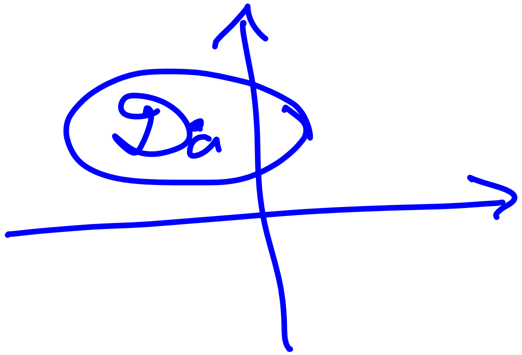
Outline

Canonical problem

$\mathcal{D}_a = \{x, \|x-c\|^2 = r^2\}$

$\mathcal{D}_a$ is bounded



$f(x) = \|x\|^2$ on $\mathbb{R}^2$

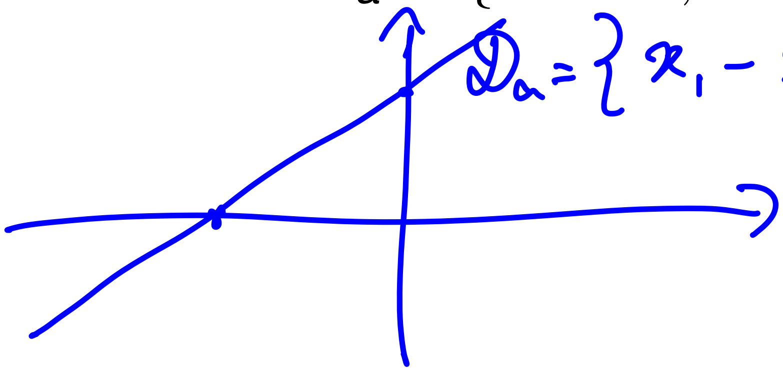
$c^E(x) = \|x-c\|^2 - r^2$

$$\inf_{\begin{cases} c^E(x) = 0 \\ c^I(x) \leq 0 \\ x \in \mathbb{R}^n \end{cases}} f(x) \quad \text{with} \quad \begin{cases} f : \mathbb{R}^n \longrightarrow \mathbb{R}, \\ c^E : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \\ c^I : \mathbb{R}^n \longrightarrow \mathbb{R}^p, \\ f, c, \text{ smooth.} \end{cases}$$

$$D_a = \{x \in \mathbb{R}^n,\ c^E(x) = 0,\ c^I(x) \leq 0\}$$

$\mathcal{D}_a = \{x_1 - x_2 = -1\}$ line

$\mathcal{D}_a$ is unbounded



General existence theorem

We consider f continuous from $C \subset \mathbb{R}^n$ into $\mathbb{R}$ with C closed.
If one of the following hypotheses is satisfied

- C bounded
- C not bounded and f coercive

$f(x) \longrightarrow +\infty$
$\|x\| \longrightarrow +\infty$

then f has a minimum on C

$c^E : \mathbb{R}^n \to \mathbb{R}^m$ $\qquad \nabla c_i^E = a_i^E \in \mathbb{R}^n$

$c^I : \mathbb{R}^n \to \mathbb{R}^p$

Notations for the gradient and the hessian of the i^{th} constraint

$$a_i^E(x) = \nabla c_i^E(x), \quad H_i^E(x) = Hess\ c_i^E(x),$$
$$a_i^I(x) = \nabla c_i^I(x), \quad H_i^I(x) = Hess\ c_i^I(x).$$

Jacobian matrices of the constraints :

$$A^E(x) = \nabla c^E(x) = \begin{pmatrix} a_1^E(x)^T \\ \vdots \\ a_m^E(x)^T \end{pmatrix}, \quad A^I(x) = \nabla c^I(x) = \begin{pmatrix} a_1^I(x)^T \\ \vdots \\ a_p^I(x)^T \end{pmatrix}.$$

$$JC^E = A^E = \underbrace{\begin{pmatrix} \nabla c_1^{E\,T} \\ \\ \nabla_m^{E\,T} \end{pmatrix}}_{n \text{ columns}} m \text{ rows}$$

$\ell : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$

Lagrangian and Lagrange multipliers (P)

