

MATH 20800: Honors Analysis in \mathbb{R}^n II

Problem Set 1

Hung Le Tran

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Textbook: Pugh, *Real Mathematical Analysis*

Problem 1.1 (5.57 done)

Show that $d : \Omega^k \rightarrow \Omega^{k+1}$ is a linear vector space homomorphism.

Solution

Let $\alpha, \beta \in \Omega^k; c \in \mathbb{R}$. WLOG, $\alpha = f dx_I, \beta = g dx_J$ where I, J are increasing k -tuples. The linearity of d for simple forms trivially implies linearity for general forms.^f

To show that d is a linear transformation, WTS $d(\alpha + c\beta) = d\alpha + cd\beta$.

If I, J are the same tuple, then:

$$\begin{aligned} d(\alpha + c\beta) &= d((f + cg)dx_I) \\ &= d(f + cg) \wedge dx_I \\ &= \left(\sum_{i=1}^n \frac{\partial(f + cg)}{\partial x_i}(x) dx_i \right) \wedge dx_I \\ &= \left(\sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) dx_i + \sum_{i=1}^n c \frac{\partial g}{\partial x_i}(x) dx_i \right) \wedge dx_I \\ &= \left(\sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) dx_i \right) \wedge dx_I + \left(\sum_{i=1}^n c \frac{\partial g}{\partial x_i}(x) dx_i \right) \wedge dx_I \quad (\text{wedge product distributes}) \\ &= \left(\sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) dx_i \right) \wedge dx_I + c \left(\sum_{i=1}^n \frac{\partial g}{\partial x_i}(x) dx_i \right) \wedge dx_I \\ &= df \wedge dx_I + cdg \wedge dx_I \\ &= d\alpha + cd\beta \end{aligned}$$

Otherwise,

$$\begin{aligned}
d(\alpha + c\beta) &= d(fdx_I + cgd x_J) \\
&= df \wedge dx_I + d(cg) \wedge dx_J \\
&= d\alpha + \left(\sum_{i=1}^n \frac{\partial(cg)}{\partial x_i}(x) dx_i \right) \wedge dx_J \\
&= d\alpha + \left(\sum_{i=1}^n c \frac{\partial g}{\partial x_i}(x) dx_i \right) \wedge dx_J \\
&= d\alpha + c \left(\sum_{i=1}^n \frac{\partial g}{\partial x_i}(x) dx_i \right) \wedge dx_J \\
&= d\alpha + cdg \wedge dx_J \\
&= d\alpha + cd\beta
\end{aligned}$$

□

Problem 1.2 (5.61 done)

Assume $d^2f = 0$ for all smooth functions f , and prove that $d^2\omega = 0$ for all smooth k -forms ω .

Solution

Let ω be any smooth k -form. If $k = 0$, then ω is a smooth function so $d^2\omega = 0$ by assumption. Otherwise, WLOG, let $\omega = f dx_I$ for some increasing k -tuple I . Then

$$\begin{aligned}
d^2\omega &= d(d\omega) = d(df \wedge dx_I) \\
&= d^2f \wedge dx_I + (-1)^{l+1} df \wedge d(dx_I) \\
&= (-1)^{l+1} df \wedge d(dx_I)
\end{aligned}$$

We compute

$$d(dx_I) = \left(\sum_{i=1}^n \frac{\partial(1)}{\partial x_i}(x) dx_i \right) \wedge dx_I = 0$$

Therefore $d^2\omega = (-1)^{l+1} df \wedge 0 = 0$ as required.

□

Problem 1.3 (5.62 done)

Does there exist a continuous mapping from the circle to itself that has no fixed-point? What about the 2-torus? The 2-sphere?

Solution

Yes. We note that there exists a center for each of the 3 shapes. Taking each point to its reflection across that center is a continuous mapping, and there is no fixed point for that mapping.

□

Problem 1.4 (5.63 done)

Show that a smooth map $T : U \rightarrow V$ induces a linear map of cohomology groups

$H^k(V) \rightarrow H^k(U)$ defined by

$$T^* : [\omega] \mapsto [T^*\omega]$$

Here, $[\omega]$ denotes the equivalence class of $\omega \in Z^k(V)$ in $H^k(V)$. The question amounts to showing that the pullback of a closed form ω is closed and that its cohomology class depends only on the cohomology class of ω .

