Functional depths and related topics

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Outline

definition of depth in $\mathbb{R}^p$

Definition (statistical depth in $\mathbb{R}^p$, (Zuo and Serfling, 2000b))

Let $\mathcal{P}$ be some class of of distributions. The bounded and non-negative mapping $D(.,.) : \mathbb{R}^p \times \mathcal{P} \to \mathbb{R}$ is called a statistical depth function if it satisfies the following properties:

- Affine invariance
  $D(Ax + b, P_{AX+b}) = D(X, P_X)$ holds for any $\mathbb{R}^p$-valued random vector X, any $p \times p$ nonsingular matrix A and any $b \in \mathbb{R}^p$.
- Maximality at centre
  $D(\theta, P) = sup_{x \in \mathbb{R}^p} D(x, P)$ holds for any $P \in \mathcal{P}$ having a unique centre of symmetry θ w.r.t. some notion of symmetry.
- Monotonicity relative to the deepest point
  For any $P \in \mathcal{P}$ having deepest point θ, $D(x, P) \leq D(\theta + \alpha(x - \theta), P)$ holds for all $\alpha \in [0, 1]$.
- Vanishing at infinity $D(x, P) \to 0$ as $||x||_{\mathbb{R}^p} \to \infty$ for each $P \in \mathcal{P}$.

definition of depth in $\mathbb{R}^p$

(Serfling(2006)) Not necessary, but desirable property when setting D:

- Symmetry
  If P is symmetric about θ, then so is $D(x, P)$.
- Continuity of D(x,P) as a function of x
  (or just have upper semi-continuity)
- Continuity of D(x,P) as a function of P
- Quasi-concavicity as a function of x
  The set $\{x : D(x, P) \geq c\}$ is convex for each real c.

Example (on $\mathbb{R}^1$)

If we denote F_P as cdf corresponding distribution measure P, then

- (By Fraiman, Muniz(2001)) $D(x, P) = 1/2 - [1/2 - F_P(x)]$
- (Halfspace depth, By Tukey(1975)) $D(x, P) = min\{F_P(x), lim_{v \to x-} F_P(v)\}$
- (Simplical depth, By Liu(2001)) $D(x, P) = F_P(x)\{1 - lim_{v \to x-} F_P(v)\}$
- (Modified band depth, By Cuevas, Fraiman(2009))
  $D(x, P) = \frac{1}{J-1} \sum_{j=2}^{J} P(x \in [min(X_1, ..., X_j), max(X_1, ..., X_j)])$

definition of depth in $\mathcal{F}$

Definition (statistical depth in $\mathcal{F}$, (Nieto-Reyes and Battey, 2016))

Let $(\mathcal{F}, A, P)$ be probability space and $\mathcal{P}$ be class of all distribution measures on $\mathcal{F}$. The bounded and non-negative mapping $D(.,.) : \mathcal{F} \times \mathcal{P} \to \mathbb{R}$ is called a statistical functional depth function if it satisfies the following properties:

- distance invariance
  $D(f(x), P_{f(X)}) = D(X, P_X)$ for any $x \in \mathcal{F}$ and $f : \mathcal{F} \to \mathcal{F}$ such that for any $y \in \mathcal{F}$,
  $d(f(x), f(y)) = a_f d(x, y)$, $a_f \in \mathbb{R} - \{0\}$.
- Maximality at centre
  For any $P \in \mathcal{P}$ with unique centre of symmetry θ w.r.t. some notion of symmetry,
  $D(\theta, P) = sup_{x \in \mathcal{F}} D(x, P)$.
- Monotonicity (strictly decreasing) relative to the deepest point
  For any $P \in \mathcal{P}$ s.t. $D(z, P) = max_{x \in \mathcal{F}} D(x, P)$ exists (:deepest point z), for $x, y \in \mathcal{F}$,
  $D(x, P) < D(y, P) < D(z, P)$ s.t. $min\{d(y, z), d(y, x)\} > 0$ and $max\{d(y, z), d(y, x)\} < d(x, z)$.

Definition ((continue.))