$\inf f(x)$
$c^E(x) = 0$
$c^I(x) \leq 0$

Let y a vector of $\mathbb{R}^m$, z a vector of $\mathbb{R}^p$, Lagrange multipliers.
The Lagrangien is defined by associated to the problem (P)

$$\ell(x, y, z) = f(x) + y \cdot c^E(x) + z \cdot c^I(x)$$

$$= f(x) + \sum_{i=1}^{m} y_i\, c_i^E(x) + \sum_{j=1}^{p} z_j\, c_j^I(x)$$

The gradient and the hessian of the Lagrangien with respect to x are

$$g(x, y, z) = \nabla_x \ell(x, y, z) = \nabla f(x) + \sum_{i=1}^{m} y_i a_i^E(x) + \sum_{i=1}^{p} z_i a_i^I(x),$$

$$H(x, y, z) = Hess_x \ell(x, y, z) = Hf(x) + \sum_{i=1}^{m} y_i H_i^E(x) + \sum_{i=1}^{p} z_i H_i^I(x).$$

Example 1: $f : \mathbb{R}^2 \to \mathbb{R} \quad f(x) = x_1 + x_2, \inf_{x_1^2+x^2=2} f(x)$

$$\ell : \mathbb{R}^2 \times \mathbb{R} \longrightarrow \mathbb{R}$$

$$(x, y) \longrightarrow \ell(x,y) = x_1 + x_2 + y\,(x_1^2 + x_2^2 - 2)$$

Example 2: $f : \mathbb{R}^n \to \mathbb{R}$ $\quad f(x) = ||x||^2, \quad \inf_{\substack{x_{i+1} - x_i \leq 2 \\ i = 1, \dots, n-1}} f(x)$

$$\mathbb{R}^n \times \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$$

$$(x, z) \longrightarrow \ell(x,z) = ||x||^2 + z \cdot c^I(x) = ||x||^2 + \sum_{j=1}^{n-1} z_j (x_{j+1} - x_j - 2)$$

primal

$$\nabla_x \ell(x, z) = 2x + \sum_{j=1}^{n-1} z_j \nabla(x_{j+1} - x_j - 2)$$

$$= 2x + z_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \dots + z_j \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \dots + z_{n-1} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$= 2x + \begin{pmatrix} -1 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & -1 & 1 \end{pmatrix} z$$

