FDA Homework 2

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1 Chapter 10

1.1 Problem 2

Show that in any inner product space, the function $y \to < x, y >$ is continuous where x is arbitary element of that inner product space.

Let $\mathcal{H}$ be an inner product space, $\{f_n\}$ be a sequence in $\mathcal{H}$ such that converges to $f \in \mathcal{H}$ in norm sense. For $x \in \mathcal{H}$, consider below relation.

$$| < x, f_n > - < x, f > |^2 = | < x, f_n - f > |^2 \leq ||x||^2 ||f_n - f||^2$$

Last inequality comse from Cauchy-Schwartz inequality. Then when $n \to \infty$, by our setting $||f_n - f|| \to 0$, so

$$lim_{n\to\infty} | < x, f_n > - < x, f > |^2 \leq 0$$

Then, since square term has always nonnegative value,

$$lim_{n\to\infty} < x, f_n > - < x, f >= 0$$

Thus $lim_{n\to\infty} < x, f_n >=< x, f >$. From this, we show that the inner product operator preserves the limit. It is equivalent statement to that inner product operator is continuous.

1.2 Problem 6

Suppose $\{e_j, j \geq 1\}$ is a complete orthonormal sequence in a Hilbert space. Show that if $\{f_j, j \geq 1\}$ is an orthonormal sequence satisfying

$$\sum_{j=1}^{\infty} ||e_j - f_j||^2 < 1$$

then $\{f_j, j \geq 1\}$ is also complete.

Firstly, I claim that e_j and f_j are not orthorgonal.
Consider a simple case that only one component of fixed j index is different e_j from f_j. then, If e_j, f_j are orthogonal, by the Pythagorean theorem, $||e_j - f_j||^2 = ||e_j||^2 + ||f_j||^2 = 1 + 1 = 2$ since e_j and f_j are unit elements. But under our assumption, $||e_j - f_j||^2$ cannot reach the value 2.

More formally,

$$||e_j - f_j||^2 =< e_j - f_j, e_j - f_j >=< e_j, e_j > - < e_j, f_j > - < f_j, e_j > + < f_j, f_j >$$
$$= ||e_j||^2 + ||f_j||^2 - (< e_j, f_j > + < f_j, e_j >) = 2 - (< e_j, f_j > + < f_j, e_j >) < 1$$

so

$$< e_j, f_j > + < f_j, e_j >=< e_j, f_j > + \overline{< e_j, f_j >} = 2Re(< e_j, f_j >)$$

cannot become 0, neither can $< e_j, f_j >$, so they are not orthogonal. Thus the claim is proved for simple case.

Next, consider many j-th components are different e_j from f_j. But above claim still holds, because $\sum_{j=1}^{\infty} ||e_j - f_j||^2 < 1$ and all norm values must be nonnegative, none of j-th component of $||e_j - f_j||$ can be greater then 1 like the simple cases. It means that $< e_j, f_j > + < f_j, e_j >= 2Re(< e_j, f_j >)$ cannot be 0 for all j. So above claim is proved for whole cases.

Then as what we get from above claim, if we express f_j as the linear combination of $\{e_i\}$, f_j always has e_j component term with nonzero coefficient. (More precisely, use Gram-Schmidt process, or just project f_j onto e_j orthogonally. Then because f_j is not orthogonal to e_j, we get e_j component term of nonzero coefficient in f_j's expression of linear combination using $\{e_i\}$.)

So, if we express whole space $\mathcal{H}$ as $\mathcal{H} = span\{e_1, e_2, ..., e_j, ...\}$, then we can also re-express $\mathcal{H}$ as $\mathcal{H} = span\{e_1, e_2, ..., f_j, ...\}$. Then for each j, we replace e_j to f_j inductively from $j = 1$ to ∞ for above span expression. Then as a result we get $\mathcal{H} = span\{f_1, f_2, ..., f_j, ...\}$ for $\{f_j, j \geq 1\}$, it's what we want.

