FDA Homework 2

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1 Chapter 10

1.1 Problem 2

Show that in any inner product space, the function $y \to < x, y >$ is continuous where x is arbitary element of that inner product space.

Let $\mathcal{H}$ be an inner product space, $\{f_n\}$ be a sequence in $\mathcal{H}$ such that converges to $f \in \mathcal{H}$ in norm sense. For $x \in \mathcal{H}$, consider below relation.

$$| < x, f_n > - < x, f > |^2 = | < x, f_n - f > |^2 \leq ||x||^2 ||f_n - f||^2$$

Last inequality comse from Cauchy-Schwartz inequality. Then when $n \to \infty$, by our setting $||f_n - f|| \to 0$, so

$$lim_{n\to\infty} | < x, f_n > - < x, f > |^2 \leq 0$$

Then

$$lim_{n\to\infty} < x, f_n > - < x, f >= 0$$

Thus, $lim_{n\to\infty} < x, f_n >=< x, f >$ and the inner product operator is preserve the limit. It is equivalent statement that inner product operator is continuous.

1.2 Problem 6

Suppose $\{e_j, j \geq 1\}$ is a complete orthonormal sequence in a Hilbert space. Show that if $\{f_j, j \geq 1\}$ is an orthonormal sequence satisfying

$$\sum_{j=1}^{\infty} ||e_j - f_j||^2 < 1$$

then $\{f_j, j \geq 1\}$ is also complete.

Firstly, I claim that e_j and f_j are not orthorgonal.
Consider a simple case that only one component of fixed j index are different e_j from f_j. then, If e_j, f_j are orthonormal, the by Pythagorean theorem, $||e_j - f_j||^2 = ||e_j||^2 + ||f_j||^2 = 1 + 1 = 2$ since e_j and f_j are unit elements. But under our assumption

$$||e_j - f_j||^2 =< e_j - f_j, e_j - f_j >=< e_j, e_j > - < e_j, f_j > - < f_j, e_j > + < f_j, f_j >$$

$$= ||e_j||^2 + ||f_j||^2 - (< e_j, f_j > + < f_j, e_j >) = 2 - (< e_j, f_j > + < f_j, e_j >) < 1$$

so in our case, $||e_j - f_j||^2$ always cannot have the value 2, and

$$< e_j, f_j > + < f_j, e_j >=< e_j, f_j > + \overline{< e_j, f_j >} = 2Re(< e_j, f_j >)$$

cannot become 0, neither can $< e_j, f_j >$. Both viewing norm value and inner product value show that they are not orthogonal. Thus the claim is proved for simple case.

Next, consider many j-th components can be different e_j from f_j. But above claim still holds, because $\sum_{j=1}^{\infty}||e_j - f_j||^2 < 1$ and all norm values must be nonnegative, none of j-th component of $||e_j - f_j||$ can be greater then 1 like the simple case, and $< e_j, f_j > + < f_j, e_j >= 2Re(< e_j, f_j >)$ cannot be 0 for all j. So above claim is proved for whole case.

Then from the result of claim, if we express f_j as the linear combination of $\{e_i\}$, then f_j always has e_j component with nonzero coefficient. (More precisely, use Gram-Schmidt process, or just project f_j onto e_j. then since f_j is not orthogonal to e_j, by property of separable Hilbert space's orthonormal basis, we get e_j component of nonzero coefficient.)

So, if we express whole space as $\mathcal{H} = span\{e_1, e_2, ..., e_j, ...\}$, then we can also re-express $\mathcal{H}$ as $\mathcal{H} = span\{e_1, e_2, ..., f_j, ...\}$. Then for each j, we replace e_j to f_j inductively from $j = 1$ to ∞ for above span expression. Then consquently we get $\mathcal{H} = span\{f_1, f_2, ..., f_j, ...\}$ for $\{f_j, j \geq 1\}$, it's what we want.

1.3 Problem 10

Suppose $\{e_j, j >= 1\}$ and $\{f_i, i >= 1\}$ are orthonormal bases in $\mathcal{H}$. Show that for any Hilbert-Schmidt operators Ψ, Φ

$$\sum_{i=1}^{\infty} < \Psi(f_i), \Phi(f_i) >= \sum_{j=1}^{\infty} < \Psi(e_j), \Phi(e_j) >$$