n rows

n-1 columns

Examples

Actives constraints

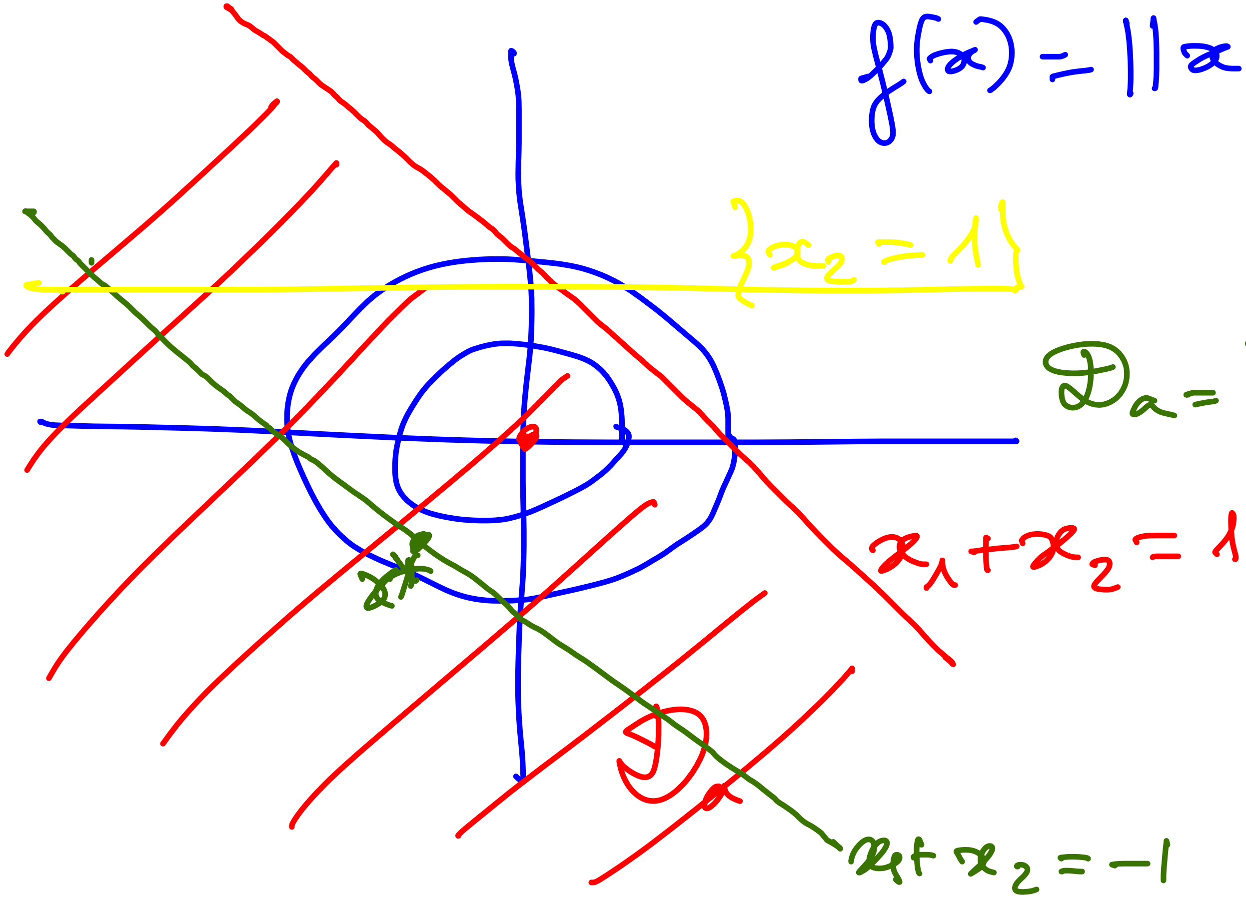
$(P) \inf f(x)$

$\mathcal{D}_a = \{c^E(x) = 0,\ c^I(x) \leq 0\}$

Let $x^\star$ a minimizer of f. on $\mathcal{D}_a$

The i^{th} inequality constraint is active if $c_i^I(x^\star) = 0$.

$f(x) = \|x\|^2$



$\mathcal{D}_a = \{x_1 + x_2 \leq 1\}$

$x^* = (0, 0)$

c^I is not active

$\mathcal{D}_a = \{x_1 + x_2 \leq -1\}$

$x^* = \left(\frac{-1}{2}, \frac{-1}{2}\right)$

c^I is active

$c^I(x) = \begin{pmatrix} x_1 + x_2 + 1 \\ x_2 - 1 \end{pmatrix}$

Actives constraints

Let $x^\star$ a minimizer of f.

The i^{th} inequality constraint is active if $c_i^I(x^\star) = 0$.

1. Exemple $f : \mathbb{R}^2 \to \mathbb{R} \quad f(x) = ||x||^2, \inf_{x_1+x_2 \leq 1} f(x)$

Actives constraints

Let $x^\star$ a minimizer of f.
The i^{th} inequality constraint is active if $c_i^I(x^\star) = 0$.

2. Exemple $f : \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x) = ||x||^2$, $\inf_{x_1+x_2\leq -1} f(x)$

proof of next slide

$$g(y,z) = \inf_{\mathcal{D}_a} f(x) + y \cdot c^E(x) + z \cdot c^I(x)$$

$$\leq \ell(x^*, y, z) = f(x^*) + \underbrace{z}_{\geq 0} \cdot \underbrace{c^I(x^*)}_{\leq 0}$$

$$\leq f(x^*) = p^*$$

Lagrange dual function

$\ell(x,y,z) = f(x) + y \cdot c^E(x) + z \cdot c^I(x)$

for each x

$g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$

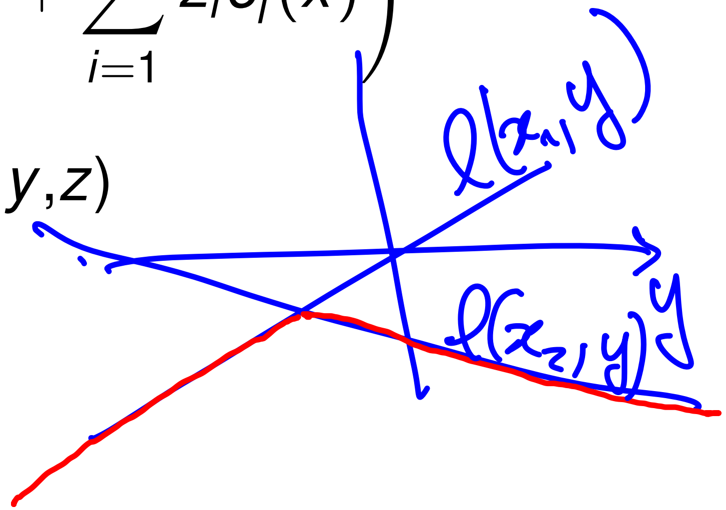
$(y,z) \to g(y,z)$ $\quad$ $(y,z) \to \ell(x,y,z)$ is affine

$$
\begin{aligned}
g(y,z) &= \inf_{x \in D_a} \ell(x,y,z) \\
&= \inf_{x \in D_a} \left(f(x) + \sum_{i=1}^{m} y_i c_i^E(x) + \sum_{i=1}^{p} z_i c_i^I(x) \right)
\end{aligned}
$$

g is concave (can be unbounded for some y,z)