1.3 Problem 10

Suppose $\{e_j, j >= 1\}$ and $\{f_i, i >= 1\}$ are orthonormal bases in $\mathcal{H}$. Show that for any Hilbert-Schmidt operators Ψ, Φ

$$\sum_{i=1}^{\infty} < \Psi(f_i), \Phi(f_i) >= \sum_{j=1}^{\infty} < \Psi(e_j), \Phi(e_j) >$$

Firstly note that $f_i = \sum_{j=1}^{\infty} < f_i, e_j > e_j$, and since Φ are Hilbert-Schmidt, there exists adjoint operator Φ^*. Using these facts,

$$\sum_{i=1}^{\infty} < \Psi(f_i), \Phi(f_i) >= \sum_{i=1}^{\infty} < \Phi^*\Psi(f_i), f_i >= \sum_{i=1}^{\infty} < \Phi^*\Psi \sum_{j=1}^{\infty} < f_i, e_j > e_j, \sum_{k=1}^{\infty} < f_i, e_k > e_j >$$

then

$$= \sum_{i=1} \sum_{j=1} \sum_{k=1} < f_i, e_j > \overline{< f_i, e_k >} < \Phi^*\Psi(e_j), e_k >= \sum_{i=1} \sum_{j=1} \sum_{k=1} < f_i, e_j >< e_k, f_i >< \Phi^*\Psi(e_j), e_k >$$

Since the operator $\Phi^*\Psi$ is also Hilbert-Schmidt, the value of absolute summation $\sum_{i=1} \sum_{j=1} \sum_{k=1} | < f_i, e_j >< e_k, f_i >< \Phi^*\Psi(e_j), e_k > | < \infty$ and we can interchange the summation order. then

$$= \sum_{j=1} \sum_{k=1} < \Phi^*\Psi(e_j), e_j > (\sum_{i=1} < f_i, e_j >< e_k, f_i >)$$

If we think the (another) linear operator $L : \mathcal{H} \to \mathcal{H}$ such that maps each f_i to e_i, then since both $\{f_i\}, \{e_i\}$ are orthonormal, $LL^* = L^*L$ is identity operator. It means that, for $j \neq k$, the $\sum_j \sum_k \sum_i < f_i, e_j >< e_k, f_i >$ terms become 0, and for only $j = k$ cases remain with value 1 because of using orthonormal set.

So we rewrite above equation (with $j = k$, using only 1 summation symbol) as

$$= \sum_{j=1} < \Phi^*\Psi(e_j), e_j >= \sum_{j=1} < \Psi(e_j), \Phi(e_j) >$$

from the assumption that $\{f_i\}$ is in orthornormal set.

1.4 Problem 12

Show that if L is bounded then L^* is also bounded, and

$$||L^*||_{\mathcal{L}} = ||L||_{\mathcal{L}}, \quad ||L^*L||_{\mathcal{L}} = ||L||_{\mathcal{L}}^2$$

Firstly, I construct a lemma.

$$||L||_{\mathcal{L}} = \sup\{| < Lx, y > | : ||x|| \leq 1, ||y|| \leq 1\}$$

For proof of the lemma, (1) $\geq$ direction is simple consquence of Cauchy-Schwartz inequality. $| < Lx, y > | \leq ||Lx|| ||y|| \leq ||Lx||$ for $||x|| \leq 1$. Take supremum(within $||x|| \leq 1$) both side. (2) To get $\leq$ direction, for $x \in \mathcal{H}$ s.t.$||x|| \leq 1$, let $x' = x/||x||, y' = Lx/||Lx||$. Then if $\sup | < Lx, y > | = M$ for some $M \in \mathcal{R}$, then $| < Lx', y' > | \leq M$ by assumption, and observe that $| < Lx', y' > | = | < \frac{Lx}{||x||}, \frac{Lx}{||Lx||} >= \frac{||Lx||^2}{||x|| ||Lx||} = \frac{||Lx||}{||x||}$, so $||Lx|| \leq M||x||$. Take supremum(within $||x|| \leq 1$) both side. Note that the argument which gives supremum value always satisfies $||x|| = 1$.

Let's start our main goal.

For boundedness and first identity, consider

$$||L||_{\mathcal{L}} = \sup\{| < Lx, y > | : ||x|| \leq 1, ||y|| \leq 1\}$$

$$= \sup\{| < x, L^*y > | : ||x|| \leq 1, ||y|| \leq 1\} = ||L^*||$$

So if L is bounded($||L||_{\mathcal{L}} < \infty$), then L^* is also bounded($||L^*||_{\mathcal{L}} < \infty$).

