Solving a Log-Linearized RBC Model Algebraically

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These notes describe how to construct a log-linear approximation of an RBC model around a non-stochastic steady state and how to solve the linearized model algebraically using the method of undetermined coefficients. The procedure described below is used to produce the simulations on this webpage: https://www.briancjenkins.com/simulations/centralized-rbc.html. The exercise is tedious and demonstrates the great value provided by the many computational toolsd and methods available for approximating and solving DSGE models.

The Model   The equilibrium conditions of the model are given by:

$$\varphi \left(1 - L_t\right)^{-\eta} = (1 - \alpha)C_t^{-\sigma}Y_t/L_t \tag{1}$$

$$C_t^{-\sigma} = \beta E_t \left[C_{t+1}^{-\sigma}(\alpha Y_{t+1}/K_{t+1} + 1 - \delta)\right] \tag{2}$$

$$K_{t+1} = I_t + (1 - \delta)K_t \tag{3}$$

$$Y_t = C_t + I_t \tag{4}$$

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} \tag{5}$$

$$\log A_{t+1} = \rho_a \log A_t + \epsilon_{t+1} \tag{6}$$

Equation (1) is the representative household's first-order condition for the optional choice of labor. Equation (2) is the household's Euler equation reflecting an optimal choice of capital for period $t + 1$. Equations (3), (4), and (5) describe the evolution of the aggregate capital stock, the goods market clearing condition, and the production function. Finally, equation (6) indicates that log TFP follows an AR(1) process where $\epsilon_{t+1} \sim \mathcal{N}\left(0, \sigma_\epsilon^2\right)$ is an exogenous shock process.

Steady State   Let capital letters without time subscripts denote non-stochastic steady state values of the model variables. The steady state is given by the following system:

$$\varphi \left(1 - L\right)^{-\eta} = (1 - \alpha)C^{-\sigma}Y/L \tag{7}$$

$$C^{-\sigma} = \beta E \left[C^{-\sigma}(\alpha Y/K + 1 - \delta)\right] \tag{8}$$

$$K = I + (1 - \delta)K \tag{9}$$

$$Y = C + I \tag{10}$$

$$Y = AK^{\alpha}L^{1-\alpha} \tag{11}$$

$$\log A = \rho_a \log A \tag{12}$$

In general, the nonlinear terms in equation (7) prevent this system from being solved algebraically. However, most of the work can be done with algebra. First, note that steady state TFP is trivial:

$$\boxed{A = 1} \tag{13}$$

Next, use the Euler equation to solve for the steady state capital-to-labor ratio:

$$\frac{K}{L} = \left(\frac{\alpha A}{\beta^{-1} + \delta - 1}\right)^{\frac{1}{1-\alpha}} \tag{14}$$

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Next, combine equation (14) with (7) to obtain an equation that characterizes steady state labor:

$$\varphi(1-L)^{-\eta}L^{\sigma} = (1-\alpha)\left[A(K/L)^{\alpha} - \delta(K/L)\right]^{-\sigma}A(K/L)^{\alpha} \tag{15}$$

Note that the right side of equation (15) is a constant. Solve (15) numerically.

The remaining steady state values are readily obtained:

$$K = (K/L)L \tag{16}$$

$$Y = AK^{\alpha}L^{1-\alpha} \tag{17}$$

$$I = \delta K \tag{18}$$

$$CY - I \tag{19}$$

Log-linear Approximation Let lowercase letters represent the log deviations of variables from their steady states, e.g.: $y_t = \log(Y_t) - \log(Y)$. The log-linearized equilibrium conditions are:

$$0 = -\sigma c_t + y_t - \frac{1+(\eta-1)L}{1-L}l_t \tag{20}$$

$$-\left[1-\beta(1-\delta)\right]k_{t+1} - \sigma E_t c_{t+1} + \left[1-\beta(1-\delta)\right]E_t y_{t+1} = -\sigma c_t \tag{21}$$

$$k_{t+1} = \delta i_t + (1-\delta)k_t \tag{22}$$

$$0 = \frac{C}{Y}c_t + \frac{I}{Y}i_t - y_t \tag{23}$$

$$0 = a_t + \alpha k_t + (1-\alpha)l_t - y_t \tag{24}$$

$$a_{t+1} = \rho a_t + \epsilon_{t+1} \tag{25}$$

Note that all forward-looking variables have been moved to the left sides of the equations and contemporaneous variables to the left. Next, use equations (23) and (24) to eliminate i_t and y_t from the rest of the system:

$$0 = a_t + \alpha k_t - \sigma c_t - \left(\frac{\alpha+(\eta-\alpha)L}{1-L}\right)l_t \tag{26}$$

$$\left[1-\beta(1-\delta)\right]E_t a_{t+1} + (\alpha-1)\left[1-\beta(1-\delta)\right]k_{t+1} - \sigma E_t c_{t+1} + (1-\alpha)\left[1-\beta(1-\delta)\right]E_t l_{t+1} = -\sigma c_t \tag{27}$$

$$k_{t+1} = \frac{Y}{K}a_t + \left(\alpha\frac{Y}{K}+1-\delta\right)k_t - \frac{C}{K}c_t + (1-\alpha)\frac{Y}{K}l_t \tag{28}$$

$$a_{t+1} = \rho a_t + \epsilon_{t+1} \tag{29}$$

Note that $E_t a_{t+1} = \rho a_t$. Set aside the law of motion for a_{t+1} and replace $E_t a_{t+1}$ with ρa_t in equation(27):

$$0 = a_t + \alpha k_t - \sigma c_t - \left(\frac{\alpha+(\eta-\alpha)L}{1-L}\right)l_t \tag{30}$$

$$(\alpha-1)\left[1-\beta(1-\delta)\right]k_{t+1} - \sigma E_t c_{t+1} + (1-\alpha)\left[1-\beta(1-\delta)\right]E_t l_{t+1} = -\left[1-\beta(1-\delta)\right]E_t \rho a_t - \sigma c_t \tag{31}$$

$$k_{t+1} = \frac{Y}{K}a_t + \left(\alpha\frac{Y}{K}+1-\delta\right)k_t - \frac{C}{K}c_t + (1-\alpha)\frac{Y}{K}l_t \tag{32}$$

Now we have a system of three linear equations in three variables: k_{t+1}, c_t, and l_t.[1] The system can be represented more concisely as:

$$0 = a_t + \alpha k_t - \sigma c_t + \phi_1 l_t \tag{33}$$

$$\phi_2 k_{t+1} - \sigma E_t c_{t+1} + \phi_3 E_t l_{t+1} = \phi_4 a_t - \sigma c_t \tag{34}$$

$$k_{t+1} = \phi_5 a_t + \phi_6 k_t + \phi_7 c_t + \phi_8 l_t \tag{35}$$

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[1]Note that a_{t+1} is determined outside of this system.

where:

$$\phi_1 = -\frac{\alpha + (\eta - \alpha)L}{1 - L} \tag{36}$$

$$\phi_2 = (\alpha - 1)\left[1 - \beta(1 - \delta)\right] \tag{37}$$

$$\phi_3 = (1 - \alpha)\left[1 - \beta(1 - \delta)\right] \tag{38}$$

$$\phi_4 = -\rho\left[1 - \beta(1 - \delta)\right] \tag{39}$$

$$\phi_5 = Y/K \tag{40}$$

$$\phi_6 = Y/K + 1 - \delta \tag{41}$$

$$\phi_7 = -C/K \tag{42}$$

$$\phi_8 = (1 - \alpha)Y/L \tag{43}$$

Solution I solve the model using the method of undetermined coefficients. Guess that the solution takes the following form:

$$k_{t+1} = \pi_1 a_t + \pi_2 k_t \tag{44}$$

$$c_t = \pi_3 a_t + \pi_4 k_t \tag{45}$$

$$l_t = \pi_5 a_t + \pi_6 k_t \tag{46}$$

$$\tag{47}$$

where the coefficients $\pi_1, \pi_2, \ldots, \pi_6$ are to be found. The guess implies the following expectations for c_{t+1} and l_{t+1}:

$$k_{t+1} = \pi_1 a_t + \pi_2 k_t \tag{48}$$

$$E_t c_{t+1} = (\rho \pi_3 + \pi_4 \pi_1) a_t + \pi_4 \pi_2 k_t \tag{49}$$

$$E_t l_{t+1} = (\rho \pi_5 + \pi_6 \pi_1) a_t + \pi_6 \pi_2 k_t \tag{50}$$

Plug the proposed solution into equations (33), (34), and (35) and find the following system of 6 equations in 6 unknowns:

