# Solving a Log-Linearized RBC Model Algebraically

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These notes describe how to construct a log-linear approximation of an RBC model around a non-stochastic steady state and how to solve the linearized model algebraically using the method of undetermined coefficients. The procedure described below is used to produce the simulations on this webpage: https://www.briancjenkins.com/simulations/centralized-rbc.html. The exercise is tedious and demonstrates the great value provided by the many computational toolsd and methods available for approximating and solving DSGE models.

**The Model** The equilibrium conditions of the model are given by:

$$\begin{aligned}
\varphi\left(1-L_t\right)^{-\eta} &= (1-\alpha) C_t^{-\sigma} Y_t / L_t && (1)\\
C_t^{-\sigma} &= \beta E_t\left[C_{t+1}^{-\sigma}\left(\alpha Y_{t+1} / K_{t+1}+1-\delta\right)\right] && (2)\\
K_{t+1} &= I_t+(1-\delta) K_t && (3)\\
Y_t &= C_t+I_t && (4)\\
Y_t &= A_t K_t^\alpha L_t^{1-\alpha} && (5)\\
\log A_{t+1} &= \rho_a \log A_t+\epsilon_{t+1} && (6)
\end{aligned}$$

Equation (1) is the representative household's first-order condition for the optional choice of labor. Equation (2) is the household's Euler equation reflecting an optimal choice of capital for period $t+1$. Equations (3), (4), and (5) describe the evolution of the aggregate capital stock, the goods market clearing condition, and the production function. Finally, equation (6) indicates that log TFP follows an AR(1) process where $\epsilon_{t+1} \sim \mathcal{N}\left(0, \sigma_\epsilon^2\right)$ is an exogenous shock process.

**Steady State** Let capital letters without time subscripts denote non-stochastic steady state values of the model variables. The steady state is given by the following system:

$$\begin{aligned}
\varphi(1-L)^{-\eta} &= (1-\alpha) C^{-\sigma} Y / L && (7)\\
C^{-\sigma} &= \beta E\left[C^{-\sigma}(\alpha Y / K+1-\delta)\right] && (8)\\
K &= I+(1-\delta) K && (9)\\
Y &= C+I && (10)\\
Y &= A K^\alpha L^{1-\alpha} && (11)\\
\log A &= \rho_a \log A && (12)
\end{aligned}$$

In general, the nonlinear terms in equation (7) prevent this system from being solved algebraically. However, most of the work can be done with algebra. First, note that steady state TFP is trivial:

$$\boxed{A=1} \tag{13}$$

Next, use the Euler equation to solve for the steady state capital-to-labor ratio:

$$\frac{K}{L}=\left(\frac{\alpha A}{\beta^{-1}+\delta-1}\right)^{\frac{1}{1-\alpha}} \tag{14}$$

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Next, combine equation (14) with (7) to obtain an equation that characterizes steady state labor:

$$\boxed{\varphi(1-L)^{-\eta}L^{\sigma}=(1-\alpha)\left[A(K/L)^{\alpha}-\delta(K/L)\right]^{-\sigma}A(K/L)^{\alpha}} \tag{15}$$

Note that the right side of equation (15) is a constant. Solve (15) numerically.

The remaining steady state values are readily obtained:

$$\boxed{\begin{aligned} K &= (K/L)L && (16)\\ Y &= AK^{\alpha}L^{1-\alpha} && (17)\\ I &= \delta K && (18)\\ CY &- I && (19)\end{aligned}}$$

**Log-linear Approximation** Let lowercase letters represent the log deviations of variables from their steady states, e.g.: $y_t=\log(Y_t)-\log(Y)$. The log-linearized equilibrium conditions are:

$$0=-\sigma c_t+y_t-\frac{1+(\eta-1)L}{1-L}l_t \tag{20}$$
$$-\left[1-\beta(1-\delta)\right]k_{t+1}-\sigma E_tc_{t+1}+\left[1-\beta(1-\delta)\right]E_ty_{t+1}=-\sigma c_t \tag{21}$$
$$k_{t+1}=\delta i_t+(1-\delta)k_t \tag{22}$$
$$0=\frac{C}{Y}c_t+\frac{I}{Y}i_t-y_t \tag{23}$$
$$0=a_t+\alpha k_t+(1-\alpha)l_t-y_t \tag{24}$$
$$a_{t+1}=\rho a_t+\epsilon_{t+1} \tag{25}$$

