1. Let B be a Banach space.

   (a) Prove that if $T \in \mathcal{B}(B, B)$ and $\|I - T\| < 1$ where I is the identity operator, then T is invertible and in fact $\sum_{n=0}^{\infty}(I - T)^n$ converges in $\mathcal{B}(B, B)$ to T^{-1}.

   (b) Prove that the set of invertible operators is open in $\mathcal{B}(B, B)$.

2. Let V be a normed vector space and $W \subset V$ a proper closed subspace.

   (a) Prove that $\|v + W\| := \inf_{w \in W} \|v + w\|$ is a norm on V/W.

   (b) Prove that for any $\epsilon > 0$ there exists $v \in V$ such that $\|v\| = 1$ and $\|v + W\| \geq 1 - \epsilon$.
   Hint: Let $u \in V \backslash W$. Then $\|u + W\| > 0$ and there exists $w \in W$ such that $\|u + W\| \leq \|u + w\|$ and

   $$\|u + w\| \leq \|u + W\| + \epsilon \|u + W\|.$$

   Now consider $\frac{u+w}{\|u+w\|}$.

3. Let V be a Banach space and $W \subset V$ a proper closed subspace. Prove that V/W with the norm defined in problem 2 is a Banach space.

   Hint: Suppose that the series $\sum_n (v_n + W)$ is absolutely summable, i.e. $\sum_n \|v_n + W\|$ converges. We wish to prove that $\sum_n (v_n + W)$ converges in V/W. For each $n \in \mathbb{N}$, there exists $w_n \in W$ such that

   $$\|v_n + w_n\| \leq \|v_n + W\| + 2^{-n}.$$

   Then $\sum_n (v_n + w_n)$ is absolutely summable, and since V is a Banach space, there exists $v \in V$ such that $v = \sum_n (v_n + w_n)$. Prove that $v + W = \sum_n (v_n + W)$, i.e.

   $$\lim_{N \to \infty} \quad v + W - \sum_{n=1}^{N} (v_n + W) \quad = 0.$$

4. Suppose V and W are Banach spaces, $T \in \mathcal{B}(V, W)$ and recall the following subspaces

   $$\ker(T) = \{v \in V \mid Tv = 0\}, \quad \operatorname{range}(T) = \{Tv \in W \mid v \in V\}.$$

   (a) Prove that $\ker(T)$ is a closed subspace of V.

   (b) If V_1 and V_2 are normed linear spaces, we say a bijective linear operator $S : V_1 \to V_1$ is an *isomorphism* if $S \in \mathcal{B}(V_1, V_2)$ and $S^{-1} \in \mathcal{B}(V_2, V_1)$. We say V_1 and V_2 are *isomorphic* if there exists an isomorphism $S : V_1 \to V_2$.
   Prove that $V/\ker(T)$ is isomorphic to $\operatorname{range}(T)$ if and only if $\operatorname{range}(T)$ is closed.
   Hint: Consider the map $S : V/\ker T \to \operatorname{range}(T)$ given by

   $$S(v + \ker T) = Tv,$$

   and first show that S is a well-defined, bijective bounded linear operator.

5. The following exercise shows we cannot drop certain hypotheses in the closed graph theorem and open mapping theorem. Let

$$W = \left\{ a = \{a_k\}_k \mid \sum_k k|a_k| < \infty \right\},$$

equipped with the ℓ^1 norm.

   (a) Prove that W is a proper, dense subspace of ℓ^1 (hence, W is not complete).
   Hint: Show that if $b = \{b_k\}_k \in \ell^1$ and $\epsilon > 0$, then there exists $N \in \mathbb{N}$ such that if

   $$a := \{b_1, b_2, \ldots, b_N, 0, 0, \ldots\} \in W,$$

   then $\|a - b\|_1 < \epsilon$.

   (b) Define $T : W \to \ell^1$ by $(Ta)_k = ka_k$. Prove that the graph of T is closed but T is not bounded.

   (c) Let $S = T^{-1} : \ell^1 \to W$. Prove that S is bounded and surjective but is not an open mapping.