Math 20250
Abstract Linear Algebra

Cong Hung Le Tran

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Professor: Zijian Yao
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Course materials: Linear Algebra by Hoffman and Kunze (2nd Edition), Linear Algebra Done Wrong by Treil

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Lecture 5

Span, Linear Independence, Basis

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Recall. Linear Combination: Let $V=$ $\mathbb{K}$-vector space with $v_1, v_2, \ldots, v_r \in V$ then

$$\mathbb{K}\langle v_1, v_2, \ldots, v_r\rangle \coloneqq \{w \in W \mid= w = a_1 v_1 + \cdots + a_r v_r; a_i \in \mathbb{K}\} \subseteq V (\text{ is a subspace of } V)$$

Definition 5.1 (Span).
$\{v_1, v_2, \ldots, v_r\}$ span V if

$$\mathbb{K}\langle v_1, v_2, \ldots, v_r\rangle = V$$

i.e. equality is achieved: every vector in V can be written as linear combinations of $\{v_1, v_2, \ldots, v_r\}$

Connecting to the previous lecture, let $\psi : \mathbb{K}^r \to V$ then $\psi \in \operatorname{Hom}_{\mathbb{K}}(\mathbb{K}^r, V) \xrightarrow{\sim} V^{\oplus r}$, i.e. ψ corresponds to $(v_1, v_2, \ldots, v_r)$ in V.
In particular, $(v_1, v_2, \ldots, v_r) \in V^{\oplus r}$ determines the map:

$$\begin{aligned}
\psi : (1, 0, \ldots, 0) \in \mathbb{K}^r &\to v_1 \\
(0, 1, \ldots, 0) \in \mathbb{K}^r &\to v_2 \\
&\vdots \\
(0, 0, \ldots, 1) \in \mathbb{K}^r &\to v_r \\
(\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{K}^r &\to \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_r v_r
\end{aligned}$$

Lemma 5.1.

1. Let $\psi : \mathbb{K}^r \to V$ be a linear transformation determined by $v_1, v_2, \ldots, v_r \in V$, i.e. $\psi(\alpha_1, \alpha_2, \ldots, \alpha_r) \coloneqq \sum_{i=1}^r \alpha_i v_i$, then

$$\operatorname{im}(\psi) = \mathbb{K}\langle v_1, v_2, \ldots, v_r\rangle$$

   is a subspace of V

2. $\{v_1, v_2, \ldots, v_r\}$ span $V \Leftrightarrow \psi$ is surjective

   i.e. a surjection $\mathbb{K}^r \to V$ corresponds to r vectors $v_1, v_2, \ldots, v_r \in V$ that span V

Remark. V is finite dimensional when $\exists$ surjection $\mathbb{K}^d \to V$
$\Leftrightarrow$ $\exists\, d$ vectors $v_1, v_2, \ldots, v_r$ that span V.
Recall: $\dim V = \min\{r \in \mathbb{Z}_{\geq 0}$ such that $\exists$ surjective $\mathbb{K}^r \to V\}$.
Next, what does it mean for ψ to be injective?

Definition 5.2 (Linear Independence).
$v_1, v_2, \ldots, v_r \in V$ are **linearly independent** if

$$a_1 v_1 + a_2 v_2 + \cdots + a_r v_r = 0; a_i \in \mathbb{K} \Rightarrow a_1 = a_2 = \cdots = a_r = 0$$

i.e. there doesn't exist non-trivial relations between the vectors.

Example. In $\mathbb{R}^2$, (0, 1) and (0, 2) are not linearly independent because

$$(-2)(0,1) + (0,2) = (0,0)$$

But (0, 1) and (1,0) are linearly independent.

Consequentially, they are **linearly dependent** otherwise, i.e.

$$\exists\, a_i \text{ not all } 0 \text{ such that } \sum a_i v_i = 0$$

Lemma 5.2. Let $\varphi : V \to W$ be a linear transformation then φ is injective if and only if

$$\ker(\psi) = \{0\} \subseteq V$$

Proof (Lemma).
($\Rightarrow$) We assume that φ is injective,
($\Leftarrow$) Suppose $\ker \psi = 0$ then we want to show if

$$a_1 v_1 + a_2 v_2 + \cdots + a_r v_r = 0$$

$\square$

Lemma 5.3. Given $\psi : \mathbb{K}^r \to V$ corresponds to $v_1, v_2, \ldots, v_r$ then $v_1, v_2, \ldots, v_r$ are linearly independent if and only if ψ is injective