207 Homework #3 Due Wednesday, Oct 18.

I. Read Pugh, Chapter 2, Section 8, and Chapter 3, Sections 1-3 (these should largely be review).

II. Chap 2: [116] (warmup), 118 (don't forget Theorem 20!), 152

III. Chap 3: [9,14,15, 49] (warmup), 17 (a,c,d), 19, 20, 21, 31, 33

IV. This exercise is about the middle thirds Cantor set C.

(a) Let I_L be the interval $[0, 1/3]$ and let I_R be the interval $[2/3, 1]$. Let $C_L = C \cap I_L$ and $C_R = C \cap I_R$. Define maps $f_L : I_L \to [0, 1]$ and $f_R : I_R \to [0, 1]$ by $f_L(x) = 3x$ and $f_R(x) = 3x - 2$. Prove that $f_L(C_L) = f_R(C_R) = C$. How can you use these maps to systematically label points in C?

(b) Find 4 squares $R_1, \ldots, R_4 \subset [0, 1]^2$ and 4 affine maps $f_i \colon R_i \to [0, 1]^2$, $i = 1, \ldots 4$ such that, if we set $(C \times C)_i = (C \times C) \cap R_i$, then $C \times C = (C \times C)_1 \cup \cdots \cup (C \times C)_4$, and $f_i((C \times C)_i) = C \times C$. How can you use these maps to systematically label points in $C \times C$? See Figure 1.

(c) Prove that $C \times C$ is homeomorphic to C.

(d) Suppose you divide $[0, 1]^2$ into 9 equal smaller squares, remove the interior of the middle ninth square from $[0, 1]^2$, repeat removing the inner ninth from the remaining squares, etc., and intersect to obtain a compact set M. Why is M not homeomorphic to $C \times C$?

(e) Prove that every real number $r \in [0, 2]$ can be written as a sum $x + y$, where $x, y \in C$. (Equivalently, $C + C = [0, 2]$). **Hint:** there are two ways to do this – algebraically and geometrically. To do algebraically, think of base 3 representations of real numbers. To do geometrically, look at the images in Figure 2.

(f) **Extra credit.** Fix $a \in [1, 3]$. For which b does the equation $ax + y = b$ always have a solution with $x, y \in C$? What happens when $a > 3$?

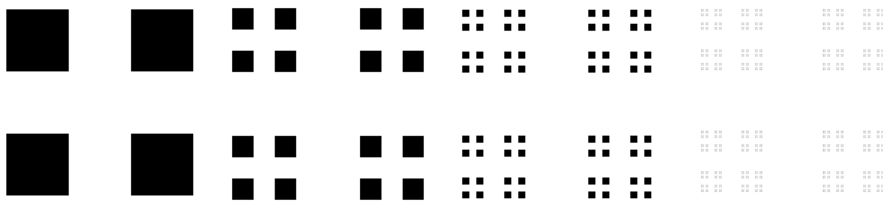

Figure 1: Constructing $C \times C$

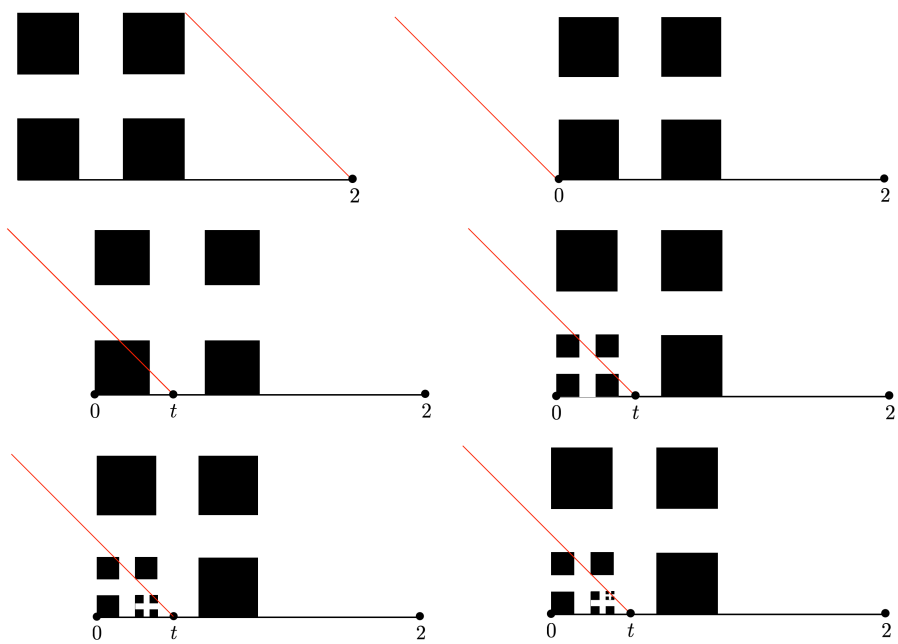

Figure 2: Hint for IV. (e)

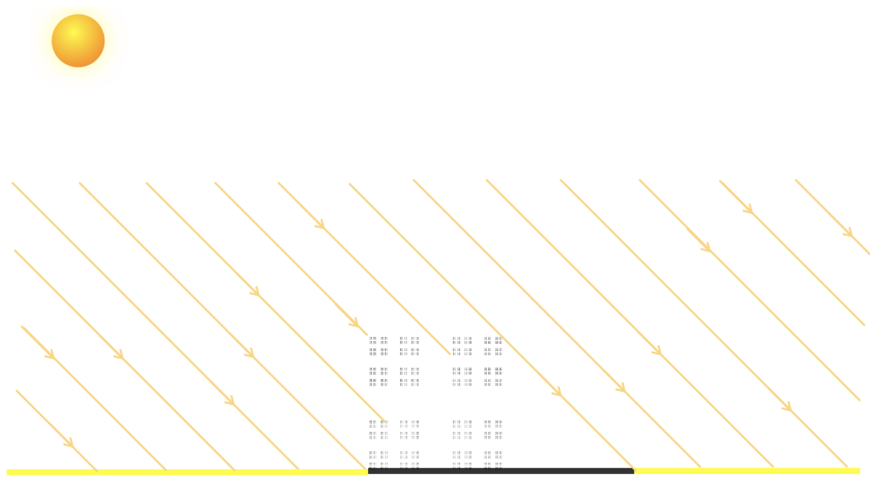

Figure 3: $C \times C$ casts big shadows for such a skinny set!