MATH 20700: Honors Analysis in Rn I

Problem Set 3

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Textbook: Pugh's Real Mathematical Analysis

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Problem 3.1 (2.118*)

The implications of compactness are frequently equivalent to it. Prove

(a) If every continuous function $f : M \to \mathbb{R}$ is bounded then M is compact.

(b) If every continuous bounded function $f : M \to \mathbb{R}$ achieves a maximum or minimum then M is compact.

(c) If every continuous function $f : M \to \mathbb{R}$ has compact range fM then M is compact.

(d) If every nested decreasing sequence of nonempty closed subsets of M has nonempty intersection then M is compact.

Toegether with Theorems 63 and 65, (a)-(d) give seven equivalent definitions of compactness. [Hint: Reason contrapositively. If M is not compact then it contains a sequence $\{p_n\}$ that has no convergent subsequence. It is fair to assume that the points p_n are distinct. Find radii $r_n > 0$ such that the neighborhoods $M_{r_n}(p_n)$ are disjoint and no sequence $q_n \in M_{r_n}(p_n)$ has a convergent subsequence. Using the metric define a function $f_n : M_{r_n}(p_n) \to \mathbb{R}$ with a spike at p_n, such as

$$f_n(x) = \frac{r_n - d(x, p_n)}{a_n + d(x, p_n)}$$

where $a_n > 0$. Set $f(x) = f_n(x)$ if $x \in M_{r_n}(p_n)$, and $f(x) = 0$ if x belongs to no $M_{r_n}(p_n)$. Show that f is continuous. With the right choice of a_n show that f is unbounded. With a different choice of a_n, it is bounded but achieves no maximum, and so on.]

Solution

(a) Suppose M is not compact. Then there exists sequence $\{p_n\} \subseteq M$ such that it has no convergent subsequence.

We first prove that there exists radii $r_n > 0$ such that $B(x_n, r_n)$ are pairwise disjoint.

Let $d_n = \inf_{i\neq n}\{d(p_n, p_i)\}$ and set $r_n = d_n/3$, which requires to show that $d_n \neq 0$.

Suppose that it is. Then

$$\forall\, \varepsilon > 0,\ \exists\, N_\varepsilon \text{ such that } d(p_n, p_{N_\varepsilon}) < \varepsilon$$

We derive a contradiction by constructing a converging subsequence of $\{p_n\}$.

Choose $\varepsilon_1 = 1$, then there exists N_1 such that $d(p_n, p_{N_1}) < 1$.

Choose $\varepsilon_2 = \min\{\varepsilon_1/2, \min_{j\leq N_1}\{d(p_n, p_j)\}/2\}$, then there exists N_2 such that $d(p_n, p_{N_2}) < \varepsilon_2$. Note that $N_2 > N_1$ due to the construction of ε_2, and that $\varepsilon_2 \leq 1/2$.

Continue constructing this way, since $N_{j+1} > N_j$, we get a subsequence $\{p_{N_j}\}_{j\in\mathbb{N}}$ that satisfies

$$d(p_{N_j}, p_n) < \varepsilon_j \leq 1/2^j$$

which therefore converges to p_n. $\Rightarrow\Leftarrow$

It follows that $d_n > 0$. Coming back, if we set $r_n = d_n/3$ then

$$d(p_n, p_m) > d(p_n, p_m)/3 + d(p_m, p_n)/3 \leq r_n + r_m$$

so the balls $B(p_n, r_n)$ and $B(p_m, r_m)$ are disjoint for all $m \neq n$.

A fact that we won't rigorously prove is that there can't be any sequence $q_n \in B(p_n, r_n)$ that has a convergent subsequence. Simply put, if there exists a convergent subsequence of $\{q_n\}$, since $r_n \xrightarrow{n\to\infty} 0$, there ought to exist a corresponding subsequence of $\{p_n\}$ too.

Then, define

$$f_n(x) = \frac{r_n - d(x, p_n)}{a_n + d(x, p_n)}$$

on $B(p_n, r_n)$, for some $a_n > 0$. We define $f(x) = f_n(x)$ on $B(p_n, r_n)$ and 0 elsewhere.

We now prove that f is **continuous**. Let $R = M \setminus \bigsqcup_{n\in\mathbb{N}} B(p_n, r_n)$.

Since d is continuous, $f_n(x)$ is continuous too, so f is continuous on $B(p_n, r_n)$. Take $y \in \partial B(p_n, r_n)$ then $f(y) = 0$. And if $y_n \in M \xrightarrow{n\to\infty} y$, $f(y_n) \xrightarrow{n\to\infty} 0$ too, since d is continuous and $f \equiv 0$ on R. It follows that f is continuous on M.

So far we have not specified which a_n; f is continuous for all $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$. For each of the following sections we choose a_n so as to bring about a contradiction, to then conclude that M has to be compact.