For second identity, let $x \in \mathcal{H}$ such that $||x|| \leq 1$, be argument of realizing supremum of definition of operator norm, $||L||_L = ||Lx||$. then consider

$$||L||_{\mathcal{L}} = \sup\{| < Lx/||Lx||, Ly > | : ||y|| \leq 1\}$$

$$= \sup\{|\frac{< x, L^*Ly >}{||Lx||}| : ||y|| \leq 1\}$$

$$= \frac{1}{||L||_{\mathcal{L}}} \sup\{| < x, L^*Ly > | : ||y|| \leq 1\} = \frac{||L^*L||_{\mathcal{L}}}{||L||_{\mathcal{L}}}$$

Note that, by our assumption over x, the supremum would be realized at $y = x$. So I can omit $||x|| \leq 1$ on condition of supremum like above expression.

Then, Multiply $||L||_{\mathcal{L}}$ both side.

2 Chapter 11

2.1 Problem 5

Suppose for each $k = 1, 2, ..., M$, $Y_{k,n}, Y_k$ are random variables such that for every $M \geq 1$,

$$[Y_{1,n}, Y_{2,n}, ..., Y_{M,n}]^T \xrightarrow{d} [Y_1, Y_2, ..., Y_M]^T$$

in the Euclidean space $\mathcal{R}^M$. Suppose $\{w_k, k \geq 1\}$ is a sequence of numbers such that

$$\sum_{k=1}^{\infty} |w_k| E(|Y_k|) < \infty \textbf{ and } \sum_{k=1}^{\infty} |w_k| \sup_{n \geq 1} E|Y_{k,n} - Y_k| < \infty$$

Using Theorem 11.1.3, show that $\sum_{k=1}^{\infty} w_k Y_{k,n} \xrightarrow{d} \sum_{k=1}^{\infty} w_k Y_k$.

To overview my procedure solving this problem easier to reader, especially the form of using the theorem 11.1.3, I will start to match our case to the notation of the theorem's assumption on the book.
our case $\leftrightarrow$ theorem11.1.3
$\sum_{k=1}^{u} w_k Y_{k,n} \leftrightarrow X_n(u)$
$\sum_{k=1}^{u} w_k Y_k \leftrightarrow X(u)$
$\sum_{k=1}^{\infty} w_k Y_{k,n} \leftrightarrow X_n$
$\sum_{k=1}^{\infty} w_k Y_k \leftrightarrow X$

More formally, since $[Y_{1,n}, Y_{2,n}, ..., Y_{M,n}]^T \xrightarrow{d} [Y_1, Y_2, ..., Y_M]^T$ for every $M \geq 1$ and $\{w_k\}$ are just real number, if I momently fix M as u, I can get a condition that $\sum_{k=1}^{u} w_k Y_{k,n} \xrightarrow{d} \sum_{k=1}^{u} w_k Y_k$. and by letting u increase to ∞, also can get $\sum_{k=1}^{u} w_k Y_k \xrightarrow{d} \sum_{k=1}^{\infty} w_k Y_k$. Note that the last term's convergence is guaranteed by our problem's assumption, $\sum_{k=1}^{\infty} |w_k| E(|Y_k|) < \infty$.

Nextly, from our problem's another assumption $\sum_{k=1}^{\infty} |w_k| \sup_{n \geq 1} E|Y_{k,n} - Y_k| < \infty$,

$$\sup_{n \geq 1} E|w_k Y_{k,n} - w_k Y_k| \to 0 \text{ as } k \to \infty$$

Since it converges to the constant with integral operator E, it implies that (with some assumption) an analogue of convergence holds in almost everywhere(almost surely) sense without E,

$$\sup_{n \geq 1} |w_k Y_{k,n} - w_k Y_k| \to 0 \text{ almost everywhere as } k \to \infty$$

(Some comment about this part: Because E is not just integral operator but some distribution-sense linear integral operator, for getting above result it may need more assumption: $Y_{k,n}$ and Y_k have distributions of complete class or some restrictions like completeness-similar type for guaranteeing $Y_* = 0$ when $E(Y_*) = 0$. But, on the other view, now since we simultaneously deal with $\{w_k\}$, the non-random real number sequence satisfying this problem's assumptions, by considering what includes the case such that $w_k \to 0$ fast enough (relatively to $E|Y_{k,n}|$), it seems that the completeness type assumption's absence has no big problem. (but in this case, may need more assumption that make sure the relative convergence rate for which above boundednesses don't come from the case that $E|Y_n| \to 0$.) Frankly whether does or does not this problem gives one of these assumptions (either one) already, I cannot sure. If not, this solution shows of only special case's and I have should taken another way to get third condition of theorem 11.1.3 in general.)