Solution

1. WTS if $\omega \in Z^k(V)$ then $T^*\omega \in Z^k(U)$, so that the mapping T^* is indeed $H^k(V) \rightarrow H^k(U)$ and thus the cohomology class $[T^*\omega]$ is well-defined.

Let $\omega \in Z^k(V)$, which means $d\omega = 0$.

Then $dT^*\omega = T^*d\omega = T^*0 = 0$ so $dT^*\omega \in Z^k(U)$ as required.

2. WTS if $\omega_1 \in [\omega]$ then $T^*\omega_1 \in [T^*\omega]$, i.e., that the mapping between the equivalence classes is independent of representative.

Since $\omega_1 \in [\omega] \in H^k(V) = B^k(V)/Z_k(V)$, there exists $\lambda \in \Omega^{k-1}(V)$ such that

$$\omega_1 = \omega + d\lambda$$

Then

$$\begin{aligned} T^*\omega_1 &= T^*(\omega + d\lambda) \\ &= T^*\omega + T^*d\lambda \\ &= T^*\omega + dT^*\lambda \end{aligned}$$

But $T^*\omega \in Z^k(U)$, $dT^*\lambda \in B^k(U)$ so $T^*\omega + dT^*\lambda \in [T^*\omega]$. Thus $T^*\omega_1 \in [T^*\omega]$ as required.

The linearity of T^* on cohomology classes is trivial from the linearity of T^* on forms. \square

Problem 1.5 (Problem 1 done)

If ω and λ are k - and m -forms respectively, prove that

$$\omega \wedge \lambda = (-1)^{km} \lambda \wedge \omega$$

Solution

WLOG, let $\omega = f dx_I$, $\lambda = g dx_J$ where I and J are k - and m -tuples respectively.

Then

$$\begin{aligned} \omega \wedge \lambda &= (f dx_I) \wedge (g dx_J) \\ &= (fg) dx_{i_1} \wedge dx_{i_2} \dots \wedge dx_{i_k} \wedge dx_{j_1} \wedge dx_{j_2} \dots \wedge dx_{j_m} \\ &= (fg)(-1)^m dx_{i_1} \wedge dx_{i_2} \dots \wedge dx_{i_{k-1}} \wedge dx_{j_1} \wedge dx_{j_2} \dots \wedge dx_{j_m} \wedge dx_{i_k} \\ &\quad (\text{perform the same permutation } (k-1) \text{ more times}) \\ &= (fg)(-1)^{km} dx_J \wedge dx_I \\ &= (-1)^{km} (g dx_J) \wedge (f dx_I) = (-1)^{km} \lambda \wedge \omega \end{aligned}$$

The linearity of wedge product promotes the result for simple forms to general forms. \square

Problem 1.6 (Problem 2 done)

Consider the 1-form $\eta = \frac{xdy - ydx}{x^2 + y^2}$ on $\mathbb{R}^2 \setminus \{0\}$.

(a) Prove that $d\eta = 0$.

(b) Let $\gamma = (r \cos t, r \sin t)$ for some $r > 0$, and let Γ be C^1 -curve in $\mathbb{R}^2 \setminus \{0\}$ with parameter interval $[0, 2\pi]$ and $\Gamma(0) = \Gamma(2\pi)$, such that the intervals $[\gamma(t), \Gamma(t)]$ do not contain $(0, 0)$ for any $t \in [0, 2\pi]$.

Prove that

$$\int_{\Gamma} \eta = 2\pi$$

(Hint: For $t \in [0, 2\pi]$, $u \in [0, 1]$, define $\Phi(t, u) = (1 - u)\Gamma(t) + u\gamma(t)$. Then Φ is a 2-surface in $\mathbb{R}^2 \setminus \{0\}$ with domain $[0, 2\pi] \times [0, 1]$. Show $\partial\Phi = \Gamma - \gamma$. Deduce that $\int_{\Gamma} \eta = \int_{\gamma} \eta$ and compute $\int_{\gamma} \eta$.)

Solution

(a) Let $f(x, y) = \frac{-y}{x^2 + y^2}$, $g(x, y) = \frac{x}{x^2 + y^2}$ then $\eta = fdx + gdy$.