$$0 = 1 - \sigma \pi_3 + \phi_1 \pi_5 \tag{51}$$

$$0 = \alpha - \sigma \pi_4 + \phi_1 \pi_6 \tag{52}$$

$$\phi_2 \pi_1 - \sigma \left(\rho \pi_3 + \pi_4 \pi_1\right) + \phi_3 \left(\rho \pi_5 + \pi_6 \pi_1\right) = \phi_4 - \sigma \pi_3 \tag{53}$$

$$\phi_2 \pi_2 - \sigma \pi_4 \pi_2 + \phi_3 \pi_6 \pi_2 = -\sigma \pi_4 \tag{54}$$

$$\pi_1 = \phi_5 + \phi_7 \pi_3 + \phi_8 \pi_5 \tag{55}$$

$$\pi_2 = \phi_6 + \phi_7 \pi_4 + \phi_8 \pi_6 \tag{56}$$

Now, let's solve the system. First, find π_5 as a function of π_3:

$$\pi_5 = \frac{\sigma \pi_3 - 1}{\phi_1} \tag{57}$$

and π_6 as a function of π_4:

$$\pi_6 = \frac{\sigma \pi_4 - \alpha}{\phi_1} \tag{58}$$

Use the previous equation to eliminate π_6 from equation (56):

$$\pi_2 = \phi_9 + \phi_{10} \pi_4 \tag{59}$$

where ϕ_9 and ϕ_9 are new constants introduced to keep the algebra manageable:

$$\phi_9 = \phi_6 - \frac{\phi_8 \alpha}{\phi_1} \tag{60}$$

$$\phi_{10} = \phi_7 + \frac{\phi_8 \sigma}{\phi_1} \tag{61}$$

Also, because it will make the algebra easier in a bit, let's note that:

$$\pi_2 \pi_6 = \phi_{11} \pi_4^2 + \phi_{12} \pi_4 + \phi_{13} \tag{62}$$

where:

$$\phi_{11} = \frac{\phi_{10}\sigma}{\phi_1} \tag{63}$$

$$\phi_{12} = \frac{\phi_9 \sigma - \phi_{10}\alpha}{\phi_1} \tag{64}$$

$$\phi_{13} = -\frac{\phi_9 \alpha}{\phi_1} \tag{65}$$

Now eliminate π_2 and π_6 from equation (54) to obtain a quadratic equation in π_4:

$$\underbrace{(\phi_3 \phi_{11} - \sigma \phi_{10})}_{a} \pi_4^2 + \underbrace{(\phi_2 \phi_{10} - \sigma \phi_9 + \phi_3 \phi_{12} + \sigma)}_{b} \pi_4 + \underbrace{(\phi_2 \phi_9 + \phi_3 \phi_{13})}_{c} = 0 \tag{66}$$

Set π_4 equal to the larger of the two roots. Now, we can obtain π_6 and π_2:

$$\pi_6 = \frac{\sigma \pi_4 - \alpha}{\phi_1} \tag{67}$$

$$\pi_2 = \phi_9 + \phi_{10}\pi_4 \tag{68}$$

Next, from equation (55) find:

$$\pi_1 = \phi_{14} + \phi_{15}\pi_3 \tag{69}$$

where:

$$\phi_{14} = \phi_5 - \frac{\phi_8}{\phi_1} \tag{70}$$

$$\phi_{15} = \phi_7 + \frac{\phi_8 \sigma}{\phi_1} \tag{71}$$

And plugging-in to equation (53), find:

$$\pi_3 = \frac{\phi_4 - \phi_{16}\phi_{14} + \phi_3 \rho \phi_1^{-1}}{\phi_{16}\phi_{15} + \sigma(1 - \rho) + \phi_3 \rho \sigma \phi_1^{-1}} \tag{72}$$

where:

$$\phi_{16} = \phi_2 - \sigma \pi_4 + \phi_3 \pi_6 \tag{73}$$

Next solve for π_1 and π_5:

$$\pi_1 = \phi_{14} + \phi_{15}\pi_3 \tag{74}$$

$$\pi_5 = \frac{\sigma \pi_3 - 1}{\phi_1} \tag{75}$$

Finally, the solutions for the variables i_t and y_t are found to be:

$$i_t = \left(\frac{Y}{I}[1 + (1 - \alpha)\pi_5] - \frac{C}{I}\pi_3 \right) a_t + \left(\frac{Y}{I}[\alpha + (1 - \alpha)\pi_6] - \frac{C}{I}\pi_4] \right) k_t \tag{76}$$

$$y_t = [1 + (1 - \alpha)\pi_5]a_t + [\alpha + (1 - \alpha)\pi_6]k_t \tag{77}$$