For our next work, we only need to work in measure(probability) sense bound. So just ignoring zero-measure part, we can get from above result,

$$\lim_{u \to \infty} \limsup_{n \to \infty} P(d(\sum_{k=1}^{u} w_k Y_{k,n}, \sum_{k=1}^{\infty} w_k Y_{k,n}) > \epsilon) = 0$$

So, theorem 11.1.3's three conditions are all set. Let's apply the theorem! Then as a result, we get $\sum_{k=1}^{\infty} w_k Y_{k,n} \xrightarrow{d} \sum_{k=1}^{\infty} w_k Y_k$.

2.2 Problem 9

Suppose $\mathcal{H}$ is an infinite dimensional separable Hilbert space and $\{e_j, j \geq 1\}$ is an orthonormal system. Define the operator Ψ by

$$\Psi(x) = \sum_{j=1}^{\infty} j^{-1} < x, e_j > e_j$$

Show that Ψ is bounded, symmetric and nonnegative definite, but it is not a covariance operator.

Firstly, Ψ is clearly linear because of the form of inner product (with multiplier).

Boundness is straightforward. To reach the supremum of a definition of operator norm(of $\mathcal{H}$), we should choose simply e_1 because j^{-1} decreases as j goes to ∞. And we can get $||\Psi|| = 1 < \infty$ incidently from the form of Ψ.

Next, since Ψ is linear and bounded as we showed, there exists an adjoint $||\Psi^*||$. Then, with the fact that j has no imaginary part,

$$< \Psi(x), y > = < \sum_{j=1}^{\infty} j^{-1} < x, e_j > e_j, y > = \sum_{j=1}^{\infty} j^{-1} << x, e_j > e_j, < y, e_j > e_j >$$

$$= \sum_{j=1}^{\infty} << x, e_j > e_j, j^{-1} < y, e_j > e_j > = \sum_{j=1}^{\infty} < x, j^{-1} < y, e_j > e_j > = \sum_{j=1}^{\infty} < x, \Psi(y) >$$

So $\Psi = \Psi^*$, and it means that Ψ is symmetric operator.

To show Ψ is nonnegative,

$$< \Psi(x), x > = \sum_{j=1}^{\infty} j^{-1} << x, e_j > e_j, x >$$

and just note that $j^{-1} > 0, < x, x > \geq 0$ for all x. (In fact, positive operator is also symmetric in Hilbert space.)

But, Ψ is not Hilbert-Schmidt, nor covariance operator. For seeing the reason, consider below.

Since Ψ is symmetric, we can apply spectral theorem to Ψ. For simplicity for our discussion, Without loss of generality(because Hilbert-Schmidt norm is basis-invariant) choose the original basis of $\mathcal{H}, \{e_j\}$ to coincide to eigenvector(or eigenfunctions, etc. corresponded some eigen-element type consisting $\mathcal{H}$) of Ψ. then, naturally $\{j^{-1}\}$ become eigenvalues. And if we denote the set of Hilbert-Schmidt operator on $\mathcal{H}$ as $\mathcal{S}$,

$$||\Psi||_{\mathcal{S}}^2 = \sum_{j=1}^{\infty} (j^{-1})^2 = \infty$$

so $\Psi \notin \mathcal{S}$.

2.3 Problem 14

Suppose X satisfies Definition 11.3.2 and let L be a bounded operator. Show that $L(X)$ is Gausian; find its expected value and covariance operator.

Maybe in the problem, a condition L is linear map from $\mathcal{H}$ to $\mathcal{H}$' is missed. I'll assume it.

Firstly, since L is bounded linear operator, there exists an adjoint operator L^*. Then, let's consider the characteristic functional of $L(X)$. For $y \in \mathcal{H}$,

$$\phi_{L(X)}(y) = Eexp\{i < y, LX >\} = Eexp\{i < L^*y, X >\} = \phi_X(L^*y)$$

Then, since X follows Gaussian by assumption(definition 11.3.2),

$$\phi_{L(X)}(y) = \phi_X(L^*y) = exp\{i < \mu, L^*y > -\frac{1}{2} < C(L^*y), L^*y >\} = exp\{i < L\mu, y > -\frac{1}{2} < LC(L^*y), y >\}$$

From above form of characteristic functional of $L(X)$, we can observe that $L(X)$ follows Gaussian again, with mean function $L\mu$, covariance operator LCL^*.