MATH 26200: Point-Set Topology

Problem Set 3

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Textbook: Munkres, *Topology*

Problem 3.1 (20.1 done)

(a) In $\mathbb{R}^n$, define

$$d'(x,y) = |x_1 - y_1| + \cdots + |x_n - y_n|$$

Show that d' is a metric that induces the usual topology on $\mathbb{R}^n$. Sketch the basis elements under d' when $n = 2$.

(b) More generally, given $p \geq 1$, define

$$d'(x,y) = \left[\sum_{i=1}^{n} |x_i - y_i|^p\right]^{1/p}$$

for $x, y \in \mathbb{R}^n$. Assume that d' is a metric. Show that it induces the usual topology on $\mathbb{R}$.

Solution

(a) Recall that the usual topology on $\mathbb{R}^n$ is the topology induced by the metric

$$d(x,y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2}$$

We prove that d' and d are comparable, i.e., that

$$\begin{aligned} d'(x,y)^2 &= \left(\sum_{i=1}^{n} |x_i - y_i|\right)^2 \\ &\geq \sum_{i=1}^{n} (x_i - y_i)^2 = d(x,y)^2 \\ \Rightarrow d'(x,y) &\geq d(x,y) \end{aligned}$$

and

$$
\begin{aligned}
d'(x,y)^2 &= \left(\sum_{i=1}^{n} |x_i - y_i|\right)^2 \\
&\leq \left(\sum_{i=1}^{n} d(x,y)\right)^2 \\
&\leq n^2 d(x,y)^2 \\
\Rightarrow d'(x,y) &\leq n d(x,y)
\end{aligned}
$$

Therefore we can exhibit basis elements for the d' metric topology and the usual topology:

$$
B_{d,\varepsilon}(x) \subset B_{d',\varepsilon}(x) \subset B_{d,\frac{\varepsilon}{n}}(x)
$$

thus d' induces the same topology.

Basis elements under d' when $n = 2$ are squares rotated by $\pi/4$.

(b) In the general case, we employ the same method. Call the d' in part (a) d_1. Then

$$
\begin{aligned}
d_1(x,y)^p &= \left(\sum_{i=1}^{n} |x_i - y_i|\right)^p \\
&\geq d(x,y)^p \\
\Rightarrow d_1(x,y) &\geq d(x,y)
\end{aligned}
$$

and

$$
\begin{aligned}
d_1(x,y)^p &= \left(\sum_{i=1}^{n} |x_i - y_i|\right)^p \\
&\leq \left(\sum_{i=1}^{n} d(x,y)\right)^p \\
\Rightarrow d_1(x,y) &\leq n d(x,y)
\end{aligned}
$$

then by the same argument, d' induces the same metric topology as d_1, which is the usual topology on $\mathbb{R}^n$. $\square$

Problem 3.2 (20.4 done)

Consider the product, uniform and box topologies on $\mathbb{R}^\omega$.

(a) In which topologies are the following functions from $\mathbb{R}$ to $\mathbb{R}^\omega$ continuous?

$$
\begin{aligned}
f(t) &= (t, 2t, 3t, \ldots) \\
g(t) &= (t, t, t, \ldots) \\
h(t) &= (t, \frac{1}{2}t, \frac{1}{3}t, \ldots)
\end{aligned}
$$

(b) In which topologies do the following sequences converge?

$$\begin{aligned} w_1 &= (1,1,1,1,\dots) & x_1 &= (1,1,1,1,\dots) \\ w_2 &= (0,2,2,2,\dots) & x_2 &= (0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\dots) \\ w_3 &= (0,0,3,3,\dots) & x_3 &= (0,0,\frac{1}{3},\frac{1}{3},\dots) \\ &\dots \\ y_1 &= (1,0,0,0,\dots) & z_1 &= (1,1,0,0,\dots) \\ y_2 &= (\frac{1}{2},\frac{1}{2},0,0,\dots) & z_2 &= (\frac{1}{2},\frac{1}{2},0,0,\dots) \\ y_3 &= (\frac{1}{3},\frac{1}{3},\frac{1}{3},0,\dots) & z_3 &= (\frac{1}{3},\frac{1}{3},0,0,\dots) \end{aligned}$$

Solution

(a) 1. Product topology. We have, by Theorem 20.5, that the metric

$$D(\mathbf{x},\mathbf{y}) = \sup\left\{\frac{\bar{d}(x_i,y_i)}{i}\right\}$$

where $\bar{d}(x_i,y_i) = \min\{|x_i - y_i|, 1\}$ induces the product topology on $\mathbb{R}^\omega$.

