FDA Homework 2

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1 Chapter 10

1.1 Problem 2

Show that in any inner product space, the function $y \to < x, y >$ is continuous where x is arbitary element of that inner product space.

Let $\mathcal{H}$ be an inner product space, $\{f_n\}$ be a sequence in $\mathcal{H}$ such that converges to $f \in \mathcal{H}$ in norm sense. For $x \in \mathcal{H}$, consider below relation.

$$| < x, f_n > - < x, f > |^2 = | < x, f_n - f > |^2 \leq ||x||^2 ||f_n - f||^2$$

Last inequality comse from Cauchy-Schwartz inequality. Then when $n \to \infty$, by our setting $||f_n - f|| \to 0$, so

$$lim_{n\to\infty} | < x, f_n > - < x, f > |^2 \leq 0$$

Then, since square term has always nonnegative value,

$$lim_{n\to\infty} < x, f_n > - < x, f >= 0$$

Thus $lim_{n\to\infty} < x, f_n >=< x, f >$. From this, we show that the inner product operator preserves the limit. It is equivalent statement to that inner product operator is continuous.

1.2 Problem 6

Suppose $\{e_j, j \geq 1\}$ is a complete orthonormal sequence in a Hilbert space. Show that if $\{f_j, j \geq 1\}$ is an orthonormal sequence satisfying

$$\sum_{j=1}^{\infty} ||e_j - f_j||^2 < 1$$

then $\{f_j, j \geq 1\}$ is also complete.

Firstly, I claim that e_j and f_j are not orthorgonal.
Consider a simple case that only one component of fixed j index is different e_j from f_j. then, If e_j, f_j are orthogonal, by the Pythagorean theorem, $||e_j - f_j||^2 = ||e_j||^2 + ||f_j||^2 = 1 + 1 = 2$ since e_j and f_j are unit elements. But under our assumption, $||e_j - f_j||^2$ cannot reach the value 2.

More formally,

$$||e_j - f_j||^2 =< e_j - f_j, e_j - f_j >=< e_j, e_j > - < e_j, f_j > - < f_j, e_j > + < f_j, f_j >$$
$$= ||e_j||^2 + ||f_j||^2 - (< e_j, f_j > + < f_j, e_j >) = 2 - (< e_j, f_j > + < f_j, e_j >) < 1$$

so

$$< e_j, f_j > + < f_j, e_j >=< e_j, f_j > + \overline{< e_j, f_j >} = 2Re(< e_j, f_j >)$$

cannot become 0, neither can $< e_j, f_j >$, so they are not orthogonal. Thus the claim is proved for simple case.

Next, consider many j-th components are different e_j from f_j. But above claim still holds, because $\sum_{j=1}^{\infty} ||e_j - f_j||^2 < 1$ and all norm values must be nonnegative, none of j-th component of $||e_j - f_j||$ can be greater then 1 like the simple cases. It means that $< e_j, f_j > + < f_j, e_j >= 2Re(< e_j, f_j >)$ cannot be 0 for all j. So above claim is proved for whole cases.

Then as what we get from above claim, if we express f_j as the linear combination of $\{e_i\}$, f_j always has e_j component term with nonzero coefficient. (More precisely, use Gram-Schmidt process, or just project f_j onto e_j orthogonally. Then because f_j is not orthogonal to e_j, we get e_j component term of nonzero coefficient in f_j's expression of linear combination using $\{e_i\}$.)

So, if we express whole space $\mathcal{H}$ as $\mathcal{H} = span\{e_1, e_2, ..., e_j, ...\}$, then we can also re-express $\mathcal{H}$ as $\mathcal{H} = span\{e_1, e_2, ..., f_j, ...\}$. Then for each j, we replace e_j to f_j inductively from $j = 1$ to ∞ for above span expression. Then as a result we get $\mathcal{H} = span\{f_1, f_2, ..., f_j, ...\}$ for $\{f_j, j \geq 1\}$, it's what we want.