- Upper semi-continuity in x
  $D(x,P)$ is upper semi-continuous as a function of x.
- Receptivity to convex hull width across the domain.
  Let $C(\mathcal{F},P)$ be convex hull in $(\mathcal{F},A,P)$ defined as
  $C(\mathcal{F},P)=\{x\in\mathcal{F}:x(v)=\alpha L(v)+(1-\alpha)U(v), v\in V, \alpha\in[0,1]\}$ where $U=\{sup_{x\in E}x(v):v\in V\}$,
  $L=\{inf_{x\in E}x(v):v\in V\}$ and E is smallest set in A s.t. $P(E)=P(\mathcal{F})$.
  Then, D has a property that $D(x,P_X)<D(f(x),P_{f(X)})$ for any $x\in C(\mathcal{F},P)$ with
  $D(x,P)<sup_{y\in\mathcal{F}}D(y,P)$ and $f:\mathcal{F}\to\mathcal{F}$ s.t. $f(y(v))=\alpha(v)y(v)$ with $\alpha(v)\in(0,1)$ for all $v\in L_\delta$ and
  $\alpha(v)=1$ otherwise where $L_\delta=argsup_{H\in V}\{sup_{x,y\in C(\mathcal{F},P)}d(x(H),y(H))\le\delta\}$ for any
  $\delta\in inf_{v\in V}d(L(v),U(v)),d(L,U)$ s.t. $\lambda(L_\delta)>0$ and $\lambda(L_\delta^c)>0$.
- Continuity in P
  For all $x\in\mathcal{F}$, for all $P\in\mathcal{P}$ and for every $\epsilon>0$, there exists a $\delta(\epsilon)>0$ s.t. $|D(x,Q)-D(x-P)|\le\epsilon$
  P-almost surely for all $Q\in\mathcal{P}$ with $d_P(Q,P)<\delta$ P-almost surely, where d_P is metric on $\mathcal{P}$.

Check the validity of existing depth on $\mathcal{F}$

- h-depth (Cuevas, Febrero and Fraiman(2007))
  $D_h(x, P) = E_X(K_h(||x - X||_{\mathcal{L}^2[0,1]}))$ on $\mathcal{L}^2[0,1]^p$: FTTTTT
- random-tukey depth (Cuesta-Albertos and Nieto-Reyes(2008))
  $D_{RT}(x, P) = min_{u \in \{u_j\}_{j=1}^k} min(P_{(u)}(-\infty, \langle u, x \rangle], P_{(u)}[\langle u, x \rangle, \infty))$
  where $P_{(u)}$: marginal distribution measure of u, on $\mathcal{L}^2[0,1]^p$: TTFTFT
- band depth(Lopez-Pintado and Romo(2009))
  $D_J(x, P) = \sum_{j=2}^J P_{S_j}(x \in S_j(P))$ where $S_j(P) = \{y \in \mathcal{F} : y(v) = \alpha_1 X_1(v) + ... + \alpha_j X_j(v), \alpha_k \in$(j-th dim simplex),$v \in V, X_i \sim P\}$ on $\mathcal{C}$ with sup norm : TTFTFT
- modified band depth (Lopez-Pintado and Romo(2009))
  $D_{MJ}(x, P) = \sum_{j=2}^J E(\lambda\{v \in V : x \in S_j(P)\})$ with above notation, on $\mathcal{C}$ with sup norm : TTFTFT
- half-region depth (Lopez-Pintado and Romo(2011))
  $D_{HR}(x, P) = min\{P(X \in H_x), P(X \in E_x)\}$ where $H_x = \{y \in \mathcal{F} : y(v) \le x(v)$ for all $v \in V\}$ and $E_x = \{y \in \mathcal{F} : y(v) \ge x(v)$ for all $v \in V\}$ on $\mathcal{C}$ with sup norm : TFFTFT
- modified half-region depth (Lopez-Pintado and Romo(2011))
  $D_{MHR}(x, P) = min\{E(\lambda\{v \in V, X(v) \le x(v)\}), E(\lambda\{v \in V, X(v) \ge x(v)\})\}/\lambda(V)$ on $\mathcal{C}$ with sup norm : TTFTFT

Consistency of functional depth: classification of existing functional depth

For showing consistency, classify depths to 3 groups (Stanislav Nagy(2018)) Let D: some depth in $\mathbb{R}^p$. then

- integrated depth (Fraiman, Muniz(2001) and Cuevas, Fraiman(2009))
  form of $FD(x, P) = \int D(f(x), f(P)d\lambda(f))$
- infimal depth (Mosler(2013))
  form of $ID(x, P) = inf_f D(f(x), x(P))$
- band depth (Lopez-Pintado, Romo(2009))
  form of $BD(x, P) = P(x \in Band(X_1, ..., X_K))$ on $\mathcal{C}$ where
  $Band(x_1, x_2) = \{y \in \mathcal{C} : min\{x_1(v), x_2(v)\} \leq y(v) \leq max\{x_1(v), x_2(v)\}, v \in V\}$
  (extend to convex hull with many X_is.)

Consistency of functional depth

Definition

For given $P \in \mathcal{P}$, let $P_n \to P$ weakly. A functional depth $D(x, P)$ is uniformly consistent for P over $\mathcal{F}$, if

$$sup_{x \in \mathcal{F}} |D(x, P_n) - D(x, P)| \to 0$$

for almost every x as $n \to \infty$.

