Math 20250: Abstract Linear Algebra
Problem Set 5

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Textbook: Linear Algebra by Hoffman and Kunze (2nd Edition)

Problem 5.1 (Sec 5.4. Problem 3)

An $n \times n$ matrix A over a field $\mathbb{F}$ is **skew-symmetric** if $A^t = -A$. If A is a skew-symmetric $n \times n$ matrix with complex entries and n is odd, prove that $\det A = 0$.

Solution

We want to show that $\det A = (-1)^n \det(-A)$. We will prove the statement by induction.
When $n = 1$, it's clear that $\det a = a = -(-a) = (-1)^1 \det(-a)$.
Suppose that the statement is true for all $m \leq n-1$, we want to prove that it is also true for n. Let $A \in \mathbb{M}_n(\mathbb{F})$ and $A(i|j)$ denote the $(n-1) \times (n-1)$ matrix that is constructed by removing the $i-$th row and $j-$th row of A. Then,

$$\begin{aligned}
\det A &= \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \det(A(1 \mid j)) \\
&= \sum_{j=1}^{n} (-1)^{1+j} (-1)(-A)_{1j} [(-1)^{n-1} \det((-A)(1 \mid j))] \\
&= (-1)^n \sum_{j=1}^{n} (-1)^{1+j} (-A)_{1j} \det((-A)(1 \mid j)) \\
&= (-1)^n \det(-A) \quad \square
\end{aligned}$$

Therefore, in this case, since n is odd, $\det A = -\det(-A)$. However, $\det A = \det(A^t) = \det(-A)$ since $A^t = -A$. It follows that $2 \det A = 0 \Rightarrow \det A = 0$. $\square$

Problem 5.2 (Sec 5.4. Problem 4)

An $n \times n$ matrix A over a field $\mathbb{F}$ is called **orthogonal** if $AA^t = I$. If A is orthogonal, show that $\det A = \pm 1$. Give an example of an orthogonal matrix for which $\det A = -1$.

Solution

$$\begin{aligned}
AA^t &= I \\
\Rightarrow \det A \det A^t &= \det I = 1 \\
\Rightarrow (\det A)^2 &= 1 \\
\Rightarrow \det A &= \pm 1
\end{aligned}$$

An example is $\begin{pmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$, whose determinant is $\dfrac{-1}{4} - \dfrac{3}{4} = -1$ □

Problem 5.3 (Sec 6.2. Problem 3)

Let A be an $n \times n$ triangular matrix over the field $\mathbb{F}$. Prove that the characteristic values of A are the diagonal entries of A, i.e. the scalars A_{ii}.

Solution

We first recall that the determinant of a triangular matrix is the product of the entries on its diagonal. Therefore, the characteristic polynomial of a triangular matrix A is

$$f = \det(xI_n - A) = \prod_{i=1}^{n}(x - A_{ii})$$

Since the eigenvalues of A are the roots of its characteristic polynomial, they are trivially A_{ii} □

Problem 5.4 (Sec 6.2. Problem 4)

Let T be the linear operator on $\mathbb{R}^3$ which is represented in the standard ordered basis by the matrix

$$\begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix}$$

Prove that T is diagonalizable by exhibiting a basis for $\mathbb{R}^3$, each vector of which is a characteristic vector of T.

Solution

We first find the eigenvalues of T. The characteristic polynomial is

$$\begin{aligned} \det(xI_3 - A) &= \begin{vmatrix} x+9 & -4 & -4 \\ 8 & x-3 & -4 \\ 16 & -8 & x-7 \end{vmatrix} \\ &= (x+9)((x-7)(x-3) - 32) - 8(-4(x-7) - 32) + 16(16 + 4(x-3)) \\ &= x^3 - x^2 - 5x - 3 \\ &= (x-3)(x+1)^2 \end{aligned}$$

Therefore 3 and -1 are eigenvalues for T.

To find eigenvectors for 3:

$$\begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix} X = 3X$$

$$\Rightarrow \begin{pmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{pmatrix} X = 0$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & \frac{-1}{2} \\ 0 & 1 & \frac{-1}{2} \end{pmatrix} X = 0$$

Therefore the eigenspace for 3 is spanned by $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

To find eigenvectors for -1:

$$\begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix} X = -X$$

$$\Rightarrow \begin{pmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{pmatrix} X = 0$$

$$\Rightarrow \begin{pmatrix} -2 & 1 & 1 \end{pmatrix} X = 0$$

Therefore the eigenspace for -1 is spanned by $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

It remains to show that $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\}$ form a basis for $\mathbb{R}^3$, but this is true since the bases of eigenspaces of different eigenvalues are linearly independent, and there are $3 = \dim(\mathbb{R}^3)$ vectors in $\mathcal{B}$. $\square$

Problem 5.5 (Sec 6.2. Problem 6)

Let T be the linear operator on $\mathbb{R}^4$ which is represented in the standard ordered basis by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix}$$

Under what conditions on a, b, c is T diagonalizable?

Solution

The characteristic polynomial of T is:

$$\begin{aligned}\det(xI_4 - A) &= \begin{vmatrix} x & 0 & 0 & 0 \\ -a & x & 0 & 0 \\ 0 & -b & x & 0 \\ 0 & 0 & -c & x \end{vmatrix} \\ &= x^4\end{aligned}$$

whose root is 0. Therefore, T only has 1 eigenvalue $c_1 = 0$. For T to be diagonalizable, $\dim W_1 = \dim \mathbb{R}^4 = 4$, where W_1 is the nullspace of $T - c_1 I_4$. Since $c_1 = 0$, $A - c_1 I_4 = A$. Therefore for $\dim W_1 = 4$, A must be the zero matrix, i.e. $a = b = c = 0$ $\square$