T)$. Therefore $int(S) \subseteq int(T)$. $\square$

Problem 5.3 (2.41 done)

Let $\|\cdot\|$ be any norm on $\mathbb{R}^m$ and let $B = \{x \in \mathbb{R}^m : \|x\| \leq 1\}$. Prove that B is compact. [Hint: It suffices to show that B is closed and bounded with respect to the Euclidean matrix.]

Solution

Notation: Let $|\cdot|$ be the standard Euclidean norm.

$\|\cdot\|$ induces a metric d on $\mathbb{R}^m$. Consider the identity map between 2 metric spaces

$$id : (\mathbb{R}^m, d_E) \to (\mathbb{R}^m, d), x \mapsto x.$$

It is clearly a bijection.

Denote $e_i \in \mathbb{R}^m, e_i = (0, \ldots, 1, \ldots, 0)$ with its i^{th} entry as 1, and the rest as 0. Define

$$L := \max_{1\leq i\leq m} \|e_i\| < \infty$$

Note that $B = id(B) = id^{-1}(B)$.

1. Claim that B is closed in $(\mathbb{R}^m, d_E)$.

For any $a, b \in (\mathbb{R}^m, d)$, we have:

$$\begin{aligned}
d(a,b) &= \|a-b\| \\
&= \|(a_1,\ldots,a_m)-(b_1,\ldots,b_m)\| \\
&= \left\|\sum_{i=1}^{m}(a_i-b_i)e_i\right\| \\
&\leq \sum_{i=1}^{m}|a_i-b_i|\|e_i\| \\
&\leq L\sum_{i=1}^{m}|a_i-b_i| \\
&\leq Lm\sqrt{\sum_{i=1}^{m}(a_i-b_i)^2} \\
&\leq Lmd_E(a,b)
\end{aligned}$$

so id is $Lm-$Lipschitz. id is therefore continuous. B is the closed unit ball in $(\mathbb{R}^m, d)$, so its preimage, B itself, is closed in $(\mathbb{R}^m, d_E)$.

2. Claim that B is bounded in $(\mathbb{R}^m, d_E)$.

2.1. Take $S = S^{m-1}(\mathbb{R}^m, d_E) \subset \mathbb{R}^m$. It is compact. Since id is continuous, $S = id(S)$ is compact in $(\mathbb{R}^m, d)$. Clearly, $0 \notin S$.

Claim that there exists $c > 0$ such that $d(u, 0) \geq c \; \forall \; u \in S$.

Suppose not. Then for all $n \in \mathbb{N}$, there exists $u_n \in S$ such that

$$d(u_n, 0) < \frac{1}{n}$$

(u_n) is a sequence in $(\mathbb{R}^m, d)$. Trivially, $u_n \xrightarrow{n\to\infty} 0$ in $(\mathbb{R}^m, d)$. However, since S is compact in $(\mathbb{R}^m, d)$, there exists a subsequence (u_{n_j}) that converges in S. But since $u_n \xrightarrow{n\to\infty} 0$, $u_{n_j} \xrightarrow{j\to\infty} 0$ too. But $0 \notin S$. $\Rightarrow\Leftarrow$

It follows that there does exist such a $c > 0$. Which means

$$\forall \; u \in S, \|u\| = d(u, 0) \geq c$$

2.2. For any $v \in B \subseteq \mathbb{R}^m$, let $w = \frac{1}{|v|}v$. Then $|w| = \frac{|v|}{|v|} = 1 \Rightarrow w \in S$. Then

$$\begin{aligned}
|v| &= \frac{\|v\|}{\|\frac{1}{|v|}v\|} \\
&= \frac{\|v\|}{\|w\|} \leq \frac{1}{c}\|v\| = \frac{1}{c}
\end{aligned}$$

Therefore B is bounded in in $(\mathbb{R}^m, d_E)$.

3. Since B is closed and bounded in $(\mathbb{R}^m, d_E)$, it follows that B is compact in $(\mathbb{R}^m, d_E)$ (H-B). id is continuous, so $B = id(B)$ is compact in $(\mathbb{R}^m, d)$. $\square$

Problem 5.4 (2.96 done)

If $A \subseteq B \subseteq C$, A is dense in B, B is dense in C, prove that A is dense in C.

Solution

Let $c \in C$ and $\varepsilon > 0$.