Property : inferior bound:

If $z \geq 0$ then $g(y,z) \leq p^\star = \inf_{x \in D_a} f(x)$



Example : solution of a linear system with minimal norm

Solve

$$p^\star = \inf_{Ax=b} x^T x$$

$f(x) = x^T x = ||x||^2$

$c^E(x) = Ax - b$

$A \in \mathcal{M}_{m \times n}(\mathbb{R})$

$b \in \mathbb{R}^m$

- Lagrangian : $\ell(x, y) = x^T x + y^T(Ax - b)$

$= f(x) + y \cdot c^E(x)$

$g(y,z) = \inf_{x \in \mathbb{R}^n} \ell(x,y)$

$\nabla_x \ell(x,y) = 2x + A^T y$

$\nabla_x \ell(x,y) = 0$ when

$x = -\frac{1}{2} A^T y$

$g(y,z) \leq \frac{1}{4} ||A^T y||^2 + y \cdot \left(A\left(-\frac{1}{2}A^T y\right) - b\right)$

$= -\frac{1}{4} ||A^T y||^2 - \underline{y} \cdot b \leq p^\star$ for all y, z

Example : solution of a linear system with minimal norm

Solve

$$p^\star = \inf_{Ax=b} x^T x$$

- Lagrangian : $\ell(x,y) = x^T x + y^T (Ax - b)$
- In order to minimize $\ell(x,y)$ with respect to x we seek gradient zeros

$$\nabla_x \ell(x,y) = 2x + A^T y = 0 \;\Leftarrow\; x = -A^T y/2$$

Example : solution of a linear system with minimal norm

Solve

$$p^\star = \inf_{Ax=b} x^T x$$

- Lagrangian : $\ell(x, y) = x^T x + y^T (Ax - b)$
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- Inject in the definition of the dual function

$$g(y) = \ell(-A^T y/2, y) = -\frac{1}{4} y^T A A^T y - b^T y$$

concave in y

Example : solution of a linear system with minimal norm

Solve

$$p^\star = \inf_{Ax=b} x^T x$$

the primal is on $\mathbb{R}^n$

the dual is on $\mathbb{R}^m$

- Lagrangian : $\ell(x, y) = x^T x + y^T(Ax - b)$
- In order to minimize $\ell(x, y)$ with respect to x we seek gradient zeros

$$\nabla_x \ell(x, y) = 2x + A^T y = 0 \Leftarrow x = -A^T y/2$$

- Inject in the definition of the dual function

$$g(y) = \ell(-A^T y/2, y) = -\frac{1}{4} y^T A A^T y - b^T y$$

concave in y

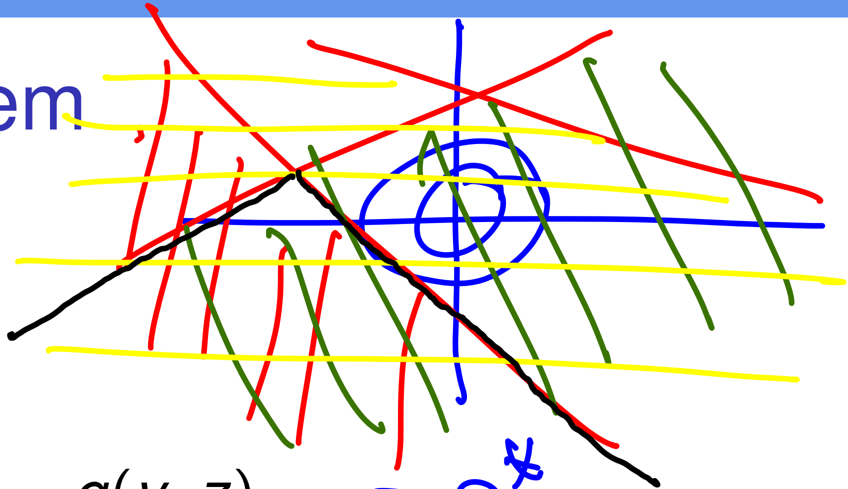
- Inferior bound property

$$\boxed{p^\star \geq -\frac{1}{4} y^T A A^T y - b^T y \; \forall y}$$

easier to solve this than p

Resolution of the dual problem

$$JC(x^*) = \begin{cases} \nabla c_1^T(x^*) \\ \vdots \\ \nabla c_p^T(x^*) \end{cases} \quad n=2$$



