Math 20250
Abstract Linear Algebra

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Lecture 5

Span, Linear Independence, Basis

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Recall. Linear Combination: Let $V =$ $\mathbb{K}$-vector space with $v_1, v_2, \ldots, v_r \in V$ then

$$\mathbb{K}\langle v_1, v_2, \ldots, v_r\rangle \coloneqq \{w \in W \mid = w = a_1v_1 + \cdots + a_rv_r; a_i \in \mathbb{K}\} \subseteq V (\text{ is a subspace of } V)$$

Definition 5.1 (Span).
$\{v_1, v_2, \ldots, v_r\}$ span V if

$$\mathbb{K}\langle v_1, v_2, \ldots, v_r\rangle = V$$

i.e. equality is achieved: every vector in V can be written as linear combinations of $\{v_1, v_2, \ldots, v_r\}$

Connecting to the previous lecture, let $\psi : \mathbb{K}^r \to V$ then $\psi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^r, V) \xrightarrow{\sim} V^{\oplus r}$, i.e. ψ corresponds to $(v_1, v_2, \ldots, v_r)$ in V.
In particular, $(v_1, v_2, \ldots, v_r) \in V^{\oplus r}$ determines the map:

$$\begin{aligned}
\psi : (1, 0, \ldots, 0) \in \mathbb{K}^r &\to v_1 \\
(0, 1, \ldots, 0) \in \mathbb{K}^r &\to v_2 \\
&\vdots \\
(0, 0, \ldots, 1) \in \mathbb{K}^r &\to v_r \\
(\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{K}^r &\to \alpha_1v_1 + \alpha_2v_2 + \cdots + \alpha_rv_r
\end{aligned}$$

Lemma 5.1.

1. Let $\psi : \mathbb{K}^r \to V$ be a linear transformation determined by $v_1, v_2, \ldots, v_r \in V$, i.e. $\psi(\alpha_1, \alpha_2, \ldots, \alpha_r) \coloneqq \sum_{i=1}^r \alpha_iv_i$, then

$$\mathrm{im}(\psi) = \mathbb{K}\langle v_1, v_2, \ldots, v_r\rangle$$

   is a subspace of V

2. $\{v_1, v_2, \ldots, v_r\}$ span $V \Leftrightarrow \psi$ is surjective
   i.e. a surjection $\mathbb{K}^r \to V$ corresponds to r vectors $v_1, v_2, \ldots, v_r \in V$ that span V

Remark. V is finite dimensional when $\exists$ surjection $\mathbb{K}^d \to V$
$\Leftrightarrow$ $\exists\, d$ vectors $v_1, v_2, \ldots, v_r$ that span V.
Recall: $\dim V = \min\{r \in \mathbb{Z}_{\geq 0}$ such that $\exists$ surjective $\mathbb{K}^r \to V\}$.
Next, what does it mean for ψ to be injective?

Definition 5.2 (Linear Independence).
$v_1, v_2, \ldots, v_r \in V$ are **linearly independent** if

$$a_1v_1 + a_2v_2 + \cdots + a_rv_r = 0; a_i \in \mathbb{K} \Rightarrow a_1 = a_2 = \cdots = a_r = 0$$

i.e. there doesn't exist non-trivial relations between the vectors.

Example. In $\mathbb{R}^2$, (0, 1) and (0, 2) are not linearly independent because

$$(-2)(0,1) + (0,2) = (0,0)$$

But (0, 1) and (1,0) are linearly independent.

Consequentially, they are **linearly dependent** otherwise, i.e.

$$\exists\ a_i \text{ not all } 0 \text{ such that } \sum a_i v_i = 0$$

Lemma 5.2. Given $\psi : \mathbb{K}^r \to V$ corresponds to $v_1, v_2, \ldots, v_r$ then $v_1, v_2, \ldots, v_r$ are linearly independent if and only if ψ is injective

In order to prove the lemma above, we shall make use of a more convenient test for whether a map $\varphi : \mathbb{K}^r \to V$ is injective.