Since B is dense in C, there exists $b \in B$ such that $d(b, c) < \varepsilon/2$.

Since A is dense in B, there exists $a \in A$ such that $d(a, b) < \varepsilon/2$. Thus, we can always pick $a \in A$ satisfying:

$$d(a, c) \leq d(a, b) + d(b, c) < \varepsilon$$

Thus A is dense in C. $\square$

Problem 5.5 (3.37 done)

Suppose that $f : \mathbb{R} \to [-M, M]$ has no jump discontinuities. Does f have the intermediate value property? (Proof or counterexample)

Solution

No. Define

$$f(x) = \begin{cases} \sin(1/x) & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ 2 & \text{for } x < 0 \end{cases}$$

Then

1. $|\sin(1/x)|, |0|, |2| \leq 2 \Rightarrow |f| \leq 2$.
2. $f = \sin(1/x)$ is continuous on $x > 0$ and $f = 2$ is continuous on $x < 0$.
3. Claim that f is discontinuous at 0, but it is not a jump discontinuity. We want to show that $\lim_{x \to 0+} f(x)$ doesn't exist.

   Suppose that it does, that $\lim_{x \to 0+} f(x) = L$. Choose $\varepsilon = 1$, then there exists $\delta > 0$ such that $x \in (0, \delta) \Rightarrow |f(x) - L| < \varepsilon$.

   Choose $N \in \mathbb{N}$ sufficiently large such that $\frac{1}{N} < \delta$, then we can construct $x_1, x_2 \in (0, \delta)$

   $$x_1 = \frac{1}{2N\pi + \frac{\pi}{2}}, x_2 = \frac{1}{2N\pi - \frac{\pi}{2}}$$

   which yields

   $$f(x_1) = \sin(1/x_1) = 1, f(x_2) = \sin(1/x_2) = -1$$

   But

   $$2 = |f(x_1) - f(x_2)| \leq |f(x_1) - L| + |f(x_2) - L| < 2\varepsilon = 2 \Rightarrow\Leftarrow$$

   Therefore $\lim_{x \to 0+} f(x)$ doesn't exist.

   The non-existence of the right limit of f at 0 implies that it is a nonjump discontinuous. (A jump discontinuity requires both right and left limits to exist).

4. f does not have the intermediate value property. There doesn't exist $x_0 \in \mathbb{R}$ such that $f(x_0) = 1.5$.

$\square$

Problem 5.6 (4.34a done)

Consider the ODE $y' = 2\sqrt{|y|}$ where $y \in \mathbb{R}$. Show that there are many solutions to this ODE, all with the same initial condition $y(0) = 0$. Not only does $y(t) = 0$ solve the ODE, but also $y(t) = t^2$ does for $t \geq 0$.

Solution

WTS that every member of the family of functions

$$\mathcal{F} := \left\{ y_{a,b} : (-1,1) \to \mathbb{R}, y_{a,b}(x) = \begin{cases} -(t-a)^2 & \text{for } t \in (-1,a) \\ 0 & \text{for } t \in [a,b] \\ (t-b)^2 & \text{for } t \in (b,1) \end{cases} \middle| a \in (-1,0), b \in (0,1) \right\}$$

is a solution to the ODE $y' = 2\sqrt{|y|}$. Take any $y = y_{a,b}$. Then

1. $y(0) = 0$ by definition.
2. On $(-1,a), y' = 2(a-t) = 2\sqrt{(t-a)^2} = 2\sqrt{|y|}$
3. On $(b,1), y' = 2(t-b) = 2\sqrt{(t-b)^2} = 2\sqrt{|y|}$
4. On $(a,b), y' = 0 = 2\sqrt{|y|}$
5. We have
$$\lim_{t\to a-} \frac{y(t)-y(a)}{t-a} = \lim_{t\to a-} \frac{-(t-a)^2}{t-a} = \lim_{t\to a-} (a-t) = 0$$
$$\lim_{t\to a+} \frac{y(t)-y(a)}{t-a} = 0$$
It follows that $y'(a) = 0 = \sqrt{|y(a)|}$.
6. We have
$$\lim_{t\to b+} \frac{y(t)-y(b)}{t-b} = \lim_{t\to b+} \frac{(t-b)^2}{t-b} = \lim_{t\to b+} (t-b) = 0$$
$$\lim_{t\to b-} \frac{y(t)-y(b)}{t-b} = 0$$
It follows that $y'(b) = 0 = \sqrt{|y(b)|}$.

