MATH 26200: Point-Set Topology

Take-home Final

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Textbook: Munkres, *Topology*.

Problem 8.1 (done)

Give examples (and justification) for each of the following:

(a) A topological space X with a subspace A that is compact but not closed.

(b) A connected space that is not path connected.

(c) A compact Hausdorff space that is not second countable. (Hint: first find a locally compact Hausdorff space that is not second countable)

(d) A connected space that is not locally connected.

(e) A Hausdorff space that is not regular.

Solution

(a) $X = \{1, 2\}$ with topology $\mathcal{T} = \{\varnothing, \{1\}, \{1, 2\}\}$. Then $\{1\}$ is clearly compact, but it is not closed.

(b) (The topologist's sine curve) Let $X = \{(x, \sin\left(\frac{1}{x}\right)) : x \in (0, 1]\} \subset \mathbb{R}^2$. It has closure $\overline{X} = X \cup \{0\} \times [-1, 1]$. Since X is a continuous image of a connected set, namely $(0, 1]$, it is connected. So $\overline{X}$ is also connected. However, it is not path connected. Suppose there exists some $f : [0, 1] \to \overline{S}$ such that $f(0) = (0, 0)$ and $f(1) = (1, \sin(1))$. We have that $\{t : f(t) \in \{0\} \times [-1, 1]\} \subset [0, 1]$ is preimage of a closed set so it is closed, so it has a max t_0. For f to be continuous, then $f(t_0) = (0, 0)$. Then consider $f : [t_0, 1] \to \overline{S}$, then $\forall\, t > t_0, f(t) \in X$. WLOG, $t_0 = 0$.

But then for all n, choose $u_n \in (0, f_1(\frac{1}{n}))$ such that $\sin(1/u_n) = (-1)^n$. By Intermediate Value Theorem, f_1 is continuous so there exists some $0 < t_n < 1/n$ such that $f_1(t_n) = u_n$. Then it follows that $f_1(t_n) = (-1)^n$. It's clear that $t_n \xrightarrow{n\to\infty} 0 \Rightarrow f_2(t_n) \xrightarrow{n\to\infty} 0$. But $f_2(t_n) = (-1)^n$ which doesn't converge. $\Rightarrow\Leftarrow$

(c) Take $\mathbb{R}$. It is Hausdorff, locally compact and not second countable. So its one-point compactification (S^1) is then compact, Hausdorff and not second countable.

(d) $\overline{X}$ from (b). It is connected as previously shown. However, at $(0, 0)$, it is not locally connected, since for every $U \subset \mathbb{R}^2$ open, one can exhibit the separation $U \cap \{0\} \times [1, 1] \sqcup U - (U \cap \{0\} \times [1, 1])$.

(e) Consider $\mathbb{R}_K$. Then there doesn't exist any open, disjoint U, V such that $0 \in U, K \subset V$. Suppose there does exist. Then U contains some basis element that contains 0. It can't be of the form (a, b) since all (a, b) around 0 intersect K. So it has to be some $(a, b) - K$. But there exists some $\frac{1}{n} \in (a, b)$ still. V also contains some basis element that contains $\frac{1}{n}$, which has to be of the form (c, d). Then $(a, b) \cap (c, d) \neq \varnothing$ trivially, so $U \cap V \neq \varnothing$. $\square$

Problem 8.2 (done)

Show that if X is a compact metric space, then the metric topology is second countable (i.e., it has a countable basis).

Solution

Fix $n \in \mathbb{N}$. Then $\mathcal{U}_n = \{B(x, \frac{1}{n}) : x \in X\}$ is an open cover for X, so it reduces to some finite subcover $\mathcal{V}_n$. Take $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$, which is countable. WTS $\mathcal{V}$ is a basis for the metric topology on X.

$\mathcal{V}$ is a collection of open sets in X, so it remains to show that for every $x \in U \subset X$ open, there exists some $V \in \mathcal{V}$ such that $x \in V \subset U$.

