FDA Homework 3

Seokjun Choi

November 8th, 2019

1 Chapter 12

1.1 Problem 2

Show that for every eigenvalue λ of a bounded operator L, we have $|\lambda| \leq ||L||_{\mathcal{L}}$.

In problem's statement, L is being assumed that it can be spectral decomposable (i.e. self-adjoint, compact(or, Hilbert-Schmidt) operator), so I will work with these assumptions.

Firstly I claim that $||L||$ or $-||L||$ is eigenvalue of L. Without loss of generality, assume first case and denote $\lambda_1 = ||L|| = \sup\{< Lf, f >: ||f|| = 1\}$. Let $\{f_n\} \in \mathcal{H}$ such that $||f_n|| = 1$, $< Tf_n, f_n > \to \lambda_1$ and $Tf_n \to g$ for some $g \in \mathcal{H}$. (such $\{f_n\}, g$ exist because L is compact and $\mathcal{H}$ is complete.) Then

$$\begin{aligned} ||Lf_n - \lambda_1 f_n||^2 &= ||Lf_n||^2 - 2\lambda_1 < Lf_n, f_n > + \lambda_1^2 ||f_n||^2 \\ &\leq ||L||^2 ||f_n||^2 - 2\lambda_1 < Lf_n, f_n > + \lambda_1^2 ||f_n||^2 \\ &\leq \lambda_1^2 - 2\lambda_1 < Lf_n, f_n > + \lambda_1^2 \\ &\leq 2\lambda_1^2 - 2\lambda_1^2 \to 0 \end{aligned}$$

So $Lf_n \to g$, $\lambda_1 f_n \to g$, and under continuity of L from the assumptions, we get $\lambda_1 g = Lg$. And we also verify $g \neq 0$ because if 0, it becomes $\lambda_1 = ||L|| = 0$, contradiction. thus λ_1 is eigenvalue of L. (The proof for $||L|| = -\lambda_1$ case is similar.)

Next I claim one more thing that above $\lambda_1 = \max |\lambda|$ over all λs which are eigenvalues of L. Without loss of generality, consider only the case all eigenvalues of L is nonnegative. (if not, change sign of it and its pair eigenfunction together.) Assume the claim is false, then there are λ^* and the pair eigenfuction v^* whose norm is 1. Then, $Lv^* = \lambda^* > \lambda_1 = ||L|| = \sup_{||v||=1} ||Lv||$, we have contradiction. So, all eigenvalues of L are smaller then λ_1, and it is what we want, $|\lambda| \leq ||L|| = \lambda_1$.

1.2 Problem 5

Assume that $X_1, ..., X_N$ are iid element of $L^2[0,1]$ with $E||X_n||^4 < \infty$ and whose first p eigenvalue are distinct. Prove that

$$|N < \hat{v}_j - v_j, v_j > | = O_P(1) \textbf{ for } j = 1, ..., p$$

Why is this a seemingly unusual convergence rate? (Hint: $| < \hat{v}_j - v_j, v_j > | = \frac{1}{2}||\hat{v}_j - v_j||$)

With hint of the problem, I'll show that $N||\hat{v_j} - v_j||^2$ is bounded in probability sense, or $O_P(1)$. (constant 1/2 does not matter in this context.)

Then by theorem 12.2.1 on our book, under our assumptions, we know that

$$\limsup_{N \to \infty} NE[||\hat{v}_j - v_j||^2] < C$$

for some C and $j = 1, ..., p$. On other hand, by Chebyshev's inequality,

$$Pr(N||(\hat{v_j} - v_j)||^2 > \alpha) < \frac{NE[||\hat{v}_j - v_j||^2]}{\alpha}$$

The result of above theorem says that right-hand side is bounded in probability sense for all $\alpha > 0$. Then using definition of boundedness in probability to left-hand side, we get $N||\hat{v_j} - v_j||^2$ is bounded in probability, which we want.

1.3 Problem 6

Prove Theorem 12.1.3: Let $x, y \in \mathcal{H}$. Then $||x \otimes y||_{\mathcal{H}\otimes\mathcal{H}} = || < y, . > x||_{\mathcal{S}}$

When we view tensor as operator, $x \otimes y(.) =< y, . > x$ holds. Using this fact, when $\{e_i\}$ are orthonormal basis of $\mathcal{H}$, by the definition of Hilbert-Schmidt norm equipped on $\mathcal{S}$,

$$|| < y, . > x||_{\mathcal{S}}^2 = \sum_{i=1}^{\infty} ||(< y, z > x)e_i||^2 \text{ for } \forall z \in \mathcal{H}$$

then by Parseval's identity,

$$= || < y, z > x||_{\mathcal{H}}^2 \text{ for } \forall z \in \mathcal{H}$$

then by above fact and z is arbitrary,

$$= ||x \otimes y||_{\mathcal{H}\otimes\mathcal{H}}^2$$

1.4 Problem 7

Suppose that the data $X_n(t) : t \in [0, 1], 1 \leq n \leq N$ are expressed using an orthonormal basis $e_1, ..., e_J$:

$$X_n(t) = \sum_{j=1}^{J} x_{n_j} e_j(t)$$

In this case, the EFPC's, $\hat{v}_i(t)$ can also be expressed as

$$\hat{v}_i(t) = \sum_{j=1}^{J} \hat{v}_{ij} e_j(t)$$

Explain how to obtain the coefficient $\hat{v}_{ij}$ from the x_{nj}. Justify your answer.

I cannot ensure the intention of this problem.

1.5 Problem 12

Under the same assumptions as in Problem 12.8.5, shows that, for $j \neq k$ and $1 \leq j \leq p$,

$$< \hat{v}_j - v_j, v_k >= \frac{< \hat{C} - C, \hat{v}_j \otimes v_k >}{\hat{\lambda}_j - \lambda_k}$$

By theorem 12.3.2 on our book, under our assumptions we know that

$$N^{1/2}(\hat{v}_j - v_j) = \sum_{i \neq j} \frac{1}{\lambda_j - \lambda_i} < \sqrt{N}(\hat{C} - C), v_i \otimes v_j > v_i + o_P(1)$$

Take both side to $< ., v_k >$ and neglect ignorable term, then

$$N^{1/2} < \hat{v}_j - v_j, v_k >= \sum_{i \neq j} \frac{1}{\lambda_j - \lambda_i} < \sqrt{N}(\hat{C} - C), v_i \otimes v_j >< v_i, v_k >$$

then, the last inner product term becomes 1 only $i = k$, and otherwise 0. So we can rewrite it without summation symbol as

$$N^{1/2} < \hat{v}_j - v_j, v_k >= \frac{\sqrt{N} < (\hat{C} - C), v_k \otimes v_j >}{\lambda_j - \lambda_k}$$

multiply $\sqrt{N}$ to both sides. And, because $\hat{\lambda}_j \to \lambda_j$ and $\hat{v}_j \to v_j$ as $N \to \infty$ in probability sense by corollary 12.3.1.

(Or for eigenfunction part convergence, using problem 5's result considering the form dividing problem's expression by N. And for eigenvalue part convergence, do like procedure of solving problem 5 using the

eigenvalue part of theorem 12.2.1 and Chebyshev's inequality.)
So we can replace λ_j, v_j with $\hat{\lambda}_j, \hat{v}_j$ with adding only ignorable terms in right-hand side, and if we disregard them, we get

$$< \hat{v}_j - v_j, v_k >= \frac{< \hat{C} - C, \hat{v}_j \otimes v_k >}{\hat{\lambda}_j - \lambda_k}$$

which we want