- f is continuous: For all ε, we can choose $\delta = \varepsilon$, then $|u-v| < \delta$ implies:

$$D(f(u),f(v)) \le \sup\left\{\frac{|iu - iv|}{i}\right\} = \sup\{|u-v|\} < \delta = \varepsilon$$

- g is continuous. For all ε, we can choose $\delta = \varepsilon$, then $|u-v| < \delta$ implies:

$$D(g(u),g(v)) \le \sup\left\{\frac{|u-v|}{i}\right\} = |u-v| < \delta = \varepsilon$$

- h is continuous. For all ε, we can choose $\delta = \varepsilon$, then $|u-v| < \delta$ implies:

$$D(g(u),g(v)) \le \sup\left\{\frac{|u-v|}{i^2}\right\} = |u-v| < \delta = \varepsilon$$

2. Uniform topology. The uniform topology is the one induced by the metric

$$\bar{\rho}(\mathbf{x},\mathbf{y}) = \sup\{\bar{d}(x_i,y_i)\}$$

- f is not continuous at 1. $f(1) = (1,2,\dots)$. Let $\varepsilon = 0.5$. Then we want to show that for all $\delta > 0$, there exists some $u \in (1-\delta, 1+\delta)$ such that $\bar{\rho}(f(u), f(1)) > \varepsilon = 0.5$.

  Indeed, for any δ, pick $u = 1 + \frac{\delta}{2}$. Then

$$f(u) = (1 + \frac{\delta}{2}, 2 + \delta, \dots)$$

and there exists N such that $N\delta/2 > 1$. Then

$$\bar{\rho}(f(u), f(1)) = \sup\{\bar{d}(f(u)_i, f(1)_i)\} \geq \bar{d}(f(u)_N, f(1)_N) = \bar{d}(N+N\delta/2, N) = 1 > 0.5$$

as required.

- g is continuous. For all ε, we can choose $\delta = \varepsilon$, then $|u - v| < \delta$ implies

$$\bar{\rho}(g(u), g(v)) = \sup\{\bar{d}(g(u)_i, g(v)_i)\} = \sup\{\bar{d}(u, v)\} \leq |u - v| < \delta = \varepsilon$$

- h is continuous. For all ε, we can choose $\delta = \varepsilon$, then $|u - v| < \delta$ implies

$$\bar{\rho}(h(u), h(v)) = \sup\{\bar{d}(h(u)_i, h(v)_i)\} = \bar{d}(u, v) \leq |u - v| < \delta = \varepsilon$$

3. Box topology.

- The uniform topology is coarser than the box topology. f is not continuous in the uniform topology, so it is also not continuous in the box topology.
- g is not continuous in the box topology. Because

$$g^{Pre}\left((-1,1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{3}, \frac{1}{3}\right) \times \ldots\right) = \{0\}$$

is not open in $\mathbb{R}$.

- h is not continuous in the box topology. Because

$$h^{Pre}\left((-1,1) \times \left(-\frac{1}{2^2}, \frac{1}{2^2}\right) \times \left(-\frac{1}{3^2}, \frac{1}{3^2}\right) \times \ldots\right) = \{0\}$$

is not open in $\mathbb{R}$.

(b) 1. Product topology.

- $w_n \xrightarrow{n\to\infty} (0, 0, \ldots)$, since

$$D(w_n, (0,0,\ldots)) = \sup\left\{\frac{\bar{d}(n, 0)}{i} : i \geq n\right\} = \frac{1}{n} \xrightarrow{n\to\infty} 0$$

- $x_n \xrightarrow{n\to\infty} (0, 0, \ldots)$, since

$$D(x_n, (0,0,\ldots)) = \sup\left\{\frac{\bar{d}(\frac{1}{n}, 0)}{i} : i \geq n\right\} = \frac{1}{n^2} \xrightarrow{n\to\infty} 0$$

- $y_n \xrightarrow{n\to\infty} (0, 0, \ldots)$, since

$$D(y_n, (0,0,\ldots)) = \sup\left\{\frac{\bar{d}(\frac{1}{n}, 0)}{i} : i \leq n\right\} = \frac{1}{n} \xrightarrow{n\to\infty} 0$$