Firstly note that $f_i = \sum_{j=1}^{\infty} < f_i, e_j > e_j$, and since Φ are Hilbert-Schmidt, there are adjoint operator Φ^*. Using these facts,

$$\sum_{i=1}^{\infty} < \Psi(f_i), \Phi(f_i) >= \sum_{i=1}^{\infty} < \Phi^*\Psi(f_i), f_i >= \sum_{i=1}^{\infty} < \Phi^*\Psi \sum_{j=1}^{\infty} < f_i, e_j > e_j, \sum_{k=1}^{\infty} < f_i, e_k > e_j >$$

then

$$= \sum_{i=1} \sum_{j=1} \sum_{k=1} < f_i, e_j > \overline{< f_i, e_k >} < \Phi^*\Psi(e_j), e_k >= \sum_{i=1} \sum_{j=1} \sum_{k=1} < f_i, e_j >< e_k, f_i >< \Phi^*\Psi(e_j), e_k >$$

then when $j \neq k$, the term becomes 0.(why?) so, only $j = k$ cases remain, and we rewrite above equation as

$$= \sum_{i=1} \sum_{j=1} < f_i, e_j >< e_j, f_i >< \Phi^*\Psi(e_j), e_j >$$

Since the operaters are Hilbert-Schmidt, the value of absolute summation is bounded and we can interchange the summation order. then

$$= \sum_{j=1} \sum_{i=1} < f_i, e_j >< e_j, f_i >< \Phi^*\Psi(e_j), e_j >= \sum_{j=1} \sum_{i=1} | < f_i, e_j > |^2 < \Phi^*\Psi(e_j), e_j >$$

$$= \sum_{j=1} ||e_j||^2 < \Phi^*\Psi(e_j), e_j >= \sum_{j=1} < \Phi^*\Psi(e_j), e_j >$$

At last part, I use the relation that $||e_j||^2 = \sum_{i=1} | < f_i, e_j > |^2 = 1$ from the assumption that $\{f_i\}$ is in orthornormal set.

$$= \sum_{j=1} < \Psi(e_j), \Phi(e_j) >$$

1.4 Problem 12

Show that if L is bounded then L^* is also bounded, and

$$||L^*||_{\mathcal{L}} = ||L||_{\mathcal{L}}, \quad ||L^*L||_{\mathcal{L}} = ||L||_{\mathcal{L}}^2$$

Firstly, I construct a lemma.

$$||L||_{\mathcal{L}} = \sup\{| < Lx, y > | : ||x|| \leq 1, ||y|| \leq 1\}$$

For proof, (1) $\geq$ direction is simple consquence of Cauchy-Schwartz inequality. $| < Lx, y > | \leq ||Lx||||y|| \leq ||Lx||$ for $||x|| \leq 1$. Take supremum(within $||x|| \leq 1$) both side. (2) To get $\leq$ direction, for $x \in \mathcal{H}, ||x|| \leq 1$, let $x' = x/||x||, y' = Lx/||Lx||$. Then if $\sup | < Lx, y > | = M$ for some $M \in \mathcal{R}$, then $| < Lx', y' > | \leq M$ by assumption, and observe that $| < Lx', y' > | = | < \frac{Lx}{||x||}, \frac{Lx}{||Lx||} >= \frac{||Lx||^2}{||x||||Lx||} = \frac{||Lx||}{||x||}$, so $||Lx|| \leq M||x||$. Take supremum(within $||x|| \leq 1$) both side.

Note that the argument which gives supremum value always satisfies $||x|| = 1$.

Let's start our main goal. Let $x \in \mathcal{H}$ such that $||x|| \leq 1$, be argument of realizing supremum of definition of operator norm, $||L||_L = ||Lx||$)

For boundedness and first identity, consider

$$||L||_{\mathcal{L}} = \sup\{| < Lx, y > | :: ||x|| \leq 1, ||y|| \leq 1\}$$

$$= \sup\{| < x, L^*y > | :: ||x|| \leq 1, ||y|| \leq 1\} = ||L^*||$$

So if L is bounded($||L||_{\mathcal{L}} < \infty$), then L^* is also bounded($||L^*||_{\mathcal{L}} < \infty$).

For second identity, consider

$$||L||_{\mathcal{L}} = \sup\{| < Lx, Ly/||Ly|| > | : ||x|| \leq 1, ||y|| \leq 1\}$$

$$= \sup\{| \frac{< L^*Lx, y >}{||Ly||} | : ||x|| \leq 1, ||y|| \leq 1\}$$

$$= \frac{1}{||L||_{\mathcal{L}}} \sup\{| < L^*Lx, y > | : ||x|| \leq 1, ||y|| \leq 1\} = \frac{||L^*L||_{\mathcal{L}}}{||L||_{\mathcal{L}}}$$

At third line, I use the definition of operater norm. Multiply $||L||_{\mathcal{L}}$ both side.