(a) Choose $a_n = \frac{r_n}{n} \Rightarrow f(p_n) = n$ which is unbounded. $\Rightarrow\Leftarrow$

(b) Choose $a_n = \frac{r_n}{2-\frac{1}{n}}$ then (denote $d(x, p_n)$ as d)

$$\begin{aligned} f_n(x) &= \frac{r_n - d}{a_n + d} \\ &= \frac{r_n + a_n}{a_n + d} - 1 \\ &\leq \frac{r_n + a_n}{a_n} - 1 = \frac{r_n}{a_n} = 2 - \frac{1}{n} < 2 \end{aligned}$$

is therefore bounded but never achieves its maximum.

Because if it does, it will be at some point p_n (not other points in M), but $f(p_n) = 2 - \frac{1}{n}$. $\Rightarrow\Leftarrow$

(c) Choose $a_n = \frac{r_n}{n} \Rightarrow f(p_n) = n \Rightarrow \mathbb{N} \subset fM$. It follows that fM is unbounded, and therefore can't be compact. $\Rightarrow\Leftarrow$

(d) Choose the same a_n. Define $K_n = \mathbb{N} \backslash \{k \in \mathbb{N} \mid k \leq n\}$, i.e., the natural numbers from n onwards. Since for all n, there exists $f_n(p_n) = n$ so $f^{Pre}(K_n)$ is nonempty.

f is continuous and K_n is closed in $\mathbb{R}$, so $f^{Pre}(K_n)$ is also closed.

Since $K_1 \subset K_2 \subset K_3 \cdots \Rightarrow f^{Pre}(K_1) \subset f^{Pre}(K_2) \subset \ldots$ is a nested decreasing sequence.

WTS $\bigcap_{n \in \mathbb{N}} f^{Pre}(K_n) = \varnothing$. Suppose there exists $y \in \bigcap_{n \in \mathbb{N}} f^{Pre}(K_n)$.

Then $y \in f^{Pre}(K_n)\ \forall\ n$. It follows that $f(y) \in K_n\ \forall\ n \in \mathbb{N}$. Specifically $f(y) \in K_1 \Rightarrow f(y) \in \mathbb{N}, f(y) \geq 2$. Thus $f(y) = m \in \mathbb{N}, m \geq 2$.

But then $f(y) \notin K_m$. $\Rightarrow\Leftarrow$ $\square$

Problem 3.2 (2.152)

Write jingles at least as good as the following. Pay attention to the meter as well as the rhyme.

When a set in the plane

is closed and bounded

you can always draw

a curve around it.

Solution

$\square$

Problem 3.3 (3.17)

Define $\mathfrak{e} : \mathbb{R} \to \mathbb{R}$ by

$$\mathfrak{e}(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

(a) Prove that $\mathfrak{e}$ is smooth; that is, $\mathfrak{e}$ has derivatives of all orders at all points x. [Hint: L'Hopital and induction. Feel free to use the standard differentiation formulas about e^x from calculus.]

(b) Is $\mathfrak{e}$ analytic?

(c) *[Omitted in pset]* The bump function

$$\beta(x) = e^2 \mathfrak{e}(1 - x)\mathfrak{e}(x + 1)$$

(d) For $|x| < 1$, show that the bump function

$$\beta(x) = e^{2x^2/(x^2-1)}$$

Bump functions have wide use in smooth function theory and differential topology. The graph of β looks like a bump.

Solution

(a) Let P_r be the proposition that there exists $r-$th order derivative of $\mathfrak{e}$ on $\mathbb{R}$, namely:

$$\mathfrak{e}^{(r)}(x) = \begin{cases} A(1/x)e^{-1/x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

where $A(1/x)$ is a polynomial in $1/x$.

Base case: $r = 1$.

When $x < 0$, $\mathfrak{e}'(x) = 0$.

When $x > 0$, $\mathfrak{e}'(x) = x^{-2}e^{-1/x}$.

Let's consider $x = 0$ and estimate:

$$\begin{aligned} \lim_{x\to 0+} \frac{e^{-1/x} - 0}{x - 0} &= \lim_{x\to 0+} \frac{1/x}{e^{1/x}} \\ &= \lim_{p\to\infty} \frac{p}{e^p} = 0 \end{aligned}$$

since e^p grows faster than any polynomial p^m. Meanwhile, it's trivial that

$$\lim_{x\to 0-} \frac{\mathfrak{e}(x) - 0}{x - 0} = 0$$

so $\mathfrak{e}'(0) = 0$. Therefore

$$\mathfrak{e}'(x) = \begin{cases} x^{-2}e^{-1/x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

P_1 is therefore true.

Inductive case: Suppose that P_k is true:

$$\mathfrak{e}^{(k)}(x) = \begin{cases} A(x)e^{-1/x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

We want to show that P_{k+1} is also true.

Then for $x > 0$,

$$\begin{aligned} \mathfrak{e}^{(k+1)}(x) &= A'(1/x)e^{-1/x} + A(x)[x^{-2}e^{-1/x}] \\ &= B(x)e^{-1/x} \end{aligned}$$

since if $A(1/x) = \sum_{j=0}^{l} a_j x^{-j}$ then

$$A'(1/x) = \sum_{j=0}^{l} (-j)a_j x^{-j-1}$$

is another polynomial in $1/x$.