- $z_n \xrightarrow{n\to\infty} (0,0,\ldots)$, since

$$D(z_n,(0,0,\ldots)) = \sup\left\{\frac{\bar{d}(\frac{1}{n},0)}{i} : i \le 2\right\} = \frac{1}{n} \xrightarrow{n\to\infty} 0$$

2. Uniform topology.

- w_n doesn't converge. Suppose that the sequence does: $w_n \xrightarrow{n\to\infty} a = (a_1, a_2, \ldots)$. Then for all N, we trivially have that

$$|a_{N+2} - N| \ge 1 \text{ or } |a_{N+2} - (N+2)| \ge 1$$

which implies

$$\bar{\rho}(a, w_N) \ge 1 \text{ or } \bar{\rho}(a, w_{N+2}) \ge 1$$

so $\bar{\rho}(a, w_n)$ does not go to 0 as $n \to \infty$.

- $x_n \xrightarrow{n\to\infty} (0,0,\ldots)$, since

$$\bar{\rho}(x_n,(0,0,\ldots)) = \sup\left\{\bar{d}(\frac{1}{n},0) : i \ge n\right\} = \frac{1}{n} \xrightarrow{n\to\infty} 0$$

- $y_n \xrightarrow{n\to\infty} (0,0,\ldots)$, since

$$\bar{\rho}(y_n,(0,0,\ldots)) = \sup\left\{\bar{d}(\frac{1}{n},0) : i \le n\right\} = \frac{1}{n} \xrightarrow{n\to\infty} 0$$

- $y_n \xrightarrow{n\to\infty} (0,0,\ldots)$, since

$$\bar{\rho}(z_n,(0,0,\ldots)) = \sup\left\{\bar{d}(\frac{1}{n},0) : i \le 2\right\} = \frac{1}{n} \xrightarrow{n\to\infty} 0$$

3. Box topology.

- w_n doesn't converge in the uniform topology, and the uniform topology is coarser than the box topology, so it doesn't converge in the box topology either.
- x_n doesn't converge. Suppose it does to a. Then $a_1 = 0$, because otherwise, WLOG, $a_1 > 0$, the neighborhood $(a_1/2, 3a_1/2) \times \mathbb{R} \times \ldots$ only has 1 term, x_1. Similarly, $a_2 = a_3 = \ldots = 0$.

  But there exists neighborhood of a:

$$\left(-\frac{1}{2},\frac{1}{2}\right) \times \left(-\frac{1}{4},\frac{1}{4}\right) \times \ldots \times \left(-\frac{1}{2n},\frac{1}{2n}\right) \times \ldots$$

  that does not contain any x_n.

- y_n doesn't converge. Using the same reasoning as above, if y_n does converge, then it must converge to $a = (0, 0, \ldots)$. However, the same neighborhood as above also does not contain any y_n.
- $z_n \xrightarrow{n\to\infty} (0, 0, \ldots)$. Convergence in coordinates after the 3rd is clear, while $1/n \xrightarrow{n\to\infty} 0$ takes care of the first 2.

$\square$

Problem 3.3 (21.3 done)

Let X_n be a metric space with metric d_n for $n \in \mathbb{Z}_+$.

(a) Show that

$$\rho(x, y) = \max\{d_1(x_1, y_1), \ldots, d_n(x_n, y_n)\}$$

is a metric for the product space $X_1 \times \ldots \times X_n$.

(b) Let $\bar{d}_i = \min\{d_i, 1\}$. Show that

$$D(x, y) = \sup\left\{\frac{\bar{d}_i(x_i, y_i)}{i}\right\}$$

is a metric for the product space $\prod X_i$

Solution

(a) We check the conditions for it to be a metric:

(1) $\rho(x, y) = \max\{d_1(x_1, y_1), \ldots, d_n(x_n, y_n)\} \geq d_1(x_1, y_1) \geq 0$.

And $0 = \rho(x, y) = \max\{d_1(x_1, y_1), \ldots, d_n(x_n, y_n)\} \Rightarrow 0 = d_1(x_1, y_1) = d_2(x_2, y_2) = \ldots = d_n(x_n, y_n) \Rightarrow x = y$.

(2) $d_1, d_2, \ldots$ are symmetric so ρ is symmetric.