1.3 Problem 10

Suppose $\{e_j, j >= 1\}$ and $\{f_i, i >= 1\}$ are orthonormal bases in $\mathcal{H}$. Show that for any Hilbert-Schmidt operators Ψ, Φ

$$\sum_{i=1}^{\infty} < \Psi(f_i), \Phi(f_i) >= \sum_{j=1}^{\infty} < \Psi(e_j), \Phi(e_j) >$$

Firstly note that $f_i = \sum_{j=1}^{\infty} < f_i, e_j > e_j$, and since Φ are Hilbert-Schmidt, there exists adjoint operator Φ^*. Using these facts,

$$\sum_{i=1}^{\infty} < \Psi(f_i), \Phi(f_i) >= \sum_{i=1}^{\infty} < \Phi^*\Psi(f_i), f_i >= \sum_{i=1}^{\infty} < \Phi^*\Psi \sum_{j=1}^{\infty} < f_i, e_j > e_j, \sum_{k=1}^{\infty} < f_i, e_k > e_j >$$

then

$$= \sum_{i=1} \sum_{j=1} \sum_{k=1} < f_i, e_j > \overline{< f_i, e_k >} < \Phi^*\Psi(e_j), e_k >= \sum_{i=1} \sum_{j=1} \sum_{k=1} < f_i, e_j >< e_k, f_i >< \Phi^*\Psi(e_j), e_k >$$

Since the operator $\Phi^*\Psi$ is also Hilbert-Schmidt, the value of absolute summation $\sum_{i=1} \sum_{j=1} \sum_{k=1} | < f_i, e_j >< e_k, f_i >< \Phi^*\Psi(e_j), e_k > | < \infty$ and we can interchange the summation order. then

$$= \sum_{j=1} \sum_{k=1} < \Phi^*\Psi(e_j), e_j > (\sum_{i=1} < f_i, e_j >< e_k, f_i >)$$

If we think the linear operator $L : \mathcal{H} \to \mathcal{H}$ such that maps each f_i to e_i, then since both $\{f_i\}, \{e_i\}$ are orthonormal, $LL^* = L^*L$ is identity operator. It means that, for $j \neq k$, the last summation's term (with i) becomes 0, and for only $j = k$ cases remain with value 1 because of using orthonormal set.

So we rewrite above equation (with $j = k$, using only 1 summation symbol) as

$$= \sum_{j=1} < \Phi^*\Psi(e_j), e_j >= \sum_{j=1} < \Psi(e_j), \Phi(e_j) >$$

from the assumption that $\{f_i\}$ is in orthornormal set.

1.4 Problem 12

Show that if L is bounded then L^* is also bounded, and

$$||L^*||_{\mathcal{L}} = ||L||_{\mathcal{L}}, \quad ||L^*L||_{\mathcal{L}} = ||L||_{\mathcal{L}}^2$$

Firstly, I construct a lemma.

$$||L||_{\mathcal{L}} = \sup\{| < Lx, y > | : ||x|| \leq 1, ||y|| \leq 1\}$$

For proof of the lemma, (1) $\geq$ direction is simple consquence of Cauchy-Schwartz inequality. $| < Lx, y > | \leq ||Lx|| ||y|| \leq ||Lx||$ for $||x|| \leq 1$. Take supremum(within $||x|| \leq 1$) both side. (2) To get $\leq$ direction, for $x \in \mathcal{H}$ s.t.$||x|| \leq 1$, let $x' = x/||x||, y' = Lx/||Lx||$. Then if $\sup | < Lx, y > | = M$ for some $M \in \mathcal{R}$, then $| < Lx', y' > | \leq M$ by assumption, and observe that $| < Lx', y' > | = | < \frac{Lx}{||x||}, \frac{Lx}{||Lx||} >= \frac{||Lx||^2}{||x|| ||Lx||} = \frac{||Lx||}{||x||}$, so $||Lx|| \leq M ||x||$. Take supremum(within $||x|| \leq 1$) both side. Note that the argument which gives supremum value always satisfies $||x|| = 1$.

Let's start our main goal.