For $x < 0$, it's trivial that $\mathfrak{e}^{k+1}(x) = 0$.

Consider when $x = 0$, estimate:

$$\begin{aligned}\lim_{x \to 0+} \frac{\mathfrak{e}^{(k)}(x) - 0}{x - 0} &= \lim_{x \to 0+} \frac{A(1/x)e^{-1/x}}{x} \\ &= \lim_{p \to \infty} \frac{pA(p)}{e^p} = 0\end{aligned}$$

since e^p grows faster than polynomial $pA(p)$.

The limit from 0- is also 0, so we can conclude

$$\mathfrak{e}^{(k+1)}(x) = \begin{cases} B(1/x)e^{-1/x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

so P_{k+1} is also true.

Conclusion: By mathematical induction, P_n is true for all $n \in \mathbb{N}$. $\mathfrak{e}$ therefore has derivatives of all orders on $\mathbb{R}$. $\square$

(b) It is not analytic around $x = 0$.

Suppose that it is, then there exists $\delta > 0$ such that if $|h| < \delta$ then

$$\sum_{r=0}^{\infty} \frac{\mathfrak{e}^r(0)}{r!} h^r = f(h)$$

(the series converges to $f(h)$).

But $LHS = 0$ and $f(h) > 0 \; \forall \; h > 0$. $\Rightarrow\Leftarrow$ $\square$

(d) For $|x| < 1$, we have $1 - x > 0, x + 1 > 0$. Therefore

$$\begin{aligned}e^2\mathfrak{e}(1-x)\mathfrak{e}(x+1) &= e^2 e^{-1/(1-x)} e^{-1/(x+1)} \\ &= e^{2 - \frac{1}{1-x} - \frac{1}{x+1}} \\ &= e^{2x^2/(x^2-1)}\end{aligned}$$

$\square$

Problem 3.4 (3.19)

Recall that the oscillation of an arbitrary function $f : [a, b] \to \mathbb{R}$ at x is

$$\operatorname{osc}_x f = \limsup_{t \to x} f(t) - \liminf_{t \to x} f(t)$$

In the proof of the Riemann-Lebesgue Theorem, D_k refers to the set of points with oscillation $\geq 1/k$.

(a) Prove that D_k is closed.

(b) Infer that the discontinuity set of f is a countable union of closed sets. (This is called an $\mathbf{F}_\sigma$-set)

(c) Infer from (b) that the set of continuity points is a countable intersection of open sets. (This is called a $\mathbf{G}_\sigma$-set)

Solution

Alternatively,

$$\text{osc}_x f = \lim_{r\to 0} diam f([x-r, x+r])$$

(a) Let $\{p_n\} \subset D_k$ such that $p_n \xrightarrow{n\to\infty} p \in \mathbb{R}$. WTS $p \in D_k$, i.e., $\text{osc}_p f \geq 1/k$.

First, note that $diam f([x-r, x+r]) \geq diam f([x-r', x+r'])$ if $r \geq r'$. In other words, $diam f([x-r, x+r])$ is increasing in r.

It follows that $diam f([x-r, x+r]) \geq osc_x f \ \forall\, x, r$.

For all $r > 0$, since $p_n \xrightarrow{n\to\infty} p$, there exists $N \in \mathbb{N}$ such that $p_N \in (p - r/2, p + r/2)$. This implies $[p_N - r/2, p_N + r/2] \subset [p - r, p + r]$.

Then,

$$diam f([p-r, p+r]) \geq diam f([p_N - r/2, p_N + r/2]) \geq 1/k$$

This is true for all $r > 0$. It follows that

$$\text{osc}_p f = \lim_{r\to 0} diam f([p-r, p+r]) \geq 1/k$$

Thus $p \in D_k$. D_k is therefore closed. $\square$

(b)

$$D = \bigcup_{k\in\mathbb{N}} D_k$$

with each D_k closed. $\square$

(c) The set of continuity points is

$$D^C = \left(\bigcup_{k\in\mathbb{N}} D_k\right)^C = \bigcap_{k\in\mathbb{N}} D_k^C$$

D_k^C is open since D_k is closed. $\square$

Problem 3.5 (3.20*)

Baire's Theorem (page 256) asserts that if a complete metric space is the countable union of closed subsets then at least one of them as nonempty interior. Use Baire's Theorem to show that the set of irrational numbers is not the countable union of closed subsets of $\mathbb{R}$.

Solution

Suppose $\mathbb{R}\backslash\mathbb{Q} = \bigcup_{k\in\mathbb{N}} U_k$, U_k closed in $\mathbb{R}$.

Then

$$\mathbb{R} = \bigcup_{k\in\mathbb{N}} U_k \sqcup \mathbb{Q} = \bigcup_{k\in\mathbb{N}} U_k \sqcup \bigcup_{k\in\mathbb{N}} \{q_k\} = \bigcup_{k\in\mathbb{N}} (U_k \sqcup \{q_k\})$$

The singleton set $\{q_k\}$ is closed, so $U_k \sqcup \{q_k\}$ is also closed.