(3) $\rho(x, y) + \rho(y, z) \geq d_i(x_i, y_i) + d_i(y_i, z_i) \geq d_i(x_i, z_i)\ \forall\ i \Rightarrow \rho(x, y) + \rho(y, z) \geq \max\{d_i(x_i, y_i)\} = \rho(x, z)$

(b) To show that D induces the same topology as the product topology on $\prod X_i$, we show that it is both finer and coarser.

1. WTS the D-metric topology is finer than the product topology.

Take a typical basis element B of the product topology: the Cartesian product of X_i at all index i except for $U_{i_1}, U_{i_2}, \ldots, U_{i_N}$ at indices $i_1, i_2, \ldots, i_N$; this basis element contains the point $(x_1, x_2, \ldots)$. It is therefore necessary for $x_{i_j} \in U_{i_j}\ \forall\ j \in [N]$.

Then we can take ball $B_{d_{i_j}}(x_{i_j}, \varepsilon_{i_j}), \varepsilon_{i_j} < 1\ \forall\ j \in [N]$.

Select $\varepsilon = \min\left\{\frac{\varepsilon_{i_j}}{i_j} \in [N]\right\}$ then

$$y \in B_D(x, \varepsilon) \Rightarrow\ \forall\ j \in [N], \frac{\bar{d}_{i_j}(x_{i_j}, y_{i_j})}{i_j} < \varepsilon < \frac{\varepsilon_{i_j}}{i_j} \Rightarrow\ \forall\ j \in [N], \bar{d}_{i_j}(x_{i_j}, y_{i_j}) < \varepsilon_{i_j} < 1$$

which implies

$$d_{i_j}(x_{i_j}, y_{i_j}) < \varepsilon_{i_j}\ \forall\ j \in [N] \Rightarrow y \in B$$

Therefore we can find $B_D(x,\varepsilon) \subset B$.

2. WTS the product topology is finer than the D-metric topology.

Take any point x, and typical basis element $B_D(x, 2\varepsilon)$ (we can always do in reverse order). Then there exists N such that $(N+1)\varepsilon > 1$.

Then we construct a basis element of the product topology that contains x, namely,

$$U = B_{d_1}(x_1, \varepsilon) \times B_{d_2}(x_2, 2\varepsilon) \times \ldots \times B_{d_N}(x_N, N\varepsilon) \times X_{N+1} \times X_{N+2} \times \ldots$$

and WTS $U \subset B_D(x, \varepsilon)$.

For any $y \in U$, we have that $\forall\, i \in [N], d_i(x_i, y_i) < i\varepsilon \Rightarrow \frac{\bar{d}_i(x_i,y_i)}{i} < i\varepsilon/i = \varepsilon$.

Meanwhile, $\forall\, i \geq N, \frac{\bar{d}_i(x_i,y_i)}{i} < \frac{1}{i} \leq \frac{1}{N+1} < \varepsilon$

It follows that

$$D(x,y) = \sup\left\{\frac{\bar{d}_i(x_i, y_i)}{i}\right\} \leq \varepsilon < 2\varepsilon$$

and therefore $U \subset B_D(x, 2\varepsilon)$.

3. From **1.**, **2.**, it follows that D is a metric for the product space $\prod X_i$. $\square$

Problem 3.4 (26.8 done)

Let $f : X \to Y$, Y is compact Hausdorff. Then f is continuous if and only if the graph of f,

$$G_f = \{x \times f(x) \mid x \in X\}$$

is closed in $X \times Y$.

Hint: If G_f is closed and V is a neighborhood of $f(x_0)$, then the intersection of G_f and $X \times (Y - V)$ is closed. Apply Exercise 7.

Solution

$\boxed{\Rightarrow}$ Suppose that f is continuous. WTS $X \times Y - G_f$ is open.

Take $x \times y \in (X \times Y - G_f)$, i.e., $y \in Y$ such that $y \neq f(x)$. Since Y is Hausdorff, there exists disjoint open $V_{fx} \ni f(x), U_y \ni y$. Since f is continuous, $f^{Pre}(V_{fx}) = V_x$ is open in X.

Then, $V_x \times U_y$ contains $x \times y$ and is a basis element of the topology on $X \times Y$. And $f(V_x) \cap U_y = V_{fx} \cap U_y = \varnothing$ so $V_x \times U_y \cap G_f = \varnothing$, i.e., $V_x \times U_y \subset (X \times Y - G_f)$. It follows that G_f is closed.