For boundedness and first identity, consider

$$||L||_{\mathcal{L}} = \sup\{| < Lx, y > | : ||x|| \leq 1, ||y|| \leq 1\}$$

$$= \sup\{| < x, L^* y > | : ||x|| \leq 1, ||y|| \leq 1\} = ||L^*||$$

So if L is bounded($||L||_{\mathcal{L}} < \infty$), then L^* is also bounded($||L^*||_{\mathcal{L}} < \infty$).

For second identity, let $x \in \mathcal{H}$ such that $||x|| \leq 1$, be argument of realizing supremum of definition of operator norm, $||L||_L = ||Lx||$. then consider

$$||L||_{\mathcal{L}} = \sup\{| < Lx/||Lx||, Ly > | : ||y|| \leq 1\}$$

$$= \sup\{|\frac{< x, L^* L y >}{||Lx||}| : ||y|| \leq 1\}$$

$$= \frac{1}{||L||_{\mathcal{L}}} \sup\{| < x, L^* L y > | : ||y|| \leq 1\} = \frac{||L^* L||_{\mathcal{L}}}{||L||_{\mathcal{L}}}$$

Note that, by our assumption over x, the supremum would be realized at $y = x$. So if $||x|| \leq 1$ is omitted on condition of supremum like above expression, it does not become problem.

Then, Multiply $||L||_{\mathcal{L}}$ both side.

2 Chapter 11

2.1 Problem 5

Suppose for each $k = 1, 2, ..., M$, $Y_{k,n}, Y_k$ are random variables such that for every $M \geq 1$,

$$[Y_{1,n}, Y_{2,n}, ..., Y_{M,n}]^T \xrightarrow{d} [Y_1, Y_2, ..., Y_M]^T$$

in the Euclidean space $\mathcal{R}^M$. Suppose $\{w_k, k \geq 1\}$ is a sequence of numbers such that

$$\sum_{k=1}^{\infty} |w_k| E(|Y_k|) < \infty \text{ and } \sum_{k=1}^{\infty} |w_k| \sup_{n\geq 1} E|Y_{k,n} - Y_k| < \infty$$

Using Theorem 11.1.3, show that $\sum_{k=1}^{\infty} w_k Y_{k,n} \xrightarrow{d} \sum_{k=1}^{\infty} w_k Y_k$.

To overview my procedure solving this problem easier to reader, especially the form of using the theorem 11.1.3, I will start to match our case to the notation of the theorem's assumption on the book.
our case $\leftrightarrow$ theorem11.1.3
$\sum_{k=1}^{u} w_k Y_{k,n} \leftrightarrow X_n(u)$
$\sum_{k=1}^{u} w_k Y_k \leftrightarrow X(u)$
$\sum_{k=1}^{\infty} w_k Y_{k,n} \leftrightarrow X_n$
$\sum_{k=1}^{\infty} w_k Y_k \leftrightarrow X$

More formally, since $[Y_{1,n}, Y_{2,n}, ..., Y_{M,n}]^T \xrightarrow{d} [Y_1, Y_2, ..., Y_M]^T$ for every $M \geq 1$ and $\{w_k\}$ are just real number, if I momently fix M as u, I can get a condition that $\sum_{k=1}^{u} w_k Y_{k,n} \xrightarrow{d} \sum_{k=1}^{u} w_k Y_k$. and by letting u increase to ∞, also can get $\sum_{k=1}^{u} w_k Y_k \xrightarrow{d} \sum_{k=1}^{\infty} w_k Y_k$. Note that the last term's convergence is guaranteed by our problem's assumption, $\sum_{k=1}^{\infty} |w_k| E(|Y_k|) < \infty$. Nextly, from our problem's another assumption $\sum_{k=1}^{\infty} |w_k| \sup_{n\geq 1} E|Y_{k,n} - Y_k| < \infty$,

$$\sup_{n\geq 1} E|w_k Y_{k,n} - w_k Y_k| \to 0 \text{ as } k \to \infty$$

Since it converges to the constant, it directly implies some analogues without E - in this case a thing of integral operator - and with measure sense. If I write concretely,

$$\lim_{u\to\infty} \limsup_{n\to\infty} P(d(\sum_{k=1}^{u} w_k Y_{k,n}, \sum_{k=1}^{\infty} w_k Y_{k,n}) > \epsilon) = 0$$

So theorem 11.1.3's condition are all set. Let's apply the theorem! Then we get $\sum_{k=1}^{\infty} w_k Y_{k,n} \xrightarrow{d} \sum_{k=1}^{\infty} w_k Y_k$.