Take $x \in U \subset X$ open. Since U is open, there exists some $B(x, r) \subset U \subset X$ (WLOG, ball centered at x). Then there exists some N such that $\frac{1}{N} < \frac{r}{2}$. Consider the finite subcover $\mathcal{V}_N$, then there has to exist some $B(x', \frac{1}{N}) \ni x \Rightarrow d(x, x') < \frac{1}{N} < \frac{r}{2}$. But $\frac{1}{N} < \frac{r}{2}$ so in fact $B(x', \frac{1}{N}) \subset B(x, r) \subset U$, and $B(x', \frac{1}{N}) \subset \mathcal{V}_n \subset \mathcal{V}$.

Using Lemma 26.4, it follows that $\mathcal{V}$ is a basis for X. So X is 2nd countable. $\square$

Problem 8.3 (25.7 done)

Consider the "infinite broom" X pictured in Figure 25.1. Show that X is not locally connected at p, but is weakly locally connected at p. [Hint: Any connected neighborhood of p must contain all the points a_i.]

(You may use 25.6 without proving it)

Solution

We view the "infinite broom" as a subspace of $\mathbb{R}^2$.

1. WTS X is not locally connected at p. WTS for any $V \ni p$ to be connected, it has to contain all points $(0, a_i)$.

If V doesn't contain all the points a_i, then consider $I = \{i : (0, a_i) \in V\} \neq \varnothing$ has some minimum element $n > 1$, which implies $(0, a_n) \in V$ but $(0, a_{n-1}) \notin V$. $V \subset \mathbb{R}^2$ open so $V \cap 0 \times \mathbb{R}$ is open, so there exists some b such that $a_n < b < a_{n-1}$ and $(0, b) \in V$. $(0, b) \in V$ implies that there exists some $B((0, b), r) \subset V$, and we know from construction of the infinite broom that there exists some broom segment from a_{n-1} that intersects this $B((0, b), r)$, hence intersecting V. One can then exhibit a separation of V with this intersection and its complement in V.

So V has to contain all $(0, a_i)$.

But then if one take a small enough open neighborhood U around p such that it doesn't contain all $(0, a_i)$, then there can't exist $p \in V \subset U$ that is connected. So X is not locally connected at p.

2. WTS X is weakly locally connected at p.

Take any open $U \ni p$, then there exists some open ball $p \in B(p, r) \subset U$. Since the broom segments decrease in height (y-coordinate) as n increases, there exists some N such that for all $n \geq N$, all of the broom segments originating from a_n is contained in $B(p, r)$. Then consider V to be the union of $[p, a_N]$ and all broom segments of a_n of $n \geq N$. Then $V \subset B(p, r) \subset U$. Furthermore, this V contains a smaller neighborhood $B(p, \frac{h}{2})$ where h is the y-coordinate of the tallest broom segment from a_N. $\square$

Problem 8.4 (33.9 done)

Show that $\mathbb{R}^J$ in the box topology is completely regular. [Hint: Show that it suffices to consider the case where the box neighborhood $(-1, 1)^J$ is disjoint from A and the point is the origin. Then use the fact that a function is continuous in the uniform topology is also continuous in the box topology.]

Solution

It suffices to show that given closed set $A \subset \mathbb{R}^J$ that does not contain 0, there is a continuous $f : \mathbb{R}^J \to [0, 1]$ such that $f(0) = 1$ and $f(A) = \{0\}$. If the point of concern is not 0, translate it there.

Since A is closed, there exists $\prod_{\alpha \in J}(-r_\alpha, r_\alpha) \cap A = \varnothing$.

We first show that one can separate 0 and $\mathbb{R}^J - (-1, 1)^J$ by a continuous function. $(-1, 1)^J$ is exactly $B(0, 1)$ of $\mathbb{R}^J$ in the uniform topology. So $\mathbb{R}^J - (-1, 1)^J$ is closed. $\mathbb{R}^J$ with the uniform topology is metrizable, so it is definitely completely regular, so there exists some continuous $f : \mathbb{R}^J \to [0, 1]$ such that $f(0) = 1$ and $f(\mathbb{R}^J - (-1, 1)^J) = \{0\}$. However, the uniform topology is coarser than the box topology, so this f is continuous in the box topology too.