$\mathbb{R}$ is a complete metric space. Baire's Theorem then implies that there exists $k \in \mathbb{N}$ such that

$$\operatorname{int}(U_k \sqcup \{q_k\}) \neq \varnothing$$

Then there exists an interior point $p \in U_k \sqcup \{q_k\}$ around which we can draw an open ball $B \subset U_k \sqcup \{q_k\}$. But any open ball around p contains at least 2 rationals, so $B \not\subset U_k \sqcup \{q_k\}$ $\Rightarrow\Leftarrow$

It follows that the set of irrationals is not a countable union of closed subsets of $\mathbb{R}$. $\square$

Problem 3.6 (3.21)

Use Exercises 19 and 20 to show there is no function $f : \mathbb{R} \to \mathbb{R}$ which is discontinuous at every irrational number and continuous at every rational number.

Solution

We know that

$$\mathbb{R} = \bigcup_{n \in \mathbb{N}} [-n, n]$$

so the set of discontinuities of f on $\mathbb{R}$ is the countable union of the set of discontinuities of f on $[-n, n]$. From (19), the set of discontinuities on $[-n, n]$ has to be a countable union of closed sets. A countable union of a countable union of closed sets is a countable union of closed sets. From (20), $\mathbb{R}\backslash\mathbb{Q}$ can't be a countable union of closed sets. It follows that the set of discontinuities on $\mathbb{R}$ is not $\mathbb{R}\backslash\mathbb{Q}$.

Taking complements, it also follows that the set of continuities is not $\mathbb{Q}$. $\square$

Problem 3.7 (3.31)

Define a Cantor set by removing from [0,1] the middle interval of length 1/4. From the remaining two intervals F^1 remove the middle intervals of length 1/16. From the remaining four intervals F^2 remove the middle intervals of length 1/64, and so on. At the $n^t h$ step in the construction F^n consists of 2^n subintervals of F^{n-1}.

(a) Prove that $F = \bigcap F^n$ is a Cantor set but not a zero set. It is referred to as a **fat Cantor set**.

(b) Infer that being a zero set is not a topological property: If 2 sets are homeomorphic and one is a zero set then the other need not be a zero set. [Hint: To get a sense of this fat Cantor set, calculate the total length of the intervals which comprise its complement. See Figure 52, Exercise 35]

Solution

(a) The length of the complement of F is

$$\sum_{k \in \mathbb{N}} 2^{k-1} \left(\frac{1}{4}\right)^k = \frac{1}{2} \sum_{k \in \mathbb{N}} \left(\frac{1}{2}\right)^k = \frac{1}{2} \left(\frac{1/2}{1 - 1/2}\right) = \frac{1}{2} < 1$$

so F is not a zero set.

We now show that F is a Cantor set by showing it is compact, nonempty, perfect and totally disconnected.

Each closed interval is compact, F is the countable intersection of compacts so it is compact.

F is clearly nonempty and infinite. Let E be the set of endpoints of the closed intervals in $\{F_n\}$.

We want to show F is perfect. Let $x \in F$. WTS x is a cluster point of F. Fix $\varepsilon > 0$.

Then pick n sufficiently large such that $(3/8)^n < \varepsilon$. Then x lies in one of the 2^n intervals of length $(3/8)^n$ that comprise F^n. Call this interval I. Then $I \subset B_{\mathbb{R}}(x, \varepsilon) = (x-\varepsilon, x+\varepsilon)$.

But $E \cap I$ is infinite (E has infinitely endpoints from its infinitely many subintervals) and $E \cap I \subset I$. It follows that $E \cap I \subset B_{\mathbb{R}}(x, \varepsilon)$. So x is a cluster point of F. Thus F is perfect.

Lastly, we prove that F is totally disconnected. With a similar set up, we know that $x \in I$ where I is one of 2^n intervals of length $(3/8)^n$ whose union is F^n.

I is closed in $\mathbb{R}$, so it is closed in F^n. $J = F^n \backslash I$ is a finite $(2^n - 1)$ union of closed intervals, so it is also closed in $\mathbb{R}$, so it is closed in F^n. It follows that I is clopen in F^n.

I is clopen in F^n and $F \subset F_n$, so $F \cap I$ is clopen in F. Therefore $F \cap I \subset I \subset B_{\mathbb{R}}(x, \varepsilon) = (x - \varepsilon, x + \varepsilon)$. It follows that F is totally disconnected.

From all the above, it follows that F is a Cantor set. $\square$

(b) Using 33b, C and F are homeomorphic. C is a zero set but F is not a zero set. Therefore it is not a topological property. $\square$

Problem 3.8 (3.33)

(a) Prove that the characteristic function f of the middle-thirds Cantor set C is Riemann integrable but the characteristic function g of the fat Cantor set F (Exercise 31) is not.

(b) Why is there a homeomorphism $h : [0, 1] \to [0, 1]$ sending C onto F?

(c) Infer that the composite of Riemann integrable functions need not be Riemann integrable. How is this example related to Corollaries 28 and 32 of the Riemann-Lebesgue Theorem? See also Exercise 35.