Solve

$$d^\star = \sup_{y\in\mathbb{R}^m, z\in\mathbb{R}^p, z\geq 0} g(y,z) = p^*$$

- Best inferior bound for $p^\star \geq d^\star$
- The dual problem is concave : existence of an optimal problem $d^\star$

Weak duality $d^\star \leq p^\star$ ← always true I.B.P.

$\boxed{\text{Strong duality } d^\star = p^\star}$

Qualified constraints : condition for $d^\star = p^\star$

$\max_{y,\, z \geq 0} g(y, z)$ $\inf_{x \in D_a} f(x)$

$JC(x^*) = \begin{matrix} JC^E(x^*) \,|\, m \text{ rows} \\ JC^I(x^*) \,|\, q \text{ rows} \end{matrix}$

- Slater condition for a convex problem f , C^I_i convex and C^E: $Ax - b$ affine : there exists an interior point in D_a. *i.e.* There exists x such that $C^E(x) = 0$ and $C^I_k < 0$ for all $k = 1, \cdots, p$.
- Linear independence constraint qualification : the rank of the matrix formed by the union of the Jacobian matrix of equality constraints and the Jacobian matrix of q constraints of active inequality in $x^\star$ is equal to $m + q$, then called maximal rank. $\leq n$

$C^I : \mathbb{R}^n \to \mathbb{R}^p$ at $C^I(x^*) = \begin{pmatrix} 0 \\ <0 \\ \vdots \\ \leq 0 \\ 0 \end{pmatrix}$ p constraints and $q \leq p$ are 0 at x^*

if $m > n$ $JC^E(x^*)$ cannot be max rank.

Case of equality constraints

$$\left\{ \begin{array}{ll} \inf & f(x) \\ s.c. & C(x) = 0 \\ & x \in \mathbb{R}^n \end{array} \right.$$

with

$$\begin{array}{ccl} f & : & \mathbb{R}^n \longrightarrow \mathbb{R}, \\ C & : & \mathbb{R}^n \longrightarrow \mathbb{R}^m, \\ f & C & \text{smooth.} \end{array}$$

Lagrange multipliers Theorem

equality constraint

Let f and C in C^1, and $x^\star$ a local minimizer of f satisfying

$$C(x^\star) = 0 \quad \text{primal feasability}$$

rank $\nabla C(x^*) = m$

If the constraints are qualified, there exists a vector of Lagrange multipliers $y^\star \in \mathbb{R}^m$ s. t. $\nabla \ell(x^*, y^*) = 0$

$$\nabla f(x^\star) + \sum_{i=1}^{m} y_i^\star \nabla C_i(x^\star) = 0 \quad \text{dual feasability}$$

- Linear constraints special case
- $n = 2$, $m = 1$ special case

Linear constraints special case

$c : \mathbb{R}^n \to \mathbb{R}^m$
$x \mapsto Cx$ with $C \in \mathcal{M}_{m \times n}(\mathbb{R})$

$$c(x) = (c_1 \cdot x, \ldots, c_m \cdot x)^{\perp}$$

$$\underline{(c_i)_{i=1,\ldots,m} \text{independant vectors family in } R^n} \quad m \leq n$$

$$\mathcal{D}_a \; \cancel{K} = \{x, c_i \cdot x = 0, i = 1, \ldots, m\} \quad \text{linear subspace}$$

$$\mathcal{D}_a = \text{span}((c_i)_{i=1,m})^{\perp}$$

$$\inf_{x \in \cancel{K}\, \mathcal{D}_a} f(x) \Leftrightarrow \inf_{\alpha \in \mathbb{R}^p} g(\alpha)$$

$$\text{with } g(\alpha) = f\left(\sum_{i=1}^{p} \alpha_i k_i\right), \; (k_i)_{i=1,\ldots,p}, \text{ basis of } \cancel{K}\, \mathcal{D}_a$$