$\boxed{\Leftarrow}$ We use the following lemma

Lemma 3.1 (Exercise 7)

If Y is compact, then the projection $\pi_1 : X \times Y \to X$ is a closed map.

Proof

Take $K \subset X \times Y$ closed. WTS $\pi_1(K)$ is closed in X, i.e., $X - \pi_1(K)$ is open in X.

Take any $x \in (X - \pi_1(K))$, i.e., there doesn't exist y such that $x \times y \in K$. This means that $\{x\} \times Y \cap K = \varnothing$. Since K is closed in $(X \times Y)$, it follows that $(X \times Y - K)$ is an open set containing the slice $\{x\} \times Y$. By the tube lemma, there exists open $W \ni x$ such that $W \times Y \subset (X \times Y - K)$, which means $W \cap \pi_1(K) = \varnothing \Rightarrow W \subset (X - \pi_1(K))$. It follows that $(X - \pi_1(K))$ is open as required. $\square$

Now the main proof. Suppose that G_f is closed in $X \times Y$. To show that f is continuous, we want to show that $f^{Pre}(V)$ is open in X for all V open. Take $f(x_0) \in V$ open. It follows that $X \times V$ is open in $X \times Y$, and therefore $X \times (Y - V)$ is closed. Then $G_f \cap X \times (Y - V)$ is also closed. Using Lemma (since Y is compact), then $\pi_1(G_f \cap X \times (Y - V))$ is also closed. Then $U \coloneqq X - \pi_1(G_f \cap X \times (Y - V))$ is open.

But then $U = \pi_1(G_f) - \pi_1(G_f \cap X \times (Y - V)) = \pi_1(G_f \cap X \times V) = f^{Pre}(V)$. It is open. $\square$

Problem 3.5 (26.10 done)

(a) Prove the following partial converse to the uniform limit theorem:

Let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions, with $f_n(x) \xrightarrow{n\to\infty} f(x)$ for each $x \in X$. If f is continuous, and if the sequence f_n is monotone increasing, and if X is compact, then the convergence is uniform.

(f_n is monotone increasing if $f_n(x) \leq f_{n+1}(x)$ for all n and x.)

(b) Give examples to show that this theorem fails if you delete the requirement that X be compact, or if you delete the requirement that the sequence be monotone.

Hint: Exercises of Chapter 21.

Solution

(a) To show that $f_n \xrightarrow{n\to\infty} f$ uniformly, WTS for all $\varepsilon > 0$, there exists N such that

$$|f(x) - f_n(x)| < \varepsilon$$

Fix $\varepsilon > 0$. Then let $U_n \coloneqq \{x \in X : f(x) - f_n(x) < \varepsilon\} = (f - f_n)^{Pre}(-\infty, \varepsilon)$. Since f, f_n are continuous, so is $(f - f_n)$ and U_n is therefore open.

Also, for all $x \in X$, since $f_n(x) \xrightarrow{n\to\infty} f(x)$, there exists N

$$|f(x) - f_N(x)| < \varepsilon \Rightarrow x \in U_N$$

It follows that $X = \bigcup_{n\in\mathbb{N}} U_n$. So $\{U_n\}_{n\in\mathbb{N}}$ is an open cover of X. X is compact so there exists a finite subcover $\{U_{n_1}, \dots, U_{n_m}\}$ for some m finite.

Take $N = \max\{n_1, \dots, n_m\}$. Since f_n is monotone increasing, $f(x) = \sup\{f_n(x)\}_{n\in\mathbb{N}}$ for every x. This means $|f(x) - f_n(x)| = f(x) - f_n(x)$, and for any $x, n_k, n \geq N$, since $N \geq n_k$, we have that $f(x) - f_n(x) \leq f(x) - f_N(x) \leq f(x) - fn_k(x)$.

Then for any x, since $x \in U_{n_k}$ for some $k \in [m]$, we have that for all $n \geq N$,

$$f(x) - f_n(x) \leq f(x) - f_N(x) \leq f(x) - f_{n_k}(x) < \varepsilon$$

(b) If X is not compact: Define on $X = (0, 1)$ not compact,

$$f_n(x) = x^n$$

then for all $x \in (0, 1)$, $f_n(x) \xrightarrow{n\to\infty} 0$ so $f \equiv 0$ on $(0, 1)$.

However, for any n, $f_n((1/2)^{1/n}) = 1/2 \not< \varepsilon$, so convergence is not uniform.