Therefore $y = y_{a,b}$ solves the ODE with initial condition $y(0) = 0$. $\square$

Problem 5.7 (5.1 done)

Let $T : V \to W$ be a linear transformation, and let $p \in V$ be given. Prove that the following are equivalent:

(a) T is continuous at the origin.

(b) T is continuous at p.

(c) T is continuous at at least one point of V.

Solution

WTS (c) implies (a). Then suppose T is continuous at some $q \in V$. Then for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|u - q| < \delta \Rightarrow |Tu - Tq| < \varepsilon$$

Use the same δ. Then

$$|v - 0| < \delta \Rightarrow |(q + v) - q| < \delta \Rightarrow |T(q + v) - Tq| < \varepsilon \Rightarrow |T(v)| < \varepsilon$$

And $T(0) = 0$ so that implies $|T(v) - T(0)| < \varepsilon$.

T is therefore continuous at the origin. Therefore (c) implies (a).

(b) implies (c) and (a) implies (c), since $0, p \in V$. By Theorem 2, (a) implies that f is continuous everywhere, which implies (b) and (c). $\square$

Problem 5.8 (5.4 done)

The **conorm** of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is

$$\mathfrak{m}(T) = \inf \left\{ \frac{|Tv|}{|v|} : v \neq 0 \right\}$$

It is the **minimum stretch** that T imparts to vectors in $\mathbb{R}^n$. Let U be the unit ball in $\mathbb{R}^n$.

- **(a)** Show that the norm and conorm of T are the radii of the smallest ball that contains TU and, when $n = m$, the largest ball contained in TU.
- **(b)** If T is an isomorphism, prove that $\mathfrak{m}(T) = \|T^{-1}\|^{-1}$.
- **(c)** If $m = n, T = I + S$, and $\|S\| < 1$, prove that $\mathfrak{m}(T) > 0$. [Hint: The inequality $|u + v| \geq |u| - |v|$ is useful because it implies $|Tu| \geq |u| - |Su|$.] How can you infer that T is an isomorphism?
- **(d)** If the norm and conorm of T are equal, what can you say about T?

Solution

Reiterate that U is the closed unit ball (Chapter 1):

$$U = \{v : |v| \leq 1\}$$

Observe that for a fixed $k > 0$, for all $v \in \mathbb{R}^n, v \neq 0$, there exists $u \in \mathbb{R}^n$ such that $|u| = k$ and

$$\frac{|Tv|}{|v|} = \frac{|Tu|}{|u|}$$

with

$$u := k\frac{v}{|v|}(\Rightarrow |u| = k|v|/|v| = k)$$

It follows that

$$\left\{ \frac{|Tv|}{|v|} : v \neq 0 \right\} = \left\{ \frac{|Tv|}{|v|} : v \in U \right\} = \left\{ \frac{|Tv|}{|v|} : |v| = 1 \right\} = \{|Tv| : |v| = 1\}$$

Intuitively, one can always project any $v \neq 0$ or $v \in U$ onto the unit sphere (and vice versa). T being a linear transformation keeps the quotient $\frac{|Tv|}{|v|}$ invariant.

Then,

$$\|T\| = \sup \left\{ \frac{|Tv|}{|v|} : v \neq 0 \right\} = \sup_{|u|=1} \{|Tu|\}$$

Similarly,

$$\mathfrak{m}(T) = \inf_{|u|=1} \{|Tu|\}$$

(a) Take $u \in \mathbb{R}^n$ with $|u| = 1$. Then $u, -u \in U$. Therefore any ball that contains TU has to have diameter:

$$diam \geq |Tu - (T(-u))| = 2|Tu|$$

Therefore the radius R of the smallest ball that contains TU has to satisfy:

$$R \geq \frac{1}{2} \sup_{|u|=1} \{2|Tu|\} = \sup_{|u|=1} \{|Tu|\} = \|T\|$$

Similarly, any ball that is contained in TU has to have diameter:

$$diam \leq 2|Tu|$$

so the radius r of the largest ball that is contained in TU has to satisfy:

$$r \leq \frac{1}{2} \inf_{|u|=1} \{2|Tu|\} = \inf_{|u|=1} \{|Tu|\} = \mathfrak{m}(T)$$

We show that equality can be achieved with closed ball B_1 of radius $\|T\|$ and closed ball B_2 of radius $\mathfrak{m}(T)$, centered at the origin, i.e., B_1 is sufficient to contain TU, and B_2 is sufficient to be contained in TU.