Then $f_y = g_x = \frac{y^2 - x^2}{(x^2 + y^2)^2}$, so

$$\begin{aligned} d\eta &= d(fdx + gdy) \\ &= df \wedge dx + dg \wedge dy \\ &= (f_x dx + f_y dy) \wedge dx + (g_x dx + g_y dy) \wedge dy \\ &= f_y dy \wedge dx + g_x dx \wedge dy \\ &= (g_x - f_y) dx \wedge dy = 0 \quad \square \end{aligned}$$

(b) $\partial\Phi$ is the 1-surface:

$$\begin{aligned} \partial\Phi(x) &= \delta^1\Phi(x) - \delta^2\Phi(x) \\ &= ((1 - x)\Gamma(2\pi) + x\gamma(2\pi)) - ((1 - x)\Gamma(0) + x\gamma(0)) \\ &\quad - (\gamma(x) - \Gamma(x)) \\ &= (1 - x)(\Gamma(2\pi) - \Gamma(0)) + x(\gamma(2\pi) - \gamma(0)) + \Gamma(x) - \gamma(x) \\ &= \Gamma(x) - \gamma(x) \end{aligned}$$

By Stokes' Theorem, since $d\eta = 0$, we therefore get:

$$0 = \int_{\Phi} d\eta = \int_{\partial\Phi} \eta = \int_{\Gamma - \gamma} \eta = \int_{\Gamma} \eta - \int_{\gamma} \eta$$

which implies

$$\int_{\Gamma} \eta = \int_{\gamma} \eta$$

We can compute:

$$\begin{aligned}\int_{\gamma} \eta &= \int_0^{2\pi} \frac{-r \sin t}{r^2} (-r \sin t) + \frac{r \cos t}{r^2} (r \cos t) dt \\ &= \int_0^{2\pi} 1 dt = 2\pi\end{aligned}$$

Therefore

$$\int_{\Gamma} \eta = \int_{\gamma} \eta = 2\pi$$

as required. □

Problem 1.7 (Problem 3 done)

Define ζ on $\mathbb{R}^3 \setminus \{0\}$ by

$$\zeta = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{r^3} \quad (r = (x^2 + y^2 + z^2)^{1/2}),$$

Let $D = [0, \pi] \times [0, 2\pi]$ and let Σ be the 2-surface in \mathbb{R}^3 defined on D given by

$$x = \sin u \cos v, y = \sin u \sin v, z = \cos u \quad (u \in [0, \pi], v \in [0, 2\pi])$$

(a) Prove $d\zeta = 0$ in $\mathbb{R}^3 \setminus \{0\}$

(b) Let S denote the restriction of Σ to $E \subset D$. Prove

$$\int_S \zeta = \int_E \sin u du dv$$

(c) Suppose g, h_1, h_2, h_3 are C^2 -functions on $[0, 1]$ and let $(x, y, z) = \Phi(s, t)$ be the 2-surface $x = g(t)h_1(s), y = g(t)h_2(s), z = g(t)h_3(s)$. Prove, using the definition of forms, that

$$\int_{\Phi} \zeta = 0$$

Solution

(a) We note that all basic 3-forms with overlapping indices are 0, so the only relevant basic 3-forms in this computation are those in which all 3 indices appear. Preliminarily,

$$\frac{\partial}{\partial x} \left(\frac{x}{(x^2 + y^2 + z^2)^{3/2}} \right) = \frac{y^2 + z^2 - 2x^2}{r^2}$$

and similarly for y, z . Then,

$$\begin{aligned}
d\zeta &= \frac{y^2 + z^2 - 2x^2}{r^2} dx \wedge dy \wedge dz + \frac{z^2 + x^2 - 2y^2}{r^2} dy \wedge dz \wedge dx \\
&\quad + \frac{x^2 + y^2 - 2z^2}{r^2} dz \wedge dx \wedge dy \\
&= \left(\frac{y^2 + z^2 - 2x^2}{r^2} + \frac{z^2 + x^2 - 2y^2}{r^2} + \frac{x^2 + y^2 - 2z^2}{r^2} \right) dx \wedge dy \wedge dz \\
&= 0 \quad \square
\end{aligned}$$