Solution

(a) 1. We show that χ_C is continuous on $[0, 1]\backslash C$. Let $x \in [0, 1]\backslash C$. Then $x \in (a, b)$ where (a, b) is the (removed) middle third of some interval. $f \equiv 0$ on (a, b), so it is continuous at x.

It follows that the set of discontinuities is a subset of C, a zero set, and therefore χ_C is RI (Riemann-Lebesgue Theorem).

2. We show that χ_F is discontinuous on F.

Suppose not, that χ_F is continuous at some $x \in F$. Fix $\varepsilon = 1/2$, then there exists δ such that

$$|x' - x| < \delta \Rightarrow |\chi_F(x') - \chi_F(x)| < 1/2$$

But χ_F only take values of 0 and 1 so

$$|x' - x| < \delta \Rightarrow \chi_F(x') = \chi_F(x) = 1 \Rightarrow x' \in F$$

This implies $(x-\delta, x+\delta) \subset F$. Which implies $(x-\delta, x+\delta) \subset F_n$ for all n. Since each F_n is the disjoint union of a finite number of intervals, it implies that $(x-\delta, x+\delta) \subset I_n \subset F_n$ for some interval I_n in F_n, for all n. But I_m with m sufficiently large has length $(3/8)^m << 2\delta$, so $(x-\delta, x+\delta) \not\subset I_m$. $\Rightarrow\Leftarrow$

By contradiction, it follows that χ_F is discontinuous on F, a non-zero set. It follows that the set of discontinuities is a non-zero set. By Riemann-Lebesgue Theorem, it is not RI.

(b) We can always construct a continuous bijection mapping $[a,b] \to [c,d], x \mapsto c + \frac{x-a}{b-a}(d-c)$, which maps $a \mapsto c, b \mapsto d$.

From this, we can construct continuous bijection $f(x,n) : [a,b] \to [c,d]$ that maps $[a, \frac{a+b}{2} - \frac{1}{3^n}]$ to $[c, \frac{c+d}{2} - \frac{1}{4^n}]$, $[\frac{a+b}{2} - \frac{1}{3^n}, \frac{a+b}{2} + \frac{1}{3^n}]$ to $[\frac{c+d}{2} - \frac{1}{4^n}, \frac{c+d}{2} + \frac{1}{4^n}]$, $[\frac{a+b}{2} + \frac{1}{3^n}, b]$ to $[\frac{c+d}{2} + \frac{1}{4^n}, d]$. $f(x)$ is continuous because all of its pieces are continuous and have the same values at endpoints. Specifically denote this map as $\Phi_{[a,b],[c,d],n}(x)$

Each C_k and F_k consists of 2^k intervals, which we denote by $\{C_{k,j}\}$ and $\{F_{k,j}\}$ for $1 \le j \le 2^k$ with the natural ordering.

Following this, we construct a family of homeos $f_n(x) : [0,1] \to [0,1]$ as follows:

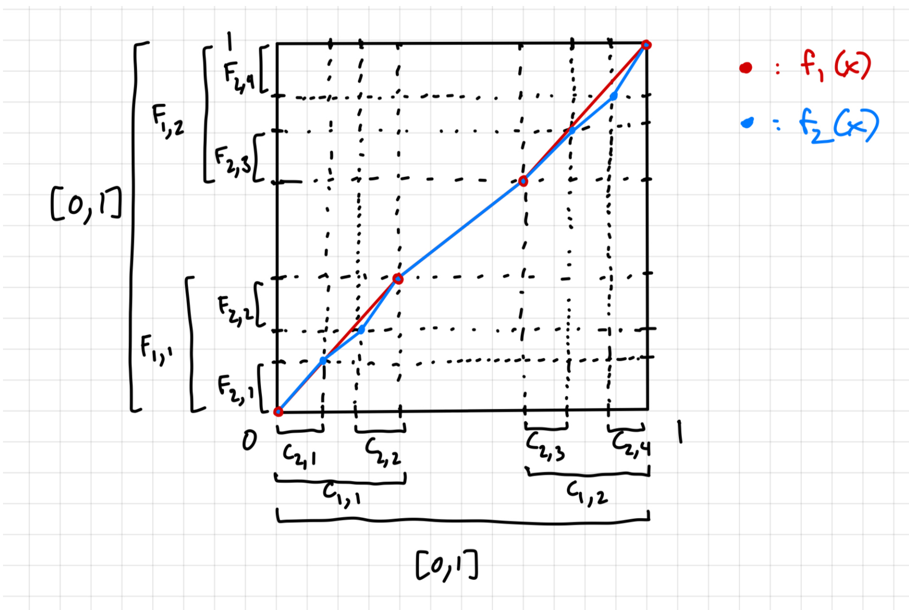
$f_1(x)$ is $\Phi_{[0,1],[0,1],1}(x)$, which by definition of Φ maps $C_{1,j}$ to $F_{1,j}$ for $1 \le j \le 2^1$, and C_1^C to F_1^C.