1. If $\|T\| = \infty \Rightarrow TU \subseteq \mathbb{R}^m = B_1$ and we're done.

If $\|T\| < \infty$, WTS

$$\sup_{|u|=1} \{|Tu|\} \geq \sup_{v \in U} \{|Tv|\}$$

For all $v \in U$, there exists $u = v/|v|$, which satisfies $|u| = 1$. Then

$$|Tu| = |Tv||u|/|v| = |Tv|/|v| \geq |Tv|$$

It follows that

$$\|T\| = \sup_{|u|=1} \{|Tu|\} \geq \sup_{v \in U} \{|Tv|\}$$

so B_1 is sufficient to contain TU. $\square$

2. If $m = n$, then $T : \mathbb{R}^n \to \mathbb{R}^n$.

Case 1: $\ker(T) \neq \{0\}$, i.e., T has a non-trivial kernel. That means there exists $w \in \mathbb{R}^n, w \neq 0$ such that $Tw = 0 \Rightarrow |Tw| = 0$. Then there exists $u = w/|w|$ with $|u| = 1 \Rightarrow |Tu| = 0$ too. Thus

$$\mathfrak{m}(T) = \inf_{|u|=1} \{|Tu|\} = 0$$

Ball B_2 with radius 0 is trivially contained in TU.

Case 2: $\ker(T) = \{0\}$, i.e., T has a trivial kernel. Then since $T : \mathbb{R}^n \to \mathbb{R}^n$, it is an isomorphism. Therefore, if w satisfies $|Tw| \leq \mathfrak{m}(T)$ (i.e., $w \in B_2$), then

$$|Tw| \leq \inf_{v \neq 0} \left\{ \frac{|Tv|}{|v|} \right\} \leq \frac{|Tw|}{|w|} \Rightarrow |w| \leq 1$$

therefore $w \in U \Rightarrow Tw \in TU \Rightarrow B_2 \subseteq TU$. $\square$

(b) T is an isomorphism. We first claim for a set $A = \{x : x > 0\}$ that

$$\inf A = 1/\sup\{1/x : x \in A\} =: 1/S$$

First, $S \geq 1/x \ \forall \ x \in A \Rightarrow 1/S \leq x \ \forall \ x \in A$ so $1/S$ is a lower bound of A. Suppose $\inf A < 1/S$ then there exists $y \in A : 0 \leq \inf A < y < 1/S$. Then

$$y < 1/S \Rightarrow 1/y > S \geq 1/y \Rightarrow\Leftarrow$$

and we are done with our claim. Since T is an isomorphism, $\ker(T) = \{0\} \Rightarrow |Tv|/|v| > 0 \ \forall \ v \neq 0$.

Then,

$$\begin{aligned}
\mathfrak{m}(T) &= \inf_{v \neq 0} \{|Tv|/|v|\} \\
&= \frac{1}{\sup_{v \neq 0}\{|v|/|Tv|\}} \\
&= \frac{1}{\sup_{w \neq 0}\{|T^{-1}w|/|w|\}} \\
&= \|T^{-1}\|^{-1} \quad \square
\end{aligned}$$

(c) Since $T = I + S$, we have

$$Tu = Iu + Su = u + Su \Rightarrow |u| = |Tu - Su| \leq |Tu| + |Su| \Rightarrow |Tu| \geq |u| - |Su|$$

Therefore,

$$\mathfrak{m}(T) = \inf_{|u|=1} \{|Tu|\} \geq \inf_{|u|=1} \{|u| - |Su|\} = 1 - \mathfrak{m}(S) \geq 1 - \|S\| > 0$$

as required.

Then if $\ker(T) \neq \{0\} \Rightarrow \ \exists \ v \neq 0 : Tv = 0 \Rightarrow \mathfrak{m}(T) = 0$, a contradiction. So $\ker(T) = \{0\}$.

$m = n \Rightarrow T$ is an isomorphism.

(d) $\|T\| = \mathfrak{m}(T) \Rightarrow \frac{|Tv|}{|v|} = c \in \mathbb{R} \;\forall\; v \neq 0$, i.e.

$$|Tv| = c|v|$$

T scales norm of all vectors with constant c. $\square$