Note that all forward-looking variables have been moved to the left sides of the equations and contemporaneous variables to the left. Next, use equations (23) and (24) to eliminate $i_t$ and $y_t$ from the rest of the system:

$$0=a_t+\alpha k_t-\sigma c_t-\left(\frac{\alpha+(\eta-\alpha)L}{1-L}\right)l_t \tag{26}$$
$$\left[1-\beta(1-\delta)\right]E_ta_{t+1}+(\alpha-1)\left[1-\beta(1-\delta)\right]k_{t+1}-\sigma E_tc_{t+1}+(1-\alpha)\left[1-\beta(1-\delta)\right]E_tl_{t+1}=-\sigma c_t \tag{27}$$
$$k_{t+1}=\frac{Y}{K}a_t+\left(\alpha\frac{Y}{K}+1-\delta\right)k_t-\frac{C}{K}c_t+(1-\alpha)\frac{Y}{K}l_t \tag{28}$$
$$a_{t+1}=\rho a_t+\epsilon_{t+1} \tag{29}$$

Note that $E_ta_{t+1}=\rho a_t$. Set aside the law of motion for $a_{t+1}$ and replace $E_ta_{t+1}$ with $\rho a_t$ in equation(27):

$$0=a_t+\alpha k_t-\sigma c_t-\left(\frac{\alpha+(\eta-\alpha)L}{1-L}\right)l_t \tag{30}$$
$$(\alpha-1)\left[1-\beta(1-\delta)\right]k_{t+1}-\sigma E_tc_{t+1}+(1-\alpha)\left[1-\beta(1-\delta)\right]E_tl_{t+1}=-\left[1-\beta(1-\delta)\right]E_t\rho a_t-\sigma c_t \tag{31}$$
$$k_{t+1}=\frac{Y}{K}a_t+\left(\alpha\frac{Y}{K}+1-\delta\right)k_t-\frac{C}{K}c_t+(1-\alpha)\frac{Y}{K}l_t \tag{32}$$

Now we have a system of three linear equations in three variables: $k_{t+1}$, $c_t$, and $l_t$.[1] The system can be represented more concisely as:

$$0=a_t+\alpha k_t-\sigma c_t+\phi_1l_t \tag{33}$$
$$\phi_2k_{t+1}-\sigma E_tc_{t+1}+\phi_3E_tl_{t+1}=\phi_4a_t-\sigma c_t \tag{34}$$
$$k_{t+1}=\phi_5a_t+\phi_6k_t+\phi_7c_t+\phi_8l_t \tag{35}$$

[1]Note that $a_{t+1}$ is determined outside of this system.

where:

$$\phi_1 = -\frac{\alpha + (\eta - \alpha)L}{1 - L} \tag{36}$$
$$\phi_2 = (\alpha - 1)\left[1 - \beta(1-\delta)\right] \tag{37}$$
$$\phi_3 = (1 - \alpha)\left[1 - \beta(1-\delta)\right] \tag{38}$$
$$\phi_4 = -\rho\left[1 - \beta(1-\delta)\right] \tag{39}$$
$$\phi_5 = Y/K \tag{40}$$
$$\phi_6 = Y/K + 1 - \delta \tag{41}$$
$$\phi_7 = -C/K \tag{42}$$
$$\phi_8 = (1-\alpha)Y/L \tag{43}$$

**Solution** I solve the model using the method of undetermined coefficients. Guess that the solution takes the following form:

$$k_{t+1} = \pi_1 a_t + \pi_2 k_t \tag{44}$$
$$c_t = \pi_3 a_t + \pi_4 k_t \tag{45}$$
$$l_t = \pi_5 a_t + \pi_6 k_t \tag{46}$$
$$\tag{47}$$

where the coefficients $\pi_1$, $\pi_2$, ..., $\pi_6$ are to be found. The guess implies the following expectations for $c_{t+1}$ and $l_{t+1}$:

$$k_{t+1} = \pi_1 a_t + \pi_2 k_t \tag{48}$$
$$E_t c_{t+1} = (\rho\pi_3 + \pi_4\pi_1)\, a_t + \pi_4\pi_2 k_t \tag{49}$$
$$E_t l_{t+1} = (\rho\pi_5 + \pi_6\pi_1)\, a_t + \pi_6\pi_2 k_t \tag{50}$$
$$\tag{51}$$

