MATH 20800: Honors Analysis in Rn II
Problem Set 4

Hung Le Tran

14 Feb 2024

Problem 4.1 (done)

(a) (Hölder's inequality) Suppose that $n \in \mathbb{N}$, and let $a_k, b_k \in \mathbb{N}, 1 \leq k \leq n$. Prove that if $1 < p < \infty$ and $1/p + 1/q = 1$ then

$$\sum_{k=1}^{n} |a_k b_k| \leq \left[\sum_{k=1}^{n} |a_k|^p\right]^{1/p} \left[\sum_{k=1}^{n} |b_k|^q\right]^{1/q}$$

Hint: Prove that if $A, B > 0$ and $t \in (0,1)$ then $A^t B^{1-t} \leq tA + (1-t)B$ by showing the function

$$f(x) \coloneqq tx + (1-t)B - x^t B^{1-t}, \quad x > 0$$

has a minimum at $x = B$.

(b) (Minkowski's inequality) Suppose that $n \in \mathbb{N}$, and let $a_k, b_k \in \mathbb{R}, 1 \leq k \leq n$. Prove that if $1 \leq p \leq \infty$ then

$$\left[\sum_{k=1}^{n} |a_k + b_k|^p\right]^{1/p} \leq \left[\sum_{k=1}^{n} |a_k|^p\right]^{1/p} + \left[\sum_{k=1}^{n} |b_k|^p\right]^{1/p}$$

Hint: By the triangle inequality

$$\sum_{k=1}^{n} |a_k + b_k|^p \leq \sum_{k=1}^{n} |a_k||a_k + b_k|^{p-1} + \sum_{k=1}^{n} |b_k||a_k + b_k|^{p-1}$$

Now apply Hölder's inequality.

Solution

(a) We first prove that for $A, B > 0$ and $t \in (0,1)$, we have

$$A^t B^{1-t} \leq tA + (1-t)B$$

Indeed, if we define $f(x) \coloneqq tx + (1-t)B - x^t B^{1-t}$ on $(0, \infty)$ then we find the critical point x_0:

$$f'(x_0) = t - t x_0^{t-1} B^{1-t}$$
$$\Rightarrow x_0 = B$$

and it is a minimum point since

$$f''(x_0) = -t(t-1)B^{t-2}B^{1-t} > 0$$

Therefore for any $A > 0$, we have that $f(A) \geq f(B) = 0 \Rightarrow A^t B^{1-t} \leq tA + (1-t)B$.

To prove Holder's inequality, take divide both sides by the RHS, then we have:

$$\begin{aligned}
\sum_{k=1}^{n}\left[\frac{|a_k|^p}{\sum_{k=1}^{n}|a_k|^p}\right]^{1/p}\left[\frac{|b_k|^q}{\sum_{k=1}^{n}|b_k|^q}\right]^{1/q} &\le \sum_{k=1}^{n}\frac{1}{p}\left[\frac{|a_k|^p}{\sum_{k=1}^{n}|a_k|^p}\right]+\frac{1}{q}\left[\frac{|b_k|^q}{\sum_{k=1}^{n}|b_k|^q}\right] \\
&= \frac{1}{p}\sum_{k=1}^{n}\left[\frac{|a_k|^p}{\sum_{k=1}^{n}|a_k|^p}\right]+\frac{1}{q}\sum_{k=1}^{n}\left[\frac{|b_k|^q}{\sum_{k=1}^{n}|b_k|^q}\right] \\
&= 1
\end{aligned}$$

hence LHS $\le$ RHS as required.

(b) Set $q = \frac{p}{p-1}$ then $1/p + 1/q = 1$.

We have, first by triangle inequality then Holder's on $\frac{1}{p} + \frac{1}{q} = 1$ then

$$\begin{aligned}
\sum_{k=1}^{n}|a_k+b_k|^p &\le \sum_{k=1}^{n}|a_k||a_k+b_k|^{p-1}+\sum_{k=1}^{n}|b_k||a_k+b_k|^{p-1} \\
&\le \left(\left[\sum_{k=1}^{n}|a_k|^p\right]^{1/p}+\left[\sum_{k=1}^{n}|b_k|^p\right]^{1/p}\right)\left[\sum_{k=1}^{n}|a_k+b_k|^{(p-1)q}\right]^{1/q} \\
&\le \left(\left[\sum_{k=1}^{n}|a_k|^p\right]^{1/p}+\left[\sum_{k=1}^{n}|b_k|^p\right]^{1/p}\right)\left[\sum_{k=1}^{n}|a_k+b_k|^{p}\right]^{1-1/p} \\
\Rightarrow \left[\sum_{k=1}^{n}|a_k+b_k|^p\right]^{1/p} &\le \left[\sum_{k=1}^{n}|a_k|^p\right]^{1/p}+\left[\sum_{k=1}^{n}|b_k|^p\right]^{1/p}
\end{aligned}$$

as required. □

Problem 4.2 (done)