If f_n is not monotone: Define on $X = [0, 1]$ compact,

$$f_n(x) = \begin{cases} 2nx & \text{on } [0, \frac{1}{2n}] \\ 2 - 2nx & \text{on } [\frac{1}{2n}, 1/n] \\ 0 & \text{otherwise} \end{cases}$$

i.e. f_n is a linear "spike", connecting $(0, 0)$, $\left(\frac{1}{2n}, 1\right)$ and $\left(\frac{1}{n}, 0\right)$, then flat 0 on the rest.

Evidently, $f_2(\frac{1}{4}) = 1 > f_1(\frac{1}{4}) = \frac{1}{2}$ so f_n is not monotone increasing.

But $f_n(x) \xrightarrow{n\to\infty} 0$ for all $x \in [0, 1]$, so $f \equiv 0$ on $[0, 1]$.

But convergence is not uniform, since for all n, $f_n(\frac{1}{2n}) - f(\frac{1}{2n}) = 1 - 0 = 1 \not< \varepsilon$, so convergence is not uniform. $\square$

Problem 3.6 (27.2 done)

Let X be a metric space with metric d; let $A \subset X$ be nonempty.

(a) Show that $d(x, A) = 0$ if and only if $x \in \overline{A}$.

(b) Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$.

(c) Define the ε-neighborhood of A in X to be the set

$$U(A, \varepsilon) = \{x \mid d(x, A) < \varepsilon\}$$

Show that $U(A, \varepsilon)$ equals the union of the open balls $B_d(a, \varepsilon)$ for $a \in A$.

(d) Assume that A is compact; let U be an open set containing A. Show that some ε-neighborhood of A is contained in U.

(e) Show the result in (d) need not hold if A is closed but not compact.

Solution

(a) $\boxed{\Rightarrow}$ Assume $d(x, A) = 0$. Take any basis $B_d(x, \varepsilon) \ni x$. Since $0 = d(x, A) = \inf\{d(x, a) : a \in A\}$, there exists $a \in A$ such that $d(x, a) < \varepsilon \Rightarrow a \in B_d(x, \varepsilon)$. Since any basis element containing x intersects A, any open neighborhood of x also intersects A. So $x \in \overline{A}$.

$\boxed{\Leftarrow}$ Assume $x \in \overline{A}$. Take basis elements $B_d(x, \frac{1}{n})$ for all $n \in \mathbb{N}$. Then there exists $a_n \in A \cap B_d(x, \frac{1}{n})$, i.e., $d(x, a_n) < \frac{1}{n}$. It then follows that $0 \leq d(x, A) = \inf\{d(x, a) : a \in$

$A\} \leq \inf\{d(x,a_n) : n \in \mathbb{N}\} = 0 \Rightarrow d(x,A) = 0$. □

(b) For any x, we can define $d_x : A \to \mathbb{R}, d_x(a) = d(x,a) = d(a,x)$. Then d_x is a continuous function. A is compact, so d_x achieves its minimum at some $a' \in A$. Therefore

$$d(x,A) = \inf\{d(x,a) : a \in A\} = d(x,a')$$

(c) WTS $U(A,\varepsilon) = \bigcup_{a\in A} B_d(a,\varepsilon)$.

Take $x \in U(A,\varepsilon)$. That means $d(x,A) < \varepsilon$. Since $d(x,A) = \inf\{d(x,a) : a \in A\}$, this means that there exists some $a' \in A$ such that $d(x,a') < \frac{d(x,A)+\varepsilon}{2} < \varepsilon \Rightarrow x \in B_d(a',\varepsilon)$. It follows that $U(A,\varepsilon) \subset \bigcup_{a\in A} B_d(a,\varepsilon)$.

Take $x \in \bigcup_{a\in A} B_d(a,\varepsilon)$, which means $d(x,a') < \varepsilon$ for some $a' \in A$. Then this means $d(x,A) = \inf\{d(x,a) : a \in A\} \leq d(x,a') < \varepsilon \Rightarrow x \in U(A,\varepsilon)$. It then follows that $U(A,\varepsilon) \supset \bigcup_{a\in A} B_d(a,\varepsilon)$.

Therefore $U(A,\varepsilon) = \bigcup_{a\in A} B_d(a,\varepsilon)$.