Lemma 5.3. Let $\varphi : V \to W$ be a linear transformation then φ is injective if and only if

$$\ker(\varphi) = \{0\} \subseteq V$$

Proof (Lemma 5.3).
$\boxed{\Rightarrow}$ We assume that φ is injective, want to show that $\ker(\varphi) = \{0\}$.
We know that $\varphi(0) = 0 \Rightarrow 0 \in \ker(\varphi)$ but since φ is injective, $\nexists v \neq 0 \in V$ such that $\varphi(v) = 0$.
It follows that $\ker(\varphi) = 0$
$\boxed{\Leftarrow}$ We want to show that $x, y \in V$ such that $\varphi(x) = \varphi(y) \Rightarrow x = y$
Since $\varphi(x - y) = \varphi(x + (-y)) = \varphi(x) - \varphi(y) = 0$, combined with $\ker(\varphi) = 0$

$$\Rightarrow x - y = 0 \Rightarrow x = y$$

□

Proof (Lemma 5.2).
Applying Lemma 5.3, we want to show: $\ker(\varphi) = 0$ iff $v_1, v_2, \ldots, v_r$ are linearly independent.
$\boxed{\Rightarrow}$ Suppose $\ker(\varphi) = \{0\}$ then want to show

$$a_1v_1 + a_2v_2 + \cdots + a_rv_r = 0 \Rightarrow a_i = 0\ \forall\ i$$

But $LHS = \varphi((a_1, a_2, \ldots, a_r)) \Rightarrow (a_1, a_2, \ldots, a_r) \in \ker(\varphi) \Rightarrow (a_1, a_2, \ldots, a_r) = 0$.
Therefore $a_i = 0\ \forall\ i$.
$\boxed{\Leftarrow}$ Suppose that $v_1, v_2, \ldots, v_r$ are linearly independent.
Then for $v \in \ker(\varphi) \Rightarrow \varphi(v) = 0$, with $v = (a_1, a_2, \ldots, a_r)$

$$\begin{aligned}\Rightarrow 0 &= \varphi(v)\\ &= \varphi((a_1, a_2, \ldots, a_r))\\ &= a_1v_1 + a_2v_2 + \cdots + a_rv_r\end{aligned}$$

But since $v_1, v_2, \ldots, v_r$ are linearly independent

$$\Rightarrow a_i = 0\ \forall\ i \Rightarrow v = 0 \Rightarrow \ker(\varphi) = 0$$

□

Corollary 5.1. If V has dimension d over $\mathbb{K}$ then there exists isomorphic $\varphi : \mathbb{K}^d \xrightarrow{\sim} V$ i.e. φ is a bijective linear transformation

Proof (Corollary). Since $d = \dim V$, by definition there exists surjective linear transformation $\pi : \mathbb{K}^d \twoheadrightarrow V$
We then claim that π is also injective.
Proving by contradiction, we suppose that π is not injective.
let $v_1, v_2, \ldots, v_d$ be the d vectors that correspond to π, i.e.

$$\pi((a_1, a_2, \ldots, a_d)) = a_1v_1 + \cdots + a_dv_d$$

By Lemma 5.2, π being not injective implies that $v_1, v_2, \ldots, v_d$ are linearly dependent.
i.e. there exists $b_1, b_2, \ldots, b_d \in \mathbb{K}$ not identically 0 such that

$$b_1v_1 + b_2v_2 + \cdots + b_dv_d = 0$$

WLOG, assume $b_1 \neq 0$.

$$\begin{aligned} \Rightarrow b_1v_1 &= -(b_2v_2 \ldots b_dv_d) \\ \Rightarrow v_1 &= -b^{-1}(b_2v_2 \ldots b_dv_d)(\,\exists\, b^{-1} \because b_1 \neq 0) \\ &= c_2v_2 + c_3v_3 + \cdots + c_dv_d \end{aligned}$$

We already know that since π is surjective, thus $v_1, v_2, \ldots, v_d$ span V. However, the above equality implies that $v_2, \ldots, v_d$ already span V!
It follows that there must exist a surjective linear transformation $\pi' : \mathbb{K}^{d-1} \twoheadrightarrow V$
$\Rightarrow\Leftarrow$, since $d = \min\{r \mid \exists \text{ surjective } \pi^r : \mathbb{K}^r \twoheadrightarrow V\}$
Therefore π is injective. It is already surjective, and therefore bijective, making it an isomorphism. $\square$

Recall. $\psi : \mathbb{K}^d \to V$ as determined by $v_1, v_2, \ldots, v_d$ is

1. **injective** when $v_1, v_2, \ldots, v_d$ are linearly independent
2. **surjective** when $v_1, v_2, \ldots, v_d$ span V

This naturally leads to our next definition.

Definition 5.3 (Basis).
$\{v_1, v_2, \ldots, v_r\}$ is called a **basis** of V if they span V and are linearly independent, i.e. $\psi_{(v_1, v_2, \ldots, v_r)} : \mathbb{K}^r \to V$ is an isomorphism.

Corollary 5.2. $\dim_{\mathbb{K}} V = d \Leftrightarrow \exists$ basis $\{v_1, v_2, \ldots, v_d\}$ for V

Corollary 5.3. If $\{v_1, v_2, \ldots, v_d\}$ and $\{w_1, w_2, \ldots, v_{d'}\}$ are basis for V then $d = d'$.