207 Homework #7 Due Wednesday, Nov 15.

I. Read Pugh, Chapter 5, Sections 4 and 6 and Appendix E.

II. Chap 5: [30], 32 (Google "group"), 33.

III. Let $f\colon B^m(0,1) \to \mathbb{R}^m$ be a C^1 map with the property that for every $p \in \mathbb{R}^m$,

$$\det(J_p f) \neq 0.$$

(where $J_p f$ is the Jacobian matrix of f at p).

(a) Prove that if $\det(J_0 f) > 0$, then $\det(J_p f) > 0$, for all $p \in B^m(0,1)$.

(b) Prove that there exist $p, q \in B^m(0,1)$ such that $|f(p)| \neq |f(q)|$.

IV. Let $U \subset \mathbb{R}^n$, and let $f\colon U \to \mathbb{R}$ be a C^1 function. Let $F\colon U \to \mathbb{R}^n$ be the gradient vector field of f:

$$F(p) = \mathrm{grad}_p(f),$$

for $p \in U$.

(a) Suppose that f is twice differentiable at some $p \in U$, and let $H_p(f) \in \mathcal{M}_{n\times n}$ be the Jacobian matrix for F at p, with respect to the standard basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ (i.e., $H_p(f)$ is DF_p, expressed with respect to the standard basis of $\mathbb{R}^n$). Show that

$$(H_p(f))_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}(p).$$

The matrix $H_p(f)$ is called the *Hessian* of f at p.

(b) Show that, with respect to the standard basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$, we have

$$D^2 f_p(v, w) = v^t H_p(f) w,$$

for all $v, w \in \mathbb{R}^n$, and that $H_p(f)$ is a symmetric matrix.

(c) Show that

$$f(p+h) = f(p) + \langle \mathrm{grad}_p(f), h\rangle + \frac{1}{2} h^t H_p(f) h + R^{(2)}(p,h),$$

where

$$\lim_{h\to 0} \frac{R^{(2)}(p,h)}{|h|^2} = 0.$$

(d) We say that p is a *critical point of* f if f is differentiable at p and $F(p) = 0$. If p is a critical point of f, then

$$f(p+h) = f(p) + \frac{1}{2} h^t H_p(f) h + R^{(2)}(p,h), \tag{1}$$

for all h close to $0 \in \mathbb{R}^n$. We say that a critical point p is *non-degenerate* if f is twice differentiable at p, and $H_p(f)$ is invertible (or equivalently, $\det(H_p(f)) \neq 0$).
Find all critical points of the function

$$f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2,$$

and compute the Hessian at these points. Determine whether they are nondegenerate.

(e) Let $A \in \mathcal{M}_{n \times n}$. A nonzero vector $v \in \mathbb{R}^n$ is called a *real eigenvector* of A if there exists a real number $\lambda \in \mathbb{R}$ such that $Av = \lambda v$. The number λ is called a *real eigenvalue of* A. Prove that if A is a symmetric matrix (meaning $A^t = A$) and $v_1, v_2 \in \mathbb{R} \setminus \{0\}$ are real eigenvectors of A with eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$, respectively, then

$$\lambda_1 \neq \lambda_2 \implies \langle v_1, v_2 \rangle = 0;$$

that is, eigenvectors for different eigenvalues are orthogonal. (Hint: use the fact that $\langle v_1, v_2 \rangle = v_1^t v_2$).

(f) For the function f in part (d), characterize each nondegnerate critical point p:

(i) Is p a local maximum? If so, in what directions (moving away from p) does the function decrease most sharply (on an infinitesimal level)?

(ii) Is p a local minimum? If so, in what directions (moving away from p) does the function increase most sharply (on an infinitesimal level)?

(iii) If p is neither a local max nor a local min, determine the directions of maximal increase/decrease of f at p.

Note that you cannot use the gradient to answer these questions, since the gradient is 0 at these points. Instead, you will need to use equation (1) above, plus some linear algebra. If you have not seen eigenvectors and diagonalization before, talk to your linear algebra buddy!

V. By drawing pictures, maximize and minimize the following functions subject to the given constraints (without using calculus).

(a) $f(x, y) = xy$; constraint: $x^2 + y^2 = 1$.

(b) $f(x, y, z) = x^2 + 2y^2 + 3z^2$; constraint $x^2 + y^2 + z^2 = 1$

(c) Find the closest and furthest points on the sphere $x^2+y^2+z^2 = 36$ from the point $(1, 2, 2)$.