(d) For every $a \in A \subset U$, since U is open, we can draw $a \in B(a,\varepsilon_a) \subset U$. Then $\{B(a,\varepsilon_a)\}$ is an open cover of A compact. Using the Lebesgue covering lemma, then there exists δ such that for any a, $B(a,\delta)$ is contained in some element of the covering, and therefore contained in U in particular. It follows that $\bigcup_{a\in A} B(a,\delta) \subset U$. From (c), $\bigcup_{a\in A} B(a,\delta) = U(A,\delta)$.

(e) Counter example: $X = [-1,0) \cup (0,1], A = U = (0,1]$. A and U are clopen in the subspace topology. But any ε-neighborhood of A, WLOG $\varepsilon < 1$, would contain the point $-\varepsilon/2$, for

$$d(-\varepsilon/2, A) = \inf\{d(-\varepsilon/2, a) : a \in A\} = \varepsilon/2 < \varepsilon$$

Clearly $-\varepsilon/2 \notin U$. □

Problem 3.7 (27.5 done)

Let X be a compact Hausdorff space, let $\{A_n\}$ be a countable collection of closed sets of X. Show that if each set A_n has empty interior in X, then the union $\bigcup A_n$ has empty interior in in X.

Hint: Imitate the proof of Theorem 27.7. This is a special case of the Baire category theorem.

Solution

(With Otto Reed)

Suppose, for sake of contradiction, that $U_0 = \mathrm{int}(\bigcup A_n) \neq \varnothing$ is open.

We claim that for any non-empty, open U and any A_n, there exists non-empty V such that $\overline{V} \subset U$ and $\overline{V} \cap A_n = \varnothing$. Take such U and A_n. Since $\mathrm{int}(A_n) = \varnothing$, U is not a subset of A_n (if $U \subset A_n$ then $U \subset \mathrm{int}(A_n) \Rightarrow \mathrm{int}(A_n) \neq \varnothing$), i.e., there exists some $a \in U$ such that $a \in U - A_n$. Then, take $K = A_n \cup (X - U)$; K is closed since A_n is closed and U is open. K is closed in compact X, so K is compact. $a \in U - A_n \Rightarrow a \notin K$. It follows from Lemma 26.4 (K is compact, $x \notin K$, X is Hausdorff) that there exists disjoint open sets $V \ni a, W \supset K$. Then, since $V \cap W = \varnothing$,

$$V \subset (X - W)$$

It follows that $\overline{V} \subset \overline{X - W}$. But $X - W$ is closed, so $\overline{X - W} = X - W$. It follows that $\overline{V} \subset X - W$.

In turn,

$$X - W \subset X - K = X - A_n \cup (X - U) = (X - A_n) \cap U = U - A_n \subset U$$

so

$$\overline{V} \subset U$$

and recall that $a \in V$, so V is non-empty. We have thus proven our claim.

Apply the claim onto U_0 and A_1, it follows that there exists non-empty U_1 such that $\overline{U_1} \subset U_0$ and $\overline{U_1} \cap A_1 = \varnothing$. Iteratively, apply the claim onto U_k and A_{k+1}, it follows that there exists non-empty U_{k+1} such that $\overline{U_{k+1}} \subset U_k$ and $\overline{U_{k+1}} \cap A_{k+1} = \varnothing$.

Each $\overline{U_k}$ is closed in compact X, so is compact. We therefore get a sequence of nested, non-empty compact $\overline{U_n}$:

$$\overline{U_1} \supset \overline{U_2} \supset \dots$$

It follows that there exists $p \in \bigcap_{k \in \mathbb{N}} \overline{U_k}$.

However, $\overline{U_k} \cap A_k = \varnothing$ for all $k \geq 1$. On top of that, $\overline{U_{k+1}} \subset U_k \subset \overline{U_k}$ so $\overline{U_j} \cap A_k = \varnothing$ for all $j \geq k \geq 1$ too. It follows that

$$\left(\bigcap_{k \in \mathbb{N}} \overline{U_k} \right) \cap \left(\bigcup_{k \in \mathbb{N}} A_k \right) = \varnothing$$

which implies $p \notin \bigcup_{k \in \mathbb{N}} A_k$.

But $p \in \overline{V_1} \subset U = \operatorname{int} \bigcup A_k \subset \bigcup A_k$, $\Rightarrow\Leftarrow$.

By contradiction, it follows that $\operatorname{int} \bigcup A_n = \varnothing$. $\square$