Then let us look at $h : \mathbb{R}^J \to \mathbb{R}^J, (x_\alpha)_{\alpha \in J} \mapsto (x_\alpha / r_\alpha)_{\alpha \in J}$. It is continuous in the box topology (both domain and codomain), since the preimage of a basic open set $\prod_{\alpha \in J}(v_\alpha - \varepsilon_\alpha, v_\alpha + \varepsilon_\alpha)$ is $\prod_{\alpha \in J}(rv_\alpha - r\varepsilon_\alpha, rv_\alpha + r\varepsilon_\alpha)$ is open.

Finally, consider $g = f \circ h : \mathbb{R}^J \to [0,1]$ is continuous. $g(0) = f(h(0)) = 1$, and $h(A) \subset \mathbb{R}^J - (-1,1)^J$ so $g(A) \subset f(\mathbb{R}^J - (-1,1)^J) = \{0\}$, as required.

Hence $\mathbb{R}^J$ in the box topology is completely regular. $\square$

Problem 8.5 (38.3 done)

Under what conditions does a metrizable space have a metrizable compactification?

Solution

A compact metric space is second countable, so for the compactification of X to be metrizable, it is *necessary* for X to be second countable.

It remains for us to show that this is the sufficient condition. Suppose X is second countable. Then from the proof of the Urysohn Metrization Theorem, we already constructed an embedding $F : X \to \mathbb{R}^\omega$ with $\mathbb{R}^\omega$ in the product topology, which we know is metrizable with metric D. Then $\overline{F(X)}$ in $\mathbb{R}^\omega$ is a metrizable compactification of X. $\square$

Problem 8.6 (46.5 done)

Consider the sequence of functions $f_n : (-1,1) \to \mathbb{R}$, defined by

$$f_n(x) = \sum_{k=1}^{n} kx^k$$

- **(a)** Show that (f_n) converges in the topology of compact convergence; conclude that the limit function is continuous. (This is a standard fact about power series.)
- **(b)** Show that (f_n) does not converge in the uniform topology.

Solution

(a) Denote $\mathcal{F} = \{f_n\}$. Take any $K \subset (-1,1)$ compact, then it must be closed and bounded, say $x \in K \Rightarrow |x| \leq M$. Clearly, $M < 1$.

Let $f(x) = \sum_{k=1}^{\infty} kx^k$ on $(-1,1)$. It is well-defined, since the series actually converges to $\frac{x}{(1-x)^2}$, in fact, absolutely:

$$\sum_{k=1}^{\infty} k|x|^k = \frac{|x|}{(1-|x|)^2}$$

Then

$$\begin{aligned}
\sup_{x \in K} (f_n - f) &= \sup_{x \in K} \sum_{k=n+1}^{\infty} kx^k \\
&\leq \sum_{k=n+1}^{\infty} kM^k \\
&= \frac{1}{(1-M)^2} M^{n+1} [n(1-M) + 1] \xrightarrow{n \to \infty} 0
\end{aligned}$$

since M^{n+1} is exponential so it diminishes quicker than $n(1-M)$ grows.

It follows that f_n converges to f in the topology of compact convergence.

Now $(-1,1)$ is locally compact, so it is compactly generated, so $\mathcal{C}((-1,1),\mathbb{R})$ is closed in Y^X in the topology of compact convergence. $f_n \xrightarrow{n \to \infty} f \Rightarrow f \in \mathcal{C}((-1,1),\mathbb{R})$, so f is continuous.

(b) We have for all $1 \gg \varepsilon > 0$, for any $n > 2$, choose $a_n = 1 - \frac{1}{n^2} \in (-1, 1)$, then

$$\begin{aligned} d(f_n, f) &\geq |f_n(a_n) - f(a_n)| \\ &= n^4 \left(1 - \frac{1}{n^2}\right)^{n+1} \left(\frac{n}{n^2} + 1\right) \\ &\geq n^4 \left(1 - \frac{n+1}{n^2}\right) \text{ (Bernoulli's inequality)} \\ &= n^4 - n^3 - n^2 \not< \varepsilon \end{aligned}$$

so the convergence is not uniform. □