$p = n - m$

$\inf_{\alpha \in \mathbb{R}^p} g(\alpha)$ is unconstrained minimization in $\mathbb{R}^p$ $\leftarrow \nabla g(\alpha) = 0$ $\qquad g(\alpha) = f\left(\sum_{i=1}^{p} \alpha_i k_i\right)$

$$\nabla g(\alpha) = \left(\nabla f\left(\sum \alpha_i k_i\right) \cdot k_i\right)_{i=1,\dots p}$$

$$\Rightarrow \nabla f(x^*) \cdot k_i = 0 \quad \text{for } i=1,\dots p$$

$$\text{span } k_i = \mathcal{D}_a \quad \Rightarrow \nabla f(x^*) \in \mathcal{D}_a^{\perp}$$

$$\mathcal{D}_a = \{(c_i)_i\}^{\perp} \quad \Rightarrow \nabla f(x^*) \in \text{span}\{(c_i)_{i=1,\dots m}\}$$

$$\Rightarrow \exists \lambda_i \quad \nabla f(x^*) = \sum \lambda_i c_i$$

$$y_i^* = -\lambda_i \qquad \nabla f(x^*) + \sum y_i^* c_i = 0$$

$$\nabla \ell(x^*, y^*) = 0$$

Special case $n = 2$, $m = 1$

Qualification condition for one single constraint $m = 1$:
$\nabla_x c_1(x^\star) \neq 0$, we can suppose $\partial_{x_2} c_1(x^\star) \neq 0$.
Implicit function theorem : $\exists V_1 \times V_2$ containing $x^\star$ and φ unique and differentiable in $x^\star$ s. t. $\forall x_1 \in V_1$ $c_1([x_1, \varphi(x_1)]) = 0$ and $x_2^\star = \varphi(x_1^\star)$ with

$$\varphi'(x_1) = \frac{-1}{\partial_{x_2} c_1(x)} \partial_{x_1} c_1(x).$$

Proof

$$\inf_{c_1(x)=0} f(x) \quad \Leftrightarrow \quad \inf_{x_1 \in V_1} \tilde{f}(x_1), \qquad \text{with } \tilde{f}(x_1) = f([x_1, \varphi(x_1)])$$

First order optimality conditions for $\tilde{f}$ (without constraints since V_1 is an open set)

$$\tilde{f}'(x_1^\star) = 0 \Leftrightarrow \frac{\partial f}{\partial x_1}([x_1^\star, \varphi(x_1^\star)]) + \varphi'(x_1^\star)\frac{\partial f}{\partial x_2}([x_1^\star, \varphi(x_1^\star)]) = 0.$$

$$y = -\frac{\partial_{x_2} f(x^\star)}{\partial_{x_2} c_1(x^\star)}$$

Example 1

$\mathcal{D}_a$ = circle of radius 1, center 0,0

$$\inf_{x_1^2+x_2^2=1} x_1^4 + x_2^4.$$

Resolution by changing variables in polar coordinates

Set $x_1 = \cos(\theta)$, $x_2 = \sin(\theta)$, problem (4) becomes $\inf_{\theta\in[0,2\pi]}(\cos\theta^4 + \sin\theta^4)$ whose solution is obtained by finding the zero of the derivative:

$$4\cos\theta\sin\theta(-\cos\theta^2 + \sin\theta^2) = -2\sin(2\theta)\cos(2\theta) = 0,$$

4 local minima $(\pm\sqrt{2}/2, \pm\sqrt{2}/2)$, where $f(x) = 1/2$,
4 local maxima $\{(1,0),(0,1),(-1,0),(0,-1))\}$, where $f(x) = 1$.

Resolution using Lagrange multipliers

$$\ell : \mathbb{R}^2 \times \mathbb{R} \longrightarrow \mathbb{R}$$
$$(x, y) \longmapsto x_1^4 + x_2^4 + y\,(x_1^2 + x_2^2 - 1)$$

We seek $x^\star \in \mathbb{R}^2$ and $y^\star \in \mathbb{R}$ s. t.