Prove that if $1 \le p < \infty$, then ℓ^p is a Banach space (you must show it is a normed space and it is complete)

Solution

Let us first have the definition of the ℓ^p space:

$$\ell^p = \left\{ a = (a_1, a_2, \ldots) \mid a_k \in \mathbb{C}, \left(\sum_{k=1}^{\infty}|a_k|^p\right)^{1/p} < \infty \right\}$$

Define the norm $\|a\| = \|a\|_p = \left(\sum_{k=1}^{\infty}|a_k|^p\right)^{1/p}$.

To show that ℓ^p is a normed space, we have to show that it is first a vector space (over $\mathbb{C}$), and the norm $\|\cdot\|$ as above is indeed a norm.

- On this space, define addition and scalar multiplication as pointwise addition and pointwise scalar multiplication. Then $(0) = (0, \cdots) \in \ell^p$ is trivially the identity.
- If $a, b \in \ell^p; \lambda \in \mathbb{C}$ then $a + \lambda b = (a_1 + \lambda b_1, \cdots)$ has:

  $$\left[\sum_{k=1}^{\infty}|a_k+\lambda b_k|^p\right]^{1/p} \le \left[\sum_{k=1}^{\infty}|a_k|^p\right]^{1/p}+\left[\sum_{k=1}^{\infty}|\lambda b_k|^p\right]^{1/p} < \infty$$

  by Minkowski's (apply for n, then $n \to \infty$ implies LHS converges and is thus well-defined) hence $a + \lambda b \in \ell^p$ too.
- $|a_k| \ge 0 \;\forall\, k \Rightarrow \|a\| \ge 0 \;\forall\, a$

- $\|a\| = 0 \Rightarrow \sum_{k=1}^{\infty}|a_k|^p = 0 \Rightarrow a_k = 0 \,\forall\, k \Rightarrow a = 0$
- Triangle inequality: Minkowski's tells us that

$$\left[\sum_{k=1}^{n}|a_k + b_k|^p\right]^{1/p} \leq \left[\sum_{k=1}^{n}|a_k|^p\right]^{1/p} + \left[\sum_{k=1}^{n}|b_k|^p\right]^{1/p} \leq \left[\sum_{k=1}^{\infty}|a_k|^p\right]^{1/p} + \left[\sum_{k=1}^{\infty}|b_k|^p\right]^{1/p} = \|a\|+\|b\| < \infty$$

  The series on the LHS then is monotonic increasing in n and bounded above and so converges, so $\|a+b\|$. When taken to the limit, the inequality still holds, so $\|a+b\| \leq \|a\| + \|b\|$.

It remains to show that that ℓ^p is complete (with respect to norm $\|\cdot\|$).

Take a Cauchy sequence $\{a^{(i)}\}_{i\in\mathbb{N}} \subset \ell^p$. Fix $\varepsilon > 0$, then there exists $N = N_\varepsilon \in \mathbb{N}$ such that $i, j \geq N$ implies

$$\|a^{(i)} - a^{(j)}\| < \varepsilon$$

Write out $a^{(i)} - a^{(j)} = (a_1^{(i)} - a_1^{(j)}, a_2^{(i)} - a_2^{(j)}, \dots)$ then it follows that for all $i, j \geq N; k \in \mathbb{N}$, we have

$$|a_k^{(i)} - a_k^{(j)}| \leq \|a^{(i)} - a^{(j)}\| < \varepsilon$$

so for each $k \in \mathbb{N}$, the sequence $\{a_k^{(i)}\}_{i\in\mathbb{N}} \subset \mathbb{C}$ is Cauchy. $\mathbb{C}$ is complete, so $a_k^{(i)} \xrightarrow{i\to\infty} b_k \in \mathbb{C}$. Define $b = (b_k)_{k\in\mathbb{N}}$. Then WTS $b \in \ell^p$ and $a^{(i)} \xrightarrow{i\to\infty} b$.

We first show that $\|a^{(i)} - b\|_p \xrightarrow{i\to\infty} 0$. A priori, this "norm" might not exist, but by showing that it gets arbitrarily small, we in the process also show that it is well-defined.