Suppose homeo $f_k(x)$ is defined for some $k \in \mathbb{N}$, that maps $C_{k,j}$ to $F_{k,j}$ for $1 \le j \le 2^k$, and C_k^C to F_k^C.

Then recursively define $f_{k+1}(x)$ as the piecewise combination of $\Phi_{C_{k,j},F_{k,j},(k+1)}$ for $1 \le j \le 2^k$, which maps $C_{k+1,j'}$ to $F_{k+1,j'}$ for $1 \le j' \le 2^{k+1}$ and $C_{k+1}^C \backslash C_k^C$ (the newly removed middle intervals of C) to $F_{k+1}^C \backslash F_k^C$ (the newly removed middle intervals of F).

The interval C_k^C remains to be mapped from and F_k^C remains to be mapped to, we use $f_k(x)$ to map them.

Pictorially,



We now WTS that $\{f_n(x)\}$ uniformly converges, i.e., converges in $(C^0([0,1]), d_{\text{sup}})$.

Note that for $m > n$, f_m agrees with f_n on C_n^C, so

$$d_{\text{sup}}(f_m, f_n) = \sup_{x\in[0,1]} |f_m(x) - f_n(x)| = \sup_{x\in C_n} |f_m(x) - f_n(x)| \leq (3/8)^n$$

Therefore, for any $\varepsilon > 0$, we can choose N sufficiently large such that $(3/8)^N < \varepsilon$. Then for $m, n \geq N$

$$d(f_m, f_n) = \sup_{x\in[0,1]} |f_m(x) - f_n(x)| < \varepsilon$$

so $\{f_n\}$ is Cauchy, and converges to some h. Each f_n is continuous so h must also be continuous. h is also a bijection, and maps from [0, 1] compact to [0, 1]. It is therefore a homeomorphism.

This h maps C to F, because C^C is mapped to F^C. This is because

$$C^C = \bigcup_{k\in\mathbb{N}} C_k^C$$

and for any $k \in \mathbb{N}$, $f_l(x)$ for $l \geq k$ agree on C_k^C (send to the same points in F_k^C). It follows that $h(x)$ also sends C_k^C to F_k^C (uniform convergence implies pointwise convergence). This is true for all $k \in \mathbb{N}$.

We therefore have a homeo $h : [0,1] \to [0,1]$ that maps C to F. $\square$

(c) It is clear that $\chi_C \circ h^{-1} = \chi_F$.

h^{-1} is a homeo on [0, 1] so it is clearly RI. From (a), χ_C is also RI, but χ_F is not.

Corollary 28 doesn't apply here because χ_C is RI and h^{-1} is continuous, not the other way around.

Corollary 32 doesn't apply because $(h^{-1})^{-1} = h$ is not Lipschitz. It sends a zero set (C) to a non-zero set (F) $\square$

Problem 3.9 (IV)

This exercise is about the middle thirds Cantor set C.

(a) Let I_L be the interval [0, 1/3] and let I_R be the interval [2/3, 1]. Let $C_L = C \cap I_L$ and $C_R = C \cap I_R$. Define maps $f_L : I_L \to [0,1]$ and $f_R : I_R \to [0,1]$ by $f_L(x) = 3x$ and $f_R(x) = 3x - 2$. Prove that

$$f_L(C_L) = f_R(C_R) = C$$

How can you use these maps to systematically label points in C?

(b) Find 4 squares $R_1, \ldots, R_4 \subset [0,1]^2$ and 4 affine maps $f_i : R_i \to [0,1]^2, i = 1, \ldots, 4$ such that, if we set $(C \times C)_i = (C \times C) \cap R_i$, then

$$C \times C = (C \times C)_1 \cup \ldots (C \times C)_4$$

and $f_i((C \times C)_i) = C \times C$. How can you use these maps to systematically label points in $C \times C$? See Figure 1.

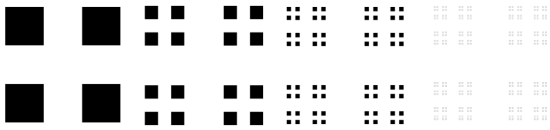


Figure 1: Constructing $C \times C$

(c) Prove that $C \times C$ is homeomorphic to C.

(d) Suppose you divide $[0,1]^2$ into 9 equal smaller squares, remove the interior of the middle ninth square from $[0,1]^2$, repeat removing the inner ninth from the remaining squares, etc., and intersect to obtain a compact set M. Why is M not homeomorphic to $C \times C$?