$C(x) = x_1^2 + x_2^2 - 1$

$JC(x^*) = 2x^*$

rank $JC(x^*) = 1$

because $(0,0) \notin \mathcal{D}_a$

$$C(x^*) = 0 \Leftarrow \quad (x_1^\star)^2 + (x_2^\star)^2 = 1$$

$$\left.\begin{aligned} 4(x_1^\star)^3 + y^\star 2x_1^\star &= 0 \\ 4(x_2^\star)^3 + y^\star 2x_2^\star &= 0 \end{aligned}\right\}$$

$$\nabla_x \ell(x, y) = \begin{pmatrix} 4x_1^3 \\ 4x_2^3 \end{pmatrix} + y \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Resolution using Lagrange multipliers

We seek $x^\star \in \mathbb{R}^2$ and $y^\star \in \mathbb{R}$ s. t.

$$\begin{aligned}
(x_1^\star)^2 + (x_2^\star)^2 &= 1\\
4(x_1^\star)^3 + y^\star 2x_1^\star &= 0\\
4(x_2^\star)^3 + y^\star 2x_2^\star &= 0
\end{aligned}$$

Resolution using Lagrange multipliers

We seek $x^\star \in \mathbb{R}^2$ and $y^\star \in \mathbb{R}$ s. t.

$$\begin{aligned}
& (x_1^\star)^2 + (x_2^\star)^2 &=& \quad 1 \\
2x_1(2x_1^2 + y) = \; & 4(x_1^\star)^3 + y^\star 2x_1^\star &=& \quad 0 \\
2x_2(2x_2^2 + y) = \; & 4(x_2^\star)^3 + y^\star 2x_2^\star &=& \quad 0
\end{aligned}$$

	$x_1^\star = 0$	$y^\star = -2(x_1^\star)^2$
$x_2^\star = 0$	$(x_1^\star)^2 + (x_2^\star)^2 \neq 1$	$(x_1^\star)^2 = 1$ et $y^\star = -2$ $f(x^\star) = 1$
$y^\star = -2(x_2^\star)^2$	$(x_2^\star)^2 = 1$ et $y^\star = -2$ $f(x^\star) = 1$	$(x_1^\star)^2 = (x_2^\star)^2 = 1/2$ et $y^\star = -1$, $f(x^\star) = 1/2$

Second order optimality conditions

$$\nabla_x \ell(x^*, y^*) = 0_{\mathbb{R}^n}$$

Let f and c in C^2, and $x^\star$ be a local minimizer of f verifying the constraints of equality $c(x^\star) = 0$. If the constraints are qualified, there exists a vector of Lagrange multipliers $y^\star \in \mathbb{R}^m$ such that

$$s \cdot H(x^\star, y^\star)s \geq 0 \quad \text{for all } s \in \mathcal{N}$$

where

$$\mathcal{N} = \{s \in \mathbb{R}^n, A(x^\star)s = 0\}.$$

$$A = JC(x^*)$$

$$H(x^*, y^*) = Hf(x^*) + \sum_{i=1}^{m} y_i^* HC_i^E(x^*)$$

Interpretation of Lagrange multipliers

The Lagrange multiplier y_i measures the sensitivity of the minimum $x^\star$ with respect to the corresponding constraint.

Initial primal and dual problems

$\inf_{c(x)=0} f(x)$	$\sup_y g(y)$ avec $g(y) = \inf_x f(x) + y^t c(x)$

Perturbated primal and dual problems

$\inf_{c(x)=\varepsilon} f(x)$	$\sup_y g(y) - \varepsilon^T y$

- x is the primal variable, ε a parameter
- $p^\star(\varepsilon)$ the optimal value when ε varies

Global interpretation of Lagrange multipliers

Hyp: strong duality for the undisturbed problem, that is $y^\star$ t.q. $g(y^\star) = d^\star = p^\star(0)$

For the perturbated problem we have

$$\begin{aligned} p^\star(\varepsilon) &\geq \max_y g(y) - \varepsilon^T y \\ &\geq g(y^\star) - \varepsilon^T y^\star \\ &\geq p^\star(0) - \varepsilon^T y^\star \end{aligned}$$

hence

- if $y_i^\star > 0$ and large, $p^\star$ increases a lot if $\varepsilon_i < 0$
- if $y_i^\star < 0$ and large, $p^\star$ disminishes a lot if $\varepsilon_i > 0$

Local interpretation of Lagrange multipliers

$\mathbb{R}^n = \text{span}(e_i)$

$e_i = (\delta_{ij})_{j=1,\dots n}$

$p^*(\varepsilon) = \inf_{C(x)=\varepsilon e_i} f(x)$

$$y_i^\star = -\frac{\partial p^\star(0)}{\partial \varepsilon_i}$$

Proof : $\varepsilon = te_i$ in the global sensitivity
$p^\star(te_i) \geq p^\star(0) - ty_i^\star$

$$\lim_{t \searrow 0} \frac{p^\star(te_i) - p^\star(0)}{t} \geq -y_i^\star$$

$$\lim_{t \nearrow 0} \frac{p^\star(te_i) - p^\star(0)}{t} \leq -y_i^\star$$

Example 2 : Diagonalization of a symetric matrix

$\ell : \mathbb{R}^2 \times \mathbb{R} \longrightarrow \mathbb{R}$ $\quad \ell(x, y) = Ax.x + y\,(\|x\|^2 - 1)$

$$\inf_{\|x\|=1} Ax \cdot x$$

$A \in S^n$

with A a symetric matrix in $\cancel{\mathbb{R}^{n\times n}}$.