We know that $\{a^{(i)}\}_{i\in\mathbb{N}}$ in Cauchy wrt $\|\|_p$, so for any $n \in \mathbb{N}$, we have that for all $i, j \geq N$,

$$\sum_{k=1}^{n}|a_k^{(i)} - a_k^{(j)}|^p < \varepsilon^p$$

Let $j \to \infty$, then

$$\sum_{k=1}^{n}|a_k^{(i)} - b_k|^p \leq \varepsilon^p$$

This holds for all $n \in \mathbb{N}$, so it follows that

$$\sum_{k=1}^{\infty}|a_k^{(i)} - b_k|^p \leq \varepsilon^p$$

since the sequence of partial sums is increasing and bounded. It follows that for all $i \geq N$,

$$\|a^{(i)} - b\|_p \xrightarrow{i\to\infty} 0$$

It remains to show that $b \in \ell^p$. The triangle inequality then implies that

$$\|b\|_p \leq \|a^{(i)}\|_p + \|a^{(i)} - b\|_p < \infty$$

for i sufficiently large ($\geq N$), so $b \in \ell^p$.

Hence $a^{(i)} \xrightarrow{i\to\infty} b \in \ell^p$, so ℓ^p is indeed a complete normed vector space, i.e., a Banach space. □

Problem 4.3 (done)

The set of all bounded sequences, ℓ^∞, can be identified with $C_\infty(\mathbb{N})$, the set of all bounded continuous functions on the metric space $(\mathbb{N}, d_{disc})$ where d_{disc} is the discrete metric. Thus, ℓ^∞ is a Banach space. Prove that

$$c_0 := \{\{a_k\}_k \in \ell^\infty \mid \lim_{k\to\infty} a_k = 0\}$$

is a closed subspace of ℓ^∞ (and is thus, a Banach space).

Solution

Take sequence $\{a^{(i)}\}_{i\in\mathbb{N}} \subset c_0$ such that $a^{(i)} \xrightarrow{i\to\infty} b \in \ell^\infty$. WTS $b \in c_0$.

To show that $b \in c_0$, we show that

$$\lim_{k\to\infty} b_k = 0$$

Fix $\varepsilon > 0$. Since $a^{(i)} \xrightarrow{i\to\infty} b$, there exists $N = N_\varepsilon \in \mathbb{N}$ such that $i \geq N$ implies

$$\|a^{(i)} - b\|_\infty < \varepsilon/2$$

In particular, we have that

$$\|a^{(N)} - b\|_\infty < \varepsilon/2$$

Since $a^{(N)} \in c_0$, there exists $K = K_{N,\varepsilon} = K_\varepsilon$ such that

$$k \geq K \Rightarrow |a_k^{(N)}| < \varepsilon/2$$

It then follows that for $k \geq K$, we have

$$|b_k| \leq |b_k - a_k^{(N)}| + |a_k^{(N)}| \leq \|a^{(N)} - b\|_\infty + |a_k^{(N)}| < \varepsilon$$

hence $\lim_{k\to\infty} b_k = 0$ as required.

It follows that c_0 is closed. □

Problem 4.4 (done)

Let $1 \leq p \leq \infty$ and

$$S := \{a = \{a_k\}_k \in \ell^p \mid \|a\|_p = 1\}$$

- **(a)** Prove that S is a closed subset of ℓ^p.
- **(b)** Prove that S is not compact. Hint: Let $e_n := \{\delta_{kn}\}_k \in S$ where δ_{kn} is the Kronecker delta. Show that $\{e_n\}_n$ does not have a convergent subsequence in S.

Solution

(a) Note that the norm as a function from a normed vector space to $\mathbb{R}$ is always continuous, since it is 1-Lipschitz.

In this case, $\|\cdot\|_p : \ell^p \to \mathbb{R}$ is therefore continuous. It follows that

$$S = \|\cdot\|_p^{-1}(\{1\})$$

is closed in ℓ^p since $\{1\}$ is closed in $\mathbb{R}$.

(b) To show that S is not compact, we demonstrate a sequence in S that has does not have a convergent subsequence in S.

For each $n \in \mathbb{N}$, let $e_n = \{\delta_{kn}\}_{k\in\mathbb{N}}$, i.e., e_n is the sequence of all zeros except for 1 at its nth index. Clearly, $e_n \in S \; \forall \; n \in \mathbb{N}$.

Suppose that $\{e_n\}_{n\in\mathbb{N}}$ has a convergent subsequence $\{e_{n_j}\}_{j\in\mathbb{N}}$ that converges to some $a = \{a_k\}_{k\in\mathbb{N}} \in S$.

