FDA Homework 2

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1 Chapter 10

1.1 Problem 2

Show that in any inner product space, the function $y \rightarrow < x, y >$ is continuous where x is arbitary element of that inner product space.

Let $\mathcal{H}$ be an inner product space, $\{f_n\}$ be a sequence in $\mathcal{H}$ such that converges to $f \in \mathcal{H}$ in norm sense. For $x \in \mathcal{H}$, consider below relation.

$$| < x, f_n > - < x, f > |^2 = | < x, f_n - f > |^2 \leq ||x||^2 ||f_n - f||^2$$

Last inequality comse from Cauchy-Schwartz inequality. Then when $n \rightarrow \infty$, by our setting $||f_n - f|| \rightarrow 0$, so

$$lim_{n\rightarrow\infty} | < x, f_n > - < x, f > |^2 \leq 0$$

Then

$$lim_{n\rightarrow\infty} < x, f_n > - < x, f >= 0$$

Thus, $lim_{n\rightarrow\infty} < x, f_n >=< x, f >$ and the inner product operator is preserve the limit. It is equivalent statement that inner product operator is continuous.

1.2 Problem 6

Suppose $\{e_j, j >= 1\}$ is a complete orthonormal sequence in a Hilbert space. Show that if $\{f_j, j >= 1\}$ is an orthonormal sequence satisfying

$$\sum_{j=1}^{\infty} ||e_j - f_j||^2 < 1$$

then $\{f_j, j >= 1\}$ is also complete.

1.3 Problem 10

Suppose $\{e_j, j >= 1\}$ and $\{f_i, i >= 1\}$ are orthonormal bases in $\mathcal{H}$. Show that for any Hilbert-Schmidt operators Ψ, Φ

$$\sum_{i=1}^{\infty} < \Psi(f_i), \Phi(f_i) >= \sum_{j=1}^{\infty} < \Psi(e_j), \Phi(e_j) >$$

Firstly note that $f_i = \sum_{j=1}^{\infty} < f_i, e_j > e_j$, and since Φ are Hilbert-Schmidt, there are adjoint operator Φ^*. Using these facts,

$$\sum_{i=1}^{\infty} < \Psi(f_i), \Phi(f_i) >= \sum_{i=1}^{\infty} < \Phi^*\Psi(f_i), f_i >= \sum_{i=1}^{\infty} < \Phi^*\Psi \sum_{j=1}^{\infty} (f_i, e_j)e_j, \sum_{k=1}^{\infty} (f_k, e_k)e_j >$$

then

$$= \sum_{i=1} \sum_{j=1} \sum_{k=1} < f_i, e_j > < f_i, \bar{e}_k > < \Phi^* \Psi(e_j), e_k >= \sum_{i=1} \sum_{j=1} \sum_{k=1} < f_i, e_j >< e_k, f_i >< \Phi^* \Psi(e_j), e_k >$$

then when $j \neq k$, the term becomes 0.(why?) so, only $j = k$ cases remain, so we rewrite above equation as

$$= \sum_{i=1} \sum_{j=1} < f_i, e_j >< e_j, f_i >< \Phi^* \Psi(e_j), e_j >$$

Since the operaters are Hilbert-Schmidt, the value of absolute summation is bounded, and we can interchange the summation order. then

$$= \sum_{j=1} \sum_{i=1} < f_i, e_j >< e_j, f_i >< \Phi^* \Psi(e_j), e_j >= \sum_{j=1} \sum_{i=1} | < f_i, e_j > |^2 < \Phi^* \Psi(e_j), e_j >$$

$$= \sum_{j=1} < \Phi^* \Psi(e_j), e_j >= \sum_{j=1} < \Phi^* \Psi(e_j), e_j >$$

$\sum_{i=1} | < f_i, e_j > |^2 = 1$ since it coincide the definition of norm square, and each element is in orthornormal set.

$$= \sum_{j=1} < \Psi(e_j), \Phi(e_j) >$$

1.4 Problem 12

Show that if L is bounded then L^* is also bounded, and

$$||L^*||_{\mathcal{L}} = ||L||_{\mathcal{L}}, \quad ||L^* L||_{\mathcal{L}} = ||L||^2_{\mathcal{L}}$$