Plug the proposed solution into equations (33), (34), and (35) and find the following system of 6 equations in 6 unknowns:

$$0 = 1 - \sigma\pi_3 + \phi_1\pi_5 \tag{52}$$
$$0 = \alpha - \sigma\pi_4 + \phi_1\pi_6 \tag{53}$$
$$\phi_2\pi_1 - \sigma(\rho\pi_3 + \pi_4\pi_1) + \phi_3(\rho\pi_5 + \pi_6\pi_1) = \phi_4 - \sigma\pi_3 \tag{54}$$
$$\phi_2\pi_2 - \sigma\pi_4\pi_2 + \phi_3\pi_6\pi_2 = -\sigma\pi_4 \tag{55}$$
$$\pi_1 = \phi_5 + \phi_7\pi_3 + \phi_8\pi_5 \tag{56}$$
$$\pi_2 = \phi_6 + \phi_7\pi_4 + \phi_8\pi_6 \tag{57}$$

Now, let's solve the system. First, find $\pi_5$ as a function of $\pi_3$:

$$\pi_5 = \frac{\sigma\pi_3 - 1}{\phi_1} \tag{58}$$

and $\pi_6$ as a function of $\pi_4$:

$$\pi_6 = \frac{\sigma\pi_4 - \alpha}{\phi_1} \tag{59}$$

Use the previous equation to eliminate $\pi_6$ from equation (57):

$$\pi_2 = \phi_9 + \phi_{10}\pi_4 \tag{60}$$

where $\phi_9$ and $\phi_9$ are new constants introduced to keep the algebra manageable:

$$\phi_9 = \phi_6 - \frac{\phi_8\alpha}{\phi_1} \tag{61}$$
$$\phi_{10} = \phi_7 + \frac{\phi_8\sigma}{\phi_1} \tag{62}$$

Also, because it will make the algebra easier in a bit, let's note that:

$$\pi_2\pi_6 = \phi_{11}\pi_4^2 + \phi_{12}\pi_4 + \phi_{13} \tag{63}$$

where:

$$\phi_{11} = \frac{\phi_{10}\sigma}{\phi_1} \tag{64}$$

$$\phi_{12} = \frac{\phi_9\sigma - \phi_{10}\alpha}{\phi_1} \tag{65}$$

$$\phi_{13} = -\frac{\phi_9\alpha}{\phi_1} \tag{66}$$

Now eliminate $\pi_2$ and $\pi_6$ from equation (55) to obtain a quadratic equation in $\pi_4$:

$$\boxed{\underbrace{(\phi_3\phi_{11} - \sigma\phi_{10})}_{a}\pi_4^2 + \underbrace{(\phi_2\phi_{10} - \sigma\phi_9 + \phi_3\phi_{12} + \sigma)}_{b}\pi_4 + \underbrace{(\phi_2\phi_9 + \phi_3\phi_{13})}_{c} = 0} \tag{67}$$

Set $\pi_4$ equal to the larger of the two roots. Now, we can obtain $\pi_6$ and $\pi_2$:

$$\pi_6 = \frac{\sigma\pi_4 - \alpha}{\phi_1} \tag{68}$$

$$\pi_2 = \phi_9 + \phi_{10}\pi_4 \tag{69}$$

Next, from equation (56) find:

$$\pi_1 = \phi_{14} + \phi_{15}\pi_3 \tag{70}$$

where:

$$\phi_{14} = \phi_5 - \frac{\phi_8}{\phi_1} \tag{71}$$

$$\phi_{15} = \phi_7 + \frac{\phi_8\sigma}{\phi_1} \tag{72}$$

And plugging-in to equation (54), find:

$$\boxed{\pi_3 = \frac{\phi_4 - \phi_{16}\phi_{14} + \phi_3\rho\phi_1^{-1}}{\phi_{16}\phi_{15} + \sigma(1-\rho) + \phi_3\rho\sigma\phi_1^{-1}}} \tag{73}$$

where:

$$\phi_{16} = \phi_2 - \sigma\pi_4 + \phi_3\pi_6 \tag{74}$$

Next solve for $\pi_1$ and $\pi_5$:

$$\pi_1 = \phi_{14} + \phi_{15}\pi_3 \tag{75}$$

$$\pi_5 = \frac{\sigma\pi_3 - 1}{\phi_1} \tag{76}$$

Finally, the solutions for the variables $i_t$ and $y_t$ are found to be:

$$i_t = \left(\frac{Y}{I}[1 + (1-\alpha)\pi_5] - \frac{C}{I}\pi_3\right)a_t + \left(\frac{Y}{I}[\alpha + (1-\alpha)\pi_6] - \frac{C}{I}\pi_4]\right)k_t \tag{77}$$

$$y_t = [1 + (1-\alpha)\pi_5]a_t + [\alpha + (1-\alpha)\pi_6]k_t \tag{78}$$