For $p = \infty$, then the limit is the pointwise limit, so, $a_k = \lim_{j\to\infty} e_{n_j}[k] = 0$. But then $\|a\| = 0 \neq 1, \Rightarrow\Leftarrow$.

We then now consider only $1 \leq p < \infty$. Then for $\varepsilon = 0.1$, there exists some N such that $j \geq J$ implies

$$\|e_{n_j} - a\|_p < \varepsilon$$

Then for all $j \geq J$,

$$\begin{aligned}\varepsilon^p > \text{LHS}^p &= \sum_{k=1}^{\infty} |a_k - \delta_{kn_j}|^p \\ &= \|a\|_p^p + (|a_{n_j} - 1|^p - |a_{n_j}|^p) \\ &= 1 + |a_{n_j} - 1|^p - |a_{n_j}|^p\end{aligned}$$

It follows that

$$|a_{n_j}|^p - |a_{n_j} - 1|^p > 1 - \varepsilon^p > 0 \Rightarrow |a_{n_j}| > |a_{n_j} - 1| \geq 1 - |a_{n_j}|$$

therefore

$$|a_{n_j}| \geq 1/2$$

This is true for all $j \geq J$, so

$$1 = \|a\|_p^p \geq \sum_{j=J}^{J+\lceil 3^p \rceil} |a_{n_j}|^p \geq 3^p \frac{1}{2^p} > 1, \Rightarrow\Leftarrow$$

Therefore, for both cases of $p = \infty$ and $1 \leq p < \infty$, there exists a sequence in S that does not have a convergent subsequence. So S is not compact. □

Problem 4.5 (done)

Let $1 \leq p < \infty$ and $1/p + 1/q = 1$.

(a) Prove that if $a = \{a_k\}_k \in \ell^p$ and $b = \{b_k\}_k \in \ell^q$ then

$$\sum_{k=1}^{\infty} |a_k b_k| \leq \|a\|_p \|b\|_q$$

(b) Let $b \in \ell^q$. Prove that $F_b : \ell^p \to \mathbb{C}$ defined via

$$F_b(a) := \sum_{k=1}^{\infty} a_k b_k, \quad a \in \ell^p,$$

is an element of $(\ell^p)^*$, the dual space of ℓ^p, and $\|F_b\| = \|b\|_{\ell^q}$.

(c) Prove that $F : \ell^q \to (\ell^p)^*, b \mapsto F_b$ is a bijective bounded linear operator.

Solution

(a) From Holder's, we know that

$$\sum_{k=1}^{n} |a_k b_k| \leq \left[\sum_{k=1}^{n} |a_k|^p\right]^{1/p} \left[\sum_{k=1}^{n} |b_k|^q\right]^{1/q} \leq \|a\|_p \|b\|_q$$

The partial sums are monotonically increasing and bounded above, so they converge and the limit is bounded by the same upper bound, hence

$$\sum_{k=1}^{\infty} |a_k b_k| \leq \|a\|_p \|b\|_q$$

as required. □

(b) We have $F_b : \ell^p \to \mathbb{C}$ with definition

$$F_b(a) := \sum_{k=1}^{\infty} a_k b_k$$

To show that $F_b \in (\ell^p)^*$, we have to show that it is a bounded linear functional on ℓ^p.

To show linearity, take any $\alpha, \beta \in \ell^p, \lambda \in \mathbb{C}$ then

$$\begin{aligned} F_b(\alpha + \lambda\beta) &= \sum_{k=1}^{\infty}((\alpha + \lambda\beta)_k b_k) \\ &= \sum_{k=1}^{\infty}(\alpha_k + \lambda\beta_k)b_k \\ &= F_b(\alpha) + \lambda F_b(\beta) \end{aligned}$$

so it is indeed linear. It is also bounded, since

$$|F_b(a)| \leq \sum_{k=1}^{\infty}|a_k b_k| \leq \|a\|_p \|b\|_q$$

so $\|F_b\| \leq \|b\|_q < \infty$. It follows that $F_b \in (\ell^p)^*$.