2.2 Problem 9

Suppose $\mathcal{H}$ is an infinite dimensional separable Hilbert space and $\{e_j, j \geq 1\}$ is an orthonormal system. Define the operator Ψ by

$$\Psi(x) = \sum_{j=1}^{\infty} j^{-1} < x, e_j > e_j$$

Show that Ψ is bounded, symmetric and nonnegative definite, but it is not a covariance operator.

Boundness is straightforward. To reach the supremum of a definition of operator norm(of $\mathcal{H}$), we should choose simply e_1 because j^{-1} decreases as j goes to ∞. And we can get $||\Psi|| = 1 < \infty$ incidently from the form of Ψ.

Next, since Ψ is clearly linear and bounded as we showed, there exists an adjoint $||\Psi^*||$. Then, with the fact that j has no imaginary part,

$$< \Psi(x), y >=< \sum_{j=1}^{\infty} j^{-1} < x, e_j > e_j, y >= \sum_{j=1}^{\infty} j^{-1} << x, e_j > e_j, < y, e_j > e_j >$$

$$= \sum_{j=1}^{\infty} << x, e_j > e_j, j^{-1} < y, e_j > e_j >= \sum_{j=1}^{\infty} < x, j^{-1} < y, e_j > e_j >= \sum_{j=1}^{\infty} < x, \Psi(y) >$$

So $\Psi = \Psi^*$, and it means that Ψ is symmetric operator.

To show Ψ is nonnegative,

$$< \Psi(x), x >= \sum_{j=1}^{\infty} j^{-1} << x, e_j > e_j, x >$$

and just note that $j^{-1} > 0, < x, x > \geq 0$ for all x. (In fact, positive operator is also symmetric in Hilbert space.)

But, Ψ is not Hilbert-Schmidt, nor covariance operator. For seeing the reason, consider below.

Since Ψ is symmetric, we can apply spectral theorem to Ψ. For simplicity for our discussion, Without loss of generality(because Hilbert-Schmidt norm is basis-invariant) choose the original basis of $\mathcal{H}, \{e_j\}$ to coincide to eigenvector(or eigenfunctions, etc. corresponded some eigen-element type consisting $\mathcal{H}$) of Ψ. then, naturally $\{j^{-1}\}$ become eigenvalues. And if we denote the set of Hilbert-Schmidt operator on $\mathcal{H}$ as $\mathcal{S}$,

$$||\Psi||_{\mathcal{S}}^2 = \sum_{j=1}^{\infty} (j^{-1})^2 = \infty$$

so $\Psi \notin \mathcal{S}$.

2.3 Problem 14

Suppose X satisfies Definition 11.3.2 and let L be a bounded operator. Show that $L(X)$ is Gausian; find its expected value and covariance operator.

Maybe in the problem, a condition L is linear map from $\mathcal{H}$ to $\mathcal{H}$' is missed. I'll assume it.

Firstly, since L is bounded linear operator, there exists an adjoint operator L^*. Then, let's consider the characteristic functional of $L(X)$. For $y \in \mathcal{H}$,

$$\phi_{L(X)}(y) = Eexp\{i < y, LX >\} = Eexp\{i < L^*y, X >\} = \phi_X(L^*y)$$

Then, since X follows Gaussian by assumption(definition 11.3.2),

$$\phi_{L(X)}(y) = \phi_X(L^*y) = exp\{i < \mu, L^*y > -\frac{1}{2} < C(L^*y), L^*y >\} = exp\{i < L\mu, y > -\frac{1}{2} < LC(L^*y), y >\}$$

From above form of characteristic functional of $L(X)$, we can observe that $L(X)$ follows Gaussian again, with mean function $L\mu$, covariance operator LCL^*.