Definition

If D is uniformly consistent for any $P \in \mathcal{P}$, then we say D is universally consistent over $\mathcal{F}$.

Theorem (Varadarajan(1952?))

Let (S, d) be a sparable metric space and μ be any distribution (Borel probability measure) on S. Then the empirical measure μ_n converges to μ almost surely:

$$P(\{w : \mu_n(.)(w) \to \mu\}) = 1$$

Consistency of functional depth

Theorem (Consistency of functional band depth (Gijbels, Nagy(2015)))

$BD(x, P)$ is not uniformly consistent over compact subset of $\mathcal{C}$.

Possible remedy: smoothing with integration and decreasing function $w : [0, \infty) \to [0, 1], w(0) = 1, w(\infty) \to 0$
Adjusted band depth: $aBD(x, P) = Ew(inf_{y \in Band(X_1, \ldots, X_k)} ||x - y||)$ for all $x \in \mathcal{C}, P \in \mathcal{P}$. Then, aBD is universally consistent over $\mathcal{C}$.

Theorem (Consistency of functional infimal depth (Gijbels, Nagy(2015)))

$ID(x, P)$ is uniformly consistent over $\mathcal{C}$ for P
when P is mixture of P_1, P_2 s.t.

- *all marginal distribution of P_1 have continuous dist. functions.*
- *P_2 is concentrated in finite-dimensional subspace of $\mathcal{C}$.*

Note that the conditions are too restrictive. (Wiener measure fails to satisfy them.) And it means that $ID(x, P)$ is not universally consistent over $\mathcal{C}$.

Consistency of functional depth

Theorem (Consistency of functional integrated depth (Nagy, Gijbels, Omelka, Hlubinka(2016)))

$FD(x, P)$ is uniformly consistent over $\mathcal{C}$.

Note that, using the definition of integration, $\mathcal{C}$ can be extend to Borel-measurable (may be discontinuous) functions, include $\mathcal{L}^2$.

Consistency of functional depth: In practice

Theorem (Consistency over partial observability, (Nagy,Ferraty(2018))

Let $P \in \mathcal{P}$ on $\mathcal{L}^2[0,1]$ and $\tilde{P}_n$ be empirical distribution of fitted n curves. Then (under some assumptions,)

$$sup_{x \in \mathcal{L}^2} |D(x, \tilde{P}_n) - D(x, P)| \to 0$$

almost every x as $n \to \infty$ when D is adjust band depth type, h-depth type. If all marginal distribution of P is absolutely continuous, then also true for integrated depth type.

Proof:
step1: show $\tilde{P}_n \to P$ weakly almost every $\omega \in \Omega$ using Varadarajan theorem and with good property of fitting kernel.
step2: using inner D with good convergence property, show outer D converges weakly.

Consistency of functional depth: In practice

Theorem (convergence rate of FD (Nagy,Ferraty(2018)))

Let P_n be empirical distribution of (true) n curves, and $\tilde{P}_n$ be one of fitted n curves.
Suppose $P(|X(s) - X(t)| \leq L|s - t|^{\beta}) = 1$ for all $s, t \in [0, 1]$.
Then, for any $P \in \mathcal{P}$ on $\mathcal{L}^2[0, 1]$, under some conditions,

$$sup_{x \in \mathcal{L}^2[0,1]} |FD(x, P_n) - FD(x, P)| = O_p(n^{-1/2})$$

Moreover, if number of data points of n-th curve is comparable to n^r and $sup_{v \in [0,1]} sup_{|s-s'| \leq \epsilon} |F_{(v)}(s) - F_{(v)}(s')| \leq K\epsilon^{\alpha}$ for some $\alpha \in (0, 1]$ where $F_{(v)}$: marginal cdf of P at v, then under some conditions,

$$\begin{aligned} &sup_{x \in \mathcal{L}^2[0,1]} |FD(x, \tilde{P}_n) - FD(x, P)| \\ &= O_p(n^{-r\alpha\beta/\{(1+\alpha)(2\beta+1)\}}) \text{ if } r < (2\beta + 1)/\beta \\ &= O_p(\{ln(n)/n\}^{\alpha/(1+\alpha)}) \text{ if } r = (2\beta + 1)/\beta \\ &= O_p(n^{-\alpha/(1+\alpha)}) \text{ if } r > (2\beta + 1)/\beta \end{aligned}$$

Note that last case is dense setting, and the rate is similar to full observing case. In other cases, become slower.

Application

More need to develop this theory and related application,

- Robust and Nonparametric functional statistics
  procedure of with rank, nonparametric estimation of distribution, ...
- Exploratory Data Analysis (EDA)
  outlier finding, data detective works, ... (Center? Cluster? Symmetry? range(width)? gap(separation)? other irregularities?)
- classification if data can be classified by relation to the center.
- (and other things...)