We've shown that $\|F_b\| \leq \|b\|_q$. To show equality, we exhibit a particular a such that $|F_b(a)| = \|a\|_p\|b\|_q$. The crux lies in that we construct a such that the equality in Holder's inequality holds:

$$\forall\ k \in \mathbb{N}, \frac{|a_k|^p}{\|a\|_p^p} = \frac{|b_k|^q}{\|b\|_q^q}$$

so that

$$\sum_{k=1}^{\infty}|a_k b_k| = \|a\|_p\|b\|_q$$

We also want $|F_b(a)| = |\sum_{k=1}^{\infty} a_k b_k| = \sum_{k=1}^{\infty}|a_k b_k|$, so we choose $a_k = c_k\overline{b_k}$ where $\overline{b_k}$ is the complex conjugate of b_k, and c_k real, nonnegative. It would then follow that

$$|F_b(a)| = \left|\sum_{k=1}^{\infty} c_k|b_k|^2\right| = \sum_{k=1}^{\infty} c_k|b_k|^2 = \sum_{k=1}^{\infty}|a_k b_k|$$

so that $|F_b(a)| = \|a\|_p\|b\|_q$, forcing $\|F_b\| = \|b\|_q$.

It remains for us to show a choice of $\{c_k\}$ so that $a \in \ell^p$ and satisfies the equal conditions of Holder's inequality so that all statements (especially those regarding convergence) are valid. Indeed, if

$$c_k := |b_k|^{(q-p)/p} \geq 0$$

then

$$\begin{aligned} \|a\|_p^p &= \sum_{k=1}^{\infty}\left(|b_k|^{(q-p)}|\overline{b_k}|^p\right) \\ &= \sum_{k=1}^{\infty}|b_k|^q = \|b\|_q^q \end{aligned}$$

so $a \in \ell^p$ and for all $k \in \mathbb{N}$:

$$\frac{|a_k|^p}{\|a\|_p^p} = \frac{|b_k|^{q-p}|\overline{b_k}|^p}{\|b\|_q^q} = \frac{|b_k|^q}{\|b\|_q^q}$$

as required, and we're done. □

(c) Consider $F : \ell^q \to (\ell^p)^*, b \mapsto F(b) = F_b$.

We proved from above that $F_b \in (\ell^p)^*$ for all b. If $b_1 \neq b_2$ then with a that is all zeros except for 1 at the coordinate they differ, $F_{b_1}(a) \neq F_{b_2}(a) \Rightarrow F_{b_1} \neq F_{b_2}$, so F is injective.

To show that F is surjective, take $f \in (\ell^p)^*$ then we construct b by

$$b_k \coloneqq f((0, \cdots, 0, 1, 0 \cdots))$$

i.e., the value when f is applied to the sequence of all zeros except for 1 at the kth index.

It is then clear that $F(b)(a) = \sum_{k=1}^{\infty} a_k b_k = f(a)$ for all $a \in \ell^p$ since f is linear, so $F(b) = f$, but this is only valid if $b \in \ell^q$. It remains for us to prove so.

Since $f \in (\ell^p)^*$, $\|f\| < \infty$.

For $a = (a_1, \ldots, a_n, 0, \ldots) \in \ell^p$ such that $\|a\|_p \leq 1$ then

$$\|f\| \geq f(a) = \sum_{k=1}^{n} a_k b_k$$

But Holder's gives us that

$$\sum_{k=1}^{n} a_k b_k \leq \sum_{k=1}^{n} |a_k b_k| \leq \left[\sum_{k=1}^{n} |a_k|^p\right]^{1/p} \left[\sum_{k=1}^{n} |b_k|^q\right]^{1/q} = \left[\sum_{k=1}^{n} |b_k|^q\right]^{1/q}$$

with equality achievable, but $\|f\| \geq f(a)$ for all qualifying a, so it follows that

$$\|f\| \geq \left[\sum_{k=1}^{n} |b_k|^q\right]^{1/q}$$

This is true for all n, so it follows that

$$\|b\|_q \leq \|f\| < \infty$$

so indeed $b \in \ell^q$ as required.

It is linear, since for all $\alpha, \beta \in \ell^q; \lambda \in \mathbb{C}$ and $a \in \ell^p$, we have

$$\begin{aligned} F(\alpha + \lambda\beta)(a) &= \sum_{k=1}^{\infty} a_k (\alpha + \lambda\beta)_k \\ &= \sum_{k=1}^{\infty} a_k \alpha_k + \sum_{k=1}^{\infty} \lambda a_k \beta_k \\ &= F(\alpha)(a) + \lambda F(\beta)(a) \end{aligned}$$

so $F(\alpha + \lambda\beta) = F(\alpha) + \lambda F(\beta)$ (expanding series of sum as sum of series makes sense, since we know a priori that each component series converges).

It remains to show that F is bounded.

From (b), we saw that $\|F_b\| = \|b\|_q$, i.e., $\|F(b)\| = \|b\|$. Theefore, $\|F\| \leq 1$, so it is a bounded linear operator as required. $\square$