Math 20250: Abstract Linear Algebra

Problem Set 3

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Textbook: Linear Algebra by Hoffman and Kunze (2nd Edition)

Problem 3.1 (Sec 2.4. Problem 4)
Let W be the subspace of $\mathbb{C}^3$ spanned by $\alpha_1 = (1, 0, i), \alpha_2 = (1 + i, 1, -1)$.

(a) Show that α_1 and α_2 form a basis for W

(b) Show that the vectors $\beta_1 = (1, 1, 0)$ and $\beta_2 = (1, i, 1 + i)$ are in W and form another basis for W

(c) What are the coordinates of α_1, α_2 in the ordered basis $\{\beta_1, \beta_2\}$ for W?

Solution (a) Suppose there exists c_1, c_2 s.t. $c_1\alpha_1 + c_2\alpha_2 = 0$

It follows that $c_1(1) + c_2(1 + i) = 0 + 0i \Rightarrow c_2 = 0 \Rightarrow c_1 = 0$.

α_1, α_2 are therefore linearly independent. Since they also span W, it can be concluded that they form a basis for W

(b) It can be observed that:

$$\beta_1 = (1, 1, 0) = -i(1, 0, i) + (1 + i, 1, -1) = -i\alpha_1 + \alpha_2$$
$$\beta_2 = (1, i, 1 + i) = (2 - i)(1, 0, i) + i(1 + i, 1, -1) = (2 - i)\alpha_1 + i\alpha_2$$

Therefore $\beta_1, \beta_2 \in W$.

We now prove that β_1, β_2 span W and are linearly independent.

First, we can rewrite α_1, α_2 as linear combinations of β_1, β_2:

$$\alpha_1 = \frac{1 - i}{2}\beta_1 + \frac{1 + i}{2}\beta_2, \alpha_2 = \frac{3 + i}{2}\beta_1 + \frac{-1 + i}{2}\beta_2$$

Since α_1, α_2 span W, it can be concluded that β_1, β_2 also span W.

Second, there exists c_1, c_2 s.t. $c_1\beta_1 + c_2\beta_2 = 0$, it follows that $c_1 + c_2 = 0, c_2(1 + i) = 0 \Rightarrow c_1 = c_2 = 0$. Thus, β_1, β_2 are linearly independent.

Therefore, β_1, β_2 form another basis for W.

(c) Following our expressions of α_1, α_2 as linear combinations of β_1, β_2, their coordinates in the ordered basis $\{\beta_1, \beta_2\}$ is

$$\alpha_1 : \left(\frac{1 - i}{2}, \frac{1 + i}{2}\right), \alpha_2 : \left(\frac{3 + i}{2}, \frac{-1 + i}{2}\right)$$

□

Problem 3.2 (Sec 3.1. Problem 1)
Which of the following functions T from $\mathbb{R}^2$ into $\mathbb{R}^2$ are linear transformations?

(a) $T(x_1, x_2) = (1 + x_1, x_2)$

(b) $T(x_1, x_2) = (x_2, x_1)$

(c) $T(x_1, x_2) = (x_1^2, x_2)$

(d) $T(x_1, x_2) = (\sin x_1, x_2)$

(e) $T(x_1, x_2) = (x_1 - x_2, 0)$

Solution (a) $T(x_1, x_2) = (1 + x_1, x_2)$

No. $T((0,0)) = (1+0,0) = (1,0) \neq 0 \in \mathbb{R}^2$

(b) $T(x_1, x_2) = (x_2, x_1)$

Yes. Let $\alpha = (x_1, x_2), \beta = (y_1, y_2)$ then:

$$\begin{aligned} T(c\alpha + \beta) &= T((cx_1 + y_1, cx_2 + y_2)) \\ &= (cx_2 + y_2, cx_1 + y_1) \\ &= c(x_2, x_1) + (y_2, y_1) \\ &= cT(\alpha) + T(\beta) \end{aligned}$$

(c) $T(x_1, x_2) = (x_1^2, x_2)$

No. Counter-example:

$T((1,0) + (-1,0)) = T((0,0)) = (0,0)$ while $T(1,0) + T(-1,0) = (1,0) + (1,0) = (2,0)$

(d) $T(x_1, x_2) = (\sin x_1, x_2)$

No. Counter-example:

$T((\frac{\pi}{2}, 0) + (\frac{3\pi}{2}, 0)) = T(2\pi, 0) = (0,0)$ while $T((\frac{\pi}{2}, 0)) + T((\frac{3\pi}{2}, 0)) = (1,0) + (1,0) = (2,0)$

(e) $T(x_1, x_2) = (x_1 - x_2, 0)$

Yes. Let $\alpha = (x_1, x_2), \beta = (y_1, y_2)$ then:

$$\begin{aligned} T(c\alpha + \beta) &= T((cx_1 + y_1, cx_2 + y_2)) \\ &= (cx_1 + y_1 - cx_2 - y_2, 0) \\ &= c(x_1 - x_2, 0) + (y_1 - y_2, 0) \\ &= cT(\alpha) + T(\beta) \end{aligned}$$

□

Problem 3.3 (Sec 3.2. Problem 2)

Let T be the (unique) linear operator on $\mathbb{C}^3$ for which:

$$\begin{aligned} T\varepsilon_1 &= (1, 0, i) \\ T\varepsilon_2 &= (0, 1, 1) \\ T\varepsilon_3 &= (i, 1, 0) \end{aligned}$$

Is T invertible?

Solution

From the textbook, T is invertible iff $\{T\varepsilon_1, T\varepsilon_2, T\varepsilon_3\}$ forms a basis for $\mathbb{C}^3$. However, it can be observed that they are not linearly independent:

$$\begin{aligned} i(1,0,i) + (0,1,1) - (i,1,0) &= (0,0,0) \\ \Rightarrow iT\varepsilon_1 + T\varepsilon_2 - T\varepsilon_3 &= 0 \in \mathbb{C}^3 \end{aligned}$$

It follows that $\{T\varepsilon_1, T\varepsilon_2, T\varepsilon_3\}$ does not form a basis for $\mathbb{C}^3$, and therefore T is not invertible. □

Problem 3.4 (Sec 3.2. Problem 7)

Find two linear operators T, U on $\mathbb{R}^2$ s.t. $TU = 0$ but $UT \neq 0$

Solution

We define operators T, U by their associated matrices:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

where for $X \in \mathbb{R}^{2\times 1}, T(X) = AX, U(X) = BX$. Then,

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ while } BA = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \neq 0$$

It follows that $TU(X) = ABX = 0$ but $UT \neq 0$ $\square$

Problem 3.5 (Sec 3.3. Problem 3)

Let W be the set of all 2×2 complex Hermitian matrices, that is, the set of 2×2 complex matrices A s.t. $A_{ij} = \overline{A_{ji}}$ (the bar denoting complex conjugation). As we pointed out in Example 6 of Chapter 2, W is a vector space over the field of *real* numbers, under the usual operations. Verify that

$$(x, y, z, t) \rightarrow \begin{bmatrix} t+x & y+iz \\ y-iz & t-x \end{bmatrix}$$

is an isomorphism of $\mathbb{R}^4$ onto W.

Solution

We first recognize that W is trivially isomorphic to $W' \subseteq \mathbb{C}^4 : (t+x, y+iz, y-iz, t-x)$. We now want to prove that there exists a linear transformation from $\mathbb{R}^4 \rightarrow W'$, that is characterized by an invertible T:

$$T \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} t+x \\ y+iz \\ y-iz \\ t-x \end{bmatrix}$$

This implies the characteristic matrix:

$$T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

Row-reducing T:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{(3)=\frac{(2)-(3)}{2i},(4)=\frac{1}{2}((1)+(4))} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & i & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{(2)=(2)-i(3),(1)=(1)-(4)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore T is invertible, implying that $\mathbb{R}^4$ is isomorphic to W', which is isomorphic to W. $\square$