$$\inf_{c(x)=0} f(x) \quad \text{with } f(x) = Ax \cdot x \text{ and } c(x) = \|x\|^2 - 1$$

ensure differentiability

Existence of a minimum since f is continuous and $\{x, \|x\| = 1\}$ bounded closed set.

f differentiable and $\{c(x) = 0\}$ Lagrange multipliers $\Rightarrow \exists y^\star \in \mathbb{R}$ s.t. $2Ax^\star + 2y^\star x^\star = 0$ $\quad \lambda = -y^*$ is a real eigenvalue

$\Rightarrow \exists(\lambda, v) \in \mathbb{R} \times \mathbb{R}^n$, $Av = \lambda v$ and $f(v) = \inf_{\|x\|=1} f(x)$.

Induction hypothesis H_n : existence of an orthonormal eigenvector basis of A with n related eigenvalues

$n = 1$ easy

Suppose H_n true

We know that there is one eigenvalue

$\nabla \ell(x^*, y^*) = 0 \qquad A x^* = -y^* x^*$

For $A \in \mathbb{R}^{n+1 \times n+1}$ we consider the subspace $H = \{\text{vect}(x^\star)\}^\perp$.
dim $H = n$
H is stable by A. Indeed

$$\text{if } x^\star \cdot x = 0 \text{ then } x^\star \cdot Ax = Ax^\star \cdot x = -y^\star x^\star \cdot x = 0$$

The restriction of A to H is a matrix $n \times n$ therefore using H_n existence of a orthonormal eigenvector basis of the restriction of A to H.
We divide $x^\star$ by $\|x^\star\|$ in order to complete this basis on $\mathbb{R}^{n+1}$.

Example 3 : Minimization of a quadratic function under linear constraints of equality

$$f(x) = \frac{1}{2}Ax\cdot x + b\cdot x$$
$$c(x) = Bx - C$$

symmetric definite positive.

with A ~~defined symetric positive~~ matrix in $\mathbb{R}^{n\times n}$, b vector in $\mathbb{R}^n$, B matrix in $\mathbb{R}^{m\times n}$ and C vector in $\mathbb{R}^m$.

Qualified constraints $\Leftrightarrow$ rank$(B) = m$.

Lagrangian : $\ell : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$

$$\ell(x,y) = \underbrace{\frac{1}{2}Ax\cdot x + b\cdot x}_{f(x)} + \underbrace{y\cdot(Bx - C)}_{y\cdot c(x)}$$

Theorem of Lagrange multipliers

$y \in \mathbb{R}^m$ $\quad \ell(x,y) = \frac{1}{2} Ax \cdot x + (Bx - C) \cdot y$

$A \in S_n^{++}$

$$\nabla_x \ell(x,y) = \quad Ax + b + B^t y = 0$$
$$Bx = C$$

A defined symetric positive matrix $\Rightarrow x = -A^{-1}(b + B^t y)$.

$B(-A^{-1}(b + B^t y)) = C$ $\quad - BA^{-1}b - BA^{-1}B^t y = C$

rank$(B) = m \Rightarrow BA^{-1}B^t$ is invertible

$BA^{-1}B^t y = -(BA^{-1}b + C)$ from which we get y then x.

$y = -(BA^{-1}B^t)^{-1}(BA^{-1}b + C)$

Application

write f as a quadratic function

$f(x) = \frac{1}{2} A x \cdot x + b \cdot x$

Find

$$\inf_{C(x)=0} f(x) \quad \text{with} \begin{cases} f(x) = 3x_1^2 + 5x_2^2 - 3x_1 x_2, \\ C(x) = x_1 + x_2 - 1 \end{cases}$$

- Lagrangian $\ell(x, y) = 3x_1^2 + 5x_2^2 - 3x_1 x_2 + y(x_1 + x_2 - 1)$

to check your computations:

$x_2 = 1 - x_1$

$g(x) = f(x, 1 - x)$