Math 20250: Abstract Linear Algebra

Problem Set 4

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Textbook: Linear Algebra by Hoffman and Kunze (2nd Edition)

Problem 4.1 (Sec 3.4. Problem 3)

Let T be a linear operator on $\mathbb{F}^n$, let A be the matrix of T in the standard ordered basis for $\mathbb{F}^n$, and let W be the subspace of $\mathbb{F}^n$ spanned by the column vectors of A. What does W have to do with T?

Solution

Let $\mathcal{B} = \{e_1, e_2, \ldots e_n\}$ be the standard ordered basis for $\mathbb{F}^n$.
Since A is the matrix of T in $\mathcal{B}$, the columns $A_j = T(e_j)$.
Since W is the subspace of $\mathbb{F}^n$ spanned by the column vectors A_j of A, it is the subspace spanned by $T(e_j)$, which is simply the range of T. □

Problem 4.2 (Sec 3.4. Problem 5)

Let T be the linear operator on $\mathbb{R}^3$, the matrix of which in the standard ordered basis is

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

Find a basis for the range of T and a basis for the null space of T.

Solution

Per the previous problem, we know that $\{(1,0,-1),(2,1,3),(1,1,4)\}$ span the range of T. However, in order to establish the basis of the range of T, we have to check if they are linearly independent.
In fact, they are not:

$$(1,0,-1) + (-1)(2,1,3) + (1,1,4) = (0,0,0)$$

and therefore $\{(1,0,-1),(2,1,3)\}$ already span the range of T.
It is also trivial that $\{(1,0,-1),(2,1,3)\}$ are linearly independent, and thus they form a basis for the range of T.
The nullspace of $T = \{v \in \mathbb{R}^3 \mid Av = 0\}$ and we can solve for v by row-reducing A:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \xrightarrow{(1)=(1)-2(2),(3)=(1)+(3)} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \end{bmatrix} \xrightarrow{(3)=(3)-5(2)} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $v = (a, -a, a) \; \forall \; a \in \mathbb{R} \Rightarrow \{(1,-1,1)\}$ forms a basis for the nullspace of T □

Problem 4.3 (Sec 3.4. Problem 11)

Let W be the space of all $n \times 1$ column matrices over a field $\mathbb{F}$. If A is an $n \times n$ matrix over $\mathbb{F}$, then A defines a linear operator L_A on W through left multiplication: $L_A(X) = AX$. Prove that every linear operator on W is a left multiplication by some $n \times n$ matrix, i.e. is L_A for some A.
Now suppose V is an $n-$dimensional vector space over the field $\mathbb{F}$, and let $\mathcal{B}$ be an ordered basis for V. For each α in V, define $U\alpha = [\alpha]_{\mathcal{B}}$. Prove that U is an isomorphism of V onto W.

If T is a linear operator on V, then UTU^{-1} is a linear operator on W. Accordingly, UTU^{-1} is left multiplication by some $n \times n$ matrix A. What is A?

Solution

For the first part, let $\mathcal{B} = \{e_1, e_2, \ldots, e_n\}$ be the analogous standard ordered basis for W, in which $e_i \in \mathbb{M}_{n\times 1}(\mathbb{F})$ has all entries as 0, other than its $i-$th as 1. Then let T be some linear operator on W:

$$\begin{aligned} T(X) &= T(x_1e_1 + x_2e_2 + \cdots + x_ne_n) \\ &= x_1T(e_1) + x_2T(e_2) + \cdots + x_nT(e_n) \\ &= AX \end{aligned}$$

where $A \in \mathbb{M}_{n\times n}(\mathbb{F})$ and the column vector $A_i := T(e_i)$. T therefore corresponds to $L_A(X) = AX$.

For the second part, we want to show that U is an isomorphism of V onto W by showing that U is a linear transformation, is surjective and injective. Indeed,

$$U(c\alpha + \beta) = [c\alpha + \beta]_{\mathcal{B}} = c[\alpha]_{\mathcal{B}} + [\beta]_{\mathcal{B}} = cU(\alpha) + U(\beta)$$

U is surjective since $\forall\ [\alpha]_{\mathcal{B}} \in W$, $\exists\ \alpha \in V$ that is retrievable via $\mathcal{B}$.

U is injective since $\ker(U) = \{\alpha \in V \mid [\alpha]_{\mathcal{B}} = 0\} = \{0\}$

It follows that U is indeed an isomorphism.

To find the matrix A that corresponds to UTU^{-1}: let $w = [\alpha]_{\mathcal{B}} \in W$, then:

$$\begin{aligned} UTU^{-1} &= UTU^{-}1([\alpha]_{\mathcal{B}}) \\ &= UT(\alpha) \\ &= [T(\alpha)]_{\mathcal{B}} \\ &= [T]_{\mathcal{B}}[\alpha]_{\mathcal{B}} \\ &= [T]_{\mathcal{B}}w \end{aligned}$$

It follows that $A = [T]_{\mathcal{B}}$ $\square$

Problem 4.4 (Sec 5.3. Problem 1)

If K is a commutative ring with identity and A is the matrix over $\mathbb{K}$ given by:

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

show that $\det A = 0$

Solution

$$\begin{aligned} \det A &= 0 \begin{vmatrix} 0 & c \\ -c & 0 \end{vmatrix} - (-a) \begin{vmatrix} a & b \\ -c & 0 \end{vmatrix} + (-b) \begin{vmatrix} a & b \\ 0 & c \end{vmatrix} \\ &= a(bc) - b(ac) = 0 \end{aligned}$$

$\square$

Problem 4.5 (Sec 5.3. Problem 2)

Prove that the determinant of the Vandemonde matrix

$$\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$$

is $(b-a)(c-a)(c-b)$

Solution

Let A be the above matrix. Then

$$\begin{aligned}
\det A &= \begin{vmatrix} b & b^2 \\ c & c^2 \end{vmatrix} - \begin{vmatrix} a & a^2 \\ c & c^2 \end{vmatrix} + \begin{vmatrix} a & a^2 \\ b & b^2 \end{vmatrix} \\
&= bc(c-b) - ca(c-a) + ab(b-a) \\
&= bc(c-b) - ca(c-a) + ab[(c-a) - (c-b)] \\
&= (c-b)(bc-ab) - (c-a)(ca-ab) \\
&= (c-b)(c-a)b - (c-a)(c-b)a \\
&= (c-b)(c-a)(b-a)
\end{aligned}$$

$\square$