VI. Fix n, and let

$$\Delta = \{(p_1, \ldots, p_n) : p_i \geq 0, \text{ for } i = 1, \ldots n, \text{ and } \sum_{i=1}^{n} p_i = 1\}$$

be the space of probability vectors in $\mathbb{R}^n$. Define the *entropy function* $H\colon \Delta \to \mathbb{R}_{\geq 0}$ by the formula

$$H(p) = \sum_{i=1}^{n} \phi(p_i),$$

where $\phi(x) = -x \log x$, for $x \in (0, 1)$, and 0, for $x \in \{0, 1\}$.

(a) Using Lagrange multipliers, find the maximum value of H on Δ.

(b) Using convexity of the negative logarithm function, find the maximum value of H on Δ.

VII. The *triple product* $\alpha\colon \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is defined by the formula

$$\alpha(u, v, w) := \det(u\, v\, w),$$

where $(u\, v\, w)$ is the matrix whose columns are u, v and w. This exercise will make multiple use of the properties of determinants;

– the determinant is multilinear in the columns of A

– swapping two columns changes the sign of the determinant;

– the absolute value of $\det(A)$ is the volume of the (possibly degenerate) parallelepiped spanned by the columns of A (which is the volume of the image of the unit cube under A);

- the sign of $\det(A)$ is the orientation of the columns relative to the standard basis $e_1, \ldots, e_n$ (which for $n = 3$ is given by the "right hand rule" – google it!)

(a) Show that for all $v, w \in \mathbb{R}^3$, there exists a unique vector $z \in \mathbb{R}^3$ such that

$$\alpha(u, v, w) = \langle u, z \rangle,$$

for all $u \in \mathbb{R}^3$. Define the *cross product of* v *and* w to be this vector $v \times w := z$.

(b) Derive the following formula for $v \times w$:

$$v \times w = \det \begin{pmatrix} \mathbf{i} & v_1 & w_1 \\ \mathbf{j} & v_2 & w_2 \\ \mathbf{k} & v_3 & w_3 \end{pmatrix},$$

where the v_i's and w_i's are the components of v and w, respectively, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are symbolic placeholders for the coordinate vectors e_1, e_2, e_3. Compute the cross product of the vectors $(6, 1, 0), (0, 2, 2)$.

(c) [Warmup – don't turn in] Verify the following properties of the cross product:

- $v \times w$ is bilinear in v, w;
- $v \times w = -w \times v$ (*antisymmetry* of the cross-product);
- $e_1 \times e_2 = e_3$, $e_2 \times e_3 = e_1$, and $e_3 \times e_1 = e_2$;
- $u \times (v \times w) + v \times (w \times u) + w \times (u \times v) = 0$ (this is known as a *Jacobi identity*, and together with bilinearity and antisymmetry, shows that $\times$ is an example of a *Lie bracket*);
- $\langle v, v \times w \rangle = \langle w, v \times w \rangle = 0$, and hence $v \times w$ is orthogonal to both v and w;
- $|\langle u, v \times w \rangle|$ is the volume of the (possibly degenerate) parallelepiped spanned by u, v, w;
- $|v \times w|$ is the area of the parallelogram spanned by v and w in $\mathbb{R}^3$, and hence:

  $$|v \times w| = |v|\,|w| \sin(\theta),$$

  where $\theta \in (-\pi/2, \pi/2)$ is the angle between v and w;

– the orientation of $v \times w$ relative to v and w is determined by the "right hand rule."

(d) Prove the product formula for cross product: If $f, g\colon \mathbb{R}^n \to \mathbb{R}^3$ are differentiable at p, then for every $v \in \mathbb{R}^n$, we have

$$D(f \times g)_p(v) = f(p) \times Dg_p(v) + Df_p(v) \times g(p).$$

In particular, when $n = 1$, we have

$$(f \times g)'(t) = f(t) \times g'(t) + f'(t) \times g(t).$$

(Please make use of the general properties of derivative here).

(e) [Warmup – don't turn in] Show that if v, w span a plane P (through the origin) in $\mathbb{R}^3$, then

$$P = \{x \in \mathbb{R}^3 : \alpha(x, v, w) = \langle x, v \times w\rangle = 0\}.$$

Use this to find the equation $ax_1 + bx_2 + cx_3 = d$ of the plane through the point $(2, 1, 2)$ and spanned by the vectors $(6, 1, 0), (0, 2, 2)$.

(f) Find the equation of the tangent plane to the *parametrized surface* $\sigma\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\sigma(r, s) = (2r^3, rs^2, 2s),$$

through the point $\sigma(1, 1) = (2, 1, 2)$. Consult your MV Calc buddy if this seems completely foreign to you. We will be discussing parametrized surfaces very soon!

(g) Now using the gradient, find the equation of the tangent plane to the *implicitly-defined* surface $x^2 - xyz + z^2 = 1$ through the point $(1, 1, 1)$.