(e) Prove that every real number $r \in [0,2]$ can be written as a sum $x+y$, where $x, y \in C$. (Equivalently, $C + C = [0,2]$). [Hint: there are two ways to do this — algebraically and geometrically. To do algebraically, think of base 3 representations of real numbers. To do geometrically, look at the images in Figure 2]

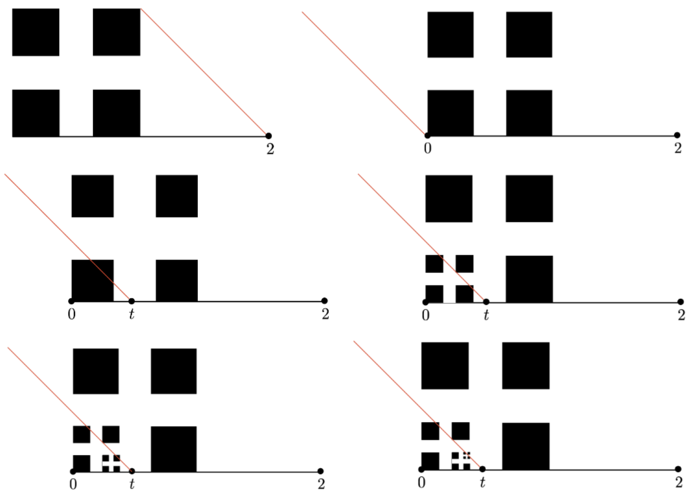


Figure 2: Hint for IV. (e)

(f) (EC) Fix $a \in [1,3]$. For which b does the equation $ax + y = b$ always have a solution with $x, y \in C$? What happens when $a > 3$?

Solution

(a) WTS $f_L(C_L) = C$.

The construction of C yields an alternative expression for C:

$$C = [0,1] \backslash \left\{ \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m} \right) \mid k, m \in \mathbb{N} \right\}$$

since these are the intervals removed from C_{m-1} to form C_m ($C_0 \equiv [0,1]$).

Let $x \in C_L$. Suppose $f_L(x) = 3x \notin C$ then

$$3x \in \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$$

for some $k, m \in \mathbb{N}$. But this implies

$$x \in \left(\frac{3k+1}{3^{m+1}}, \frac{3k+2}{3^{m+1}}\right)$$

so $x \notin C$, $\Rightarrow\Leftarrow$

It follows that $3x \in C$. This implies $f_L(C_L) \subseteq C$

Let $y \in C$, WTS $\frac{y}{3} \in C_L$. This direction is similar, so $C \subseteq f_L(C_L)$.

It follows that $f_L(C_L) = C$.

The proof for $f_R(C_R) = C$ is similar, since we can reflect [0, 1] and translate 1 unit in the positive $x-$direction. Then $f_R^{flipped}(x') = 3x'$. $\square$

It's trivial that $C \cap (1/3, 2/3) = \varnothing$. It follows that

$$C = (C \cap I_L) \sqcup (C \cap I_R)$$

Given $x_0 \in C$. Label x_0 as an infinite string of 'L' and 'R' as follows:

1. Since $x_0 \in C = (C \cap I_L) \sqcup (C \cap I_R)$, we have $x_0 \in C \cap I_L$ or $x_0 \in C \cap I_R$. Initialize $x' = x_0$.
2. If $x' \in C_L$, add to string of x_0 'L'. Assign $f_L(x') \to x'$.
3. Else ($x' \in C_R$), add to string of x_0 'R'. Assign $f_R(x') \to x'$.
4. Repeat (2) and (3) with new value of x'.

(b) Define $R_1 := I_L \times I_L, R_2 := I_L \times I_R, R_3 := I_R \times I_L, R_4 := I_R \times I_R$ and

$$\begin{aligned}
f_1 : R_1 \to [0,1]^2 \quad (x, y) \mapsto (f_L(x), f_L(y)) \\
f_2 : R_2 \to [0,1]^2 \quad (x, y) \mapsto (f_L(x), f_R(y)) \\
f_3 : R_3 \to [0,1]^2 \quad (x, y) \mapsto (f_R(x), f_L(y)) \\
f_4 : R_4 \to [0,1]^2 \quad (x, y) \mapsto (f_R(x), f_R(y))
\end{aligned}$$

Then $(C \times C)_1 = (C \times C) \cap R_1 = C_L \times C_L$. Similar for $i = 2, 3, 4$. Since $C = C_L \sqcup C_R$, it follows that

$$C \times C = \bigcup_{i=1}^{4} (C \times C)_i$$

Furthermore,

$$f_1((C \times C)_1) = f_1(C_L \times C_L) = C \times C$$

since $f_L(C_L)$ as proven in (a). The same proof applies for $i = 2, 3, 4$.

We can now label $(x_0, y_0) \in C \times C$ as a infinite string of 'L' and 'R' as follows:

1. Initialize $x' = x_0, y' = y_0$.

2. If $x' \in C_L$, add 'L' to string. Assign $f_L(x') \to x'$.

3. Else ($x' \in C_R$), add 'R' to string. Assign $f_R(x') \to x'$.

4. If $y' \in C_L$, add 'L' to string. Assign $f_L(y') \to y'$.

5. Else ($y' \in C_R$), add 'R' to string. Assign $f_R(y') \to y'$.

6. Repeat from (2) with new values of x' and y'.

(c) Construct function $h : C \times C \to C$, $(x_0, y_0) \in C \times C$ gets mapped to z_0 of the same L, R label sequence.