(b) Since $x = \sin u \cos v, y = \sin u \sin v, z = \cos u$, we can preliminarily compute:

$$\begin{aligned}
\frac{\partial \Sigma_{(2,3)}}{\partial(u,v)} &= \det \begin{bmatrix} \cos u \sin v & \sin u \cos v \\ -\sin u & 0 \end{bmatrix} = \sin^2 u \cos v \\
\frac{\partial \Sigma_{(3,1)}}{\partial(u,v)} &= \det \begin{bmatrix} -\sin u & 0 \\ \cos u \cos v & -\sin u \sin v \end{bmatrix} = \sin^2 u \sin v \\
\frac{\partial \Sigma_{(1,2)}}{\partial(u,v)} &= \det \begin{bmatrix} \cos u \cos v & -\sin u \sin v \\ \cos u \sin v & \sin u \cos v \end{bmatrix} = \sin u \cos u (\cos^2 v + \sin^2 v) = \sin u \cos u
\end{aligned}$$

therefore

$$\begin{aligned}
\int_S \zeta &= \int_S \frac{x}{r^3} dy \wedge dz + \frac{y}{r^3} dz \wedge dx + \frac{z}{r^3} dx \wedge dy \\
&= \int_E \left(\sin u \cos v \frac{\partial \Sigma_{(2,3)}}{\partial(u,v)} + \sin u \sin v \frac{\partial \Sigma_{(3,1)}}{\partial(u,v)} + \cos u \frac{\partial \Sigma_{(1,2)}}{\partial(u,v)} \right) du dv \\
&= \int_E (\sin u \cos v \sin^2 u \cos v + \sin u \sin v \sin^2 u \sin v + \cos u \sin u \cos v) du dv \\
&= \int_E (\sin^3 u (\cos^2 v + \sin^2 v) + \cos^2 u \sin u) du dv \\
&= \int_E (\sin u (\sin^2 u + \cos^2 u)) du dv \\
&= \int_E \sin u du dv
\end{aligned}$$

as required. \square

(c) Restate that Φ is the 2-surface in \mathbb{R}^3 , mapping $(s, t) \mapsto (g(t)h_1(s), g(t)h_2(s), g(t)h_3(s))$.

First computing the Jacobians (abuse of notation: suppressing the arguments for con-

cisiness):

$$\begin{aligned}\frac{\partial \Phi_{(2,3)}}{\partial(u,v)} &= \det \begin{bmatrix} gh'_2 & g'h_2 \\ gh'_3 & g'h_3 \end{bmatrix} = gg'(h_3h'_2 - h_2h'_3) \\ \frac{\partial \Phi_{(3,1)}}{\partial(u,v)} &= \det \begin{bmatrix} gh'_3 & g'h_3 \\ gh'_1 & g'h_1 \end{bmatrix} = gg'(h_1h'_3 - h_3h'_1) \\ \frac{\partial \Phi_{(1,2)}}{\partial(u,v)} &= \det \begin{bmatrix} gh'_1 & g'h_1 \\ gh'_2 & g'h_2 \end{bmatrix} = gg'(h_2h'_1 - h_1h'_2) \\ r^3 &= (g^2(h_1^2 + h_2^2 + h_3^2))^{3/2}\end{aligned}$$

therefore

$$\begin{aligned}\int_{\Phi} \zeta &= \int_{I^2} \frac{1}{(g^2(h_1^2 + h_2^2 + h_3^2))^{3/2}} \left(gh_1 \frac{\partial \Phi_{(2,3)}}{\partial(u,v)} + gh_2 \frac{\partial \Phi_{(3,1)}}{\partial(u,v)} + gh_3 \frac{\partial \Phi_{(1,2)}}{\partial(u,v)} \right) dudv \\ &= \int_{I^2} \frac{gh_1gg'(h_3h'_2 - h_2h'_3) + gh_2gg'(h_1h'_3 - h_3h'_1) + gh_3gg'(h_2h'_1 - h_1h'_2)}{(g^2(h_1^2 + h_2^2 + h_3^2))^{3/2}} dudv \\ &= \int_{I^2} \frac{g^2g'(h_1h_3h'_2 - h_1h_2h'_3 + h_2h_1h'_3 - h_2h_3h'_1 + h_3h_2h'_1 - h_3h_1h'_2)}{(g^2(h_1^2 + h_2^2 + h_3^2))^{3/2}} dudv \\ &= \int_{I^2} 0 dudv = 0\end{aligned}$$

□