More concretely, let S be the set of all infinite sequence of 'L' and 'R'. Then there exists the labeling bijection as constructed in (a) $l_1 : C \times S$, and the label bijection as constructed in (b) $l_2 : C \times C \to S$. Then $h = l_2^{-1} \circ l_1$ is a bijection.

By construction of l_1, we have

$$l_1^{-1}(s) : S \to C$$

that maps

$$s \mapsto \sum_{k \in \mathbb{N}} \frac{2}{3^k} \delta(s[k], \text{`}R')$$

where $s[k]$ is the $k-$th letter of string s, and $\delta(s_k, \text{`}R')$ returns 1 if $s[k]$ is 'R', 0 otherwise. We now endow S with the metric

$$d_S(s_1, s_2) = d_C(l_1^{-1}(s_1), l_1^{-1}(s_2))$$

Given $s_0 \in S, \varepsilon > 0$, then there exists $N \in \mathbb{N}$ such that $1/3^N < \varepsilon$.

Let $x_0 = l_1^{-1}(s_0)$. Then for any $x' \in B_C(x_0, 1/3^N)$, $s' = l_1(x') = \text{`}s_0[1]s_0[2] \dots s_0[N] \dots'$

This is because $d_C(x', x_0) < 1/3^N$ so

$$|\sum_{k \in \mathbb{N}} \frac{2}{3^k} \delta(s'[k], 'R') - \sum_{k \in \mathbb{N}} \frac{2}{3^k} \delta(s_0[k], 'R')| < 1/3^N$$

which forces s' to agree with s_0 for the first N letters.

It follows that $d_S(s_0, l_1(x')) < 1/3^N < \varepsilon$. Thus, $x' \in B_C(x_0, 1/3^N) \Rightarrow d_S(s_0, l_1(x')) < \varepsilon$.

l_1 is therefore continuous. C is also compact. So l_1 is homeo.

A similar proof for l_2 applies. It follows that h is also a homeo. $\square$

(d) In short, M is connected, but $C \times C$ is disconnected, so they can't be homeomorphic.

$\square$

(e) Rephrasing the question, we want to show that the line

$$l : x + y = r$$

satisfies $l \cap (C \times C) \neq \varnothing$ for all $r \in [0, 2]$.

Case 1: $r \in C$.

Then there is a trivial intersection $(r, 0) \in l \cap C \times C$.

Case 2: $r \notin C$.

Then r was in the middle third of some subinterval $C_{N,k} \subseteq C_N$. Call the thirds of $C_{N,K} = I_L \sqcup I_M \sqcup I_R$, then $r \in I_M$. Note that the length of I_L, I_M, I_R is $1/3^N$. For convenience, let $q = 1/3^N$.

We now WTS that $l \cap (I_L \times [0, 1/3^N]) \neq \varnothing$.

Denote $I_L = [a, a+q], I_M = (a+q, a+2q), I_R = [a+2q, a+3q]$ then $r = a+q+p$ where $p \in (0, q)$.

Then $a+q \in I_L, p \in (0, q) \subset [0, 1/3^N] \Rightarrow$ Point $(a+q, p) \in (I_L \times [0, 1/3^N])$. Furthermore, $(a+q)+p = r$ trivially, so point $(a+q, p) \in l$.

Thus, $l \cap (I_L \times [0, 1/3^N]) \neq \varnothing$. And $(I_L \times [0, 1/3^N]) \subset (C_N \times C_N) \Rightarrow l \cap (C_N \times C_N) \neq \varnothing$.

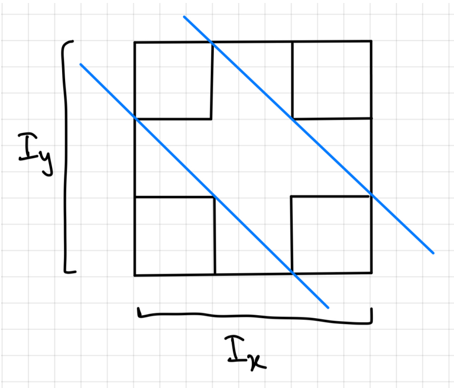
An easy consequence of this is that since $C_N \subset C_n \; \forall \, n \leq N \Rightarrow l \cap (C_n \times C_n) \neq \varnothing \; \forall \, n \leq N$.

We want to show that $l \cap (C_{N+1} \times C_{N+1}) \neq \varnothing$. Since $l \cap (C_N \times C_N) \neq \varnothing$,

$$l \cap (I_x \times I_y) \neq \varnothing$$

where $I_x, I_y \subset C_N$, are some of the 2^N intervals that make up C_N.

The follow pictorial proof (which can easily be rigorized) shows that l must lie in 1 of 3 zones of $I_x \times I_y$ divided by the blue lines, and therefore intersect one of the "corner squares". It follows that $l \cap (C_{N+1} \times C_{N+1}) \neq \varnothing$.



By induction, it follows that $l \cap (C_n \times C_n) \; \forall \, n \geq N$.

Combining the 2 subparts, it follows that $l \cap (C_n \times C_n) \; \forall \, n \in \mathbb{N} \Rightarrow l \cap (C \times C)$. $\square$